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New results on expansion of iterated Itô and Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The case of arbitrary CONS in $L_2[t, T]$

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1 Introduction

The importance of the problem of numerical integration of SDEs is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems under random disturbances and to the application of SDEs for solving various mathematical problems, among which we mention signal filtering, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems.

Iterated Itô and Stratonovich stochastic integrals can be used to construct high-order strong (mean-square) numerical methods for various types of systems of SDEs with non-commutative noise. For example, for

- Itô stochastic differential equations
- Itô stochastic differential equations with jumps
- McKean stochastic differential equations
- stochastic differential equations with switchings
- semilinear stochastic partial differential equations with multiplicative trace class noise

Let $(\Omega,\ F,\ \mathrm{P})$ be a complete probability space, let $\{F_t, t \in [0,\ T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{W}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,\ T]$. We assume that the components $\mathbf{W}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. As an example, consider a system of Itô SDEs with non-commutative noise

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \sum_{j=1}^m \int\limits_0^t B_j(\mathbf{x}_\tau, \tau)d\mathbf{W}_\tau^{(j)},\ \mathbf{x}_0 = \mathbf{x}(0, \omega), \qquad (1)$$

where $\mathbf{x}_t \in \mathbf{R}^n$, the nonrandom functions $\mathbf{a},\ B_j : \mathbf{R}^n \times [0,\ T] \to \mathbf{R}^n$ guarantee the existence and uniqueness up to stochastic equivalence of a strong solution of (1), $\mathbf{x}_0$ is F_0-measurable, $\mathbf{E}\left|\mathbf{x}_0\right|^2 < \infty$, $\mathbf{x}_0$ and $\mathbf{W}_t - \mathbf{W}_0$ are independent when $t > 0$.

• Suppose that $\mathbf{a}$ and B_j $(j = 1, \ldots, m)$ are several times continuously differentiable with respect to both arguments.

One of the effective approaches to the numerical integration of Itô SDEs is based on the Taylor–Itô and Taylor–Stratonovich expansions. These expansions contain iterated Itô and Stratonovich stochastic integrals:

$$J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (2)$$

$$J^{\,*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)}, \qquad (3)$$

where $\mathbf{W}_{\tau}^{(i)}$ $(i=1,\ldots,\ m)$ are independent standard Wiener processes, $\psi_1(\tau),\ldots,\psi_k(\tau)\ :\ [t,T]\ \to\ \mathbf{R},\ \mathbf{W}_{\tau}^{(0)}\ =\ \tau,\ i_1,\ldots,i_k\ =\ 0,\ 1,\ \ldots,\ m,$ $d\mathbf{W}_{\tau}^{(i)}$ and $\circ\, d\mathbf{W}_{\tau}^{(i)}$ denote Itô and Stratonovich differentials, respectively.

• A natural question arises: is it possible to construct a numerical scheme for Itô SDE that includes only increments of the Wiener processes but has a higher order of convergence than the Euler method? It is known that this is impossible for $m>1$ in the general case ("Clark–Cameron paradox"). This explains the need to use iterated stochastic integrals for constructing high-order strong numerical methods for Itô SDEs.

2 Expansion of Iterated Itô Stochastic Integrals

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$. Define the following function (factorized Volterra-type kernel) on the hypercube $[t, T]^k$:

$$K(t_1, \ldots, t_k) = \psi_1(t_1) \ldots \psi_k(t_k) \mathbf{1}_{\{t_1 < \ldots < t_k\}} \quad (k \geq 2), \quad K(t_1) = \psi_1(t_1). \tag{4}$$

Then

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K - K_{p_1 \ldots p_k} \right\|_{L_2([t,T]^k)} = 0,$$

where $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$,

$$K_{p_1 \ldots p_k}(t_1, \ldots, t_k) = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l), \tag{5}$$

$$C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k \tag{6}$$

is the Fourier coefficient corresponding to $K(t_1, \ldots, t_k)$.

Let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{7}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

Further, we will consider sums of the form

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}},$$

where $a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}} \in \mathbf{R}$.

Theorem 1 [Kuz1] (2023). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t, T]$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \underline{\underset{p_1,\ldots,p_k \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = \ i_{g_{2s}} \neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = \ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr),$$

where

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau) d\mathbf{W}_\tau^{(i)}$$

are i.i.d. $N(0,1)$*-r.v.'s for various* i *or* j *(if* $i \neq 0$*),* $i_1, \ldots, i_k = 0, 1, \ldots, m$, $J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)}$ *is defined by (2),* $C_{j_k \ldots j_1}$ *is the Fourier coefficient (6),* $[x]$ *is an integer part of* x, $\mathbf{1}_A$ *is the indicator of* A, $\mathbf{W}_\tau^{(0)} = \tau$, $\prod_{\emptyset} \stackrel{\text{def}}{=} 1$, $\sum_{\emptyset} \stackrel{\text{def}}{=} 0$.

• **Remark.** The case $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS in $L_2[t, T]$ such that $\phi_j(x) \in C[t, T]$ or $\phi_j(x)$ is piecewise continuous on $[t, T]$ $\forall j \in \mathbf{N}$ has been considered in **[Kuz3]** (2006–2009).

For the proof of Theorem 1, we use the multiple Wiener stochastic integral **[Ito]** (1951)

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)},$$

where $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$, $\lim\limits_{N\to\infty} \left\|\Phi - \Phi_N\right\|_{L_2([t,T]^k)} = 0$,

$$\Phi_N(t_1,\ldots,t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k),$$

where $a_{l_1\ldots l_k} \in \mathbf{R}$ and such that $a_{l_1\ldots l_k} = 0$ if $l_p = l_q$ for some $p \neq q$,

$$\mathbf{1}_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A \\ 0 & \text{otherwise} \end{cases},$$

$\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$, $\boxed{i = 0,1,\ldots,m,\ \mathbf{W}_{\tau}^{(0)} = \tau}$, $\{\tau_j\}_{j=0}^{N}$ such that $t = \tau_0 < \ldots < \tau_N = T$, $\Delta_N = \max\limits_{0\le j\le N-1}(\tau_{j+1} - \tau_j) \to 0$ if $N \to \infty$.

Note the well known estimate for the multiple Wiener stochastic integral

$$\boxed{\mathbf{E}\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2 \le C_k\left\|\Phi\right\|^2_{L_2([t,T]^k)},\ \ C_k<\infty} \tag{8}$$

• **Remark.** Theorem 1 can be reformulated as follows

$$J\,[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{9}$$

where

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{10}$$

• **Remark.** Another form (based on explicit product of Hermite polynomials) of the expansion from Theorem 1 can be found in **[Ryb1]** (2021).

• **Remark.** In **[FoxTaqqu]** (1987) an analogue of (10) for nonrandom $x_1, \ldots, x_k$ instead of $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$ is constructed using diagrams. This means that the application of the formula from **[FoxTaqqu]** (1987), unlike the formula (10), is difficult when performing algebraic transformations.

In **[FoxTaqqu]** (1987) (Proposition 5.1) an analogue of (10) is constructed for the special case $j_1 = \ldots = j_k$. Moreover, the specified analogue is based on diagrams, i.e. it is presented in implicit form.

As it turned out, a version of (10) was obtained in terms of Wick polynomials and for the case of vector valued random measures in **[Major]** (2019) (see Theorem 7.2, p. 69).

However, much earlier the formula (10) is obtained in **[Kuz3]** (2009) for the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS in $L_2[t, T]$ such that $\phi_j(x) \in C[t, T]$ or $\phi_j(x)$ is piecewise continuous on $[t, T]$ $\forall j \in \mathbf{N}$.

Let us consider particular cases of Theorem 1 for $k = 1, \ldots, 4$

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)},$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_1=i_2\neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr),$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

where $\mathbf{1}_A$ is the indicator of the set A.

3 Stratonovich Stochastic Integral

Let $M_2[t,T]$ $(0 \le t < T < \infty)$ be the class of random functions $\xi(\tau,\omega) \stackrel{\rm def}{=} \xi_\tau : [t,T]\times\Omega \to \mathbf{R}$ satisfying the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau\in[t,T]$ and

$$\mathbf{E}\left[\int\limits_t^T (\xi_\tau)^2 d\tau\right] < \infty, \quad \mathbf{E}(\xi_\tau)^2 < \infty \quad \text{for all} \quad \tau\in[t,T].$$

Let $Q[t,T]$ be the class of Itô processes $\eta_\tau^{(i)}$, $\tau\in[t,T]$ $(i=1,\ldots,m)$:

$$\eta_\tau^{(i)} = \eta_t^{(i)} + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s d\mathbf{W}_s^{(i)} \quad \text{w. p. 1},$$

where $(a_s)^4, (b_s)^4 \in M_2[t,T]$ and $\lim\limits_{s\to\tau} \mathbf{E}\left|b_s - b_\tau\right|^4 = 0$ for all $\tau\in[t,T]$.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)} + \eta_{\tau_{j+1}}^{(i)}\right), \tau_j\right)\left(\mathbf{W}_{\tau_{j+1}}^{(l)} - \mathbf{W}_{\tau_j}^{(l)}\right) \stackrel{\rm def}{=} \int\limits_t^{T} F(\eta_\tau^{(i)},\tau)\circ d\mathbf{W}_\tau^{(l)}$$

is called the Stratonovich stochastic integral, where $F \in C^{2,1}(\mathbf{R},[t, T])$, $i, l =, \ldots, m$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t, T]$ as in Sect. 2.

Note that if $F(x,\tau) = F_1(x)F_2(\tau)$, then the smoothness condition can be weakened: it suffices to replace the condition with respect to τ by continuity with respect to this variable.

It is well-known that

$$\int\limits_t^T F\left(\eta_\tau^{(i)},\tau\right) \circ d\mathbf{W}_\tau^{(l)} = \int\limits_t^T F\left(\eta_\tau^{(i)},\tau\right) d\mathbf{W}_\tau^{(l)} + \frac{1}{2}\mathbf{1}_{\{i=l\}}\int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau,\tau) b_\tau d\tau$$

w. p. 1, where $\eta_\tau^{(i)} \in Q[t, T]$, $F \in C^{2,1}(\mathbf{R},[t, T])$, $F\left(\eta_\tau^{(i)},\tau\right) \in M_2[t, T]$, $\mathbf{1}_A$ is the indicator of the set A and $i, l = 1, \ldots, m$.

he iterated Stratonovich stochastic integral will be denoted as

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ d\mathbf{W}_{t_k}^{(i_k)}, \qquad (11)$$

where $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$.

4 Expansion of iterated Stratonovich Stochastic Integrals of Multiplicities 2-6. The Case of Legendre Polynomials and Trigonometric Functions (Old Results)

Theorem 2 [Kuz3] (2018, 2022). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials or trigonometric functions in $L_2[t,T]$ and $\psi_1(\tau)$, $\psi_2(\tau), \psi_3(\tau) \in C^1[t,T]$. Then, for the iterated Stratonovich stochastic integrals of 2nd and 3rd multiplicities we have*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \quad (k=2,\ 3), \tag{12}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\} \le \frac{C}{p} \quad (k=2,\ 3), \tag{13}$$

where $i_1, i_2, i_3 = 0, 1, \ldots, m$ in (12) and $i_1, i_2, i_3 = 1, \ldots, m$ in (13), constant C is independent of p; another notations as in Theorem 1.

Theorem 3 [Kuz3] (2022). *Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a CONS of Legendre polynomials or trigonometric functions in $L_2[t,T]$ and $\psi_1(\tau),\ldots,\psi_5(\tau)\in C^1[t,T]$. Then, for the iterated Stratonovich stochastic integral of 4th and 5th multiplicities we have*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\quad (k=4,\ 5),\qquad (14)$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}\quad (k=4,\ 5),\qquad (15)$$

where $i_1,\ldots,i_5=0,1,\ldots,m$ in (14) and $i_1,\ldots,i_5=1,\ldots,m$ in (15), constant C does not depend on p, ε is an arbitrary small positive real number for the case of Legendre polynomials and $\varepsilon=0$ for the case of trigonometric functions; another notations as in Theorem 1.

Theorem 4 [Kuz3] (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric functions in* $L_2[t,T]$. *Then, for the iterated Stratonovich stochastic integral of 6th multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_6}^{(i_6)} \qquad (16)$$

we have

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)},$$

where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6 \quad \text{and} \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{W}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$), $\mathbf{W}_\tau^{(0)} = \tau$.

5 Expansion of Iterated Stratonovich Stochastic Integrals (New Results)

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be an arbitrary CONS in $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$. Denote

$$
C_{j_k\ldots j_{l+1}\ \underline{j_l\ j_l}\ j_{l-2}\ldots j_1}\Bigg|_{(j_l j_l)\curvearrowright(\cdot)} \stackrel{\rm def}{=}
$$

$$
\stackrel{\rm def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\underline{\psi_l(t_l)\psi_{l-1}(t_l)}\times
$$

$$
\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}\ \underline{dt_l}\ t_{l+1}\ldots dt_k,
$$

where we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (7)).

Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \underline{\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})}\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \underline{\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})}\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{s_1-1}}^{(i_{s_1-1})}\ \underline{dt_{s_1+1}}\ d\mathbf{W}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{W}_{t_{s_l-1}}^{(i_{s_l-1})}\ \underline{dt_{s_l+1}}\ d\mathbf{W}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (17)$$

where $(s_l, \ldots, s_1) \in A_{k,l}$,

$$A_{k,l} = \bigl\{(s_l, \ldots, s_1) : s_l > s_{l-1}+1, \ldots, s_2 > s_1+1; s_l, \ldots, s_1 = 1, \ldots, k-1\bigr\}, \tag{18}$$

$l = 1, 2, \ldots, [k/2]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k.

Theorem 5 [Kuz3] (1997). *Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals is correct*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$$

w. p. 1, where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Theorem 6 [Kuz2] (2024). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$ and the following condition*

$$\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\Biggl(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)^2=0 \tag{19}$$

is satisfied for all $r=1,2,\ldots,[k/2]$ and for all possible permutations of the set $(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})$, where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set. Then

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{20}$$

where

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in A_{k,r}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}\stackrel{\text{def}}{=}\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}.$$

Theorem 7 (Corollary from Theorem 6) [Kuz2] (2024). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$. *Furthermore, another conditions of Theorem 6 are fulfilled. Then*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)},$$

where $k \in \mathbf{N}$, $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is the Stratonovich stochastic integral of the form*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ d\mathbf{W}_{t_k}^{(i_k)};$$

another notations are the same as in Theorem 6.

Proof of Theorem 7. By Theorem 5: $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ w. p. 1 for $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$. $\square$

Proof of Theorem 6. Consider the following representation for multiple Wiener stochastic integral **[Kuz1]**

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}$$

w. p. 1, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $r=1,2,\ldots,[k/2]$, $\prod_{\emptyset}\stackrel{\rm def}{=}1$, $\sum_{\emptyset}\stackrel{\rm def}{=}0$, $[x]$ is an integer part of x, $\mathbf{1}_A$ is the indicator of A.

Further, we have w. p. 1

$$\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} -$$

$$- \sum_{r=1}^{[k/2]} (-1)^r \sum_{\substack{(\{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}\}, \{q_1, \ldots, q_{k-2r}\}) \\ \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}.$$

Iterated application of the above equality gives

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$$

w. p. 1, where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \stackrel{\text{def}}{=} 1$ for $k=2r$.

Multiplying both sides of the above equality by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+ \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{22}$$

Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (22), we have (see Theorem 1)

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+ \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{23}$$

For the red color expression on the right-hand side of (23), we have

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \boxed{\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Big|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \right)\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} +$$

$$+\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = [\textit{by } (8) \textit{ and } (19))] =$$

$$= \frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \quad \text{w. p. 1.} \qquad (24)$$

(the proof of (24) is based on Theorem 1 and is presented below)

To prove (24) it is enough to prove that

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} \qquad (25)$$

w. p. 1, where $g_2 = g_1 + 1, \ldots, g_{2r} = g_{2r-1} + 1$, $g_{2i-1} \stackrel{\text{def}}{=} s_i$; $i = 1, 2, \ldots, r$, $r = 1, 2, \ldots, [k/2]$, $(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (17) and $\mathrm{A}_{k,r}$ is defined by (18); another notations in (25) are the same as in Theorems 5 and 6.

• **Remark.** The equality (25) is proved for the case $\phi_0(x) \equiv 1/\sqrt{T-t}$ in **[Kuz3]** (2022). Let us prove (25) for the general case.

Using the Itô formula, we obtain w. p. 1

$$\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) \int\limits_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\ldots \qquad (26)$$

$$\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{l-2}}^{(i_{l-2})} dt_{l-1} d\mathbf{W}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{W}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1} \right)\times$$

$$\times \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{l-2}}^{(i_{l-2})} d\mathbf{W}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{W}_{t_k}^{(i_k)} -$$

$$- \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1} \right)\times$$

$$\times \int\limits_t^{t_{l-2}} \psi_{l-3}(t_{l-3})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{l-3}}^{(i_{l-3})} d\mathbf{W}_{t_{l-2}}^{(i_{l-2})} d\mathbf{W}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (27)$$

where $l \geq 3$. Note that the formula (27) will change in an obvious way for the case $t_{l+1} = T$. We will also assume that the transformation (27) is not carried out for $l = 2$ since the integral

$$\int\limits_t^{t_3} \psi_2(t_1)\psi_1(t_1)dt_1$$

is an internal integral on the left-hand side of (27) for this case.

• **Remark.** Obviously, under the conditions of Theorem 6, the derivation of (27) will remain valid if in (27) we replace all differentials of the form $d\mathbf{W}_{t_j}^{(i_j)}$ with dt_j and we use Fubuni's Theorem instead of the Itô formula.

Let us carry out the transformation (27) for the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ iteratively for $s_1, \ldots, s_r$. After this, apply (9) (Theorem 1) to each of the obtained iterated Itô stochastic integrals. As a result, we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} = [\text{by } (27)] = \prod_{q=1}^{r} \mathbf{1}_{\{i_{s_q}=i_{s_q+1}\neq 0\}} \times$$

$$\times \sum_{d=1}^{2^r} \left(\hat{J}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_k)[s_r,\ldots,s_1]} - \bar{J}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_k)[s_r,\ldots,s_1]} \right) =$$

$$= [\text{by Theorem 1}] = \prod_{q=1}^{r} \mathbf{1}_{\{i_{s_q}=i_{s_q+1}\neq 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_{s_1-1},j_{s_1+2},\ldots,j_{s_r-1},j_{s_r+2},\ldots,j_k=0}^{p} \sum_{d=1}^{2^r} \left(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k} - \right.$$

$$\left. -\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k} \right) \times$$

$$\times J'[\phi_{j_1} \ldots \phi_{j_{s_1-1}} \phi_{j_{s_1+2}} \ldots \phi_{j_{s_r-1}} \phi_{j_{s_r+2}} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_{s_1-1} i_{s_1+2}\ldots i_{s_r-1} i_{s_r+2}\ldots i_k)}, \quad (28)$$

where some terms in the sum $\sum\limits_{d=1}^{2^r}$ can be identically equal to zero due to the remark to (27).

• **Remark.** Taking into account that the integrals $\hat{J}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_k)[s_r,\ldots,s_1]}$ and the Fourier coefficients $\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$ are formed on the basis of the same kernels (the same applies to the integrals $\bar{J}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_k)[s_r,\ldots,s_1]}$ and the Fourier coefficients $\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$), as well as a remark about the relationship of the transformation (27) based on the Itô formula and on the basis of Fubuni's Theorem, we obtain using Fubini's theorem (applying the inverse transformation from (27) to (26) in which all differentials of the form $d\mathbf{W}_{t_j}^{(i_j)}$ are replaced with dt_j)

$$\sum_{d=1}^{2^r}\left(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}-\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}\right)=$$
$$=C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (29)$$

Combining (29) and (28), we get (25).

Using (23) and (24), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]} =$$

$$= \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$ is defined by (17). Theorems 6 and 7 are proved. □

Theorem 8 [Kuz3] (2024). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$ $(k=2,3,4,5)$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ $(k=2,3,4,5)$ *of iterated Itô stochastic integrals defined by (21) we have*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where notations are the same as in Theorem 1.

Theorem 9 (Corollary from Theorem 8) [Kuz3] (2024). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$ $(k=2,3,4,5)$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ $(k=2,3,4,5)$ *defined by (3) we have*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where notations are the same as in Theorem 1.

Proof of Theorem 8 is based on verification of the condition (19) for $k = 2, 3, 4, 5$. At that, cases $k = 2, 3, 4, 5$ are reduced to verification of 1, 3, 9 and 25 conditions, respectively, for different values of r and different pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$. Let us list these conditions.

Case k=2:

$$\sum_{j=0}^{\infty} \int_t^T \psi_2(t_2)\phi_j(t_2) \int_t^{t_2} \psi_1(\tau)\phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_t^T \psi_1(\tau)\psi_2(\tau) d\tau,$$

Case k=3:

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_3 j_2 j_1}\Big|_{j_1=j_2} - \frac{1}{2} C_{j_3 j_2 j_1}\Big|_{(j_1 j_2)\curvearrowright(\cdot), j_1=j_2} \right)^2 = 0,$$

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_2 j_1}\Big|_{j_2=j_3} - \frac{1}{2} C_{j_3 j_2 j_1}\Big|_{(j_2 j_3)\curvearrowright(\cdot), j_2=j_3} \right)^2 = 0,$$

$$\lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_3 j_2 j_1}\Big|_{j_1=j_3} \right)^2 = 0.$$

Case k=4:

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4j_3j_1j_1}-\frac{1}{2}C_{j_4j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4j_1j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_4j_2j_2j_1}-\frac{1}{2}C_{j_4j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}\right)^2=0,\qquad \boxed{\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=0,}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_2j_1}-\frac{1}{2}C_{j_3j_3j_2j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0,$$

$$\boxed{\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}=\frac{1}{4}C_{j_3j_3j_1j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)(j_1j_1)\curvearrowright(\cdot)},\qquad \lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=0,}$$

Case k=5:

$$\lim_{p\to\infty}\sum_{j_3,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_4j_3j_1j_1}-\frac{1}{2}C_{j_5j_4j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_4j_1j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2,j_3,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_1j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2,j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_4j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_2j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_4,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_4j_2j_2j_1}-\frac{1}{2}C_{j_5j_4j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_4j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_4j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_5=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_5j_3j_3j_2j_1}-\frac{1}{2}C_{j_5j_3j_3j_2j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\left(\sum_{j_4=0}^{p}C_{j_4j_4j_3j_2j_1}-\frac{1}{2}C_{j_4j_4j_3j_2j_1}\Bigg|_{(j_4j_4)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_5j_3j_3j_1j_1}-\frac{1}{4}C_{j_5j_3j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot),(j_3j_3)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p}C_{j_2j_3j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_2j_1j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_3j_1j_1}-\frac{1}{4}C_{j_4j_4j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_4j_1j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_1j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_1j_2j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_4j_2j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_3j_2j_1}\right)^2=0,\quad \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_2j_1}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_4=0}^{p}C_{j_4j_4j_2j_2j_1}-\frac{1}{4}C_{j_4j_4j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0,$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2j_1}\right)^2=0,$$

$$\underline{\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_4j_3j_1j_1}\right)^2=0.} \tag{30}$$

• **Example. Case k=5.** Proof of (30) (two pairs). Let $\psi_1(\tau),\ldots,\psi_5(\tau) \equiv 1$ for simplicity. Using Fubini's Theorem, we get

$$\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_3 j_4 j_3 j_1 j_1}\right)^2 =$$

$$= \sum_{j_4=0}^{p}\left(\int\limits_t^T \phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\times\right.$$

$$\left.\times\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2 \le$$

$$\le \sum_{j_4=0}^{\infty}\left(\int\limits_t^T \phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\times\right.$$

$$\left.\times\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2 = \text{[Parseval's equality]} =$$

$$= \int_t^T \left(\sum_{j_1,j_3=0}^{p} \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \int_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 =$$

$$= \int_t^T \left(\sum_{j_3=0}^{p} \int_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 \mp \frac{t_3 - t}{2} \right) dt_3 \int_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \le$$

$$\le 2 \int_t^T \left(\sum_{j_3=0}^{p} \int_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \int_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 +$$

$$+ 2 \int_t^T \left(\sum_{j_3=0}^{p} \int_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \le$$

[Cauchy-Bunyakovsky's inequality and Parseval's equality]

$$\leq K_1 \int\limits_t^T \sum_{j_3=0}^{p} \left(\int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \right)^2 dt_4 +$$

$$+2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \leq$$

$$\leq K_1 \int\limits_t^T \sum_{j_3=0}^{\infty} \left(\int\limits_t^{t_4} \phi_{j_3}(t_3) \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right) dt_3 \right)^2 dt_4 +$$

$$+2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int\limits_{t_4}^{T} \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 =$$

[Parseval's equality]

$$= K_1 \int\limits_t^T \int\limits_t^{t_4} \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right)^2 dt_3 dt_4 +$$

$$+2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 =$$

$$= K_1 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3 < t_4\}} \left(\frac{1}{2} \sum_{j_1=0}^{p} \left(\int\limits_t^{t_3} \phi_{j_1}(t_2) dt_2 \right)^2 - \frac{t_3 - t}{2} \right)^2 dt_3 dt_4 +$$

$$+2 \int\limits_t^T \left(\sum_{j_3=0}^{p} \int\limits_t^{t_4} \phi_{j_3}(t_3) \frac{t_3 - t}{2} dt_3 \int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 \ \to 0 \quad \text{as} \quad p \to \infty.$$

[Parseval's equality, generalized Parseval's equality and term-by-term integration of absolutely convergent series]

6 Calculation of Matrix Traces of Volterra-Type Integral Operators

It is easy to see that the function

$$K(t_1,\ldots,t_k)=\psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}_{\{t_1<\ldots<t_k\}}\ \ (k\ge 2) \qquad (31)$$

for even $k=2r$ ($r\in\mathbf{N}$) forms a family of integral operators $\mathbb{K}: L_2([t,T]^r)\to L_2([t,T]^r)$ of the form

$$(\mathbb{K}f)(t_{g_1},\ldots,t_{g_r})=\int\limits_{[t,T]^r}K(t_1,\ldots,t_k)f(t_{g_{r+1}},\ldots,t_{g_k})dt_{g_{r+1}}\ldots dt_{g_k}, \qquad (32)$$

where $\{g_1,\ldots,g_k\}=\{1,\ldots,k\}$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T],\ t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$. For example,

$$(\mathbb{K}f)(t_3,t_4)=\int\limits_{[t,T]^2}K(t_1,\ldots,t_4)f(t_1,t_2)dt_1dt_2=$$

$$=\mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_4(t_4)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)f(t_1,t_2)dt_1dt_2.$$

• **Remark.** It is well known that the Volterra integral operator (the simplest operator from the family (32)) is not a trace class operator. On the other hand, it is known that for trace class operators the equality of matrix and integral traces holds. It is known that for the Volterra integral operator (although it is not a trace class operator), the equality of matrix and integral traces is also true. Thus, one cannot count on the fact that operators of the more general form (32) are operators of the trace class. As a result, the proof of the equalities of matrix and integral traces for Volterra–type integral operators (32) is a problem.

• **Remark.** Recall the condition (19) from Theorem 6

$$\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)^2=0.$$

The red-colored expression is the difference between the prelimit expression for the matrix trace of the operator (32) and its integral trace.

Theorem 10. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$. *Then the equality*

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \qquad (33)$$

is satisfied for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (7)) and for any fixed* $j_1,\ldots,j_q,\ldots,j_k$ $(q\neq g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$*, where* $k\geq 2r$ *and* $r=1,2,\ldots,[k/2]$.

Furthermore, the series (33) converges absolutly for $k=2r$ *and converges absolutely for any fixed* $j_1,\ldots,j_q,\ldots,j_k$*, where* $q\neq g_1,g_2,\ldots,g_{2r-1}$, g_{2r} *for* $k>2r$.

Proof of Theorem 10. **Step 1.** The case $k=2$ and $r=1$. The equality (33) for the case $k=2$ and $r=1$ looks as follows

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(\tau)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau, \qquad (34)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in L_2[t,T]$.

The equality (34) is proved in

[Kuz2] (2018) for special CONS in $L_2[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in C^1[t,T]$.

• **[Ryb3]** (2023) for an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in L_2[t,T]$.

[Kuz2] (2024) for an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in C^1[t,T]$.

Step 2. The case $k = 2r$ of (33) for

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1,$$

i.e. all pairs are formed by adjacent indices

$$\lim_{p\to\infty} \sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{p} C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2j_2} = \frac{1}{2^r}\int_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times$$

$$\times \int_t^{t_{2r}} \psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots \int_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r},$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_{2r}(\tau) \in L_2[t,T]$ and $r \in \mathbf{N}$.

- **[Ryb3]** (2023) (the proof of Step 2 on the base of trace class operators). **[Kuz3]** (2024) (the proof of Step 2 on the base of Step 1 and induction).

Step 3. The case $k = 2r$ of (33) and without the condition

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1.$$

This case is considered in **[Kuz3]** (2024). We will consider an example on Step 3 in Appendix 1 for this presentation.

Step 4. General case $k \geq 2r$ of (33). This case is proved in **[Kuz3]** (2024). The proof will be given in Appendix 2 for this presentation.

Using Theorem 10, consider some sufficient conditions under which Theorems 6 and 7 are satisfied.

Suppose that

$$\exists \quad A \stackrel{\text{def}}{=} \lim_{p,q\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{q} \left(\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right)^2 < \infty \tag{35}$$

for all $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)) and $r = 1, 2, \ldots, [k/2]$. Then by Theorem 10

$$A = \lim_{q\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{q} \lim_{p\to\infty} \left(\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Big|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right)^2 = 0.$$

Suppose that

$$\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\left(\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}\left|C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right|\right)^2<\infty \qquad (36)$$

for all $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (7)) and $r = 1, 2, \ldots, [k/2]$. Then by MCT

$$\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Big|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)^2=$$

$$=\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\lim_{p\to\infty}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Big|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)^2=0.$$

Theorem 11 [Kuz3] (2024). *Suppose that one of the conditions (35) or (36) is fulfilled, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$. Then, for the sum of iterated Itô stochastic integrals*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$$

the following expansion

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\rm l.i.m.} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

holds. If in addition $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$, then for

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\rm l.i.m.} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

holds, where notations are the same as in Theorem 1.

7 Generalization of Theorem 7 $(k = 2)$ to the Case $\psi_1(\tau),\ \psi_2(\tau) \in L_2[t, T]$

Consider another definition of the Stratonovich stochastic integral. Let $\xi_\tau, \tau \in [0, T]$ be some measurable random process such that

$$\int\limits_t^T |\xi_\tau| d\tau < \infty \quad \text{w. p. 1} \quad (t \geq 0).$$

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \frac{1}{\tau_{j+1} - \tau_j} \int\limits_{\tau_j}^{\tau_{j+1}} \xi_s ds \ \left(\mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}\right) \stackrel{\text{def}}{=} \int\limits_t^{T} \xi_\tau \circ \mathbf{W}_\tau^{(i)} \qquad (37)$$

is called the Stratonovich stochastic integral, where $i = 0, 1, \ldots, m$, $\{\tau_j\}_{j=0}^N$ is a partition of $[t, T]$ as in Sect. 2.

We will denote the iterated Stratonovich integral (37) as follows

$$J^{S}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)},$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$, $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{W}_{\tau}^{(0)}=\tau$.

Theorem 12 [Kuz2] (2024). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in L_2[t,T]$. Then*

$$J^{S}[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

where $i_1,i_2=1,\ldots,m$; another notations as in Theorem 1.

Proof. Using step functions technique, it can be shown that w. p. 1

$$J^{S}[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=J[\psi^{(2)}]_{T,t}^{(i_1 i_2)}\ \ (i_1\neq i_2),$$

where $J[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ is the iterated Itô stochastic integral.

In [BardRov] (2021) using the Malliavin calculus it was shown that w. p. 1

$$J^{S}[\psi^{(2)}]_{T,t}^{(i_1 i_1)} = J[\psi^{(2)}]_{T,t}^{(i_1 i_1)} + \frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau \quad (i_1 = i_2).$$

Then, using a generalization of Theorem 6 for the case $p_1, p_2 \to \infty$, we obtain w. p. 1

$$J^{S}[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \bar{J}[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where $\bar{J}[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ is the sum of iterated Itô stochastic integrals defined by (21). Theorem 12 is proved. □

Appendix 1. Example on Step 3 in the Proof of Theorem 10

Let us prove that

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p} C_{j_3 j_4 j_4 j_3 j_1 j_1} =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p} \int\limits_t^T \psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{t_6} \psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5} \psi_4(t_4)\phi_{j_4}(t_4)\times$$

$$\times\int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2 dt_3 dt_4 dt_5 dt_6 = 0, \tag{38}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2[t,T]$.

Step I. Using Step 2 ($k=2$) and generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{\to T}\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{\to T}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^{\to T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6= \qquad (39)$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)dt_6\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)dt_3\times$$

$$\times\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)dt_4dt_5\times$$

$$\times\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$= \int_t^T \psi_6(t_6)\psi_3(t_6)dt_6 \cdot \frac{1}{2}\int_t^T \psi_5(t_4)\psi_4(t_4)dt_4 \cdot \frac{1}{2}\int_t^T \psi_2(t_2)\psi_1(t_2)dt_2. \quad (40)$$

Rewrite (40) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty} \int_{[t,T]^6} \mathbf{1}_{\{t_1<t_2\}} 1_{\{t_4<t_5\}} \psi_6(t_6)\phi_{j_3}(t_6)\psi_5(t_5)\phi_{j_4}(t_5)\psi_4(t_4)\phi_{j_4}(t_4)\times$$

$$\times\psi_3(t_3)\phi_{j_3}(t_3)\psi_2(t_2)\phi_{j_1}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int_{[t,T]^3} \psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \qquad (41)$$

Step II. Suppose that $\psi_2(\tau) = \bar{\phi}_{l_1}(\tau), \psi_3(\tau) = \bar{\phi}_{l_2}(\tau), \psi_4(\tau) = \bar{\phi}_{l_3}(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2, t_3, t_4) = \sum_{l_1,l_2.l_3=0}^{q} C_{l_3 l_2 l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3)\bar{\phi}_{l_3}(t_4),$$

where $\{\bar{\phi}_j(x)\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials in $L_2[t, T]$ and $C_{l_3 l_2 l_1}$ are Fourier–Legendre coefficients for the function

$$g(t_2, t_3, t_4) = \psi_2(t_2)\psi_3(t_3)\psi_4(t_4)\mathbf{1}_{\{t_2<t_3\}},$$

where $\psi_2(\tau), \psi_3(\tau), \psi_4(\tau) \in L_2[t, T]$.

From (41) we obtain

$$\sum_{j_1, j_3, j_4=0}^{\infty} \int\limits_{[t,T]^6} \mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_4<t_5\}} s_q(t_2, t_3, t_4)\psi_6(t_6)\psi_5(t_5)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int\limits_{[t,T]^3} s_q(t_2, t_6, t_4)\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)dt_2dt_4dt_6. \qquad (42)$$

• **Remark.** Note that the equality (42) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

• **Remark.** The right-hand side of (42) defines (as a scalar product of $s_q(t_2,t_6,t_4)$ and $\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$ given by the function $\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$. On the left-hand side of (42) (by virtue of the equality (42)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$.

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (42)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<\underline{t_2<t_3}\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\times$$

$$\times\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6.$$

After **Step III** and **Step IV** (by analogy with **Step II**) we obtain

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<\underline{t_2<t_3<t_4}<\underline{t_5<t_6}\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\times$$

$$\times\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\underbrace{\mathbf{1}_{\{t_6<t_4\}}\mathbf{1}_{\{t_4<t_6\}}}_{=0}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\times$$

$$\times\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6=0.$$

The equality (38) is proved.

Appendix 2. Proof of Step 4 in the Proof of Theorem 10

Using Fubini's Theorem, we obtain

$$\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (43)$$

where $2 < l < k-1$ and $h_1(\tau), \ldots, h_k(\tau) \in L_2[t, T]$. The case $l = k$ is considered by analogy with (43). The case $l = 1$ is obvious.

Suppose that $k > 2r$. Let us carry out the transformation (43) for

$$C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

iteratively for $j_{q_1}, \ldots, j_{q_{k-2r}}$ $(k > 2r)$. As a result, we obtain

$$C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \left(\hat{C}^{(d)}_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \bar{C}^{(d)}_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right), \tag{44}$$

where some terms in the sum

$$\sum_{d=1}^{2^{k-2r}}$$

can be identically equal to zero.

Applying (44) and **Step 3** ($k = 2r$) in the proof of Theorem 10, we get

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}\Biggl(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$- \bar{C}^{(d)}_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr) =$$

$$= \sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\times$$

$$\times\Biggl(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowleft(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowleft(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$- \bar{C}^{(d)}_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowleft(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowleft(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr). \qquad (45)$$

Case A. The condition

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1 \qquad (46)$$

is fulfilled for

$$\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},\ \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \quad (d=1,2,\ldots,2^{k-2r}), \qquad (47)$$

$$C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (48)$$

Case B. The quantities (47) are such that the condition (46) is satisfied for (47). The expression (48) is such that the condition

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0 \qquad (49)$$

is fulfilled for (48).

Case C. The quantities (47) are such that the condition (49) is satisfied for (47). The expression (48) is such that (49) is fulfilled for (48).

For **Case A**, using transformation (43), we obtain

$$\sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \left(\hat{C}_{j_k \ldots j_1}^{(d)} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} - \right.$$

$$\left. - \bar{C}_{j_k \ldots j_1}^{(d)} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} \right) =$$

$$= \frac{1}{2^r} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} . \qquad (50)$$

For **Case B**

$$\hat{C}_{j_k \ldots j_1}^{(d)} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} =$$

$$= \bar{C}_{j_k \ldots j_1}^{(d)} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} . \qquad (51)$$

For **Case C**

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0. \tag{52}$$

Combining (50), (51), (52) and (45), we complete the proof of **Step 4** ($k > 2r$) in the proof of Theorem 10.

Thanks for your attention!