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A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process

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Abstract

The work is devoted to a new approach to the expansion of iterated Stratonovich stochastic integrals with respect to the components of a multidimensional Wiener process. This approach is based on multiple Fourier–Legendre series as well as multiple trigonometric Fourier series. The theorem on the mean-square convergent expansion for the iterated Stratonovich stochastic integrals of arbitrary multiplicity is formulated and proved under the condition of convergence of trace series. This condition has been verified for integrals of multiplicities 1 to 5 and complete orthonormal systems of Legendre polynomials and trigonometric functions in Hilbert space. The Hu–Meyer formula and multiple Wiener stochastic integral were used in the proof of the mentioned theorem. The rate of mean-square convergence of the obtained expansions is found. The results of the work can be applied to the numerical integration of Itô stochastic differential equations with non-commutative noise in the framework of the approach based on the Taylor–Stratonovich expansion.

Contents

1 Introduction

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0, T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{W}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0, T]$. We assume that the components $\mathbf{W}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{W}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \quad (1)$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n$, $B : \mathbf{R}^n \times [0, T] \to \mathbf{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1). Let $\mathbf{x}_0$ is F_0-measurable and $\mathbf{E} \left|\mathbf{x}_0\right|^2 < \infty$ ($\mathbf{E}$ denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{W}_t - \mathbf{W}_0$ are independent when $t > 0$.

It is well known that Itô SDEs are adequate mathematical models of dynamic systems of various physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions. The most important feature of such expansions is a presence in them of the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)$ are nonrandom functions on $[t, T]$, $\mathbf{W}_\tau^{(i)}$ $(i=1,\ldots, m)$ are independent standard Wiener processes, $\mathbf{W}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0, 1, \ldots, m$, $d\mathbf{W}_\tau^{(i)}$ and $\circ\, d\mathbf{W}_\tau^{(i)}$ denote Itô and Stratonovich differentials, respectively $(i=1,\ldots,m)$.

Effective solution of the problem of mean-square approximation of iterated Stratonovich stochastic integrals (3) composes the subject of the work.

Remark 1. It is well known that the following representation takes place

$$\mathbf{W}_\tau^{(i)}-\mathbf{W}_t^{(i)}=\underset{p\to\infty}{\text{l.i.m.}}\left(\mathbf{W}_\tau^{(i)p}-\mathbf{W}_t^{(i)p}\right),\quad \mathbf{W}_\tau^{(i)p}-\mathbf{W}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^\tau\phi_j(s)ds\cdot\zeta_j^{(i)},$$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{W}_s^{(i)},\quad d\mathbf{W}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau,$$

where $p\in\mathbf{N}$, $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$, $\mathbf{W}_s^{(0)}=s$, $i=0,1,\ldots,m$, $\zeta_j^{(i)}$ are i.i.d. $N(0,1)$-r.v.'s for various i or j $(i\ne 0)$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)p}\ldots d\mathbf{W}_{t_k}^{(i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\to\ ?$$

if $p\to\infty$, where $i_1,\ldots,i_k=0,1,\ldots,m$, $C_{j_k\ldots j_1}$ has the form

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k.$$

The case $i_1=\ldots=i_k\ne 0$ can be obtained from [JK-1],[BK-1],[B-1] under additional assumptions among which is the existence of limiting traces.

2 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & \text{for} \quad t_1 < \ldots < t_k \\ & \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

where $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$, and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. It is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K - K_{p_1 \ldots p_k} \right\|_{L_2([t,T]^k)} = 0,$$

where

$$K_{p_1 \ldots p_k}(t_1, \ldots, t_k) = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l),$$

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \qquad (5)$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t, T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} \Delta\tau_j \to 0 \text{ if } N \to \infty, \ \Delta\tau_j = \tau_{j+1} - \tau_j. \qquad (6)$$

Theorem 1 [1, Section 1.1.3] (2006). *Suppose that every* $\psi_l(\tau)$ $(l=1, \ldots, k)$ *is a continuous nonrandom function on* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of continuous functions in* $L_2([t, T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (7)$$

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1,\ldots,l_k): l_1,\ldots,l_k = 0,1,\ldots,N-1;\, l_g \ne l_r\, (g\ne r);\, g,r=1,\ldots,k\},$$

$i_1,\ldots,i_k = 0,1,\ldots,m$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by (2),* $\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{W}_\tau^{(i)}$ *are i.i.d.* $N(0,1)$*-r.v.'s for various i or j (if* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient (5),* $\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ \ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t, T]$, *which satisfies the condition (6).*

Let us consider particular cases of Theorem 1 (see (7)) for $k = 1, \ldots, 5$

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{8}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{9}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1, p_2, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \right.$$

$$\left. - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \right), \tag{10}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (11)$$

$$
J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}$$

$$+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}$$

$$+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \qquad (12)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the generalization of Theorem 1. In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{13}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

Consider the sum

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}}$$

Theorem 2 [1, Section 1.11] *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(14)$$

that converges in the mean-square sense is valid, *where* $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem 1.*

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3 Expansions of Iterated Stratonovich Stochastc Integrals of Multiplicities 1 to 4. Some Old Results

Let $\mathrm{M}_2[t, T]$ $(0 \le t < T < \infty)$ be the class of random functions $\xi(\tau,\omega) \stackrel{\rm def}{=} \xi_\tau : [t, T] \times \Omega \to \mathbf{R}$, which satisfy the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau \in [t, T]$, ξ_τ is independent with increments $\mathbf{W}_{s+\Delta} - \mathbf{W}_s$ for $s \ge \tau$, $\Delta > 0$, and

$$\int\limits_t^T \mathbf{E}(\xi_\tau)^2 d\tau < \infty, \quad \mathbf{E}(\xi_\tau)^2 < \infty \quad \text{for all} \quad \tau \in [t, T].$$

We introduce the class $\mathrm{Q}_4[t, T]$ of Itô processes $\eta_\tau^{(i)}$, $\tau \in [t, T]$, $i = 1, \ldots, m$ of the form

$$\eta_\tau^{(i)} = \eta_t^{(i)} + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s d\mathbf{W}_s^{(i)} \quad \text{w. p. 1}, \qquad (15)$$

where $(a_s)^4, (b_s)^4 \in \mathrm{M}_2[t, T]$ and $\lim\limits_{s\to\tau} \mathbf{E}\left|b_s - b_\tau\right|^4 = 0$ for all $\tau \in [t, T]$.

Consider a function $F(x, \tau) : \mathbf{R} \times [t, T] \rightarrow \mathbf{R}$ for fixed τ from the class $C_2(-\infty, \infty)$ consisting of twice continuously differentiable in x functions on the interval $(-\infty, \infty)$ such that the first two derivatives are bounded.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)} + \eta_{\tau_{j+1}}^{(i)}\right), \tau_j\right)\left(\mathbf{W}_{\tau_{j+1}}^{(l)} - \mathbf{W}_{\tau_j}^{(l)}\right) \stackrel{\text{def}}{=} \int_t^T F(\eta_\tau^{(i)}, \tau) \circ d\mathbf{W}_\tau^{(l)} \tag{16}$$

is called the Stratonovich stochastic integral with respect to the component $\mathbf{W}_\tau^{(l)}$ $(l = 1, \ldots, m)$ of the multidimentional Wiener process $\mathbf{W}_\tau$, where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition (6).

Under proper conditions we have

$$\int_t^T F(\eta_\tau^{(i)}, \tau) \circ d\mathbf{W}_\tau^{(l)} = \int_t^T F(\eta_\tau^{(i)}, \tau) d\mathbf{W}_\tau^{(l)} + \frac{1}{2}\mathbf{1}_{\{i=l\}} \int_t^T \frac{\partial F}{\partial x}(\eta_\tau, \tau) b_\tau d\tau \quad \text{w. p. } 1, \tag{17}$$

where $\mathbf{1}_A$ is the indicator of the set A and $i, l = 1, \ldots, m$.

A possible variant of conditions under which the formula (17) is correct, for example, consisits of the conditions

$$\eta_\tau^{(i)} \in \mathrm{Q}_4[t, T], \quad F(\eta_\tau^{(i)}, \tau) \in \mathrm{M}_2[t, T], \quad F(x, \tau) \in C_2(-\infty, \infty) \ \text{(for fixed } \tau\text{)},$$

where $i = 1, \ldots, m$.

As it turned out, approximations of the iterated Stratonovich stochastic integrals (3) are essentially simpler than the appropriate approximations of the iterated Itô stochastic integrals (2) based on Theorems 1 and 2.

According to the standard connection (17) between Itô and Stratonovich stochastic integrals, the iterated Itô and Stratonovich stochastic integrals (2) and (3) of first multiplicity are equal to each other w. p. 1. So, we begin the consideration from the multiplicity $k = 2$ (the case $k = 1$ is given by (8)).

The following three theorems adapt Theorems 1, 2 for the integrals (3) of multiplicities 2 to 4.

Theorem 3 [1, Section 2.1.2] (2012, 2018). *Suppose that* $\psi_1(\tau), \psi_2(\tau)$ *are continuously differentiable functions at the interval* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then, the iterated Stratonovich stochastic integral of second multiplicity*

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \int\limits_t^T \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \circ d\mathbf{W}_{t_2}^{(i_2)}$$

is expanded into the converging in the mean-square sense double series

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where $i_1, i_2 = 0, 1, \ldots, m$; *another notations are the same as in Theorems 1, 2.*

Theorem 4 [1, Section 2.2.5] (2013). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space $L_2([t, T])$. Furthermore, let the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t, T]$ and the functions $\psi_1(\tau), \psi_3(\tau)$ are twice continuously differentiable at the interval $[t, T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \circ d\mathbf{W}_{t_2}^{(i_2)} \circ d\mathbf{W}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \qquad (18)$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$; another notations are the same as in Theorems 1, 2.

Theorem 5 [1, Section 2.3] (2013). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_4)} = \int\limits_t^T \ldots \int\limits_t^{t_2} \circ\, d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ\, d\mathbf{W}_{t_4}^{(i_4)}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)} \ldots \zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is correct, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$; *another notations are the same as in Theorems 1, 2.*

4 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k

In this section, we prove the expansion of iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbf{N}$) under the condition of convergence of trace series.

Consider the Fourier coefficient

$$C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k \qquad (19)$$

corresponding to the function (4), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. At that we suppose

$$\phi_0(x) = \frac{1}{\sqrt{T-t}}.$$

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k =$$

$$= \hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}, \qquad (20)$$

i.e. $\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type (19) but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau)$, $\ldots$, $\psi_{l-2}(\tau)$, $\sqrt{T-t}\,\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau)$, $\ldots$, $\psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (13)).

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\rm def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

Note that the operation S_l $(l = 1, 2, \ldots, r)$ acts on the value

$$\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Big|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \qquad (21)$$

as follows: S_l multiplies (21) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k \ldots j_1}\Big|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k \ldots j_1}\Big|_{(j_{g_{2l}} j_{g_{2l-1}}) \frown (\cdot), j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (22)$$

Since (22) is again the Fourier coefficient, then the action of superposition $S_l S_m$ on (21) is obvious. For example, for $r=3$

$$S_3 S_1 \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_5, g_6} \right\} =$$

$$= \frac{1}{2^2} \mathbf{1}_{\{g_6=g_5+1\}} \mathbf{1}_{\{g_2=g_1+1\}} \sum_{j_{g_3}=p+1}^{\infty} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot)(j_{g_6} j_{g_5}) \curvearrowright (\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}},$$

$$S_2 \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_5, g_6} \right\} =$$

$$= \frac{1}{2} \mathbf{1}_{\{g_4=g_3+1\}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_5}=p+1}^{\infty} C_{j_k \ldots j_1} \Bigg|_{(j_{g_4} j_{g_3}) \curvearrowright (\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}}.$$

Theorem 6 [1, Section 2.10] (2022). *Assume that the continuously differentiable functions* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions* $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *are such that the following conditions are satisfied*:

1. The equality

$$\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j=0}^{\infty}\int\limits_t^s \Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (23)$$

holds for all $s\in(t,T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t,T]$ *and the series on the right-hand side of (23) converges absolutely.*

2. The estimates

$$\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \quad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s \in (t, T)$ *and for some* $\alpha, \beta > 0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$, $j, p \in \mathbf{N}$, *and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau < \infty, \quad \int\limits_t^T |\Psi_2(\tau)|\, d\tau < \infty.$$

3. The condition

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (13)) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ \ldots,\ r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\rm def}{=}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int_t^T \psi_k(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)} \qquad (24)$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \qquad (25)$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k=0,1,\ldots,m,$

$$C_{j_k\ldots j_1} = \int_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (26)$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense*, $\mathbf{W}_\tau^{(0)}=\tau,$

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are independent $\mathrm{N}(0,1)$ *– random variables for various* i *or* j $($*if* $i\neq 0)$.

Proof. **Step 1.** Let us find a representation of the random variable

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

Let us consider the following multiple stochastic integral

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \mathop{\rm l.i.m.}_{N\to\infty} \sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_p;\ q\ne p;\ q,p=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{W}_{\tau_{j_l}}^{(i_l)}, \qquad (27)$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\Delta\mathbf{W}_{\tau_j}^{(i)}=\mathbf{W}_{\tau_{j+1}}^{(i)}-\mathbf{W}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$, which satisfies the condition (6), $i_1,\ldots,i_k=0,1,\ldots,m$.

The stochastic integral with respect to the scalar standard Wiener process $(i_1=\ldots=i_k\ne 0)$ and similar to (27) was considered in [I-1] (1951) and is called the multiple Wiener stochastic integral.

Note that the following well known estimate

$$\mathsf{E}\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2 \le C_k \int\limits_{[t,T]^k} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k \qquad (28)$$

is true for the multiple Wiener stochastic integral, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (27) and C_k is a constant.

From the proof of Theorem 1 [1, Section 1.1.3] it follows that (7) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \ = \ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \qquad (29)$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral defined by (27) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2).

Let us consider the following multiple stochastic integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{W}_{\tau_{j_l}}^{(i_l)}, \qquad (30)$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k \to \mathbf{R}$ is a continuous nonrandom function on $[t,T]^k$. Another notations are the same as in (27).

The stochastic integral with respect to the scalar standard Wiener process ($i_1 = \ldots = i_k \neq 0$) and similar to (30) has been considered in the literature (see, for example, Remark 1.5.7 [B-1]). The integral (30) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)} = J[\varphi_1]_{T,t}^{(i_1)} \ldots J[\varphi_k]_{T,t}^{(i_k)} \quad \text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k) = \varphi_1(t_1)\ldots\varphi_k(t_k)$.

Theorem 7 [1, Section 1.9] *Suppose that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}$ is a continuous nonrandom function on $[t,T]^k$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\tag{31}$$

converging in the mean-square sense is valid, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (27),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient. Another notations are the same as in Theorems 1,2.

Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{W}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{W}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{W}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (32)$$

where $(s_l, \ldots, s_1) \in \mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l} = \big\{(s_l, \ldots, s_1) : s_l > s_{l-1}+1, \ldots, s_2 > s_1+1; s_l, \ldots, s_1 = 1, \ldots, k-1\big\}, \tag{33}$$

$l = 1, 2, \ldots, [k/2]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Itô and Stratonovich stochastic integrals (2) and (3) of arbitrary multiplicity k.

Theorem 8 [1, Section 2.4.1] (1997). *Suppose that every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a continuous nonrandom function at the interval $[t, T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals (3) and (2) is correct*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)[s_r, \ldots, s_1]} \tag{34}$$

w. p. 1, where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Consider Theorem 7 (see (31)) for $\Phi(t_1,\ldots,t_k)=K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and without passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$

$$J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}-\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\ J[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\qquad(35)$$

w. p. 1, where $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (27), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral (30),

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\qquad(36)$$

$$K^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}_{p_1\ldots p_k}(t_{q_1},\ldots,t_{q_{k-2r}}) = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}). \tag{37}$$

By iteratively applying the formula (35), we obtain a representation of the multiple Stratonovich stochastic integral of multiplicity k as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$J[K_{p_1\ldots p_k}]^{(i_1\ldots i_k)}_{T,t} = J'[K_{p_1\ldots p_k}]^{(i_1\ldots i_k)}_{T,t} +$$

$$+\sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times J'[K^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}_{p_1\ldots p_k}]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\quad \text{w. p. 1.} \tag{38}$$

From (38) we have

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\,\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \qquad (39)$$

The formulas (38), (39) can be considered as new representations of the Hu-Meyer formula for the case of a multidimensional Wiener process [R-2] (also see [B-1], [JK-1]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (36)).

Further, we will use the representation (39) for $p_1=\ldots=p_k=p$, i.e.

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\,\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.}\qquad (40)$$

Step 2. Under the Condition 2 of Theorem 6, we get

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = 0 \qquad (41)$$

or

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}, \qquad (42)$$

where $l-1 \geq s+1$.

Step 3. Using Conditions 1 and 2 of Theorem 6, we obtain

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} = \frac{1}{2} C_{j_k\ldots j_1}\Bigg|_{(j_l j_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}. \qquad (43)$$

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (40), we have (see Theorem 1 and (29))

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{44}$$

Using **Step 2** and **Step 3**, we have for $r = 1, 2, \ldots, [k/2]$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Big|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots j_{g_{2r-1}}=j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} +$$

$$+ \underset{p\to\infty}{\text{l.i.m.}} R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \qquad (45)$$

$$R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left((-1)^r \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} + \right.$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} +$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} +$$

$$\ldots$$

$$\left. +(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} \right) \times$$

$$\times\, J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \tag{46}$$

We have for $g_2 = g_1 + 1, \ldots, g_{2r} = g_{2r-1} + 1$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}} \times$$

$$\times C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}}, j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}} \ldots \phi_{j_{m_{2r-1}}} \phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots0 i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \text{(by Theorem 1 and (29))} = \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1}, \qquad (47)$$

where the notations are the same as in (32) and (33), $g_{2i-1} = s_i$ ($i = 1, 2, \ldots, r$, $r = 1, 2, \ldots, [k/2]$) the last transition is based on (29), and

$$C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots j_1}\Bigg|_{(j_l j_l) \curvearrowright j_m} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l) \times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2}) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_{l-2} dt_l t_{l+1} \ldots dt_k =$$

$$= \bar{C}_{j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1},$$

i.e. $\bar{C}_{j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1}$ is again the Fourier coefficient of type (19) but with a new shorter multi-index $j_k \ldots j_{l+1} j_m j_{l-2} \ldots j_1$ and new weight functions $\psi_1(\tau), \ldots, \psi_{l-2}(\tau), \psi_{l-1}(\tau)\psi_l(\tau), \psi_{l+1}(\tau), \ldots, \psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (13))).

Using (42)-(47), and Theorem 8, we prove that [1, Section 2.10]

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} =$$

$$= J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \qquad (48)$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (32). Theorem 6 is proved.

[R-2] Rybakov, K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2021), 81–115.

1. An expansion similar to (25) was obtained in [R-2], where the author used a definition of the Stratonovich stochastic integral, which differs from (16). The proof from [R-2] is somewhat simpler than the proof proposed in this work. However, our proof allows us to estimate the rate of convergence in Theorem 6.

2. We also note that Conditions 1 and 2 of Theorem 6 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$ [1, Chapters 1, 2] (see (49)–(54) below).

3. Taking into account the modification of Theorem 1 for the case of integration interval $[t, s]$, $s \in (t, T]$ of iterated Itô stochastic integrals (2) [1, Section 1.8] we can formulate an analogue of Theorem 6 for the case of integration interval $[t, s]$, $s \in (t, T)$ [1, Section 2.10].

In [1, Sections 2.1.2, 2.7, 2.9] the following formulas are proved

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j=0}^{\infty} C_{jj}(s), \quad s \in (t, T], \tag{49}$$

$$\left|\sum_{j=p+1}^{\infty} C_{jj}(s)\right| \le \frac{C}{p}\left(\frac{1}{\left(1-z^2(s)\right)^{1/4}}+1\right), \quad s \in (t, T), \tag{50}$$

$$\left|\sum_{j=p+1}^{\infty} C_{jj}(s)\right| \le \frac{C}{p}, \quad s \in (t, T], \tag{51}$$

where constant C does not depend p,

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}, \quad C_{jj}(s) = \int\limits_t^s \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1)dt_1 dt_2,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials (formulas (49), (50)) or trigonometric functions (formulas (49), (51)) in the space $L_2([t, T])$, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t, T]$.

For the case of Legendre polynomials, we have [1]

$$\left|\int_t^x \psi(\tau)\phi_j(\tau)d\tau\right| + \left|\int_x^T \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + 1\right), \quad (52)$$

$$\left|\int_v^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + \frac{1}{(1-(z(v))^2)^{1/4}} + 1\right), \quad (53)$$

where $j \in \mathbf{N}$, $z(x), z(v) \in (-1, 1)$, $x, v \in (t, T)$, $v < x$, $\psi(\tau)$ is a continuously differentiable function on $[t, T]$, constant C does not depend on j.

For the case of trigonometric functions, we obtain

$$\left|\int_t^x \psi(\tau)\phi_j(\tau)d\tau\right| + \left|\int_x^T \psi(\tau)\phi_j(\tau)d\tau\right| + \left|\int_v^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \quad (54)$$

where $j \in \mathbf{N}$, $x, v \in (t, T)$, $v < x$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t, T]$, constant C does not depend on j.

5 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 5 (Polynomial and Trigonometric Cases)

Theorem 9 [1, Section 2.11] *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int_t^T \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \circ d\mathbf{W}_{t_2}^{(i_2)} \circ d\mathbf{W}_{t_3}^{(i_3)} \quad (55)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$; *another notations are the same as in Theorem 1.*

Proof. As follows from (49)–(54), Conditions 1 and 2 of Theorem 6 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Let us verify Condition 3 of Theorem 6 for the iterated Stratonovich stochastic integral (55). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_3j_1j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_1}\right)^2=0.$$

Theorem 10 [1, Section 2.12] *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \int\limits_t^T \psi_4(t_4)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_4}^{(i_4)} \qquad (56)$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_4 = 0,1,\ldots,m$; another notations are the same as in Theorem 1.

Proof. As follows from (49)–(54), Conditions 1 and 2 of Theorem 6 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Let us verify Condition 3 of Theorem 6 for the iterated Stratonovich stochastic integral (56). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0$$

Theorem 11 [1, Section 2.13] *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^T \psi_5(t_5)\ldots\int\limits_t^{t_2} \psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_5}^{(i_5)} \qquad (57)$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\mathrm{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5=0,1,\ldots,m$; *another notations are the same as in Theorem 1.*

Proof. As follows from (49)–(54), Conditions 1 and 2 of Theorem 6 are satisfied for CONS of Legendre polynomials and trigonometric functions in $L_2([t, T])$. Let us verify Condition 3 of Theorem 6 for the iterated Stratonovich stochastic integral (57). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=0,$$

where $(\{g_1,g_2\},\{g_3,g_4\},\{q_1\})$ and $(\{g_1,g_2\},\{q_1,q_2,q_3\})$ are partitions of the set $\{1,2,\ldots,5\}$ that is $\{g_1,g_2,g_3,g_4,q_1\}=\{g_1,g_2,q_1,q_2,q_3\}=\{1,2,\ldots,5\}$; braces mean an unordered set, and parentheses mean an ordered set.

In [1, Sect. 1.7.2] it is shown that under the conditions of Theorem 1 (polynomial and trigonometric cases) the following estimate

$$\mathbf{E}\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}\right)^2 \le \frac{k!P_k(T-t)^k}{p} \qquad (58)$$

holds, where $i_1,\ldots,i_k = 1,\ldots,m$, constant P_k depends only on k,

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right)$$

6 Rate of the Mean-Square Convergence for Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 5

In this section, we consider the rate of convergence for approximations of iterated Stratonovich stochastic integrals.

Theorem 12 [1, Section 2.15] *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity* $J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ *defined by (3) the following estimate*

$$\mathbf{E}\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}-\sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\le\frac{C}{p}\quad (p\in\mathbf{N})$$

is fulfilled, where $i_1,i_2,i_3=1,\ldots,m$, *constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Theorem 13 [1, Section 2.15] *Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Furthermore, let $\psi_1(\tau), \ldots, \psi_4(\tau)$ be continuously differentiable nonrandom functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity $J^*[\psi^{(4)}]_{T,t}^{(i_1 \ldots i_4)}$ defined by (3) the following estimate*

$$\mathsf{E}\left(J^*[\psi^{(4)}]_{T,t}^{(i_1 \ldots i_4)} - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4 \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_4}^{(i_4)}\right)^2 \le \frac{C}{p^{1-\varepsilon}} \quad (p \in \mathbf{N})$$

holds, where $i_1, \ldots, i_4 = 1, \ldots, m$, constant C does not depend on p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ and $\varepsilon = 0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t, T])$; another notations are the same as in Theorem 1.

Theorem 14 [1, Section 2.15] *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity $J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}$ defined by (3) the following estimate*

$$\mathbf{E}\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\le\frac{C}{p^{1-\varepsilon}}\quad(p\in\mathbf{N})$$

is valid, where $i_1,\ldots,i_5=1,\ldots,m$, constant C is independent of p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t,T])$; another notations are the same as in Theorem 1.

We should also note the following theorem for the case $k = 2$.

Theorem 15 [1, Sect. 2.8.1] *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$*. Then, for the iterated Stratonovich stochastic integral of second multiplicity* $J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ *defined by (3) the following estimate*

$$\mathbf{E}\left(J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} - \sum_{j_1, j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}\right)^2 \le \frac{C}{p} \quad (p \in \mathbf{N})$$

is fulfilled, where $i_1, i_2 = 1, \ldots, m$*, constant* C *is independent of* p*; another notations are the same as in Theorem 1.*

Note that the analogue of Theorem 15 for the case $k = 1$ follows from (58) [1, Sect. 1.7.2].

Thanks for your attention!