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Recent results on a new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process

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1 Introduction

The importance of the problem of numerical integration of stochastic differential equations is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems of various physical nature under random disturbances and to the application of these equations for solving various mathematical problems, among which we mention signal filtering in the background of random noise, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems, etc.

Iterated Itô and Stratonovich stochastic integrals can be used to construct high-order strong (mean-square) numerical methods for various types of systems of stochastic differential equations with non-commutative noise. For example, for systems of

Itô stochastic differential equations,

Itô stochastic differential equations with jumps,

McKean stochastic differential equations,

stochastic differential equations with switchings,

semilinear stochastic partial differential equations with multiplicative trace class noise.

Let $(\Omega,\ F,\ \mathrm{P})$ be a complete probability space, let $\{F_t, t \in [0, T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{W}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0, T]$. We assume that the components $\mathbf{W}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. As an example, consider a system of Itô stochastic differential equations (SDEs) with non-commutative noise in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \sum_{j=1}^{m} \int\limits_0^t B_j(\mathbf{x}_\tau, \tau) d\mathbf{W}_\tau^{(j)}, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a}$, $B_j : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n$ guarantee the existence and uniqueness up to stochastic equivalence of a strong solution of the equation (1). Let $\mathbf{x}_0$ is F_0-measurable and $\mathbf{E}\left|\mathbf{x}_0\right|^2 < \infty$ ($\mathbf{E}$ denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{W}_t - \mathbf{W}_0$ are independent when $t > 0$.

One of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions. The most important feature of such expansions is a presence in them of the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (2)$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)}, \qquad (3)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)$ are nonrandom functions on $[t, T]$, $\mathbf{W}_\tau^{(i)}$ $(i=1,\ldots,$ $m)$ are independent standard Wiener processes, $\mathbf{W}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,$ $1,\ \ldots,\ m$, $d\mathbf{W}_\tau^{(i)}$ and $\circ\, d\mathbf{W}_\tau^{(i)}$ denote Itô and Stratonovich differentials, respectively $(i=1,\ldots,m)$.

Effective solution of the problem of mean-square approximation of iterated Stratonovich stochastic integrals (3) composes the subject of the work.

Remark 1.

$$\mathbf{W}_{\tau}^{(i)}-\mathbf{W}_{t}^{(i)}=\underset{p\to\infty}{\text{l.i.m.}}\left(\mathbf{W}_{\tau}^{(i)p}-\mathbf{W}_{t}^{(i)p}\right),\ \ \mathbf{W}_{\tau}^{(i)p}-\mathbf{W}_{t}^{(i)p}=\sum_{j=0}^{p}\int\limits_{t}^{\tau}\phi_j(s)ds\cdot\zeta_j^{(i)},$$

$$\zeta_j^{(i)}=\int\limits_{t}^{T}\phi_j(s)d\mathbf{W}_s^{(i)},\quad d\mathbf{W}_{\tau}^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $p\in\mathbf{N}$, $\tau\in[t,T]$, $t\ge 0$, $\zeta_j^{(i)}$ are i.i.d. $N(0,1)$-r.v.'s for various i or j, $i=1,\ldots,m$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_{t}^{T}\psi_k(t_k)\ldots\int\limits_{t}^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)p_1}\ldots d\mathbf{W}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\to ?$$

if $p_1,\ldots,p_k\to\infty$, where $i_1,\ldots,i_k=0,1,\ldots,m$,

$$C_{j_k\ldots j_1}=\int\limits_{t}^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k.$$

The case $i_1=\ldots=i_k\neq 0$, $p_1=\ldots=p_k=p$ can be obtained from [JK-1], [BK-1],[B-1] under hard-to-verify condition of existence of limiting traces.

2 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series. Preliminary Results

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$. Define the following function (factorized Volterra-type kernel) on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & \text{for} \quad t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases}, \qquad (4)$$

where $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$, and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be an arbitrary CONS in $L_2[t, T]$.

Then we have

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K-K_{p_1\ldots p_k}\right\|_{L_2([t,T]^k)}=0,$$

where $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$,

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \qquad (5)$$

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (6)$$

is the Fourier coefficient.

Let $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\ \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0 \text{ if } N\to\infty,\ \Delta\tau_j=\tau_{j+1}-\tau_j. \qquad (7)$$

Theorem 1 [1, Sect. 1.1] (2006). *Let $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS in $L_2[t,T]$ such that $\phi_j(x)\in C[t,T]$ or $\phi_j(x)$ is piecewise continuous on $[t,T]$ $\forall j\in\mathbf{N}$. Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\Biggr),$$

where $G_k=H_k\backslash L_k,\quad H_k=\bigl\{(l_1,\ldots,l_k):l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$

$$L_k=\bigl\{(l_1,\ldots,l_k):l_1,\ldots,l_k=0,1,\ldots,N-1;\,l_g\ne l_r\,(g\ne r);\,g,r=1,\ldots,k\bigr\},$$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are i.i.d. $N(0,1)$-r.v.'s for various i or j $(i\ne 0)$, $i_1,\ldots,i_k=0,1,\ldots,m$, $C_{j_k\ldots j_1}$ is the Fourier coefficient (6), $\Delta\mathbf{W}_{\tau_j}^{(i)}=\mathbf{W}_{\tau_{j+1}}^{(i)}-\mathbf{W}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\mathbf{W}_\tau^{(0)}=\tau$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (2), $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$ satisfying the condition (7).

For the proof of Theorem 1, we used the following multiple stochastic integral

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \mathop{\rm l.i.m.}_{N\to\infty} \sum_{\substack{j_1,\ldots,j_k=0\\ j_q\neq j_p;\ q\neq p;\ q,p=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{W}_{\tau_{j_l}}^{(i_l)}, \qquad (8)$$

where $\Phi(t_1,\ldots,t_k) = K(t_1,\ldots,t_k)$ (factorized Volterra-type kernel defined by (4)) and $\Phi(t_1,\ldots,t_k) = \phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)$, $\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$ $(i = 0,1,\ldots,m)$, $i_1,\ldots,i_k = 0,1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t, T]$ satisfying the condition (7).

A stochastic integral similar to (8) with respect to the scalar standard Wiener process $(i_1 = \ldots = i_k \neq 0)$ and $\Phi(t_1,\ldots,t_k) \in L_2([t, T]^k)$ has been considered in [I-1] (1951) and is called the multiple Wiener stochastic integral.

Let us consider particular cases of Theorem 1 for $k=1,\ldots,4$

$$J[\psi^{(1)}]_{T,t}^{(i_1)}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)},$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{9}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

Consider the sum

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}}$$

Theorem 2 [1, Sect. 1.1] (2009). *Let* $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS in* $L_2[t,T]$ *such that* $\phi_j(x)\in C[t,T]$ *or* $\phi_j(x)$ *is piecewise continuous on* $[t,T]$ $\forall j\in\mathbf{N}$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are i.i.d. $N(0,1)$*-r.v.'s for various* i *or* j $(if\ i\neq 0)$, $i_1,\ldots,i_k=0,1,\ldots,m$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by (2),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient (6),* $[x]$ *is an integer part of a real number* x, $\mathbf{W}_\tau^{(0)}=\tau$.

To generalize Theorem 2 for an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$, we use the multiple Wiener stochastic integral [I-1] (1951)

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \Delta \mathbf{W}_{\tau_{l_1}}^{(i_1)} \ldots \Delta \mathbf{W}_{\tau_{l_k}}^{(i_k)},$$

where $\Phi(t_1, \ldots, t_k) \in L_2([t, T]^k)$, $\lim\limits_{N\to\infty} \left\|\Phi - \Phi_N\right\|_{L_2([t,T]^k)} = 0$,

$$\Phi_N(t_1, \ldots, t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1}, \tau_{l_1+1})}(t_1) \ldots \mathbf{1}_{[\tau_{l_k}, \tau_{l_k+1})}(t_k),$$

where $a_{l_1\ldots l_k} \in \mathbf{R}$ and such that $a_{l_1\ldots l_k} = 0$ if $l_p = l_q$ for some $p \neq q$,

$$\mathbf{1}_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A \\ 0 & \text{otherwise} \end{cases},$$

$\Delta \mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$, $i = 0, 1, \ldots, m$, $\mathbf{W}_\tau^{(0)} = \tau$, $\{\tau_j\}_{j=0}^N$ is a partition of $[t, T]$ satisfying the condition (7).

Note the well known estimate for the multiple Wiener stochastic integral

$$\mathbf{E}\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2 \leq C_k \left\|\Phi\right\|_{L_2([t,T]^k)}^2, \ \ C_k < \infty. \tag{10}$$

Theorem 3 (Generalization of Theorem 2) [1, Sect. 1.11]. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$. Then*

$$
J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),
\tag{11}
$$

where notations are the same as in Theorem 2.

Remark 2. The expression in parentheses is the Hermite polynomial of degree k of random vector argument. Moreover, (11) has the form

$$
J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J\,'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}.
$$

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Proof. Step 1.

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{12}$$

where $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$, permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Step 2. From (12) we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J'[K]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} + J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1},$$

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) = K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l).$$

Step 3. Applying (12), we have

$$J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Biggl(K(t_1,\ldots,t_k) -$$

$$-\sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \Biggr) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}.$$

Then

$$\mathsf{M}\left\{ \left(J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \right)^2 \right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k$$

$$= C_k \left\| K - K_{p_1\ldots p_k} \right\|_{L_2([t,T]^k)}^2 \to 0$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on k. Thus

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}.$$

Step 4. Applying the Itô formula, we obtain

$$J'[\phi_{j_1}\ldots\phi_{j_m}]_{T,t}^{(i_1\ldots i_m)} \cdot J'[\phi_{j'_1}\ldots\phi_{j'_n}]_{T,t}^{(1\ldots 1)} = J'[\phi_{j_1}\ldots\phi_{j_m}\phi_{j'_1}\ldots\phi_{j'_n}]_{T,t}^{(i_1\ldots i_m 1\ldots 1)} \tag{13}$$

w. p. 1, where $i_1,\ldots,i_m \neq 1$. The equality (13) follows from (12) and

$$\sum_{(j_1,\ldots,j_m)} \int_t^T \phi_{j_m}(t_m)\ldots\int_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_m}^{(i_m)}\times$$

$$\times\sum_{(j'_1,\ldots,j'_n)} \int_t^T \phi_{j'_n}(t'_n)\ldots\int_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots\mathbf{w}_{t'_n}^{(1)} =$$

$$=\sum_{(j_1,\ldots,j_m,j'_1,\ldots,j'_n)} \int_t^T \phi_{j_m}(t_m)\ldots\int_t^{t_2}\phi_{j_1}(t_1)\int_t^{t_1}\phi_{j'_n}(t'_n)\ldots\int_t^{t'_2}\phi_{j'_1}(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_n}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_m}^{(i_m)} \quad \text{w. p. 1},$$

where $i_1, \ldots, i_m \neq 1$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1, \ldots, j_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$.

Using the equality (13) and Theorem 3.1 from [I-1] (1951), we get w. p. 1 for an arbitrary CONS $\{\phi_j(x)\}_{j=0}^{\infty}$ in $L_2([t, T])$:

$$J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} = \begin{matrix} \textit{product of Hermite polynomials} \\ \textit{of arguments } \zeta_{j_1}^{(i_1)}, \ \ldots \ , \ \zeta_{j_k}^{(i_k)} \end{matrix} = \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}.$$

We also note the following usefull estimate [1, Sect. 1.12]

$$\mathsf{E}\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}\right)^2\le$$

$$\le C\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),$$

where $i_1,\ldots,i_k=0,1,\ldots,m$ and constant C depends only on k and $T-t$. Moreover, $C\le k!$ for the following two cases:

1) $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$,

2) $i_1,\ldots,i_k=0,1,\ldots,m,\quad i_1^2+\ldots+i_k^2>0$ and $T-t\in(0,1)$;

here $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}$ is the expression on the right-hand side of (11) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; other notations as in Theorem 3.

3 Stratonovich Stochastic Integral

Let $M_2[t, T]$ $(0 \le t < T < \infty)$ be the class of random functions $\xi(\tau,\omega) \stackrel{\rm def}{=} \xi_\tau : [t, T] \times \Omega \to \mathbf{R}$ satisfying the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau \in [t, T]$ and

$$\int\limits_t^T \mathbf{E}(\xi_\tau)^2 d\tau < \infty, \quad \mathbf{E}(\xi_\tau)^2 < \infty \quad \text{for all} \quad \tau \in [t, T].$$

Let $Q[t, T]$ be the class of Itô processes $\eta_\tau^{(i)}$, $\tau \in [t, T]$ $(i = 1, \ldots, m)$:

$$\eta_\tau^{(i)} = \eta_t^{(i)} + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s d\mathbf{W}_s^{(i)} \quad \text{w. p. 1},$$

where $(a_s)^4, (b_s)^4 \in M_2[t, T]$ and $\lim\limits_{s\to\tau} \mathbf{E} \left| b_s - b_\tau \right|^4 = 0$ for all $\tau \in [t, T]$.

The mean-square limit

$$\mathop{\rm l.i.m.}_{N\to\infty} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)} + \eta_{\tau_{j+1}}^{(i)}\right), \tau_j\right)\left(\mathbf{W}_{\tau_{j+1}}^{(l)} - \mathbf{W}_{\tau_j}^{(l)}\right) \stackrel{\rm def}{=} \int\limits_t^T F(\eta_\tau^{(i)}, \tau) \circ d\mathbf{W}_\tau^{(l)}$$

is called the Stratonovich stochastic integral, where $i, l =, \ldots, m$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t, T]$ satisfying the condition (7), $F : \mathbf{R} \times [t, T] \to \mathbf{R}$ and $F \in C^{2,1}(\mathbf{R}, [t, T])$ (here $C^{2,1}(\mathbf{R}, [t, T])$ is the space of functions that are twice differentiable with respect to x and once with respect to τ. Moreover, all these derivatives are bounded).

It is well-known that

$$\int\limits_t^T F\left(\eta_\tau^{(i)}, \tau\right) \circ d\mathbf{W}_\tau^{(l)} = \int\limits_t^T F\left(\eta_\tau^{(i)}, \tau\right) d\mathbf{W}_\tau^{(l)} + \frac{1}{2}\mathbf{1}_{\{i=l\}} \int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau, \tau) b_\tau d\tau$$

w. p. 1, where $\eta_\tau^{(i)} \in Q[t, T]$, $F \in C^{2,1}(\mathbf{R}, [t, T])$, $F\left(\eta_\tau^{(i)}, \tau\right) \in M_2[t, T]$, $\mathbf{1}_A$ is the indicator of the set A and $i, l = 1, \ldots, m$. For $F(x, \tau) \equiv F_1(x) F_2(\tau)$, smoothness with respect to τ can be replaced by continuity.

The iterated Stratonovich stochastic integral will be denoted as

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ d\mathbf{W}_{t_k}^{(i_k)}, \qquad (14)$$

where $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$.

4 Expansion of Iterated Stratonovich Stochastic Integrals

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be an arbitrary CONS in $L_2[t,T]$ ($\phi_0(x)=1/\sqrt{T-t}$). Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright(\cdot)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k=$$

$$=\hat C_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\hat C_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type (6) but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau)$, $\ldots$, $\psi_{l-2}(\tau)$, $\sqrt{T-t}\ \psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau)$, $\ldots$, $\psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (9))).

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright j_m} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k =$$

$$= \bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1},$$

i.e. $\bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1}$ is again the Fourier coefficient of type (6) but with a new shorter multi-index $j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \psi_{l-1}(\tau)\psi_l(\tau),\ \psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (9))).

Theorem 4 [1, Sect.2.10] (2023). *Assume that a CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ *in* $L_2([t,T])$ $(\phi_0(x)=1/\sqrt{T-t})$ *and continuous functions* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are such that*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl.\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}- \qquad (15)$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\right|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)^2=0$$

for all $r=1,2,\ldots,[k/2]$. *Then*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by (14), other notations as in Theorem 3.*

Proof. Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{W}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{W}_{t_{s_l-1}}^{(i_{s_l-1})} dt_{s_l+1} d\mathbf{W}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (16)$$

where $(s_l, \ldots, s_1) \in A_{k,l}$,

$$A_{k,l} = \Bigl\{(s_l, \ldots, s_1) : s_l > s_{l-1}+1, \ldots, s_2 > s_1+1; s_l, \ldots, s_1 = 1, \ldots, k-1\Bigr\}, \tag{17}$$

$l = 1, 2, \ldots, [k/2]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k.

Theorem 5 [1, Sect. 2.4] (1997). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$. *Then, the following relation between iterated Stratonovich and Itô stochastic integrals is correct*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$$

w. p. 1, where $\sum\limits_{\emptyset}$ *is supposed to be equal to zero.*

Consider the following representation for multiple Wiener stochastic integral [1, Sect. 1.11]

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\, i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\, j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}$$

w. p. 1, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\prod_{\emptyset}\stackrel{\rm def}{=}1.$

Further, we have w. p. 1

$$\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} -$$

$$- \sum_{r=1}^{[k/2]} (-1)^r \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}.$$

Iterated application of the above equality gives

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$$

w. p. 1, where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\rm def}{=}1$ for $k=2r$.

Multiplying both sides of the above equality by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \tag{18}$$

The equality (18) is a version of the Hu-Meyer formula for the kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ of the form (5) and the multidimensional Wiener process. Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in the equality (18), we have (see Theorem 3 and Remark 2)

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{19}$$

For the red color expression on the right-hand side of (19), we have (let $p_1=\ldots=p_k=p$ for simplicity)

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}} \times
$$

$$
\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}} \times
$$

$$
\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$
+\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right)\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\frac{1}{2^r}\prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1.} \qquad (20)$$

(by Theorem 3 and Remark 2; see explanation on the next page)

We have w. p. 1 for $g_2=g_1+1,\ \ldots,\ g_{2r}=g_{2r-1}+1$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\times$$

$$\times C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots0i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\ \text{(by Theorem 3 and Remark 2)}\ =\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]},\qquad (21)$$

where the notations are the same as in Theorem 5, $g_{2i-1} = s_i$ $(i = 1, 2, \ldots, r,\ r = 1, 2, \ldots, [k/2])$.

Using conditions of Theorem 4, the estimate (10) for multiple Wiener stochastic integral, (19)-(21) and Theorem 5, we obtain

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)} =$$

$$= J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} =$$

$$= J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$$

w. p. 1, where $J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (16). Theorem 4 is proved.

Consider a generalization of Theorem 4. Denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \stackrel{\rm def}{=} \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)},$$

where $\psi_1(\tau),\ldots,\psi_k(\tau),\psi_l(\tau)\psi_{l-1}(\tau)\in L_2[t,T]$ $(l=2,3,\ldots,k)$ and other notations as in Theorem 5.

From the proof of Theorem 4 we get

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\ldots,\psi_k(\tau)$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2[t,T]$ $(l=2,3,\ldots,k)$, other notations as in Theorem 4.

Let $\xi_\tau, \tau\in[0,T]$ be some measurable random process such that

$$\int_t^T |\xi_\tau| d\tau < \infty \quad \text{w. p. 1} \quad (t\ge 0).$$

The mean-square limit (if it exists) [SU-1]

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \frac{1}{\tau_{j+1}-\tau_j} \int\limits_{\tau_j}^{\tau_{j+1}} \xi_s ds \ \left(\mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}\right) \stackrel{\text{def}}{=} \int\limits_t^T \xi_\tau \circ \mathbf{W}_\tau^{(i)} \qquad (22)$$

is called the Stratonovich stochastic integral, where $i = 0, 1, \ldots, m$, $\{\tau_j\}_{j=0}^N$ is a partition of $[t, T]$ satisfying the condition (7).

We will denote the iterated Stratonovich integral (22) as follows

$$J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ d\mathbf{W}_{t_k}^{(i_k)},$$

where $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\mathbf{W}_\tau^{(0)} = \tau$.

From [BR-1] (2021) it follows that for the case $i_1 = \ldots = i_k \neq 0$ (the case of a scalar standard Wiener process) we have w. p. 1

$$J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}. \qquad (23)$$

Using Malliavin calculus for diffusion processes it is probably possible to prove (23) for the case $i_1, \ldots, i_k = 0, 1, \ldots, m$.

5 Problem of Verification of Condition (15)

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\rm def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),\, j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

Theorem 6 [1, Sect. 2.10] (2022-2023). *Assume that a CONS* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ *in* $L_2[t,T]$ *and continuous functions* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are such that the following condition holds*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (9)) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\text{def}}{=}\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

$<<$ Remark 3. The above condition is stronger than the condition (15) and it gives the way of the verification of (15) $>>$

Then

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)},$$

where $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by (14),* $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$C_{j_k\ldots j_1} = \int_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense*, $\mathbf{W}_\tau^{(0)} = \tau$,

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are i.i.d. $N(0,1)$*-r.v.'s for various* i *or* j $($*if* $i \neq 0)$.

Proof.

Step 1. Under the Conditions of Theorem 6, we get

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1},$$

where $l-1 \ge s+1$.

Step 2. Using Conditions of Theorem 6, we obtain

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1} = \frac{1}{2} C_{j_k\ldots j_1}\Big|_{(j_lj_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}.$$

Step 3. Recall the equality (19) for $p_1 = \ldots = p_k = p$:

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{24}$$

For the red color expression on the right-hand side of (24), we have using Step 1 and Step 2

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1}, \qquad (25)$$

where the notations are the same as in (16) and (17), $g_{2i-1}=s_i$ ($i=1,2,\ldots,r,\ \ r=1,2,\ldots,[k/2]$) and

$$R_{T,t}^{(p)g_1,g_2,\ldots,g_{2r-1},g_{2r}(i_{q_1}\ldots i_{q_{k-2r}})} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl((-1)^r \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} +$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} +$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} +$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} \Biggr) \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}.$$

Using conditions of Theorem 6, (24), (25) and Theorem 5, we obtain

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= J\left[\psi^{(k)}\right]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J\left[\psi^{(k)}\right]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} =$$

$$= J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$$

w. p. 1. Theorem 6 is proved.

6 Expansion of iterated Stratonovich Stochastic Integrals of Multiplicities $1-6$ Without Condition (15). The Cases of Legendre Polynomials and Trigonometric Functions

Theorem 7 [1, Sect. 2.1] (2011). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where $i_1,i_2=0,1,\ldots,m$.

Proof is based on the equality (according to Theorem 6)

$$\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau=\sum_{j_1=0}^{\infty}\int\limits_t^T\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)dsd\tau=\sum_{j_1=0}^{\infty}C_{j_1 j_1}.$$

Theorem 8 [1, Sect. 2.17] (2023). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t, T]$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

where $i_1, i_2, i_3 = 0, 1, \ldots, m$.

Proof is based on the equalities (according to Theorem 6)

$$\lim_{p_1,p_2,p_3\to\infty}\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3 j_1 j_1}\right)^2=0, \quad \lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3 j_3 j_1}\right)^2=0,$$

$$\lim_{p_1,p_2,p_3\to\infty}\sum_{j_2=0}^{p_2}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty} C_{j_1 j_2 j_1}\right)^2=0.$$

Theorem 9 [1, Sect. 2.12] (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then*

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)},$$

where $i_1,\ldots,i_4=0,1,\ldots,m$.

Proof is based on the equalities (according to Theorem 6)

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0,\quad \lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0.$$

Theorem 10 [1, Sect. 2.13] (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t, T]$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then*

$$J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)},$$

where $i_1, \ldots, i_5 = 0, 1, \ldots, m$.

Proof is based on the equalities (according to Theorem 6)

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}, j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Big|_{j_{g_1}=j_{g_2}} \right)^2 = 0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=0,$$

where $(\{g_1,g_2\},\{g_3,g_4\},\{q_1\})$ and $(\{g_1,g_2\},\{q_1,q_2,q_3\})$ are partitions of the set $\{1,2,\ldots,5\}$ that is

$$\{g_1,g_2,g_3,g_4,q_1\}=\{g_1,g_2,q_1,q_2,q_3\}=\{1,2,\ldots,5\};$$

braces mean an unordered set, and parentheses mean an ordered set.

Theorem 11 [1, Sect. 2.14] (2023). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $(\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1)$

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\circ\, d\mathbf{w}_{t_1}^{(i_1)}\ldots\circ\, d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$.

Proof is based on the equalities (according to Theorem 6)

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3},j_{q_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_6=g_5+1}\right)^2=0,$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_4=g_3+1,g_6=g_5+1}\right)^2=0,$$

where the expressions

$$(\{g_1, g_2\}, \{g_3, g_4\}, \{g_5, g_6\}), \quad (\{g_1, g_2\}, \{g_3, g_4\}, \{q_1, q_2\}),$$

$$(\{g_1, g_2\}, \{q_1, q_2, q_3, q_4\})$$

are partitions of the set $\{1, 2, \ldots, 6\}$ that is

$$\{g_1, g_2, g_3, g_4, g_5, g_6\} = \{g_1, g_2, g_3, g_4, q_1, q_2\} =$$

$$= \{g_1, g_2, q_1, q_2, q_3, q_4\} = \{1, 2, \ldots, 6\};$$

braces mean an unordered set, and parentheses mean an ordered set.

All the above equalities (instead of the red color expression) are proved similarly to the proof of Theorem 10). The red color expression is equivalent to the following 15 equalities:

$$\lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_2 j_1 j_3 j_2 j_1} = 0, \quad \text{(I)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_3j_2j_3j_2j_1} = 0, \quad \text{(II)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_1j_2j_1} = 0, \quad \text{(III)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_2j_3j_3j_2j_1} = 0, \quad \text{(IV)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_2j_2j_3j_3j_1} = 0, \quad \text{(V)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1} = 0, \quad \text{(VI)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_3j_2j_1j_1} = 0, \quad \text{(VII)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1} = 0, \quad \text{(VIII)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1} = 0, \quad \text{(IX)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1} = 0, \quad \text{(X)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_1j_3j_3j_2j_1} = 0, \quad \text{(XI)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_2j_3j_2j_1} = 0, \quad \text{(XII)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_1j_3j_2j_1} = 0, \quad \text{(XIII)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1} = 0, \quad \text{(XIV)}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_3j_1j_2j_1} = 0. \quad \text{(XV)}$$

For the proof of the above equalities we use the following interesting property for the Fourier coefficients

$$C_{j_6 j_5 j_4 j_3 j_2 j_1} + C_{j_1 j_2 j_3 j_4 j_5 j_6} = C_{j_6} C_{j_5 j_4 j_3 j_2 j_1} - C_{j_5 j_6} C_{j_4 j_3 j_2 j_1} +$$

$$+ C_{j_4 j_5 j_6} C_{j_3 j_2 j_1} - C_{j_3 j_4 j_5 j_6} C_{j_2 j_1} + C_{j_2 j_3 j_4 j_5 j_6} C_{j_1},$$

where

$$C_{j_6 \ldots j_1} = \int\limits_t^T \psi_6(t_6) \phi_{j_6}(t_6) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_6$$

Recall that $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$ in Theorem 11.

6 Rate of Convergence in Theorems 7-10

Theorem 12 [1, Sect. 1.7.2] (2020). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)$ $\in C^1[t,T]$. *Then* $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} \to J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *w. p. 1* $(p\to\infty)$ *and*

$$\mathsf{E}\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}\right)^2 \le \frac{C}{p},$$

where $i_1,\ldots,i_k = 0,1,\ldots,m$, *constant* C *depends only on* k *and* $T-t$,

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right).$$

Theorem 13 [1, Sect. 2.8.1] (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t, T]$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then*

$$\mathbf{E}\left(J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2 \le \frac{C}{p} \quad (p \in \mathbf{N}),$$

where $i_1, i_2 = 1, \ldots, m$, *constant* C *is independent of* p.

Theorem 14 [1, Sect. 2.15, 2.16] (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in the space* $L_2[t, T]$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then*

$$\mathbf{E}\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2 \le \frac{C}{p} \quad (p \in \mathbf{N}),$$

where $i_1, i_2, i_3 = 1, \ldots, m$, *constant* C *is independent of* p.

Theorem 15 [1, Sect. 2.15, 2.16] (2022). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then*

$$\mathbf{E}\left(J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}\right)^2\le\frac{C}{p^{1-\varepsilon}}\quad(p\in\mathbf{N}),$$

where $i_1,\ldots,i_4=1,\ldots,m$, *constant* C *does not depend on* p, $\varepsilon>0$ *is an arbitrary small positive real number for the case of CONS of Legendre polynomials in* $L_2[t,T]$ $(\psi_1(\tau),\ldots,\psi_4(\tau)\not\equiv 1)$ *and* $\varepsilon=0$ *for the case of CONS of Legendre polynomials in* $L_2[t,T]$ $(\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1)$ *or for the case of CONS of trigonometric functions in* $L_2[t,T]$ $(\psi_1(\tau),\ldots,\psi_4(\tau)\not\equiv 1$ *or* $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1)$.

Theorem 16 [1, Sect. 2.15, 2.16] (2022). *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$ *and* $\psi_1(\tau),\ \ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then*

$$\mathsf{E}\left(J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\le\frac{C}{p^{1-\varepsilon}}\quad(p\in\mathbf{N}),$$

where $i_1,\ldots,i_5=1,\ldots,m$, *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of CONS of Legendre polynomials in* $L_2[t,T]$ *and* $\varepsilon=0$ *for the case of CONS of trigonometric functions in* $L_2[t,T]$.

Remark 4. In Theorems 15 and 16, $\varepsilon>0$ appears due to the following estimate for the Legendre polynomials (since we need to eliminate divergent integrals)

$$|P_j(x)|<\frac{K}{\sqrt{j+1}(1-x^2)^{1/4}},\quad x\in(-1,1),\quad j\in\mathbf{N},\quad K<\infty.$$

7 An Iterated Passage to the Limit Removes Condition (15) on Trace Series

Theorem 17 [1, Sect. 2.4] (1997, 2021). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *are continuously differentiable nonrandom functions at the interval* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t, T]$. *Then*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \qquad (26)$$

where notations are the same as in Theorem 1 and (26) means the following

$$\lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \mathbf{E}\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}\right)^{2n} = 0,$$

where $n \in \mathbf{N}$.

8 New Result Based on Multiple Fourier-Walsh and Fourier-Haar Series

The mean-square limit (if it exists) [SU-1]

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}\ldots\sum_{l_k=0}^{N-1}\frac{1}{\Delta\tau_{l_1}\ldots\Delta\tau_{l_k}}\int\limits_{[\tau_{l_1},\tau_{l_1+1}]\times\ldots\times[\tau_{l_k},\tau_{l_k+1}]}\Phi(t_1,\ldots,t_k)dt_1\ldots dt_k\times$$

$$\times\,\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\stackrel{\text{def}}{=}\bar{J}^{S}[\Phi]_{T,t}^{(i_1\ldots i_k)} \qquad (27)$$

is called the multiple Stratonovich stochastic integral of $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$, where $\Delta\mathbf{W}_{\tau_j}^{(i)}=\mathbf{W}_{\tau_{j+1}}^{(i)}-\mathbf{W}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\Delta\tau_j=\tau_{j+1}-\tau_j$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (7), $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{W}_{\tau}^{(0)}=\tau$.

Theorem 18 [R-2] (2023). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Haar or Walsh functions in* $L_2[t,T]$ *and* $\Phi(t_1,\ldots,t_k)\in L_2[t,T]^k$ *is such that the multiple Stratonovich stohastic integral* $\bar{J}^S[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *(defined by (27)) exists. Then*

$$\bar{J}^S[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{2^p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where $i_1,\ldots,i_k=1,\ldots,m$,

$$\zeta_j^{(i)}=\int_t^T\phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are $i.i.d.$ $N(0,1)$*-r.v.'s for various* i *or* j,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient.

Thanks for your attention!