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EXPANSION OF THE STRATONOVICH MULTIPLE STOCHASTIC INTEGRALS BASED ON THE FOURIER MULTIPLE SERIES

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An expansion of multiple Stratonovich stochastic integrals of multiplicity k, $k \in N$, into multiple series of products of Gaussian random variables is obtained. The coefficients of this expansion are the coefficients of multiple Fourier-series expansion of a function of several variables relative to a complete orthonormal system in the space $L_2([t,T])$. The convergence in mean of order n, $n \in N$, is established. Some expansions of multiple Stratonovich stochastic integrals with the help of polynomial and trigonometric systems are considered. Bibliography: 8 titles.

§1. Introduction

Consider a fundamental probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a Wiener random process $\mathbf{w}_t \in \mathbb{R}^n$ with independent components $\mathbf{w}_t^{(i)}$, $i = 1, \ldots, m$. We consider a nondecreasing family of σ-algebras $\{\mathcal{F}_t,\, t \in [0,T]\}$ of subsets of $(\Omega, \mathcal{F}, \mathbf{P})$ such that, for every $t \in [0,T]$, the random variable $\mathbf{w}_t$ is $\mathcal{F}_t$-measurable.

We introduce the class $\mathcal{M}_2([0,T])$ of functions

$$\xi_t \stackrel{\text{def}}{=} \xi(t,\omega) : [0,T] \times \Omega \to \mathbb{R}^1$$

measurable with respect to the pair (t,ω) of variables and $\mathcal{F}_t$-measurable for all $t \in [0,T]$. It is assumed that

$$\int_0^T \mathbf{E}\left\{(\xi(t,\omega))^2\right\} dt < \infty \quad \text{and} \quad \mathbf{E}\left\{(\xi(t,\omega))^2\right\} < \infty$$

for all $t \in [0,T]$. The class $\mathcal{M}_2([0,T])$ is endowed with the norm

$$\|\xi\|_{2,T} = \left(\int_0^T \mathbf{E}\left\{(\xi(t,\omega))^2\right\} dt\right)^{\frac{1}{2}}.$$

Consider the process $\left\{\eta_\tau^{(i)},\, \tau \in [t,T]\right\}$ of the form

$$\eta_\tau^{(i)} = \eta_t^{(i)} + \int_t^\tau a_s ds + \int_t^\tau b_s d\mathbf{w}_s^{(i)}, \tag{1}$$

where a_s^2, $b_s^2 \in \mathcal{M}_2([t,T])$, the inequality $\mathbf{E}\left\{(b_s - b_\tau)^4\right\} \le C|s-\tau|^\gamma$ holds for all s, $\tau \in [t,T]$ and some $C, \gamma \in (0,\infty)$, $\mathbf{w}_s^{(0)} = s$, and $\int_t^\tau b_s d\mathbf{w}_s^{(i)}$ is the stochastic Itô integral ($i = 1, \ldots, m$).

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The symbol $\int\limits_t^{*T} \eta_s^{(i)} d\mathbf{w}_s^{(j)}$ denotes the following sum:

$$\int\limits_t^{*T} \eta_s^{(i)} d\mathbf{w}_s^{(j)} = \int\limits_t^{T} \eta_s^{(i)} d\mathbf{w}_s^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\neq 0\}} \int\limits_t^{T} b_s ds \ \text{a.s.} \qquad (2)$$

Here and in what follows, $\mathbf{1}_A$ is the indicator function of a set A. The integral $\int\limits_t^{*T} \eta_s^{(i)} d\mathbf{w}_s^{(j)}$ is called the Stratonovich stochastic integral.

We consider a system of Itô stochastic differential equations (SDE) of the form

$$d\mathbf{x}_t = \mathbf{a}(\mathbf{x}_t, t)dt + A(\mathbf{x}_t, t)d\mathbf{w}_t, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \qquad (3)$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is a solution of the SDE (3). Assume that nonrandom functions $\mathbf{a}(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and $A(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness (up to stochastic equivalence) of a solution of the SDE (3) [8]. Let the random variables $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{w}_t - \mathbf{w}_0$ be independent for any $t \in [0, T]$.

It follows from results of [1, 2] that, for $s > t$, under certain conditions, a solution of the Itô SDE (3) admits the following unified Taylor–Itô expansions:

$$\mathbf{x}_s = \mathbf{x}_t + \sum_{(k,j,l_1,\dots,l_k,i_1,\dots,i_k)\in\mathcal{A}_r} C_{jl_1\dots l_k}^{(i_1\dots i_k)}(\mathbf{x}_t, t)\frac{(s-t)^j}{j!} J_{(l_1\dots l_k)s,t}^{(i_1\dots i_k)} + R_{s,t}^{(r+1)} \ \text{a.s.}, \qquad (4)$$

$$\mathbf{x}_s = \mathbf{x}_t + \sum_{(k,j,l_1,\dots,l_k,i_1,\dots,i_k)\in\mathcal{A}_r} \tilde{C}_{jl_1\dots l_k}^{(i_1\dots i_k)}(\mathbf{x}_t, t)\frac{(s-t)^j}{j!} \tilde{J}_{(l_1\dots l_k)s,t}^{(i_1\dots i_k)} + R_{s,t}^{(r+1)} \text{a.s.}, \qquad (5)$$

where $\mathcal{A}_r = \{(k, j, l_1, \dots, l_k, i_1, \dots, i_k) : 1 \le k + j + \sum\limits_{p=1}^{k} l_p \le r;\ i_1, \dots, i_k = 1, \dots m;\ k,\ j,\ l_1, \dots, l_k = 0,\ 1, \dots\}$, $R_{s,t}^{(r+1)}$ is the remainder term written in the integral form (see [1, 2]),

$$J_{(l_1\dots l_k)s,t}^{(i_1\dots i_k)} = \int\limits_t^{s} (t - t_k)^{l_k} \dots \int\limits_t^{t_2} (t - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (6)$$

and

$$\tilde{J}_{(l_1\dots l_k)s,t}^{(i_1\dots i_k)} = \int\limits_t^{s} (s - t_k)^{l_k} \dots \int\limits_t^{t_2} (s - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}. \qquad (7)$$

For $s > t$, this solution also admits (see [3–5]) the following Taylor–Itô expansions:

$$\mathbf{x}_s = \mathbf{x}_t + \sum_{(\lambda_1,\dots,\lambda_k,i_1,\dots,i_k)\in\mathcal{M}_r} G_{\lambda_1\dots\lambda_k}^{(i_1\dots i_k)}(\mathbf{x}_t, t) I_{(\lambda_1\dots\lambda_k)s,t}^{(i_1\dots i_k)} + D_{s,t}^{(r+1)} \ \text{a.s.} \qquad (8)$$

and the Taylor–Stratonovich expansions:

$$\mathbf{x}_s = \mathbf{x}_t + \sum_{(\lambda_1,\ldots,\lambda_k,i_1,\ldots,i_k)\in\mathcal{M}_r} G^{*(i_1\ldots i_k)}_{\lambda_1\ldots\lambda_k}(\mathbf{x}_t,t) I^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)s,t} + D^{*(r+1)}_{s,t} \text{ a.s.,} \tag{9}$$

where $\mathcal{M}_r = \{(\lambda_1,\ldots,\lambda_k,i_1,\ldots,i_k) : \lambda_l \in \{0,1\},\ i_l = 0$ for $\lambda_l = 0$ and $i_l = 1,\ldots,m$ for $\lambda_l = 1;\ l = 1,\ldots,k;\ k = 1,\ldots,r\}$, $D^{(r+1)}_{s,t}$ and $D^{*(r+1)}_{s,t}$ are the remainder terms in the integral form (see [3–5]),

$$I^{(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)s,t} = \int\limits_t^s \ldots \int\limits_t^{t_2} d\mathbf{w}^{(i_k)}_{t_1} \ldots d\mathbf{w}^{(i_1)}_{t_k}, \tag{10}$$

and

$$I^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)s,t} = \int\limits_t^{*s} \ldots \int\limits_t^{*t_2} d\mathbf{w}^{(i_k)}_{t_1} \ldots d\mathbf{w}^{(i_1)}_{t_k}. \tag{11}$$

The functions $C^{(i_1\ldots i_k)}_{jl_1\ldots l_k}(\mathbf{x}_t,t)$, $\tilde{C}^{(i_1\ldots i_k)}_{jl_1\ldots l_k}(\mathbf{x}_t,t)$, $G^{(i_1\ldots i_k)}_{\lambda_1\ldots\lambda_k}(\mathbf{x}_t,t)$, and $G^{*(i_1\ldots i_k)}_{\lambda_1\ldots\lambda_k}(\mathbf{x}_t,t)$ in (4), (5), (8), and (9) are determined (see [1, 2, 3–5]) by the diffusion and drift operators of the Itô formula and the processes $\mathbf{a}(\mathbf{x}_t,t)$ and $A(\mathbf{x}_t,t)$.

Consider the problem of numerical solution of the Itô SDE arising in a wide class of applications (see, for example, [5]). This problem is reduced to a joint numerical simulation of multiple stochastic integrals of the form (6), (7), (10), or (11).

A method of expansion of multiple stochastic integrals of the form (11) based on the Fourier-series expansion with random coefficients for the process $\left\{\mathbf{w}_t - \frac{t}{\Delta}\mathbf{w}_\Delta, t \in [0,\Delta], \Delta > 0\right\}$ was proposed in [3]. Following this method, to obtain the needed expansion, one has to substitute the Fourier-series expansions of the processes $\mathbf{w}^{(i_l)}_t$, $l = 1,\ldots,k$, into integral (11) and perform some transformations. Within the framework of this approach, no convenient expression was given for the general term of the expansion of a Stratonovich multiple stochastic integral of arbitrary multiplicity k. In [3, 5, 6], expansions of multiple stochastic integrals of the form (11) were obtained only for multiplicities 1, 2, and 3. In addition, this method allows us to use only the trigonometric system for expansion of a multiple stochastic integral. As we show below, this system is not optimal from the point of view of the mean-square convergence of the series and of their simplicity.

Consider a multiple Stratonovich integral of the form

$$\int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k}, \tag{12}$$

where $\psi_l(\tau) \in C^1_{[t,T]}$, $l = 1,\ldots,k$. (Here and below, $C^1_{[t,T]}$ is the space of functions continuously differentiable on $[t,T]$.) In this paper, a more powerful new method of expansion of this type of multiple Stratonovich integrals is proposed. This method is based on the representation of the multiple Stratonovich integral (12) as a multiple stochastic integral and subsequent multiple orthogonal expansion of the integrand. Various orthonormal systems in the space $L_2([t,T])$ can be used for these purposes. This enables us to obtain a general form of expansion of the multiple stochastic integral (12) of multiplicity k and to use various complete orthonormal systems.

Later we show that expansions of multiple stochastic integrals of the form (12) obtained by the proposed method using the polynomial system of functions

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \tag{13}$$

where $P_j(x)$ are the Legendre polynomials, converge really faster and have a simpler form than similar expansions obtained by the same methods with the help of the trigonometric system, i.e., with respect to functions of the form

$$\phi_j(s)=\frac{1}{\sqrt{T-t}}\begin{cases} 1 & \text{for} \quad j=0,\\ \sqrt{2}\sin\dfrac{2\pi r(s-t)}{T-t} & \text{for} \quad j=2r-1;\ r=1,2,\dots,\\ \sqrt{2}\cos\dfrac{2\pi r(s-t)}{T-t} & \text{for} \quad j=2r. \end{cases} \tag{14}$$

Indeed, let

$$I^{*(i_1\dots i_k)}_{l_1\dots l_k T,t}=\int\limits_t^{*T}(t-t_k)^{l_k}\dots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{w}^{(i_1)}_{t_1}\dots d\mathbf{w}^{(i_k)}_{t_k}, \tag{15}$$

and let $I^{*(i_1\dots i_k)q}_{l_1\dots l_k T,t}$ be an approximation of $I^{*(i_1\dots i_k)}_{l_1\dots l_k T,t}$. Our method with respect to system (14) gives us the following expressions:

$$I^{*(i_1)}_{0_{T,t}}=\sqrt{T-t}\zeta^{(i_1)}_0,$$

$$I^{*(i_1)q}_{1_{T,t}}=-\frac{(T-t)^{\frac{3}{2}}}{2}\left[\zeta^{(i_1)}_0+\frac{\sqrt{2}}{\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta^{(i_1)}_{2r-1}\right],$$

$$I^{*(i_2 i_1)q}_{00_{T,t}}=\frac{1}{2}(T-t)\Biggl[\zeta^{(i_1)}_0\zeta^{(i_2)}_0+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Bigl\{\zeta^{(i_1)}_{2r}\zeta^{(i_2)}_{2r-1}-\zeta^{(i_1)}_{2r-1}\zeta^{(i_2)}_{2r}+$$
$$+\sqrt{2}\left(\zeta^{(i_1)}_{2r-1}\zeta^{(i_2)}_0-\zeta^{(i_1)}_0\zeta^{(i_2)}_{2r-1}\right)\Bigr\}\Biggr], \tag{16}$$

and

$$\mathbf{E}\left\{\left(I^{*(i_2 i_1)}_{00_{T,t}}-I^{*(i_2 i_1)q}_{00_{T,t}}\right)^2\right\}=\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)\ (i_1\neq i_2), \tag{17}$$

where $\zeta^{(i)}_j=\int\limits_t^T\phi_j(s)d\mathbf{w}^{(i)}_s$, $i_1,\ i_2=1,\dots,m$, and the functions $\phi_j(s)$ have the form (14).

The same method with respect to system (13) gives us the following expressions:

$$I^{*(i_1)}_{0_{T,t}}=\sqrt{T-t}\zeta^{(i_1)}_0,$$

$$I_{1_{T,t}}^{*(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{00_{T,t}}^{*(i_2i_1)q}=\frac{T-t}{2}\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i-1}^{(i_1)}\right\}\right],$$

and

$$\mathbf{E}\left\{\left(I_{00_{T,t}}^{*(i_2i_1)}-I_{00_{T,t}}^{*(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^2}{4}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\ (i_1\neq i_2), \tag{18}$$

where $\zeta_j^{(i)}=\int_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$, $i_1,\ i_2=1,\dots,m$, and the functions $\phi_j(s)$ have the form (13).

§2. Relations between multiple Stratonovich and Itô integrals

We introduce the following notation:

$$J^*\left(\psi^{(k)}\right)_{T,t}=\int_t^{*T}\psi_k(t_k)\dots\int_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\dots d\mathbf{w}_{t_k}^{(i_k)}, \tag{19}$$

$$J\left(\psi^{(k)}\right)_{T,t}=\int_t^{T}\psi_k(t_k)\dots\int_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\dots d\mathbf{w}_{t_k}^{(i_k)}, \tag{20}$$

$$J\left(\psi^{(k)}\right)_{T,t}^{s_l,\dots,s_1}\stackrel{\text{def}}{=}\prod_{q=1}^{l}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\neq 0\}}$$

$$\times\int_t^{T}\psi_k(t_k)\dots\int_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})$$

$$\times\int_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\dots\int_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})$$

$$\times\int_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\dots\int_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\dots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}$$

$$\dots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\dots d\mathbf{w}_{t_k}^{(i_k)}, \tag{21}$$

where $(s_l,\dots,s_1)\in\mathcal{A}_{kl}$; $\psi_p(\tau)\in C_{[t,T]}^1$, $i_p=0,1,\dots,m$ $(p=1,\dots,k)$; $\mathcal{A}_{kl}=\{(s_l,\dots,s_1)$: $s_q>s_{q-1}+1;\ q=2,\dots,l;\ s_l,\dots,s_1=1,\dots,k-1\}$; $l=1,\dots,\left[\frac{k}{2}\right]$; $[x]$ is the integer part of the number x; $\psi^{(k)}=(\psi_k,\dots,\psi_1)$.

The following lemma establishes a relationship between the multiple stochastic integrals $J^*\left(\psi^{(k)}\right)_{T,t}$ and $J\left(\psi^{(k)}\right)_{T,t}$.

Lemma 1.

$$J^*\left(\psi^{(k)}\right)_{T,t}=J\left(\psi^{(k)}\right)_{T,t}+\sum_{r=1}^{\left[\frac{k}{2}\right]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{kr}}J\left(\psi^{(k)}\right)_{T,t}^{s_r,\ldots,s_1}\ \ \text{a.s.}\tag{22}$$

Proof. The proof is by induction.

For $k=1$, the following equality is obvious: $J\left(\psi^{(1)}\right)_{T,t}=J^*\left(\psi^{(1)}\right)_{T,t}$ a.s. For $k=2$, relation (22) implies the relation

$$J^*\left(\psi^{(2)}\right)_{T,t}=J\left(\psi^{(2)}\right)_{T,t}+\frac{1}{2}J\left(\psi^{(2)}\right)_{T,t}^{1}\ \ \text{a.s.}\tag{23}$$

To prove equality (23), we consider the process $\eta_\tau^{(i_1)}=\psi_2(\tau)\int\limits_t^\tau\psi_1(s)d\mathbf{w}_s^{(i_1)}$. By the Itô formula, its stochastic differential is

$$d\eta_\tau^{(i_1)}=\int\limits_t^\tau\psi_1(s)d\mathbf{w}_s^{(i_1)}d\psi_2(\tau)+\psi_1(\tau)\psi_2(\tau)d\mathbf{w}_\tau^{(i_1)}.\tag{24}$$

It follows from (24) that the diffusion coefficient of the process $\eta_\tau^{(i_1)}$ equals $\mathbf{1}_{\{i_1\neq 0\}}\psi_1(\tau)\psi_2(\tau)$. Further, relation (2) implies (23). Thus, relation (22) is proved for $k=1,2$. Now, by the induction hypothesis, the following equality holds a.s.:

$$J^*\left(\psi^{(k+1)}\right)_{T,t}=\int\limits_t^{*T}\psi_{k+1}(\tau)J\left(\psi^{(k)}\right)_{\tau,t}d\mathbf{w}_\tau^{(i_{k+1})}$$

$$+\sum_{r=1}^{\left[\frac{k}{2}\right]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{kr}}\int\limits_t^{*T}\psi_{k+1}(\tau)J\left(\psi^{(k)}\right)_{\tau,t}^{s_r,\ldots,s_1}d\mathbf{w}_\tau^{(i_{k+1})}.\tag{25}$$

Using the Itô formula and relation (2), we establish similarly to (23) that the following equalities hold a.s.:

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J\left(\psi^{(k)}\right)_{\tau,t}d\mathbf{w}_\tau^{(i_{k+1})}=J\left(\psi^{(k+1)}\right)_{T,t}+\frac{1}{2}J\left(\psi^{(k+1)}\right)_{T,t}^{k},\tag{26}$$

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J\left(\psi^{(k)}\right)_{\tau,t}^{s_r,\ldots,s_1}d\mathbf{w}_\tau^{(i_{k+1})}$$

$$=\begin{cases}J\left(\psi^{(k+1)}\right)_{T,t}^{s_r,\ldots,s_1} & \text{for}\quad s_r=k-1,\\ J\left(\psi^{(k+1)}\right)_{T,t}^{s_r,\ldots,s_1}+\frac{1}{2}J\left(\psi^{(k+1)}\right)_{T,t}^{k,s_r,\ldots,s_1} & \text{for}\quad s_r<k-1.\end{cases}\tag{27}$$

Substituting relations (25) and (27) into (25) and regrouping the terms, we come to the following relations valid a.s.:

$$J^* \left(\psi^{(k+1)}\right)_{T,t} = J \left(\psi^{(k+1)}\right)_{T,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k+1,r}} J \left(\psi^{(k+1)}\right)_{T,t}^{s_r,\ldots,s_1} \tag{28}$$

for k even and

$$J^* \left(\psi^{(k'+1)}\right)_{T,t} = J \left(\psi^{(k'+1)}\right)_{T,t} + \sum_{r=1}^{\left[\frac{k'}{2}\right]+1} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k'+1,r}} J \left(\psi^{(k'+1)}\right)_{T,t}^{s_r,\ldots,s_1} \tag{29}$$

for $k = k' + 1$ odd. Relations (28) and (29) prove that the following equality holds a.s.:

$$J^* \left(\psi^{(k+1)}\right)_{T,t} = J \left(\psi^{(k+1)}\right)_{T,t} + \sum_{r=1}^{\left[\frac{k+1}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k+1,r}} J \left(\psi^{(k+1)}\right)_{T,t}^{s_r,\ldots,s_1}. \tag{30}$$

□

§3. An expansion of the Stratonovich multiple stochastic integral

Define the function

$$K(t_1,\ldots,t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l<t_{l+1}\}}, \ k \geq 2, \tag{31}$$

on the set $[t,T]^k$. Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system in the space $L_2([t,T])$ consisting of continuously differentiable functions.

Let $H([t,T]) \subset L_2([t,T])$ be the space of functions $f(x)$ bounded on $[t,T]$, piecewise smooth on (t,T), and such that their orthogonal (Fourier-series) expansions $\sum\limits_{j=0}^{\infty} C_j\phi_j(x)$, $C_j = \int\limits_t^T f(x)\phi_j(x)dx$, converge at any interior point $x \in [t,T]$ to the values $\frac{1}{2}\big(f(x+0) + f(x-0)\big)$, converge uniformly to $f(x)$ on any closed interval of continuity, and converge at the points $x = t$ and $x = T$.

Remark 1. Here and below, convergence of orthogonal expansions in the space $L_2([t,T])$ is understood as convergence in the norm.

Define functions $C_{j_{q-1}\ldots j_1}(t_q,\ldots,t_k)$, $q = 1,\ldots,k$, as follows:

$$C_{j_{q-1}\ldots j_1}(t_q,\ldots,t_k) \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^T K(t_1,\ldots,t_k) \prod_{l=1}^{q-1} \phi_{j_l}(t_l)dt_1 \ldots dt_{q-1}, \quad 2 \leq q \leq k.$$

The following theorem is the main result of this paper.

Theorem. *Assume that the following conditions are satisfied.*

(1) $\psi_i(\tau) \in C^1_{[t,T]}$, $i = 1, \dots, k$.

(2) $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuously differentiable functions in the space* $L_2([t,T])$.

(3) $G_{j_{q-1}\dots j_1}(t_q, \dots, t_k) \in H([t,T])$ *with respect to the variable* t_q, $q = 2, \dots, k$; $K(t_1, \dots, t_k) \in H([t,T])$ *with respect to the variable* t_1.

Then the Stratonovich multiple stochastic integral $J^*\left(\psi^{(k)}\right)_{T,t}$ *admits the following multiple orthogonal expansion:*

$$J^*\left(\psi^{(k)}\right)_{T,t} = \sum_{j_1=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1} \prod_{l=1}^{k} \zeta_{(j_l)T,t}^{(i_l)} \tag{32}$$

converging in average of any degree $n \in \mathbb{N}$.

In the formulas above, if i_l *are* j_l *different and if* $i_l \neq 0$, *then the variables* $\zeta_{(j_l)T,t}^{(i_l)} = \int\limits_t^T \phi_{j_l}(s) d\mathbf{w}_s^{(i_l)}$ *are the standard Gaussian variables and*

$$C_{j_k \dots j_1} = \int\limits_t^T \dots \int\limits_t^T K(t_1, \dots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \dots dt_k. \tag{33}$$

To prove the theorem, we establish some auxiliary assertions.

Let a function $\mathcal{B}_{k-1}^{\pm}(t_1, \dots, t_k)$ be defined on the set $[t,T]^k$ as follows:

$$\mathcal{B}_{k-1}^{\pm}(t_1, \dots, t_k) = K(t_1, \dots, t_k)$$

$$+ \prod_{l=1}^{k} \psi_l(t_l) \sum_{r=1}^{k-1} \sum_{\substack{s_r, \dots, s_1 = 1 \\ s_r > \dots > s_1}}^{k-1} \frac{1}{g(s_1, \dots, s_r)} \prod_{l=1}^{r} \mathbf{1}_{\{t_{s_l} = t_{s_l+1}\}} \prod_{\substack{l=1 \\ l \neq s_1, \dots, s_r}}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}}, \tag{34}$$

where $g(s_1, \dots, s_r) = (q_1+1)! \dots (q_{p_k}+1)!$; $p_k = 1, 2, \dots, [\frac{k}{2}]$; $q_1, \dots, q_{p_k}$ are the lengths of all possible subsequences of the form $g, g-1, \dots, g-m$ with $g = 1, \dots, k-1$, $m = 0, 1, \dots, k-2$, chosen from the sequence $s_r, \dots, s_1$; $s_r > \dots > s_1$; $s_r, \dots, s_1 = 1, \dots, k-1$.

In particular, if $k = 2$ or $k = 3$, then formula (34) implies that

$$\mathcal{B}_1^{\pm}(t_1, t_2) = \prod_{l=1}^{2} \psi_l(t_l) \left(\mathbf{1}_{\{t_1 < t_2\}} + \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \right)$$

and

$$\mathcal{B}_2^{\pm}(t_1, t_2, t_3) = \prod_{l=1}^{3} \psi_l(t_l) \Bigg(\mathbf{1}_{\{t_1 < t_2\}} \mathbf{1}_{\{t_2 < t_3\}}$$

$$+ \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \mathbf{1}_{\{t_2 < t_3\}} + \frac{1}{2} \mathbf{1}_{\{t_1 < t_2\}} \mathbf{1}_{\{t_2 = t_3\}} + \frac{1}{6} \mathbf{1}_{\{t_1 = t_2\}} \mathbf{1}_{\{t_2 = t_3\}} \Bigg).$$

Lemma 2. *Under the assumptions of the above theorem, the function $\mathcal{B}^{\pm}_{k-1}(t_1,\dots,t_k)$ is extendable into the multiple orthogonal (Fourier) series*

$$\mathcal{B}^{\pm}_{k-1}(t_1,\dots,t_k)=\sum_{j_1=0}^{\infty}\dots\sum_{j_k=0}^{\infty}C_{j_k\dots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) \tag{35}$$

at any interior point of the hypercube $[t,T]^k$. Here the coefficients $C_{j_k\dots j_1}$ have the form (33)*. The multiple series* (35) *converges on the boundary Γ_k of the hypercube $[t,T]^k$ and converges uniformly to the function $\mathcal{B}^{\pm}_{k-1}(t_1,\dots,t_k)$ in any closed domain of its continuity.*

Proof. We consider the cases $k=2$ and $k=3$. Introduce the following functions:

$$K'(t_1,t_2)=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2;\\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2;\end{cases}\qquad t_1,t_2\in[t,T],$$

$$K_1(t_1,t_2,t_3)=\begin{cases}\psi_3(t_3)\psi_2(t_2)\psi_1(t_1), & t_1\le t_2\le t_3;\\ \psi_3(t_3)\psi_1(t_2)\psi_2(t_1), & t_2\le t_1\le t_3;\\ \psi_1(t_3)\psi_3(t_2)\psi_2(t_1), & t_3\le t_1\le t_2;\\ \psi_2(t_3)\psi_3(t_2)\psi_1(t_1), & t_1\le t_3\le t_2;\\ \psi_1(t_3)\psi_2(t_2)\psi_3(t_1), & t_3\le t_2\le t_1;\\ \psi_2(t_3)\psi_1(t_2)\psi_3(t_1), & t_2\le t_3\le t_1;\end{cases}\qquad t_1,t_2,t_3\in[t,T],$$

$$K_2(t_1,t_2,t_3)=\begin{cases}\psi_3(t_3)\psi_2(t_2)\psi_1(t_1), & t_1\le t_2\le t_3;\\ \psi_3(t_3)\psi_1(t_2)\psi_2(t_1), & t_2\le t_1\le t_3;\\ 0 \quad \text{otherwise};\end{cases}\qquad t_1,t_2,t_3\in[t,T],$$

$$K_3(t_1,t_2,t_3)=\begin{cases}\psi_2(t_3)\psi_3(t_2)\psi_1(t_1), & t_1\le t_3\le t_2;\\ \psi_3(t_3)\psi_2(t_2)\psi_1(t_1), & t_1\le t_2\le t_3;\\ 0 \quad \text{otherwise};\end{cases}\qquad t_1,t_2,t_3\in[t,T].$$

First we prove our lemma for $k=2$. Obviously, the Fourier series

$$S(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l)$$

converges uniformly to the function $K(t_1,t_2)$ on any closed subset belonging to the interior of the hypercube and disjoint with the set $\{t_1\neq t_2\}$ (the function $K(t_1,t_2)$ is continuous on such a set). The sum of the series is finite on the boundary of the square. Thus, it remains to show that $S(t_1,t_1)=\frac{1}{2}\prod_{l=1}^{2}\psi_l(t_1)$, $t_1\in(t,T)$. Consider the following Fourier-series expansion $S'(t_1,t_2)$ of the function $K'(t_1,t_2)$:

$$S'(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C'_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l), \tag{36}$$

where $C'_{j_2j_1} = \int\limits_{[t,T]^2} K'(t_1,t_2)\prod\limits_{l=1}^{2}\phi_{j_l}(t_l)dt_1dt_2$. It is clear that $S'(t_1,t_2)$ converges uniformly to $K'(t_1,t_2)$ in any closed subdomain of the open square $(t,T)^2$ since the function $K'(t_1,t_2)$ is continuous there. Furthermore, changing the order of integration, we obtain the equality

$$C'_{j_2j_1} = C_{j_2j_1} + C_{j_1j_2}. \tag{37}$$

Substituting (37) into (36) and putting $t_1 = t_2$ in (36), we come to the relation $S'(t_1,t_1) = \psi_1(t_1)\psi_2(t_1) = 2S(t_1,t_1)$, i.e., $S(t_1,t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1)$. This completes the proof for the case $k = 2$.

Now we consider the case $k = 3$. Obviously, the Fourier series

$$S(t_1,t_2,t_3) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_3j_2j_1}\prod_{l=1}^{3}\phi_{j_l}(t_l)$$

converges uniformly to the function $K(t_1,t_2,t_3)$ on any closed subset belonging to the open cube $(t,T)^3$ and disjoint from the subsets $t_1 = t_2, t_2 = t_3, t_1 = t_3$ (the function $K(t_1,t_2,t_3)$ is continuous there). The sum of this Fourier series is finite on the boundary of the cube. Now we have to show that

$$S(t_1,t_1,t_1) = \frac{1}{6}\prod_{l=1}^{3}\psi_l(t_1), \tag{38}$$

$$S(t_2,t_2,t_3) = \frac{1}{2}\psi_3(t_3)\prod_{l=1}^{2}\psi_l(t_2), \tag{39}$$

and

$$S(t_1,t_3,t_3) = \frac{1}{2}\psi_1(t_1)\prod_{l=2}^{3}\psi_l(t_3). \tag{40}$$

Consider the Fourier-series expansions $S_i(t_1,t_2,t_3)$ of the functions $K_i(t_1,t_2,t_3)$, $i = 1,2,3$:

$$S_i(t_1,t_2,t_3) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C^{(i)}_{j_3j_2j_1}\prod_{l=1}^{3}\phi_{j_l}(t_l), \tag{41}$$

where $C^{(i)}_{j_3j_2j_1} = \int\limits_{[t,T]^3} K_i(t_1,t_2,t_3)\prod\limits_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3$. The series $S_1(t_1,t_2,t_3)$ converges uniformly to $K_1(t_1,t_2,t_3)$ on any closed subdomain of $(t,T)^3$ since the function $K_1(t_1,t_2,t_3)$ is continuous there. The series $S_2(t_1,t_2,t_3)$ and $S_3(t_1,t_2,t_3)$ converge uniformly to $K_2(t_1,t_2,t_3)$ and $K_3(t_1,t_2,t_3)$ in any closed subdomain of the set of their continuity.

Changing the order of integration, we obtain the equalities

$$C^{(1)}_{j_3j_2j_1} = C_{j_3j_2j_1} + C_{j_3j_1j_2} + C_{j_2j_1j_3} + C_{j_2j_3j_1} + C_{j_1j_2j_3} + C_{j_1j_3j_2}, \tag{42}$$

$$C^{(2)}_{j_3j_2j_1} = C_{j_3j_2j_1} + C_{j_3j_1j_2}, \tag{43}$$

and

$$C^{(3)}_{j_3j_2j_1} = C_{j_3j_2j_1} + C_{j_2j_3j_1}. \tag{44}$$

Substituting equalities (42), (43), and (44) into (41) for $i = 1, 2, 3$, respectively, and putting $t_1 = t_2 = t_3$ for $i = 1$, $t_1 = t_2$ for $i = 2$, and $t_2 = t_3$ for $i = 3$ in (41), we obtain equalities (38)–(40) after some simple transformations. The proof of Lemma 2 for the case $k = 3$ is completed. The general case can be considered similarly. □

Consider a partition of the interval $[t, T]$ of the form

$$t = \tau_0 < \ldots < \tau_N = T$$

such that

$$\Delta_N = \max_{0 \le j \le N-1} |\tau_{j+1} - \tau_j| \to 0 \quad \text{as} \quad N \to \infty. \tag{45}$$

Lemma 3. *Under assumptions of the theorem formulated above,*

$$J\left(\psi^{(k)}\right)_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}} \ \text{a.s.}, \tag{46}$$

where $\Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}} = \mathbf{w}^{(i_l)}_{\tau_{j_l+1}} - \mathbf{w}^{(i_l)}_{\tau_{j_l}}$, $i_l = 0, 1, \ldots, m$, *and* $\{\tau_{j_l}\}_{j_l=0}^{N-1}$ *is a partition of* $[t, T]$ *satisfying condition* (45).

Proof. It is easy to see that, under the assumptions of our lemma, the integral sum of the integral $J\left(\psi^{(k)}\right)_{T,t}$ can be represented as the sum of a prelimit expression of the left-hand side of (46) and a variable tending in average to zero as $N \to \infty$. □

Remark 2. If, for some $l \in \{1, \ldots, k\}$, one replaces $\Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}}$ by $\left(\Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}}\right)^p$ in expression (46), then the differential $d\mathbf{w}^{(i_l)}_{t_l}$ in $J\left(\psi^{(k)}\right)_{T,t}$ for $p = 2$ transforms to dt_1. If $p = 3, 4, \ldots$, the right-hand side of (46) vanishes a.s. If, for some $l \in \{1, \ldots, k\}$, one replaces $\Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}}$ by $(\Delta \tau_{j_l})^p$, $p = 2, 3, \ldots$, in (46), then the right-hand side of (46) also vanishes a.s.

We define the following stochastic integrals:

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}}$$

$$\overset{\text{def}}{=} \int_t^T \ldots \int_t^T \Phi(t_1, \ldots, t_k) d\mathbf{w}^{(i_k)}_{t_k} \ldots d\mathbf{w}^{(i_1)}_{t_1} = J[\Phi]^{(k)}_{T,t}, \tag{47}$$

$$\underset{N\to\infty}{\text{l.i.m.}} \left\{ \sum_{j_1,\ldots,j_k=0}^{N-1} - \sum_{j_1,\ldots,j_k=1}^{N-2} \right\} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}} \overset{\text{def}}{=} J[\Phi]^{\Gamma_k}_{T,t}. \tag{48}$$

Let $\mathcal{D}_k = \{(t_1, \ldots, t_k) : t \le t_1 < \ldots < t_k \le T\}$ and let $\Gamma_{\mathcal{D}_k}$ be the boundary of $\mathcal{D}_k$. We write $\Phi(t_1, \ldots, t_k) \in C^1(\mathcal{D}_k)$ if the following conditions are satisfied:

(AI) the function $\Phi(t_1,\ldots,t_k)$ is continuously differentiable in $\mathcal{D}_k$;
(AII) the function $\Phi(t_1,\ldots,t_k)$ is bounded on $\Gamma_{\mathcal{D}_k}$.

Similarly to the proof of Lemma 3, one can show that if $\Phi(t_1,\ldots,t_k)\in C^1(\mathcal{D}_k)$, then

$$I[\Phi]_{T,t}^{(k)} \stackrel{\rm def}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

$$= \underset{N\to\infty}{\rm l.i.m.} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\Phi(\tau_{j_1},\ldots,\tau_{j_k})\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \ \text{a.s.}, \qquad (49)$$

where $I[\Phi]_{T,t}^{(k)}$ is understood as the Itô multiple integral.

We introduce the following notation:

$$\mathbf{1}_{j_l,j_{l+1}}(j_{q_1},\ldots,j_{q_2},j_l,j_{q_3},\ldots,j_{q_{k-2}},j_l,j_{q_{k-1}},\ldots,j_{q_k})$$

$$\stackrel{\rm def}{=} (j_{q_1},\ldots,j_{q_2},j_{l+1},j_{q_3},\ldots,j_{q_{k-2}},j_{l+1},j_{q_{k-1}},\ldots,j_{g_k}), \qquad (50)$$

where $l\in N$; $l\ne q_1,\ldots,q_2,q_3,\ldots,q_{k-2},q_{k-1},\ldots,q_k\in N$;

$$S_N^{(k)}(a)=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\sum_{(j_1,\ldots,j_k)} a_{(j_1,\ldots,j_k)};$$

$$\mathcal{C}_{s_r}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\}=\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}$$

$$\times\sum_{\prod\limits_{l=1}^{r}\mathbf{1}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)} a_{\prod\limits_{l=1}^{r}\mathbf{1}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)};$$

$$\mathcal{C}_{s_r}^{*}\mathcal{C}_{s_{r-1}}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\}=\mathcal{C}_{s_{r-1}}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\}+\mathcal{C}_{s_r}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\};$$

$$\prod_{l=1}^{0}\mathbf{1}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)=(j_1,\ldots,j_k);$$

$$\prod_{l=1}^{r}\mathbf{1}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)=\mathbf{1}_{j_{s_r},j_{s_r+1}}\ldots\mathbf{1}_{j_{s_1},j_{s_1+1}}(j_1\ldots j_k);$$

$$\mathcal{C}_{s_0}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\}=S_N^{(k)}(a).$$

In the formulas above, $s_1,\ldots,s_r=1,\ldots,k-1$; $s_r>\ldots>s_1$; $r=1,\ldots,k-1$; $\sum\limits_{(j_{q_1},\ldots,j_{q_k})}$ means the sum over all the permutations $(j_{q_1},\ldots,j_{q_k})$; $q_1,\ldots,q_k\in\{1,\ldots,k\}$; $a_{(j_{q_1},\ldots,j_{q_k})}$ are scalars.

The following equality is easily proved by induction:

$$\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{N-1}a_{(j_1,\ldots,j_k)}=\mathcal{C}_{k-1}^{*}\ldots\mathcal{C}_{1}^{*}\{S_N^{(k)}(a)\}$$

$$=\sum_{r=0}^{k-1}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\mathcal{C}_{s_r}^{+}\ldots\mathcal{C}_{s_1}^{+}\{S_N^{(k)}(a)\},\quad k\ge 1;\quad \sum_{\emptyset}\stackrel{\rm def}{=}1. \tag{51}$$

Put $a_{(j_1,\ldots,j_k)}=\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod\limits_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$. Then relations (47) and (51) imply that

$$J[\Phi]_{T,t}^{(k)}=\sum_{r=0}^{k-1}\sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{kr}}I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\ \text{a.s.}, \tag{52}$$

where

$$I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}=\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\sum_{\prod\limits_{l=1}^{r}\mathbf{1}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}$$

$$\times\Biggl[\Phi\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},\ldots,t_k\right)$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1+1})}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}$$

$$\ldots d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r+1})}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Biggr], \tag{53}$$

and

$$\prod_{l=1}^{0}\mathbf{1}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)\stackrel{\rm def}{=}(t_1,\ldots,t_k);\qquad \sum_{(s_0,\ldots,s_1)\in\mathcal{A}_{k0}}\stackrel{\rm def}{=}1;\quad k\ge 2.$$

Remark 3. The terms on the right-hand side of (53) have to be understood as follows. For any permutation from the collection $\prod\limits_{l=1}^{r}\mathbf{1}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)$ on the right-hand side of (53), one has to replace every pair of differentials with coinciding lower indices of the form $d\mathbf{w}_{t_p}^{(i)}d\mathbf{w}_{t_p}^{(j)}$ (there exist r such pairs) by the values $\mathbf{1}_{\{i=j\neq 0\}}dt_p$.

Lemma 4. *Let $\Phi(t_1,\ldots,t_k)\in C^1(\mathcal{D}_k)$. Then*

$$\mathbf{E}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^{2n}\right\}\le C_{nk}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^{2n}(t_1,\ldots,t_k)dt_1\ldots dt_k,\quad \textit{where}\quad C_{nk}<\infty,\quad n\in N. \quad (54)$$

Proof. Using standard inequalities for stochastic integrals at $(\xi_\tau)^n\in\mathcal{M}_2([t_0,t])$, we obtain the estimates

$$\mathbf{E}\left\{\left|\int\limits_{t_0}^{t}\xi_\tau df_\tau\right|^{2n}\right\}\le (t-t_0)^{n-1}\left(n(2n-1)\right)^n\int\limits_{t_0}^{t}\mathbf{E}\left\{|\xi_\tau|^{2n}\right\}d\tau \quad (55)$$

and

$$\mathbf{E}\left\{\left|\int\limits_{t_0}^{t}\xi_\tau d\tau\right|^{2n}\right\}\le (t-t_0)^{2n-1}\int\limits_{t_0}^{t}\mathbf{E}\left\{|\xi_\tau|^{2n}\right\}d\tau. \quad (56)$$

Let $\xi^{(l)}_{t_{l+1},\ldots,t_k,t}=\int\limits_t^{t_{l+1}}\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_l}^{(i_l)}$ for $l=1,\ldots,k-1$, and let $\xi^{(0)}_{t_1,\ldots,t_k,t}\stackrel{\rm def}{=}\Phi(t_1,\ldots,t_k)$. It can be shown by induction that $\left(\xi^{(l)}_{t_{l+1},\ldots,t_k,t}\right)^n\in\mathcal{M}_2([t,T])$. Multiple application of inequalities (55) and (56) proves our lemma. $\square$

Using the Minkowski inequality and Lemma 4, we obtain from (52) the following estimates:

$$\left(\mathbf{E}\left\{\left(J[\Phi]_{T,t}^{(k)}\right)^{2n}\right\}\right)^{\frac{1}{2n}}\le\sum_{r=0}^{k-1}\sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{kr}}\left(\mathbf{E}\left\{\left(I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right)^{2n}\right\}\right)^{\frac{1}{2n}}, \quad (57)$$

$$\left(\mathbf{E}\left\{\left(I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right)^{2n}\right\}\right)^{\frac{1}{2n}}$$
$$\le C_{nk}^{s_1\ldots s_r}\left[\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\sum_{\prod\limits_{l=1}^{r}\mathbf{1}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\right.$$
$$\times\Phi^{2n}\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},\ldots,t_k\right)$$
$$\left.\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k\right]^{\frac{1}{2n}}, \quad (58)$$

where permutations during summation in (58) take place only in $\Phi^{2n}(\ldots)$, $C_{nk}^{s_1\ldots s_r}<\infty$, and

$$\prod_{l=1}^{0}\mathbf{1}_{t_{s_l},t_{s_l}+1}(t_1,\ldots,t_k)\stackrel{\rm def}{=}(t_1,\ldots,t_k).$$

Lemma 5. *Under the assumptions of our theorem,*

$$J[\mathcal{B}^{\pm}_{k-1}\{K\}]^{(k)}_{T,t} = J^*\left(\psi^{(k)}\right)_{T,t} \ \text{ a.s.} \tag{59}$$

Proof. Substituting relation (34) into (59) and using Lemma 3 and Remark 2, we get the equality

$$J[\mathcal{B}^{\pm}_{k-1}\{K\}]^{(k)}_{T,t} = J\left(\psi^{(k)}\right)_{T,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{kr}} J\left(\psi^{(k)}\right)^{s_r,\ldots,s_1}_{T,t} \ \text{ a.s.}$$

Now Lemma 5 is a consequence of Lemma 1. □

Lemma 6. *Let* $|\Phi(t_1,\ldots,t_k)| < \infty$ *for* $(t_1,\ldots,t_k) \in [t,T]^k$*. Then* $J[\Phi]^{\Gamma_k}_{T,t} = 0$ *a.s.*

Proof. First assume that $\Phi(t_1,\ldots,t_k) \equiv 1$. In this case, the integral sum $J[1]^{\Gamma_k}_{T,t}$ consists of a finite sum of random variables $\alpha^N_p \beta^N_{k-p}$, where

$$\alpha^N_p = \sum_{s_1,\ldots,s_p=1}^{N-2} \prod_{l=1}^{p} \Delta\mathbf{w}^{(r_l)}_{\tau_{s_l}} \quad \text{and} \quad \beta^N_{k-p} = \prod_{l=1}^{k-p} \left(\Delta\mathbf{w}^{(r_{p+l})}_{\tau_0} + \Delta\mathbf{w}^{(r_{p+l})}_{\tau_{N-1}}\right).$$

In the formula above, $\alpha^N_0 \stackrel{\text{def}}{=} 1$; $\{r_1,\ldots,r_k\} = \{i_1,\ldots,i_k\}$; $p = 0, 1, \ldots, k-1$; $k-p = 1,\ldots,k$; $i_1,\ldots,i_k = 0, 1, \ldots, m$, and the inequality $\mathbf{E}\left\{\left(\alpha^N_p\right)^2\right\} < \infty$ holds. Using the Minkowski inequality and taking into account the independence of the random variables α^N_p and β^N_{k-p}, one can show that $J[1]^{\Gamma_k}_{T,t} = 0$ a.s. For an arbitrary bounded function $\Phi(t_1,\ldots,t_k)$ defined on the set $[t,T]^k$, the proof is similar. □

Lemma 7. *Let* $\varphi_i(s) \in C^1_{[t,T]}$*,* $i = 1,\ldots,k$*. Then*

$$\prod_{l=1}^{k} \int_t^T \varphi_l(s) d\mathbf{w}^{(i_l)}_s = \int_t^T \ldots \int_t^T \prod_{l=1}^{k} \varphi_l(t_l) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k} \ \text{ a.s.} \tag{60}$$

Proof. First let $i_l \neq 0$, $l = 1,\ldots,k$. Denote $J^l_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} \varphi_l(\tau_j)\Delta\mathbf{w}^{(i_l)}_{\tau_j}$ and $J^l = \int_t^T \varphi_l(s) d\mathbf{w}^{(i_l)}_s$. Since

$$\prod_{l=1}^{k} J^l_N - \prod_{l=1}^{k} J^l = \sum_{l=1}^{k} \left(\prod_{g=1}^{l-1} J^g\right) \left(J^l_N - J^l\right) \left(\prod_{g=l+1}^{k} J^g_N\right),$$

the Minkowski and Cauchy–Bunyakovskii inequalities show that

$$\left(\mathbf{E}\left\{\left(\prod_{l=1}^{k} J^l_N - \prod_{l=1}^{k} J^l\right)^2\right\}\right)^{\frac{1}{2}} \le C_k \sum_{l=1}^{k} \left(\mathbf{E}\left\{\left(J^l_N - J^l\right)^4\right\}\right)^{\frac{1}{4}}, \quad C_k < \infty. \tag{61}$$

It is clear that $J_N^l - J^l = \sum\limits_{g=0}^{N-1} \zeta_g^l$, where $\zeta_g^l = \int\limits_{\tau_g}^{\tau_{g+1}} (\varphi_l(\tau_g) - \varphi_l(s))\, d\mathbf{w}_s^{(i_l)}$. Since the values ζ_g^l are independent for different g, the following equality holds [7]:

$$\mathbf{E}\left\{\left(\sum_{j=0}^{N-1} \zeta_j^l\right)^4\right\} = \sum_{j=0}^{N-1} \mathbf{E}\left\{\left(\zeta_j^l\right)^4\right\} + 6\sum_{j=0}^{N-1} \mathbf{E}\left\{\left(\zeta_j^l\right)^2\right\} \sum_{q=0}^{j-1} \mathbf{E}\left\{\left(\zeta_q^l\right)^2\right\}. \tag{62}$$

Since the values ζ_j^l are Gaussian and the functions $\varphi_i(s)$ are continuously differentiable, the right-hand side of equality (62) tends to zero as $N \to \infty$. This fact and inequality (61) prove equality (60). If $\mathbf{w}_{t_l}^{(i_l)} = t_l$ for some $l \in \{1, \dots, k\}$, the proof of the lemma is similar. □

Lemmas 2 and 5–7 show that

$$J^*\left(\psi^{(k)}\right)_{T,t} = \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \prod_{l=1}^{k} \int_t^T \phi_{j_l}(s) d\mathbf{w}_s^{(i_l)} + J[R_{p_1 \dots p_k}]_{T,t}^{(k)} \quad \text{a.s.}, \tag{63}$$

where the value $J[R_{p_1 \dots p_k}]_{T,t}^{(k)}$ is defined by (47) and

$$R_{p_1 \dots p_k}(t_1, \dots, t_k) = \left\{\sum_{j_1=0}^{\infty} \dots \sum_{j_k=0}^{\infty} - \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k}\right\} C_{j_k \dots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l). \tag{64}$$

If $(t_1, \dots, t_k) \in [t,T]^k \setminus \Gamma_k$, then Lemma 2 implies that

$$\lim_{p_1 \to \infty} \dots \lim_{p_k \to \infty} R_{p_1 \dots p_k}(t_1, \dots, t_k) = 0. \tag{65}$$

Remark 4. By Lemma 2, condition (65) holds uniformly on any closed subdomain of continuity of the function $R_{p_1 \dots p_k}(t_1, \dots, t_k)$, and the limit on the left-hand side of (65) is finite on the sets Γ_k.

Lemma 8. *Under the assumptions of our theorem,*

$$\lim_{p_1 \to \infty} \dots \lim_{p_k \to \infty} \mathbf{E}\left\{\left(J[R_{p_1 \dots p_k}]_{T,t}^{(k)}\right)^n\right\} = 0, \quad n \in N. \tag{66}$$

Proof. Relations (34), (35), and (64) imply that

$$R_{p_1 \dots p_k}(t_1, \dots, t_k) = K(t_1, \dots, t_k) + \prod_{l=1}^{k} \psi_l(t_l) \sum_{r=1}^{k-1} \sum_{\substack{s_r, \dots, s_1 = 1 \\ s_r > \dots > s_1}}^{k-1} \frac{1}{g(s_1, \dots, s_r)}$$

$$\times \prod_{l=1}^{r} \mathbf{1}_{\{t_{s_l} = t_{s_l+1}\}} \prod_{\substack{l=1 \\ l \neq s_1, \dots, s_r}}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} - \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l). \tag{67}$$

Equality (67) shows that the function $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ satisfies conditions (AI)–(AII) in the domain of integration of the multiple stochastic integral on the right-hand side of (52). Replacing $\Phi(t_1,\ldots,t_k)$ by $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ in expressions (57) and (58), passing to the limit in the integrands in (57) and (58), and taking into account relation (65) and Remark 4, we establish the desired conclusion. Lemma 4 is proved. This completes the proof of the theorem. □

In conclusion, we show that if the indices $i_1,\ldots,i_k\in\{1,\ldots,m\}$ are pairwise different, then the approximation of the integral $J^*\left(\psi^{(k)}\right)_{T,t}$ of the form

$$J^*\left(\psi^{(k)}\right)^q_{T,t}=\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta^{(i_l)}_{(j_l)T,t},\quad q<\infty,$$

satisfies the following relation:

$$\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)^q_{T,t}-J^*\left(\psi^{(k)}\right)_{T,t}\right)^2\right\}$$

$$=\int\limits_t^T\ldots\int\limits_t^T K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{q}C^2_{j_k\ldots j_1}. \tag{68}$$

By Lemma 2, we have the equality $J^*\left(\psi^{(k)}\right)_{T,t}=J\left(\psi^{(k)}\right)_{T,t}$ a.s for pairwise different nonzero indices $i_1,\ldots,i_k$. Hence,

$$\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)_{T,t}\right)^2\right\}=\mathbf{E}\left\{\left(J\left(\psi^{(k)}\right)_{T,t}\right)^2\right\}$$

$$=\int\limits_t^T\ldots\int\limits_t^T K^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{69}$$

In addition, under the same conditions, the following relations hold:

$$\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)_{T,t}-J^*\left(\psi^{(k)}\right)^q_{T,t}\right)^2\right\}$$

$$=\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)_{T,t}\right)^2\right\}-\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)^q_{T,t}\right)^2\right\} \tag{70}$$

and

$$\mathbf{E}\left\{\left(J^*\left(\psi^{(k)}\right)^q_{T,t}\right)^2\right\}=\sum_{j_1,\ldots,j_k=0}^{q}C^2_{j_k\ldots j_1}. \tag{71}$$

Now relations (69)–(71) imply formula (68).

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REFERENCES

1. O. Yu. Kul'chitskij and D. F. Kuznetsov, "The Taylor–Itô expansion of the Itô processes in a neighborhood of a fixed time moment," Deposited at VINITI, No. 2637-B93, 1–26 (1993).
2. O. Yu. Kul'chitskij and D. F. Kuznetsov, "The unified Taylor–Itô expansion," *Zap. Nauchn. Semin. POMI*, **244**, 186–204 (1997).
3. G. N. Mil'shteyn, *Numerical Solution of Stochastic Differential Equations* [in Russian], Sverdlovsk (1988).
4. E. Platen and W. Wagner, "On a Taylor formula for a class of Itô processes," *Prob. Math. Statist.*, **3**, 37–51 (1982).
5. P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag (1992).
6. P. E. Kloeden, E. Platen, and I. W. Wright, "Approximation of multiple stochastic integrals," *Stoch. Anal. Applic.*, **10**, 431–441 (1992).
7. A. V. Skorokhod, *Random Processes with Independent Increments* [in Russian], Moscow (1964).
8. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations* [in Russian], Kiev (1968).