Journal of Automation and Information Sciences, 33(7), 2001

Finite-Difference Strong Numerical Methods of Order 1.5 and 2.0 for Stochastic Differential Ito Equations with Nonadditive Multidimensional Noise

D.F. KUZNETSOV

The paper is devoted to construction of strong numerical methods of the Runge-Kutta type of order 1.5 and 2.0 for Ito stochastic differential equations. Explicit and implicit one-step and implicit two-step strong numerical schemes are presented.

Key words: stochastic differential equation, numerical methods, Taylor-Ito expansion, finite differences, Runge-Kutta method, explicit scheme, implicit scheme.

Introduction

Let $(\Omega, \mathfrak{F}, \mathrm{P})$ be a complete probabilistic space; $\{\mathfrak{F}_t\}_{t\in[0,T]}$ is a non-decreasing continuous from the right family of σ-subalgebras $\mathfrak{F}$; $\{W_t, t \in [0, T]\}$ is an m-dimensional $\mathfrak{F}_t$-measurable for each $t \in [0, T]$ standard Wiener process with independent components $W_t^{(i)}$ $(i = 1, \ldots, m)$.

Consider an Ito stochastic differential equation of the form

$$d\mathrm{x}_t = \mathrm{a}(\mathrm{x}_t, t)\,dt + \Sigma(\mathrm{x}_t, t)\,dW_t,\ \mathrm{x}_0 = \mathrm{x}(0, \omega), \tag{1}$$

where $\mathrm{x}_t \in \mathfrak{R}^n$ is a solution of equation (1); $\mathrm{a}(\mathrm{x}, t) : \mathfrak{R}^n \times [0, T] \rightarrow \mathfrak{R}^n$, $\Sigma(\mathrm{x}, t) : \mathfrak{R}^n \times [0, T] \rightarrow \mathfrak{R}^{n\times m}$ are non-random functions that together with $\mathrm{x}_0 \in \mathfrak{R}^n$ satisfy standard conditions [1] of existence and uniqueness of solution of (1) in the meaning of stochastic equivalence.

One of known approaches to construction of numerical methods for Ito stochastic differential equations is based on the Taylor-Ito expansion of their solutions [2]. For their realization, numerical methods based on this expansion require calculation of partial derivatives of different order of functions $\mathrm{a}(\mathrm{x}, t)$ and $\Sigma(\mathrm{x}, t)$ at each step of integration. Calculation of these derivatives requires additional calculation effort which results in inconvenience when numerical methods are used. Note that this was the reason due to which numerical methods for ordinary differential equations, based directly on the Taylor formula, have not been widely used in practice. Instead of them, as a rule, finite-difference numerical methods of the Runge-Kutta type are used. They do not require calculation of partial derivatives of the right-hand side of an ordinary differential equation.

Authors of some works (see, for instance, [2]), theoretically and practically demonstrated that heuristic generalizations of well-familiar numerical methods of the Runge-Kutta type as a rule appear low-efficient for stochastic differential equations or do not converge to solutions of these equations at all. The cause of this lies in essential distinctions between rules of differentiation of complex function in deterministic and stochastic calculi. In other words, a number of terms is absent in formulae of numerical methods for ordinary differential equations. However, taking them into account is obligatory for attaining required degree of accuracy of a numerical method as applied to a stochastic differential equation.

Thus a necessity arises to construct "correct", i.e., taking into account the rule of differentiating of complex function in stochastic calculation (the Ito formula), finite-difference numerical methods of the Runge-Kutta type for Ito stochastic differential equations.

ISSN 1064-2315

Consider a partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0, T]$, for which $0 = \tau_0 < \tau_1 < \ldots < \tau_N = T$; $\Delta_N = \max_{0 \le p \le N-1} |\tau_{p+1} - \tau_p|$.

Discrete approximation y_{τ_p}, $p = 0, 1, \ldots, N$, of solution x_t, $t \in [0, T]$, to equation (1) corresponding to Δ_N is called [2] to be strongly converging with the order $\gamma > 0$ at the time instant T to the process x_t, $t \in [0, T]$, if there exists a constant $C > 0$ independent of Δ_N and a number $\delta_0 > 0$ such, that $M\{|x_T - y_T|\} \le C\,\Delta_N^{\gamma}\ \forall\, \Delta_N \in (0, \delta_0)$.

Strong numerical methods (discrete approximations) for Ito stochastic differential equations are applied [2] in solutions of problems of constructing new mathematical models of the type (1), in Markov chains filtering with continuous time and a finite discrete state space, in testing procedures of parameter estimation of dynamic systems described by equations of the type (1) etc.

This paper is devoted to derivation of finite-difference strong numerical methods of the Runge-Kutta type of orders of accuracy 1.5 and 2.0 for (1). At that we present explicit and implicit one-step and also implicit two-step numerical schemes.

Taylor approximations of partial derivatives of deterministic functions

Let $f(x, t), g(x, t) : \mathfrak{R}^n \times [0, T] \to \mathfrak{R}^n$ and function f is twice continuously differentiable with respect to x and it is continuously differentiable with respect to t, and g is bounded. Then for any $\Delta > 0$ we obtain by Taylor formula

$$\Delta \sum_{i=1}^{n} g^{(i)} f'_i = \sum_{i=1}^{2} \pi_i f(x + \sigma_i \Delta g, t) + O(\Delta^2), \tag{2}$$

$$\Delta \sum_{i=1}^{n} g^{(i)} f'_i = \sum_{i=1}^{2} \alpha_i f(x + \beta_i \Delta g, t) + O(\Delta^3), \tag{3}$$

$$\Delta \sum_{i=1}^{n} g^{(i)} f'_i = \sum_{i=1}^{4} \eta_i f(x + \omega_i \Delta g, t) + O(\Delta^4), \tag{4}$$

$$\Delta^2 \sum_{i,j=1}^{n} g^{(i)} g^{(j)} f''_{ij} = \sum_{i=1}^{3} \varphi_i f(x + \psi_i \Delta g, t) + O(\Delta^3), \tag{5}$$

$$\Delta^2 \sum_{i,j=1}^{n} g^{(i)} g^{(j)} f''_{ij} = \sum_{i=1}^{4} \mu_i f(x + \lambda_i \Delta g, t) + O(\Delta^4), \tag{6}$$

$$\Delta \left(\dot{f} + \sum_{i=1}^{n} g^{(i)} f'_i \right) = \sum_{i=1}^{2} \chi_i f(x + \delta_i \Delta g, t + \gamma_i \Delta) + O(\Delta^2), \tag{7}$$

$$2f + \Delta \left(\dot{f} + \sum_{i=1}^{n} g^{(i)} f'_i \right) = \sum_{i=1}^{2} \rho_i f(x + \nu_i \Delta g, t + \xi_i \Delta) + O(\Delta^2), \tag{8}$$

where $u(x, t) \stackrel{\text{def}}{=} u$; $u^{(i)}(x, t) \stackrel{\text{def}}{=} u^{(i)}$; $\frac{\partial f}{\partial t} \stackrel{\text{def}}{=} \dot{f}$; $\frac{\partial f}{\partial x^{(i)}} \stackrel{\text{def}}{=} f'_i$; $\frac{\partial^2 f}{\partial x^{(i)} \partial x^{(j)}} \stackrel{\text{def}}{=} f''_{ij}$; $u^{(i)}$ is the i-th component of u (u denotes any of functions f, g); coefficients $\pi_i, \sigma_i, \ldots, \xi_i$ satisfy the following systems of equations:

$$\begin{cases} \sum_{i=1}^{3} \mu_i = 0, \\ \sum_{i=1}^{3} \mu_i \lambda_i = 0, \\ \sum_{i=1}^{3} \mu_i \frac{\lambda_i^2}{2} = 1, \\ \sum_{i=1}^{3} \mu_i \lambda_i^3 = 0, \end{cases} \quad \begin{cases} \sum_{i=1}^{4} \eta_i = 0, \\ \sum_{i=1}^{4} \eta_i \omega_i = 1, \\ \sum_{i=1}^{4} \eta_i \omega_i^2 = 0, \\ \sum_{i=1}^{4} \eta_i \omega_i^3 = 0, \end{cases} \quad \begin{cases} \sum_{i=1}^{2} \alpha_i = 0, \\ \sum_{i=1}^{2} \alpha_i \beta_i = 1, \\ \sum_{i=1}^{2} \alpha_i \beta_i^2 = 0, \end{cases} \tag{9}$$

$$\begin{cases} \sum_{i=1}^{2} \chi_i = 0, \\ \sum_{i=1}^{2} \chi_i \gamma_i = 1, \\ \sum_{i=1}^{2} \chi_i \delta_i = 1, \end{cases} \quad \begin{cases} \sum_{i=1}^{2} \rho_i = 2, \\ \sum_{i=1}^{2} \xi_i \rho_i = 1, \\ \sum_{i=1}^{2} \nu_i \rho_i = 1, \end{cases} \quad \begin{cases} \sum_{i=1}^{3} \varphi_i = 0, \\ \sum_{i=1}^{3} \varphi_i \psi_i = 0, \\ \sum_{i=1}^{3} \varphi_i \frac{\psi_i^2}{2} = 1, \end{cases} \tag{10}$$

$$\begin{cases} \sum_{i=1}^{2} \pi_i = 0, \\ \sum_{i=1}^{2} \pi_i \sigma_i = 1. \end{cases} \tag{11}$$

It is easy to check up that one of possible solutions of systems (9)–(11) is the following one: $\alpha_1 = -\alpha_2 = 1$, $\beta_1 = -\beta_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = 1$, $\mu_3 = -2$, $\lambda_1 = -\lambda_2 = 1$, $\lambda_3 = 0$, $\eta_1 = -\eta_2 = \frac{1}{48}$, $\eta_3 = -\eta_4 = \frac{27}{48}$, $\omega_1 = -\omega_2 = -3$, $\omega_3 = -\omega_4 = 1$, $\rho_1 = \rho_2 = 1$, $\xi_1 = 0$, $\xi_2 = 1$, $\nu_1 = 0$, $\nu_2 = 1$, $\chi_1 = -\chi_2 = 1$, $\gamma_1 = -\gamma_2 = \frac{1}{2}$, $\delta_1 = -\delta_2 = \frac{1}{2}$, $\pi_1 = -\pi_2 = \frac{1}{2}$, $\sigma_1 = 2$, $\sigma_2 = 0$, $\varphi_1 = \varphi_2 = 1$, $\varphi_3 = -2$, $\psi_1 = 0$, $\psi_2 = 2$, $\psi_3 = 1$.

From this point onwards, in numerical experiments we shall use only this solution of systems (9)–(11).

Explicit one-step strong finite-difference methods of orders of accuracy 1.5 and 2.0

Introduce notations:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathbf{a}^{(i)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(i)}} + \frac{1}{2} \sum_{j=1}^{m} \sum_{l,i=1}^{n} \Sigma^{(lj)}(\mathbf{x}, t) \Sigma^{(ij)}(\mathbf{x}, t) \frac{\partial^2}{\partial \mathbf{x}^{(l)} \partial \mathbf{x}^{(i)}},$$

$$G_0^{(i)} = \sum_{j=1}^{n} \Sigma^{(ji)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(j)}},\ i = 1, \ldots, m,$$

where $\mathbf{a}(\mathbf{x}, t) = \| \mathbf{a}^{(i)}(\mathbf{x}, t) \|_{i=1}^{n}$; $\Sigma(\mathbf{x}, t) = \| \Sigma^{(ij)}(\mathbf{x}, t) \|_{i,j=1}^{n,m}$.

Denote by $I_{l_1 \ldots l_k\, s,t}^{(i_1 \ldots i_k)}$ ($s > t$; $i_1, \ldots, i_k = 1, \ldots, m$; $l_1, \ldots, l_k = 0, 1, 2, \ldots$; $k \geqslant 1$) the repeated stochastic Ito integral of the form

$$I_{l_1 \ldots l_k\ s,t}^{(i_1 \ldots i_k)} = \int_t^s (t - t_k)^{l_k} \ldots \int_t^{t_2} (t - t_1)^{l_1}\, dW_{t_1}^{(i_1)} \ldots dW_{t_k}^{(i_k)},$$

and denote its approximation by $\hat{I}_{l_1 \ldots l_k\ s,t}^{(i_1 \ldots i_k)}$.

Consider first an auxiliary explicit one-step strong numerical scheme of order of accuracy 1.0 based on the unified Taylor-Ito expansion [3]:

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} \Sigma_{i_1} \hat{I}_{0\ \tau_{p+1},\tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{00\ \tau_{p+1},\tau_p}^{(i_2 i_1)} +$$

$$+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{0\ \tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{1\ \tau_{p+1},\tau_p}^{(i_1)} \right) - L \Sigma_{i_1} \hat{I}_{1\ \tau_{p+1},\tau_p}^{(i_1)} \right] + \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{000\ \tau_{p+1},\tau_p}^{(i_3 i_2 i_1)} . \tag{12}$$

Here and further $\mathbf{y}_{\tau_p} \overset{\text{def}}{=} \mathbf{y}_p$; $\tau_p = p\Delta$; $\Sigma_i(\mathbf{x}, t)$ is the i-th column of matrix $\Sigma(\mathbf{x}, t)$; the functions Σ_{i_1}, $\mathbf{a}, \ldots, G_0^{(i_3)} G_0^{(i_2)} \Sigma_{i_1}$ in the right-hand side of (12) are calculated at the point $(\mathbf{y}_p, \tau_p)$.

Despite the right-hand side of (12) contains terms of $\Delta^{\frac{1}{2}}$, Δ and $\Delta^{\frac{3}{2}}$ orders of smallness in the mean-square sense at $\Delta \to 0$, the numerical method (12) has strong order of accuracy 1.0, not 1.5 as it might have appeared at first sight. This conclusion follows from [2, Theorem 11.5.2, Pages 391–392] and is confirmed by the numerical experiment 1 (see below).

Put $\mathbf{y}_{pr}^{(ji)} = \mathbf{y}_p + \sqrt{\Delta}\, r_j \Sigma_i(\mathbf{y}_p, \tau_p)$; $\mathbf{z}_{pr}^{(j)} = \mathbf{y}_p + \Delta r_j \mathbf{a}(\mathbf{y}_p, \tau_p)$; $\mathbf{u}_{pr}^{(ji)} = \mathbf{y}_p + \Delta r_j \Sigma_i(\mathbf{y}_p, \tau_p)$; $\Delta_{pr}^{(j)} = \tau_p + r_j \Delta$, where r_j denotes any coefficient appearing in the systems (9)–(11).

Using formulae (2)–(8), let us construct on the basis of numerical scheme (12) the following auxiliary explicit one-step strong finite-difference method of order of accuracy 1.0:

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \Delta \mathbf{a}(\mathbf{y}_p, \tau_p) + \sum_{i_1=1}^{m} \Sigma_{i_1}(\mathbf{y}_p, \tau_p) \hat{I}_{0\ \tau_{p+1},\tau_p}^{(i_1)} +$$

$$+ \frac{1}{\sqrt{\Delta}} \sum_{i_1,i_2=1}^{m} \sum_{j=1}^{2} \alpha_j \Sigma_{i_1}(\mathbf{y}_{p\beta}^{(ji_2)}, \tau_p) \hat{I}_{00\ \tau_{p+1},\tau_p}^{(i_2 i_1)} +$$

$$+ \frac{1}{\Delta} \sum_{i_1=1}^{m} \left[\sum_{j=1}^{2} \pi_j \mathbf{a}(\mathbf{u}_{p\sigma}^{(ji_1)}, \tau_p) \left(\Delta \hat{I}_{0\ \tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{1\ \tau_{p+1},\tau_p}^{(i_1)} \right) - \right. \tag{13}$$

$$\left. - \left(\frac{1}{2} \sum_{r=1}^{m} \sum_{j=1}^{3} \varphi_j \Sigma_{i_1}(\mathbf{y}_{p\psi}^{(jr)}, \tau_p) + \sum_{j=1}^{2} \chi_j \Sigma_{i_1}(\mathbf{z}_{p\delta}^{(j)}, \Delta_{p\gamma}^{(j)}) \right) \hat{I}_{1\ \tau_{p+1},\tau_p}^{(i_1)} \right] +$$

$$+ \frac{1}{\Delta} \sum_{i_1,i_2,i_3=1}^{m} \sum_{j=1}^{2} \alpha_j \mathbf{k}_{i_1 i_2}(\mathbf{y}_{p\beta}^{(ji_3)}, \tau_p) \hat{I}_{000\ \tau_{p+1},\tau_p}^{(i_3 i_2 i_1)} ,$$

where

$$\mathbf{k}_{i_1 i_2}(\mathbf{y}_p, \tau_p) = \sum_{j=1}^{2} \pi_j \Sigma_{i_1}(\mathbf{y}_{p\sigma}^{(ji_2)}, \tau_p). \tag{14}$$

By making use of Taylor formula the right-hand side of (13) can be transformed to a representation that differs from the right-hand side of (12) by a quantity of higher order of smallness, in the mean square sense, compared to $\Delta^{\frac{3}{2}}$. Then, strong convergence of order of accuracy 1.0 of the numerical scheme (13) can be substantiated using [2, Theorem 11.5.2, Pages 391–392] or [4, Theorem 5.4, Pages 293–294].

Using standard relationships between Ito and Stratonovich repeated stochastic integrals [2] we obtain

$$I_{0\,\tau_{p+1},\tau_p}^{(i_1)} = I_{0\,\tau_{p+1},\tau_p}^{*(i_1)}, \quad I_{1\,\tau_{p+1},\tau_p}^{(i_1)} = I_{1\,\tau_{p+1},\tau_p}^{*(i_1)} \quad \text{almost surely (a. s.)}, \tag{15}$$

$$I_{00\,\tau_{p+1},\tau_p}^{(i_2 i_1)} = I_{00\,\tau_{p+1},\tau_p}^{*(i_2 i_1)} - \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\Delta \quad \text{a. s.}, \tag{16}$$

$$I_{000\,\tau_{p+1},\tau_p}^{(i_3 i_2 i_1)} = I_{000\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)} + \frac{1}{2}\mathbf{1}_{\{i_3=i_2\}} I_{1\,\tau_{p+1},\tau_p}^{*(i_1)} - \frac{1}{2}\mathbf{1}_{\{i_2=i_1\}}\left(\Delta I_{0\,\tau_{p+1},\tau_p}^{*(i_3)} + I_{1\,\tau_{p+1},\tau_p}^{*(i_3)}\right) \quad \text{a. s.}, \tag{17}$$

where

$$I_{l_1\ldots l_k\,s,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*s} (t-t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t-t_1)^{l_1}\, dW_{t_1}^{(i_1)} \ldots dW_{t_k}^{(i_k)};$$

$s > t$; $i_1, \ldots, i_k = 1, \ldots, m$; $l_1, \ldots, l_k = 0, 1, 2, \ldots$; $k \geqslant 1$; $\int^{*}$ denotes Stratonovich integral; $\mathbf{1}_A$ is the indicator of the set A.

The stochastic integrals in the right-hand sides of (15)–(17) can be approximated by the method that makes use of multiple Fourier series in Legendre polynomials [4]. In the result, using (15)–(17) we obtain approximations of repeated Ito stochastic integrals appearing in (12), (13).

Consider an Ito stochastic differential equation of the form

$$dx_t = ax_t\,dt + bx_t\,dw_t, \tag{18}$$

where a, b are constants, w_t is a standard scalar Wiener process. It is known [2] that the solution of equation (18) at $t = T$ has the form

$$x_T = x_0 e^{\left(a-\frac{1}{2}b^2\right)T + bw_T}. \tag{19}$$

Let us test numerical method (12) by the scheme suggested in [2]. This scheme [2] consists in calculation of estimate $\hat{\varepsilon}_{L,M}$ of error $\varepsilon = \mathrm{M}\{|\mathrm{x}_T - \mathrm{y}_T|\}$ by formula

$$\hat{\varepsilon}_{L,M} = \frac{1}{LM}\sum_{k=1}^{L}\sum_{j=1}^{M} |\mathrm{x}_T^{(k,j)} - \mathrm{y}_T^{\Delta(k,j)}| \tag{20}$$

for different values of the integration step Δ; $\mathrm{y}_T^{\Delta(k,j)}$ ($k = 1, \ldots, L$, $j = 1, \ldots, M$) are independent realizations of random variable $\mathrm{y}_T = \mathrm{y}_T^{\Delta}$ obtained by the studied numerical method at constant integration step Δ. The random variable is the approximate value of x_T; $\mathrm{x}_T^{(k,j)}$ ($k = 1, \ldots, L$, $j = 1, \ldots, M$) are independent realizations of random variable x_T of the form (19); $\mathrm{x}_T^{(k,j)}$ and $\mathrm{y}_T^{\Delta(k,j)}$ at fixed k and j correspond to the same realization of the Wiener process w_t, $t \in [0, T]$.

Numerical experiment 1 (Figure 1). To model 2000 ($M = 20$ groups with $L = 100$ items in each of them) of independent realizations of random variable x_T of the form (19) by formula

$$x_0 e^{\left(a-\frac{1}{2}b^2\right)T + b\sqrt{\Delta}\sum\limits_{p=1}^{N}\zeta_{0,p}^{(1)}} \tag{21}$$

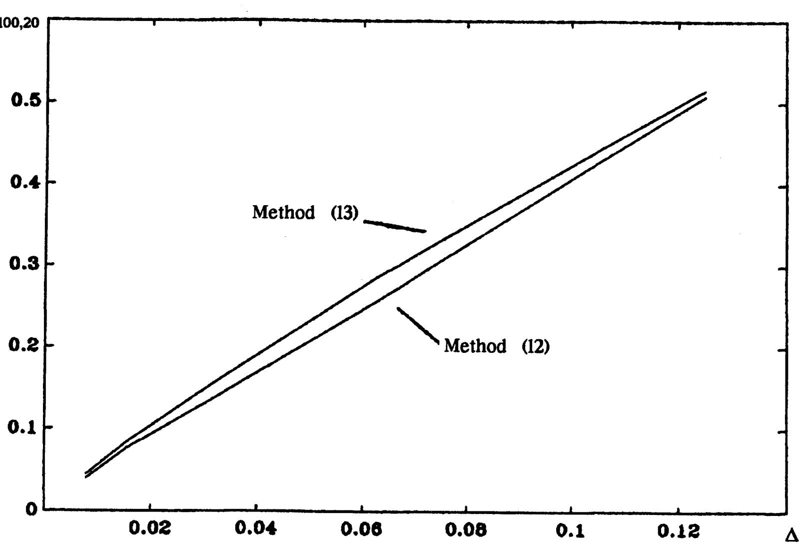

Figure 1

at $x_0 = 1; T = 1; a = b = 1.5; N = T/\Delta; \Delta = 2^{-3}$, and also by the numerical method (12) for the same input data and $y_0 = 1$. For approximation of integrals from (12), use relationships

$$I^{*(1)}_{0\,\tau_{p+1},\tau_p} = \sqrt{\Delta}\,\zeta_0^{(1)},\ I^{*(11)}_{00\,\tau_{p+1},\tau_p} = \frac{\Delta}{2}(\zeta_0^{(1)})^2 \ \text{a. s.,} \tag{22}$$

$$I^{*(1)}_{1\,\tau_{p+1},\tau_p} = -\frac{\Delta^{\frac{3}{2}}}{2}\left(\zeta_0^{(1)} + \frac{1}{\sqrt{3}}\zeta_1^{(1)}\right),\ I^{*(111)}_{000\,\tau_{p+1},\tau_p} = \frac{\Delta^{\frac{3}{2}}}{6}(\zeta_0^{(1)})^3 \ \text{a. s.} \tag{23}$$

and formulae (15)–(17), here $\zeta^{(1)}_{0,p} \overset{\text{def}}{=} \zeta_0^{(1)}$, $\zeta^{(1)}_{1,p} \overset{\text{def}}{=} \zeta_1^{(1)}$ $(p = 1, 2, \ldots, N)$ are independent in totality standard Gaussian random variables. Calculate $\hat{\varepsilon}_{100,20}$ by formula (20), repeat calculations for $\Delta = 2^{-j}$, $j = 4, 5, 6, 7$, plot dependence $\hat{\varepsilon}_{100,20}(\Delta)$, $\Delta = 2^{-j}, j = 3, 4, \ldots, 7$. Repeat calculations for the numerical method (13).

Consider numerical method of the form

$$y_{p+1} = q_{p+1,p} + \frac{\Delta^2}{2} La(y_p, \tau_p), \tag{24}$$

where $q_{p+1,p}$ is the right-hand side of (12); $\frac{\Delta^2}{2} La$ is the systematic addend of the second order of smallness from the Taylor-Ito expansion [2].

The finite-difference numerical method corresponding to the numerical scheme (24) has the form

$$y_{p+1} = \bar{q}_{p+1,p} + \frac{\Delta}{2}\left[\frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{3}\mu_j a(y^{(jr)}_{p\lambda}, \tau_p) + \sum_{j=1}^{2}\rho_j a(z^{(j)}_{p\nu}, \Delta^{(j)}_{p\xi})\right], \tag{25}$$

where $\bar{q}_{p+1,p}$ is the right-hand side of (13) without the addend $\Delta\, a(y_p, \tau_p)$.

Numerical experiment 2 (Figure 2). Repeat numerical experiment 1 for numerical methods (24), (25).

According to [2, Theorem 10.6.3, Pages 361–364; Theorem 11.5.2, Pages 391–392], numerical methods (24), (25) have strong order of accuracy 1.5 and their errors appear to be significantly smaller than errors of numerical schemes (12), (13) (see Figures 1, 2).

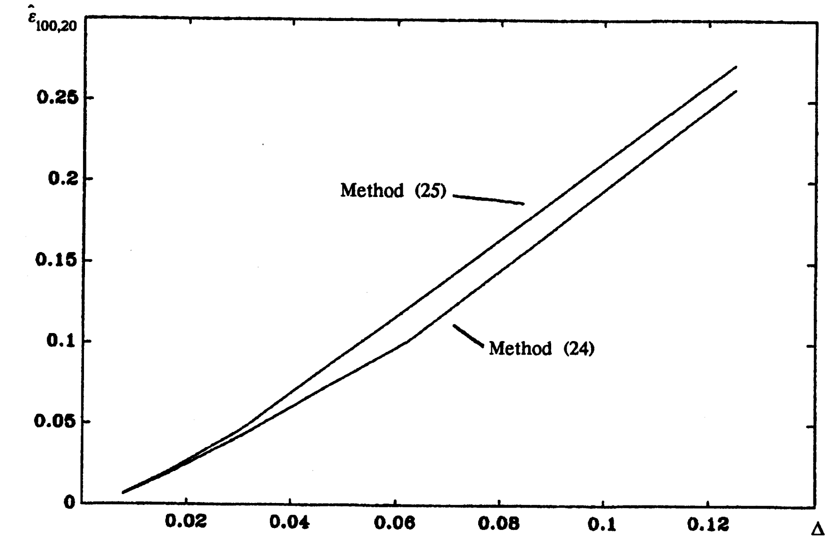

Figure 2

Consider an explicit one-step strong numerical method of order of accuracy 2.0 of the form [3]

$$
\begin{aligned}
\mathbf{y}_{p+1} &= \mathbf{q}_{p+1,p} + \frac{\Delta^2}{2} L\mathbf{a} + \sum_{i_1,i_2=1}^{m} \Bigg[G_0^{(i_2)} L \Sigma_{i_1} \left(\hat{I}^{(i_2 i_1)}_{10\,\tau_{p+1},\tau_p} - \hat{I}^{(i_2 i_1)}_{01\,\tau_{p+1},\tau_p} \right) - \\
&- L G_0^{(i_2)} \Sigma_{i_1} \hat{I}^{(i_2 i_1)}_{10\,\tau_{p+1},\tau_p} + G_0^{(i_2)} G_0^{(i_1)} \mathbf{a} \left(\hat{I}^{(i_2 i_1)}_{01\,\tau_{p+1},\tau_p} + \Delta \hat{I}^{(i_2 i_1)}_{00\,\tau_{p+1},\tau_p} \right) \Bigg] + \\
&+ \sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} \Sigma_{i_1} \hat{I}^{(i_4 i_3 i_2 i_1)}_{0000\,\tau_{p+1},\tau_p},
\end{aligned}
\tag{26}
$$

where $\mathbf{q}_{p+1,p}$ is the right-hand side of (12); the functions in the right-hand side of (26) are calculated at the point $(\mathbf{y}_p, \tau_p)$.

Using formulae (2)–(8), let us construct on the basis of the numerical method (26) the following explicit one-step strong finite-difference numerical scheme of order of accuracy 2.0:

$$
\mathbf{y}_{p+1} = \mathbf{y}_p + \frac{\Delta}{2} \left[\frac{1}{2} \sum_{r=1}^{m} \sum_{j=1}^{3} \mu_j \mathbf{a}(\mathbf{y}^{(jr)}_{p\lambda}, \tau_p) + \sum_{j=1}^{2} \rho_j \mathbf{a}(\mathbf{z}^{(j)}_{p\nu}, \Delta^{(j)}_{p\xi}) \right] + \mathbf{v}_{p+1,p},
\tag{27}
$$

where

$$
\begin{aligned}
\mathbf{v}_{p+1,p} &= \sum_{i_1=1}^{m} \Sigma_{i_1}(\mathbf{y}_p, \tau_p) \hat{I}^{(i_1)}_{0\,\tau_{p+1},\tau_p} + \frac{1}{\sqrt{\Delta}} \sum_{i_1,i_2=1}^{m} \sum_{j=1}^{4} \eta_j \Sigma_{i_1}(\mathbf{y}^{(ji_2)}_{p\omega}, \tau_p) \hat{I}^{(i_2 i_1)}_{00\,\tau_{p+1},\tau_p} + \\
&+ \frac{1}{\Delta} \sum_{i_1=1}^{m} \Bigg[\sum_{j=1}^{2} \pi_j \mathbf{a}(\mathbf{u}^{(ji_1)}_{p\sigma}, \tau_p) \left(\Delta \hat{I}^{(i_1)}_{0\,\tau_{p+1},\tau_p} + \hat{I}^{(i_1)}_{1\,\tau_{p+1},\tau_p} \right) - \\
&- \left(\frac{1}{2} \sum_{r=1}^{m} \sum_{j=1}^{3} \mu_j \Sigma_{i_1}(\mathbf{y}^{(jr)}_{p\lambda}, \tau_p) + \sum_{j=1}^{2} \chi_j \Sigma_{i_1}(\mathbf{z}^{(j)}_{p\delta}, \Delta^{(j)}_{p\gamma}) \right) \hat{I}^{(i_1)}_{1\,\tau_{p+1},\tau_p} \Bigg] +
\end{aligned}
$$

$$+\frac{1}{\Delta}\sum_{i_1,i_2,i_3=1}^{m}\sum_{j=1}^{4}\eta_j \mathbf{b}_{i_1 i_2}(\mathbf{y}_{p\omega}^{(ji_3)},\tau_p)\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}+$$

$$+\frac{1}{\Delta\sqrt{\Delta}}\sum_{i_1,i_2=1}^{m}\left[\sum_{j=1}^{4}\eta_j \mathbf{g}_{i_1}(\mathbf{y}_{p\omega}^{(ji_2)},\tau_p)\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)+\right.$$

$$+\sum_{j=1}^{4}\eta_j \mathbf{h}_{i_1}(\mathbf{y}_{p\omega}^{(ji_2)},\tau_p)\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)- \tag{28}$$

$$-\left.\left(\frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{3}\mu_j \mathbf{k}_{i_1 i_2}(\mathbf{y}_{p\lambda}^{(jr)},\tau_p)+\sum_{j=1}^{2}\chi_j \mathbf{k}_{i_1 i_2}(\mathbf{z}_{p\delta}^{(j)},\Delta_{p\gamma}^{(j)})\right)\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right]+$$

$$+\frac{1}{\Delta\sqrt{\Delta}}\sum_{i_1,i_2,i_3,i_4=1}^{m}\sum_{j=1}^{4}\eta_j \mathbf{c}_{i_1 i_2 i_3}(\mathbf{y}_{p\omega}^{(ji_4)},\tau_p)\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4 i_3 i_2 i_1)},$$

$$\mathbf{b}_{i_1 i_2}(\mathbf{y}_p,\tau_p)=\sum_{j=1}^{2}\alpha_j\Sigma_{i_1}(\mathbf{y}_{p\beta}^{(ji_2)},\tau_p),\ \mathbf{h}_{i_1}(\mathbf{y}_p,\tau_p)=\sum_{j=1}^{2}\pi_j \mathbf{a}(\mathbf{u}_{p\sigma}^{(ji_1)},\tau_p),$$

$$\mathbf{g}_{i_1}(\mathbf{y}_p,\tau_p)=\frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{3}\varphi_j\Sigma_{i_1}(\mathbf{y}_{p\psi}^{(jr)},\tau_p)+\sum_{j=1}^{2}\chi_j\Sigma_{i_1}(\mathbf{z}_{p\delta}^{(j)},\Delta_{p\gamma}^{(j)}),$$

$$\mathbf{c}_{i_1 i_2 i_3}(\mathbf{y}_p,\tau_p)=\sum_{j=1}^{2}\alpha_j \mathbf{k}_{i_1 i_2}(\mathbf{y}_{p\beta}^{(ji_3)},\tau_p);$$

$\mathbf{k}_{i_1 i_2}(\mathbf{y}_p,\tau_p)$ is defined by equality (14).

Using standard relationships between Ito and Stratonovich repeated stochastic integrals [2] we obtain

$$I_{\alpha_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}=I_{\alpha_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}\Delta^2 \text{ a. s. } (\alpha=01 \text{ or } 10), \tag{29}$$

$$\begin{aligned}I_{0000_{\tau_{p+1},\tau_p}}^{(i_4 i_3 i_2 i_1)}&=I_{0000_{\tau_{p+1},\tau_p}}^{*(i_4 i_3 i_2 i_1)}+\frac{1}{2}\mathbf{1}_{\{i_4=i_3\}}I_{10_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)}-\frac{1}{2}\mathbf{1}_{\{i_3=i_2\}}\left(I_{10_{\tau_{p+1},\tau_p}}^{*(i_4 i_1)}-I_{01_{\tau_{p+1},\tau_p}}^{*(i_4 i_1)}\right)-\\&-\frac{1}{2}\mathbf{1}_{\{i_2=i_1\}}\left(\Delta I_{00_{\tau_{p+1},\tau_p}}^{*(i_4 i_3)}+I_{01_{\tau_{p+1},\tau_p}}^{*(i_4 i_3)}\right)+\frac{1}{8}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_4=i_3\}}\Delta^2 \text{ a. s.}\end{aligned} \tag{30}$$

The repeated Stratonovich stochastic integrals in the right-hand sides of (15)–(17), (29), (30) can be approximated by the method described in [4]. Then using the above-mentioned formulae we obtain approximations of Ito repeated stochastic integrals appearing in the right-hand sides of (26), (27).

Numerical experiment 3 (Figure 3). Repeat the numerical experiment 1 for numerical methods (26), (27) using for approximations of repeated stochastic integrals formulae (15)–(17), (22), (23), (29), (30) and also relationships [4]

$$\begin{aligned}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(11)}&=-\frac{\Delta^2}{4}\left(\frac{2}{3}(\zeta_0^{(1)})^2+\frac{1}{\sqrt{3}}\zeta_0^{(1)}\zeta_1^{(1)}-\frac{1}{3\sqrt{5}}\zeta_2^{(1)}\zeta_0^{(1)}+\right.\\&+\left.\sum_{i=1}^{q}\left(\frac{1}{(2i-1)(2i+3)}(\zeta_i^{(1)})^2-\frac{1}{(2i+3)\sqrt{(2i+1)(2i+5)}}\zeta_i^{(1)}\zeta_{i+2}^{(1)}\right)\right),\end{aligned}$$

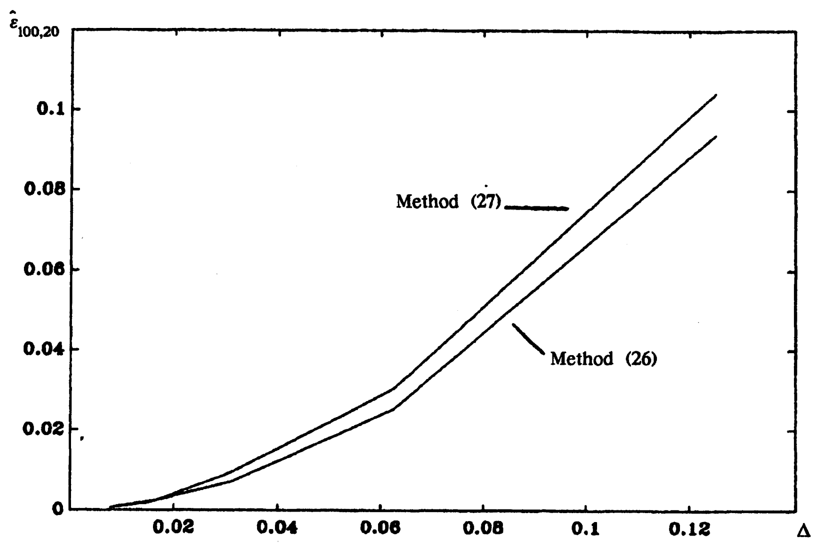

Figure 3

$$I^{*(11)}_{01\,\tau_{p+1},\tau_p} = I^{*(1)}_{0\,\tau_{p+1},\tau_p} I^{*(1)}_{1\,\tau_{p+1},\tau_p} - I^{*(11)}_{10\,\tau_{p+1},\tau_p},\ I^{*(1111)}_{0000\,\tau_{p+1},\tau_p} = \frac{\Delta^2}{24}\left(\zeta_0^{(1)}\right)^4 \ \text{a. s.}$$

at $q = 1$; $\zeta_{i,p}^{(1)} \overset{\text{def}}{=} \zeta_i^{(1)}$ $(i = 0, 1, \ldots, q + 2, p = 1, 2, \ldots, N)$ are independent in totality Gaussian standard random variables and $\zeta_{0,p}^{(1)} \overset{\text{def}}{=} \zeta_0^{(1)}$ $(p = 1, 2, \ldots, N)$ are the same random variables as in (21), (22); $\hat{I}^{*(11)}_{10\,\tau_{p+1},\tau_p}$ is an approximation of the integral $I^{*(11)}_{10\,\tau_{p+1},\tau_p}$.

Note that convergence of the numerical method (27) is substantiated analogously to convergence of the numerical method (13).

Implicit one-step strong finite-difference methods of orders of accuracy 1.5 and 2.0

The need in implicit numerical methods for Ito stochastic differential equations is conditioned by the following reason. These methods possess a higher degree of stability [2, 5] than explicit methods.

Consider a family of implicit one-step strong numerical schemes of order of accuracy 1.5 [2, 5] based on the unified Taylor-Ito expansion [4]

$$\begin{aligned} \mathbf{y}_{p+1}^{(l)} &= \mathbf{y}_p^{(l)} + \{\tilde{\alpha}_l \mathbf{a}^{(l)}(\mathbf{y}_{p+1}, \tau_{p+1}) + (1 - \tilde{\alpha}_l)\, \mathbf{a}^{(l)}\}\, \Delta + \\ &+ \left(\frac{1}{2} - \tilde{\alpha}_l\right) \{\tilde{\beta}_l L \mathbf{a}^{(l)}(\mathbf{y}_{p+1}, \tau_{p+1}) + (1 - \tilde{\beta}_l)\, L\mathbf{a}^{(l)}\}\, \Delta^2 + \mathbf{v}_{p+1,p}^{(l)}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathbf{v}_{p+1,p}^{(l)} = &\sum_{i_1=1}^{m} \Sigma^{(li_1)} \hat{I}^{(i_1)}_{0\,\tau_{p+1},\tau_p} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} \Sigma^{(li_1)} \hat{I}^{(i_2 i_1)}_{00\,\tau_{p+1},\tau_p} + \\ &+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \mathbf{a}^{(l)} \left((1 - \tilde{\alpha}_l)\, \Delta \hat{I}^{(i_1)}_{0\,\tau_{p+1},\tau_p} + \hat{I}^{(i_1)}_{1\,\tau_{p+1},\tau_p} \right) - L\Sigma^{(li_1)} \hat{I}^{(i_1)}_{1\,\tau_{p+1},\tau_p} \right] + \\ &+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)} G_0^{(i_2)} \Sigma^{(li_1)} \hat{I}^{(i_3 i_2 i_1)}_{000\,\tau_{p+1},\tau_p}\,; \end{aligned} \tag{32}$$

$\tilde{\alpha}_l, \tilde{\beta}_l \in [0, 1]$; $l = 1, 2, \ldots, n$; index (l) denotes the l-th component of a vector; functions in the right-hand sides of (31), (32) without arguments shown are calculated at the point $(\mathbf{y}_p, \tau_p)$.

Put $\tilde{\alpha}_l = \frac{1}{2}$ in the right-hand side of (31) and approximate the partial derivatives appearing into the right-hand side (31) by finite differences using relationships (2)–(8). In the result we obtain the following numerical method:

$$\mathbf{y}_{p+1}^{(l)} = \mathbf{y}_p^{(l)} + \frac{1}{2}\{\mathbf{a}^{(l)}(\mathbf{y}_{p+1}, \tau_{p+1}) + \mathbf{a}^{(l)}\}\Delta + \mathbf{v}_{p+1,p}^{(l)}, \tag{33}$$

where

$$\begin{aligned}\mathbf{v}_{p+1,p}^{(l)} = &\sum_{i_1=1}^{m} \Sigma^{(li_1)}(\mathbf{y}_p, \tau_p)\, \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \frac{1}{\sqrt{\Delta}} \sum_{i_1,i_2=1}^{m} \sum_{j=1}^{2} \alpha_j \Sigma^{(li_1)}(\mathbf{y}_{p\beta}^{(ji_2)}, \tau_p)\, \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} + \\ &+ \frac{1}{\Delta} \sum_{i_1=1}^{m} \Bigg[\sum_{j=1}^{2} \pi_j \mathbf{a}^{(l)}(\mathbf{u}_{p\sigma}^{(ji_1)}, \tau_p) \left(\frac{1}{2}\Delta \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \right) - \\ &- \left(\frac{1}{2} \sum_{r=1}^{m} \sum_{j=1}^{3} \varphi_j \Sigma^{(li_1)}(\mathbf{y}_{p\psi}^{(jr)}, \tau_p) + \sum_{j=1}^{2} \chi_j \Sigma^{(li_1)}(\mathbf{z}_{p\delta}^{(j)}, \Delta_{p\gamma}^{(j)}) \right) \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \Bigg] + \\ &+ \frac{1}{\Delta} \sum_{i_1,i_2,i_3=1}^{m} \sum_{j=1}^{2} \alpha_j \mathbf{k}_{i_1 i_2}^{(l)}(\mathbf{y}_{p\beta}^{(ji_3)}, \tau_p)\, \hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}\,;\end{aligned} \tag{34}$$

and the meaning of notation appearing in (13), (31) is preserved; $l = 1, 2, \ldots, n$.

Numerical experiment 4 (Figure 4). Repeat numerical experiment 1 for numerical methods (31) at $\tilde{\alpha}_1 = \frac{1}{2}$ and (33).

Consider a family of implicit one-step strong numerical methods of order of accuracy 2.0 [2] based on the Taylor-Ito unified expansion [3]

$$\mathbf{y}_{p+1}^{(l)} = \mathbf{g}_{p+1,p}^{(l)} + \mathbf{r}_{p+1,p}^{(l)}, \tag{35}$$

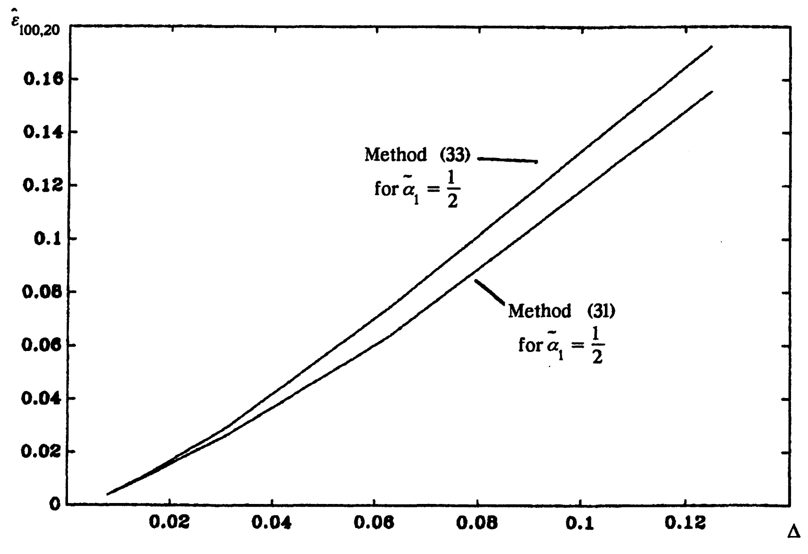

Figure 4

where

$$\mathrm{r}_{p+1,p}^{(l)} = \sum_{i_1,i_2=1}^{m} \left[G_0^{(i_2)} L \Sigma^{(li_1)} \left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} - \hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} \right) - L G_0^{(i_2)} \Sigma^{(li_1)} \hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} + \right.$$

$$\left. + G_0^{(i_2)} G_0^{(i_1)} \mathrm{a}^{(l)} \left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} + \Delta (1 - \tilde{\alpha}_l) \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} \right) \right] + \tag{36}$$

$$+ \sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} \Sigma^{(li_1)} \hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4 i_3 i_2 i_1)} \ ;$$

$\mathrm{g}_{p+1,p}^{(l)}$ is the right-hand side of (31); $l = 1, 2, \ldots, n$.

After approximation partial derivatives appearing in the right-hand side of (35) by finite differences using equations (2)–(8), and taking $\tilde{\alpha}_l = \frac{1}{2}$, $l = 1, 2, \ldots, n$, we obtain the following strong numerical method of order of accuracy 2.0:

$$\mathrm{y}_{p+1} = \mathrm{y}_p + \frac{\Delta}{2} \{ \mathrm{a}(\mathrm{y}_{p+1}, \tau_{p+1}) + \mathrm{a}(\mathrm{y}_p, \tau_p) \} + \mathrm{q}_{p+1,p}, \tag{37}$$

where

$$\mathrm{q}_{p+1,p} = \mathrm{v}_{p+1,p} - \frac{1}{2} \sum_{i_1=1}^{m} \sum_{j=1}^{2} \pi_j \mathrm{a}(\mathrm{u}_{p\sigma}^{(ji_1)}, \tau_p) \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} -$$

$$- \frac{1}{2\sqrt{\Delta}} \sum_{i_1,i_2=1}^{m} \sum_{j=1}^{4} \eta_j \mathrm{h}_{i_1} (\mathrm{y}_{p\omega}^{(ji_2)}, \tau_p) \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} \ ; \tag{38}$$

and the meaning of notations appearing in (27) is preserved; $\mathrm{v}_{p+1,p}$ is defined by formula (28).

Numerical experiment 5 (Figure 5). Repeat numerical experiment 3 for numerical methods (35) at $\tilde{\alpha}_1 = \frac{1}{2}$ and (37).

It follows from [2, Section 12.6, Pages 420–425] that the numerical schemes (35), (37) are strong schemes of order of accuracy 2.0.

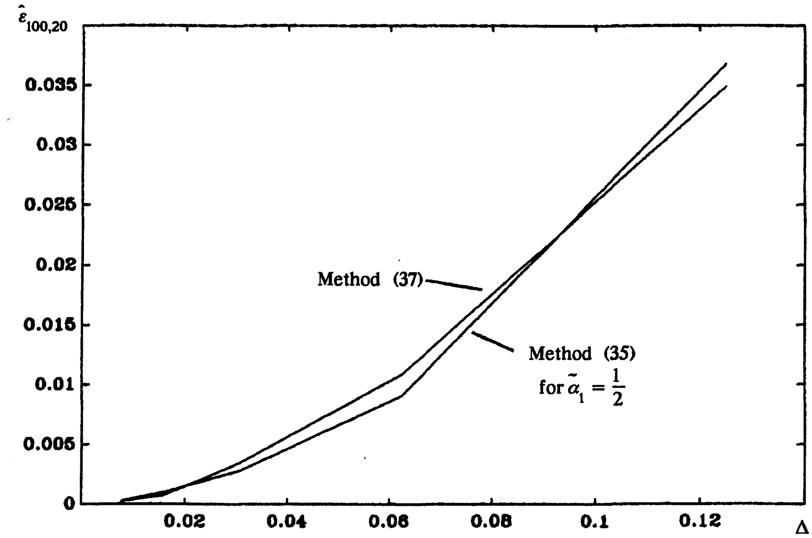

Figure 5

Implicit two-step strong finite-difference methods of orders of accuracy 1.5 and 2.0

Consider a two-step strong numerical scheme of the order of accuracy 1.5 [2] based on the unified Taylor-Ito expansion [3]

$$y^{(l)}_{p+1} = (1 - \tilde{\gamma}_l)\, y^{(l)}_p + \tilde{\gamma}_l y^{(l)}_{p-1} + \frac{1}{2}\Delta \{a^{(l)}(y_{p+1}, \tau_{p+1}) +$$

$$+ (1 + \tilde{\gamma}_l)\, a^{(l)}(y_p, \tau_p) + \tilde{\gamma}_l a^{(l)}(y_{p-1}, \tau_{p-1})\} + v^{(l)}_{p+1,\,p} + \tilde{\gamma}_l v^{(l)}_{p,p-1}, \tag{39}$$

where $v^{(l)}_{p+1,p}$ is defined by formula (32) at $\tilde{\alpha}_l = \frac{1}{2}$; $\tilde{\gamma}_l \in [0, 1]$; $l = 1, 2, \ldots, n$.

Approximate partial derivatives appearing in $v^{(l)}_{p+1,p}$ from (39) by finite differences using relationships (2)–(8). In the result the quantity $v^{(l)}_{p+1,p}$ from (39) is transformed to the form (34).

Numerical experiment 6 (Figure 6). Repeat numerical experiment 1 for numerical methods (39), (32) and (39), (34) at $\tilde{\gamma}_1 = 1$; $\tilde{\alpha}_1 = \frac{1}{2}$. Perform the initial step of the method (39), (32) by the scheme (12) and the initial step of the method (39), (34) by the scheme (13).

Consider a two-step implicit strong numerical method of order of accuracy 2.0 of the form [2, 3]

$$y_{p+1} = y_{p-1} + \frac{1}{2}\Delta \{a(y_{p+1}, \tau_{p+1}) + 2a(y_p, \tau_p) + a(y_{p-1}, \tau_{p-1})\} + \sum_{k=p}^{p+1} q_{k,k-1}, \tag{40}$$

where

$$q^{(l)}_{p+1,p} = v^{(l)}_{p+1,p} + r^{(l)}_{p+1,p}; \tag{41}$$

$v^{(l)}_{p+1,p}$ has the form (32) at $\tilde{\alpha}_l = \frac{1}{2}$; $r^{(l)}_{p+1,p}$ is defined by equality (36) at $\tilde{\alpha}_l = \frac{1}{2}$.

Approximate partial derivatives appearing in $q_{p+1,p}$ of the form (41) by finite differences using relationship (2)–(8). In the result $q_{p+1,p}$ is transformed to the form (38).

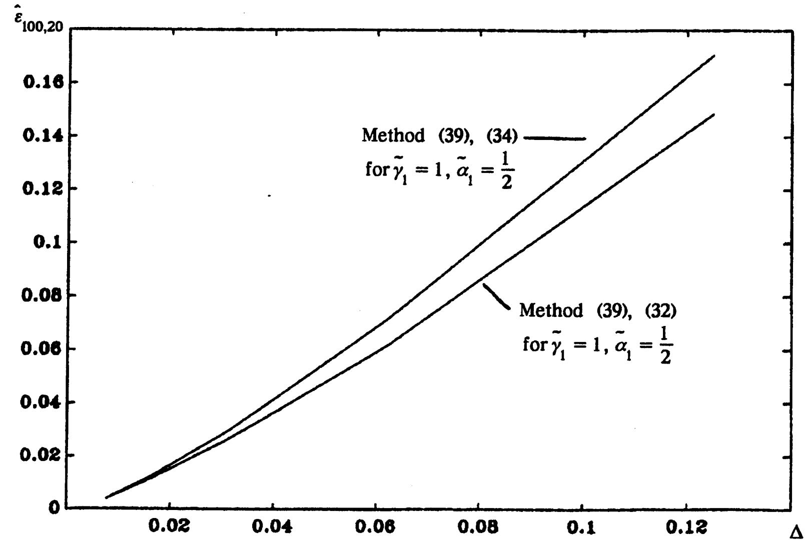

Figure 6

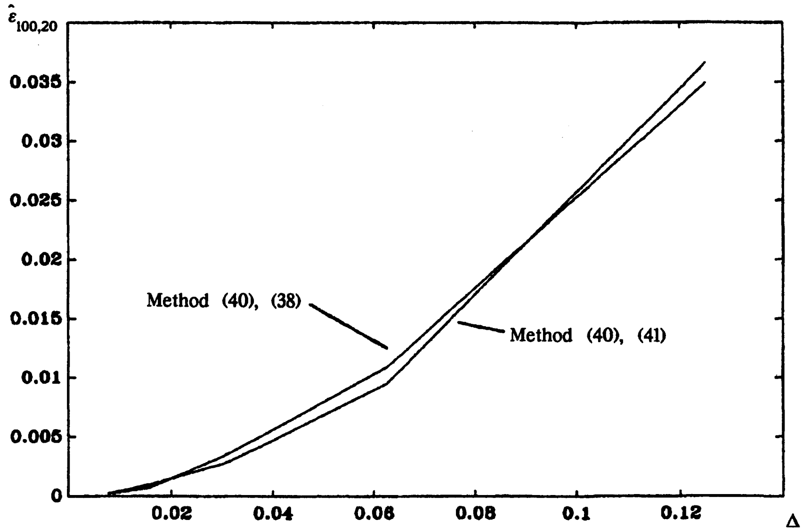

Figure 7

Numerical experiment 7 (Figure 7). Repeat numerical experiment 3 for numerical methods (40), (41) and (40), (38). Perform the initial step of the method (40), (41) by the scheme (26) and the initial step of the method (40), (38) by the scheme (37).

Conclusion

It should be noted that authors of [2] presented finite-difference numerical methods of order of accuracy 1.5 analogues to those reported in this paper, however the methods [2] have somewhat more complex form. As for finite-difference numerical methods (27), (37) and the method (40), (38) having order of accuracy 2.0, they do not have analogues of the same degree of generality. So, for instance, authors of [2] presented explicit and implicit finite-difference one-step numerical schemes of orders of accuracy 2.0 only for the Ito stochastic differential equation (1) with scalar ($m = 1$) and additive ($\Sigma(x, t) \equiv \Sigma(t)$: $[0, T] \rightarrow \mathfrak{R}^n$) noise. At that, the result of numerical experiment in [2, Page 389] demonstrates that the least error of the first of these schemes is approximately equal to $2^{-3.25}$, it increases when Δ decreases, starting from $\Delta = 2^{-3}$ and attains value $2^{-3.1}$ at $\Delta = 2^{-4}$ (cf. Figure 3).

References

1. Gikhman I.I. Skorokhod A.V., Introduction to the theory of random processes [in Russian], Nauka, Moscow, 1977.
2. Kloeden P.E., Platen E., Numerical solution of stochastic differential equations, Springer-Verlag, Berlin, 1992.
3. Kuznetsov D.F., Numerical integration of stochastic differential equations [in Russian], Izdat. St.-Petersburg GU, St.-Petersburg, 2001.
4. Kuznetsov D.F., Application of Legendre polynomials to mean-square approximation of solutions of stochastic differential equations, *Problemy upravleniya i informatiki*, 2000, No. 5, 84–104.
5. Milshtein G.N., Numerical integration of stochastic differential equations [in Russian], Izdat. Ural. universiteta, Sverdlovsk, 1988.