Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs

Dmitriy F. Kuznetsov

Russia, 195251, Saint-Petersburg, Polytechnicheskaya st., 29
Peter the Great Saint-Petersburg Polytechnic University
Institute of Applied Mathematics and Mechanics
Department of Higher Mathematics

e-mail: sde_kuznetsov@inbox.ru

Dedicated to My Family

The first, second, and third editions of this monograph were published in the Journal
"Differencialnie Uravnenia i Protsesy Upravlenia"
(Differential Equations and Control Processes),
no. 4 (2020), A.1-A.606, no. 4 (2021), A.1-A.788, and no. 1 (2023), A.1-A.947
Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

Preface

God does not care about our mathematical difficulties. He integrates empirically

— *Albert Einstein*

The book is devoted to the problem of strong (mean-square) approximation of iterated Itô and Stratonovich stochastic integrals in the context of numerical integration of Itô stochastic differential equations (SDEs) and non-commutative semilinear stochastic partial differential equations (SPDEs) with nonlinear multiplicative trace class noise. The presented monograph opens up a new direction in researching of iterated stochastic integrals and summarizes the author's research on the mentioned problem carried out in the period 1994–2025.

The basis of this book composes on the monographs [1]-[17] and recent author's results [18]-[71].

This monograph (also see books [6]-[11], [14]-[17]) is the first monograph where the problem of strong (mean-square) approximation of iterated Itô and Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process is systematically analyzed in application to the numerical solution of SDEs.

For the first time we successfully use the generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ for the expansion and strong approximation of iterated Itô stochastic integrals of arbitrary multiplicity k, $k \in \mathbf{N}$ as well as for the expansion of some other types of iterated stochastic integrals (Chapter 1).

The above result has been adapted for iterated Stratonovich stochastic integrals of multiplicities 1 to 8 for the following two cases (Chapter 2).

1. The case of continuously differentiable weight functions (multiplicities 1 to 5) and weight functions identically equal to one (multiplicities 6 to 8). In this case, we use a complete orthonormal system of Legendre polynomials or

trigonometric functions in $L_2([t,T])$.

2. The case of continuous weight functions (multiplicities 1 and 2), binomial weight functions (multiplicities 3 and 4) and weight functions identically equal to one (multiplicities 5 and 6). In this case, we use an arbitrary complete orthonormal system of functions in $L_2([t,T])$.

Recently (in 2024), the mentioned adaptation has also been carried out for iterated Stratonovich stochastic integrals of multiplicity k, $k \in \mathbf{N}$ (Chapter 2, Theorems 2.59, 2.61) but under one additional condition (the case of continuous weight functions and an arbitrary complete orthonormal system of functions in $L_2([t,T])$).

Two theorems on expansions of iterated Stratonovich stochastic integrals of multiplicity $k, k \in \mathbf{N}$ based on iterated Fourier series with the pointwise convergence are formulated and proved (Chapter 2).

The integration order replacement technique for the class of iterated Itô stochastic integrals has been introduced (Chapter 3). This result is generalized for the class of iterated stochastic integrals with respect to martingales.

Four new forms of the Taylor–Itô and Taylor–Stratonovich expansions (the so-called unified Taylor–Itô and Taylor–Stratonovich expansions) are presented (Chapter 4).

Exact expressions are obtained for the mean-square approximation error of iterated Itô stochastic integrals of arbitrary multiplicity k, $k \in \mathbf{N}$ (Chapter 1) and iterated Stratonovich stochastic integrals of multiplicities 1 to 4 (Chapter 5). Furthermore, we provided a significant practical material (Chapter 5) devoted to the expansions and approximations of specific iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich expansions (Chapter 4) using the system of Legendre polynomials and the system of trigonometric functions.

The methods formulated in this book have been compared with some existing methods of strong approximation of iterated Itô and Stratonovich stochastic integrals (Chapter 6).

The results of Chapter 1 were applied (Chapter 7) to the approximation of iterated stochastic integrals with respect to the finite-dimensional approximation $\mathbf{W}_t^M$ of the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of arbitrary multiplicity k, $k \in \mathbf{N}$) and to the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of multiplicities 1 to 3).

This book will be interesting for specialists dealing with the theory of stochastic processes, applied and computational mathematics as well as senior students and postgraduates of technical institutes and universities.

Exact solutions of Itô SDEs and semilinear SPDEs are known in rather rare cases. Therefore, the need arises to construct numerical procedures for solving these equations.

The importance of the problem of numerical integration of Itô SDEs and semilinear SPDEs is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems of various physical nature under random disturbances and to the application of these equations for solving various mathematical problems, among which we mention signal filtering in the background of random noise, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems, etc.

It is well known that one of the effective and perspective approaches to the numerical integration of Itô SDEs and semilinear SPDEs is an approach based on the stochastic analogues of the Taylor formula for solutions of these equations. This approach uses the finite discretization of temporal variable and performs numerical modeling of solutions of Itô SDEs and semilinear SPDEs in discrete moments of time using stochastic analogues of the Taylor formula.

Speaking about Itô SDEs, note that the most important feature of the mentioned stochastic analogues of the Taylor formula for solutions of Itô SDEs is a presence in them of the so-called iterated Itô and Stratonovich stochastic integrals which are the functionals of a complex structure with respect to components of a multidimensional Wiener process. These iterated Itô and Stratonovich stochastic integrals are subject for study in this book and are defined by the following formulas

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{(Itô integrals)},$$

$$\int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{(Stratonovich integrals)},$$

where $\psi_1(\tau),\ldots,\psi_k(\tau):[t,T]\to\mathbf{R}$ are nonrandom functions (as a rule, in the applications they are identically equal to 1 or have a binomial form (see Chapter 4)), $\mathbf{w}_\tau$ is a random vector with an $m+1$ components: $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for

$i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Apparently, one of the first who began the study of such stochastic integrals (the case $k = 2$, $m = 2$, $\psi_1(\tau), \psi_2(\tau) \equiv 1$, $i_1 = 1$, $i_2 = 2$) was Lévy, who introduced the concept of the so-called Lévy stochastic area and studied its properties.

The above iterated stochastic integrals are the specific objects in the theory of stochastic processes. From the one side, nonrandomness of weight functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is the factor simplifying their structure. From the other side, nonscalarity of the Wiener process $\mathbf{f}_\tau$ with independent components $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ and the fact that the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are different for various l $(l = 1, \ldots, k)$ are essential complicating factors of the structure of iterated stochastic integrals. Taking into account features mentioned above, we suppose that the systems of iterated Itô and Stratonovich stochastic integrals play the extraordinary and perhaps the key role for solving the problem of numerical integration of Itô SDEs.

A natural question arises: is it possible to construct a numerical scheme for Itô SDE that includes only increments of the Wiener processes $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$, but has a higher order of convergence than the Euler method? It is known that this is impossible for $m > 1$ in the general case. This fact is called the "Clark–Cameron paradox" [72] and explains the need to use iterated stochastic integrals for constructing high-order numerical methods for Itô SDEs.

We want to mention in short that there are two main criteria of numerical methods convergence for Itô SDEs: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Itô SDE, simply stated, but the distribution of Itô SDE solution. Both mentioned criteria are independent, i.e. in general it is impossible to state that from the execution of strong criterion follows the execution of weak criterion and vice versa. Each of two convergence criteria is oriented on the solution of specific classes of mathematical problems connected with Itô SDEs.

Numerical integration of Itô SDEs based on the strong convergence criterion of approximation is widely used for the numerical simulation of sample trajectories of solutions to Itô SDEs (which is required for constructing new mathematical models based on such equations and for the numerical solution of different mathematical problems connected with Itô SDEs). Among these problems, we note the following: signal filtering under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis.

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Itô or Stratonovich stochastic integrals is very important and difficult from theoretical and computing point of view.

Seems that iterated stochastic integrals may be approximated by multiple integral sums. However, this approach implies the partitioning of the interval of integration $[t, T]$ for iterated stochastic integrals. The length $T - t$ of this interval is already fairly small (because it is a step of integration of numerical methods for Itô SDEs) and does not need to be partitioned. Computational experiments show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors.

The problem of effective decreasing of the mentioned cost (in several times or even in several orders) is very difficult and requires new complex investigations. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case allows the investigation with using of the Itô formula. In the more general case, when not all numbers $i_1, \ldots, i_k$ are equal, the mentioned problem turns out to be more complex (it cannot be solved using the Itô formula and requires more deep and complex investigation). Note that even for the case $i_1 = \ldots = i_k \neq 0$, but for different functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ the mentioned difficulties persist and simple sets of iterated Itô and Stratonovich stochastic integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (with respect to the mean-square approximation) using the system of standard Gaussian random variables. The Itô formula is also useless in this case and as a result we need to use more complex but effective expansions.

Why the problem of the mean-square approximation of iterated stochastic integrals is so complex?

Firstly, the mentioned stochastic integrals (in the case of fixed limits of integration) are the random variables, whose density functions are unknown in the general case. The exception is connected with the narrow particular case which is the simplest iterated Itô stochastic integral with multiplicity 2 and $\psi_1(\tau), \psi_2(\tau) \equiv 1$; $i_1, i_2 = 1, \ldots, m$. Nevertheless, the knowledge of this density function not gives a simple way for approximation of iterated Itô stochastic integral of multiplicity 2.

Secondly, we need to approximate not only one stochastic integral, but several iterated stochastic integrals that are complexly dependent in a probabilistic sense.

Often, the problem of combined mean-square approximation of iterated Itô and Stratonovich stochastic integrals occurs even in cases when the exact solution of Itô SDE is known. It means that even if you know the solution of Itô SDE exactly, you cannot model it numerically without the combined numerical modeling of iterated stochastic integrals.

Note that for a number of special types of Itô SDEs the problem of approximation of iterated stochastic integrals may be simplified but cannot be solved. Equations with additive vector noise, with non-additive scalar noise, with additive scalar noise, with a small parameter are related to such types of equations. In these cases, simplifications are connected to the fact that some members from stochastic Taylor expansions are equal to zero or we may neglect some members from these expansions due to the presence of a small parameter.

Furthermore, the problem of combined numerical modeling (with respect to the mean-square convergence criterion) of iterated Itô and Stratonovich stochastic integrals is rather new.

One of the main and unexpected achievements of this book is the successful usage of functional analysis methods (more concretely, we mean generalized multiple Fourier series in various systems of basis functions that converge in the sense of the norm in $L_2([t, T]^k)$) in this scientific field.

The problem of combined numerical modeling (with respect to the mean-square convergence criterion) of systems of iterated Itô and Stratonovich stochastic integrals was analyzed in the context of the problem of numerical integration of Itô SDEs in the following monographs:

[I] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Kluwer Academic Publishers. Dordrecht. 1995 (Russian Ed. 1988).

[II] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer-Verlag. Berlin. 1992 (2nd Ed. 1995, 3rd Ed. 1999).

[III] Milstein G.N., Tretyakov M. V. Stochastic Numerics for Mathematical

Physics. Springer-Verlag. Berlin. 2004.

[IV] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. Polytechnical University Publ. St.-Petersburg. 2007 [2] (2nd Ed. 2007 [3], 3rd Ed. 2009 [4], 4th Ed. 2010 [5], 5th Ed. 2017 [12], 6th Ed. 2018 [13]).

Note that the initial version of the book [IV] has been published in 2006 [1]. Also we mention the books [6] (2010), [7] (2011), [8] (2011), [9] (2012), [10] (2013), [11] (2017) and [14] (2020), [15] (2021), [16] (2023), [17] (2024).

The books [I] and [III] analyze the problem of the mean-square approximation of iterated stochastic integrals only for two simplest iterated Itô stochastic integrals of 1st and 2nd multiplicities ($k = 1$ and 2, $\psi_1(\tau)$ and $\psi_2(\tau) \equiv 1$) for the multidimensional case: $i_1, i_2 = 0, 1, \ldots, m$. In addition, the main idea is based on the expansion of the so-called Brownian bridge process into the trigonometric Fourier series (version of the so-called Karhunen–Loève expansion). This method is called in [I] and [III] as the Fourier method[1].

In [II] using the Fourier method [I], the attempt was made to obtain the mean-square approximation of elementary iterated Stratonovich stochastic integrals of multiplicities 1 to 3 ($k = 1, \ldots, 3$, $\psi_1(\tau), \ldots, \psi_3(\tau) \equiv 1$) for the multidimensional case: $i_1, \ldots, i_3 = 0, 1, \ldots, m$. However, as we can see in the presented book, the results of the monograph [II], related to the mean-square approximation of iterated Stratonovich stochastic integrals of 3rd multiplicity, cause a number of critical remarks (see discussions in Sect. 2.41, 2.42, 6.2).

The main purpose of this book is to construct and develop newer and more effective methods (than presented in the books [I]–[III]) of combined mean-square approximation of iterated Itô and Stratonovich stochastic integrals.

Talking about the history of solving the problem of combined mean-square approximation of iterated stochastic integrals, the idea to find a basis of random variables using which we may represent iterated stochastic integrals turned out to be useful. This idea was transformed several times during last decades.

Attempts to approximate the iterated stochastic integrals using various integral sums were made until 1980s and later, i.e. the interval of integration $[t, T]$ of the stochastic integral was divided into n parts and the iterated stochastic integral was represented approximately by the multiple integral sum, which included the system of independent standard Gaussian random variables, whose

[1] To date, there is confusion in the literature about who first proposed the Fourier method [I], [III]. As far as the author of this book knows, the mentioned method first appeared in the Russian edition of the monograph by G.N. Milstein [82] (pp. 121–135), which was published in 1988.

numerical modeling is not a problem.

However, as we noted above, it is obvious that the length $T-t$ of integration interval $[t,T]$ of the iterated stochastic integrals is a step of integration of numerical methods for Itô SDEs, which is already a rather small value even without the additional splitting. Numerical experiments demonstrate that such approach results in drastic increasing of computational costs accompanied by the growth of multiplicity of the stochastic integrals (beginning from 2nd and 3rd multiplicity) that is necessary for construction of high-order strong numerical methods for Itô SDEs or in the case of decrease of integration step of numerical methods, and thereby it is almost useless for practice.

The new step for solution of the problem of combined mean-square approximation of iterated stochastic integrals was made by Milstein G.N. in his monograph [I] (1988). For the expansion of iterated stochastic integrals, he proposed to use the trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karhunen–Loève expansion). Using this method, expansions of two simplest iterated Itô stochastic integrals of multiplicities 1 and 2 are obtained and their mean-square convergence is proved.

As we noted above, the attempt to develop this idea together with the Wong–Zakai approximation [73]-[75] was made in the monograph [II] (1992), where the expansions of simplest iterated Stratonovich stochastic integrals of multiplicities 1 to 3 were obtained. However, due to a number of limitations and technical difficulties which are typical for the method [I], in [II] and following publications this problem was not solved more completely. In addition, the author has reasonable doubts about application of the Wong–Zakai results [73]-[75] to approximation of iterated Stratonovich stochastic integrals of 3rd multiplicity in the monograph [II] (see discussions in Sect. 2.41, 2.42, 6.2).

It is necessary to note that the computational cost for the method [I] is significantly less than for the method of multiple integral sums.

Regardless of the method [I] positive features, the number of its limitations are also outlined. Among them let us mention the following.

1. The absence of explicit formula for calculation of expansion coefficients for iterated stochastic integrals.

2. The practical impossibility of exact calculation of the mean-square approximation error of iterated stochastic integrals with the exception of simplest integrals of 1st and 2nd multiplicity (as a result, it is necessary to consider redundant terms of expansions and it results to the growth of computational

cost and complication of the numerical methods for Itô SDEs).

3. There is a hard limitation on the system of basis functions — it may be only the trigonometric functions.

4. There are some technical problems if we use this method for iterated stochastic integrals whose multiplicity is greater than 2nd.

Nevertheless, it should be noted that the analyzed method is a concrete step forward in this scientific field.

The author thinks that the method presented by him in [IV] (for the first time this method is appeared in the final form in [1] (2006)) and in this book (hereafter this method is reffered to as the method of generalized multiple Fourier series) is a breakthrough in solution of the problem of combined mean-square approximation of iterated Itô stochastic integrals.

The idea of this method is as follows: the iterated Itô stochastic integral of multiplicity k, $k \in \mathbf{N}$ is represented as the multiple stochastic integral from the certain nonrandom discontinuous function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Itô stochastic integral. Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Itô stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Itô stochastic integral.

As a result, we obtain the following new possibilities and advantages in comparison with the Fourier method [I].

1. There is an explicit formula for calculation of expansion coefficients of iterated Itô stochastic integral with any fixed multiplicity k. In other words, we can calculate (without any preliminary and additional work) the expansion coefficient with any fixed number in the expansion of iterated Itô stochastic integral of the preset fixed multiplicity. At that, we do not need any knowledge about coefficients with other numbers or about other iterated Itô stochastic integrals included in the considered set.

2. We have new possibilities for obtainment the exact and approximate expressions for the mean-square approximation errors of iterated Itô stochastic

integrals. These possibilities are realized by the exact and estimate formulas for the mentioned mean-square approximation errors. As a result, we would not need to consider redundant terms of expansions that may complicate approximations of iterated Itô stochastic integrals.

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T]^k)$, we have new possibilities for approximation — we can use not only the trigonometric functions as in [I] but the Legendre polynomials as well as the systems of Haar and Rademacher–Walsh functions.

4. As it turned out, it is more convenient to work with Legendre polynomials for approximation of iterated Itô stochastic integrals. The approximations themselves are simpler than their analogues based on the system of trigonometric functions. Probably for the systems of Haar and Rademacher–Walsh functions the expansions of iterated stochastic integrals become more complex and less effective for practice [IV]. Expansions based on Haar functions for $k = 2$ were also considered in [87], [95], [222]. Note that the multiple Fourier–Walsh and Fourier–Haar series ($k \in \mathbf{N}$) were applied to the mean-square approximation of multiple Stratonovich stochastic integrals (defined as in [143], [144]) in [221]. The convergence of these approximations was proved with respect to the special subsequence $n_m = 2^m$ $(m \to \infty)$ [221].

5. The question about what kind of functions (polynomial or trigonometric) is more convenient in the context of computational costs for approximation turns out to be nontrivial, since it is necessary to compare approximations not for one stochastic integral but for several stochastic integrals at the same time. At that there is a possibility that computational costs for some integrals will be smaller for the system of Legendre polynomials and for others — for the system of trigonometric functions. The author proved [21] (also see Sect. 5.3 in this book) that the computational costs are significantly less for the system of Legendre polynomials at least in the case of approximation of the special set of iterated Itô stochastic integrals, which are necessary for the implementation of strong numerical methods for Itô SDEs with the order of convergence $\gamma = 1.5$. In addition, the author supposes that this effect will be more impressive when analyzing more complex sets of iterated Itô stochastic integrals ($\gamma = 2.0$, 2.5, $3.0, \ldots$). This supposition is based on the fact that the polynomial system of functions has a significant advantage (in comparison with the trigonometric system of functions) in the mean-square approximation of iterated Itô stochastic integrals for which not all weight functions are equal to 1.

6. The Milstein approach [I] for approximation of iterated Itô stochastic integrals leads to iterated applicaton of the operation of limit transition (in contrast with the method of generalized multiple Fourier series, for which the operation of limit transition is implemented only once) starting at least from the second or third multiplicity of iterated Itô stochastic integrals (we mean at least double or triple integration with respect to components of a multidimensional Wiener process). Multiple series are more preferential for approximation than the iterated ones, since the partial sums of multiple series converge for any possible case of joint converging to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [II] the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein approach [I] together with the Wong–Zakai approximation [73]-[75] (see discussions in Sect. 2.41, 2.42, 6.2).

7. The convergence in the mean of degree $2n$, $n \in \mathbf{N}$ as well as the convergence with probability 1 of approximations from the method of generalized multiple Fourier series are proved. The convergence rate for these two types of convergence is estimated.

8. The method of generalized multiple Fourier series has been applied for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson random measures and iterated stochastic integrals with respect to martingales) as well as for approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process.

9. Another modification of the method of generalized multiple Fourier series is connected with the application of complete orthonormal with weight $r(t_1) \ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t, T]^k)$.

10. As it turned out, the method of generalized multiple Fourier series can be adapted for iterated Stratonovich stochastic integrals. This adaptation is carried out in Chapter 2 for the following two cases.

1). The case of continuously differentiable weight functions (multiplicities 1 to 5) and weight functions identically equal to one (multiplicities 6 to 8). In this case, we use a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t, T])$.

2). The case of continuous weight functions (multiplicities 1 and 2), binomial weight functions (multiplicities 3 and 4) and weight functions identically equal to one (multiplicities 5 and 6). In this case, we use an arbitrary complete orthonormal system of functions in $L_2([t,T])$.

Recently (in 2024), the mentioned adaptation has also been carried out for iterated Stratonovich stochastic integrals of multiplicity k, $k \in \mathbf{N}$ but under one additional condition (the case of continuous weight functions and an arbitrary complete orthonormal system of functions in $L_2([t,T])$ (Chapter 2, Theorems 2.59, 2.61)). The rate of mean-square convergence of approximations of iterated Stratonovich stochastic integrals is found (Sect. 2.8, 2.15, 2.16).

11. The method of generalized multiple Fourier series is reformulated using Hermite polynomials in Sect. 1.10 and generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ in Sect. 1.11, 1.12, 1.14, 1.15. At that, in Sect. 1.11, 1.12 we use the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process.

12. The results of Chapter 1 (Theorems 1.1, 1.2, 1.14, 1.16) and Chapter 2 (Theorems 2.1–2.10, 2.14, 2.17, 2.30, 2.32–2.36, 2.41–2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.65) can be considered from the point of view of the Wong–Zakai approximation [73]-[75] for the case of a multidimensional Wiener process and the Wiener process approximation based on its series expansion using various complete orthonormal systems of functions in the space $L_2([t,T])$ (see discussions in Sect. 2.41, 2.42, 6.2). These results overcome a number of difficulties that were noted above and relate to the Fourier method [I].

The theory presented in this book was realized [53], [54] in the form of a software package in the Python programming language. The mentioned software package implements the strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs (with multidimensional noncommutative noise) based on the unified Taylor–Itô and Taylor–Stratonovich expansions (Chapter 4). At that for the numerical simulation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 we applied the formulas based on multiple Fourier–Legendre series (Chapter 5). Moreover, we used [53], [54] the database with 270,000 exactly calculated Fourier–Legendre coefficients.

Throughout the book, special attention is paid to two systems of basis functions in the space $L_2([t,T])$. Namely, we mainly use the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. This is due to two reasons. The first of these is that the trigonometric basis system has already been used to approximate iterated stochastic integrals in the 1980s-1990s (see above), and the author needed to compare his results with the results of other authors. The second reason is that the system of Legendre polynomials is optimal (see Sect. 5.3) for the implementation of strong numerical methods with convergence order 1.5 and higher for Itô SDEs with multidimensional non-commutative noise. The system of Legendre polynomials was first applied to the approximation of iterated stochastic integrals in the author's work [76] in 1997 (also see [77]-[79]). According to the author's opinion, other complete orthonormal systems of functions in the space $L_2([t,T])$ (for example, systems of Haar and Rademacher–Walsh functions) turn out to be less efficient for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals.

The attentive reader will notice that Chapters 1 and 2 of this book can be somewhat shortened since Theorem 1.16 is a generalization of Theorems 1.1, 1.2 and Theorems 2.3, 2.33, 2.34, 2.41 are generalizations of Theorems 2.1, 2.2, 2.4–2.9. However, the author did not make the appropriate changes in Chapters 1, 2 for a number of reasons. In particular, the application of the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process to the expansion of iterated Itô stochastic integrals (Theorem 1.16) and a new approach to the expansion of iterated Stratonovich stochastic integrals (Theorems 2.30–2.65) were obtained by the author recently (in 2021–2025), while Theorems 1.1, 1.2, 2.4–2.9 were obtained by the author in the period from 2005 to 2013. In addition, the proof of each of the mentioned theorems contains some original ideas that the author would like to keep in Chapters 1 and 2. Moreover, a significant part of Chapter 2 is devoted to the proof of Hypothesis 2.5 (Sect. 2.28) for various special cases (Theorems 2.1–2.9, 2.30, 2.33–2.36, 2.41, 2.45–2.48, 2.50, 2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.65). In order to prove these theorems, we developed a number of approaches to the expansion of iterated Stratonovich stochastic integrals.

Thus, the results of Chapters 1, 2 are presented primarily in the order in which they were obtained by the author.

Dmitriy F. Kuznetsov July, 2025

Acknowledgements

I would like to thank the Deputy Editor of the Journal "Differencialnie Uravnenia i Protsesy Upravlenia" Dr. Nataly B. Ampilova for her timely administrative support and encouragement and Dr. Konstantin A. Rybakov for useful discussion of some presented results.

Basic Notations

$\mathbf{N}$	set of natural numbers
$\mathbf{R}$, $\mathbf{R}^1$	set of real numbers
$\mathbf{R}^n$	n-dimensional Euclidean space
$(a_1, \ldots, a_n)$	ordered set with elements $a_1, \ldots, a_n$
$\{a_1, \ldots, a_n\}$	unordered set with elements $a_1, \ldots, a_n$
$n!$	$1 \cdot 2 \cdot \ldots \cdot n$ for $n \in \mathbf{N}$ $(0! = 1)$
$(2n-1)!!$	$1 \cdot 3 \cdot \ldots \cdot (2n-1)$ for $n \in \mathbf{N}$
$\stackrel{\sf def}{=}$	equal by definition
$\equiv$	identically equal to
C_n^m	binomial coefficient $n!/(m!(n-m)!)$
$\emptyset$	empty set
$\mathbf{1}_A$	indicator of the set A
$x \in X$	x is an element of the set X
$X \bigcup Y$	union of sets X and Y
$X \times Y$	Cartesian product of sets X and Y
$\overline{\lim\limits_{n\to\infty}}$	$\limsup\limits_{n\to\infty}$
$\underline{\lim\limits_{n\to\infty}}$	$\liminf\limits_{n\to\infty}$
$x \ll y$	x much less than y
$[x]$	largest integer number not exceeding x

$\lvert x\rvert$	absolute value of the real number x
$F:\ X\to Y$	function F from X into Y
$A^{(ij)}$	ijth element of the matrix A
A_i	ith colomn of the matrix A
$\mathbf{x}^{(i)}$	ith component of the vector $\mathbf{x}\in\mathbf{R}^n$
$O(x)$	expression being divided by x remains bounded as $x\to 0$
$\sum\limits_{(i_1,\ldots,i_k)}$	sum with respect to all possible permutations $(i_1,\ldots,i_k)$
$\mathsf{M}\{\xi\}$	expectation of ξ
$\mathsf{M}\{\xi\vert\mathrm{F}\}$	conditional expectation of ξ with respect to F
$\xi\sim\mathrm{N}(m,\sigma^2)$	Gaussian random variable ξ with expectation m and variance σ^2
$\underset{n\to\infty}{\text{l.i.m.}}$	limit in the mean-square sense
$\mathcal{B}(X)$	σ-algebra of Borel subsets of X
f_t	scalar standard Wiener process
$\mathbf{f}_t$	vector standard Wiener process with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$
w. p. 1	with probability 1
$\mathbf{w}_t$	vector with components $\mathbf{w}_t^{(i)}$, $i=0,1,\ldots,m$ and property $\mathbf{w}_t^{(i)}=\mathbf{f}_t^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_t^{(0)}=t$
$\dfrac{\partial F}{\partial\mathbf{x}^{(i)}}$	partial derivative of $F:\mathbf{R}^n\to\mathbf{R}$
$\dfrac{\partial^2 F}{\partial\mathbf{x}^{(i)}\partial\mathbf{x}^{(j)}}$	2nd order partial derivative of $F:\mathbf{R}^n\to\mathbf{R}$
$\int\limits_t^T\ldots d\mathbf{w}_\tau^{(i)}$	Itô stochastic integral

$\int\limits_t^{*T} \ldots d\mathbf{w}_\tau^{(i)}$	Stratonovich stochastic integral
$\int\limits_t^{T} \ldots \circ\, d\mathbf{w}_\tau^{(i)}$	Stratonovich stochastic integral [143]
$\mathbf{W}_t$	Q-Wiener process
$J[\psi^{(k)}]_{T,t}, \quad I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$	iterated Itô stochastic integrals
$J^*[\psi^{(k)}]_{T,t}, \quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$	iterated Stratonovich stochastic integrals
$J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$	iterated Stratonovich stochastic integral [144]
$\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$	multiple Stratonovich stochastic integral [144]
$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}, \quad I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$	approximations of iterated Itô stochastic integrals
$J^*[\psi^{(k)}]_{T,t}^{p}, \quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}$	approximations of iterated Stratonovich stochastic integrals
$J[\Phi]_{T,t}^{(k)}, \quad J[\Phi]_{T,t}^{(i_1\ldots i_k)}$	multiple Stratonovich stochastic integrals
$J'[\Phi]_{T,t}^{(k)}, \quad J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$	multiple Wiener stochastic integrals
$P_n(x)$	Legendre polynomials
$H_n(x),\ h_n(x)$	Hermite polynomials
$H_n(x,y)$	polynomials related to the Hermite polynomials
$L_2(D)$	Hilbert space of square integrable functions on D
$\Vert\cdot\Vert_{L_2(D)}$	norm in the Hilbert space $L_2(D)$
$\mathrm{tr}\ A$	trace of the operator A
$\Vert\cdot\Vert_H$	norm in the Hilbert space H
$\langle u,v\rangle_H$	scalar product in the Hilbert space H
$L_{HS}(U,H)$	space of Hilbert–Schmidt operators from U to H

$\Vert\cdot\Vert_{L_{HS}(U,H)}$	operator norm in the space of Hilbert–Schmidt operators from U to H
$\int\limits_t^T \ldots d\mathbf{W}_\tau$	stochastic integral with respect to the Q-Wiener process

Contents

Chapter 1

Method of Expansion and Mean-Square Approximation of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series

This chapter is devoted to the expansions of iterated Itô stochastic integrals with respect to components of the multidimensional Wiener process based on generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k),\ k \in \mathbf{N}$. The method of generalized multiple Fourier series for expansion and mean-square approximation of iterated Itô stochastic integrals of arbitrary multiplicity k, $k \in \mathbf{N}$ is proposed and developed. The obtained expansions contain only one operation of the limit transition in contrast to existing analogues. In this chapter it is also obtained the generalization of the proposed method for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T]^k)$, $k \in \mathbf{N}$ as well as for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbf{N}$. It is shown that in the case of a scalar Wiener process the proposed method leads to the well known expansion of iterated Itô stochastic integrals based on the Itô formula and Hermite polynomials. The convergence in the mean of degree $2n$, $n \in \mathbf{N}$ as well as the convergence with probability 1 of the proposed method are proved. The exact and approximate expressions for the mean-square approximation error of iterated Itô stochastic integrals of multiplicity k, $k \in \mathbf{N}$ have been derived. The considered method has been applied for other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson random measures and iterated stochastic integrals with respect to martingales).

1.1 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series Converging in the Mean

1.1.1 Introduction

The idea of representing the iterated Itô and Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using multiple and iterated Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of author's publications [1]-[70] (also see early publications [76] (1997), [77] (1998), [78] (2000), [79] (2001), [80] (1994), [81] (1996)). Note that another approaches to the mean-square approximation of iterated Itô and Stratonovich stochastic integrals can be found in [71], [82]-[99].

Specifically, the approach [1]-[70] appeared for the first time in [80], [81]. In these works the mentioned idea is formulated more likely at the level of guess (without any satisfactory grounding), and as a result the papers [80], [81] contain rather fuzzy formulations and a number of incorrect conclusions. Note that in [80], [81] we used the trigonometric multiple Fourier series converging in the sense of norm in the space $L_2([t, T]^k)$, $k = 1, 2, 3$. It should be noted that the results of [80], [81] are correct for a sufficiently narrow particular case when the numbers $i_1, ..., i_k$ are pairwise different, $i_1, \ldots, i_k = 1, \ldots, m$ (see Theorem 1.1 below).

Usage of Fourier series with respect to the system of Legendre polynomials for approximation of iterated stochastic integrals took place for the first time in the publications of the author [76]-[79] (also see [1]-[71]).

The question about what integrals (Itô or Stratonovich) are more suitable for expansions within the frames of distinguished direction of researches has turned out to be rather interesting and difficult.

On the one side, the results of Chapter 1 (see Theorems 1.1, 1.2, 1.16) conclusively demonstrate that the structure of iterated Itô stochastic integrals is rather convenient for expansions into multiple series with respect to the system of standard Gaussian random variables regardless of the multiplicity k of the iterated Itô stochastic integral.

On the other side, the results of Chapter 2 [6]-[23], [26], [28], [30], [32]-

[39], [42], [43], [45]-[47], [52], [64], [65], [76]-[79] convincingly demontrate that the final formulas for expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 8 (the case of continuously differentiable weight functions and a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t,T])$) and iterated Stratonovich stochastic integrals of multiplicity k, $k \in \mathbf{N}$ (the case of continuous weight functions and an arbitrary complete orthonormal system of functions in $L_2([t,T])$) are more compact than their analogues for iterated Itô stochastic integrals.

1.1.2 Itô Stochastic Integral

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega): [0,T]\times\Omega \to \mathbf{R}$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$. Further, we will use the following notation: $f(t,\omega) \stackrel{\text{def}}{=} f_t$.

Let us consider the right-continous family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s < t$.

2. The Wiener process f_t is F_t-measurable for all $t \in [0,T]$.

3. The process $f_{t+\Delta} - f_t$ for all $t \ge 0$, $\Delta > 0$ is independent with the events of σ-algebra F_t.

Let us introduce the class $\mathrm{M}_2([0,T])$ of functions $\xi : [0,T]\times\Omega \to \mathbf{R}$, which satisfy the conditions:

1. The function $\xi(t,\omega)$ is measurable with respect to the pair of variables (t,ω).

2. The function $\xi(t,\omega)$ is F_t-measurable for all $t \in [0,T]$ and $\xi(\tau,\omega)$ is independent with increments $f_{t+\Delta} - f_t$ for $t \ge \tau$, $\Delta > 0$.

3. The following relation is fulfilled

$$\int\limits_0^T \mathsf{M}\left\{(\xi(t,\omega))^2\right\} dt < \infty.$$

4. $\mathsf{M}\left\{(\xi(t,\omega))^2\right\} < \infty$ for all $t \in [0,T]$.

For any partition $\tau_j^{(N)}$, $j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$
0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty \tag{1.1}
$$

we will define the sequence of step functions

$$
\xi^{(N)}(t,\omega)=\xi_j\left(\omega\right)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),
$$

where $\xi^{(N)}(t,\omega)\in \mathrm{M}_2([0,T])$, $j=0,1,\ldots,N-1$, $N=1,2,\ldots$ Here and further, w. p. 1 means with probability 1.

Let us define the Itô stochastic integral for $\xi(t,\omega)\in \mathrm{M}_2([0,T])$ as the following mean-square limit [100], [101] (also see [84])

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{j=0}^{N-1}\xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(f\left(\tau_{j+1}^{(N)},\omega\right)-f\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\xi_\tau df_\tau, \tag{1.2}
$$

where $\xi^{(N)}(t,\omega)$ is any step function from the class $\mathrm{M}_2([0,T])$, which converges to the function $\xi(t,\omega)$ in the following sense

$$
\lim_{N\to\infty}\int\limits_0^T \mathsf{M}\left\{\left|\xi^{(N)}(t,\omega)-\xi(t,\omega)\right|^2\right\}dt=0. \tag{1.3}
$$

Further, we will denote $\xi(\tau,\omega)$ as ξ_τ.

It is well known [100] that the Itô stochastic integral exists as the limit (1.2) and it does not depend on the selection of sequence $\xi^{(N)}(t,\omega)$. Furthermore, the Itô stochastic integral satisfies w. p. 1 to the following properties [100]

$$
\mathsf{M}\left\{\int\limits_0^T\xi_t df_t\Bigg|\mathrm{F}_0\right\}=0,
$$

$$
\mathsf{M}\left\{\left|\int\limits_0^T\xi_t df_t\right|^2\Bigg|\mathrm{F}_0\right\}=\mathsf{M}\left\{\int\limits_0^T\xi_t^2 dt\Bigg|\mathrm{F}_0\right\},
$$

$$
\int\limits_0^T(\alpha\xi_t+\beta\psi_t)df_t=\alpha\int\limits_0^T\xi_t df_t+\beta\int\limits_0^T\psi_t df_t,
$$

where $\xi_t,\ \phi_t \in \mathrm{M}_2([0,T]),\ \alpha,\ \beta \in \mathbf{R}^1$.

Let us define the stochastic integral for $\xi_\tau \in \mathrm{M}_2([0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(\tau_{j+1}^{(N)}-\tau_j^{(N)}\right) \stackrel{\text{def}}{=} \int\limits_0^T \xi_\tau d\tau, \tag{1.4}$$

where $\xi^{(N)}(t,\omega)$ is any step function from the class $\mathrm{M}_2([0,T])$, which converges in the sense (1.3) to the function $\xi(t,\omega)$.

1.1.3 Theorem on Expansion of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$)

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider the following iterated Itô stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.5}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau,\ i_1,\ldots,i_k=0,1,\ldots,m.$

Let us consider the approach to expansion of the iterated Itô stochastic integrals (1.5) [1]-[70] (the so-called method of generalized multiple Fourier series). The idea of this method is as follows: the iterated Itô stochastic integral (1.5) of multiplicity k, $k\in\mathbf{N}$ is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$. Here $[t,T]$ is the interval of integration of the iterated Itô stochastic integral (1.5). Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Itô stochastic integral (1.5) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Itô stochastic integral (1.5).

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ (we will also consider the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ in Sect. 1.11, 1.12). Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{1.6}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{1.7}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{1.8}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{1.9}$$

Theorem 1.1[2] [1] (2006) (also see [2]-[70]). *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (1.10)$$

where

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k, \quad \mathrm{H}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r = 1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \qquad (1.11)$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (1.8), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of $[t,T]$, which satisfies the condition* (1.9).

Proof. At first, let us prove preparatory lemmas.

Lemma 1.1. *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1}, \qquad (1.12)$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval $[t,T]$ satisfying the condition* (1.9).

[2]Theorem 1.1 will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$ in Sect. 1.11 (see Theorem 1.16). Theorem 1.1 marked the beginning of a systematic study of the problem of strong approximation of iterated Itô and Stratonovich stochastic integrals that have been most fully studied to date in this book.

Proof. It is easy to notice that using the property of stochastic integrals additivity, we can write

$$J[\psi^{(k)}]_{T,t}=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}+\varepsilon_N\qquad \text{w. p. 1,}\tag{1.13}$$

where

$$J[\psi_l]_{s,\theta}=\int\limits_{\theta}^{s}\psi_l(\tau)d\mathbf{w}_{\tau}^{(i_l)}$$

and

$$\varepsilon_N=\sum_{j_k=0}^{N-1}\int\limits_{\tau_{j_k}}^{\tau_{j_k+1}}\psi_k(s)\int\limits_{\tau_{j_k}}^{s}\psi_{k-1}(\tau)J[\psi^{(k-2)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k-1})}d\mathbf{w}_{s}^{(i_k)}+$$

$$+\sum_{r=1}^{k-3}G[\psi_{k-r+1}^{(k)}]_N\times$$

$$\times\sum_{j_{k-r}=0}^{j_{k-r+1}-1}\int\limits_{\tau_{j_{k-r}}}^{\tau_{j_{k-r}+1}}\psi_{k-r}(s)\int\limits_{\tau_{j_{k-r}}}^{s}\psi_{k-r-1}(\tau)J[\psi^{(k-r-2)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k-r-1})}d\mathbf{w}_{s}^{(i_{k-r})}+$$

$$+G[\psi_3^{(k)}]_N\sum_{j_2=0}^{j_3-1}J[\psi^{(2)}]_{\tau_{j_2+1},\tau_{j_2}},$$

where

$$G[\psi_m^{(k)}]_N=\sum_{j_k=0}^{N-1}\sum_{j_{k-1}=0}^{j_k-1}\ldots\sum_{j_m=0}^{j_{m+1}-1}\prod_{l=m}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$(\psi_m,\psi_{m+1},\ldots,\psi_k)\stackrel{\text{def}}{=}\psi_m^{(k)},\qquad(\psi_1,\ldots,\psi_k)=\psi_1^{(k)}\stackrel{\text{def}}{=}\psi^{(k)}.$$

Using the standard estimates (1.26), (1.27) (see below) for the moments of stochastic integrals, we obtain w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N=0.\tag{1.14}$$

Comparing (1.13) and (1.14), we get

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}\qquad\text{w. p. 1.}\tag{1.15}$$

Let us rewrite $J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}$ in the form

$$J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}=\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}+\int\limits_{\tau_{j_l}}^{\tau_{j_l+1}}(\psi_l(\tau)-\psi_l(\tau_{j_l}))d\mathbf{w}_{\tau}^{(i_l)}$$

and substitute it into (1.15). Then, due to the moment properties of stochastic integrals and continuity (which means uniform continuity) of the functions $\psi_l(s)$ $(l=1,\ldots,k)$ it is easy to see that the prelimit expression on the right-hand side of (1.15) is a sum of the prelimit expression on the right-hand side of (1.12) and the value which tends to zero in the mean-square sense if $N\to\infty$. Lemma 1.1 is proved.

Remark 1.1. *It is easy to see that if* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.12) *for some* $l\in\{1,\ldots,k\}$ *is replaced with* $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p=2,\ i_l\ne 0)$, *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\psi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p=3,4,\ldots,$ *then the right-hand side of the formula* (1.12) *will become zero w. p.* 1. *If we replace* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.12) *for some* $l\in\{1,\ldots,k\}$ *with* $(\Delta\tau_{j_l})^p$ $(p=2,3,\ldots)$, *then the right-hand side of the formula* (1.12) *also will be equal to zero w. p.* 1.

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\rm def}{=}J[\Phi]_{T,t}^{(k)},\tag{1.16}$$

where $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}^1$ is a nonrandom function (the properties of this function will be specified further).

Denote

$$D_k=\{(t_1,\ldots,t_k):\ t\le t_1<\ldots<t_k\le T\}.\tag{1.17}$$

We will use the same symbol D_k to denote the open and closed domains corresponding to the domain D_k defined by (1.17). However, we always specify what domain we consider (open or closed).

Also we will write $\Phi(t_1,\ldots,t_k)\in C(D_k)$ if $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function of k variables in the closed domain D_k.

Let us consider the iterated Itô stochastic integral

$$I[\Phi]_{T,t}^{(k)}\stackrel{\rm def}{=}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{1.18}$$

where $\Phi(t_1,\ldots,t_k)\in C(D_k)$.

Using the arguments which similar to the arguments used in the proof of Lemma 1.1 it is easy to demonstrate that if $\Phi(t_1,\ldots,t_k)\in C(D_k)$, then the following equality is fulfilled

$$I[\Phi]_{T,t}^{(k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\Phi(\tau_{j_1},\ldots,\tau_{j_k})\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\quad \text{w. p. 1.} \tag{1.19}$$

In order to explain this, let us check the correctness of the equality (1.19) when $k=3$. For definiteness we will suppose that $i_1,i_2,i_3=1,\ldots,m$. We have

$$I[\Phi]_{T,t}^{(3)}\stackrel{\text{def}}{=}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\int\limits_t^{\tau_{j_3}}\int\limits_t^{t_2}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_t^{t_2}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\left(\int\limits_t^{\tau_{j_2}}+\int\limits_{\tau_{j_2}}^{t_2}\right)\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_2}}^{t_2}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \tag{1.20}$$

Let us demonstrate that the second limit on the right-hand side of (1.20) equals to zero.

Actually, for the second moment of its prelimit expression we get

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_2}}^{t_2}\Phi^2(t_1,t_2,\tau_{j_3})dt_1dt_2\Delta\tau_{j_3}\le M^2\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\frac{1}{2}\left(\Delta\tau_{j_2}\right)^2\Delta\tau_{j_3}\to 0$$

when $N \to \infty$. Here M is a constant, which restricts the module of the function $\Phi(t_1, t_2, t_3)$ due to its continuity, $\Delta\tau_j = \tau_{j+1} - \tau_j$.

Considering the obtained conclusions, we have

$$I[\Phi]_{T,t}^{(3)} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1, t_2, t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1, t_2, \tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, \tau_{j_2}, \tau_{j_3}) - \Phi(\tau_{j_1}, \tau_{j_2}, \tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1}, \tau_{j_2}, \tau_{j_3}) \Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (1.21)$$

In order to get the sought result, we just have to demonstrate that the first two limits on the right-hand side of (1.21) equal to zero. Let us prove that the first one of them equals to zero (proof for the second limit is similar).

The second moment of prelimit expression of the first limit on the right-hand side of (1.21) equals to the following expression

$$\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3}. \qquad (1.22)$$

Since the function $\Phi(t_1, t_2, t_3)$ is continuous in the closed bounded domain D_3, then it is uniformly continuous in this domain. Therefore, if the distance between two points of the domain D_3 is less than $\delta(\varepsilon)$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on mentioned points), then the corresponding oscillation of the function $\Phi(t_1, t_2, t_3)$ for these two points of the domain D_3 is less than ε.

If we assume that $\Delta\tau_j < \delta(\varepsilon)$ $(j = 0, 1, \ldots, N-1)$, then the distance between points (t_1, t_2, τ_{j_3}), $(t_1, \tau_{j_2}, \tau_{j_3})$ is obviously less than $\delta(\varepsilon)$. In this case

$$|\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})| < \varepsilon.$$

Consequently, when $\Delta\tau_j < \delta(\varepsilon)$ $(j = 0,\ 1, \ldots, N-1)$ the expression (1.22) is estimated by the following value

$$\varepsilon^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1} \Delta\tau_{j_2} \Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.$$

Therefore, the first limit on the right-hand side of (1.21) equals to zero. Similarly, we can prove that the second limit on the right-hand side of (1.21) equals to zero.

Consequently, the equality (1.19) is proved for $k = 3$. The cases $k = 2$ and $k > 3$ are analyzed absolutely similarly.

It is necessary to note that the proof of correctness of (1.19) is similar when the nonrandom function $\Phi(t_1, \ldots, t_k)$ is continuous in the open domain D_k and bounded at its boundary.

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_k=0 \\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J'[\Phi]_{T,t}^{(k)}, \tag{1.23}$$

where $\Phi(t_1, \ldots, t_k):\ [t, T]^k \to \mathbf{R}^1$ is the same function as in (1.16).

According to (1.19), we get the following equality

$$J'[\Phi]_{T,t}^{(k)} = \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \Bigl(\Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \Bigr) \quad \text{w. p. 1}, \tag{1.24}$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1, \ldots, t_k)$. At the same time permutations $(t_1, \ldots, t_k)$ when summing are performed in (1.24) only in the expression, which is enclosed in parentheses. Moreover, the nonrandom function $\Phi(t_1, \ldots, t_k)$ is assumed to be continuous in the corresponding closed domains of integration. The case when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domains of integration and bounded at their boundaries is also possible.

It is not difficult to see that (1.24) can be rewritten in the form

$$J'[\Phi]_{T,t}^{(k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},\tag{1.25}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Lemma 1.2. *Suppose that $\Phi(t_1,\ldots,t_k)\in C(D_k)$ or $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function in the open domain D_k and bounded at its boundary. Then*

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^2\right\}\le C_k\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k,\quad C_k<\infty,$$

where $I[\Phi]_{T,t}^{(k)}$ is defined by the formula (1.18).

Proof. Using standard properties and estimates of stochastic integrals for $\xi_\tau\in \mathrm{M}_2([t,T])$, we have [101]

$$\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau df_\tau\right|^2\right\}=\int\limits_t^T\mathsf{M}\{|\xi_\tau|^2\}d\tau,\tag{1.26}$$

$$\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau d\tau\right|^2\right\}\le (T-t)\int\limits_t^T\mathsf{M}\{|\xi_\tau|^2\}d\tau.\tag{1.27}$$

Let us denote

$$\xi[\Phi]_{t_{l+1},\ldots,t_k,t}^{(l)}=\int\limits_t^{t_{l+1}}\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_l}^{(i_l)},$$

where $l=1,\ldots,\ k-1$ and $\xi[\Phi]_{t_1,\ldots,t_k,t}^{(0)}\stackrel{\rm def}{=}\Phi(t_1,\ldots,t_k)$.

By induction it is easy to demonstrate that

$$\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t} \in {\rm M}_2([t,T])$$

with respect to the variable t_{l+1}. Further, using the estimates (1.26), (1.27) repeatedly we obtain the statement of Lemma 1.2.

It is not difficult to see that in the case $i_1,\ldots,i_k=1,\ldots,m$ from the proof of Lemma 1.2 we obtain

$$\mathsf{M}\left\{\left|I[\Phi]^{(k)}_{T,t}\right|^2\right\}=\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{1.28}$$

Lemma 1.3. *Suppose that every $\varphi_l(s)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$. Then*

$$\prod_{l=1}^k J[\varphi_l]_{T,t}=J[\Phi]^{(k)}_{T,t}\quad \hbox{w. p. 1}, \tag{1.29}$$

where

$$J[\varphi_l]_{T,t}=\int\limits_t^T\varphi_l(s)d{\bf w}_s^{(i_l)},\qquad \Phi(t_1,\ldots,t_k)=\prod_{l=1}^k\varphi_l(t_l),$$

and the integral $J[\Phi]^{(k)}_{T,t}$ is defined by the equality (1.16).

Proof. Let at first $i_l\ne 0,\ l=1,\ldots,k$. Let us denote

$$J[\varphi_l]_N\stackrel{\rm def}{=}\sum_{j=0}^{N-1}\varphi_l(\tau_j)\Delta{\bf w}_{\tau_j}^{(i_l)}.$$

Since

$$\prod_{l=1}^k J[\varphi_l]_N-\prod_{l=1}^k J[\varphi_l]_{T,t}=$$

$$=\sum_{l=1}^k\left(\prod_{g=1}^{l-1}J[\varphi_g]_{T,t}\right)\left(J[\varphi_l]_N-J[\varphi_l]_{T,t}\right)\left(\prod_{g=l+1}^{k}J[\varphi_g]_N\right), \tag{1.30}$$

then due to the Minkowski inequality and the inequality of Cauchy–Bunyakovsky we obtain

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^k J[\varphi_l]_N-\prod_{l=1}^k J[\varphi_l]_{T,t}\right|^2\right\}\right)^{1/2}\le$$

$$\le C_k \sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|J[\varphi_l]_N - J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{1.31}$$

where C_k is a constant.

Note that

$$J[\varphi_l]_N - J[\varphi_l]_{T,t} = \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j},$$

$$J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} = \int\limits_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(\tau_j) - \varphi_l(s)\right) d\mathbf{w}_s^{(i_l)}.$$

Since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ are independent for various j, then [102]

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \tag{1.32}$$

Moreover, since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ is a Gaussian random variable, we have

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(\tau_j) - \varphi_l(s)\right)^2 ds,$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(\tau_j) - \varphi_l(s)\right)^2 ds\right)^2.$$

Using these relations and continuity (which means uniform continuity) of the functions $\varphi_l(s)$, we get

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4\left(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\right) <$$

$$< 3\varepsilon^4\left(\delta(\varepsilon)(T-t) + (T-t)^2\right),$$

where $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$). Then the right-hand side of the formula (1.32) tends to zero when $N \to \infty$. Considering this fact as well as (1.31), we obtain (1.29).

If $\mathbf{w}_{t_l}^{(i_l)} = t_l$ for some $l \in \{1, \ldots, k\}$, then the proof of Lemma 1.3 becomes obviously simpler and it is performed similarly. Lemma 1.3 is proved.

Remark 1.2. *It is easy to see that if $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (1.29) for some $l \in \{1, \ldots, k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ ($p = 2,\ i_l \ne 0$), then the differential $d\mathbf{w}_{t_l}^{(i_l)}$ in the integral $J[\Phi^{(k)}]_{T,t}$ will be replaced with dt_l. If $p = 3, 4, \ldots,$ then the right-hand side of the formula (1.29) will become zero w. p. 1.*

Let us consider the case $p = 2$ in detail. Let $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (1.29) for some $l \in \{1, \ldots, k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^2$ ($i_l \ne 0$) and

$$
J[\varphi_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2, \quad J[\varphi_l]_{T,t} \stackrel{\text{def}}{=} \int_t^T \varphi_l(s)ds.
$$

We have

$$
\left(\mathsf{M}\left\{\left|J[\varphi_l]_N - J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4} =
$$

$$
= \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \int_t^T \varphi_l(s)ds\right|^4\right\}\right)^{1/4} =
$$

$$
= \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \int_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(s) - \varphi_l(\tau_j) + \varphi_l(\tau_j)\right)ds\right)\right|^4\right\}\right)^{1/4} \le
$$

$$
\le \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \Delta\tau_j\right)\right|^4\right\}\right)^{1/4} +
$$

$$
+ \left|\sum_{j=0}^{N-1} \int_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(\tau_j) - \varphi_l(s)\right)ds\right|. \qquad (1.33)
$$

From the relation, which is similar to (1.32), we obtain

$$
\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)\right|^4\right\}=
$$

$$
=\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^4\right\}+
$$

$$
+6\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^2\right\}\times
$$

$$
\times\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_q}^{(i_l)}\right)^2-\Delta\tau_q\right)^2\right\}=60\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\left(\Delta\tau_j\right)^4+
$$

$$
+24\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\left(\Delta\tau_j\right)^2\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\left(\Delta\tau_q\right)^2\le C\left(\Delta_N\right)^2\ \to 0 \tag{1.34}
$$

if $N\to\infty$, where constant C does not depend on N.

The second term on the right-hand side of (1.33) tends to zero if $N\to\infty$ due to continuity (which means uniform continuity) of the function $\varphi_l(s)$ at the interval $[t,T]$. Then, taking into account (1.30), (1.31), (1.33), (1.34), we come to the affirmation of Remark 1.2.

Let us prove Theorem 1.1. According to Lemma 1.1, we have

$$
J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}= \tag{1.35}
$$

$$
= \int\limits_{t}^{T} \ldots \int\limits_{t}^{t_2} \sum_{(t_1,\ldots,t_k)} \left(K(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \right) \quad \text{w. p. 1}, \tag{1.36}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression enclosed in parentheses.

It is easy to see that (1.36) can be rewritten in the form

$$
J[\psi^{(k)}]_{T,t} = \sum_{(t_1,\ldots,t_k)} \int\limits_{t}^{T} \ldots \int\limits_{t}^{t_2} K(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{1.37}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Since integration of a bounded function with respect to the set with measure zero for Riemann or Lebesgue integrals gives zero result, then the following formula is correct for these integrals

$$
\int\limits_{[t,T]^k} |G(t_1,\ldots,t_k)| dt_1 \ldots dt_k = \sum_{(t_1,\ldots,t_k)} \int\limits_{t}^{T} \ldots \int\limits_{t}^{t_2} |G(t_1,\ldots,t_k)| dt_1 \ldots dt_k, \tag{1.38}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $dt_1 \ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals are changed correspondently and $|G(t_1,\ldots,t_k)|$ is the integrable function on the hypercube $[t,T]^k$.

According to Lemmas 1.1, 1.3 and (1.24), (1.25), (1.36), (1.37), we get the following representation

$$
J[\psi^{(k)}]_{T,t} =
$$

$$
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \int\limits_{t}^{T} \ldots \int\limits_{t}^{t_2} \sum_{(t_1,\ldots,t_k)} \left(\phi_{j_1}(t_1) \ldots \phi_{j_k}(t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \right) +
$$

$$
+ R_{T,t}^{p_1,\ldots,p_k} =
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}= \tag{1.39}
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.
$$

$$
\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times
$$

$$
\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}\quad\text{w. p. 1,} \tag{1.40}
$$

where

$$
R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)\times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.41}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series. According to Lemma 1.2 and (1.38), we have

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=
$$

$$
=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k\to 0 \tag{1.42}
$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}$. Theorem 1.1 is proved.

Note that from (1.39) and (1.42) it follows that

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1}, \tag{1.43}
$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.23).

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=0,1,\ldots,m$ from Theorem 1.1 we obtain

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}. \tag{1.44}
$$

1.1.4 Expansions of Iterated Itô Stochastic Integrals with Multiplicities 1 to 7 Based on Theorem 1.1

In order to evaluate the significance of Theorem 1.1 for practice we will demonstrate its transformed particular cases (see Remark 1.2) for $k=1,\ldots,7$ [1]-[63]

$$
J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{1.45}
$$

$$
J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \qquad (1.46)
$$

$$
J[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (1.47)
$$

$$
J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (1.48)
$$

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{1.49}
$$

$$
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$\left.-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\right), \qquad (1.50)$$

$$J[\psi^{(7)}]_{T,t} = \underset{p_1,\ldots,p_7\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_7=0}^{p_7} C_{j_7\ldots j_1}\Biggl(\prod_{l=1}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_6\ne 0, j_1=j_6\}}\prod_{\substack{l=1\\ l\ne 1,6}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_2=i_6\ne 0, j_2=j_6\}}\prod_{\substack{l=1\\ l\ne 2,6}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_3=i_6\ne 0, j_3=j_6\}}\prod_{\substack{l=1\\ l\ne 3,6}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_4=i_6\ne 0, j_4=j_6\}}\prod_{\substack{l=1\\ l\ne 4,6}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_5=i_6\ne 0, j_5=j_6\}}\prod_{\substack{l=1\\ l\ne 5,6}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2\}}\prod_{\substack{l=1\\ l\ne 1,2}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3\}}\prod_{\substack{l=1\\ l\ne 1,3}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_4\ne 0, j_1=j_4\}}\prod_{\substack{l=1\\ l\ne 1,4}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_5\ne 0, j_1=j_5\}}\prod_{\substack{l=1\\ l\ne 1,5}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0, j_2=j_3\}}\prod_{\substack{l=1\\ l\ne 2,3}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_2=i_4\ne 0, j_2=j_4\}}\prod_{\substack{l=1\\ l\ne 2,4}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_2=i_5\ne 0, j_2=j_5\}}\prod_{\substack{l=1\\ l\ne 2,5}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_3=i_4\ne 0, j_3=j_4\}}\prod_{\substack{l=1\\ l\ne 3,4}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_3=i_5\ne 0, j_3=j_5\}}\prod_{\substack{l=1\\ l\ne 3,5}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_4=i_5\ne 0, j_4=j_5\}}\prod_{\substack{l=1\\ l\ne 4,5}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_7=i_1\ne 0, j_7=j_1\}}\prod_{\substack{l=1\\ l\ne 1,7}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_7=i_2\ne 0, j_7=j_2\}}\prod_{\substack{l=1\\ l\ne 2,7}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_7=i_3\ne 0, j_7=j_3\}}\prod_{\substack{l=1\\ l\ne 3,7}}^{7}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_7=i_4\ne 0, j_7=j_4\}}\prod_{\substack{l=1\\ l\ne 4,7}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_7=i_5\ne 0, j_7=j_5\}}\prod_{\substack{l=1\\ l\ne 7,5}}^{7}\zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_7=i_6\ne 0, j_7=j_6\}}\prod_{\substack{l=1\\ l\ne 7,6}}^{7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_3=i_4\ne 0, j_3=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)} + \mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_3=i_5\ne 0, j_3=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_4=i_5\ne 0, j_4=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)} + \mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_2=i_4\ne 0, j_2=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_2=i_5\ne 0, j_2=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)} + \mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_4=i_5\ne 0, j_4=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0, j_1=j_4, i_2=i_3\ne 0, j_2=j_3\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)} + \mathbf{1}_{\{i_1=i_4\ne 0, j_1=j_4, i_2=i_5\ne 0, j_2=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0, j_1=j_4, i_3=i_5\ne 0, j_3=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)} + \mathbf{1}_{\{i_1=i_5\ne 0, j_1=j_5, i_2=i_3\ne 0, j_2=j_3\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)} +$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_7=j_1,i_7=i_1\ne 0,j_5=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=4,2,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}-$$

$$
-\Biggl(\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_1}^{(i_1)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_6\ne 0,j_1=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_2}^{(i_2)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_5\ne 0,j_2=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_6\ne 0,j_2=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_3}^{(i_3)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5\}}\Biggr)\zeta_{j_4}^{(i_4)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_5}^{(i_5)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_6}^{(i_6)}-
$$

$$-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_5=i_6\ne 0,j_5=j_6\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_4=i_5\ne 0,j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6\}}+$$
$$+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5\}}+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6\}}+$$
$$+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_7}^{(i_7)}\Biggr), \tag{1.51}$$

where $\mathbf{1}_A$ is the indicator of the set A.

1.1.5 Expansion of Iterated Itô Stochastic Integrals of Multiplicity k ($k\in\mathbf{N}$) Based on Theorem 1.1

Consider a generalization of the formulas (1.45)–(1.51) for the case of arbitrary multiplicity k for $J[\psi^{(k)}]_{T,t}$. In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{1.52}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (1.52) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{1.53}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbf{R}$.

Below there are several examples of sums in the form (1.53)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2,g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$

$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+a_{12,54,3}+a_{15,24,3}+$$

$$+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can formulate Theorem 1.1 (see (1.10)) using alternative form.

Theorem 1.2[3] [4] (2009) (also see [5]-[17], [24], [29], [39], [48], [49]). *Under the conditions of Theorem* 1.1 *the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(1.54)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x, $\prod_{\emptyset}\stackrel{\rm def}{=}1$, $\sum_{\emptyset}\stackrel{\rm def}{=}0$; *another notations are the same as in Theorem* 1.1.

Proof. The equality (1.54) will be proved by induction in Sect. 1.14 (see the proof of Theorem 1.23).

[3]The connection of formulas (1.45)–(1.51), (1.54) with Hermite polynomials is studied in Sect. 1.10, 1.11 (see Theorems 1.14–1.17).

In particular, from (1.54) for $k = 5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_5=0}^{p_5} C_{j_5\ldots j_1} \Biggl(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\}) \\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \prod_{l=1}^{3} \zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\}) \\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}} \mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}} \zeta_{j_{q_1}}^{(i_{q_1})} \Biggr).$$

The last equality obviously agrees with (1.49).

It is now appropriate to make a remark about the structure of the formulas (1.45)–(1.51) and (1.54). Using (1.39), (1.43), (1.45)–(1.51), (1.54), we obtain

$$J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr) \quad (1.55)$$

w. p. 1, where the multiple stochastic integral $J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.23); another notations in (1.55) are the same as in Theorem 1.2.

The stochastic integral with respect to the scalar standard Wiener process ($i_1 = \ldots = i_k \ne 0$) and similar to (1.23) was considered in [106] (1951) and is called the multiple Wiener stochastic integral [106]. Note that $\Phi(t_1, \ldots, t_k) \in L_2([t, T]^k)$ in [106] (this case will be considered in Sect. 1.11–1.14).

As we will see in Sect. 1.10, 1.11, 1.14, the expression on the right-hand side of (1.55) is the Wick polynomial with arguments $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$. Moreover, the given expression is an explicit representation of the Wick polynomial, in contrast to its representation in the form of a product of Hermite polynomials (see Sect. 1.10, 1.11, 1.14) or its another representation (or definition) using a recurrence relation (see (1.391)).

To best of our knowledge, the representation of the multiple Wiener stochastic integral in the form of a Wick polynomial (see (1.55)) for the case of a multidimensional Wiener process ($i_1,\ldots,i_k = 0,1,\ldots,m$) and the case $j_1,\ldots,j_k = 0,1,2,\ldots$ was first obtained in our monographs [1] (2006), [3] (2007), and [4] (2009). More precisely, the formula (1.55) is obtained in our monograph [4] (2009) as part of the formula (5.30) (see [4], p. 220). Moreover, partiular cases $k = 1,\ldots,5$ (see (1.45)–(1.49)) of the formula (1.55) were obtained in [1] (2006) as parts of the formulas on the pages 243-244 and partiular cases $k = 1,\ldots,7$ (see (1.45)–(1.51)) of the formula (1.55) were obtained in [3] (2007) as parts of the formulas on the pages 208-218.

The indicated formulas are obtained for the case when $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous nonrandom functions on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of piecewise continuous functions in the space $L_2([t,T])$ (see Sect. 1.1.7 and [1] (2006), [3] (2007), and [4] (2009)). Note that the generality of the above results is even too great when applied to the numerical integration of Itô stochastic differential equations.

It should be noted that in [110] (1987) an L_2–version of the formula (1.55) was obtained, but only for the special case $j_1 = \ldots = j_k$. The above result in [110] (Proposition 5.1) is obtained using diagrams, i.e. (unlike our results) in an implicit form (see Sect. 1.14 (below Remark 1.18) for details).

Let us turn to the comparison of the formula (1.55) with another interesting work [113] (2019). An L_2-version of (1.55) was obtained in [113] in terms of Wick polynomials and for the case of vector valued random measures (see [113], Theorem 7.2, p. 69). In earlier works of this author (see for example [112]) only the case of scalar valued random measures was considered (see Sect. 1.14 (below Remark 1.18) for details).

In Sect. 1.14 (Theorems 1.22, 1.23) we consider L_2–versions of the formula (1.55). At that, to prove Theorems 1.22 and 1.23 we use only the Itô formula, in contrast to the diagram method from [113].

1.1.6 Comparison of Theorem 1.2 with the Representations of Iterated Itô Stochastic Integrals Based on Hermite Polynomials

Note that the correctness of the formulas (1.45)–(1.51) can be verified in the following way. If $i_1 = \ldots = i_7 = i = 1,\ldots,m$ and $\psi_1(s),\ldots,\psi_7(s) \equiv \psi(s)$, then we can derive from (1.45)–(1.51) [2]-[17], [29] the well known equalities

$$J[\psi^{(1)}]_{T,t} = \frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t} = \frac{1}{2!}\left(\delta_{T,t}^2 - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t} = \frac{1}{3!}\left(\delta_{T,t}^3 - 3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t} = \frac{1}{4!}\left(\delta_{T,t}^4 - 6\delta_{T,t}^2\Delta_{T,t} + 3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t} = \frac{1}{5!}\left(\delta_{T,t}^5 - 10\delta_{T,t}^3\Delta_{T,t} + 15\delta_{T,t}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t} = \frac{1}{6!}\left(\delta_{T,t}^6 - 15\delta_{T,t}^4\Delta_{T,t} + 45\delta_{T,t}^2\Delta_{T,t}^2 - 15\Delta_{T,t}^3\right),$$

$$J[\psi^{(7)}]_{T,t} = \frac{1}{7!}\left(\delta_{T,t}^7 - 21\delta_{T,t}^5\Delta_{T,t} + 105\delta_{T,t}^3\Delta_{T,t}^2 - 105\delta_{T,t}\Delta_{T,t}^3\right)$$

w. p. 1, where

$$\delta_{T,t} = \int\limits_t^T \psi(s)d\mathbf{f}_s^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(s)ds,$$

which can be independently obtained using the Itô formula and Hermite polynomials [108].

When $k = 1$ everything is evident. Let us consider the cases $k = 2$ and $k = 3$ in detail. When $k = 2$ and $p_1 = p_2 = p$ we have (see (1.46)) [2]-[17], [29]

$$J[\psi^{(2)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)} - \sum_{j_1=0}^{p} C_{j_1j_1}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\Bigl(C_{j_2j_1} + C_{j_1j_2}\Bigr)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)} + \sum_{j_1=0}^{p} C_{j_1j_1}\left(\left(\zeta_{j_1}^{(i)}\right)^2 - 1\right)\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1} C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)} + \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2 - 1\right)\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\sum_{\substack{j_1,j_2=0\\ j_1\ne j_2}}^{p} C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)} + \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2 - 1\right)\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^2 - \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2\right)$$

$$= \frac{1}{2!}\left(\delta_{T,t}^2 - \Delta_{T,t}\right) \quad \text{w. p. 1.} \tag{1.56}$$

Let us explain the last step in (1.56). For the Itô stochastic integral the following estimate [103] is valid

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^q\right\} \le K_q \mathsf{M}\left\{\left(\int\limits_t^T |\xi_\tau|^2 d\tau\right)^{q/2}\right\}, \tag{1.57}$$

where $q > 0$ is a fixed number, f_τ is a scalar standard Wiener process, $\xi_\tau \in \mathrm{M}_2([t,T])$, K_q is a constant depending only on q,

$$\int\limits_t^T |\xi_\tau|^2 d\tau < \infty \quad \text{w. p. 1},$$

$$\mathsf{M}\left\{\left(\int\limits_t^T |\xi_\tau|^2 d\tau\right)^{q/2}\right\} < \infty.$$

Since

$$\delta_{T,t} - \sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)} = \int\limits_t^T \left(\psi(s) - \sum_{j_1=0}^{p} C_{j_1}\phi_{j_1}(s)\right) d\mathbf{f}_s^{(i)},$$

then applying the estimate (1.57) to the right-hand side of this expression and considering that

$$\int\limits_t^T \left(\psi(s) - \sum_{j_1=0}^{p} C_{j_1}\phi_{j_1}(s)\right)^2 ds \ \to 0$$

if $p \to \infty$, we obtain

$$\int\limits_t^T \psi(s) d\mathbf{f}_s^{(i)} = q\ \text{-}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}, \quad q > 0. \tag{1.58}$$

Here $q\text{ - }\underset{p\to\infty}{\text{l.i.m.}}$ is a limit in the mean of degree q. Hence, if $q=4$, then it is easy to conclude that w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^2=\delta_{T,t}^2.$$

This equality as well as Parseval's equality were used in the last step of the formula (1.56).

When $k=3$ and $p_1=p_2=p_3=p$ we obtain (see (1.47)) [2]-[17], [29]

$$J[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\right.$$

$$\left.-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i)}-\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1}\zeta_{j_1}^{(i)}-\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}\zeta_{j_2}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\sum_{j_1,j_3=0}^{p}\Bigl(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\Bigr)\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\Bigl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Bigr)\times\right.$$

$$\times\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Bigl(C_{j_3j_1j_3}+C_{j_1j_3j_3}+C_{j_3j_3j_1}\Bigr)\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Bigl(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\Bigr)\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\sum_{j_1=0}^{p}C_{j_1j_1j_1}\left(\zeta_{j_1}^{(i)}\right)^3-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p} C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p} C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{\substack{j_1,j_2,j_3=0\\ j_1\ne j_2,j_2\ne j_3,j_1\ne j_3}}^{p} C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.
$$

$$
+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+
$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p} C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p} C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\right.
$$

$$
-\frac{1}{6}\left(3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\right.
$$

$$
\left.+\sum_{j_1=0}^{p} C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right)+
$$

$$
+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+
$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p} C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p} C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2\sum_{j_3=0}^{p} C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right)\quad \text{w. p. 1.} \tag{1.59}
$$

The last step in (1.59) follows from Parseval's equality, Theorem 1.1 for $k=1$, and the equality

$$
\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^3=\delta_{T,t}^3\quad \text{w. p. 1,}
$$

which can be obtained easily when $q = 8$ (see (1.58)).

In addition, we used the following relations between Fourier coefficients for the considered case

$$C_{j_1j_2} + C_{j_2j_1} = C_{j_1}C_{j_2}, \quad 2C_{j_1j_1} = C_{j_1}^2, \tag{1.60}$$

$$C_{j_1j_2j_3} + C_{j_1j_3j_2} + C_{j_2j_3j_1} + C_{j_2j_1j_3} + C_{j_3j_2j_1} + C_{j_3j_1j_2} = C_{j_1}C_{j_2}C_{j_3}, \tag{1.61}$$

$$2\left(C_{j_1j_1j_3} + C_{j_1j_3j_1} + C_{j_3j_1j_1}\right) = C_{j_1}^2 C_{j_3}, \tag{1.62}$$

$$6C_{j_1j_1j_1} = C_{j_1}^3. \tag{1.63}$$

1.1.7 On Usage of Discontinuous Complete Orthonormal Systems of Functions in Theorem 1.1

Analyzing the proof of Theorem 1.1, we can ask the question: can we weaken the continuity condition for the functions $\phi_j(x)$, $j = 1, 2, \ldots$?[4]

We will say that the function $f(x) : [t, T] \to \mathbf{R}$ satisfies the condition $(\star)$, if it is continuous at the interval $[t, T]$ except may be for the finite number of points of the finite discontinuity as well as it is right-continuous at the interval $[t, T]$.

Furthermore, let us suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$, each function $\phi_j(x)$ of which for $j < \infty$ satisfies the condition $(\star)$.

It is easy to see that continuity of the functions $\phi_j(x)$ was used substantially for the proof of Theorem 1.1 in two places. More precisely, we mean Lemma 1.3 and the formula (1.19). It is clear that without the loss of generality the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t, T]$ in Lemma 1.3 and (1.19) can be taken so "dense" that among the points τ_j of this partition there will be all points of jumps of the functions $\varphi_1(\tau) = \phi_{j_1}(\tau), \ldots, \varphi_k(\tau) = \phi_{j_k}(\tau)$ $(j_1, \ldots, j_k < \infty)$ and among the points $(\tau_{j_1}, \ldots, \tau_{j_k})$ for which $0 \le j_1 < \ldots < j_k \le N - 1$ there will be all points of jumps of the function $\Phi(t_1, \ldots, t_k)$.

Let us demonstrate how to modify the proofs of Lemma 1.3 and the formula (1.19) in the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$, each function $\phi_j(x)$ of which for $j < \infty$ satisfies the condition $(\star)$.

[4] The results of this section will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ in Sect. 1.11 (see Theorem 1.16).

At first, appeal to Lemma 1.3. From the proof of this lemma it follows that

$$
\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +
$$

$$
+6\sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}, \tag{1.64}
$$

$$
\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds,
$$

$$
\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds\right)^2.
$$

Suppose that the functions $\varphi_l(s)$ $(l = 1,\ldots,k)$ satisfy the condition $(\star)$ and the partition $\{\tau_j\}_{j=0}^N$ includes all points of jumps of the functions $\varphi_l(s)$ $(l = 1,\ldots,k)$. It means that for the integral

$$
\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds
$$

the integrand function is continuous at the interval $[\tau_j, \tau_{j+1}]$, except possibly the point τ_{j+1} of finite discontinuity.

Let $\mu \in (0, \Delta\tau_j)$ be fixed. Due to continuity (which means uniform continuity) of the functions $\varphi_l(s)$ $(l = 1,\ldots,k)$ at the interval $[\tau_j, \tau_{j+1} - \mu]$ we have

$$
\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds =
$$

$$
= \int\limits_{\tau_j}^{\tau_{j+1}-\mu} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds + \int\limits_{\tau_{j+1}-\mu}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds <
$$

$$
< \varepsilon^2(\Delta\tau_j - \mu) + M^2\mu. \tag{1.65}
$$

When obtaining the inequality (1.65) we supposed that $\Delta\tau_j < \delta(\varepsilon)$ for all $j = 0, 1, \ldots, N-1$ (here $\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on s),

$$|\varphi_l(\tau_j) - \varphi_l(s)| < \varepsilon$$

for $s \in [\tau_j, \tau_{j+1} - \mu]$ (due to uniform continuity of the functions $\varphi_l(s)$, $l = 1, \ldots, k$),

$$|\varphi_l(\tau_j) - \varphi_l(s)| < M$$

for $s \in [\tau_{j+1} - \mu, \tau_{j+1}]$, M is a constant (potential discontinuity point of the function $\varphi_l(s)$ is the point τ_{j+1}).

Performing the passage to the limit in the inequality (1.65) when $\mu \to +0$, we get

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds \le \varepsilon^2 \Delta\tau_j. \tag{1.66}$$

Using (1.66) to estimate the right-hand side of (1.64), we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4 \left(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\right) <$$

$$< 3\varepsilon^4 \left(\delta(\varepsilon)(T-t) + (T-t)^2\right). \tag{1.67}$$

This implies that

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \to 0$$

when $N \to \infty$ and Lemma 1.3 remains correct.

Now, let us present explanations concerning the correctness of (1.19), when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$, each function $\phi_j(x)$ of which for $j < \infty$ satisfies the condition $(\star)$.

Consider the case $k = 3$ and the representation (1.21). Let us demonstrate that in the studied case the first limit on the right-hand side of (1.21) equals to zero (similarly, we can demonstrate that the second limit on the right-hand side of (1.21) equals to zero; proof of the second limit equality to zero on the right-hand side of the formula (1.20) is the same as for the case of continuous functions $\phi_j(x),\ j=0,1,\ldots$).

The second moment of the prelimit expression of first limit on the right-hand side of (1.21) looks as follows

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2\Delta\tau_{j_3}.$$

Further, for the fixed $\mu\in(0,\Delta\tau_{j_2})$ and $\rho\in(0,\Delta\tau_{j_1})$ we have

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2=$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\right)\left(\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2=$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\times$$

$$\times\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2<$$

$$<\varepsilon^2\left(\Delta\tau_{j_2}-\mu\right)\left(\Delta\tau_{j_1}-\rho\right)+M^2\rho\left(\Delta\tau_{j_2}-\mu\right)+$$

$$+M^2\mu\left(\Delta\tau_{j_1}-\rho\right)+M^2\mu\rho, \tag{1.68}$$

where M is a constant, $\Delta\tau_j<\delta(\varepsilon)$ for $j=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$). We suppose here that the partition $\{\tau_j\}_{j=0}^N$ contains all discontinuity points of the function $\Phi(t_1,t_2,t_3)$ as points τ_j (for each variable with fixed remaining two variables). When obtaining the inequality (1.68) we also supposed that potential discontinuity points of this function (for each variable with fixed remaining two variables) are contained among the points $\tau_{j_1+1},\tau_{j_2+1},\tau_{j_3+1}$.

Let us explain in detail how we obtained the inequality (1.68). Since the function $\Phi(t_1, t_2, t_3)$ is continuous at the closed bounded set

$$Q_3 = \left\{(t_1, t_2, t_3) : t_1 \in [\tau_{j_1}, \tau_{j_1+1} - \rho], t_2 \in [\tau_{j_2}, \tau_{j_2+1} - \mu], t_3 \in [\tau_{j_3}, \tau_{j_3+1} - \nu]\right\},$$

where ρ, μ, ν are fixed small positive numbers such that

$$\nu \in (0, \Delta\tau_{j_3}), \quad \mu \in (0, \Delta\tau_{j_2}), \quad \rho \in (0, \Delta\tau_{j_1}),$$

then this function is also uniformly continous at this set. Moreover, the function $\Phi(t_1, t_2, t_3)$ is supposed to be bounded at the closed set D_3 (see the proof of Theorem 1.1).

Since the distance between points (t_1, t_2, τ_{j_3}), $(t_1, \tau_{j_2}, \tau_{j_3}) \in Q_3$ is obviously less than $\delta(\varepsilon)$ ($\Delta\tau_j < \delta(\varepsilon)$ for $j = 0, 1, \ldots, N-1$), then

$$|\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})| < \varepsilon.$$

This inequality was used to estimate the first double integral in (1.68). Estimating the three remaining double integrals in (1.68) we used the boundedness property for the function $\Phi(t_1, t_2, t_3)$ in the form of inequality

$$|\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})| < M.$$

Performing the passage to the limit in the inequality (1.68) when $\mu, \rho \to +0$, we obtain the estimate

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right)^2 dt_1 dt_2 \le \varepsilon^2 \Delta\tau_{j_2}\Delta\tau_{j_1}.$$

This estimate provides

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3} \le$$

$$\le \varepsilon^2 \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.$$

The last inequality means that in the considered case the first limit on the right-hand side of (1.21) equals to zero (similarly, we can demonstrate that the second limit on the right-hand side of (1.21) equals to zero).

Consequently, the formula (1.19) is correct when $k = 3$ in the studied case. Similarly, we can perform the argumentation for the cases $k = 2$ and $k > 3$.

Therefore, in Theorem 1.1 we can use complete orthonormal systems of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j < \infty$ satisfies the condition $(\star)$.

One of the examples of such systems of functions is a complete orthonormal system of Haar functions in the space $L_2([t,T])$

$$\phi_0(x) = \frac{1}{\sqrt{T-t}}, \quad \phi_{nj}(x) = \frac{1}{\sqrt{T-t}}\,\varphi_{nj}\left(\frac{x-t}{T-t}\right),$$

where $n = 0, 1, \ldots,\ \ j = 1, 2, \ldots, 2^n$, and the functions $\varphi_{nj}(x)$ are defined as

$$\varphi_{nj}(x) = \begin{cases} 2^{n/2}, & x \in [(j-1)/2^n,\ (j-1)/2^n + 1/2^{n+1}) \\ \\ -2^{n/2}, & x \in [(j-1)/2^n + 1/2^{n+1},\ j/2^n) \\ \\ 0, & \text{otherwise} \end{cases},$$

$n = 0, 1, \ldots,\ j = 1, 2, \ldots, 2^n$ (we choose the values of Haar functions in the points of discontinuity in such a way that these functions will be right-continuous).

The other example of similar system of functions is a complete orthonormal system of Rademacher–Walsh functions in the space $L_2([t,T])$

$$\phi_0(x) = \frac{1}{\sqrt{T-t}},$$

$$\phi_{m_1\ldots m_k}(x) = \frac{1}{\sqrt{T-t}}\,\varphi_{m_1}\left(\frac{x-t}{T-t}\right)\ldots\varphi_{m_k}\left(\frac{x-t}{T-t}\right),$$

where $0 < m_1 < \ldots < m_k,\ \ m_1, \ldots, m_k = 1, 2, \ldots,\ \ k = 1, 2, \ldots,$

$$\varphi_m(x) = (-1)^{[2^m x]},$$

$x \in [0,1],\ \ m = 1, 2, \ldots,\ \ [y]$ is an integer part of a real number y.

1.1.8 Remark on Usage of Complete Orthonormal Systems of Functions in Theorem 1.1

Note that actually the functions $\phi_j(s)$ from the complete orthonormal system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ depend not only on s, but on t and T.

For example, the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ have the following form

$$\phi_j(s,t,T)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}\right),$$

$$P_j(y)=\frac{1}{2^j j!}\frac{d^j}{dy^j}\left(y^2-1\right)^j,$$

where $P_j(y)$ $(j=0,1,2,\ldots)$ is the Legendre polynomial,

$$\phi_j(s,t,T)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(s-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(s-t)/(T-t)\right), & j=2r\end{cases} \qquad (1.69)$$

where $r=1,2,\ldots$

Note that the specified systems of functions are assumed to be used in the context of implementation of numerical methods for Itô SDEs (see Chapter 4) for the sequences of time intervals

$$[T_0,T_1],\ [T_1,T_2],\ [T_2,T_3],\ \ldots$$

and Hilbert spaces

$$L_2([T_0,T_1]),\ L_2([T_1,T_2]),\ L_2([T_2,T_3]),\ \ldots$$

We can explain that the dependence of functions $\phi_j(s,t,T)$ on t and T (hereinafter these constants will mean fixed moments of time) will not affect on the main properties of independence of random variables

$$\zeta_{(j)T,t}^{(i)}=\int\limits_t^T\phi_j(s,t,T)d\mathbf{w}_s^{(i)},$$

where $i = 1, \ldots, m$ and $j = 0, 1, 2, \ldots$

Indeed, for fixed t and T due to orthonormality of the mentioned systems of functions we have

$$\mathsf{M}\left\{\zeta_{(j)T,t}^{(i)}\zeta_{(g)T,t}^{(r)}\right\} = \mathbf{1}_{\{i=r\}}\mathbf{1}_{\{j=g\}},$$

where $i, r = 1, \ldots, m,\ \ j, g = 0, 1, 2, \ldots$

This means that $\zeta_{(j)T,t}^{(i)}$ and $\zeta_{(g)T,t}^{(r)}$ are independent for $j \neq g$ or $i \neq r$ (since these random variables are Gaussian).

From the other side, the random variables

$$\zeta_{(j)T_1,t_1}^{(i)} = \int\limits_{t_1}^{T_1} \phi_j(s, t_1, T_1) d\mathbf{w}_s^{(i)}, \qquad \zeta_{(j)T_2,t_2}^{(i)} = \int\limits_{t_2}^{T_2} \phi_j(s, t_2, T_2) d\mathbf{w}_s^{(i)}$$

are independent if $[t_1, T_1] \cap [t_2, T_2] = \emptyset$ (the case $T_1 = t_2$ is possible) according to the properties of the Itô stochastic integral.

Therefore, the important properties of random variables $\zeta_{(j)T,t}^{(i)}$, which are the basic motive of their usage, are saved.

1.1.9 Convergence in the Mean of Degree $2n$ ($n \in \mathbf{N}$) of Expansions of Iterated Itô Stochastic Integrals from Theorem 1.1

Constructing the expansions of iterated Itô stochastic integrals from Theorem 1.1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge not only in the mean-square sense but in the stronger probabilistic senses.

We will obtain the general estimate which proves convergence in the mean of degree $2n$ ($n \in \mathbf{N}$) of expansions from Theorem 1.1.

According to the notations of Theorem 1.1 (see (1.41)), we have

$$R_{T,t}^{p_1,\ldots,p_k} = J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1, \ldots, t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (1.70)$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) \stackrel{\rm def}{=} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),$$

$J[\psi^{(k)}]_{T,t}$ is the stochastic integral (1.5), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (1.10) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}$.

Note that for definiteness we consider in this section the case $i_1,\ldots,i_k = 1,\ldots,m.$ Another notations from this section are the same as in the formulation and proof of Theorem 1.1.

When proving Theorem 1.1 we obtained the following estimate (see (1.42))

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k,$$

where C_k is a constant.

Assume that

$$\eta_{t_l,t}^{(l-1)} \stackrel{\rm def}{=} \int\limits_t^{t_l}\ldots\int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_{l-1}}^{(i_{l-1})}, \quad l=2,\ 3,\ldots,k+1,$$

$$\eta_{T,t}^{(k)} \stackrel{\rm def}{=} \int\limits_t^{T}\ldots\int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}, \quad \eta_{t_{k+1},t}^{(k)} \stackrel{\rm def}{=} \eta_{T,t}^{(k)}.$$

Using the Itô formula it is easy to demonstrate that [101]

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\} = n(2n-1)\int\limits_{t_0}^{t}\mathsf{M}\left\{\left(\int\limits_{t_0}^{s}\xi_u df_u\right)^{2n-2}\xi_s^2\right\}ds.$$

Using the Hölder inequality (under the integral sign on the right-hand side of the last equality) for $p=n/(n-1),\ q=n\ (n>1)$ and using the increasing of the value

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}$$

with the growth of t, we get

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\le n(2n-1)\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\right)^{(n-1)/n}\times$$

$$\times\int\limits_{t_0}^{t}\left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds.$$

After raising to power n the obtained inequality and dividing the result by

$$\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\right)^{n-1},$$

we get the following estimate

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\le (n(2n-1))^n\left(\int\limits_{t_0}^{t}\left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds\right)^n. \qquad (1.71)$$

Using the estimate (1.71) repeatedly, we have

$$\mathsf{M}\left\{\left(\eta_{T,t}^{(k)}\right)^{2n}\right\}\le (n(2n-1))^n\left(\int\limits_{t}^{T}\left(\mathsf{M}\left\{\left(\eta_{t_k,t}^{(k-1)}\right)^{2n}\right\}\right)^{1/n}dt_k\right)^n\le$$

$$\le (n(2n-1))^n\times$$

$$\times\left(\int\limits_{t}^{T}\left((n(2n-1))^n\left(\int\limits_{t}^{t_k}\left(\mathsf{M}\left\{\left(\eta_{t_{k-1},t}^{(k-2)}\right)^{2n}\right\}\right)^{1/n}dt_{k-1}\right)^n\right)^{1/n}dt_k\right)^n=$$

$$=(n(2n-1))^{2n}\left(\int\limits_{t}^{T}\int\limits_{t}^{t_k}\left(\mathsf{M}\left\{\left(\eta_{t_{k-1},t}^{(k-2)}\right)^{2n}\right\}\right)^{1/n}dt_{k-1}dt_k\right)^n\le\ldots$$

$$\ldots\le (n(2n-1))^{n(k-1)}\left(\int\limits_{t}^{T}\int\limits_{t}^{t_k}\ldots\int\limits_{t}^{t_3}\left(\mathsf{M}\left\{\left(\eta_{t_2,t}^{(1)}\right)^{2n}\right\}\right)^{1/n}dt_3\ldots dt_{k-1}dt_k\right)^n=$$

$$
= (n(2n-1))^{n(k-1)}(2n-1)!! \left(\int\limits_t^T \ldots \int\limits_t^{t_2} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n \le
$$

$$
\le (n(2n-1))^{n(k-1)}(2n-1)!!\times
$$

$$
\times \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n .
$$

The penultimate step was obtained using the formula

$$
\mathsf{M}\left\{ \left(\eta^{(1)}_{t_2,t}\right)^{2n} \right\} = (2n-1)!! \left(\int\limits_t^{t_2} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1 \right)^n ,
$$

which follows from Gaussianity of

$$
\eta^{(1)}_{t_2,t} = \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}^{(i_1)}_{t_1}.
$$

Similarly, we estimate each summand on the right-hand side of (1.70). Then, from (1.70) using the Minkowski inequality, we finally get

$$
\mathsf{M}\left\{ \left(R^{p_1,\ldots,p_k}_{T,t}\right)^{2n} \right\} \le
$$

$$
\le \left(k! \left((n(2n-1))^{n(k-1)}(2n-1)!! \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n \right)^{1/2n} \right)^{2n}
$$

$$
= (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times
$$

$$
\times \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n . \qquad (1.72)
$$

Using the orthonormality of the functions $\phi_j(s)$ $(j=0,1,2,\ldots)$, we obtain

$$\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k =$$

$$=\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k =$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-2\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k+$$

$$+\int\limits_{[t,T]^k}\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k =$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-2\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k+$$

$$+\sum_{j_1=0}^{p_1}\sum_{j_1'=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\sum_{j_k'=0}^{p_k}C_{j_k\ldots j_1}C_{j_k'\ldots j_1'}\prod_{l=1}^{k}\int\limits_t^T\phi_{j_l}(t_l)\phi_{j_l'}(t_l)dt_l =$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-2\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}. \qquad (1.73)$$

Let us substitute (1.73) into (1.72)

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times$$

$$\times\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n. \qquad (1.74)$$

Due to Parseval's equality

$$\int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\ \to\ 0 \qquad (1.75)$$

if $p_1,\ldots,p_k\to\infty$. Therefore, the inequality (1.72) (or (1.74)) means that the expansions of iterated Itô stochastic integrals obtained using Theorem 1.1 converge in the mean of degree $2n$ ($n\in\mathbf{N}$) to the appropriate iterated Itô stochastic integrals.

1.1.10 Conclusions

Thus, we obtain the following useful possibilities and modifications of the approach based on Theorem 1.1.[5]

1. There is an explicit formula (see (1.8)) for calculation of expansion coefficients of the iterated Itô stochastic integral (1.5) with any fixed multiplicity k ($k\in\mathbf{N}$).

2. We have possibilities for exact calculation of the mean-square approximation error of the iterated Itô stochastic integral (1.5) [14]-[18], [31] (see Sect. 1.2).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only the trigonometric functions as in [82]-[85], [92], [93], [96], [97], but the Legendre polynomials.

[5]Theorem 1.1 will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ in Sect. 1.11 (see Theorem 1.16).

4. As it turned out [1]-[63], it is more convenient to work with Legendre polynomials for approximation of the iterated Itô stochastic integrals (1.5) (see Chapter 5). Approximations based on Legendre polynomials essentially simpler than their analogues based on trigonometric functions [1]-[63]. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [21], [40] (see Sect. 5.3).

5. The Milstein approach [82] (see Sect. 6.2 in this book) to expansion of iterated stochastic integrals based on the Karhunen–Loève expansion of the Brownian bridge process (also see [83]-[85], [92], [93], [96], [97]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1.1) starting from the second or third multiplicity of the iterated Itô stochastic integral (1.5). Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [83]-[85], [93] the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein approach [82] together with the Wong–Zakai approximation [73]-[75] (see discussions in Sect. 2.41, 2.42, 6.2).

6. As we mentioned above, constructing the expansions of iterated Itô stochastic integrals from Theorem 1.1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge with probability 1. The convergence with probability 1 in Theorem 1.1 has been proved for some particular cases in [3]-[17], [32] (see Sect. 1.7.1) and for the general case of iterated Itô stochastic integrals of multiplicity k ($k \in \mathbf{N}$) in [14]-[17], [27], [29], [31], [32] (see Sect. 1.7.2).

7. The generalizations of Theorem 1.1 for an arbitrary complete orthonormal system of functions in $L_2([t,T]^k)$ [29] and complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in $L_2([t,T]^k)$ [12]-[17], [41] as well as for iterated stochastic integrals with respect to martingale Poisson measures and iterated stochastic integrals with respect to martingales [1]-[17], [41] are presented in Sect. 1.3–1.6, 1.11.

8. The adaptation of Theorem 1.1 for iterated Stratonovich stochastic integrals was carried out in [6]-[23], [28], [30], [32]-[39], [43], [45]-[47], [50], [52],

[64], [65] (see Chapter 2).

9. Application of Theorem 1.1 for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in [14]-[17], [24], [25], [48], [49] (see Chapter 7).

1.2 Exact Calculation of the Mean-Square Error in the Method of Approximation of Iterated Itô Stochastic integrals Based on Generalized Multiple Fourier Series

This section is devoted to the obtainment of exact and approximate expressions for the mean-square approximation error in Theorem 1.1 for iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$). As a result, we do not need to use redundant terms of expansions of iterated Itô stochastic integrals.

1.2.1 Introduction

Recall that we called the method of expansion and mean-square approximation of iterated Itô stochastic integrals based on Theorem 1.1 as the method of generalized multiple Fourier series. The question about how estimate or even calculate exactly the mean-square approximation error of iterated Itô stochastic integrals for the method of generalized multiple Fourier series composes the subject of Sect. 1.2. From the one side the mentioned question is essentially difficult in the case of a multidimensional Wiener process, because of we need to take into account all possible combinations of the components of a multidimensional Wiener process. From the other side an effective solution of the mentioned problem allows to construct more simple expansions of iterated Itô stochastic integrals than in [82]-[87], [92]-[94], [96], [97].

Sect. 1.2.2 is devoted to the formulation and proof of Theorem 1.3, which allows to calculate exacly the mean-square approximation error of iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) for the method of generalized multiple Fourier series. The particular cases ($k = 1, \ldots, 5$) of Theorem 1.3 are considered in detail in Sect. 1.2.3. In Sect. 1.2.4 we prove an effective estimate for the mean-square approximation error of iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) for the method of generalized multiple Fourier series.

1.2.2 Theorem on Exact Calculation of the Mean-Square Approximation Error for Iterated Itô Stochastic integrals

Theorem 1.3[6] [12]-[18], [31]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-
$$

$$
-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{1.76}
$$

where

$$
J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},
$$

$$
J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right), \tag{1.77}
$$

$$
S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)}, \tag{1.78}
$$

the Fourier coefficient $C_{j_k\ldots j_1}$ has the form (1.8),

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)} \tag{1.79}
$$

are independent standard Gaussian random variables for various i or j $(i=1,\ldots,m)$,

$$
\sum_{(j_1,\ldots,j_k)}
$$

[6]Theorem 1.3 will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ in Sect. 1.12 (see Theorem 1.18).

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$ (*see* (1.76)); *another notations are the same as in Theorem* 1.1.

Remark 1.3. *Note that*

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=$$

$$=\mathsf{M}\left\{\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k=C_{j_k\ldots j_1}. \qquad (1.80)$$

Therefore, in the case of pairwise different numbers $i_1,\ldots,i_k$ *from Theorem* 1.3 *we obtain*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^p\ldots\sum_{j_k=0}^p C_{j_k\ldots j_1}^2. \qquad (1.81)$$

Moreover, if $i_1=\ldots=i_k$, *then from Theorem* 1.3 *we get*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^p\ldots\sum_{j_k=0}^p C_{j_k\ldots j_1}\Biggl(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\Biggr),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

For example, for the case $k=3$ *we have*

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^{p}\right)^2\right\}=\int\limits_t^T\psi_3^2(t_3)\int\limits_t^{t_3}\psi_2^2(t_2)\int\limits_t^{t_2}\psi_1^2(t_1)dt_1dt_2dt_3-$$

$$-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Bigl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_3j_1}+C_{j_2j_1j_3}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Bigr).$$

Proof. Using Theorem 1.1 for the case $i_1,\ldots,i_k=1,\ldots,m$ and $p_1=\ldots=p_k=p$, we obtain

$$J[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right). \tag{1.82}$$

For $n>p$ we can write

$$J[\psi^{(k)}]_{T,t}^{n}=\left(\sum_{j_1=0}^{p}+\sum_{j_1=p+1}^{n}\right)\ldots\left(\sum_{j_k=0}^{p}+\sum_{j_k=p+1}^{n}\right)C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)=$$

$$=J[\psi^{(k)}]_{T,t}^{p}+\xi[\psi^{(k)}]_{T,t}^{p+1,n}. \tag{1.83}$$

Let us prove that due to the special structure of random variables $S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}$ (see (1.45)–(1.51), (1.54), (1.78)) the following relations are correct

$$\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right\}=0, \tag{1.84}$$

$$\mathsf{M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j_l'}^{(i_l)}-S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)}\right)\right\}=0, \tag{1.85}$$

where

$$(j_1,\ldots,j_k)\in\mathrm{K}_p,\quad (j_1',\ldots,j_k')\in\mathrm{K}_n\backslash\mathrm{K}_p$$

and

$$\mathrm{K}_n=\left\{(j_1,\ldots,j_k):\ 0\le j_1,\ldots,j_k\le n\right\},$$

$$\mathrm{K}_p = \left\{(j_1, \ldots, j_k) : \ 0 \le j_1, \ldots, j_k \le p\right\}.$$

For the case $i_1, \ldots, i_k = 1, \ldots, m$ and $p_1 = \ldots = p_k = p$ from (1.39), (1.40) (see the proof of Theorem 1.1) we obtain

$$\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)} =$$

$$= \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \qquad (1.86)$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1, \ldots, j_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$; another notations are the same as in Theorem 1.1.

So, we obtain (1.84) from (1.86) due to the moment property of the Itô stochastic integral.

Let us prove (1.85). From (1.86) we have

$$0 \le \left|\mathsf{M}\left\{\left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k} \zeta_{j_l'}^{(i_l)} - S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)}\right)\right\}\right| =$$

$$= \left|\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.\right.$$

$$\left.\left.\times \int\limits_t^T \phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2} \phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}\right| \le$$

$$\le \sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k}(t_k)\phi_{j_k'}(t_k)dt_k\ldots\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1 =$$

$$
= \sum_{(j_1',\ldots,j_k')} \mathbf{1}_{\{j_1=j_1'\}} \ldots \mathbf{1}_{\{j_k=j_k'\}}, \qquad (1.87)
$$

where $\mathbf{1}_A$ is the indicator of the set A. From (1.87) we obtain (1.85).

First, let us prove (1.87) for the cases $k=2$ and $k=3$. We have

$$
\mathsf{M}\left\{\sum_{(j_1,j_2)}\sum_{(j_1',j_2')}\int\limits_t^T \phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\int\limits_t^T \phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\right\} =
$$

$$
= \int\limits_t^T \phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T \phi_{j_1}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \phi_{j_2}(s)\phi_{j_1'}(s)ds\int\limits_t^T \phi_{j_1}(s)\phi_{j_2'}(s)ds =
$$

$$
= \mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}} + \mathbf{1}_{\{i_1=i_2\}}\cdot\mathbf{1}_{\{j_2=j_1'\}}\mathbf{1}_{\{j_1=j_2'\}}, \qquad (1.88)
$$

$$
\mathsf{M}\left\{\sum_{(j_1,j_2,j_3)}\sum_{(j_1',j_2',j_3')}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\times\right.
$$

$$
\left.\times\int\limits_t^T \phi_{j_3'}(t_3)\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\right\} =
$$

$$
= \int\limits_t^T \phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T \phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T \phi_{j_1}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T \phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T \phi_{j_2}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T \phi_{j_1}(s)\phi_{j_1'}(s)ds\int\limits_t^T \phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T \phi_{j_3}(s)\phi_{j_2'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T \phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T \phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T \phi_{j_3}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_1'}(s)ds+
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds=
$$

$$
=\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}}+\mathbf{1}_{\{i_1=i_2\}}\cdot\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}+\mathbf{1}_{\{i_1=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}. \tag{1.89}
$$

From (1.88) and (1.89) we get

$$
\left|\mathsf{M}\left\{\sum_{(j_1,j_2)}\sum_{(j_1',j_2')}\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\times\right.\right.
$$

$$
\left.\left.\times\int\limits_t^T\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\right\}\right|\le
$$

$$
\le\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}+\mathbf{1}_{\{j_2=j_1'\}}\mathbf{1}_{\{j_1=j_2'\}}=
$$

$$
=\sum_{(j_1',j_2')}\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}},
$$

$$
\left|\mathsf{M}\left\{\sum_{(j_1,j_2,j_3)}\sum_{(j_1',j_2',j_3')}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\times\right.\right.
$$

$$
\left.\left.\times\int\limits_t^T\phi_{j_3'}(t_3)\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\right\}\right|\le
$$

$$
\le\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}}+\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+
$$

$$
+\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}+\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+
$$

$$
+\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}=
$$

$$
= \sum_{(j_1',j_2',j_3')} \mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_3'\}},
$$

where we used the relation

$$
\int\limits_t^T \phi_i(\tau)\phi_j(\tau)d\tau = \mathbf{1}_{\{i=j\}}, \quad i,j=0,1,2\ldots
$$

Now consider the case of an arbitrary $k\in\mathbf{N}$. We have

$$
\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.
$$

$$
\left.\times\int\limits_t^T \phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=
$$

$$
=\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.
$$

$$
\left.\times\int\limits_t^T \phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1')}\ldots d\mathbf{f}_{t_k}^{(i_k')}\right\}=
$$

$$
=\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\times
$$

$$
\times\int\limits_t^T \phi_{j_k}(t_k)\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1\ldots dt_k=
$$

$$
=\sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\int\limits_t^T \phi_{j_k}(t_k)\phi_{j_k'}(t_k)dt_k\ldots\int\limits_t^{T}\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1=
$$

$$
=\sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\mathbf{1}_{\{j_k=j_k'\}}\ldots\mathbf{1}_{\{j_1=j_1'\}}, \tag{1.90}
$$

where $(i_1',\ldots,i_k')=(i_1,\ldots,i_k)$. However, if j_r' swapped with j_q' in the permutation $(j_1',\ldots,j_k')$, then i_r' swapped with i_q' in the permutation $(i_1',\ldots,i_k')$ and if

j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

From (1.90) we obtain (1.87). The equality (1.85) is proved.

Note that the formula (1.85) (in the light of the results of Sect. 1.10, 1.11) can be interpreted as a consequence of the orthogonality of two random variables that are Hermite polynomials of vector random arguments.

From (1.85) we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{p}\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right\}=0.$$

Due to (1.77), (1.82), and (1.83) we can write

$$\xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}^{n}-J[\psi^{(k)}]_{T,t}^{p},$$

$$\underset{n\to\infty}{\text{l.i.m.}}\ \xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\stackrel{\text{def}}{=}\xi[\psi^{(k)}]_{T,t}^{p+1}.$$

We have

$$0\le\left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$=\left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1}-\xi[\psi^{(k)}]_{T,t}^{p+1,n}+\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$\le\left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1}-\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|+\left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1,n}J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$=\left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|\le$$

$$\le\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^{2}\right\}}\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}}\le$$

$$\le \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\times$$

$$\times\left(\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}-J[\psi^{(k)}]_{T,t}\right)^2\right\}}+\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}}\right)\le$$

$$\le K\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\to 0\quad \text{if}\quad n\to\infty, \tag{1.91}$$

where K is a constant.

From (1.91) it follows that

$$\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}=0$$

or

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}=0.$$

The last equality means that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}. \tag{1.92}$$

Taking into account (1.92), we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}-2\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}-$$

$$-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}. \tag{1.93}$$

Let us consider the value

$$
\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^p\right\}.
$$

The relations (1.77) and (1.86) imply that

$$
J[\psi^{(k)}]_{T,t}^p=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}. \tag{1.94}
$$

After substituting (1.94) into (1.93), we finally get

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-
$$

$$
-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}.
$$

Theorem 1.3 is proved.

1.2.3 Exact Calculation of the Mean-Square Approximation Errors for the Cases $k=1,\ldots,5$

Let us denote

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}\stackrel{\rm def}{=}E_k^p,
$$

$$
\|K\|_{L_2([t,T]^k)}^2=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\rm def}{=}I_k.
$$

The case $k=1$

In this case from Theorem 1.3 we obtain

$$
E_1^p=I_1-\sum_{j_1=0}^{p}C_{j_1}^2.
$$

The case $k = 2$

In this case from Theorem 1.3 we have

(I). $i_1 \neq i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2, \tag{1.95}$$

(II). $i_1 = i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}C_{j_1j_2}. \tag{1.96}$$

Note that from (1.77), (1.86), (1.88), (1.92), and (1.93) we obtain

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 - \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1,j_2=0}^{p} C_{j_2j_1}C_{j_1j_2}. \tag{1.97}$$

Obviously, the relation (1.97) is consistent with (1.95) and (1.96).

Example 1.1. Let us consider the following iterated Itô stochastic integral

$$I_{(00)T,t}^{(i_1i_2)} = \int_t^T \int_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)}, \tag{1.98}$$

where $i_1, i_2 = 1, \ldots, m$.

Approximation of the iterated Itô stochastic integral (1.98) based on the expansion (1.10) (Theorem 1.1, the case of Legendre polynomials) has the following form

$$I_{(00)T,t}^{(i_1i_2)p} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{p} \frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right). \tag{1.99}$$

Note that (1.99) has been derived for the first time in [76] (1997) (also see [77]-[79]) with using the another approach. This approach will be considered in Sect. 2.4. Later (1.99) was obtained [1] (2006), [2]-[63] on the base of Theorem 1.1.

Using (1.95), we get

$$
\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)p}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{p}\frac{1}{4i^2-1}\right), \qquad (1.100)
$$

where $i_1 \neq i_2$.

It should also be noted that the formula (1.100) has been obtained for the first time in [76] (1997) by direct calculation.

The case $k = 3$

In this case from Theorem 1.3 we obtain

(I). $i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3$:

$$
E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2, \qquad (1.101)
$$

(II). $i_1 = i_2 = i_3$:

$$
E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_3 j_2 j_1}\right), \qquad (1.102)
$$

(III).1. $i_1 = i_2 \neq i_3$:

$$
E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1}, \qquad (1.103)
$$

(III).2. $i_1 \neq i_2 = i_3$:

$$
E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1}, \qquad (1.104)
$$

(III).3. $i_1 = i_3 \neq i_2$:

$$
E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3}. \qquad (1.105)
$$

It is not difficult to see that from (1.77), (1.86), (1.89), (1.92), and (1.93) we obtain

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 -$$

$$-\mathbf{1}_{\{i_1=i_2\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_3j_1j_2} -$$

$$-\mathbf{1}_{\{i_2=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_2j_3j_1} -$$

$$-\mathbf{1}_{\{i_1=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_1j_2j_3} -$$

$$-\mathbf{1}_{\{i_1=i_2=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \left(C_{j_2j_1j_3} + C_{j_1j_3j_2}\right). \qquad (1.106)$$

Obviously, the relation (1.106) is consistent with (1.101)–(1.105).

Note that the cases $k = 2$ and $k = 3$ (excepting the formula (1.102)) were investigated for the first time in [2] (2007) using the direct calculation.

Example 1.2. Let us consider the following iterated Itô stochastic integral

$$I_{(000)T,t}^{(i_1i_2i_3)} = \int_t^T \int_t^{t_3} \int_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}, \qquad (1.107)$$

where $i_1, i_2, i_3 = 1, \ldots, m$.

Approximation of the iterated Itô stochastic integral (1.107) based on Theorem 1.1 (the case of Legendre polynomials and $p_1 = p_2 = p_3 = p$) has the following form [1] (2006), [2]-[63]

$$I_{(000)T,t}^{(i_1i_2i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \qquad (1.108)$$

where

$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar C_{j_3j_2j_1}, \qquad (1.109)$$

$$\bar C_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_i(x)$ is the Legendre polynomial $(i=0,1,2,\ldots)$.

For example, using (1.103) and (1.104), we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,J_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1},$$

where $i_1=i_2\ne i_3$,

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,J_3=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1},$$

where $i_1\ne i_2=i_3$.

The exact values of Fourier–Legendre coefficients $\bar C_{j_3j_2j_1}$ can be calculated for example using computer algebra system Derive [1]-[17], [32] (see Sect. 5.1, Tables 5.4–5.36). For more details on calculating of $\bar C_{j_3j_2j_1}$ using Python programming language see [53], [54].

For the case $i_1=i_2=i_3$ it is convenient to use the following well known formula

$$I_{(000)T,t}^{(i_1i_1i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right)\quad \text{w. p. 1.}$$

The case $k=4$

In this case from Theorem 1.3 we have

(I). $i_1,\ldots,i_4$ are pairwise different:

$$E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}^2,$$

(II). $i_1 = i_2 = i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,\ldots,j_4)} C_{j_4\ldots j_1} \Biggr),$$

(III).1. $i_1 = i_2 \ne i_3, i_4;\ \ i_3 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_2)} C_{j_4\ldots j_1} \Biggr), \tag{1.110}$$

(III).2. $i_1 = i_3 \ne i_2, i_4;\ \ i_2 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_3)} C_{j_4\ldots j_1} \Biggr), \tag{1.111}$$

(III).3. $i_1 = i_4 \ne i_2, i_3;\ \ i_2 \ne i_3:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_4)} C_{j_4\ldots j_1} \Biggr), \tag{1.112}$$

(III).4. $i_2 = i_3 \ne i_1, i_4;\ \ i_1 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \Biggr), \tag{1.113}$$

(III).5. $i_2 = i_4 \ne i_1, i_3;\ \ i_1 \ne i_3:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_2,j_4)} C_{j_4\ldots j_1} \Biggr), \tag{1.114}$$

(III).6. $i_3 = i_4 \ne i_1, i_2;\ \ i_1 \ne i_2:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \Biggr), \tag{1.115}$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1} \Biggr), \qquad (1.116)$$

(IV).2. $i_2 = i_3 = i_4 \neq i_1$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1} \Biggr), \qquad (1.117)$$

(IV).3. $i_1 = i_2 = i_4 \neq i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1} \Biggr), \qquad (1.118)$$

(IV).4. $i_1 = i_3 = i_4 \neq i_2$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1} \Biggr), \qquad (1.119)$$

(V).1. $i_1 = i_2 \neq i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_2)} \Biggl(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \Biggr) \Biggr), \qquad (1.120)$$

(V).2. $i_1 = i_3 \neq i_2 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_3)} \Biggl(\sum_{(j_2,j_4)} C_{j_4\ldots j_1} \Biggr) \Biggr), \qquad (1.121)$$

(V).3. $i_1 = i_4 \neq i_2 = i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \Biggl(\sum_{(j_1,j_4)} \Biggl(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \Biggr) \Biggr). \qquad (1.122)$$

The case $k = 5$

In this case from Theorem 1.3 we obtain

(I). $i_1, \ldots, i_5$ are pairwise different:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}^2,$$

(II). $i_1 = i_2 = i_3 = i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,\ldots,j_5)} C_{j_5\ldots j_1} \right),$$

(III).1. $i_1 = i_2 \neq i_3, i_4, i_5$ (i_3, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} C_{j_5\ldots j_1} \right),$$

(III).2. $i_1 = i_3 \neq i_2, i_4, i_5$ (i_2, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1} \right),$$

(III).3. $i_1 = i_4 \neq i_2, i_3, i_5$ (i_2, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \right),$$

(III).4. $i_1 = i_5 \neq i_2, i_3, i_4$ (i_2, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right),$$

(III).5. $i_2 = i_3 \neq i_1, i_4, i_5$ (i_1, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right),$$

(III).6. $i_2 = i_4 \ne i_1, i_3, i_5$ (i_1, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \right),$$

(III).7. $i_2 = i_5 \ne i_1, i_3, i_4$ (i_1, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} C_{j_5\ldots j_1} \right),$$

(III).8. $i_3 = i_4 \ne i_1, i_2, i_5$ (i_1, i_2, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(III).9. $i_3 = i_5 \ne i_1, i_2, i_4$ (i_1, i_2, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right),$$

(III).10. $i_4 = i_5 \ne i_1, i_2, i_3$ (i_1, i_2, i_3 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(IV).1. $i_1 = i_2 = i_3 \ne i_4, i_5$ ($i_4 \ne i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1} \right),$$

(IV).2. $i_1 = i_2 = i_4 \ne i_3, i_5$ ($i_3 \ne i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1} \right),$$

(IV).3. $i_1 = i_2 = i_5 \neq i_3, i_4 \ (i_3 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).4. $i_2 = i_3 = i_4 \neq i_1, i_5 \ (i_1 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1}\Biggr),$$

(IV).5. $i_2 = i_3 = i_5 \neq i_1, i_4 \ (i_1 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).6. $i_2 = i_4 = i_5 \neq i_1, i_3 \ (i_1 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).7. $i_3 = i_4 = i_5 \neq i_1, i_2 \ (i_1 \neq i_2)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).8. $i_1 = i_3 = i_5 \neq i_2, i_4 \ (i_2 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).9. $i_1 = i_3 = i_4 \neq i_2, i_5 \ (i_2 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1}\Biggr),$$

(IV).10. $i_1 = i_4 = i_5 \neq i_2, i_3$ $(i_2 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1}\right),$$

(V).1. $i_1 = i_2 = i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_3,j_4)} C_{j_5\ldots j_1}\right),$$

(V).2. $i_1 = i_2 = i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_3,j_5)} C_{j_5\ldots j_1}\right),$$

(V).3. $i_1 = i_2 = i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_4,j_5)} C_{j_5\ldots j_1}\right),$$

(V).4. $i_1 = i_3 = i_4 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1}\right),$$

(V).5. $i_2 = i_3 = i_4 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3,j_4,j_5)} C_{j_5\ldots j_1}\right),$$

(VI).1. $i_5 \neq i_1 = i_2 \neq i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).2. $i_5 \ne i_1 = i_3 \ne i_2 = i_4 \ne i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).3. $i_5 \ne i_1 = i_4 \ne i_2 = i_3 \ne i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).4. $i_4 \ne i_1 = i_2 \ne i_3 = i_5 \ne i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).5. $i_4 \ne i_1 = i_5 \ne i_2 = i_3 \ne i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).6. $i_4 \ne i_2 = i_5 \ne i_1 = i_3 \ne i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).7. $i_3 \ne i_2 = i_5 \ne i_1 = i_4 \ne i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).8. $i_3 \ne i_1 = i_2 \ne i_4 = i_5 \ne i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).9. $i_3 \ne i_2 = i_4 \ne i_1 = i_5 \ne i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).10. $i_2 \ne i_1 = i_4 \ne i_3 = i_5 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).11. $i_2 \ne i_1 = i_3 \ne i_4 = i_5 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).12. $i_2 \ne i_1 = i_5 \ne i_3 = i_4 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VI).13. $i_1 \ne i_2 = i_3 \ne i_4 = i_5 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3)} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).14. $i_1 \ne i_2 = i_4 \ne i_3 = i_5 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).15. $i_1 \ne i_2 = i_5 \ne i_3 = i_4 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VII).1. $i_1 = i_2 = i_3 \neq i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VII).2. $i_1 = i_2 = i_4 \neq i_3 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_5)}\left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).3. $i_1 = i_2 = i_5 \neq i_3 = i_4$:

$$E_p = I - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).4. $i_2 = i_3 = i_4 \neq i_1 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).5. $i_2 = i_3 = i_5 \neq i_1 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).6. $i_2 = i_4 = i_5 \neq i_1 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).7. $i_3 = i_4 = i_5 \neq i_1 = i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).8. $i_1 = i_3 = i_5 \neq i_2 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_4)}\Biggl(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\Biggr)\Biggr),$$

(VII).9. $i_1 = i_3 = i_4 \neq i_2 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_5)}\Biggl(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1}\Biggr)\Biggr),$$

(VII).10. $i_1 = i_4 = i_5 \neq i_2 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_3)}\Biggl(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1}\Biggr)\Biggr).$$

Let us make a remark about Theorem 1.3. It is easy to see that the right-hand side of the formula (1.76) consists of two parts. The first part tends to zero when $p \to \infty$ by Parseval's equality. At the same time the second part also tends to zero when $p \to \infty$, but due to the generalized Parseval equality. Let us explain the above reasoning in more detail for the case $k = 3$.

For the case $k = 3$ we have (see (1.106))

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 -$$

$$-\mathbf{1}_{\{i_1=i_2\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_3j_1j_2} -$$

$$-\mathbf{1}_{\{i_2=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_2j_3j_1} -$$

$$-\mathbf{1}_{\{i_1=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} C_{j_1j_2j_3} -$$

$$-\mathbf{1}_{\{i_1=i_2=i_3\}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \left(C_{j_2j_1j_3} + C_{j_1j_3j_2}\right). \qquad (1.123)$$

Applying the Parseval equality, we obtain

$$\lim_{p\to\infty}\left(I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2\right)=0. \tag{1.124}$$

The generalized Parseval equality gives

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_1j_2j_3}=0,\quad \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_3j_1j_2}=0, \tag{1.125}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_1j_3j_2}=0,\quad \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_2j_1j_3}=0, \tag{1.126}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_2j_3j_1}=0. \tag{1.127}$$

Let us explain in more detail the first equality in (1.125). Using the generalized Parseval equality, we have

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_2j_3}C_{j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_3}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T\psi_1(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\psi_3(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$
= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \psi_3(t_1)\psi_2(t_2)\psi_1(t_3) \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 \times
$$

$$
\times \int\limits_{[t,T]^3} \mathbf{1}_{\{t_1<t_2<t_3\}} \psi_1(t_1)\psi_2(t_2)\psi_3(t_3) \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 =
$$

$$
= \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \mathbf{1}_{\{t_1<t_2<t_3\}} \psi_3(t_1)\psi_1(t_1) \left(\psi_2(t_2)\right)^2 \psi_3(t_3)\psi_1(t_3) dt_1 dt_2 dt_3 = 0.
\tag{1.128}
$$

Applying (1.123), (1.124), and (1.125)–(1.127), we get (see (1.76) for the case $k=3$)

$$
\lim_{p\to\infty} E_3^p = 0.
$$

1.2.4 Estimate for the Mean-Square Approximation Error of Iterated Itô Stochastic Integrals Based on Theorem 1.1

In this section, we prove the useful estimate for the mean-square approximation error in Theorem 1.1.

Theorem 1.4 [12]-[17], [31]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Then the estimate*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le
$$

$$
\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right)
\tag{1.129}
$$

is valid for the following cases:

1. $i_1,\ldots,i_k = 1,\ldots,m$ *and* $0 < T-t < \infty$,
2. $i_1,\ldots,i_k = 0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2 > 0,$ *and* $0 < T-t < 1$,

where $J[\psi^{(k)}]_{T,t}$ is the iterated Itô stochastic integral (1.5)*, $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of* (1.10) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 1.1.

Proof. In the proof of Theorem 1.1 we obtained w. p. 1 the following representation (see (1.40))

$$J[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}+R_{T,t}^{p_1,\ldots,p_k},$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (1.10) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}$ and

$$R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.130}$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$, which are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

The stochastic integrals on the right-hand side of (1.130) will be dependent in a stochastic sense ($i_1,\ldots,i_k=1,\ldots,m,\ k\in\mathbf{N}$). Let us estimate the second moment of

$$J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}.$$

Using (1.26), (1.38), (1.130), the orthonormality of the system $\{\phi_j(x)\}_{j=0}^{\infty}$ (see the relation (1.73)), and the elementary inequality

$$\left(a_1+a_2+\ldots+a_p\right)^2\le p\left(a_1^2+a_2^2+\ldots+a_p^2\right),\quad p\in\mathbf{N}, \tag{1.131}$$

we obtain the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$
= k! \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= k! \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \right), \qquad (1.132)
$$

where $T-t \in (0,\infty)$ and $i_1,\ldots,i_k = 1,\ldots,m.$

From (1.26), (1.27), (1.38), (1.130), (1.131), and the orthonormality of the system $\{\phi_j(x)\}_{j=0}^{\infty}$ we obtain

$$
\mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} \right)^2 \right\} \le
$$

$$
\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= C_k \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \right),
$$

where $i_1,\ldots,i_k = 0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0$, and C_k is a constant.

It is not difficult to see that the constant C_k depends on k (k is the multiplicity of the iterated Itô stochastic integral) and $T-t$ ($T-t$ is the length of integration interval of the iterated Itô stochastic integral). Moreover, C_k has the following form

$$
C_k = k! \cdot \max\Bigl\{ (T-t)^{\alpha_1},\ (T-t)^{\alpha_2},\ \ldots,\ (T-t)^{\alpha_{k!}} \Bigr\},
$$

where $\alpha_1,\alpha_2,\ldots,\alpha_{k!} = 0,\ 1,\ldots,\ k-1.$

However, $T-t$ is an integration step of numerical procedures for Itô SDEs (see Chapter 4), which is a rather small value. For example, $0 < T-t < 1.$ Then $C_k \le k!$

It means that for the case $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ and $0<T-t<1$ we get (1.129). Theorem 1.4 is proved.

Example 1.3. The particular case of the estimate (1.129) for the iterated Itô stochastic integral $I_{(000)T,t}^{(i_1i_2i_3)}$ (see (1.107)) has the following form

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2\right),
$$

where $i_1,i_2,i_3=1,\ldots,m$ and $C_{j_3j_2j_1}$ is defined by the formula (1.109).

Let us consider the case of pairwise different $i_1,\ldots,i_k=1,\ldots,m$ and prove the following equality

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}=
$$

$$
=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2, \tag{1.133}
$$

where notations are the same as in Theorem 1.4.

The stochastic integrals on the right-hand side of (1.130) are uncorrelated for the case of pairwise different $i_1,\ldots,i_k=1,\ldots,m$. Moreover, these integrals have zero expectations. Then

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}=
$$

$$
={\sf M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times\right.\right.
$$

$$
\left.\left.\times d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right)^2\right\}=
$$

$$
=\sum_{(t_1,\ldots,t_k)}{\sf M}\left\{\left(\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times\right.\right.
$$

$$
\left.\left.\times d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right)^2\right\}=
$$

$$
= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}.
$$

1.3 Expansion of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Complete Orthonormal with Weight $r(t_1)\ldots r(t_k)$ Systems of Functions in the Space $L_2([t,T]^k)$

In this section, we consider a modification of Theorem 1.1 for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbf{N}$.[7]

Let $\{\Psi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$. It is well known that the Fourier series of the function $f(x)$ $\left(f(x)\sqrt{r(x)} \in L_2([t,T])\right)$ with respect to the system $\{\Psi_j(x)\}_{j=0}^{\infty}$ converges to the function $f(x)$ in the mean-square sense with weight $r(x)$, i.e.

$$
\lim_{p\to\infty} \int\limits_t^T \left(f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \right)^2 r(x) dx = 0, \tag{1.134}
$$

where

$$
\tilde{C}_j = \int\limits_t^T f(x)\Psi_j(x) r(x) dx \tag{1.135}
$$

is the Fourier coefficient.

The relations (1.134), (1.135) can be obtained if we will expand the function $f(x)\sqrt{r(x)} \in L_2([t,T])$ into a usual Fourier series with respect to the complete

[7] The results of this section are generalized to the case of an arbitrary complete orthonormal with weight $r(x) \ge 0$ system of functions $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(x)\sqrt{r(x)}, \ldots, \psi_k(x)\sqrt{r(x)} \in L_2([t,T])$ in Sect. 1.13 (see Theorems 1.20, 1.21).

orthonormal with weight 1 system of functions

$$\left\{\Psi_j(x)\sqrt{r(x)}\right\}_{j=0}^{\infty}$$

in the space $L_2([t,T])$. Then

$$\lim_{p\to\infty}\int\limits_t^T\left(f(x)\sqrt{r(x)}-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\sqrt{r(x)}\right)^2 dx =$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(f(x)-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\right)^2 r(x)dx = 0, \qquad (1.136)$$

where $\tilde C_j$ is defined by (1.135).

Let us consider an obvious generalization of this approach to the case of k variables. Let us expand the function $K(t_1,\ldots,t_k)$ such that

$$K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}\in L_2([t,T]^k)$$

using the complete orthonormal system of functions

$$\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)},\quad j_l=0,1,2,\ldots,\quad l=1,\ldots,k$$

in the space $L_2([t,T]^k)$ into the generalized multiple Fourier series.

It is well known that the mentioned generalized multiple Fourier series converges in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)^2\times$$

$$\times dt_1\ldots dt_k =$$

$$=\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\prod_{l=1}^{k}r(t_l)\times$$

$$\times dt_1\ldots dt_k = 0, \qquad (1.137)$$

where

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Biggl(\Psi_{j_l}(t_l)r(t_l)\Biggr)dt_1\ldots dt_k.$$

Let us consider the following iterated Itô stochastic integrals

$$\tilde{J}[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (1.138)$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$.

So, we obtain the following version of Theorem 1.1.

Theorem 1.5 [13]-[17], [29], [41]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then*

$$\tilde{J}[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \quad (1.139)$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition *of* $[t,T]$, *which satisfies the condition* (1.9),

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k \qquad (1.140)$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Proof. According to Lemmas 1.1, 1.3 and (1.24), (1.25), (1.36), (1.37), we get the following representation

$$\tilde{J}[\psi^{(k)}]_{T,t}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)\prod_{l=1}^k\sqrt{r(t_l)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\prod_{l=1}^k\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times$$

$$\times\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\times
$$

$$
\times\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.
$$

$$
\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+\tilde R_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\times
$$

$$
\times\left(\prod_{l=1}^{k}\tilde\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+\tilde R_{T,t}^{p_1,\ldots,p_k}\quad \text{w. p. 1},
$$

where

$$
\tilde R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}-\right.
$$

$$
\left.-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $\tilde{R}_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 1.2 and (1.38), we have

$$\mathsf{M}\left\{\left(\tilde{R}_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\prod_{l=1}^k \sqrt{r(t_l)} -\right.$$

$$\left. - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^k \left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)^2 dt_1\ldots dt_k = \tag{1.141}$$

$$= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^k \Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \left(\prod_{l=1}^k r(t_l)\right) dt_1\ldots dt_k \to 0 \tag{1.142}$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral (1.138). Theorem 1.5 is proved.

Let us formulate the version of Theorem 1.4.

Theorem 1.6 [14]-[17], [29], [41]. *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x) \ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then the estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t} - \tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^k r(t_l)\right)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}^2_{j_k\ldots j_1}\right) \tag{1.143}$$

is valid for the following cases:

1. $i_1,\ldots,i_k = 1,\ldots,m$ *and* $0 < T-t < \infty$,
2. $i_1,\ldots,i_k = 0,1,\ldots,m$, $i_1^2+\ldots+i_k^2 > 0$, *and* $0 < T-t < 1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1.138), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.139) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ *; another notations are the same as in Theorem* 1.5*.*

1.4 Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures Based on Generalized Multiple Fourier Series

In this section, we consider the version of Theorem 1.1 connected with the expansion of iterated stochastic integrals with respect to martingale Poisson measures.

1.4.1 Stochastic Integral with Respect to Martingale Poisson Measure

Let us consider the Poisson random measure on the set $[0, T] \times \mathbf{Y}$ ($\mathbf{R}^n \overset{\text{def}}{=} \mathbf{Y}$). We will denote the value of this measure at the set $\Delta \times A$ ($\Delta \subseteq [0, T]$, $A \subset \mathbf{Y}$) as $\nu(\Delta, A)$. Assume that

$$\mathsf{M}\left\{\nu(\Delta, A)\right\} = |\Delta|\Pi(A),$$

where $|\Delta|$ is the Lebesgue measure of Δ, $\Pi(A)$ is a measure on σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbf{Y}$, and $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ consisting of sets $A \subset \mathcal{B}$ that satisfy the condition $\Pi(A) < \infty$.

Let us consider the martingale Poisson measure

$$\tilde{\nu}(\Delta, A) = \nu(\Delta, A) - |\Delta|\Pi(A).$$

Let $(\Omega, \text{F}, \mathsf{P})$ be a fixed probability space, let $\{\text{F}_t,\ t \in [0, T]\}$ be a non-decreasing family of σ-algebras $\text{F}_t \subset \text{F}$.

Assume that the following conditions are fulfilled:

1. The random variables $\nu([0, t), A)$ are F_t-measurable for all $A \subseteq \mathcal{B}_0$, $t \in [0, T]$.

2. The random variables $\nu([t, t + h), A)$, $A \subseteq \mathcal{B}_0$, $h > 0$ do not depend on events of σ-algebra F_t.

Let us define the class $H_l(\Pi, [0, T])$ of random functions $\varphi : [0, T] \times \mathbf{Y} \times \Omega \to \mathbf{R}^1$ that are F_t-measurable for all $t \in [0, T]$, $\mathbf{y} \in \mathbf{Y}$ and satisfy the following condition

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\Bigl\{|\varphi(t,\mathbf{y})|^l\Bigr\}\Pi(d\mathbf{y})dt < \infty.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$, which satisfies the condition (1.9), and define the stochastic integral with respect to the martingale Poisson measure for $\varphi(t,\mathbf{y}) \in H_2(\Pi,[0,T])$ as the following mean-square limit [100]

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi^{(N)}(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}), \qquad (1.144)$$

where $\varphi^{(N)}(t,\mathbf{y})$ is any sequence of step functions from the class $H_2(\Pi,[0,T])$ such that

$$\lim_{N\to\infty} \int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\left\{\left|\varphi(t,\mathbf{y}) - \varphi^{(N)}(t,\mathbf{y})\right|^2\right\}\Pi(d\mathbf{y})dt \to 0.$$

It is well known [100] that the stochastic integral (1.144) exists, it does not depend on selection of the sequence $\varphi^{(N)}(t,\mathbf{y})$ and it satisfies w. p. 1 to the following properties

$$\mathsf{M}\left\{\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\Bigg|{\rm F}_0\right\} = 0,$$

$$\int\limits_0^T \int\limits_{\mathbf{Y}} (\alpha\varphi_1(t,\mathbf{y}) + \beta\varphi_2(t,\mathbf{y}))\tilde{\nu}(dt,d\mathbf{y}) =$$

$$= \alpha\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi_1(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) + \beta\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi_2(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}),$$

$$\mathsf{M}\left\{\left|\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\right|^2 \Bigg|{\rm F}_0\right\} = \int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\Bigl\{|\varphi(t,\mathbf{y})|^2\Bigr|{\rm F}_0\Bigr\}\Pi(d\mathbf{y})dt,$$

where $\alpha,\ \beta \in \mathbf{R}^1$ and $\varphi_1(t,\mathbf{y})$, $\varphi_2(t,\mathbf{y})$, $\varphi(t,\mathbf{y})$ from the class $H_2(\Pi,[0,T])$.

The stochastic integral

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y})$$

with respect to the Poisson measure will be defined as follows [100]

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y}) = \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) + \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\Pi(d\mathbf{y})dt, \quad (1.145)$$

where we suppose that the right-hand side of the last equality exists.

According to the Itô formula for Itô processes with jumps, we get [100]

$$(z_t)^p = \int\limits_0^t \int\limits_{\mathbf{Y}} \Bigl((z_{\tau-} + \gamma(\tau,\mathbf{y}))^p - (z_{\tau-})^p\Bigr)\nu(d\tau,d\mathbf{y}) \quad \text{w. p. 1}, \quad (1.146)$$

where $p\in\mathbf{N}$ and $z_{\tau-}$ means the left-sided limit value of the process z_τ at the point τ,

$$z_t = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\nu(d\tau,d\mathbf{y}).$$

We suppose that the function $\gamma(\tau,\mathbf{y})$ satisfies the conditions of existence of the right-hand side of (1.146) [100].

Let us consider the useful estimate for moments of stochastic integrals with respect to the Poisson measure [100]

$$a_p(T) \le \max_{j\in\{p,\ 1\}} \left\{ \left(\int\limits_0^T \int\limits_{\mathbf{Y}} \left(\left((b_p(\tau,\mathbf{y}))^{1/p} + 1 \right)^p - 1 \right) \Pi(d\mathbf{y})d\tau \right)^j \right\}, \quad (1.147)$$

where

$$a_p(t) = \sup_{0\le\tau\le t} \mathsf{M}\Bigl\{|z_\tau|^p\Bigr\}, \quad b_p(\tau,\mathbf{y}) = \mathsf{M}\Bigl\{|\gamma(\tau,\mathbf{y})|^p\Bigr\}.$$

We suppose that the right-hand side of (1.147) exists. According to (see (1.145))

$$\int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\tilde{\nu}(d\tau,d\mathbf{y}) = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\nu(d\tau,d\mathbf{y}) - \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})d\tau$$

and the Minkowski inequality, we obtain

$$\left(\mathsf{M}\left\{|\tilde{z}_t|^{2p}\right\}\right)^{1/2p} \le \left(\mathsf{M}\left\{|z_t|^{2p}\right\}\right)^{1/2p} + \left(\mathsf{M}\left\{|\hat{z}_t|^{2p}\right\}\right)^{1/2p}, \qquad (1.148)$$

where

$$\tilde{z}_t = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau, \mathbf{y})\tilde{\nu}(d\tau, d\mathbf{y})$$

and

$$\hat{z}_t \stackrel{\text{def}}{=} \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau, \mathbf{y})\Pi(d\mathbf{y})d\tau.$$

The value $\mathsf{M}\left\{|\hat{z}_\tau|^{2p}\right\}$ can be estimated using the well known inequality [100]

$$\mathsf{M}\left\{|\hat{z}_t|^{2p}\right\} \le t^{2p-1} \int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \gamma(\tau, \mathbf{y})\Pi(d\mathbf{y})\right|^{2p}\right\} d\tau, \qquad (1.149)$$

where we suppose that

$$\int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \gamma(\tau, \mathbf{y})\Pi(d\mathbf{y})\right|^{2p}\right\} d\tau < \infty.$$

1.4.2 Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures

Let us consider the following iterated stochastic integrals

$$P[\chi^{(k)}]_{T,t} =$$

$$= \int\limits_t^T \int\limits_{\mathbf{X}} \chi_k(t_k, \mathbf{y}_k) \ldots \int\limits_t^{t_2} \int\limits_{\mathbf{X}} \chi_1(t_1, \mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}_1) \ldots \tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}_k), \qquad (1.150)$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\mathbf{R}^n \stackrel{\text{def}}{=} \mathbf{X}$, $\chi_l(\tau, \mathbf{y}) = \psi_l(\tau)\varphi_l(\mathbf{y})$ $(l = 1, \ldots, k)$, every function $\psi_l(\tau) : [t, T] \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ and every function $\varphi_l(\mathbf{y})$: $\mathbf{X} \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ such that

$$\chi_l(\tau,\mathbf{y})\in H_2(\Pi,[t,T])\quad (l=1,\ldots,k),$$

where definition of the class $H_2(\Pi,[t,T])$ see above, $\nu^{(i)}(dt,d\mathbf{y})$ $(i=1,\ldots,m)$ are independent Poisson measures for various i, which are defined on $[0,T]\times\mathbf{X}$,

$$\tilde{\nu}^{(i)}(dt,d\mathbf{y})=\nu^{(i)}(dt,d\mathbf{y})-\Pi(d\mathbf{y})dt\quad (i=1,\ldots,m)$$

are independent martingale Poisson measures for various i, $\tilde{\nu}^{(0)}(dt,d\mathbf{y})\stackrel{\rm def}{=}\Pi(d\mathbf{y})dt$, $\nu^{(0)}(dt,d\mathbf{y})\stackrel{\rm def}{=}\Pi(d\mathbf{y})dt$.

Let us formulate an analogue of Theorem 1.1 for the iterated stochastic integrals (1.150).

Theorem 1.7 [1]-[17], [41]. *Suppose that the following conditions are hold*:

1. *Every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1.1.7).

3. *For* $l=1,\ldots,k$ *and* $q=2^{k+1}$ *the following condition is satisfied*

$$\int\limits_{\mathbf{X}}|\varphi_l(\mathbf{y})|^q\,\Pi(d\mathbf{y})<\infty.$$

Then, for the iterated stochastic integral with respect to martingale Poisson measures $P[\chi^{(k)}]_{T,t}$ *defined by* (1.150) *the following expansion*

$$P[\chi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{g=1}^{k}\pi_{j_g}^{(g,i_g)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\int\limits_{\mathbf{X}}\varphi_g(\mathbf{y})\tilde{\nu}^{(i_g)}([\tau_{l_g},\tau_{l_g+1}),d\mathbf{y})\Biggr)\qquad(1.151)$$

that converges in the mean-square sense is valid, where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition* (1.9),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$, *random variables*

$$
\pi_j^{(g,i_g)} = \int\limits_t^T \phi_j(\tau) \int\limits_{\mathbf{X}} \varphi_g(\mathbf{y}) \tilde{\nu}^{(i_g)}(d\tau, d\mathbf{y})
$$

are independent for various i_g *(if* $i_g \neq 0$*) and uncorrelated for various* j*,*

$$
C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k
$$

is the Fourier coefficient,

$$
K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t,T], \quad k \geq 2,
$$

and $K(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t,T]$*.*

Proof. The scheme of the proof of Theorem 1.7 is the same with the scheme of the proof of Theorem 1.1. Some differences will take place in the proof of Lemmas 1.4, 1.5 (see below) and in the final part of the proof of Theorem 1.7.

Lemma 1.4. *Suppose that every* $\psi_l(\tau)$ $(l = 1, \ldots, k)$ *is a continuous function at the interval* $[t,T]$ *and every function* $\varphi_l(\mathbf{y})$ $(l = 1, \ldots, k)$ *such that*

$$
\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^2 \, \Pi(d\mathbf{y}) < \infty.
$$

Then, the following equality

$$
P[\bar{\chi}^{(k)}]_{T,t} = \underset{N \to \infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \chi_l(\tau_{j_l}, \mathbf{y}) \bar{\nu}^{(i_l)}([\tau_{j_l}, \tau_{j_l+1}), d\mathbf{y}) \quad (1.152)
$$

is valid w. p. 1, where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition* (1.9)*,*

$$
\bar{\nu}^{(i)}([\tau, s), d\mathbf{y}) = \begin{cases} \tilde{\nu}^{(i)}([\tau, s), d\mathbf{y}) \\ \\ \nu^{(i)}([\tau, s), d\mathbf{y}) \end{cases} \quad (i = 0, 1, \ldots, m).
$$

In contrast to the integral $P[\chi^{(k)}]_{T,t}$ *defined by* (1.150)*,* $\bar{\nu}^{(i_l)}(dt_l, d\mathbf{y}_l)$ *is used in the integral* $P[\bar{\chi}^{(k)}]_{T,t}$ *instead of* $\tilde{\nu}^{(i_l)}(dt_l, d\mathbf{y}_l)$ $(l = 1, \ldots, k)$*.*

Proof. Using the moment properties of stochastic integrals with respect to the Poisson measure (see above) and the conditions of Lemma 1.4, it is easy to notice that the integral sum of the integral $P[\bar{\chi}^{(k)}]_{T,t}$ can be represented as a sum of the prelimit expression from the right-hand side of (1.152) and the value, which converges to zero in the mean-square sense if $N \to \infty$. Lemma 1.4 is proved.

Note that in the case when the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_j, $j = 0, 1, \ldots, N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$. Further, we can apply the argumentation as in Sect. 1.1.7.

Let us consider the following multiple and iterated stochastic integrals

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l(\mathbf{y}) \tilde{\nu}^{(i_l)}([\tau_{j_l}, \tau_{j_l+1}), d\mathbf{y}) \stackrel{\text{def}}{=} P[\Phi]_{T,t}^{(k)},$$

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y}) \tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}) \ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y}) \tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \hat{P}[\Phi]_{T,t}^{(k)},$$

where $\Phi(t_1, \ldots, t_k) : \ [t, T]^k \to \mathbf{R}^1$ is a bounded nonrandom function and the sense of notations of the formula (1.152) is remaining.

Note that if the functions $\varphi_l(\mathbf{y})$ $(l = 1, \ldots, k)$ satisfy the conditions of Lemma 1.4 and the function $\Phi(t_1, \ldots, t_k)$ is continuous in the domain of integration, then for the integral $\hat{P}[\Phi]_{T,t}^{(k)}$ the equality similar to (1.152) is valid w. p. 1.

Lemma 1.5. *Assume that the following representation takes place*:

$$g_l(\tau, \mathbf{y}) = h_l(\tau)\varphi_l(\mathbf{y}) \quad (l = 1, \ldots, k),$$

where the functions $h_l(\tau) : [t, T] \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ *satisfy the condition* $(\star)$ (*see Sect.* 1.1.7) *and the functions* $\varphi_l(\mathbf{y}) : \mathbf{X} \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ *satisfy the condition*

$$\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^p \, \Pi(d\mathbf{y}) < \infty \quad \textit{for} \quad p = 2^{k+1}.$$

Then

$$\prod_{l=1}^{k}\int\limits_{t}^{T}\int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}) = P[\Phi]_{T,t}^{(k)} \quad w.\ p.\ 1, \tag{1.153}$$

where $i_l=0,1,\ldots,m$ $(l=1,\ldots,k)$ *and*

$$\Phi(t_1,\ldots,t_k)=\prod_{l=1}^{k}h_l(t_l).$$

Proof. Let us introduce the following notations

$$J[\bar{g}_l]_N \stackrel{\rm def}{=} \sum_{j=0}^{N-1}\int\limits_{\mathbf{X}} g_l(\tau_j,\mathbf{y})\bar{\nu}^{(i_l)}([\tau_j,\tau_{j+1}),d\mathbf{y}),$$

$$J[\bar{g}_l]_{T,t} \stackrel{\rm def}{=} \int\limits_{t}^{T}\int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}),$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (1.9).

It is easy to see that

$$\prod_{l=1}^{k}J[\bar{g}_l]_N-\prod_{l=1}^{k}J[\bar{g}_l]_{T,t}=$$

$$=\sum_{l=1}^{k}\left(\prod_{q=1}^{l-1}J[\bar{g}_q]_{T,t}\right)\left(J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}\right)\left(\prod_{q=l+1}^{k}J[\bar{g}_q]_N\right).$$

Using the Minkowski inequality and the inequality of Cauchy–Bunyakovsky together with the estimates of moments of stochastic integrals with respect to the Poisson measure and the conditions of Lemma 1.5, we obtain

$$\left({\sf M}\left\{\left|\prod_{l=1}^{k}J[\bar{g}_l]_N-\prod_{l=1}^{k}J[\bar{g}_l]_{T,t}\right|^2\right\}\right)^{1/2}\le C_k\sum_{l=1}^{k}\left({\sf M}\left\{\left|J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{1.154}$$

where $C_k<\infty$.

We have

$$J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}=\sum_{q=0}^{N-1}J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q},$$

where

$$J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q} = \int\limits_{\tau_q}^{\tau_{q+1}} \left(h_l(\tau_q) - h_l(s)\right) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y}).$$

Let us introduce the notation

$$h_l^{(N)}(s) = h_l(\tau_q), \quad s \in [\tau_q, \tau_{q+1}), \quad q = 0, 1, \ldots, N-1.$$

Then

$$J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q} =$$

$$= \int\limits_t^T \left(h_l^{(N)}(s) - h_l(s)\right) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y}).$$

Applying the estimates (1.147) (for $p = 4$) and (1.148), (1.149) (for $p = 2$) to the value

$$\mathsf{M}\left\{\left|\int\limits_t^T \left(h_l^{(N)}(s) - h_l(s)\right) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y})\right|^4\right\},$$

taking into account (1.154), the conditions of Lemma 1.5, and the estimate

$$|h_l(\tau_q) - h_l(s)| < \varepsilon, \quad s \in [\tau_q, \tau_{q+1}], \quad q = 0, 1, \ldots, N-1, \qquad (1.155)$$

where ε is an arbitrary small positive real number and $|\tau_{q+1} - \tau_q| < \delta(\varepsilon)$, we obtain that the right-hand side of (1.154) converges to zero when $N \to \infty$. Therefore, we come to the affirmation of Lemma 1.5.

It should be noted that (1.155) is valid if the functions $h_l(s)$ are continuous at the interval $[t, T]$, i.e. these functions are uniformly continuous at this interval. So, $|h_l(\tau_q) - h_l(s)| < \varepsilon$ if $s \in [\tau_q, \tau_{q+1}]$, where $|\tau_{q+1} - \tau_q| < \delta(\varepsilon)$, $q = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$).

In the case when the functions $h_l(s)$ $(l = 1, \ldots, k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_q, $q = 0, 1, \ldots, N$ there are all points of jumps of the functions $h_l(s)$ $(l = 1, \ldots, k)$. Further, we can apply the argumentation as in Sect. 1.1.7.

Obviously, if $i_l = 0$ for some $l = 1, \ldots, k$, then we also come to the affirmation of Lemma 1.5. Lemma 1.5 is proved.

Proving Theorem 1.7 by the scheme of the proof of Theorem 1.1 using Lemmas 1.4, 1.5 and moment properties of stochastic integrals with respect to the martingale Poisson measures, we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y})\Pi(d\mathbf{y})\times
$$

$$
\times \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 \times
$$

$$
\times dt_1\ldots dt_k =
$$

$$
= C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y})\Pi(d\mathbf{y}) \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 \times
$$

$$
\times dt_1\ldots dt_k \le
$$

$$
\le \bar{C}_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0
$$

if $p_1,\ldots,p_k\to\infty$, where constant $\bar{C}_k$ depends only on k (k is the multiplicity of the iterated stochastic integral with respect to the martingale Poisson measures). Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$
R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right) \times
$$

$$
\times \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y})\ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y}), \qquad (1.156)
$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.156) are performed only in the values $\varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y})\ldots\varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y})$. At the same time, the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover,

$\varphi_r(\mathbf{y})$ swapped with $\varphi_q(\mathbf{y})$ in the permutation $(\varphi_1(\mathbf{y}),\ldots,\varphi_k(\mathbf{y}))$. Theorem 1.7 is proved.

Let us consider the application of Theorem 1.7. Let $i_1\ne i_2$ and $i_1,i_2=1,\ldots,m$. Using Theorem 1.7 and the system of Legendre polynomials, we obtain

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_2(\mathbf{y}_2)\int\limits_t^{t_2}\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)\tilde{\nu}^{(i_2)}(dt_2,d\mathbf{y}_2)=$$

$$=\frac{T-t}{2}\left(\pi_0^{(1,i_1)}\pi_0^{(2,i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\pi_{i-1}^{(1,i_1)}\pi_i^{(2,i_2)}-\pi_i^{(1,i_1)}\pi_{i-1}^{(2,i_2)}\right)\right),$$

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)=\sqrt{T-t}\,\pi_0^{(1,i_1)},$$

where

$$\pi_j^{(l,i_l)}=\int\limits_t^T\phi_j(\tau)\int\limits_{\mathbf{X}}\varphi_l(\mathbf{y})\tilde{\nu}^{(i_l)}(d\tau,d\mathbf{y})\quad (l=1,2)$$

and $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

1.5 Expansion of Iterated Stochastic Integrals with Respect to Martingales Based on Generalized Multiple Fourier Series

1.5.1 Stochastic Integral with Respect to Martingale

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a fixed probability space, let $\{\mathrm{F}_t,\, t\in[0,T]\}$ be a non-decreasing family of σ-algebras $\mathrm{F}_t\subset\mathrm{F}$, and let $\mathrm{M}_2(\rho,[0,T])$ be a class of F_t-measurable for each $t\in[0,T]$ martingales M_t satisfying the conditions

$$\mathsf{M}\left\{(M_s-M_t)^2\right\}=\int\limits_t^s\rho(\tau)d\tau, \tag{1.157}$$

$$\mathsf{M}\left\{|M_s-M_t|^p\right\}\le C_p|s-t|,\quad p=3,4,\ldots,$$

where $0 \leq t < s \leq T$, $\rho(\tau)$ is a non-negative and continuously differentiable nonrandom function at the interval $[0, T]$, $C_p < \infty$ is a constant.

Let us define the class $H_2(\rho, [0, T])$ of stochastic processes ξ_t, $t \in [0, T]$, which are F_t-measurable for all $t \in [0, T]$ and satisfy the condition

$$\int\limits_0^T \mathsf{M}\left\{|\xi_t|^2\right\}\rho(t)dt < \infty.$$

For any partition $\left\{\tau_j^{(N)}\right\}_{j=0}^{N}$ of the interval $[0, T]$ such that

$$0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0 \leq j \leq N-1}\left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0 \ \text{ if } \ N \to \infty \tag{1.158}$$

we will define the sequence of step functions $\xi^{(N)}(t, \omega)$ by the following relation

$$\xi^{(N)}(t, \omega) = \xi_j(\omega) \quad \text{w. p. 1} \quad \text{for} \quad t \in \left[\tau_j^{(N)}, \tau_{j+1}^{(N)}\right),$$

where $\xi^{(N)}(t, \omega) \in H_2(\rho, [0, T])$, $j = 0, 1, \ldots, N-1$, $N = 1, 2, \ldots$

Let us define the stochastic integral with respect to martingale from the process $\xi(t, \omega) \in H_2(\rho, [0, T])$ as the following mean-square limit [100]

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}\left(\tau_j^{(N)}, \omega\right)\left(M\left(\tau_{j+1}^{(N)}, \omega\right) - M\left(\tau_j^{(N)}, \omega\right)\right) \stackrel{\text{def}}{=} \int\limits_0^T \xi_\tau dM_\tau, \tag{1.159}$$

where $\xi^{(N)}(t, \omega)$ is any step function from the class $H_2(\rho, [0, T])$, which converges to the function $\xi(t, \omega)$ in the following sense

$$\lim_{N\to\infty} \int\limits_0^T \mathsf{M}\left\{\left|\xi^{(N)}(t, \omega) - \xi(t, \omega)\right|^2\right\}\rho(t)dt = 0.$$

It is well known [100] that the stochastic integral (1.159) exists, it does not depend on selection of the sequence $\xi^{(N)}(t, \omega)$ and it satisfies w. p. 1 to the following properties

$$\mathsf{M}\left\{\int\limits_0^T \xi_t dM_t \Bigg| \mathrm{F}_0\right\} = 0,$$

$$
\mathsf{M}\left\{\left|\int\limits_0^T \xi_t dM_t\right|^2 \Biggl| \mathrm{F}_0\right\} = \mathsf{M}\left\{\int\limits_0^T \xi_t^2 \rho(t)dt\Biggl|\mathrm{F}_0\right\},
$$

$$
\int\limits_0^T (\alpha\xi_t + \beta\psi_t)dM_t = \alpha\int\limits_0^T \xi_t dM_t + \beta\int\limits_0^T \psi_t dM_t,
$$

where $\xi_t,\ \psi_t \in H_2(\rho, [0,T])$, $\alpha, \beta \in \mathbf{R}^1$.

1.5.2 Expansion of Iterated Stochastic Integrals with Respect to Martingales

Let $Q_4(\rho, [0,T])$ be the class of martingales M_t, $t \in [0,T]$, which satisfy the following conditions:

1. M_t, $t \in [0,T]$ belongs to the class $\mathrm{M}_2(\rho, [0,T])$.
2. For some $\alpha > 0$ the following estimate is correct

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau g(s)dM_s\right|^4\right\} \le K_4\int\limits_t^\tau |g(s)|^\alpha ds, \tag{1.160}
$$

where $0 \le t < \tau \le T$, $g(s)$ is a bounded nonrandom function at the interval $[0,T]$, $K_4 < \infty$ is a constant.

Let $G_n(\rho, [0,T])$ be the class of martingales M_t, $t \in [0,T]$, which satisfy the following conditions:

1. M_t, $t \in [0,T]$ belongs to the class $\mathrm{M}_2(\rho, [0,T])$.
2. The following estimate is correct

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau g(s)dM_s\right|^n\right\} < \infty,
$$

where $0 \le t < \tau \le T$, $n \in \mathbf{N}$, $g(s)$ is the same function as in the definition of the class $Q_4(\rho, [0,T])$.

Let us remind that if $(\xi_t)^n \in H_2(\rho, [0,T])$ with $\rho(t) \equiv 1$, then the following estimate is correct [100]

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau \xi_s ds\right|^{2n}\right\} \le (\tau - t)^{2n-1}\int\limits_t^\tau \mathsf{M}\left\{|\xi_s|^{2n}\right\}ds, \quad 0 \le t < \tau \le T. \tag{1.161}
$$

Let us consider the iterated stochastic integral with respect to martingales

$$J[\psi^{(k)}]_{T,t}^{M}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)},\qquad(1.162)$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$, $M_s^{(r,i)}$ $(r=1,\ldots,k,\ i=1,\ldots,m)$ are independent martingales for various $i=1,\ldots,m$, $M_s^{(r,0)}\stackrel{\rm def}{=}s$.

Now we can formulate the following theorem.

Theorem 1.8 [1]-[17], [41]. *Suppose that the following conditions are hold:*

1. *Every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7).*

3. *$M_s^{(l,i_l)}\in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ with $n=2^{k+1}$, $i_l=1,\ldots,m$, $l=1,\ldots,k$ $(k\in\mathbf{N})$.*

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^{M}$ with respect to martingales defined by (1.162) the following expansion

$$J[\psi^{(k)}]_{T,t}^{M}=\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\rm l.i.m.}\sum_{(l_1,\ldots,l_k)\in{\rm G}_k}\phi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k=0,1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition similar to (1.158), $\Delta M_{\tau_j}^{(r,i)}=M_{\tau_{j+1}}^{(r,i)}-M_{\tau_j}^{(r,i)}$ $(i=0,1,\ldots,m,\ \ r=1,\ldots,k)$,

$${\rm G}_k={\rm H}_k\backslash{\rm L}_k,\quad {\rm H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$${\rm L}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\Bigr\},$$

l.i.m. *is a limit in the mean-square sense,*

$$\xi_j^{(l,i_l)}=\int\limits_t^T\phi_j(s)dM_s^{(l,i_l)}$$

are independent for various i_l (*if* $i_l \neq 0$) *and uncorrelated for various* j (*if* $\rho(\tau)$ *is a constant,* $i_l \neq 0$) *random variables,*

$$
C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k
$$

is the Fourier coefficient,

$$
K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases}, \quad t_1,\ldots,t_k \in [t,T], \quad k \geq 2,
$$

and $K(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t,T]$.

Remark 1.4. *Note that from Theorem* 1.8 *for the case* $\rho(\tau) \equiv 1$ *we obtain the variant of Theorem* 1.1.

Proof. The proof of Theorem 1.8 is similar to the proof of Theorem 1.1. Some differences will take place in the proof of Lemmas 1.6, 1.7 (see below) and in the final part of the proof of Theorem 1.8.

Lemma 1.6. *Assume that* $M_s^{(r,i)} \in \mathrm{M}_2(\rho,[t,T])$ $(i = 1,\ldots,m)$, $M_s^{(r,0)} = s$ $(r = 1,\ldots,k)$, *and every* $\psi_l(\tau)$ $(l = 1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$. *Then*

$$
J[\psi^{(k)}]_{T,t}^{M} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta M_{\tau_{j_l}}^{(l,i_l)} \quad w.\ p.\ 1, \tag{1.163}
$$

where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition similar to* (1.158), $i_l = 0,1,\ldots,m,$ $l = 1,\ldots,k$; *another notations are the same as in Theorem* 1.8.

Proof. According to the properties of the stochastic integral with respect to martingales, we have [100]

$$
\mathsf{M}\left\{\left(\int\limits_{t}^{\tau}\xi_s dM_s^{(l,i_l)}\right)^2\right\} = \int\limits_{t}^{\tau}\mathsf{M}\left\{|\xi_s|^2\right\}\rho(s)ds, \tag{1.164}
$$

$$
\mathsf{M}\left\{\left(\int\limits_{t}^{\tau}\xi_s ds\right)^2\right\} \leq (\tau - t)\int\limits_{t}^{\tau}\mathsf{M}\left\{|\xi_s|^2\right\}ds, \tag{1.165}
$$

where $\xi_s \in H_2(\rho, [0,T])$, $0 \le t < \tau \le T$, $i_l = 1,\ldots,m$, $l = 1,\ldots,k$. Then the integral sum for the integral $J[\psi^{(k)}]_{T,t}^M$ under the conditions of Lemma 1.6 can be represented as a sum of the prelimit expression from the right-hand side of (1.163) and the value, which converges to zero in the mean-square sense if $N \to \infty$. More detailed proof of the similar lemma for the case $\rho(\tau) \equiv 1$ can be found in Sect. 1.1.3 (see Lemma 1.1).

In the case when the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_j, $j = 0, 1, \ldots, N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$. So, we can apply the argumentation as in Sect. 1.1.7.

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \Delta M_{\tau_{j_l}}^{(l,i_l)} \stackrel{\text{def}}{=} I[\Phi]_{T,t}^{(k)}, \tag{1.166}$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition similar to (1.158) and $\Phi(t_1,\ldots,t_k):\ [t,T]^k \to \mathbf{R}^1$ is a bounded nonrandom function; another notations are the same as in Theorem 1.8.

Lemma 1.7. *Let* $M_s^{(l,i_l)} \in Q_4(\rho, [t,T])$, $G_n(\rho, [t,T])$ *with* $n = 2^{k+1}$, $k \in \mathbf{N}$ $(i_l = 1,\ldots,m,\ \ l = 1,\ldots,k)$ *and the functions* $g_1(s),\ldots,g_k(s)$ *satisfy the condition* $(\star)$ *(see Sect.* 1.1.7*). Then*

$$\prod_{l=1}^{k} \int\limits_t^T g_l(s) dM_s^{(l,i_l)} = I[\Phi]_{T,t}^{(k)} \quad w.\ p.\ 1,$$

where $i_l = 0, 1, \ldots, m,\ \ l = 1,\ldots,k,$

$$\Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} g_l(t_l).$$

Proof. Let us denote

$$J[g_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} g_l(\tau_j) \Delta M_{\tau_j}^{(l,i_l)}, \quad J[g_l]_{T,t} \stackrel{\text{def}}{=} \int\limits_t^T g_l(s) dM_s^{(l,i_l)},$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition similar to (1.158).

Note that

$$
\prod_{l=1}^{k} J[g_l]_N - \prod_{l=1}^{k} J[g_l]_{T,t} =
$$

$$
= \sum_{l=1}^{k} \left(\prod_{q=1}^{l-1} J[g_q]_{T,t} \right) \left(J[g_l]_N - J[g_l]_{T,t} \right) \left(\prod_{q=l+1}^{k} J[g_q]_N \right).
$$

Using the Minkowski inequality and the inequality of Cauchy-Bunyakovsky as well as the conditions of Lemma 1.7, we obtain

$$
\left(\mathsf{M} \left\{ \left| \prod_{l=1}^{k} J[g_l]_N - \prod_{l=1}^{k} J[g_l]_{T,t} \right|^2 \right\} \right)^{1/2} \le
$$

$$
\le C_k \sum_{l=1}^{k} \left(\mathsf{M} \left\{ \left| J[g_l]_N - J[g_l]_{T,t} \right|^4 \right\} \right)^{1/4}, \qquad (1.167)
$$

where $C_k < \infty$ is a constant.

We have

$$
J[g_l]_N - J[g_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta g_l]_{\tau_{q+1},\tau_q},
$$

$$
J[\Delta g_l]_{\tau_{q+1},\tau_q} = \int\limits_{\tau_q}^{\tau_{q+1}} \left(g_l(\tau_q) - g_l(s) \right) dM_s^{(l,i_l)}.
$$

Let us introduce the notation

$$
g_l^{(N)}(s) = g_l(\tau_q), \quad s \in [\tau_q, \tau_{q+1}), \quad q = 0, 1, \ldots, N-1.
$$

Then

$$
J[g_l]_N - J[g_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta g_l]_{\tau_{q+1},\tau_q} =
$$

$$
= \int\limits_{t}^{T} \left(g_l^{(N)}(s) - g_l(s) \right) dM_s^{(l,i_l)}.
$$

Applying the estimate (1.160), we obtain

$$
\mathsf{M}\left\{\left|\int\limits_t^T \left(g_l^{(N)}(s)-g_l(s)\right)dM_s^{(l,i_l)}\right|^4\right\}\le
$$

$$
\le K_4\int\limits_t^T \left|g_l^{(N)}(s)-g_l(s)\right|^{\alpha}ds=
$$

$$
=K_4\sum_{q=0}^{N-1}\int\limits_{\tau_q}^{\tau_{q+1}}\left|g_l(\tau_q)-g_l(s)\right|^{\alpha}ds<
$$

$$
<K_4\varepsilon^{\alpha}\sum_{q=0}^{N-1}(\tau_{q+1}-\tau_q)=K_4\varepsilon^{\alpha}(T-t). \tag{1.168}
$$

Note that we used the estimate

$$
|g_l(\tau_q)-g_l(s)|<\varepsilon,\quad s\in[\tau_q,\tau_{q+1}],\quad q=0,1,\ldots,N-1 \tag{1.169}
$$

to derive (1.168), where $|\tau_{q+1}-\tau_q|<\delta(\varepsilon)$ and ε is an arbitrary small positive real number.

The inequality (1.169) is valid if the functions $g_l(s)$ are continuous at the interval $[t,T]$, i.e. these functions are uniformly continuous at this interval. So, $|g_l(\tau_q)-g_l(s)|<\varepsilon$ if $s\in[\tau_q,\tau_{q+1}]$, where $|\tau_{q+1}-\tau_q|<\delta(\varepsilon)$, $q=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on points of the interval $[t,T]$).

Thus, taking into account (1.168), we obtain that the right-hand side of (1.167) converges to zero when $N\to\infty$. Hence, we come to the affirmation of Lemma 1.7.

In the case when the functions $g_l(s)$ $(l=1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_q, $q=0,1,\ldots,N$ there are all points of jumps of the functions $g_l(s)$ $(l=1,\ldots,k)$. So, we can apply the argumentation as in Sect. 1.1.7.

Obviously if $i_l=0$ for some $l=1,\ldots,k$, then we also come to the affirmation of Lemma 1.7. Lemma 1.7 is proved.

Proving Theorem 1.8 similar to the proof of Theorem 1.1 using Lemmas 1.6, 1.7 and moment properties of stochastic integrals with respect to martingales (see (1.164), (1.165)), we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times$$

$$\times \tilde{\rho}_1(t_1)dt_1 \ldots \tilde{\rho}_k(t_k)dt_k \le \tag{1.170}$$

$$\le \bar{C}_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= \bar{C}_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k \to 0$$

if $p_1,\ldots,p_k \to \infty$, where constant $\bar{C}_k$ depends only on k (k is the multiplicity of the iterated stochastic integral with respect to martingales) and $\tilde{\rho}_l(s) \equiv \rho(s)$ or $\tilde{\rho}_l(s) \equiv 1$ ($l = 1,\ldots,k$). Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right) \times$$

$$\times dM_{t_1}^{(1,i_1)} \ldots dM_{t_k}^{(k,i_k)}, \tag{1.171}$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.171) are performed only in the values $dM_{t_1}^{(1,i_1)} \ldots dM_{t_k}^{(k,i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover, r swapped with q in the permutation $(1,\ldots,k)$. Theorem 1.8 is proved.

1.6 One Modification of Theorems 1.5 and 1.8

1.6.1 Expansion of Iterated Stochastic Integrals with Respect to Martingales Based on Generalized Multiple Fourier Series. The Case $\rho(x)/r(x) < \infty$

Let us compare the expressions (1.141) and (1.170). If we suppose that $r(x) \ge 0$ and

$$\frac{\rho(x)}{r(x)} \le C < \infty,$$

where $\rho(x)$ as in (1.157), then

$$\int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \rho(t_1)dt_1\ldots\rho(t_k)dt_k =$$

$$= \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \frac{\rho(t_1)}{r(t_1)}r(t_1)dt_1\ldots\frac{\rho(t_k)}{r(t_k)}r(t_k)dt_k \le$$

$$\le C_k' \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \left(\prod_{l=1}^{k} r(t_l)\right) dt_1\ldots dt_k \to 0$$

if $p_1,\ldots,p_k \to \infty$ (see (1.142)), where C_k' is a constant, $\{\Psi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$, and the Fourier coefficient $\tilde{C}_{j_k\ldots j_1}$ has the form (1.140).

So, we obtain the following modification of Theorems 1.5 and 1.8.

Theorem 1.9 [13]-[17], [41]. *Suppose that the following conditions are fulfilled*:

1. *Every* $\psi_l(\tau)$ $(l = 1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$.

2. $\{\Psi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal with weight* $r(x) \ge 0$ *system of functions in the space* $L_2([t,T])$, *each function* $\Psi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Moreover,*

$$\frac{\rho(x)}{r(x)} \le C < \infty.$$

3. $M_s^{(l,i_l)} \in Q_4(\rho,[t,T]),\ G_n(\rho,[t,T])$ *with* $n = 2^{k+1}$, $i_l = 1,\ldots,m$, $l = 1,\ldots,k$ $(k \in \mathbf{N})$.

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^M$ *with respect to martingales defined by* (1.162) *the following expansion*

$$J[\psi^{(k)}]_{T,t}^M = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)} -$$

$$-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \Psi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k = 1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition similar to* (1.158), $\Delta M_{\tau_j}^{(r,i)} = M_{\tau_{j+1}}^{(r,i)} - M_{\tau_j}^{(r,i)}$ $(i = 1,\ldots,m,\ \ r = 1,\ldots,k)$,

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g \ne r);\ g,r = 1,\ldots,k\Bigr\},$$

l.i.m. *is a limit in the mean-square sense,*

$$\xi_j^{(l,i_l)} = \int\limits_t^T \Psi_j(s)dM_s^{(l,i_l)}$$

are independent for various $i_l = 1,\ldots,m$ $(l = 1,\ldots,k)$ *and uncorrelated for various* j *(if* $\rho(x) \equiv r(x)$*) random variables,*

$$\tilde{C}_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & otherwise\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Remark 1.5. *Note that if* $\rho(x), r(x)\equiv 1$ *in Theorem* 1.9, *then we obtain the variant of Theorem* 1.1.

1.6.2 Example on Application of Theorem 1.9 and the System of Bessel Functions

Let us consider the following boundary-value problem

$$(p(x)\Phi'(x))'+q(x)\Phi(x)=-\lambda r(x)\Phi(x),$$
$$\alpha\Phi(a)+\beta\Phi'(a)=0,\quad \gamma\Phi(b)+\delta\Phi'(b)=0,$$

where the functions $p(x)$, $q(x)$, $r(x)$ satisfy the well known conditions and α, β, γ, δ, λ are real numbers.

It is well known (Steklov V.A.) that the eigenfunctions $\Phi_0(x)$, $\Phi_1(x)$, $\ldots$ of this boundary-value problem form a complete orthonormal with weight $r(x)$ system of functions in the space $L_2([a,b])$. It means that the Fourier series of the function $\sqrt{r(x)}f(x)\in L_2([a,b])$ with respect to the system of functions $\sqrt{r(x)}\Phi_0(x)$, $\sqrt{r(x)}\Phi_1(x),\ldots$ converges in the mean-square sense to the function $\sqrt{r(x)}f(x)$ at the interval $[a,b]$. Moreover, the Fourier coefficients are defined by the formula

$$\tilde C_j=\int\limits_a^b f(x)\Phi_j(x)r(x)dx. \tag{1.172}$$

It is known that when solving the problem on oscillations of a circular membrane (general case), a boundary-value problem arises for the following Euler–Bessel equation

$$r^2R''(r)+rR'(r)+\left(\lambda^2r^2-n^2\right)R(r)=0\quad(\lambda\in\mathbf{R},\quad n\in\mathbf{N}). \tag{1.173}$$

The eigenfunctions of this problem, taking into account specific boundary conditions, are the following functions

$$J_n\left(\mu_j\frac{\tau}{L}\right), \tag{1.174}$$

where $\tau \in [0, L]$ and μ_j $(j = 0, 1, 2, \ldots)$ are positive roots of the Bessel function $J_n(\mu)$ $(n = 0, 1, 2, \ldots)$ numbered in ascending order.

The problem on radial oscillations of a circular membrane leads to the boundary-value problem for the equation (1.173) for $n = 0$, the eigenfunctions of which are the functions (1.174) when $n = 0$.

Let us consider the system of functions

$$\Psi_j(\tau) = \frac{\sqrt{2}}{T J_{n+1}(\mu_j)} J_n\left(\frac{\mu_j}{T}\tau\right), \quad j = 0, 1, 2, \ldots, \tag{1.175}$$

where

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{n+2m} \frac{1}{\Gamma(m+1)\Gamma(m+n+1)}$$

is the Bessel function of the first kind,

$$\Gamma(z) = \int\limits_0^{\infty} e^{-x} x^{z-1} dx$$

is the gamma-function, μ_j are positive roots of the function $J_n(x)$ numbered in ascending order, and n is a natural number or zero.

Due to the well known properties of the Bessel functions, the system $\{\Psi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal with weight τ system of continuous functions in the space $L_2([0, T])$.

Let us use the system of functions (1.175) in Theorem 1.9.

Consider the following iterated stochastic integral with respect to martingales

$$\int\limits_0^T \int\limits_0^s dM_\tau^{(1)} dM_s^{(2)},$$

where

$$M_s^{(i)} = \int\limits_0^s \sqrt{\tau} d\mathbf{f}_\tau^{(i)} \quad (i = 1, 2),$$

$\mathbf{f}_\tau^{(i)}$ $(i = 1, 2)$ are independent standard Wiener processes, $M_s^{(i)}$ $(i = 1, 2)$ are martingales (here $\rho(\tau) \equiv \tau$), $0 \le s \le T$. In addition, $M_s^{(i)}$ has a Gaussian distribution.

It is obvious that the conditions of Theorem 1.9 are fulfilled for $k = 2$. Using Theorem 1.9, we obtain

$$\int\limits_0^T\int\limits_0^s dM_\tau^{(1)}dM_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\tilde C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)}=\int\limits_0^T\Psi_j(\tau)dM_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i=1,2,$ $j=0,1,2,\ldots$),

$$\tilde C_{j_2j_1}=\int\limits_0^T s\Psi_{j_2}(s)\int\limits_0^s \tau\Psi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient.

It is obvious that we can get the same result using the another approach: we can use Theorem 1.1 for the iterated Itô stochastic integral

$$\int\limits_0^T\sqrt{s}\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)},$$

and as a system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in Theorem 1.1 we can take

$$\phi_j(s)=\frac{\sqrt{2s}}{TJ_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T}s\right),\quad j=0,1,2,\ldots$$

As a result, we obtain

$$\int\limits_0^T\sqrt{s}\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)}=\int\limits_0^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i=1,2,$ $j=0,1,2,\ldots$),

$$C_{j_2j_1}=\int\limits_0^T\sqrt{s}\phi_{j_2}(s)\int\limits_0^s\sqrt{\tau}\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient. Obviously that $C_{j_2j_1} = \tilde{C}_{j_2j_1}$.

Easy calculation demonstrates that

$$\tilde{\phi}_j(s) = \frac{\sqrt{2(s-t)}}{(T-t)J_{n+1}(\mu_j)} J_n\left(\frac{\mu_j}{T-t}(s-t)\right), \quad j = 0, 1, 2, \ldots$$

is a complete orthonormal system of functions in the space $L_2([t,T])$.

Then, using Theorem 1.1, we obtain

$$\int\limits_t^T \sqrt{s-t} \int\limits_t^s \sqrt{\tau - t} d\mathbf{f}_\tau^{(1)} d\mathbf{f}_s^{(2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)},$$

where

$$\tilde{\zeta}_j^{(i)} = \int\limits_t^T \tilde{\phi}_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i = 1, 2,$ $j = 0, 1, 2, \ldots$),

$$C_{j_2j_1} = \int\limits_t^T \sqrt{s-t}\tilde{\phi}_{j_2}(s) \int\limits_t^s \sqrt{\tau - t}\tilde{\phi}_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient.

1.7 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals in Theorem 1.1

1.7.1 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicities 1 and 2

Let us address now to the convergence with probability 1 (w. p. 1). Consider in detail the iterated Itô stochastic integral (1.98) and its expansion, which is corresponds to (1.99) for the case $i_1 \neq i_2$

$$I_{(00)T,t}^{(i_1i_2)} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \quad (1.176)$$

First, note the well known fact [104].

Lemma 1.8. *If for the sequence of random variables ξ_p and for some $\alpha>0$ the number series*

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

In our specific case ($i_1\ne i_2$)

$$I_{(00)T,t}^{(i_1i_2)}=I_{(00)T,t}^{(i_1i_2)p}+\xi_p,\quad \xi_p=\frac{T-t}{2}\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right),$$

where

$$I_{(00)T,t}^{(i_1i_2)p}=\frac{T-t}{2}\left(\zeta_{0}^{(i_1)}\zeta_{0}^{(i_2)}+\sum_{i=1}^{p}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \qquad (1.177)$$

Furthermore,

$${\sf M}\left\{|\xi_p|^2\right\}=\frac{(T-t)^2}{2}\sum_{i=p+1}^{\infty}\frac{1}{4i^2-1}\le\frac{(T-t)^2}{2}\int\limits_{p}^{\infty}\frac{1}{4x^2-1}dx=$$

$$=-\frac{(T-t)^2}{2}\frac{1}{4}\ln\left|1-\frac{2}{2p+1}\right|\le\frac{C}{p}, \qquad (1.178)$$

where constant C is independent of p.

Therefore, taking $\alpha=2$ in Lemma 1.8, we cannot prove the convergence of ξ_p to zero w. p. 1, since the series

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^{2}\right\}$$

will be majorized by the divergent Dirichlet series with the index 1. Let us take $\alpha=4$ and estimate the value ${\sf M}\left\{|\xi_p|^4\right\}$.

From (1.74) for $k=2,$ $n=2$ and (1.178) we obtain

$${\sf M}\left\{|\xi_p|^4\right\}\le\frac{K}{p^2} \qquad (1.179)$$

and

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^4\right\}\le K\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \qquad (1.180)$$

where constant K is independent of p.

Since the series on the right-hand side of (1.180) converges, then according to Lemma 1.8, we obtain that $\xi_p \to 0$ when $p \to \infty$ w. p. 1. Then

$$I_{(00)T,t}^{(i_1 i_2)p} \to I_{(00)T,t}^{(i_1 i_2)} \ \ \text{when} \ \ p \to \infty \ \ \text{w. p. 1.}$$

Let us analyze the following iterated Itô stochastic integrals

$$I_{(01)T,t}^{(i_1 i_2)} = \int\limits_t^T (t-s) \int\limits_t^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}, \quad I_{(10)T,t}^{(i_1 i_2)} = \int\limits_t^T \int\limits_t^s (t-\tau) d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)},$$

whose expansions based on Theorem 1.1 and Legendre polynomials have the following form (also see Chapter 5, Sect. 5.1)

$$I_{(01)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)p} - \frac{(T-t)^2}{4} \left(\frac{\zeta_0^{(i_1)} \zeta_1^{(i_2)}}{\sqrt{3}} + \right.$$

$$\left. + \sum_{i=0}^{p} \left(\frac{(i+2)\zeta_i^{(i_1)} \zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)} \zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right) + \xi_p^{(01)},$$

$$I_{(10)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)p} - \frac{(T-t)^2}{4} \left(\frac{\zeta_0^{(i_2)} \zeta_1^{(i_1)}}{\sqrt{3}} + \right.$$

$$\left. + \sum_{i=0}^{p} \left(\frac{(i+1)\zeta_{i+2}^{(i_2)} \zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)} \zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right) + \xi_p^{(10)},$$

where

$$\xi_p^{(01)} = -\frac{(T-t)^2}{4} \left(\sum_{i=p+1}^{\infty} \frac{1}{\sqrt{4i^2-1}} \left(\zeta_{i-1}^{(i_1)} \zeta_i^{(i_2)} - \zeta_i^{(i_1)} \zeta_{i-1}^{(i_2)} \right) + \right.$$

$$\left. + \sum_{i=p+1}^{\infty} \left(\frac{(i+2)\zeta_i^{(i_1)} \zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)} \zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right),$$

$$\xi_p^{(10)} = -\frac{(T-t)^2}{4} \left(\sum_{i=p+1}^{\infty} \frac{1}{\sqrt{4i^2-1}} \left(\zeta_{i-1}^{(i_1)} \zeta_i^{(i_2)} - \zeta_i^{(i_1)} \zeta_{i-1}^{(i_2)} \right) + \right.$$

$$
+\sum_{i=p+1}^{\infty}\left(\frac{(i+1)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+2)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr).
$$

Then for the case $i_1\ne i_2$ we obtain

$$
\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^2\right\}=\frac{(T-t)^4}{16}\times
$$

$$
\times\sum_{i=p+1}^{\infty}\left(\frac{2}{4i^2-1}+\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}+\frac{1}{(2i-1)^2(2i+3)^2}\right)\le
$$

$$
\le K\sum_{i=p+1}^{\infty}\frac{1}{i^2}\le\frac{K}{p}, \tag{1.181}
$$

where constant K is independent of p.

Analogously, we get

$$
\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^2\right\}\le\frac{K}{p}, \tag{1.182}
$$

where constant K does not depend on p.

From (1.74) for $k=2,\ n=2$ and (1.181), (1.182) we have

$$
\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\}+\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\}\le\frac{K_1}{p^2}
$$

and

$$
\sum_{p=1}^{\infty}\left(\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\}+\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\}\right)\le K_1\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \tag{1.183}
$$

where constant K_1 is independent of p.

According to (1.183) and Lemma 1.8, we obtain that $\xi_p^{(01)},\ \xi_p^{(10)}\to 0$ when $p\to\infty$ w. p. 1. Then

$$
I_{(01)T,t}^{(i_1i_2)p}\to I_{(01)T,t}^{(i_1i_2)},\quad I_{(10)T,t}^{(i_1i_2)p}\to I_{(10)T,t}^{(i_1i_2)}\quad\text{when}\quad p\to\infty\quad\text{w. p. 1},
$$

where $i_1\ne i_2$.

Let us consider the case $i_1=i_2$

$$
I_{(01)T,t}^{(i_1i_1)}=\frac{(T-t)^2}{4}-\frac{(T-t)^2}{4}\Biggl(\left(\zeta_0^{(i_1)}\right)^2+\frac{\zeta_0^{(i_1)}\zeta_1^{(i_1)}}{\sqrt{3}}+
$$

$$
+\sum_{i=0}^{p}\left(\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right)\Biggr)+\mu_p^{(01)},
$$

$$
I_{(10)T,t}^{(i_1i_1)}=\frac{(T-t)^2}{4}-\frac{(T-t)^2}{4}\Biggl(\left(\zeta_0^{(i_1)}\right)^2+\frac{\zeta_0^{(i_1)}\zeta_1^{(i_1)}}{\sqrt{3}}+
$$

$$
+\sum_{i=0}^{p}\left(-\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right)\Biggr)+\mu_p^{(10)},
$$

where

$$
\mu_p^{(01)}=-\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right),
$$

$$
\mu_p^{(10)}=-\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(-\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right).
$$

Then

$$
\mathsf{M}\left\{\left(\mu_p^{(01)}\right)^2\right\}=\mathsf{M}\left\{\left(\mu_p^{(10)}\right)^2\right\}=\frac{(T-t)^4}{16}\times
$$

$$
\times\Biggl(\sum_{i=p+1}^{\infty}\frac{1}{(2i+1)(2i+5)(2i+3)^2}+\sum_{i=p+1}^{\infty}\frac{2}{(2i-1)^2(2i+3)^2}+
$$

$$
+\left(\sum_{i=p+1}^{\infty}\frac{1}{(2i-1)(2i+3)}\right)^2\Biggr)\le\frac{K}{p^2}
$$

and

$$
\sum_{p=1}^{\infty}\left(\mathsf{M}\left\{\left|\mu_p^{(01)}\right|^2\right\}+\mathsf{M}\left\{\left|\mu_p^{(10)}\right|^2\right\}\right)\le K\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \qquad (1.184)
$$

where constant K is independent of p.

According to Lemma 1.8 and (1.184), we obtain that $\mu_p^{(01)},\ \mu_p^{(10)}\to 0$ when $p\to\infty$ w. p. 1. Then

$$
I_{(01)T,t}^{(i_1i_1)p}\to I_{(01)T,t}^{(i_1i_1)},\quad I_{(10)T,t}^{(i_1i_1)p}\to I_{(10)T,t}^{(i_1i_1)}\quad \text{when}\quad p\to\infty\quad \text{w. p. 1.}
$$

Analogously, we have

$$I_{(02)T,t}^{(i_1 i_2)p} \to I_{(02)T,t}^{(i_1 i_2)}, \quad I_{(11)T,t}^{(i_1 i_2)p} \to I_{(11)T,t}^{(i_1 i_2)}, \quad I_{(20)T,t}^{(i_1 i_2)p} \to I_{(20)T,t}^{(i_1 i_2)} \quad \text{when} \quad p \to \infty \quad \text{w. p. 1},$$

where

$$I_{(02)T,t}^{(i_1 i_2)} = \int\limits_t^T (t-s)^2 \int\limits_t^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}, \quad I_{(20)T,t}^{(i_1 i_2)} = \int\limits_t^T \int\limits_t^s (t-\tau)^2 d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)},$$

$$I_{(11)T,t}^{(i_1 i_2)} = \int\limits_t^T (t-s) \int\limits_t^s (t-\tau) d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)},$$

$i_1, i_2 = 1, \ldots, m$. This result is based on the expansions of stochastic integrals $I_{(02)T,t}^{(i_1 i_2)}$, $I_{(20)T,t}^{(i_1 i_2)}$, $I_{(11)T,t}^{(i_1 i_2)}$ (see the formulas (5.27)–(5.29) in Chapter 5).

Let us denote

$$I_{(l)T,t}^{(i_1)} = \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_1)},$$

where $l = 0, 1, 2 \ldots$

The expansions (5.7)–(5.9), (5.30), (5.38) (see Chapter 5) for stochastic integrals $I_{(0)T,t}^{(i_1)}$, $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$, $I_{(3)T,t}^{(i_1)}$, $I_{(l)T,t}^{(i_1)}$ are correct w. p. 1 (they include $1, 2, 3, 4$, and $l+1$ members of expansion, correspondently).

1.7.2 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$)

In this section, we formulate and prove the theorem on convergence with probability 1 (w. p. 1) of expansions of iterated Itô stochastic integrals in Theorem 1.1 for the case of multiplicity k ($k \in \mathbf{N}$). This section is written on the base of Sect. 1.7.2 from [14]-[17] as well as on Sect. 6 from [31] and Sect. 9 from [29].

Let us remind the well known fact from the mathematical analysis, which is connected to existence of iterated limits.

Proposition 1.1. *Let* $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$\lim_{n,m \to \infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \textit{for any } m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \textit{for any } n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Theorem 1.10 [14]-[17], [27], [29], [31], [32]. *Let $\psi_l(\tau)$ $(l = 1,\ldots,k)$ are continuously differentiable nonrandom functions on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \textit{if} \quad p\to\infty$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is the expression on the right-hand side of (1.10) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1 = \ldots = p_k = p$, i.e. (see Theorem 1.1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),$$

where $i_1,\ldots,i_k = 1,\ldots,m$, rest notations are the same as in Theorem 1.1.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2, \tag{1.185}$$

where

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k \\ 0, & \text{otherwise}\end{cases} = \prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{1.186}$$

where $t_1, \ldots, t_k \in [t,T]$ for $k \ge 2$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$, $\mathbf{1}_A$ denotes the indicator of the set A,

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \qquad (1.187)$$

is the Fourier coefficient.

Using (1.186), we obtain

$$C_{j_k\ldots j_1} = \int\limits_t^T \phi_{j_k}(t_k)\psi_k(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1) dt_1 \ldots dt_k.$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \stackrel{\rm def}{=} \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

If $p_1 = \ldots = p_k = p$, then we also write

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C^2_{j_k\ldots j_1} \stackrel{\rm def}{=} \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty} \ldots \lim_{p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \stackrel{\rm def}{=} \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty} \lim_{p_2,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \stackrel{\rm def}{=} \sum_{j_1=0}^{\infty} \sum_{j_2,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}$$

and so on.

Let us consider the following lemma.

Lemma 1.9. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_k=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} \qquad (1.188)$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Proof. Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{1.189}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Obviously, the expression (1.189) defines the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le\sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2<\infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{1.190}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{1.191}$$

and equality of the limits (1.190) and (1.191).

Taking into account the above reasoning, we have

$$
\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{1.192}
$$

Since the limit

$$
\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2
$$

exists (see the Parseval equality (1.185)), then from Proposition 1.1 we have

$$
\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{1.193}
$$

Using (1.192) and Proposition 1.1, we get

$$
\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{1.194}
$$

Combining (1.194) and (1.193), we obtain

$$
\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2.
$$

Repeating the above steps, we complete the proof of Lemma 1.9.

Further, let us show that for $s=1,\ldots,k$

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{1.195}
$$

Using the arguments which we used in the proof of Lemma 1.9, we have

$$
\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=0}^{p}\sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2 \tag{1.196}
$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\}=\{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we obtain

$$
\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\ldots=
$$

$$
=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}\sum_{j_s=0}^{p}C_{j_k\ldots j_1}^2. \tag{1.197}
$$

Using (1.196), (1.197) and Lemma 1.9, we get

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-
$$

$$
-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.
$$

So, the equality (1.195) is proved.

Using the Parseval equality and Lemma 1.9, we obtain

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} +$$

$$+ \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 = \ldots =$$

$$= \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{p}\ldots\sum_{j_{k-1}=0}^{p}\sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 \le$$

$$\le \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{s=1}^{k}\left(\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2\right). \qquad (1.198)$$

Note that we use the following

$$
\sum_{j_1=0}^{p}\ldots\sum_{j_{s-1}=0}^{p}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 \le
$$

$$
\le\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 \le
$$

$$
\le\lim_{m_{s-1}\to\infty}\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 =
$$

$$
=\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 \le
$$

$$
\le\ldots\le
$$

$$
\le\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2
$$

to derive (1.198), where $m_1,\ldots,m_{s-1}>p$.

Denote

$$
C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,
$$

where $s=1,\ldots,k-1$.

Let us remind the Dini Theorem, which we will use further.

Theorem (Dini). *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a,b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a,b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a,b]$.*

For $s<k$ due to the Parseval equality and Dini Theorem as well as (1.195) we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 =
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 =
$$

$$
= \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-1}=0}^{\infty} \int\limits_t^T \psi_k^2(t_k) \left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =
$$

$$
= \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-2}=0}^{\infty} \int\limits_t^T \psi_k^2(t_k) \sum_{j_{k-1}=0}^{\infty} \left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =
$$

$$
= \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-2}=0}^{\infty} \int\limits_t^T \psi_k^2(t_k) \int\limits_t^{t_k} \psi_{k-1}^2(t_{k-1}) \left(C_{j_{k-2}\ldots j_1}(t_{k-1})\right)^2 \times
$$

$$
\times dt_{k-1} dt_k \le
$$

$$
\le C \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-2}=0}^{\infty} \int\limits_t^T \left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =
$$

$$
= C \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-3}=0}^{\infty} \int\limits_t^T \sum_{j_{k-2}=0}^{\infty} \left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =
$$

$$
= C \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-3}=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_{k-2}^2(\theta) \left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le
$$

$$
\le K \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-3}=0}^{\infty} \int\limits_t^T \left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le
$$

$$
\le \ldots \le
$$

$$
\le C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \int\limits_t^T \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =
$$

$$
= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \sum_{j_1=0}^{\infty} \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \qquad (1.199)
$$

where constants C, K depend on $T-t$ and constant C_k depends on k and $T-t$.

Let us explane more precisely how we obtain (1.199). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$
\sum_{j=0}^{\infty}\left(\int\limits_t^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2=
$$

$$
=\int\limits_t^{T}\left(\mathbf{1}_{\{s<\tau\}}\right)^2g^2(s)ds=\int\limits_t^{\tau}g^2(s)ds. \qquad (1.200)
$$

The equality (1.200) has been applied repeatedly when we obtaining (1.199). Using the replacement of integration order in Riemann integrals, we have

$$
C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s=
$$

$$
=\int\limits_t^{\tau}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{\tau}\phi_{j_2}(t_2)\psi_2(t_2)\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1\stackrel{\sf def}{=}
$$

$$
\stackrel{\sf def}{=}\tilde C_{j_s\ldots j_1}(\tau).
$$

For $l=1,\ldots,s$ we will use the following notation

$$
\tilde C_{j_s\ldots j_l}(\tau,\theta)=
$$

$$
=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.
$$

Using the Parseval equality and Dini Theorem, from (1.199) we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\le
$$

$$
\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^{T}\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
=C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^{T}\sum_{j_1=0}^{\infty}\left(\tilde C_{j_s\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = \tag{1.201}
$$

$$
= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = \tag{1.202}
$$

$$
= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t_1}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s \ldots j_3}(\tau, t_2)\right)^2 dt_2 dt_1 d\tau \le
$$

$$
\le C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s \ldots j_3}(\tau, t_2)\right)^2 dt_2 dt_1 d\tau \le
$$

$$
\le C_k^{'} \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s \ldots j_3}(\tau, t_2)\right)^2 dt_2 d\tau \le
$$

$$
\le \ldots \le
$$

$$
\le C_k^{''} \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_{s-1}^2(t_{s-1}) \left(\tilde{C}_{j_s}(\tau, t_{s-1})\right)^2 dt_{s-1} d\tau \le
$$

$$
\le \tilde{C}_k \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \left(\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta \right)^2 du d\tau, \tag{1.203}
$$

where constants $C_k^{'}$, $C_k^{''}$, $\tilde{C}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (1.203). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$
\sum_{j=0}^{\infty} \left(\int\limits_{\theta}^{\tau} \phi_j(s) g(s) ds \right)^2 = \sum_{j=0}^{\infty} \left(\int\limits_t^T \mathbf{1}_{\{\theta < s < \tau\}} \phi_j(s) g(s) ds \right)^2 =
$$

$$
= \int\limits_t^T \left(\mathbf{1}_{\{\theta < s < \tau\}}\right)^2 g^2(s) ds = \int\limits_{\theta}^{\tau} g^2(s) ds. \tag{1.204}
$$

The equality (1.204) has been applied repeatedly when we obtaining (1.203).

Let us explane more precisely the passing from (1.201) to (1.202) (the same steps have been used when we derive (1.203)).

We have

$$\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau-$$

$$-\sum_{j_2=0}^{n}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau=$$

$$=\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau=$$

$$=\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2 dt_1\Delta\tau_j, \tag{1.205}$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (1.9).

Since the non-decreasing functional sequence $u_n(\tau_j,t_1)$ and its limit function $u(\tau_j,t_1)$ are continuous on the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1, where

$$u_n(\tau_j,t_1)=\sum_{j_2=0}^{n}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2,$$

$$u(\tau_j,t_1)=\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2=\int\limits_{t_1}^{\tau_j}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau_j,t_2)\right)^2 dt_2,$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j,t_1)$ to $u(\tau_j,t_1)$ at the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1. As a result, we obtain

$$\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2<\varepsilon,\quad t_1\in[t,\tau_j] \tag{1.206}$$

for $n>N(\varepsilon)\in\mathbf{N}$ ($N(\varepsilon)$ exists for any $\varepsilon>0$ and it does not depend on t_1).

From (1.205) and (1.206) we obtain

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2 dt_1\Delta\tau_j\le$$

$$\leq \varepsilon \lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)dt_1\Delta\tau_j = \varepsilon\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)dt_1 d\tau. \qquad (1.207)$$

From (1.207) we get

$$\lim_{n\to\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau = 0.$$

This fact completes the proof of passing from (1.201) to (1.202).

Let us estimate the integral

$$\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta \qquad (1.208)$$

from (1.203) for the cases when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the estimates for the integral

$$\int\limits_t^{\tau}\phi_j(\theta)\psi(\theta)d\theta, \quad j\geq p+1, \qquad (1.209)$$

where $\psi(\theta)$ is a continuously differentiable function on the interval $[t,T]$, have been obtained in [6]-[17], [22], [33] (also see Sect. 2.2.5).

Let us estimate the integral (1.208) using the approach from [22], [33].

First, consider the case of Legendre polynomials. Then $\phi_j(s)$ is defined as follows

$$\phi_j(\theta)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(\theta-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j\geq 0,$$

where $P_j(x)$ $(j=0,1,2\ldots)$ is the Legendre polynomial.

Further, we have

$$\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta = \frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy =$$

$$
=\frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x))-P_{j-1}(z(x)))\psi(x)-(P_{j+1}(z(v))-P_{j-1}(z(v)))\psi(v)-
$$

$$
-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y)-P_{j-1}(y))\psi'(u(y))dy\Biggr), \qquad (1.210)
$$

where $x,v\in(t,T)$, $j\ge p+1$, $u(y)$ and $z(x)$ are defined by the following relations

$$
u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t},
$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (1.210) we used the following well known property of the Legendre polynomials

$$
\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),\quad j=1,2,\ldots
$$

From (1.210) and the well known estimate for the Legendre polynomials [115] (also see [121])

$$
|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbf{N},
$$

where constant K does not depend on y and j, it follows that

$$
\left|\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+C_1\right), \quad (1.211)
$$

where $j\in\mathbf{N}$, $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$ and constants C,C_1 do not depend on j.

From (1.211) we obtain

$$
\left(\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta\right)^2<\frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}}+\frac{1}{(1-(z(v))^2)^{1/2}}+C_3\right), \quad (1.212)
$$

where $j\in\mathbf{N}$, constants C_2,C_3 do not depend on j.

Let us apply (1.212) for the estimate of the right-hand side of (1.203). We have

$$\int\limits_t^T \int\limits_t^{\tau} \left(\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta)d\theta \right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2} \left(\int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^{1} \int\limits_{-1}^{x} \frac{dy}{(1-y^2)^{1/2}} dx + K_2 \right) \le$$

$$\le \frac{K_3}{j_s^2}, \tag{1.213}$$

where $j_s \in \mathbf{N}$, constants K_1, K_2, K_3 are independent of j_s.

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta) = \frac{1}{\sqrt{T-t}} \begin{cases} 1, & j=0 \\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1, \\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r \end{cases} \tag{1.214}$$

where $r = 1, 2, \ldots$

Using the system of functions (1.214), we have

$$\int\limits_v^x \phi_{2r-1}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}} \int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$= -\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(\psi(x)\cos\frac{2\pi r(x-t)}{T-t} - \psi(v)\cos\frac{2\pi r(v-t)}{T-t} -$$

$$- \int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\Biggr), \tag{1.215}$$

$$\int\limits_v^x \phi_{2r}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}} \int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$
= \sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(\psi(x)\sin\frac{2\pi r(x-t)}{T-t} - \psi(v)\sin\frac{2\pi r(v-t)}{T-t} -
$$

$$
- \int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\Biggr), \tag{1.216}
$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (1.215) and (1.216), we obtain for the trigonometric case

$$
\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 \le \frac{C_4}{j^2}, \tag{1.217}
$$

where $j \in \mathbf{N}$, constant C_4 is independent of j.

From (1.217) we finally have

$$
\int\limits_t^T \int\limits_t^\tau \left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le \frac{K_4}{j_s^2}, \tag{1.218}
$$

where $j_s \in \mathbf{N}$, constant K_4 is independent of j_s.

Combining (1.203), (1.213), and (1.218), we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \le
$$

$$
\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2} \le L_k\int\limits_p^{\infty}\frac{dx}{x^2} = \frac{L_k}{p}, \tag{1.219}
$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (1.198) and (1.219) we obtain

$$
\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 \le \frac{G_k}{p}, \tag{1.220}
$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use the estimate (1.74). Using (1.220) and the estimate (1.74) for the case $p_1 = \ldots = p_k = p$ and $n = 2$, we obtain

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le
$$

$$
\le C_{2,k}\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2\right)^2 \le
$$

$$
\le \frac{H_{2,k}}{p^2}, \tag{1.221}
$$

where

$$
C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!
$$

and $H_{2,k} = G_k^2 C_{2,k}$.

Let α and ξ_p in Lemma 1.8 be chosen as follows

$$
\alpha = 4, \quad \xi_p = \left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|.
$$

From (1.221) we obtain

$$
\sum_{p=1}^{\infty}\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty. \tag{1.222}
$$

Using Lemma 1.8 and the estimate (1.222), we have

$$
J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p \to \infty
$$

w. p. 1, where (see Theorem 1.1)

$$
J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - \right.
$$

$$
\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right) \tag{1.223}
$$

or (see Theorem 1.2)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \tag{1.224}$$

where $i_1,\ldots,i_k = 1,\ldots,m$ in (1.223) and (1.224). Theorem 1.10 is proved.

Remark 1.6. *From Theorem* 1.4 *and Lemma* 1.9 *we obtain*

$$\lim_{p_{q_1}\to\infty} \overline{\lim_{p_{q_2}\to\infty}} \ldots \overline{\lim_{p_{q_k}\to\infty}} \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k! \cdot \lim_{p_{q_1}\to 0} \ldots \lim_{p_{q_k}\to\infty} \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2 \right) =$$

$$= k! \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 \right) = 0$$

for the following cases:

1. $i_1,\ldots,i_k = 1,\ldots,m$ *and* $0 < T-t < \infty$,
2. $i_1,\ldots,i_k = 0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2 > 0,$ *and* $0 < T-t < 1$.

At that, $(q_1,\ldots,q_k)$ *is any permutation such that* $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$, $J[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1.5), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.10) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$, $\overline{\lim}$ *means* lim sup; *another notations are the same as in Theorem* 1.1.

Remark 1.7. *Taking into account Theorem* 1.4 *and the estimate* (1.220), *we obtain the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\} \le \frac{k!P_k(T-t)^k}{p}, \tag{1.225}$$

where $i_1,\ldots,i_k = 1,\ldots,m$ *and constant* P_k *depends only on* k.

The estimate (1.225) can be written in a slightly different form. Let us consider this question in more detail.

By analogy with (1.128) we have

$$\lim_{p\to\infty}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}C_{j_{m_k}\ldots j_{m_1}}=0, \tag{1.226}$$

where $(m_1,\ldots,m_k)$ *is any permutation of the set* $\{1,\ldots,k\}$ *such that* $(m_k,\ldots,m_1)\neq(k,\ldots,1)$; *braces mean an unordered set, and parentheses mean an ordered set.*

Further, using (1.226) and the estimate (1.220), we obtain

$$\left|\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}C_{j_{m_k}\ldots j_{m_1}}\right|=\left|\sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}C_{j_{m_k}\ldots j_{m_1}}-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}C_{j_{m_k}\ldots j_{m_1}}\right|\le$$

$$\le\left(\sum_{j_1,\ldots,j_k=0}^{\infty}-\sum_{j_1,\ldots,j_k=0}^{p}\right)\left|C_{j_k\ldots j_1}C_{j_{m_k}\ldots j_{m_1}}\right|\le$$

$$\le\frac{1}{2}\left(\sum_{j_1,\ldots,j_k=0}^{\infty}-\sum_{j_1,\ldots,j_k=0}^{p}\right)\left(C^2_{j_k\ldots j_1}+C^2_{j_{m_k}\ldots j_{m_1}}\right)=\left(\sum_{j_1,\ldots,j_k=0}^{\infty}-\sum_{j_1,\ldots,j_k=0}^{p}\right)C^2_{j_k\ldots j_1}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p}C^2_{j_k\ldots j_1}\le\frac{G_k}{p}, \tag{1.227}$$

where constant G_k *depends on* k *and* $T-t$.

Combining (1.76), (1.80), (1.220), and (1.227), we get

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\}\le\frac{\tilde P_k(T-t)^k}{p},$$

where $i_1,\ldots,i_k=1,\ldots,m$ *and constant* $\tilde P_k$ *depends only on* k.

It is easy to see that from the proof of Theorem 1.4 and (1.220) we obtain the estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\}\le\frac{Q_k}{p}, \tag{1.228}$$

where $i_1,\ldots,i_k=0,1,\ldots,m$ *and constant* Q_k *depends only on* k *and* $T-t$.

Remark 1.8. *The estimates (1.74) and (1.220) imply the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\} \le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{(P_k)^n\,(T-t)^{nk}}{p^n}, \qquad (1.229)$$

where $i_1,\ldots,i_k = 1,\ldots,m,$ $n \in \mathbf{N},$ and constant P_k depends only on k.

1.7.3 Rate of Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$

Consider the question on the rate of convergence w. p. 1 in Theorem 1.10. Using the inequality (1.229), we obtain

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\}\right)^{1/2n} \le \frac{Q_{n,k}}{\sqrt{p}}, \qquad (1.230)$$

where $n \in \mathbf{N}$ and

$$Q_{n,k} = k!\ (n(2n-1))^{(k-1)/2}\,((2n-1)!!)^{1/2n}\ \sqrt{P_k}\ (T-t)^{k/2}.$$

According to the Lyapunov inequality, we have

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{n}\right\}\right)^{1/n} \le \frac{Q_{n,k}}{\sqrt{p}} \qquad (1.231)$$

for all $n \in \mathbf{N}$. Following [105] (Lemma 2.1), we get

$$\left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right| = \frac{p^{1/2-\varepsilon}}{p^{1/2-\varepsilon}}\left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right| \le$$

$$\le \frac{1}{p^{1/2-\varepsilon}}\sup_{p\in\mathbf{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right) = \frac{\eta_\varepsilon}{p^{1/2-\varepsilon}} \qquad (1.232)$$

w. p. 1, where

$$\eta_\varepsilon = \sup_{p\in\mathbf{N}} \left(p^{1/2-\varepsilon} \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right| \right)$$

and $\varepsilon > 0$ is fixed.

For $q > 1/\varepsilon$, $q \in \mathbf{N}$ we obtain (see (1.231)) [105]

$$\mathsf{M}\left\{|\eta_\varepsilon|^q\right\} = \mathsf{M}\left\{ \left(\sup_{p\in\mathbf{N}} \left(p^{1/2-\varepsilon} \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right| \right) \right)^q \right\} =$$

$$= \mathsf{M}\left\{ \sup_{p\in\mathbf{N}} \left(p^{(1/2-\varepsilon)q} \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right|^q \right) \right\} \le$$

$$\le \mathsf{M}\left\{ \sum_{p=1}^{\infty} p^{(1/2-\varepsilon)q} \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right|^q \right\} =$$

$$= \sum_{p=1}^{\infty} p^{(1/2-\varepsilon)q} \mathsf{M}\left\{ \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right|^q \right\} \le$$

$$\le \sum_{p=1}^{\infty} p^{(1/2-\varepsilon)q} \frac{(Q_{q,k})^q}{p^{q/2}} = (Q_{q,k})^q \sum_{p=1}^{\infty} \frac{1}{p^{\varepsilon q}} < \infty. \qquad (1.233)$$

From (1.232) we obtain that for all $\varepsilon > 0$ there exists a random variable η_ε such that the inequality (1.232) is fulfilled w. p. 1 for all $p \in \mathbf{N}$. Moreover, from the Lyapunov inequality and (1.233), we obtain $\mathsf{M}\left\{|\eta_\varepsilon|^q\right\} < \infty$ for all $q \ge 1$.

1.8 Modification of Theorem 1.1 for the Case of Integration Interval $[t, s]$ $(s \in (t, T])$ of Iterated Itô Stochastic Integrals

1.8.1 Formulation and Proof of Theorem 1.1 Modification

Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on $[t, T]$. Define the following function on the hypercube $[t, T]^k$

$$\bar{K}(t_1,\ldots,t_k,s)={\bf 1}_{\{t_k<s\}}K(t_1,\ldots,t_k), \qquad (1.234)$$

where the function $K(t_1,\ldots,t_k)$ is defined by (1.6), $s\in(t,T]$ (s is fixed), and ${\bf 1}_A$ is the indicator of the set A. So, we have

$$\bar{K}(t_1,\ldots,t_k,s)={\bf 1}_{\{t_1<\ldots<t_k<s\}}\psi_1(t_1)\ldots\psi_k(t_k)=$$

$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s\\ \\ 0, & \text{otherwise}\end{cases}, \qquad (1.235)$$

where $k\ge 1$, $t_1,\ldots,t_k\in[t,T]$, and $s\in(t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $\bar{K}(t_1,\ldots,t_k,s)$ defined by (1.235) is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $\bar{K}(t_1,\ldots,t_k,s)\in L_2([t,T]^k)$ is converging to this function in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|\bar{K}(t_1,\ldots,t_k,s)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \qquad (1.236)$$

where

$$C_{j_k\ldots j_1}(s)=\int\limits_{[t,T]^k}\bar{K}(t_1,\ldots,t_k,s)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k=$$

$$=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (1.237)$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Note that

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \qquad (1.238)$$

$$=\int\limits_t^T\mathbf{1}_{\{t_k<s\}}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \hbox{w. p. 1},$$

where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=0,1,\ldots,m.$

Consider the partition $\{\tau_j\}_{j=0}^N$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \hbox{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \qquad (1.239)$$

Theorem 1.11 [15]–[17], [29]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (1.240)$$

where $J[\psi^{(k)}]_{s,t}$ is defined by (1.238), *$s\in(t,T]$ (s is fixed),*

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m,$*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (1.237), *$\Delta\mathbf{w}_{\tau_j}^{(i)}=$*

$\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$, *which satisfies the condition* (1.239).

Proof. Let us consider the multiple stochastic integrals (1.16), (1.23). We will write $J[\Phi]_{s,t}^{(k)}$ and $J'[\Phi]_{s,t}^{(k)}$ $(s\in(t,T]$, s is fixed) if the function $\Phi(t_1,\ldots,t_k)$ in (1.16) and (1.23) is replaced by $\mathbf{1}_{\{t_1,\ldots,t_k<s\}}\Phi(t_1,\ldots,t_k)$.

By analogy with (1.24), we have

$$J'[\Phi]_{s,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\mathbf{1}_{\{t_k<s\}}\sum\limits_{(t_1,\ldots,t_k)}\left(\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad\text{w. p. 1,}\tag{1.241}$$

where

$$\sum\limits_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$. At the same time permutations $(t_1,\ldots,t_k)$ when summing are performed in (1.241) only in the expression, which is enclosed in parentheses. Moreover, the nonrandom function $\Phi(t_1,\ldots,t_k)$ is assumed to be continuous in the corresponding closed domains of integration. The case when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domains of integration and bounded at their boundaries is also possible.

Let us write (1.241) as

$$J'[\Phi]_{s,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum\limits_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad\text{w. p. 1,}\tag{1.242}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed in (1.242) only in the expression $\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$.

It is not difficult to notice that (1.241), (1.242) can be rewritten in the form (see (1.25))

$$J'[\Phi]_{s,t}^{(k)}=\sum\limits_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad\text{w. p. 1,}\tag{1.243}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of

integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

According to Lemma 1.1, we have

$$
J[\psi^{(k)}]_{s,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\mathbf{1}_{\{\tau_{l_k}<s\}}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=
$$

$$
=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,}\quad (1.244)
$$

where $K(t_1,\ldots,t_k)$ is defined by (1.6) and permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression $K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$.

According to Lemmas 1.1, 1.3 and (1.24), (1.25), (1.242)–(1.244), we get the following representation

$$
J[\psi^{(k)}]_{s,t}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+
$$

$$
+R_{T,t,s}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\times
$$

$$
\times\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\ +\ R_{T,t,s}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.
$$

$$
\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t,s}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\times
$$

$$
\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t,s}^{p_1,\ldots,p_k}\quad \text{w. p. 1},
$$

where

$$
R_{T,t,s}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)\times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=
$$

$$
=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}- \tag{1.245}
$$

$$
-\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{1.246}
$$

w. p. 1, where permutations $(t_1,\ldots,t_k)$ when summing in (1.245) are performed only in the values $\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time permutations $(t_1,\ldots,t_k)$ when summing in (1.246) are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. Moreover, the indices near upper limits of integration in the iterated stochastic integrals in (1.245), (1.246) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t,s}^{p_1,\ldots,p_k}$ of the series. According to Lemma 1.2, we have

$$
\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times
$$

$$
\times dt_1\ldots dt_k, \tag{1.247}
$$

where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{s,t}$ and permutations $(t_1,\ldots,t_k)$ when summing in (1.247) are performed only in the values $\mathbf{1}_{\{t_k<s\}}$ and $dt_1\ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (1.247) are changed correspondently.

Since $K(t_1,\ldots,t_k)\equiv 0$ if the condition $t_1<\ldots<t_k$ is not fulfilled, then

$$
\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times
$$

$$
\times dt_1\ldots dt_k, \tag{1.248}
$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.248) are performed only in the values $dt_1\ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (1.248) are changed correspondently.

Then from (1.38), (1.236), and (1.248) we obtain

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\prod_{l=1}^k \phi_{j_l}(t_l)\right)^2 \times$$

$$\times dt_1\ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(\bar{K}(t_1,\ldots,t_k,s) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\prod_{l=1}^k \phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{s,t}$. Theorem 1.11 is proved.

Remark 1.9. *Obviously from Theorem* 1.11 *for the case* $s=T$ *we obtain Theorem* 1.1.

Remark 1.10. *It is not difficult to see that Theorem* 1.11 *is valid for the case when* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1.1.7 *for details*).

From Theorem 1.11 for the case of pairwise different numbers $i_1,\ldots,i_k = 1,\ldots,m$ we obtain

$$J[\psi^{(k)}]_{s,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}. \qquad (1.249)$$

Note that the expression on the right-hand side of (1.249) coincides for the case $k=1$, $\psi_1(t_1)\equiv 1$ with the right-hand side of the formula (6.2) (approximation of the increment of the Wiener process based on its series expansion).

Remark 1.11. *Note that by analogy with the proof of estimate* (1.220) *we obtain the following inequality*

$$\int\limits_{[t,T]^k} \bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2(s) \le \frac{G_k(s)}{p}, \qquad (1.250)$$

where $\bar{K}(t_1,\ldots,t_k,s)$ *and* $C_{j_k\ldots j_1}(s)$ *are defined by the equalities* (1.234) *and* (1.237), *respectively; constant* $G_k(s)$ *depends on* k *and* $s-t$ ($s\in(t,T]$, s *is fixed*).

The following obvious modification of Theorem 1.4 takes place.

Theorem 1.12. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1.1.7). *Then*

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k(s)\left(\int\limits_{[t,T]^k}\bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2(s)\right), \quad (1.251)
$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, *constant* $C_k(s)$ *depends only on* k *and* $s-t$. *Moreover,* $C_k(s)\le k!$ *for the following cases:*

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{s,t}$ *is the stochastic integral* (1.238), $J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.240) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$, $\bar{K}(t_1,\ldots,t_k,s)$ *and* $C_{j_k\ldots j_1}(s)$ *are defined by the equalities* (1.234) *and* (1.237), *respectively;* $s\in(t,T]$ (s *is fixed*); *another notations are the same as in Theorem* 1.11.

Remark 1.12. *Combining the estimates* (1.250) *and* (1.251), *we obtain*

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p,\ldots,p}\right)^2\right\}\le\frac{Q_k(s)}{p}, \quad (1.252)
$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, *constant* $Q_k(s)$ *depends only on* k *and* $s-t$; *another notations are the same as in* (1.250) *and* (1.251).

Remark 1.13. *An analogue of the estimate* (1.74) *for the iterated Itô stochastic integral* (1.238) *has the following form*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le
$$

$$
\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \times
$$

$$
\times\ \left(\ \int\limits_{[t,T]^k}\bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2(s)\right)^n, \quad (1.253)
$$

where $J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.240) *before passing to the limit*

$$
\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ ,
$$

$\bar{K}(t_1,\ldots,t_k,s)$ *and* $C_{j_k\ldots j_1}(s)$ *are defined by the equalities* (1.234) *and* (1.237), *respectively;* $s\in(t,T]$ (s *is fixed*); $i_1,\ldots,i_k=1,\ldots,m.$

Remark 1.14. *The estimates* (1.250) *and* (1.253) *imply the following inequality*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p,\ldots,p}\right)^{2n}\right\}\le
$$

$$
\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{(P_k)^n\,(s-t)^{nk}}{p^n},
$$

where $i_1,\ldots,i_k=1,\ldots,m$, $n\in\mathbf{N}$, *and constant* P_k *depends only on* k.

1.8.2 Expansions of Iterated Itô Stochastic Integrals with Multiplicities 1 to 5 and Miltiplicity k Based on Theorem 1.11

Consider particular cases of Theorem 1.11 for $k=1,\ldots,5$

$$
J[\psi^{(1)}]_{s,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}(s)\zeta_{j_1}^{(i_1)}, \quad (1.254)
$$

$$
J[\psi^{(2)}]_{s,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \quad (1.255)
$$

$$J[\psi^{(3)}]_{s,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}(s)\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad (1.256)$$

$$J[\psi^{(4)}]_{s,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}(s)\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \quad (1.257)$$

$$J[\psi^{(5)}]_{s,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}(s)\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right),
$$

where $\mathbf{1}_A$ is the indicator of the set A, $C_{j_k\ldots j_1}(s)$ $(k=1,\ldots,5)$ has the form (1.237), $s\in(t,T]$ (s is fixed).

Consider a generalization of the above formulas

$$
J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),
$$

where $k\in\mathbf{N}$, $C_{j_k\ldots j_1}(s)$ has the form (1.237); another notations are the same as in Theorem 1.2.

1.9 Expansion of Multiple Wiener Stochastic Integral Based on Generalized Multiple Fourier Series

Let us consider the multiple stochastic integral (1.23)

$$
\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J'[\Phi]_{T,t}^{(k)},\qquad(1.258)
$$

where for simplicity we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}^1$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (1.9).

The stochastic integral with respect to the scalar standard Wiener process $(i_1=\ldots=i_k\ne 0)$ and similar to (1.258) was considered in [106] and is called the multiple Wiener stochastic integral [106]. Note that $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ in [106] (this case will be considered in Sect. 1.11, 1.12).

Consider the following theorem on expansion of the multiple Wiener stochastic integral (1.258) based on generalized multiple Fourier series.

Theorem 1.13.[8] *Suppose that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}^1$ is a continuous nonrandom function on $[t,T]^k$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Then the following expansions*

$$J'[\Phi]_{T,t}^{(k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),\qquad(1.259)$$

$$J'[\Phi]_{T,t}^{(k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)$$

$$(1.260)$$

converging in the mean-square sense are valid, where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k\qquad(1.261)$$

[8] Theorem 1.13 will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ (see Sect. 1.11, Theorem 1.17).

is the Fourier coefficient, $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$, *which satisfies the condition* (1.9); $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.2.

Proof. Using Lemma 1.3 and (1.24), (1.25), we get the following representation

$$J'[\Phi]_{T,t}^{(k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$
+R_{T,t}^{p_1,\ldots,p_k} \quad \text{w. p. 1},
$$

where

$$
R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(\Phi(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right) \times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series using Lemma 1.2 and (1.38). We have

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le
$$

$$
\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(\Phi(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times
$$

$$
\times dt_1 \ldots dt_k =
$$

$$
= C_k \int\limits_{[t,T]^k} \left(\Phi(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times
$$

$$
\times dt_1 \ldots dt_k \to 0
$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(k)}$. The expansion (1.259) is proved. Using (1.259) and Remark 1.2, we get the expansion (1.260) (see Theorem 1.2). Theorem 1.13 is proved.

Note that particular cases of the expansion (1.260) are determined by the equalities (1.45)–(1.51), in which the Fourier coefficient $C_{j_k\ldots j_1}$ $(k = 1,\ldots,7)$ has the form (1.261).

1.10 Reformulation of Theorems 1.1, 1.2, and 1.13 Using Hermite Polynomials

In [107] it was noted that Theorem 3.1 ([106], p. 162) can be applied to the case of multiple Wiener stochastic integral with respect to components of the multidimensional Wiener process. As a result, Theorems 1.1, 1.2, and 1.13 can be reformulated using Hermite polynomials. Consider this approach using our notations. Note that we derive the formula (1.266) (see below) in two different ways. One of them is not based on Theorem 3.1 [106] (see the proof of Theorem 1.22 below for details).

We will say that the condition $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ $(i_1,\ldots,i_k=0,1,\ldots,m)$ *if* $m_1,\ldots,m_k$ *are multiplicities of the elements* $i_1,\ldots,i_k$, *respectively, i.e.*

$$\{i_1,\ldots,i_k\}=\{\overbrace{i_1,\ldots,i_1}^{m_1},\overbrace{i_2,\ldots,i_2}^{m_2},\ldots,\overbrace{i_r,\ldots,i_r}^{m_r}\}\quad (m_{r+1}=\ldots=m_k=0),$$

where $r=1,\ldots,k$, *braces mean an unordered set, and parentheses mean an ordered set. At that,* $m_1+\ldots+m_k=k$, $m_1,\ldots,m_k=0,1,\ldots,k$, *and all elements with nonzero multiplicities are pairwise different.*

In this section, we consider the case $i_1,\ldots,i_k=0,1,\ldots,m$. Note that in [107] the case $i_1,\ldots,i_k=1,\ldots,m$ was considered.

Let the condition $(\star\star)$ is fulfilled for the multi-index $(i_1\ldots i_k)$. Then

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=J'\Biggl[\underbrace{\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}}_{m_1}\underbrace{\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}}_{m_2}\ldots$$

$$\ldots\underbrace{\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}}_{m_k}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1}\overbrace{i_2\ldots i_2}^{m_2}\ldots\overbrace{i_k\ldots i_k}^{m_k})}\tag{1.262}$$

w. p. 1, where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.23) (also see (1.258)), $\Phi(t_1,\ldots,t_k)=\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7), $\{j_{g_1},\ldots,j_{g_{m_1+m_2+\ldots+m_k}}\}=\{j_1,\ldots,j_k\}$.

From (1.262) we have

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=J'\left[\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}\right]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1})}\cdot J'\left[\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}\right]_{T,t}^{(\overbrace{i_2\ldots i_2}^{m_2})}\cdot\ldots$$

$$\ldots \cdot J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}\right]_{T,t}^{(\overbrace{i_k\ldots i_k}^{m_k})} \qquad (1.263)$$

w. p. 1, where

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})} \stackrel{\rm def}{=} 1 \quad \text{for} \quad m_l=0. \qquad (1.264)$$

The detailed proof of the equality (1.263) will be given in Sect. 1.14 (see the proof of Theorem 1.22).

Let us consider the following multiple Wiener stochastic integral

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})} \quad (m_l>0),$$

where we suppose that

$$\left\{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}},\ldots,j_{g_{m_1+m_2+\ldots+m_l}}\right\}=$$

$$=\{\underbrace{j_{h_{1,l}},\ldots,j_{h_{1,l}}}_{n_{1,l}},\underbrace{j_{h_{2,l}},\ldots,j_{h_{2,l}}}_{n_{2,l}},\ldots,\underbrace{j_{h_{d_l,l}},\ldots,j_{h_{d_l,l}}}_{n_{d_l,l}}\}, \qquad (1.265)$$

where $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$. Note that the numbers $m_1,\ldots,m_k,\ g_1,\ldots,g_k$ depend on $(i_1,\ldots,i_k)$ and the numbers $n_{1,l},\ldots,n_{d_l,l},\ h_{1,l},\ldots,h_{d_l,l},\ d_l$ depend on $\{j_1,\ldots,j_k\}$. Moreover, $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$.

Using Theorem 3.1 [106], we get w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l=0\end{cases} \quad (m_l>0), \qquad (1.266)$$

where $H_n(x)$ is the Hermite polynomial of degree n

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)=n!\sum_{m=0}^{[n/2]}\frac{(-1)^m x^{n-2m}}{m!(n-2m)!2^m} \quad (n\in\mathbf{N}), \quad (1.267)$$

and $\zeta_j^{(i)}$ $(i=0,1,\ldots,m,\ j=0,1,\ldots)$ is defined by (1.11).

For example,

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=x^2-1,$$

$$H_3(x)=x^3-3x,\quad H_4(x)=x^4-6x^2+3,$$

$$H_5(x)=x^5-10x^3+15x.$$

From (1.264) and (1.266) we obtain w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}, \tag{1.268}$$

where $\mathbf{1}_A$ denotes the indicator of the set A.

Using (1.263) and (1.268), we get w. p. 1

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right), \tag{1.269}$$

where notations are the same as in (1.265) and (1.266).

The equality (1.269) allows us to reformulate Theorems 1.1, 1.2, and 1.13 using the Hermite polynomials.[9]

[9] Theorems 1.14, 1.15 (see below) will be generalized to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ in Sect. 1.11 (see Theorems 1.16, 1.17).

Theorem 1.14 [29] (reformulation of Theorems 1.1 and 1.2). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1 \ldots i_k)$ *and the condition* (1.265) *is also fulfilled. Furthermore, let every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l\neq 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l=0\end{cases}\right) \tag{1.270}$$

converging in the mean-square sense is valid, where we denote the stochastic integral (1.5) *as* $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$; $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$; $m_1+\ldots+m_k=k$; *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (1.267); *another notations are the same as in Theorem* 1.1.

Theorem 1.15 [29] (reformulation of Theorem 1.13). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1 \ldots i_k)$ *and the condition* (1.265) *is also fulfilled. Furthermore, let* $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}^1$ *is a continuous nonrandom function on* $[t,T]^k$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l\neq 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l=0\end{cases}\right)$$

converging in the mean-square sense is valid, where we denote the multiple Wiener stochastic integral (1.258) *as* $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$; $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$; $m_1+\ldots+m_k=k$; *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (1.267); *another notations are the same as in Theorem* 1.13.

From (1.268) we have w. p. 1

$$J'[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{k}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=\begin{cases}H_k\left(\zeta_{j_1}^{(i_1)}\right), & \text{if} \quad i_1\ne 0\\ \\ \left(\zeta_{j_1}^{(0)}\right)^k, & \text{if} \quad i_1=0\end{cases}\qquad (k>0). \qquad (1.271)$$

Let us show how the relation (1.271) can be obtained from Theorem 1.2. To prove (1.271) using Theorem 1.2 we choose $i_1=\ldots=i_k$ and $j_1=\ldots=j_k$ $(i_1=0,1,\ldots,m)$ in the following formula (this formula follows from a comparison of (1.43) and (1.54) or can be obtained using the recurrence relation (1.391))

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \qquad (1.272)$$

w. p. 1, where notations are the same as in Theorem 1.2.

The case $i_1=0$ of (1.271) is obvious. Simple combinatorial reasoning shows that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}=$$

$$=\frac{C_k^2\cdot C_{k-2}^2\cdot\ldots\cdot C_{k-(r-1)2}^2}{r!}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r}, \qquad (1.273)$$

where $i_1=\ldots=i_k$, $j_1=\ldots=j_k$ $(i_1=1,\ldots,m)$, and

$$C_k^l=\frac{k!}{l!(k-l)!}$$

is the binomial coefficient.

We have

$$\frac{C_k^2\cdot C_{k-2}^2\cdot\ldots\cdot C_{k-(r-1)2}^2}{r!}=\frac{k!}{r!(k-2r)!2^r}. \tag{1.274}$$

Combining (1.272), (1.273), and (1.274), we get w. p. 1

$$J'[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{k}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=\left(\zeta_{j_1}^{(i_1)}\right)^k+k!\sum_{r=1}^{[k/2]}\frac{(-1)^r}{r!(k-2r)!2^r}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r}=$$

$$=k!\sum_{r=0}^{[k/2]}\frac{(-1)^r}{r!(k-2r)!2^r}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r}=H_k\left(\zeta_{j_1}^{(i_1)}\right).$$

The relation (1.271) is proved using (1.272).

From (1.269) and (1.272) we obtain the following equalities for multiple Wiener stochastic integral

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right) \tag{1.275}$$

w. p. 1, where notations are the same as in Theorem 1.2 and (1.265), (1.266).

Let us make a remark about how it is possible to obtain the formula (1.266) without using Theorem 3.1 [106].

Consider the set of polynomials $H_n(x,y)$, $n=0,1,\ldots$ defined by [108]

$$H_n(x,y)=\left.\left(\frac{d^n}{d\alpha^n}e^{\alpha x-\alpha^2 y/2}\right)\right|_{\alpha=0}\ \ \ \left(H_0(x,y)\stackrel{\sf def}{=}1\right). \tag{1.276}$$

It is well known that polynomials $H_n(x,y)$ are connected with the Hermite polynomials (1.267) by the formula [108]

$$H_n(x,y)=y^{n/2}H_n\left(\frac{x}{\sqrt{y}}\right)=n!\sum_{i=0}^{[n/2]}\frac{(-1)^i x^{n-2i}y^i}{i!(n-2i)!2^i}. \tag{1.277}$$

For example,

$$H_1(x,y)=x,$$
$$H_2(x,y)=x^2-y,$$
$$H_3(x,y)=x^3-3xy,$$
$$H_4(x,y)=x^4-6x^2y+3y^2,$$
$$H_5(x,y)=x^5-10x^3y+15xy^2.$$

From (1.267) and (1.277) we get

$$H_n(x,1)=H_n(x). \tag{1.278}$$

Obviously, without loss of generality, we can write

$$(j_1\ldots j_k)=\bigl(\underbrace{j_1\ldots j_1}_{m_1}\ \underbrace{j_2\ldots j_2}_{m_2}\ \ldots\ \underbrace{j_r\ldots j_r}_{m_r}\bigr), \tag{1.279}$$

where $m_1+\ldots+m_r=k$, $m_1,\ldots,m_r=1,\ldots,k$, $r=1,\ldots,k$, $k>0$, and $j_1,\ldots,j_r$ are pairwise different.

Analyzing the proof of Theorem 1.1 and using (1.338), (1.361) (see the proof of Theorem 1.22 below), we can notice that w. p. 1 (we suppose that the condition (1.279) is fulfilled)

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_g;\ q\ne g;\ q,g=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}=$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_{m_1}=0\\ l_q\ne l_g;\ q\ne g;\ q,g=1,\ldots,m_1}}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\times
$$

$$
\times \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_{m_1+1},\ldots,l_{m_1+m_2}=0\\ l_q\ne l_g;\ q\ne g;\ q,g=m_1+1,\ldots,m_1+m_2}}^{N-1} \phi_{j_2}(\tau_{l_{m_1+1}})\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\times
$$

$$
\ldots
$$

$$
\times \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_{k-m_r+1},\ldots,l_k=0\\ l_q\ne l_g;\ q\ne g;\ q,g=k-m_r+1,\ldots,k}}^{N-1} \phi_{j_r}(\tau_{l_{k-m_r+1}})\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}=
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\sum_{l_{m_1}=0}^{N-1}\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_1,\ldots,l_{m_1})\in \mathrm{G}'_{1,m_1}}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\right)\times
$$

$$
\times \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{m_1+1}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\sum_{l_{m_1+m_2}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_{m_1+1},\ldots,l_{m_1+m_2})\in \mathrm{G}'_{m_1+1,m_1+m_2}}\phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\right)\times
$$

$$
\ldots
$$

$$
\times \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{k-m_r+1}=0}^{N-1}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\sum_{l_k=0}^{N-1}\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_{k-m_r+1},\ldots,l_k)\in \mathrm{G}'_{k-m_r+1,k}}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{k-m_r+1}}^{(i_1)}\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}\right),
$$

where the set $\mathrm{G}'_{m,n}$ is defined according to the same rule as the set G_k in (1.10). However, the elements of the set $\mathrm{G}'_{m,n}$ are the numbers $l_m,\ldots,l_n$ $(n>m)$, while the elements of the set G_k are the numbers $l_1,\ldots,l_k$.

We have (see the proof of Theorem 1.1) w. p. 1 $(i_1\neq 0)$

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\sum_{l_{m_1}=0}^{N-1}\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_1,\ldots,l_{m_1})\in\mathrm{G}'_{1,m_1}}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\right)=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\ \left(\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1}+\sum_{r=1}^{[m_1/2]}(-1)^r\times\right.
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{m_1-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{m_1-2r}\}=\{1,2,\ldots,m_1\}}}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r\times
$$

$$
\left.\times\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}\right)=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\ \left(\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1}+\sum_{r=1}^{[m_1/2]}\frac{(-1)^r m_1!}{r!(m_1-2r)!2^r}\times\right.
$$

$$
\left.\times\left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}\right)=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{r=0}^{[m_1/2]}\frac{(-1)^r m_1!}{r!(m_1-2r)!2^r}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r\times
$$

$$
\times\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}=
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_1}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right),
$$

where notations are the same as in Theorems 1.1, 1.2.

Similarly we get w. p. 1

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \left(\sum_{l_{m_1+1}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\sum_{l_{m_1+m_2}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_{m_1+1},\ldots,l_{m_1+m_2})\in\mathrm{G}'_{m_1+1,m_1+m_2}}\phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\right)=
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_2}\left(\sum_{l_1=0}^{N-1}\phi_{j_2}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_2}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right),
$$

$$
\ldots
$$

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \left(\sum_{l_{k-m_r+1}=0}^{N-1}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\sum_{l_k=0}^{N-1}\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_{k-m_r+1},\ldots,l_k)\in\mathrm{G}'_{k-m_r+1,k}}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{k-m_r+1}}^{(i_1)}\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}\right)=
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_r}\left(\sum_{l_1=0}^{N-1}\phi_{j_r}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_r}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right).
$$

Then

$$
J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_1}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)\times
$$

$$\times \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_2}\left(\sum_{l_1=0}^{N-1}\phi_{j_2}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_2}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)\times\ldots$$

$$\ldots\times \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_r}\left(\sum_{l_1=0}^{N-1}\phi_{j_r}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_r}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right) \qquad (1.280)$$

w. p. 1 for $i_1\ne 0$ and

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(0\ldots 0)}=\lim_{N\to\infty}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\tau_{l_1}\right)^{m_1}\ldots\left(\sum_{l_r=0}^{N-1}\phi_{j_r}(\tau_{l_r})\Delta\tau_{l_r}\right)^{m_r}=$$

$$=\left(\int\limits_t^T\phi_{j_1}(s)ds\right)^{m_1}\ldots\left(\int\limits_t^T\phi_{j_r}(s)ds\right)^{m_r}=\left(\zeta_{j_1}^{(0)}\right)^{m_1}\ldots\left(\zeta_{j_r}^{(0)}\right)^{m_r} \qquad (1.281)$$

for $i_1=0$, where we suppose that the condition (1.279) is fulfilled; also we use in (1.280) and (1.281) the same notations as in the proof of Theorem 1.1.

Applying (1.277), (1.278), Lemma 1.3, and Remark 1.2 to the right-hand side of (1.280), we finally obtain w. p. 1

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=H_{m_1}\left(\int\limits_t^T\phi_{j_1}(s)d\mathbf{w}_s^{(i_1)},\int\limits_t^T\phi_{j_1}^2(s)ds\right)\times$$

$$\times H_{m_2}\left(\int\limits_t^T\phi_{j_2}(s)d\mathbf{w}_s^{(i_1)},\int\limits_t^T\phi_{j_2}^2(s)ds\right)\ldots H_{m_r}\left(\int\limits_t^T\phi_{j_r}(s)d\mathbf{w}_s^{(i_1)},\int\limits_t^T\phi_{j_r}^2(s)ds\right)=$$

$$=H_{m_1}\left(\zeta_{j_1}^{(i_1)},1\right)H_{m_2}\left(\zeta_{j_2}^{(i_1)},1\right)\ldots H_{m_r}\left(\zeta_{j_r}^{(i_1)},1\right)=$$

$$=H_{m_1}\left(\zeta_{j_1}^{(i_1)}\right)H_{m_2}\left(\zeta_{j_2}^{(i_1)}\right)\ldots H_{m_r}\left(\zeta_{j_r}^{(i_1)}\right)$$

for $i_1\ne 0$, where we suppose that the condition (1.279) is fulfilled. Thus, an equality similar to (1.266) is proved without using Theorem 3.1 [106].

Consider particular cases of the equality (1.275) for $k = 1, \ldots, 4$ and $i_1, \ldots, i_4 = 1, \ldots, m$ (see (1.45)–(1.48)). We have w. p. 1

$$J'[\phi_{j_1}]_{T,t}^{(i_1)} = \zeta_{j_1}^{(i_1)} = H_1\left(\zeta_{j_1}^{(i_1)}\right);$$

$$J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_2)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}} =$$

$$= \begin{cases} H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_2)}\right), & \text{if} \quad i_1 = i_2,\ j_1 = j_2 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right), & \text{otherwise} \end{cases}; \qquad (1.282)$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_1 i_1)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_1)} =$$

$$= \begin{cases} H_3\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 = j_2 = j_3 \\ \\ H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 = j_2 \neq j_3 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_2\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_2 = j_3 \neq j_1 \\ \\ H_0\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_2\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 = j_3 \neq j_2 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 \neq j_2,\ j_2 \neq j_3,\ j_1 \neq j_3 \end{cases}; \qquad (1.283)$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_3)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} = H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right),$$

where i_1, i_2, i_3 are pairwise different;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_1 i_3)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} =$$

$$= \left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}}\right)\zeta_{j_3}^{(i_3)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_3}]_{T,t}^{(i_3)} =$$

$$
= \begin{cases} H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), & \text{if} \quad j_1 = j_2 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), & \text{if} \quad j_1 \neq j_2 \end{cases},
$$

where $i_1 = i_2 \neq i_3$;

$$
J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_2)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} - \mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} =
$$

$$
= \zeta_{j_1}^{(i_1)}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} - \mathbf{1}_{\{j_2=j_3\}}\right) = J'[\phi_{j_1}]_{T,t}^{(i_1)} J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)} =
$$

$$
= \begin{cases} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_2\left(\zeta_{j_2}^{(i_2)}\right) H_0\left(\zeta_{j_3}^{(i_2)}\right), & \text{if} \quad j_2 = j_3 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_2)}\right), & \text{if} \quad j_1 \neq j_2 \end{cases},
$$

where $i_1 \neq i_2 = i_3$;

$$
J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_1)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)} =
$$

$$
= \zeta_{j_2}^{(i_2)}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right) = J'[\phi_{j_2}]_{T,t}^{(i_2)} J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} =
$$

$$
= \begin{cases} H_2\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 = j_3 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 \neq j_3 \end{cases},
$$

where $i_1 = i_3 \neq i_2$;

$$
J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1 i_1)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}\zeta_{j_4}^{(i_1)} -
$$

$$
-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_1)}\zeta_{j_4}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_1)}\zeta_{j_4}^{(i_1)} - \mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)} -
$$

$$
-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)} - \mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} +
$$

$$
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{j_2=j_4\}} + \mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{j_2=j_3\}} =
$$

$$
= \begin{cases}
H_4\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{I}) \\
H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{II}) \\
H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{III}) \\
H_0\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_2\left(\zeta_{j_3}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{IV}) \\
H_0\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right) H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{V}) \\
H_1\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_2\left(\zeta_{j_3}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VI}) \\
H_1\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right) H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VII}) \\
H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VIII}) \\
H_3\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{IX}) \\
H_1\left(\zeta_{j_1}^{(i_1)}\right) H_3\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{X}) \\
H_0\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right) H_3\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XI}) \\
H_0\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_3\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XII}) \\
H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XIII}) \\
H_2\left(\zeta_{j_1}^{(i_1)}\right) H_2\left(\zeta_{j_2}^{(i_1)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right) H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XIV}) \\
H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_2\left(\zeta_{j_3}^{(i_1)}\right) H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XV})
\end{cases},
$$

where $H_n(x)$ is the Hermite polynomial (1.267) of degree n and (I)–(XV) are the following conditions

(I). $j_1 = j_2 = j_3 = j_4$,

(II). j_1, j_2, j_3, j_4 are pairwise different,

(III). $j_1 = j_2 \neq j_3, j_4;\ j_3 \neq j_4$,

(IV). $j_1 = j_3 \neq j_2, j_4;\ j_2 \neq j_4$,

(V). $j_1 = j_4 \neq j_2, j_3;\ j_2 \neq j_3$,

(VI). $j_2 = j_3 \neq j_1, j_4;\ j_1 \neq j_4$,

(VII). $j_2 = j_4 \neq j_1, j_3;\ j_1 \neq j_3$,

(VIII). $j_3 = j_4 \neq j_1, j_2;\ j_1 \neq j_2$,

(IX). $j_1 = j_2 = j_3 \neq j_4$,

(X). $j_2 = j_3 = j_4 \neq j_1$,

(XI). $j_1 = j_2 = j_4 \neq j_3$,

(XII). $j_1 = j_3 = j_4 \neq j_2$,

(XIII). $j_1 = j_2 \neq j_3 = j_4$,

(XIV). $j_1 = j_3 \neq j_2 = j_4$,

(XV). $j_1 = j_4 \neq j_2 = j_3$.

Moreover, from (1.263) we have w. p. 1

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_4)} = H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right),$$

where i_1, i_2, i_3, i_4 are pairwise different;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_4)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_3}^{(i_3)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right), \qquad (1.284)$$

where $i_1 = i_2 \neq i_3, i_4;\ i_3 \neq i_4$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_4)} = J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right), \qquad (1.285)$$

where $i_1 = i_3 \neq i_2, i_4;\ i_2 \neq i_4$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_1)} = J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), \qquad (1.286)$$

where $i_1 = i_4 \neq i_2, i_3;\ \ i_2 \neq i_3;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_4)} = J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right), \tag{1.287}$$

where $i_2 = i_3 \neq i_1, i_4;\ \ i_1 \neq i_4;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_2)} = J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), \tag{1.288}$$

where $i_2 = i_4 \neq i_1, i_3;\ \ i_1 \neq i_3;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_3)} = J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3 i_3)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right), \tag{1.289}$$

where $i_3 = i_4 \neq i_1, i_2;\ \ i_1 \neq i_2;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1 i_4)} = J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_4}^{(i_4)}\right), \tag{1.290}$$

where $i_1 = i_2 = i_3 \neq i_4;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_2)} = J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2 i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right), \tag{1.291}$$

where $i_2 = i_3 = i_4 \neq i_1;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_1)} = J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_3}^{(i_3)}\right), \tag{1.292}$$

where $i_1 = i_2 = i_4 \neq i_3;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_1)} = J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right), \tag{1.293}$$

where $i_1 = i_3 = i_4 \neq i_2;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_3)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3 i_3)}, \tag{1.294}$$

where $i_1 = i_2 \neq i_3 = i_4;$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_2)} = J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2 i_2)}, \tag{1.295}$$

where $i_1 = i_3 \neq i_2 = i_4$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_1)} = J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)}, \tag{1.296}$$

where $i_1 = i_4 \neq i_2 = i_3$.

Note that the right-hand sides of (1.284)–(1.296) contain multiple Wiener stochastic integrals of multiplicities 2 and 3. These integrals are considered in detail in (1.282), (1.283).

It should be noted that the formulas (1.54) (Theorem 1.2) and (1.270) (Theorem 1.14) are interesting from various points of view. The formulas (1.45)–(1.50) (these formulas are particular cases of (1.54) for $k = 1, \ldots, 6$) are convenient for numerical modeling of iterated Itô stochastic integrals of multiplicities 1 to 6 (see Chapter 5). For example, in [53] and [54], approximations of iterated Itô stochastic integrals of multiplicities 1 to 6 in the Python programming language were successfully implemented using (1.45)–(1.50) and Legendre polynomials.

On the other hand, the equality (1.270) is interesting by a number of reasons. Firstly, this equality connects Itô's results on multiple Wiener stochastic integral ([106], Theorem 3.1) with the theory of mean-square approximation of iterated Itô stochastic integrals presented in this book. Secondly, the equality (1.270) is based on the Hermite polynomials, which have the orthogonality property on $\mathbf{R}$ with a Gaussian weight. This feature opens up new possibilities in the study of iterated Itô stochastic integrals. Note that the indicated orthogonality property is indirectly reflected by the formula (1.85) (see the proof of Theorem 1.3).

1.11 Generalization of Theorems 1.1, 1.2, 1.14, and 1.15 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$, $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$

In this section, we will use the definition of the multiple Wiener stochastic integral from [106], [109] to generalize Theorems 1.1, 1.2, 1.14, and 1.15 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$, $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$.

Consider the following step function on the hypercube $[t,T]^k$

$$\Phi_N(t_1,\ldots,t_k)=\sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{1.297}$$

where $a_{l_1\ldots l_k}\in\mathbf{R}$ and such that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$,

$$\mathbf{1}_A(\tau)=\begin{cases}1 & \text{if } \tau\in A\\ \\ 0 & \text{otherwise}\end{cases},$$

$N\in\mathbf{N}$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (1.9):

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{1.298}$$

Let us define the multiple Wiener stochastic integral for $\Phi_N(t_1,\ldots,t_k)$ [106], [109]

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}\stackrel{\sf def}{=}\sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{1.299}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

It is known (see [109], Lemma 9.6.4) that for any $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ there exists a sequence of step functions $\Phi_N(t_1,\ldots,t_k)$ of the form (1.297) such that

$$\lim_{N\to\infty}\int\limits_{[t,T]^k}\left(\Phi(t_1,\ldots,t_k)-\Phi_N(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k=0. \tag{1.300}$$

We have

$$\Phi_N(t_1,\ldots,t_k)=\sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k)=$$

$$=\sum_{(l_1,\ldots,l_k)}\ \sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1} a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{1.301}$$

where permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$ (recall that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$).

Using (1.301), we get

$$\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{1.302}$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}, \tag{1.303}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$ and permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$. At the same time the indices near upper limits of integration in the iterated stochastic integrals in (1.302) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$ (see (1.302)). In addition, the multiple Wiener stochastic integral $J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.299) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Itô stochastic integral.

Using (1.300), (1.303), Lemma 1.2 for $\Phi(t_1,\ldots,t_k)\in L_2(D_k)$, and (1.38) for Lebesgue integrals, we have

$$\mathsf{M}\left\{\left(J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}-J'[\Phi_M]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$= C_k \left\|\Phi_N - \Phi_M\right\|^2_{L_2([t,T]^k)} \le$$

$$\le 2C_k \left(\left\|\Phi_N - \Phi\right\|^2_{L_2([t,T]^k)} + \left\|\Phi - \Phi_M\right\|^2_{L_2([t,T]^k)}\right)^2 \to 0$$

if $N, M \to \infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

Thus, there exists the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]^{(i_1\ldots i_k)}_{T,t}.$$

We will define the multiple Wiener stochastic integral for $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$ by the formula [106], [109]

$$J'[\Phi]^{(i_1\ldots i_k)}_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]^{(i_1\ldots i_k)}_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \Delta\mathbf{w}^{(i_1)}_{\tau_{l_1}} \ldots \Delta\mathbf{w}^{(i_k)}_{\tau_{l_k}}, \tag{1.304}$$

where $\Phi_N(t_1,\ldots,t_k)$ is defined by (1.297), $\Delta\mathbf{w}^{(i)}_{\tau_j} = \mathbf{w}^{(i)}_{\tau_{j+1}} - \mathbf{w}^{(i)}_{\tau_j}$, $i = 0, 1, \ldots, m$, $\mathbf{w}^{(0)}_{\tau} = \tau$.

It is easy to see that the above definition coincides with (1.23) if the function $\Phi(t_1,\ldots,t_k):\ [t,T]^k \to \mathbf{R}$ is continuous in the hypercube $[t,T]^k$.

Let us prove the following equality

$$J'[\Phi]^{(i_1\ldots i_k)}_{T,t} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k} \quad \text{w. p. 1}, \tag{1.305}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[\Phi]^{(i_1\ldots i_k)}_{T,t}$ is defined by (1.304) and

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k}$$

is the iterated Itô stochastic integral.

The equality (1.305) has already been proved for the case $\Phi(t_1,\ldots,t_k) = \Phi_N(t_1,\ldots,t_k)$ (see (1.303)).

From (1.303) we have

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1.} \tag{1.306}$$

Passing to the limit $\underset{N\to\infty}{\text{l.i.m.}}$ in the equality (1.306), we obtain

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1.} \tag{1.307}$$

Using Lemma 1.2 for $\Phi(t_1,\ldots,t_k)\in L_2(D_k)$, (1.38) for Lebesgue integrals, and (1.300), we get

$$\mathsf{M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k\ \to 0 \tag{1.308}$$

if $N\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral. The relations (1.307) and (1.308) prove the equality (1.305).

Using (1.305) and the isometry property of the Itô stochastic integral, we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \tag{1.309}$$

where $K=K(t_1,\ldots,t_k)$ is defined by (1.6), i.e.

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\qquad (\psi_l(\tau)\in L_2([t,T]))\,, \tag{1.310}$$

where $l=1,\ldots,k$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Applying (1.309) and the linearity property of the Itô stochastic integral, we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \tag{1.311}$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)\stackrel{\rm def}{=}K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) \tag{1.312}$$

and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{1.313}$$

is the Fourier coefficient corresponding to $K(t_1,\ldots,t_k)$.

Using the Itô formula, we have

$$\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_q}^{(i_q)}\times$$

$$\times\sum_{(j_1',\ldots,j_n')}\int\limits_t^T\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(g)}\ldots\mathbf{w}_{t_n'}^{(g)}=$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_n'}(t_n')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(g)}\ldots d\mathbf{w}_{t_n'}^{(g)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} \qquad (1.314)
$$

w. p. 1, where $g=0$ or $g=1$, $n,q\in\mathbf{N}$, $i_1,\ldots,i_q\ne 0,\ 1$,

$$
\sum_{(j_1,\ldots,j_k)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_d in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_d in the permutation $(i_1,\ldots,i_k)$.

The detailed proof of (1.314) will be given in Sect. 1.14 (see the proof of Theorem 1.22). The equality (1.314) means that (see (1.305))

$$
J'[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J'[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(g\ldots g)} =
$$

$$
= J'[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q g\ldots g)} \qquad (1.315)
$$

w. p. 1, where $g=0$ or $g=1$, $n,q\in\{0\}\cup\mathbf{N}$, $i_1,\ldots,i_q\ne 0,\ 1$, and $J'[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)} \stackrel{\sf def}{=} 1$ for $q=0$.

Using the equality (1.315), we obtain (1.263) for the case of an arbitrary complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions in $L_2([t,T])$.

Suppose that the conditions $(\star\star)$ (see Sect. 1.10) and (1.265) are fulfilled. Applying Theorem 9.6.9 [109] (also see [106], Theorem 3.1) and (1.275) (also see Theorem 1.23 below), we get

$$
J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} =
$$

$$
= \prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right) =
$$

$$= \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \quad (1.316)$$

w. p. 1, where notations are the same as in Theorems 1.2 and 1.14; the multiple Wiener stochastic integral $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.304).

Again applying (1.305), we have

$$J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \quad (1.317)$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1.304).

According to Lemma 1.2 for $\Phi(t_1,\ldots,t_k) \in L_2(D_k)$, (1.7), and (1.38) for Lebesgue integrals, we have

$$\mathsf{M}\left\{ \left(J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \right)^2 \right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k \to 0 \quad (1.318)$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$. Thus (see (1.311) and (1.318)),

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} \qquad (1.319)$$

and the following theorem is proved.

Theorem 1.16 [29] (generalization of Theorems 1.1, 1.2, and 1.14). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 1.10*) and the condition* (1.265) *is also fulfilled. Furthermore, let* $\psi_l(\tau)\in L_2([t,T])$ $(l=1,\ldots,k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansions*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right), \qquad (1.320)$$

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right) \qquad (1.321)$$

converging in the mean-square sense are valid, where $[x]$ *is an integer part of a real number* x*;* $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$*;* $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$*;* $d_l=1,\ldots,m_l$*;* $l=1,\ldots,k$*;* $m_1+\ldots+m_k=k$*; the numbers* $m_1,\ldots,m_k$*,* $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$*,* $h_{1,l},\ldots,h_{d_l,l}$*,* d_l *depend on* $\{j_1,\ldots,j_k\}$*; moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$*;* $H_n(x)$ *is the Hermite polynomial* (1.267)*; another notations as in Theorems* 1.1, 1.2, and 1.14.

Replacing the function $K(t_1,\ldots,t_k)$ by $\Phi(t_1,\ldots,t_k)$ we get the following theorem.

Theorem 1.17 [29] (generalization of Theorems 1.13, 1.15). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 1.10*) and* *the condition* (1.265) *is also fulfilled. Furthermore, let* $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansions*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right),\tag{1.322}$$

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\tag{1.323}$$

converging in the mean-square sense are valid, where $[x]$ *is an integer part of a real number* x*;* $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$*;* $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$*;* $d_l=1,\ldots,m_l$*;* $l=1,\ldots,k$*;* $m_1+\ldots+m_k=k$*; the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$*; moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$*; the multiple Wiener stochastic integral* $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (1.304)*;* $H_n(x)$ *is the Hermite polynomial* (1.267)*; another notations as in Theorems* 1.13, 1.15.

It should be noted that an analogue of Theorem 1.17 (more precisely, the expansion like (1.322) for the case $i_1,\ldots,i_k\neq 0$) was considered in [107]. Also note that the proof in [107] is different from the proof given in this section. In [107], the author interprets the multiple Wiener stochastic integral from a finite-dimensional kernel as a linear operator and proves that this operator is bounded. In our proof of Theorems 1.16, 1.17 we several times use the representation (1.305) of the multiple Wiener stochastic integral as the sum (with respect to permutations) of iterated Itô stochastic integrals and then estimate the remainder of the series (see (1.318) for details).

Note that the results of work [107], as well as the results of Chapter 1 of this book, are based on our idea [1] (2006) on the expansion of the kernel (1.6) (or $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$) into a generalized multiple Fourier series (see [1], Chapter 5, Theorem 5.1, pp. 235-245 or Sect. 1.1.3 of this book for details).

1.12 Generalization of Theorems 1.3, 1.4 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$

In this section, we will use the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process to generalize Theorems 1.3, 1.4 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 1.18. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-
$$

$$
-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{1.324}
$$

where

$$
J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},
$$

$$
J[\psi^{(k)}]_{T,t}^p=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{1.325}
$$

$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ *is the multiple Wiener stochastic integral defined by* (1.304)*, the Fourier coefficient* $C_{j_k\ldots j_1}$ *has the form* (1.313)*,* $K(t_1,\ldots,t_k)$ *is defined by*

(1.310),

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i = 1, \ldots, m$),

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1, \ldots, j_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$ (see (1.324)).

Proof. First, note that the formula (1.325) appears due to the equality (1.311). Using the equality (1.305), we get

$$J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{1.326}$$

where permutations $(t_1, \ldots, t_k)$ when summing are performed only in the values $d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1, \ldots, t_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$.

It is easy to see that the equality (1.326) can be written in the form

$$J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} = \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{1.327}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1, \ldots, j_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$.

Thus, an analogue of the equality (1.86) is proved under the conditions of Theorem 1.18 (compare (1.77), (1.86) and (1.325), (1.327)). Further proof of

Theorem 1.18 is similar to the proof of Theorem 1.3. Theorem 1.18 is proved.

Consider the following obvious generalization of Theorem 1.4.

Theorem 1.19. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the estimate*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right) \qquad (1.328)$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{T,t}$ *is the iterated Itô stochastic integral* (1.5), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.321) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorems* 1.1, 1.2, 1.16.

In addition, under the conditions of Theorem 1.19 we have the estimate (also see (1.74))

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times$$

$$\times\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n. \qquad (1.329)$$

1.13 Generalization of Theorems 1.5, 1.6 to the Case of an Arbitrary Complete Orthonormal with Weight $r(x)\ge 0$ System of Functions in the Space $L_2([t,T])$ and $\psi_1(x)\sqrt{r(x)},\ \ldots,\ \psi_k(x)\sqrt{r(x)}\in L_2([t,T])$

In this section, we will use the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process to generalize Theorems

1.5, 1.6 to the case of an arbitrary complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$ and $\psi_1(x)\sqrt{r(x)},\ \ldots,\ \psi_k(x)\sqrt{r(x)}$ $\in L_2([t,T])$. From the results of Sect. 1.3, 1.11 we obtain the following two theorems.

Theorem 1.20. *Suppose that* $\psi_1(x)\sqrt{r(x)},\ldots,\psi_k(x)\sqrt{r(x)} \in L_2([t,T])$, *where* $r(x) \ge 0$. *Moreover, let*

$$\left\{\Psi_j(x)\sqrt{r(x)}\right\}_{j=0}^{\infty}$$

is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. *Then, for the iterated Itô stochastic integral*

$$\tilde{J}[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \qquad (1.330)$$

the following expansion

$$\tilde{J}[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\tilde{\zeta}_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (1.331)$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\tilde{\zeta}_j^{(i)} = \int\limits_t^T \Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*)*,

$$\tilde{C}_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient, $K(t_1,\ldots,t_k)$ *is defined by* (1.310); *another notations are the same as in Theorems* 1.1, 1.2, 1.5.

Theorem 1.21. *Under the conditions of Theorem* 1.20 *the following estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t}-\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^{k}r(t_l)\right)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}^2\right)$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1.330), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.331) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorems* 1.6, 1.20.

1.14 Proof of Theorems 1.16 and 1.17 on the Base of the Itô Formula and Without Explicit Use of the Multiple Wiener Stochastic Integral

Note that Theorems 1.16 and 1.17 can also be proved without explicit use of the multiple Wiener stochastic integral. To do this, we introduce the following sum of iterated Itô stochastic integrals

$$J''[\Phi]_{T,t}^{(i_1\ldots i_k)}\stackrel{\sf def}{=}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\qquad(1.332)$$

where $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$, $i_1,\ldots,i_k=0,1,\ldots,m$, $d\mathbf{w}_\tau^{(0)}=d\tau$; another notations are the same as in (1.305).

Further, using the isometry property of the Ito stochastic integral as well as the linearity property of this integral, we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J''[K]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1},\quad(1.333)$$

where $K(t_1,\ldots,t_k)$ and $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ are defined by (1.310) and (1.312) correspondingly. Moreover, $J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ are defined by (1.332). Obviously, we can consider an analogue of (1.333) for $\Phi(t_1,\ldots,t_k)$ instead of $K(t_1,\ldots,t_k)$.

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in (1.333) and using (1.317), (1.318), (1.332), we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{1.334}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

It is easy to see that the equality (1.334) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{1.335}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Further, using the Itô formula, we can prove the following equality

$$\sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} = \prod_{l=1}^k \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^r \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \quad (1.336)$$

w. p. 1, where notations are the same as in Theorem 1.2 and (1.335).

The main difficulty in proving (1.336) using the Itô formula is related to the need to take into account various combinations of indices $i_1,\ldots,i_k=0,1,\ldots,m$. To avoid this difficulty, consider another approach, also based on the Itô formula.

First, we prove the following modification and generalization of Theorem 3.1 from [106] (1951) for the case $i_1,\ldots,i_k=0,1,\ldots,m$ using the Itô formula and without explicit use of the multiple Wiener stochastic integral.

Theorem 1.22 [29]. *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 1.10*) and the condition* (1.265) *is also fulfilled. Furthermore, let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \prod_{l=1}^k \left(\mathbf{1}_{\{m_l=0\}} + \mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right) \quad (1.337)$$

w. p. 1, where $i_1,\ldots,i_k = 0,1,\ldots,m;$ $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l} = m_l;$ $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;$ $d_l=1,\ldots,m_l;$ $l=1,\ldots,k;$ $m_1+\ldots+m_k=k;$ *the numbers* $m_1,\ldots,m_k,$ $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l},$ $h_{1,l},\ldots,h_{d_l,l},$ d_l *depend on* $\{j_1,\ldots,j_k\}$*; moreover,* $\{j_{g_1},\ldots,j_{g_k}\} = \{j_1,\ldots,j_k\}$*;* $H_n(x)$ *is the Hermite polynomial* (1.267)*; another notations are the same as in Theorem* 1.14*.*

Proof. First, consider the case $i_1 = \ldots = i_k = 1, \ldots, m$ and $j_1, \ldots, j_k \in \{0\} \cup \mathbf{N}$. By induction, we prove the following equality

$$p! \int\limits_t^T \phi_l(t_p) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_p}^{(1)} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q, \underbrace{l,\ldots,l}_{p})} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t_p') \ldots \int\limits_t^{t_2'} \phi_l(t_1') \times$$

$$\times d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_p'}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \qquad (1.338)$$

w. p. 1, where $p \in \mathbf{N}$, $l \neq j_1, \ldots, j_q$, and

$$\sum_{(q_1,\ldots,q_n)}$$

means the sum with respect to all possible permutations $(q_1, \ldots, q_n)$.

Consider the case $p = 1$. Using the Itô formula, we get w. p. 1 for $s \in [t, T]$

$$\int\limits_t^s \phi_l(\tau) d\mathbf{w}_{\tau}^{(1)} \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \int\limits_t^s \phi_l(\tau) \phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +$$

$$+ \int\limits_t^s \phi_l(\tau) \int\limits_t^{\tau} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} d\mathbf{w}_{\tau}^{(1)} +$$

$$+ \int\limits_t^s \phi_{j_q}(\tau) \left(\int\limits_t^{\tau} \phi_l(\theta) d\mathbf{w}_{\theta}^{(1)} \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} \right) d\mathbf{w}_{\tau}^{(1)}. \qquad (1.339)$$

Hereinafter in this section always $s \in [t, T]$. Differentiating by the Itô formula the expression in parentheses on the right-hand side of equality (1.339) and combining the result of differentiation with (1.339), we obtain w. p. 1

$$
J_{(l)s,t} J_{(j_q \ldots j_1)s,t} =
$$

$$
= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +
$$

$$
+ J_{(l j_q \ldots j_1)s,t} +
$$

$$
+ \int\limits_t^s \phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_l(\theta)\phi_{j_{q-1}}(\theta) \int\limits_t^{\theta} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} d\theta d\mathbf{w}_{\tau}^{(1)} +
$$

$$
+ J_{(j_q l j_{q-1} \ldots j_1)s,t} +
$$

$$
+ \int\limits_t^s \phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(\theta) \times
$$

$$
\times \left(\int\limits_t^{\theta} \phi_l(u)\ d\mathbf{w}_u^{(1)} \int\limits_t^{\theta} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} \right) d\mathbf{w}_{\theta}^{(1)} d\mathbf{w}_{\tau}^{(1)}, \tag{1.340}
$$

where

$$
\int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \stackrel{\sf def}{=} J_{(j_q \ldots j_1)s,t}.
$$

Continuing the process of iterative application of the Itô formula, we have w. p. 1

$$
J_{(l)s,t} J_{(j_q \ldots j_1)s,t} =
$$

$$
= J_{(l j_q \ldots j_1)s,t} + J_{(j_q l j_{q-1} \ldots j_1)s,t} + \ldots + J_{(j_q \ldots j_1 l)s,t} +
$$

$$
+ \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots
$$

$$
\ldots + \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_l(\tau)\phi_{j_1}(\tau) d\tau d\mathbf{w}_{t_2}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}. \qquad (1.341)
$$

Summing the equality (1.341) over permutations $(j_1, \ldots, j_q)$, we get

$$
\sum_{(j_1,\ldots,j_q)} J_{(l)s,t} J_{(j_q\ldots j_1)s,t} = \sum_{(j_1,\ldots,j_q,l)} J_{(l j_q \ldots j_1)s,t} + S(s) \qquad (1.342)
$$

w. p. 1, where

$$
S(s) =
$$

$$
= \sum_{(j_1,\ldots,j_q)} \Biggl(\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots
$$

$$
\ldots + \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_l(\tau)\phi_{j_1}(\tau) d\tau d\mathbf{w}_{t_2}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \Biggr). \qquad (1.343)
$$

Consider

$$
\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)}.
$$

Applying the Itô formula, we get w. p. 1

$$
\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} =
$$

$$
= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +
$$

$$
+ \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \times
$$

$$\times\left(\int\limits_t^{t_{q-1}}\phi_l(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{q-1}}\phi_{j_{q-2}}(t_{q-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-2}}^{(1)}\right)d\mathbf{w}_{t_{q-1}}^{(1)}.$$

By iterative application of the Itô formula (as above), we obtain w. p. 1

$$\int\limits_t^{s}\phi_l(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{s}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}=$$

$$=\int\limits_t^{s}\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots$$

$$\ldots+\int\limits_t^{s}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(\tau)\phi_{j_q}(\tau)d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}. \qquad (1.344)$$

Summing the equality (1.344) over permutations $(j_1,\ldots,j_q)$, we get

$$\sum_{(j_1,\ldots,j_q)}\int\limits_t^{s}\phi_l(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{s}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}=S_1(s), \qquad (1.345)$$

w. p. 1, where

$$S_1(s)=$$

$$=\sum_{(j_1,\ldots,j_q)}\left(\int\limits_t^{s}\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots\right.$$

$$\left.\ldots+\int\limits_t^{s}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(\tau)\phi_{j_q}(\tau)d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}\right). \qquad (1.346)$$

It is not difficult to see that

$$S(s)=S_1(s)\quad \text{w. p. 1}. \qquad (1.347)$$

Moreover, due to the orthogonality of $\{\phi_j(x)\}_{j=0}^{\infty}$ and (1.345), (1.347), we have

$$S(T) = S_1(T) = 0 \quad \text{w. p. 1.} \tag{1.348}$$

Thus (see (1.342), (1.348)), the equality (1.338) is proved for the case $p = 1$. Let us assume that the equality (1.338) is true for $p = 2, 3, \ldots, k-1$, and prove its validity for $p = k$.

From (1.342) for the case $q = k-1$, $j_1 = \ldots = j_{k-1} = l$ we obtain

$$(J_1)_{s,t}\,(k-1)!\,(J_{k-1})_{s,t} = k!\,(J_k)_{s,t} + S_2(s) \tag{1.349}$$

w. p. 1, where

$$S_2(s) = S(s)\Biggr|_{j_1=\ldots=j_q=l,\ q=k-1} \quad (k \ge 2) \quad \text{and} \quad S_2(s) \stackrel{\sf def}{=} 0 \ \ (q = k-1,\ k = 1),$$

$$\int\limits_t^s \phi_l(t_r)\ldots\int\limits_t^{t_2}\phi_l(t_1)d{\bf w}_{t_1}^{(1)}\ldots d{\bf w}_{t_r}^{(1)} \stackrel{\sf def}{=} (J_r)_{s,t} \quad (r\in{\bf N}) \quad \text{and} \quad (J_0)_{s,t} \stackrel{\sf def}{=} 1.$$

Taking into account (1.343), (1.345)–(1.347) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we have

$$S_2(T) = (k-1)!\,(J_{k-2})_{T,t}\,. \tag{1.350}$$

Combining (1.349) and (1.350), we obtain the following recurrence relation

$$k!\,(J_k)_{T,t} = (J_1)_{T,t}\,(k-1)!\,(J_{k-1})_{T,t} - (k-1)!\,(J_{k-2})_{T,t} \tag{1.351}$$

w. p. 1.

Using (1.351) and the induction hypothesis, we get w. p. 1

$$k!\int\limits_t^T \phi_l(t_k)\ldots\int\limits_t^{t_2}\phi_l(t_1)d{\bf w}_{t_1}^{(1)}\ldots d{\bf w}_{t_k}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf w}_{t_1}^{(1)}\ldots d{\bf w}_{t_q}^{(1)} =$$

$$
= \int\limits_t^T \phi_l(\tau)\ d\mathbf{w}_\tau^{(1)} \Biggl((k-1)! \int\limits_t^T \phi_l(t_{k-1}) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-1}}^{(1)} \times
$$

$$
\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \Biggr) -
$$

$$
-(k-1)! \int\limits_t^T \phi_l(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-2}}^{(1)} \times
$$

$$
\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =
$$

$$
= \int\limits_t^T \phi_l(\tau)\ d\mathbf{w}_\tau^{(1)} \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) \times
$$

$$
\times d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} -
$$

$$
-(k-1) \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) \times
$$

$$
\times d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}. \tag{1.352}
$$

Let $\boxed{l}$ be the symbol l which does not participate in the following sum with respect to permutations

$$
\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.
$$

Using (1.342), we have w. p. 1

$$
\int\limits_t^s \phi_l(\tau)\ d\mathbf{w}_\tau^{(1)} \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) \times
$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\ d\mathbf{w}_\tau^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t_{k-1}')\ldots \int\limits_t^{t_2'} \phi_l(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(J_{(\boxed{l}j_q\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q\boxed{l}j_{q-1}\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + \ldots$$

$$\ldots + J_{(j_q\ldots j_1\boxed{l}\underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q\ldots j_1 l\boxed{l}\underbrace{l\ldots l}_{k-2})s,t} + \ldots + J_{(j_q\ldots j_1\underbrace{l\ldots l}_{k-1}\boxed{l})s,t}\Biggr) + S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} J_{(j_q\ldots j_1\underbrace{l\ldots l}_{k})s,t} + S_3(s), \qquad (1.353)$$

where

$$S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(\int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_l(t_{k-1}')\ldots \int\limits_t^{t_2'} \phi_l(t_1') d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots$$

$$+ \ldots \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{\boxed{l}}(\tau)\phi_{j_1}(\tau)\times$$

$$\times \int\limits_t^{\tau} \phi_l(t_{k-1}')\ldots \int\limits_t^{t_2'} \phi_l(t_1') d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} +$$

$$
+\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{\boxed{l}}(\tau)\phi_l(\tau)\times
$$

$$
\times\int\limits_t^{\tau}\phi_l(t_{k-2}')\ldots\int\limits_t^{t_2'}\phi_l(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}+\ldots
$$

$$
\ldots+\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_l(t_{k-1}')\ldots\int\limits_t^{t_3'}\phi_l(t_2')\int\limits_t^{t_2'}\phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\Biggr).
$$

Using (1.343), (1.345)–(1.347), we get w. p. 1

$$
S_3(s)=
$$

$$
=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}\int\limits_t^s\phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_l(t_{k-2}')\ldots\int\limits_t^{t_2'}\phi_l(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=
$$

$$
=(k-1)\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})}\int\limits_t^s\phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_l(t_{k-2}')\ldots\int\limits_t^{t_2'}\phi_l(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}+
$$

$$
+\sum_{(j_1,\ldots,j_{q-1},\underbrace{l,\ldots,l}_{k-1})}\int\limits_t^s\phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^s\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} +
$$

$$
+ \sum_{(j_1,\ldots,j_{q-2},j_q \underbrace{l,\ldots,l}_{k-1})} \int\limits_t^{s} \phi_{\boxed{l}}(\tau)\phi_{j_{q-1}}(\tau)d\tau \int\limits_t^{s} \phi_{j_q}(t_q) \int\limits_t^{t_q} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} d\mathbf{w}_{t_q}^{(1)} +
$$

$$
\ldots
$$

$$
+ \sum_{(j_2,\ldots,j_q \underbrace{l,\ldots,l}_{k-1})} \int\limits_t^{s} \phi_{\boxed{l}}(\tau)\phi_{j_1}(\tau)d\tau \int\limits_t^{s} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times
$$

$$
\times \int\limits_t^{t_2} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_2}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}. \tag{1.354}
$$

Applying (1.354) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we finally have

$$
S_3(T) = (k-1) \sum_{(j_1,\ldots,j_q, \underbrace{l,\ldots,l}_{k-2})} \int\limits_t^{T} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}. \tag{1.355}
$$

Combining (1.352), (1.353), (1.355), we obtain w. p. 1

$$
k! \int\limits_t^{T} \phi_l(t_k) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_k}^{(1)} \times
$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(\underbrace{l,\ldots,l}_{k})} \int\limits_t^T \phi_{l}(t_k)\ldots \int\limits_t^{t_2} \phi_{l}(t_1) d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{l}(t_k')\ldots \int\limits_t^{t_2'} \phi_{l}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}, \qquad (1.356)$$

where $l\ne j_1,\ldots,j_q$.

The equality (1.338) is proved. From the other hand, (1.356) means that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{q+n})} = J''[\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{n})}\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots 1}^{q})} \qquad (1.357)$$

w. p. 1, where $n,q=0,1,2\ldots;$ $l\ne j_1,\ldots,j_q$ and

$$J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots 1}^{q})} \stackrel{\sf def}{=} 1$$

for $q=0$.

Note that [108] (see Chapter 6, Sect. 6.6 of this book for details)

$$\int\limits_t^T \phi_{l}(t_n)\ldots \int\limits_t^{t_2} \phi_{l}(t_1) d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)} =$$

$$=\frac{1}{n!}H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}, \int\limits_t^T \phi_l^2(\tau)d\tau\right) =$$

$$
=\frac{1}{n!}H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)},1\right)=\frac{1}{n!}H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right) \qquad (1.358)
$$

w. p. 1, where $n\in\mathbf{N}$, $H_n(x,y)$ is defined by (1.276) (also see (1.277)), and $H_n(x)$ is the Hermite polynomial (1.267).

From (1.358) we have w. p. 1

$$
J''[\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{n})}=n!\int\limits_t^T \phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)}=
$$

$$
=n!\frac{1}{n!}H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right)=H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right), \qquad (1.359)
$$

where $n\in\mathbf{N}$.

Combining (1.357) and (1.359), we obtain

$$
J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{q+n})}=H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right)\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots 1}^{q})} \qquad (1.360)
$$

w. p. 1, where $n,q=0,1,2\ldots;\ l\ne j_1,\ldots,j_q$.

The iterated application of the formula (1.360) completes the proof of Theorem 1.22 for the case $i_1=\ldots=i_k=1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbf{N}$.

To prove Theorem 1.22 for the case $i_1=\ldots=i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbf{N}$, we need to prove the following formula in addition to the previous proof

$$
p!\int\limits_t^T \phi_l(t_p)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_p\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q=
$$

$$
=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{p})}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t'_p)\ldots\int\limits_t^{t'_2}\phi_l(t'_1)dt'_1\ldots dt'_p dt_1\ldots dt_q, \qquad (1.361)
$$

where $p \in \mathbf{N}$,

$$\sum_{(j_1,\ldots,j_d)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_d)$.

First, consider the case $p = 1$. We have

$$d\left(\int\limits_t^s \phi_l(\theta)d\theta \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_q\right) =$$

$$= \phi_l(s) \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_q ds +$$

$$+\phi_{j_q}(s)\left(\int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_{q-1} \cdot \int\limits_t^s \phi_l(\theta)d\theta\right) ds.$$

Then

$$\int\limits_t^s \phi_l(\theta)d\theta \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_q =$$

$$= I_{(lj_q\ldots j_1)s,t} +$$

$$+\int\limits_t^s \phi_{j_q}(\tau)\left(\int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_{q-1} \cdot \int\limits_t^{\tau} \phi_l(\theta)d\theta\right) d\tau,$$

where

$$\int\limits_t^s \phi_{j_r}(t_r) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_r \stackrel{\mathsf{def}}{=} I_{(j_r\ldots j_1)s,t}. \tag{1.362}$$

Continuing this process, we get

$$\int\limits_t^s \phi_l(\theta)d\theta \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} = \sum_{(j_1,\ldots,j_q,l)} I_{(lj_q\ldots j_1)s,t}, \tag{1.363}$$

where

$$\sum_{(j_1,\ldots,j_d)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_d)$.

The equality (1.361) is proved for the case $p = 1$. Let us assume that the equality (1.361) is true for $p = 2, 3, \ldots, k-1$, and prove its validity for $p = k$.

From (1.363) for $j_1 = \ldots = j_q = l$, $q = k-1$ we have

$$(I_1)_{s,t}\,(k-1)!\,(I_{k-1})_{s,t} = k!\,(I_k)_{s,t}\,, \qquad (1.364)$$

where $k \in \mathbf{N}$ and

$$\int\limits_t^s \phi_l(t_k) \ldots \int\limits_t^{t_2} \phi_l(t_1) dt_1 \ldots dt_k \stackrel{\sf def}{=} (I_k)_{s,t}\,, \quad (I_0)_{s,t} \stackrel{\sf def}{=} 1.$$

Using (1.364) and the induction hypothesis, we obtain

$$k!\,(I_k)_{s,t} \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} = (I_1)_{s,t}\,(k-1)!\,(I_{k-1})_{s,t} \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} =$$

$$= I_{(l)s,t} \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(j_q\ldots j_1\,\underbrace{l,\ldots,l}_{k-1})s,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t} I_{(j_q\ldots j_1\,\underbrace{l,\ldots,l}_{k-1})s,t}, \qquad (1.365)$$

where $I_{(j_r\ldots j_1)s,t}$ is defined by (1.362) and $\boxed{l}$ is the symbol l which does not participate in the following sum with respect to permutations

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.$$

By analogy with (1.363) we obtain

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t} I_{(j_q\ldots j_1\,\underbrace{l,\ldots,l}_{k-1})s,t} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(I_{(\boxed{l}j_q\ldots j_1\,\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\boxed{l}j_{q-1}\ldots j_1\,\underbrace{l\ldots l}_{k-1})s,t} + \ldots$$

$$\ldots + I_{(j_q\ldots j_1 \boxed{l}\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\ldots j_1 l\boxed{l}\underbrace{l\ldots l}_{k-2})s,t} + \ldots + I_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k-1}\boxed{l})s,t}\Biggr) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} I_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k})s,t}. \tag{1.366}$$

Substituting $s=T$ into (1.365), (1.366) and combining (1.365), (1.366), we conlude that the equality (1.361) is proved for $p=k$. The equality (1.361) is proved.

Note that

$$n!\int\limits_t^T \phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_n = n!\frac{1}{n!}\left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n =$$

$$= \left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n, \tag{1.367}$$

where $n\in\mathbf{N}$.

After substituting (1.367) into (1.361), we have for $p=n$

$$\left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n \sum_{(j_1,\ldots,j_q)} J_{(j_q\ldots j_1)T,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{n})} J_{(j_q\ldots j_1\underbrace{l\ldots l}_{n})T,t}. \tag{1.368}$$

The equality (1.368) means that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{0\ldots 0}^{q+n})} = \left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n \cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}, \tag{1.369}$$

where $n,q=0,1,2\ldots$ and $J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(0\ldots 0)} \stackrel{\sf def}{=} 1$ for $q=0$.

The relations (1.360) and (1.369) prove Theorem 1.22 for the case $i_1=\ldots=i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbf{N}$.

Remark 1.15. *Note that the equality* (1.361) *can be obtained in another way. Let* $D_q=\{(t_1,\ldots,t_q)\in[t,T]^q:\ \exists\ i\ne j$ *such that* $t_i=t_j\}$ *be the "diagonal set" of* $[t,T]^q$ $(q=2,3,\ldots)$ [109]. *Since the Lebesgue meashure of the set*

D_q is equal to zero [109], then (see (1.332))

$$
J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}=\int\limits_{[t,T]^q}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q. \tag{1.370}
$$

From (1.370) we have

$$
J''[\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p})}\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}=
$$

$$
=\int\limits_{[t,T]^q}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q\int\limits_{[t,T]^p}\phi_l(t_1)\ldots\phi_l(t_p)dt_1\ldots dt_p=
$$

$$
=\int\limits_{[t,T]^{p+q}}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)\phi_l(t_1')\ldots\phi_l(t_p')dt_1'\ldots dt_p'dt_1\ldots dt_q=
$$

$$
=J''[\phi_{j_1}\ldots\phi_{j_q}\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p+q})}. \tag{1.371}
$$

It is not difficult to see that the equality (1.371) is nothing but the equality (1.361) written in another form.

To complete the proof of Theorem 1.22, we need to consider the case $i_1,\ldots,i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbf{N}$.

Obviously, the proof of Theorem 1.22 will be completed if we prove the following equalities

$$
\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times
$$

$$
\times\sum_{(j_1',\ldots,j_n')}\int\limits_t^T\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_n'}^{(1)}=
$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_n'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}, \tag{1.372}
$$

$$
\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times
$$

$$
\times\sum_{(j_1',\ldots,j_n')}\int\limits_t^T \phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_n'}^{(0)} =
$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_n'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} \tag{1.373}
$$

w. p. 1, where $n,q\in\mathbf{N}$, $d\mathbf{w}_{\tau}^{(0)}\stackrel{\sf def}{=}d\tau$, $i_1,\ldots,i_q\ne 1$ in (1.372) and $i_1,\ldots,i_q\ne 0$ in (1.373),

$$
\sum_{(j_1,\ldots,j_g)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_g)$. At the same time if j_r swapped with j_d in the permutation $(j_1,\ldots,j_g)$, then i_r swapped with i_d in the permutation $(i_1,\ldots,i_g)$.

The equalities (1.372) and (1.373) mean that

$$
J''[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q 1\ldots 1)} = J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J''[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(1\ldots 1)}, \tag{1.374}
$$

$$
J''[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q 0\ldots 0)} = J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J''[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(0\ldots 0)} \tag{1.375}
$$

w. p. 1, where $i_1,\ldots,i_q\ne 1$ in (1.374) and $i_1,\ldots,i_q\ne 0$ in (1.375).

First, we prove the equality (1.372). Consider the case $n=1$. Using the Itô formula, we get w. p. 1

$$\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)} +$$

$$+\int\limits_t^s \phi_{j_q}(\tau)\left(\int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{q-1}}^{(i_{q-1})}\int\limits_t^{\tau}\phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)}\right)d\mathbf{w}_\tau^{(i_q)}$$

$$=\ldots=$$

$$= J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)} + J_{(j_qj_1'j_{q-1}\ldots j_1)s,t}^{(i_q1i_{q-1}\ldots i_1)} + \ldots + J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)}, \tag{1.376}$$

where

$$\int\limits_t^s \phi_{j_r}(t_r)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_r}^{(i_r)} \stackrel{\sf def}{=} J_{(j_r\ldots j_1)s,t}^{(i_r\ldots i_1)}, \tag{1.377}$$

$i_1,\ldots,i_r=0,1,\ldots,m.$

From (1.376) we obtain

$$\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \sum_{(j_1,\ldots,j_q)}\int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q)}\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q)}\left(J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)} + J_{(j_qj_1'j_{q-1}\ldots j_1)s,t}^{(i_q1i_{q-1}\ldots i_1)} + \ldots + J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)}\right) =$$

$$= \sum_{(j_1,\ldots,j_q,j_1')} J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)} \tag{1.378}$$

w. p. 1, where $J^{(i_r\ldots i_1)}_{(j_r\ldots j_1)s,t}$ is defined by (1.377). The equality (1.372) is proved for the case $n=1$.

Let us assume that the equality (1.372) is true for $n=2,3,\ldots,k-1$, and prove its validity for $n=k$.

Applying (1.342), (1.343), (1.345)–(1.347), we obtain w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^s\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}^{(1)}_{t_1'}\ldots d\mathbf{w}^{(1)}_{t_k'}=$$

$$=\int\limits_t^s\phi_{j_k'}(\theta)d\mathbf{w}^{(1)}_{\theta}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^s\phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}^{(1)}_{t_1}\ldots d\mathbf{w}^{(1)}_{t_{k-1}}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^s\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\theta\int\limits_t^s\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}^{(1)}_{t_1}\ldots d\mathbf{w}^{(1)}_{t_{k-2}}. \tag{1.379}$$

After substituting $s=T$ in (1.379) and applying the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we get w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}^{(1)}_{t_1'}\ldots d\mathbf{w}^{(1)}_{t_k'}=$$

$$=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}^{(1)}_{\theta}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}^{(1)}_{t_1}\ldots d\mathbf{w}^{(1)}_{t_{k-1}}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}^{(1)}_{t_1}\ldots d\mathbf{w}^{(1)}_{t_{k-2}}, \tag{1.380}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Using (1.380) and the induction hypothesis, we obtain w. p. 1

$$
\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=
$$

$$
=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-
$$

$$
-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=
$$

$$
=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-
$$

$$
-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{1.381}
$$

Further, applying the induction hypothesis, we have w. p. 1

$$\sum_{(j_1',\ldots,j_{k-1}')} \mathbf{1}_{\{j_k'=j_{k-1}'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$=\Biggl(\sum_{(j_1',\ldots,j_{k-2}')} \mathbf{1}_{\{j_k'=j_{k-1}'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}+$$

$$+\sum_{(j_1',\ldots,j_{k-3}',j_{k-1}')} \mathbf{1}_{\{j_k'=j_{k-2}'\}} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-2}) \int\limits_t^{t_{k-2}} \phi_{j_{k-3}'}(t_{k-3})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-3}}^{(1)}d\mathbf{w}_{t_{k-2}}^{(1)}+\ldots$$

$$\ldots+\sum_{(j_2',\ldots,j_{k-1}')} \mathbf{1}_{\{j_k'=j_1'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2})\ldots \int\limits_t^{t_3} \phi_{j_2'}(t_2) \int\limits_t^{t_2} \phi_{j_{k-1}'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\Biggr)\times$$

$$\times\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$=\Biggl(\mathbf{1}_{\{j_k'=j_{k-1}'\}} \sum_{(j_1',\ldots,j_{k-2}')} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}+$$

$$+\mathbf{1}_{\{j_k'=j_{k-2}'\}} \sum_{(j_1',\ldots,j_{k-3}',j_{k-1}')} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-2}) \int\limits_t^{t_{k-2}} \phi_{j_{k-3}'}(t_{k-3})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)\times$$

$$
\times d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-3}}^{(1)}d\mathbf{w}_{t_{k-2}}^{(1)}+\ldots
$$

$$
\ldots+\mathbf{1}_{\{j'_k=j'_1\}}\sum_{(j'_2,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j'_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_3}\phi_{j'_2}(t_2)\int\limits_t^{t_2}\phi_{j'_{k-1}}(t_1)\times
$$

$$
\left.\times d\mathbf{w}_{t_1}^{(1)}d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\right)\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=
$$

$$
=\mathbf{1}_{\{j'_k=j'_{k-1}\}}\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-2})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\times
$$

$$
\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\mathbf{1}_{\{j'_k=j'_{k-2}\}}\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-3},j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-2})\times
$$

$$
\times\int\limits_t^{t'_{k-2}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots
$$

$$
\ldots
$$

$$
\ldots+\mathbf{1}_{\{j'_k=j'_1\}}\sum_{(j_1,\ldots,j_q,j'_2,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)\int\limits_t^{t'_2}\phi_{j'_{k-1}}(t'_1)d\mathbf{w}_{t'_1}^{(1)}d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\stackrel{\text{def}}{=}
$$

$$\stackrel{\sf def}{=} S_4(T). \tag{1.382}$$

By analogy with (1.344) we obtain w. p. 1

$$\int\limits_t^T \phi_l(\tau)\phi_{j_r}(\tau)d\tau \int\limits_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})} =$$

$$= \int\limits_t^T \phi_l(\tau)\phi_{j_r}(\tau) \int\limits_t^{\tau} \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})}d\tau + \ldots$$

$$\ldots + \int\limits_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(\tau)\phi_{j_r}(\tau)d\tau d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})}, \tag{1.383}$$

where $i_1,\ldots,i_{r-1} = 0, 1,\ldots,m$.

Using iteratively the Itô formula, as well as (1.383) and combinatorial reasoning, we obtain w. p. 1 (see Remark 1.16 below for details)

$$\int\limits_t^T \phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$
+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\Biggl(\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}'}(t_{k-2}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\int\limits_t^{t_{k-1}'}\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}'}(t_{k-3}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_3'}\phi_{j_2'}(t_2')\int\limits_t^{t_2'}\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(0)}\times
$$

$$
\times d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr)=
$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')}\Biggl\{\int\limits_t^T\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-2}'}(t_{k-2}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_{\theta}^{(0)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\int\limits_t^{t'_1}\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta)d\mathbf{w}_\theta^{(0)}\times
$$

$$
\left.\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}+
$$

$$
+\sum\limits_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-3},j'_{k-1})}\left\{\int\limits_t^T\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\times\right.
$$

$$
\times\int\limits_t^{t'_{k-1}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_\theta^{(0)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\int\limits_t^{t'_{k-1}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\times
$$

$$
\left.\times\int\limits_t^{t'_1}\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta)d\mathbf{w}_\theta^{(0)}d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}+\ldots
$$

$$
+\sum\limits_{(j_1,\ldots,j_q,j'_2,\ldots,j'_{k-1})}\left\{\int\limits_t^T\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\ldots\right.
$$

$$
\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_\theta^{(0)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)\int\limits_t^{t'_2}\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)d\mathbf{w}_\theta^{(0)}\times
$$

$$
\left.\times d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}=
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-2}'}(t_{k-2}')\ldots
$$

$$
\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-3}',j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}')\times
$$

$$
\times \int\limits_t^{t_{k-1}'} \phi_{j_{k-3}'}(t_{k-3}')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)} d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots
$$

$$
\ldots+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_2',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}')\ldots
$$

$$
\ldots \int\limits_t^{t_3'} \phi_{j_2'}(t_2')d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+S_4(T). \tag{1.384}
$$

From (1.381), (1.382), and (1.384) we conclude that the equality (1.372) is proved for $n=k$. The equality (1.372) is proved.

Remark 1.16. *It should be noted that the sums with respect to permutations*

$$\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}$$

in (1.384)*, containing the expressions* $\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta),\ldots,\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)$*, should be understood in a special way. Let us explain this rule on the basis of the sum*

$$\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}'}(t_{k-2}')\ldots$$

$$\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \qquad (1.385)$$

More precisely, permutations $\left(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}'\right)$ *when summing in* (1.385) *are performed in such a way that if* j_r^* *swapped with* j_d^* *in the permutation* $\left(j_{q+k-1}^*,\ldots,j_1^*\right)=\left(j_q,\ldots,j_1,j_{k-1}',j_{k-2}',\ldots,j_1'\right)$*, then* i_r^* *swapped with* i_d^* *in the permutation*

$$\left(i_{q+k-1}^*,\ldots,i_1^*\right)=\left(i_q,\ldots,i_1,0,\underbrace{1,\ldots,1}_{k-2}\right).$$

Moreover, $\bar\phi_{j_r^*}$ *swapped with* $\bar\phi_{j_d^*}$ *in the permutation*

$$\left(\bar\phi_{j_{q+k-1}^*},\ldots,\bar\phi_{j_1^*}\right)=\left(\phi_{j_q},\ldots,\phi_{j_1},\ \phi_{j_k'}\cdot\phi_{j_{k-1}'},\ \phi_{j_{k-2}'},\ldots,\phi_{j_1'}\right).$$

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}$$

in (1.384) *that contain the expressions* $\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta),\ldots,\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)$*.*

Let us prove the equality (1.373). Consider the case $n=1$. By analogy with (1.376) and (1.378) we obtain

$$\int\limits_t^s\phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(0)}\sum_{(j_1,\ldots,j_q)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\sum_{(j_1,\ldots,j_q,j_1')}J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_10)}$$

w. p. 1, where $J_{(j_r\ldots j_1)s,t}^{(i_r\ldots i_1)}$ is defined by (1.377). The equality (1.373) is proved for the case $n=1$.

Let us assume that the equality (1.373) is true for $n=2,3,\ldots,k-1$, and prove its validity for $n=k$.

In complete analogy with (1.363) we get

$$\int\limits_t^s \phi_{j'_k}(\theta)d\theta\int\limits_t^s \phi_{j'_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j'_1}(t_1)dt_1\ldots dt_{k-1}=$$

$$=J_{(j'_kj'_{k-1}\ldots j'_1)s,t}^{(0\ldots 0)}+J_{(j'_{k-1}j'_kj'_{k-2}\ldots j'_1)s,t}^{(0\ldots 0)}+\ldots+J_{(j'_{k-1}\ldots j'_1j'_k)s,t}^{(0\ldots 0)}. \tag{1.386}$$

Applying (1.386), we have

$$\sum_{(j'_1,\ldots,j'_k)}\int\limits_t^T \phi_{j'_k}(t'_k)\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(0)}\ldots d\mathbf{w}_{t'_k}^{(0)}=$$

$$=\sum_{(j'_1,\ldots,j'_{k-1})}\left(J_{(j'_kj'_{k-1}\ldots j'_1)s,t}^{(0\ldots 0)}+J_{(j'_{k-1}j'_kj'_{k-2}\ldots j'_1)s,t}^{(0\ldots 0)}+\ldots+J_{(j'_{k-1}\ldots j'_1j'_k)s,t}^{(0\ldots 0)}\right)=$$

$$=\int\limits_t^T \phi_{j'_k}(\theta)d\theta\sum_{(j'_1,\ldots,j'_{k-1})}\int\limits_t^T \phi_{j'_{k-1}}(t_{k-1})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t_1)d\mathbf{w}_{t_1}^{(0)}\ldots d\mathbf{w}_{t_{k-1}}^{(0)}. \tag{1.387}$$

Using (1.387) and the induction hypothesis, we obtain w. p. 1

$$\sum_{(j'_1,\ldots,j'_k)}\int\limits_t^T \phi_{j'_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j'_1}(t_1)d\mathbf{w}_{t_1}^{(0)}\ldots d\mathbf{w}_{t_k}^{(0)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\int\limits_t^T \phi_{j'_k}(\theta)d\theta\sum_{(j'_1,\ldots,j'_{k-1})}\int\limits_t^T \phi_{j'_{k-1}}(t'_{k-1})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(0)}\ldots d\mathbf{w}_{t'_{k-1}}^{(0)}\times$$

$$
\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =
$$

$$
= \int\limits_t^T \phi_{j_k'}(\theta) d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(0)} \ldots d\mathbf{w}_{t_{k-1}'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_k'}(\theta) d\theta \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(0)} \ldots d\mathbf{w}_{t_{k-1}'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)}. \qquad (1.388)
$$

An iterative application of the Itô formula leads to the following equality

$$
\int\limits_t^T \phi_{j_k'}(\theta) d\theta \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(0)} \ldots d\mathbf{w}_{t_{k-1}'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =
$$

$$
= J_{(j_k' j_q \ldots j_1 j_{k-1}' \ldots j_1')T,t}^{(0 i_q \ldots i_1 0 \ldots 0)} + J_{(j_q j_k' j_{q-1} \ldots j_1 j_{k-1}' \ldots j_1')T,t}^{(i_q 0 i_{q-1} \ldots i_1 0 \ldots 0)} + \ldots J_{(j_q \ldots j_1 j_k' j_{k-1}' \ldots j_1')T,t}^{(i_q \ldots i_1 0 \ldots 0)} +
$$

$$
+ J_{(j_q \ldots j_1 j_{k-1}' j_k' j_{k-2}' \ldots j_1')T,t}^{(i_q \ldots i_1 0 \ldots 0)} + \ldots + J_{(j_q \ldots j_1 j_{k-1}' \ldots j_1' j_k')T,t}^{(i_q \ldots i_1 0 \ldots 0)} \qquad (1.389)
$$

w. p. 1.

Combining (1.388) and (1.389), we finally obtain w. p. 1

$$
\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times
$$

$$
\times \sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)} =
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}.
$$

The equality (1.373) is proved for $n = k$. The equality (1.373) is proved. Theorem 1.22 is proved.

To complete the proof of Theorems 1.16 and 1.17, we prove the following theorem.

Theorem 1.23. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following representation*

$$
J''[\phi_{j_1}\ldots\phi_{j_k}]^{(i_1\ldots i_k)} = \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \qquad (1.390)
$$

is valid w. p. 1, where $i_1,\ldots,i_k = 0,1,\ldots,m$, $[x]$ is an integer part of a real number x, $\prod\limits_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\sf def}{=} 0$; the sum in the second line of the formula (1.390) is the sum with respect to all possible partitions (1.53); another notations are the same as in Theorems 1.1, 1.2.

Remark 1.17. *It should be noted that the formulas (1.338), (1.371), (1.374), (1.375) follow from (1.390). It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (1.390) equal to 0 or 1.*

Proof. The proof of Theorem 1.23 is carried out by induction using the following recurrence relation

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=J''[\phi_{j_k}]_{T,t}^{(i_k)}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})}-$$

$$-\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\neq 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})}\qquad(1.391)$$

w. p. 1.

Let us prove the recurrence relation (1.391). Using iteratively the Itô formula, the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, as well as (1.383) and combinatorial reasoning, we obtain w. p. 1 (see Remark 1.18 below for details)

$$J''[\phi_{j_k}]_{T,t}^{(i_k)}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})}=$$

$$=\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(j_1,\ldots,j_{k-1})}\Biggl(\mathbf{1}_{\{i_k=i_{k-1}\neq 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}+$$

$$+\mathbf{1}_{\{i_k=i_{k-2}\neq 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}+\ldots
$$

$$
\ldots+\mathbf{1}_{\{i_k=i_1\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_k}(\theta)\phi_{j_1}(\theta)\times
$$

$$
\left.\times d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\right)=
$$

$$
=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+
$$

$$
+\sum_{(j_1,\ldots,j_{k-2})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\Biggl\{\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}+\ldots
$$

$$
\left.\ldots+\int\limits_t^T\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}\right\}+
$$

$$
+\sum_{(j_1,\ldots,j_{k-3},j_{k-1})}\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\Biggl\{\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_{k-3}}(t_{k-3})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}d\mathbf{w}_{\theta}^{(0)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\times
$$

$$
\left.\times d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\right\}+\ldots
$$

$$
\ldots + \sum_{(j_2,\ldots,j_{k-1})} \mathbf{1}_{\{i_k=i_1\ne 0\}} \Biggl\{ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_1}(\theta) \int\limits_t^{\theta} \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times
$$

$$
\times d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} d\mathbf{w}_{\theta}^{(0)} + \ldots
$$

$$
\ldots + \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_k}(\theta)\phi_{j_1}(\theta) d\mathbf{w}_{\theta}^{(0)} d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} \Biggr\} =
$$

$$
= \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} +
$$

$$
+ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta) d\theta \sum_{(j_1,\ldots,j_{k-2})} \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}} \int\limits_t^T \phi_{j_{k-2}}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})} +
$$

$$
+ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta) d\theta \sum_{(j_1,\ldots,j_{k-3},j_{k-1})} \mathbf{1}_{\{i_k=i_{k-2}\ne 0\}} \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \int\limits_t^{t_{k-1}} \phi_{j_{k-3}}(t_{k-3}) \ldots
$$

$$
\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})} d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} + \ldots
$$

$$
\ldots + \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_1}(\theta) d\theta \sum_{(j_2,\ldots,j_{k-1})} \mathbf{1}_{\{i_k=i_1\ne 0\}} \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times
$$

$$
\times d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} =
$$

$$
= J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)} + \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}} \mathbf{1}_{\{j_k=j_{k-1}\}} \cdot J''[\phi_{j_1} \ldots \phi_{j_{k-2}}]_{T,t}^{(i_1 \ldots i_{k-2})} +
$$

$$
+ \mathbf{1}_{\{i_k=i_{k-2}\ne 0\}} \mathbf{1}_{\{j_k=j_{k-2}\}} \cdot J''[\phi_{j_1} \ldots \phi_{j_{k-3}} \phi_{j_{k-1}}]_{T,t}^{(i_1 \ldots i_{k-3} i_{k-1})} + \ldots
$$

$$\ldots + \mathbf{1}_{\{i_k=i_1\ne 0\}}\mathbf{1}_{\{j_k=j_1\}}\cdot J''[\phi_{j_2}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_2\ldots i_{k-1})} =$$

$$= J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\ne 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})}. \quad (1.392)$$

The equality (1.391) is proved. Theorem 1.23 is proved.

Remark 1.18. *It should be noted that the sums with respect to permutations*

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (1.392), *containing the expressions*

$$\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\ne 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta),$$

should be understood in a special way. Let us explain this rule on the basis of the sum

$$\sum_{(j_1,\ldots,j_{k-1})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}. \quad (1.393)$$

More precisely, permutations $(j_1,\ldots,j_{k-1})$ *when summing in* (1.393) *are performed in such a way that if* j_r *swapped with* j_d *in the permutation* $(j_1,\ldots,j_{k-1})$, *then* i_r *swapped with* i_d *in the permutation* $(i_1,\ldots,i_{k-2},i_{k-1})$ (*note that* $i_{k-1}=0$). *Moreover,* $\bar{\phi}_{j_r}$ *swapped with* $\bar{\phi}_{j_d}$ *in the permutation*

$$\left(\bar{\phi}_{j_1},\ldots,\bar{\phi}_{j_{k-1}}\right)=\left(\phi_{j_1},\ldots,\phi_{j_{k-2}},\ \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\cdot\phi_{j_k}\cdot\phi_{j_{k-1}}\right),$$

where $\bar{\phi}_{j_{k-1}}(\tau)=\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\tau)\phi_{j_{k-1}}(\tau)$.

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (1.392) *that contain the expressions*

$$\mathbf{1}_{\{i_k=i_{k-2}\neq 0\}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\neq 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta).$$

The relations (1.334), (1.337), (1.390) prove Theorem 1.16. An analogue of the formula (1.334) for $\Phi(t_1,\ldots,t_k)$ instead of $K(t_1,\ldots,t_k)$ and (1.337), (1.390) prove Theorem 1.17.

We note a number of works [110]-[113] in which the properties of multiple Wiener stochastic integrals were studied using measure theory, in particular, the formulas for the product of such integrals were obtained.

First of all, let us compare Theorem 1.23 with Proposition 5.1 from [110]. An analogue of the right-hand side of (1.390) for nonrandom $x_1,\ldots,x_k$ is constructed in [110] using diagrams (see the formula (5.1) in [110]). This means that the application of the formula (5.1) from [110], unlike the formula (1.390), is difficult when performing algebraic transformations.

Further, we note that the formula (5.1) from [110] was applied to the representation of the multiple Wiener stochastic integral somewhat differently than the formula (1.390). Namely, using Proposition 5.1 [110]. Let us expain this difference in more detail.

Proposition 5.1 from [110] in our degree of generality and in our notations can be written as

$$J''\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}=$$

$$=J''\Biggl[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{m_1}\underbrace{\phi_{j_2}\ldots\phi_{j_2}}_{m_2}\ldots\underbrace{\phi_{j_p}\ldots\phi_{j_p}}_{m_p}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1}\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2}\ldots\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}=$$

$$=J''\left[\phi_{j_1}\ldots\phi_{j_1}\right]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})}\cdot J''\left[\phi_{j_2}\ldots\phi_{j_2}\right]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})}\cdot\ldots\cdot J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}\qquad(1.394)$$

w. p. 1, where

$$J''\left[\phi_{j_1}\ldots\phi_{j_1}\right]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})},\ J''\left[\phi_{j_2}\ldots\phi_{j_2}\right]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})},\ldots,\ J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}$$

are defined by the right-hand side of the formula (5.1) from [110], $m_1+\ldots+m_p = k$, $m_1,\ldots,m_p>0$, $j_q\ne j_d$ $(q\ne d,\ q,d=1,\ldots,p)$, $i_1,\ldots,i_k=1,\ldots,m$.

This actually means that in [110] an analogue of the formula (1.390) is constructed for the special case $j_1=\ldots=j_k$. Moreover, the specified analogue is based on the formula (5.1) [110] obtained using diagrams.

Comparing the formulas (1.390) and (1.394) (or (5.1) from [110]), it is easy to understand that the transition from (1.390) to (1.394) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (1.390) equal to 0 or 1. The reverse transition from the formula (1.394) to the formula (1.390) is not obvious. Note that the formula (1.390) (not the formula (1.394)) is convenient for the numerical integration of Itô stochastic differential equations (see Chapter 5 of this book for details).

Let us turn to the comparison of Theorem 1.23 with another interesting work [113] (2019). As it turned out, a version of Theorem 1.23 was obtained in terms of Wick polynomials and for the case of vector valued random measures in [113] (see Theorem 7.2, p. 69). However, much earlier the formula (1.390) (Theorem 1.23) is obtained in our monograph [4] (2009) as part of the formula (5.30) (see [4], p. 220). Moreover, particular cases of the formula (1.390) were obtained even earlier in our works [1] (2006) and [3] (2007). More precisely, partiular cases $k=1,\ldots,5$ of the formula (1.390) were obtained in [1] (2006) as parts of the formulas on the pages 243-244 and partiular cases $k=1,\ldots,7$ of the formula (1.390) were obtained in [3] (2007) as parts of the formulas on the pages 208-218.

We also note that we have found an explicit expression for the Wick polynomial of degree k of the arguments $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$ (see the formula (1.390)), which is very convenient for the numerical simulation of iterated Itô stochastic integrals (1.5) [53], [54]. Note that the representation of the Wick polynomial of the arguments $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$ in terms of the product of Hermite polynomials is less convenient for the numerical simulation of iterated Itô stochastic integrals (1.5). For example, the expression for $J''[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}$ in terms of the product of Hermite polynomials, even under the condition $i_1=i_2=i_3=i_4$, already contains 15 different expressions (see Sect. 1.10). At the same time, all these 15 expressions are contained in one formula (1.390) provided that $k=4$ and $i_1=i_2=i_3=i_4$. It is very convenient, since in computer simulation using the formula (1.390), in addition to modeling of random variables $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$, it remains only to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (1.390) equal to 0 or 1.

It should be noted that in [111] (Theorem 6.1) a diagram formula was obtained for the product of two multiple Wiener stochastic integrals with respect to vector valued random measures. The formula (1.372) can be derived from the diagram formula [111]. Although the proof of the diagram formula [111] is much more complicated than our proof of the formula (1.372).

To conclude this section, we say a few words about expansions (1.320) and (1.321). The transition from the expansion (1.321) to the expansion (1.320) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (1.321) equal to 0 or 1. The reverse transition from the formula (1.320) to the formula (1.321) is also possible but not obvious. However, Theorems 1.22 and 1.23 provide a transition from (1.320) to (1.321) and vice versa. Note that the expansion (1.320) is interesting from the point of view of studying the structure of the expansion of iterated Itô stochastic integrals. On the orther hand, the expansion (1.321) is exceptionally convenient for applications (see Chapter 5 of this book and [53], [54]).

1.15 Generalization of Theorem 1.11 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$

Suppose that $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k),$$

where the function $K(t_1,\ldots,t_k)$ has the form (1.6), $s\in(t,T]$ (s is fixed), and $\mathbf{1}_A$ is the indicator of the set A.

Further, we have (see (1.6))

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_1<\ldots<t_k<s\}}\psi_1(t_1)\ldots\psi_k(t_k)=$$

$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s\\ & \\ 0, & \text{otherwise}\end{cases},$$

where $\bar{K}(t_1,\ldots,t_k,s)\in L_2([t,T]^k)$, $k\ge 1$, $t_1,\ldots,t_k\in[t,T]$, and $s\in(t,T]$.

Note that

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\int\limits_t^T\mathbf{1}_{\{t_k<s\}}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad\text{w. p. 1,}\qquad(1.395)$$

where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=0,1,\ldots,m.$

Applying Theorem 1.16 to the iterated Itô stochastic integral (1.395), we obtain the following generalization of Theorem 1.11 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 1.24. *Suppose that* $\psi_1(\tau),\ \ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then, the following expansion*

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x*,*

$$C_{j_k\ldots j_1}(s)=\int\limits_{[t,T]^k}\bar{K}(t_1,\ldots,t_k,s)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k=$$

$$=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, $\prod\limits_{\emptyset}\stackrel{\sf def}{=}1,\ \sum\limits_{\emptyset}\stackrel{\sf def}{=}0$*; another notations are the same as in Theorem* 1.2.

Note that the estimates (1.251) and (1.253) will also be valid under the conditions of Theorem 1.24.

Chapter 2

Expansions of Iterated Stratonovich Stochastic Integrals Based on Generalized Multiple and Iterated Fourier Series

This chapter is devoted to the adaptation of Theorems 1.1, 1.16 for iterated Stratonovich stochastic integrals. The case of continuously differentiable weight functions (multiplicities 1 to 5) and weight functions identically equal to one (multiplicities 6 to 8) is considered. In this case, we use a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t,T])$. In addition, the case of continuous weight functions (multiplicities 1 and 2), binomial weight functions (multiplicities 3 and 4) and weight functions identically equal to one (multiplicities 5 and 6) is studied. In this case, we use an arbitrary complete orthonormal system of functions in $L_2([t,T])$. Recently (in 2024), the above adaptation has also been carried out for iterated Stratonovich stochastic integrals of multiplicity k, $k \in \mathbf{N}$ (Theorems 2.59, 2.61) but under one additional condition.

2.1 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicity 2 Based on Theorem 1.1. The case $p_1, p_2 \to \infty$ and Smooth Weight Functions

2.1.1 Approach Based on Theorem 1.1 and Integration by Parts

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) \stackrel{\mathsf{def}}{=} f_t : [0,T] \times \Omega \rightarrow \mathbf{R}$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$.

Consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s\subset\mathrm{F}_t\subset\mathrm{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let $\mathrm{M}_2([t,T])$ $(t\ge 0)$ be the class of random functions $\xi(\tau,\omega)\stackrel{\sf def}{=}\xi_\tau:[t,T]\times\Omega\to\mathbf{R}$ defined as in Sect. 1.1.2.

We introduce the class $\mathrm{Q}_m([t,T])$ $(t\ge 0)$ of Itô processes η_τ, $\tau\in[t,T]$ of the form

$$\eta_\tau=\eta_t+\int\limits_t^\tau a_s ds+\int\limits_t^\tau b_s df_s, \tag{2.1}$$

where $(a_\tau)^m,(b_\tau)^m\in\mathrm{M}_2([t,T])$ and $\lim\limits_{s\to\tau}\mathsf{M}\bigl\{|b_s-b_\tau|^4\bigr\}=0$ for all $\tau\in[t,T]$. The second integral on the right-hand side of (2.1) is the Itô stochastic integral (see Sect. 1.1.2).

Let $C^{2,1}(\mathbf{R}\times[t,T])$ $(t\ge 0)$ be the space of functions $F(x,\tau):\mathbf{R}\times[t,T]\to\mathbf{R}$ such that

$$\left|\frac{\partial F}{\partial x}(x,\tau)\right|\le K,\quad \left|\frac{\partial^2 F}{\partial x^2}(x,\tau)\right|\le K,\quad \left|\frac{\partial F}{\partial \tau}(x,\tau)\right|\le K,\quad \left|\frac{\partial^2 F}{\partial \tau\partial x}(x,\tau)\right|\le K$$

for all $x\in\mathbf{R}$ and $\tau\in[t,T]$, where constant K does not depend on x,τ.

Let $\tau_j^{(N)}$, $j=0,1,\ldots,N$ be a partition of the interval $[t,T]$, $t\ge 0$ such that

$$t=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max\limits_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty. \tag{2.2}$$

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}}\sum_{j=0}^{N-1}F\left(\frac{1}{2}\left(\eta_{\tau_j^{(N)}}+\eta_{\tau_{j+1}^{(N)}}\right),\tau_j^{(N)}\right)\left(f_{\tau_{j+1}^{(N)}}-f_{\tau_j^{(N)}}\right)\stackrel{\sf def}{=}\int\limits_t^{*T}F(\eta_\tau,\tau)df_\tau \tag{2.3}$$

is called [114] the Stratonovich stochastic integral of the process $F(\eta_\tau,\tau)$, $\tau\in[t,T]$, where $\tau_j^{(N)}$, $j=0,1,\ldots,N$ is a partition of the interval $[t,T]$ satisfying the condition (2.2).

It is known [114] (also see [84]) that under proper conditions, the following relation between Stratonovich and Itô stochastic integrals holds

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau + \frac{1}{2}\int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau \quad \text{w. p. 1.} \tag{2.4}$$

If the Wiener processes in (2.1) and (2.3) are independent, then

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau \quad \text{w. p. 1.} \tag{2.5}$$

A possible variant of conditions under which the formulas (2.4) and (2.5) are correct, for example, consists of the conditions: $\eta_\tau \in \mathrm{Q}_4([t,T])$, $F(\eta_\tau,\tau) \in \mathrm{M}_2([t,T])$, $F(x,\tau) \in C^{2,1}(\mathbf{R} \times [t,T])$.

Note that if $F(x,\tau) = F_1(x)F_2(\tau)$, then it suffices to require that $F(x,\tau)$ be twice differentiable with respect to x (with bounded derivatives) and continuous with respect to τ (instead of the condition $F(x,\tau) \in C^{2,1}(\mathbf{R} \times [t,T])$).

In Sect. 2.1–2.17, in most cases, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal systems of Legendre polynomials or trigonometric functions in $L_2([t,T])$. Therefore, we will pay attention on the following well known facts about these two systems of functions [115].

Suppose that the function $f(x)$ is bounded at the interval $[t,T]$. Moreover, its derivative $f'(x)$ is continuous function at the interval $[t,T]$ except may be the finite number of points of the finite discontinuity. Then the Fourier series

$$\sum_{j=0}^{\infty} C_j\phi_j(x), \quad C_j = \int\limits_t^T f(x)\phi_j(x)dx$$

converges at any internal point x of the interval $[t,T]$ to the value $(f(x+0)+f(x-0))/2$ and converges uniformly to $f(x)$ on any closed interval (of continuity of the function $f(x)$) lying inside $[t,T]$. At the same time the Fourier–Legendre series converges if $x=t$ and $x=T$ to $f(t+0)$ and $f(T-0)$ correspondently, and the trigonometric Fourier series converges if $x=t$ and $x=T$ to $(f(t+0)+f(T-0))/2$ in the case of periodic continuation of the function $f(x)$.

In Sect. 2.1 we consider the case $k=2$ of the following iterated Stratonovich and Itô stochastic integrals

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.6}$$

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.7)$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes.

Let us formulate and prove the following theorem on expansion of iterated Stratonovich stochastic integrals of multiplicity 2.

Theorem 2.1 [8] (2011), [10]-[22], [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. At the same time $\psi_2(s)$ is a continuously differentiable nonrandom function on $[t, T]$ and $\psi_1(s)$ is twice continuously differentiable nonrandom function on $[t, T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense is valid, where

$$C_{j_2 j_1} = \int\limits_t^T \psi_2(s_2) \phi_{j_2}(s_2) \int\limits_t^{s_2} \psi_1(s_1) \phi_{j_1}(s_1) ds_1 ds_2$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. In accordance to the standard relations between Stratonovich and Itô stochastic integrals (see (2.4) and (2.5)) we have w. p. 1

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \mathbf{1}_{\{i_1 = i_2\}} \int\limits_t^T \psi_1(t_1) \psi_2(t_1) dt_1, \qquad (2.8)$$

where here and further $\mathbf{1}_A$ is the indicator of the set A.

From the other side according to (1.46), we have

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \lim_{p_1,p_2\to\infty} \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1}. \tag{2.9}$$

From (2.8) and (2.9) it follows that Theorem 2.1 will be proved if

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{2.10}$$

Note that in this section and in Sect. 2.1.2 we present two different proofs (under different conditions) of the existence of a limit on the right-hand side of (2.10) for the polynomial and trigonometric cases.

Let us prove (2.10). Consider the function

$$K^*(t_1,t_2) = K(t_1,t_2) + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_1), \tag{2.11}$$

where $t_1, t_2 \in [t,T]$ and $K(t_1,t_2)$ is defined by (1.6) for $k=2$.

Let us expand the function $K^*(t_1,t_2)$ defined by (2.11) using the variable t_1, when t_2 is fixed, into the generalized Fourier series at the interval (t,T)

$$K^*(t_1,t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2)\phi_{j_1}(t_1) \quad (t_1 \ne t, T), \tag{2.12}$$

where

$$C_{j_1}(t_2) = \int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1 = \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1. \tag{2.13}$$

The equality (2.12) is satisfied pointwise in each point of the interval (t,T) with respect to the variable t_1, when $t_2 \in [t,T]$ is fixed, due to a piecewise smoothness of the function $K^*(t_1,t_2)$ with respect to the variable $t_1 \in [t,T]$ (t_2 is fixed).

Note also that due to well known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (2.12) converges when $t_1 = t, T$.

Obtaining (2.12) we also used the fact that the right-hand side of (2.12) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

The function $C_{j_1}(t_2)$ is a continuously differentiable one at the interval $[t, T]$. Let us expand it into the generalized Fourier series at the interval (t, T)

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\ne t,T), \tag{2.14}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and the equality (2.14) is satisfied pointwise at any point of the interval (t, T) (the right-hand side of (2.14) converges when $t_2 = t, T$).

Let us substitute (2.14) into (2.12)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2, \tag{2.15}$$

where the series on the right-hand side of (2.15) converges at the boundary of the square $[t, T]^2$.

It is easy to see that substituting $t_1 = t_2$ in (2.15), we obtain

$$\frac{1}{2}\psi_1(t_1)\psi_2(t_1)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1). \tag{2.16}$$

From (2.16) we formally have

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\int\limits_t^T\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$
= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\int\limits_t^T C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =
$$

$$
= \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =
$$

$$
= \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}} = \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1} = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{2.17}
$$

Let us explain the second step in (2.17) (the fourth step in (2.17) follows from the orthonormality of functions $\phi_j(s)$ at the interval $[t,T]$).

We have

$$
\left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| \le
$$

$$
\le \int\limits_t^T \left|\psi_2(t_1)G_{p_1}(t_1)\right| dt_1 \le C\int\limits_t^T \left|G_{p_1}(t_1)\right| dt_1, \tag{2.18}
$$

where $C<\infty$ and

$$
\sum_{j=p+1}^{\infty}\int\limits_t^{\tau}\psi_1(s)\phi_j(s)ds\phi_j(\tau) \stackrel{\sf def}{=} G_p(\tau).
$$

Let us consider the case of Legendre polynomials. Then

$$
\left|G_{p_1}(t_1)\right| = \frac{1}{2}\left|\sum_{j_1=p_1+1}^{\infty}(2j_1+1)\int\limits_{-1}^{z(t_1)}\psi_1(u(y))P_{j_1}(y)dyP_{j_1}(z(t_1))\right|, \tag{2.19}
$$

where

$$
u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}, \tag{2.20}
$$

and $P_j(s)$ is the Legendre polynomial.

From (2.19) and the well known formula

$$
\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j=1,2,\ldots \tag{2.21}
$$

we obtain

$$
|G_{p_1}(t_1)|=\frac{1}{2}\Biggl|\sum_{j_1=p_1+1}^{\infty}\Biggl\{\left(P_{j_1+1}(z(t_1))-P_{j_1-1}(z(t_1))\right)\psi_1(t_1)-
$$

$$
-\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(u(y))dy\Biggr\}P_{j_1}(z(t_1))\Biggr|\le
$$

$$
\le C_0\Biggl|\sum_{j_1=p_1+1}^{\infty}\left(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1))-P_{j_1-1}(z(t_1))P_{j_1}(z(t_1))\right)\Biggr|+
$$

$$
+\frac{T-t}{4}\Biggl|\sum_{j_1=p_1+1}^{\infty}\Biggl\{\psi_1'(t_1)\Biggl(\frac{1}{2j_1+3}(P_{j_1+2}(z(t_1))-P_{j_1}(z(t_1)))-
$$

$$
-\frac{1}{2j_1-1}(P_{j_1}(z(t_1))-P_{j_1-2}(z(t_1)))\Biggr)-
$$

$$
-\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}\Biggl(\frac{1}{2j_1+3}(P_{j_1+2}(y)-P_{j_1}(y))-
$$

$$
-\frac{1}{2j_1-1}(P_{j_1}(y)-P_{j_1-2}(y))\Biggr)\psi_1''(u(y))dy\Biggr\}P_{j_1}(z(t_1))\Biggr|, \tag{2.22}
$$

where C_0 is a constant, ψ_1' and ψ_1'' are derivatives of the function $\psi_1(s)$ with respect to the variable $u(y)$.

From (2.22) and the well known estimate for Legendre polynomials [115]

$$
|P_n(y)|<\frac{K}{\sqrt{n+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad n\in\mathbf{N}, \tag{2.23}
$$

where constant K does not depend on y and n, we have

$$
|G_{p_1}(t_1)|<
$$

$$
<C_0\Biggl|\lim_{n\to\infty}\sum_{j_1=p_1+1}^{n}\left(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1))-P_{j_1-1}(z(t_1))P_{j_1}(z(t_1))\right)\Biggr|+
$$

$$
+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}}+\int\limits_{-1}^{z(t_1)}\frac{dy}{(1-y^2)^{1/4}}\frac{1}{(1-(z(t_1))^2)^{1/4}}\right)<
$$

$$
< C_0 \left| \lim_{n\to\infty} \left(P_{n+1}(z(t_1))P_n(z(t_1)) - P_{p_1}(z(t_1))P_{p_1+1}(z(t_1))\right)\right| +
$$

$$
+C_1 \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + C_2 \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) <
$$

$$
< C_3 \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{p_1}\right)\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} +
$$

$$
+C_1 \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + C_2 \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) \le
$$

$$
\le C_4\left(\left(\frac{1}{p_1}+\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + \sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) \le
$$

$$
\le \frac{K}{p_1}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \tag{2.24}
$$

where $C_0, C_1, \ldots, C_4, K$ are constants, $t_1 \in (t, T)$, and

$$
\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2} \le \int\limits_{p_1}^{\infty}\frac{dx}{x^2} = \frac{1}{p_1}. \tag{2.25}
$$

From (2.18) and (2.24) we get

$$
\left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| <
$$

$$
< \frac{K}{p_1}\left(\int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/2}} + \int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/4}}\right) \to 0
$$

if $p_1 \to \infty$. So, we obtain

$$
\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 =
$$

$$
= \sum_{j_1=0}^{\infty}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty}\int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 =
$$

$$
=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\int\limits_t^T C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1}. \qquad (2.26)
$$

In (2.26) we used the fact that the Fourier–Legendre series

$$
\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)
$$

of the smooth function $C_{j_1}(t_1)$ converges uniformly to this function at the interval $[t+\varepsilon,T-\varepsilon]$ for any $\varepsilon>0$, converges to this function at the any point $t_1\in(t,T)$, and converges to $C_{j_1}(t+0)$ and $C_{j_1}(T-0)$ when $t_1=t,\ T$.

More precisely, we have

$$
\int\limits_t^T\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1=\int\limits_{t+\varepsilon}^{T-\varepsilon}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1+A_{\varepsilon}+B_{\varepsilon}=
$$

$$
=\sum_{j_2=0}^{\infty}C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1+A_{\varepsilon}+B_{\varepsilon}=
$$

$$
=\sum_{j_2=0}^{\infty}C_{j_2j_1}\left(\int\limits_t^T-\int\limits_t^{t+\varepsilon}-\int\limits_{T-\varepsilon}^{T}\right)\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1+A_{\varepsilon}+B_{\varepsilon}=
$$

$$
=\sum_{j_2=0}^{\infty}C_{j_2j_1}\Biggl(\mathbf{1}_{\{j_1=j_2\}}-\varepsilon\bigl(\phi_{j_2}(\lambda)\phi_{j_1}(\lambda)+\phi_{j_2}(\theta)\phi_{j_1}(\theta)\bigr)\Biggr)+A_{\varepsilon}+B_{\varepsilon}=
$$

$$
=C_{j_1j_1}-\varepsilon\left(\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(\lambda)\phi_{j_1}(\lambda)+\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(\theta)\phi_{j_1}(\theta)\right)+A_{\varepsilon}+B_{\varepsilon}, \qquad (2.27)
$$

where $\theta\in[t,t+\varepsilon]$, $\lambda\in[T-\varepsilon,T]$, and

$$
A_{\varepsilon}=\int\limits_t^{t+\varepsilon}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1,\quad B_{\varepsilon}=\int\limits_{T-\varepsilon}^{T}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1.
$$

In obtaining (2.27) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j=0,1,2\ldots$

Further, we have $|A_\varepsilon| + |B_\varepsilon| \le \varepsilon C$, where $C < \infty$ is a constant. Performing the passage to the limit $\lim\limits_{\varepsilon\to+0}$ in the equality (2.27), we get

$$\int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = C_{j_1 j_1}.$$

Then (see (2.26))

$$\sum_{j_1=0}^{\infty}\int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}$$

and the relation (2.10) is proved for the case of Legendre polynomials.

Let us consider the trigonometric case and suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in $L_2([t,T])$.

Denote

$$S_{p_1} \stackrel{\sf def}{=} \left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| =$$

$$= \left|\int\limits_t^T \sum_{j_1=p_1+1}^{\infty} \psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|.$$

We have

$$S_{2p_1} = \left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{2p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| =$$

$$= \left|\int\limits_t^T \sum_{j_1=2p_1+1}^{\infty} \psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right| =$$

$$= \frac{2}{T-t}\left|\int\limits_t^T \psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^{t_1}\psi_1(s)\sin\frac{2\pi j_1(s-t)}{T-t}ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.$$

$$\left.\left.+\int\limits_t^{t_1}\psi_1(s)\cos\frac{2\pi j_1(s-t)}{T-t}ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\right)dt_1\right| =$$

$$
=\frac{1}{\pi}\left|\int\limits_t^T\left(\psi_1(t)\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.
$$

$$
+\frac{T-t}{2\pi}\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\psi_1'(t_1)-\psi_1'(t)\cos\frac{2\pi j_1(t_1-t)}{T-t}-\right.
$$

$$
-\int\limits_t^{t_1}\sin\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}-
$$

$$
\left.\left.\left.-\int\limits_t^{t_1}\cos\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\right)\right)dt_1\right|\le
$$

$$
\le C_1\left|\int\limits_t^T\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right|+\frac{C_2}{p_1}=
$$

$$
=C_1\left|\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\int\limits_t^T\psi_2(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right|+\frac{C_2}{p_1}, \tag{2.28}
$$

where constants C_1, C_2 do not depend on p_1.

Here we used the fact that the functional series

$$
\sum_{j_1=1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t} \tag{2.29}
$$

converges uniformly at the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon>0$ due to Dirichlet–Abel Theorem, and converges to zero at the points t and T. Moreover, the series (2.29) (with accuracy to a linear transformation) is the trigonometric Fourier series of the smooth function $K(t_1)=t_1-t$, $t_1\in[t,T]$. Thus, (2.29) converges to the smooth function at any point $t_1\in(t,T)$.

From (2.28) we obtain

$$
S_{2p_1}\le C_3\left|\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\psi_2(T)-\psi_2(t)-\int\limits_t^T\cos\frac{2\pi j_1(s-t)}{T-t}\psi_2'(s)ds\right)\right|+\frac{C_2}{p_1}\le\frac{C_4}{p_1}, \tag{2.30}
$$

where constants C_2, C_3, C_4 do not depend on p_1.

Further,

$$S_{2p_1-1}=\left|\int\limits_t^T\sum_{j_1=2p_1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|=$$

$$=\left|S_{2p_1}+\int\limits_t^T\psi_2(t_1)\phi_{2p_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{2p_1}(\theta)d\theta dt_1\right|\le$$

$$\le S_{2p_1}+\frac{2}{T-t}\left|\int\limits_t^T\psi_2(t_1)\cos\frac{2\pi p_1(t_1-t)}{T-t}\int\limits_t^{t_1}\psi_1(\theta)\cos\frac{2\pi p_1(\theta-t)}{T-t}d\theta dt_1\right|. \quad (2.31)$$

Moreover,

$$\int\limits_t^T\psi_2(t_1)\cos\frac{2\pi p_1(t_1-t)}{T-t}\int\limits_t^{t_1}\psi_1(\theta)\cos\frac{2\pi p_1(\theta-t)}{T-t}d\theta dt_1=$$

$$=\frac{T-t}{2\pi p_1}\int\limits_t^T\psi_2(t_1)\cos\frac{2\pi p_1(t_1-t)}{T-t}\left(\psi_1(t_1)\sin\frac{2\pi p_1(t_1-t)}{T-t}-\right.$$

$$\left.-\int\limits_t^T\psi_1'(\theta)\sin\frac{2\pi p_1(\theta-t)}{T-t}d\theta\right)dt_1. \quad (2.32)$$

The relations (2.30)–(2.32) imply that

$$S_{2p_1-1}\le\frac{C_5}{p_1}, \quad (2.33)$$

where constant C_5 is independent of p_1.

From (2.30) and (2.33) we obtain

$$S_{p_1}=\left|\int\limits_t^T\sum_{j_1=p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|\le\frac{K}{p_1}\to 0 \quad (2.34)$$

if $p_1\to\infty$, where constant K does not depend on p_1 ($p_1\in\mathbf{N}$).

Further steps are similar to the proof of (2.10) for the case of Legendre polynomials. Theorem 2.1 is proved.

Note that the estimate (2.34) will be used further.

2.1.2 Approach Based on Theorem 1.1 and Double Fourier–Legendre Series Summarized by Pringsheim Method

In Sect. 2.1.1 we considered the proof of Theorem 2.1 based on Theorem 1.1 and double integration by parts (this procedure leads to the requirement of double continuous differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$). In this section, we formulate and prove an analogue of Theorem 2.1 but under the weakened conditions: the functions $\psi_1(\tau)$, $\psi_2(\tau)$ only one time continuously differentiable at the interval $[t,T]$. At that we will use the double Fourier series summarized by Pringsheim method.

Theorem 2.2 [13]-[17], [28], [47]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, $\psi_1(s)$, $\psi_2(s)$ are continuously differentiable functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\tag{2.35}$$

that converges in the mean-square sense is valid, where

$$C_{j_2j_1}=\int\limits_t^T\psi_2(s_2)\phi_{j_2}(s_2)\int\limits_t^{s_2}\psi_1(s_1)\phi_{j_1}(s_1)ds_1ds_2\tag{2.36}$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. Theorem 2.2 will be proved if we prove the equality (see the proof of Theorem 2.1)

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1},\tag{2.37}$$

where $C_{j_1 j_1}$ is defined by the formula (1.8) for $k = 2$ and $j_1 = j_2$. At that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

Firstly, consider the sufficient conditions of convergence of double Fourier–Legendre series summarized by Pringsheim method.

Let $P_j(x)$ $(j = 0, 1, 2, \ldots)$ be the Legendre polynomial. Consider the function $f(x, y)$ defined for $(x, y) \in [-1, 1]^2$. Furthermore, consider the double Fourier–Legendre series summarized by Pringsheim method and corresponding to the function $f(x, y)$

$$\lim_{n,m\to\infty} \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{1}{2}\sqrt{(2j+1)(2i+1)}C_{ij}^{*}P_i(x)P_j(y) \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \sum_{i,j=0}^{\infty} \frac{1}{2}\sqrt{(2j+1)(2i+1)}C_{ij}^{*}P_i(x)P_j(y), \qquad (2.38)$$

where

$$C_{ij}^{*} = \frac{1}{2}\sqrt{(2j+1)(2i+1)} \int\limits_{[-1,1]^2} f(x, y)P_i(x)P_j(y)dxdy. \qquad (2.39)$$

Consider the generalization for the case of two variables [120] of the theorem on equiconvergence for the Fourier–Legendre series [121].

Proposition 2.1 [120]. *Let $f(x, y) \in L_2([-1, 1]^2)$ and the function*

$$f(x, y)\left(1 - x^2\right)^{-1/4}\left(1 - y^2\right)^{-1/4}$$

is integrable on $[-1, 1]^2$. Moreover, let

$$|f(x, y) - f(u, v)| \le G(y)|x - u| + H(x)|y - v|,$$

where $G(y), H(x)$ are bounded functions on $[-1, 1]^2$. Then for all $(x, y) \in (-1, 1)^2$ the following equality is satisfied

$$\lim_{n,m\to\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{1}{2}\sqrt{(2j+1)(2i+1)}C_{ij}^{*}P_i(x)P_j(y)-\right.$$

$$\left.-(1-x^2)^{-1/4}(1-y^2)^{-1/4}S_{nm}(\arccos x, \arccos y, F)\right) = 0. \qquad (2.40)$$

At that, the convergence in (2.40) is uniform on the rectangle

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \textit{for any} \quad \varepsilon, \delta > 0,$$

$S_{nm}(\theta, \varphi, F)$ *is a partial sum of the double trigonometric Fourier series of the auxiliary function*

$$F(\theta, \varphi) = \sqrt{|\sin\theta|}\sqrt{|\sin\varphi|} f(\cos\theta, \cos\varphi), \quad \theta, \varphi \in [0, \pi],$$

and the Fourier coefficient C^*_{ij} *is defined by* (2.39).

Proposition 2.1 implies that the following equality

$$\lim_{n,m\to\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{1}{2}\sqrt{(2j+1)(2i+1)}C^*_{ij}P_i(x)P_j(y) - f(x,y)\right) = 0 \quad (2.41)$$

is fulfilled for all $(x, y) \in (-1, 1)^2$, and convergence in (2.41) is uniform on the rectangle

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \text{for any} \quad \varepsilon, \delta > 0$$

if the corresponding conditions of convergence of the double trigonometric Fourier series of the auxiliary function

$$g(x, y) = f(x, y)\left(1 - x^2\right)^{1/4}\left(1 - y^2\right)^{1/4} \quad (2.42)$$

are satisfied.

Note also that Proposition 2.1 does not imply any conclusions on the behavior of the double Fourier–Legendre series on the boundary of the square $[-1, 1]^2$.

For each $\delta > 0$ *let us call the exact upper edge of difference* $|f(\mathbf{t}') - f(\mathbf{t}'')|$ *in the set of all points* $\mathbf{t}'$, $\mathbf{t}''$ *which belong to the domain* D *as the module of continuity of the function* $f(\mathbf{t})$ $(\mathbf{t} = (t_1, \ldots, t_k))$ *in the* k*-dimentional domain* D $(k \ge 1)$ *if the distance between* $\mathbf{t}', \mathbf{t}''$ *satisfies the condition* $\rho(\mathbf{t}', \mathbf{t}'') < \delta$.

We will say that the function of k $(k \ge 1)$ *variables* $f(\mathbf{t})$ $(\mathbf{t} = (t_1, \ldots, t_k))$ *belongs to the Hölder class with the parameter* $\alpha \in (0, 1]$ $(f(\mathbf{t}) \in C^{\alpha}(D))$ *in the domain* D *if the module of continuity of the function* $f(\mathbf{t})$ $(\mathbf{t} = (t_1, \ldots, t_k))$ *in the domain* D *has orders* $o(\delta^{\alpha})$ $(\alpha \in (0, 1))$ *and* $O(\delta)$ $(\alpha = 1)$.

In 1967, Zhizhiashvili L.V. proved that the rectangular sums of multiple trigonometric Fourier series of the function of k variables in the hypercube $[t, T]^k$ converge uniformly to this function in the hypercube $[t, T]^k$ if the function belongs to $C^{\alpha}([t,T]^k)$, $\alpha>0$ (definition of the Hölder class with any parameter $\alpha>0$ can be found in the well known mathematical analysis tutorials [122]).

More precisely, the following statement is correct.

Proposition 2.2 [122]. *If the function $f(x_1,\ldots,x_n)$ is periodic with period 2π with respect to each variable and belongs in $\mathbf{R}^n$ to the Hölder class $C^{\alpha}(\mathbf{R}^n)$ for any $\alpha>0$, then the rectangular partial sums of multiple trigonometric Fourier series of the function $f(x_1,\ldots,x_n)$ converge to this function uniformly in $\mathbf{R}^n$.*

Let us back to the proof of Theorem 2.2 and consider the following Lemma.

Lemma 2.1. *Let the function $f(x,y)$ satisfies to the following condition*

$$|f(x,y)-f(x_1,y_1)|\le C_1|x-x_1|+C_2|y-y_1|,$$

where $C_1,C_2<\infty$ and (x,y), $(x_1,y_1)\in[-1,1]^2$. Then the following inequality is fulfilled

$$|g(x,y)-g(x_1,y_1)|\le K\rho^{1/4}, \tag{2.43}$$

where $g(x,y)$ in defined by (2.42),

$$\rho=\sqrt{(x-x_1)^2+(y-y_1)^2},$$

(x,y) and $(x_1,y_1)\in[-1,1]^2$, $K<\infty$.

Proof. First, we assume that $x\ne x_1$, $y\ne y_1$. In this case we have

$$|g(x,y)-g(x_1,y_1)|=$$

$$=\Bigl|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}\left(f(x,y)-f(x_1,y_1)\right)+$$

$$+f(x_1,y_1)\left(\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right)\Bigr|\le$$

$$\le C_1\left|x-x_1\right|+C_2\left|y-y_1\right|+$$

$$+C_3\left|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right|, \tag{2.44}$$

where $C_3<\infty$.

Moreover,

$$\left|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right|=$$

$$=\Bigl|\left(1-x^2\right)^{1/4}\left(\left(1-y^2\right)^{1/4}-\left(1-y_1^2\right)^{1/4}\right)+$$

$$+\left(1-y_1^2\right)^{1/4}\left(\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right)\Bigr| \le$$

$$\le \left|\left(1-y^2\right)^{1/4}-\left(1-y_1^2\right)^{1/4}\right|+\left|\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right|, \qquad (2.45)$$

$$\left|\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right| =$$

$$=\Bigl|\left((1-x)^{1/4}-(1-x_1)^{1/4}\right)(1+x)^{1/4}+$$

$$+(1-x_1)^{1/4}\left((1+x)^{1/4}-(1+x_1)^{1/4}\right)\Bigr| \le$$

$$\le K_1\left(\left|(1-x)^{1/4}-(1-x_1)^{1/4}\right|+\left|(1+x)^{1/4}-(1+x_1)^{1/4}\right|\right), \qquad (2.46)$$

where $K_1 < \infty$.

It is not difficult to see that

$$\left|(1\pm x)^{1/4}-(1\pm x_1)^{1/4}\right| =$$

$$=\frac{|(1\pm x)-(1\pm x_1)|}{\left((1\pm x)^{1/2}+(1\pm x_1)^{1/2}\right)\left((1\pm x)^{1/4}+(1\pm x_1)^{1/4}\right)}=$$

$$=|x_1-x|^{1/4}\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}}\cdot\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}} \le$$

$$\le |x_1-x|^{1/4}. \qquad (2.47)$$

The last inequality follows from the obvious inequalities

$$\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}} \le 1,$$

$$\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}} \le 1.$$

From (2.44)–(2.47) we obtain

$$|g(x,y)-g(x_1,y_1)| \le$$

$$\le C_1|x-x_1|+C_2|y-y_1|+C_4\left(|x_1-x|^{1/4}+|y_1-y|^{1/4}\right) \le$$

$$\le C_5\rho+C_6\rho^{1/4} \le K\rho^{1/4},$$

where $C_5, C_6, K < \infty$.

The cases $x = x_1$, $y \neq y_1$ and $x \neq x_1$, $y = y_1$ can be considered analogously to the case $x \neq x_1$, $y \neq y_1$. At that, the consideration begins from the inequalities

$$|g(x,y) - g(x_1,y_1)| \le K_2 \left|\left(1-y^2\right)^{1/4} f(x,y) - \left(1-y_1^2\right)^{1/4} f(x_1,y_1)\right|$$

$(x = x_1,\ y \neq y_1)$ and

$$|g(x,y) - g(x_1,y_1)| \le K_2 \left|\left(1-x^2\right)^{1/4} f(x,y) - \left(1-x_1^2\right)^{1/4} f(x_1,y_1)\right|$$

$(x \neq x_1,\ y = y_1)$, where $K_2 < \infty$. Lemma 2.1 is proved.

Lemma 2.1 and Proposition 2.2 imply that rectangular sums of double trigonometric Fourier series of the function $g(x,y)$ converge uniformly to the function $g(x,y)$ in the square $[-1,1]^2$. This means that the equality (2.41) holds.

Consider the auxiliary function

$$K'(t_1,t_2) = \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \ge t_2 \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1 \le t_2 \end{cases}, \quad t_1,t_2 \in [t,T] \tag{2.48}$$

and prove that

$$|K'(t_1,t_2) - K'(t_1^*,t_2^*)| \le L\left(|t_1 - t_1^*| + |t_2 - t_2^*|\right), \tag{2.49}$$

where $L < \infty$ and (t_1,t_2), $(t_1^*,t_2^*) \in [t,T]^2$.

By the Lagrange formula for the functions $\psi_1(t_1^*)$, $\psi_2(t_1^*)$ at the interval

$$\left[\min\left\{t_1,t_1^*\right\}, \max\left\{t_1,t_1^*\right\}\right]$$

and for the functions $\psi_1(t_2^*)$, $\psi_2(t_2^*)$ at the interval

$$\left[\min\left\{t_2,t_2^*\right\}, \max\left\{t_2,t_2^*\right\}\right]$$

we obtain

$$|K'(t_1,t_2) - K'(t_1^*,t_2^*)| \le$$

$$\le \left| \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \ge t_2 \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1 \le t_2 \end{cases} - \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \ge t_2^* \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^* \le t_2^* \end{cases} \right| +$$

$$
+L_1\left|t_1-t_1^*\right|+L_2\left|t_2-t_2^*\right|, \quad L_1, L_2<\infty. \tag{2.50}
$$

We have

$$
\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases} - \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases} =
$$

$$
=\begin{cases}\ \ \ \ 0, \quad t_1\ge t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\le t_2,\ t_1^*\le t_2^*\\ \\ \psi_2(t_1)\psi_1(t_2)-\psi_1(t_1)\psi_2(t_2), \quad t_1\ge t_2,\ t_1^*\le t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2)-\psi_2(t_1)\psi_1(t_2), \quad t_1\le t_2,\ t_1^*\ge t_2^*\end{cases}. \tag{2.51}
$$

By Lagrange formula for the functions $\psi_1(t_2)$, $\psi_2(t_2)$ at the interval

$$
[\min\{t_1,t_2\},\max\{t_1,t_2\}]
$$

we obtain the estimate

$$
\left|\ \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases} - \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}\right| \le
$$

$$
\le L_3|t_2-t_1|\begin{cases}0, \quad t_1\ge t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\le t_2,\ t_1^*\le t_2^*\\ \\ 1, \quad t_1\le t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\ge t_2,\ t_1^*\le t_2^*\end{cases}, \tag{2.52}
$$

where $L_3<\infty$.

Let us show that if $t_1\le t_2,\ t_1^*\ge t_2^*$ or $t_1\ge t_2,\ t_1^*\le t_2^*$, then the following inequality is satisfied

$$
|t_2-t_1|\le|t_1^*-t_1|+|t_2^*-t_2|. \tag{2.53}
$$

First, consider the case $t_1\ge t_2,\ t_1^*\le t_2^*$. For this case

$$
t_2+(t_1^*-t_2^*)\le t_2\le t_1.
$$

Then

$$
(t_1^*-t_1)-(t_2^*-t_2)\le t_2-t_1\le 0
$$

and (2.53) is satisfied.

For the case $t_1 \le t_2,\ t_1^* \ge t_2^*$ we obtain

$$t_1 + (t_2^* - t_1^*) \le t_1 \le t_2.$$

Then

$$(t_1 - t_1^*) - (t_2 - t_2^*) \le t_1 - t_2 \le 0$$

and also (2.53) is satisfied.

From (2.52) and (2.53) we have

$$\left| \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \ge t_2 \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1 \le t_2 \end{cases} - \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \ge t_2^* \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^* \le t_2^* \end{cases} \right| \le$$

$$\le L_3\left(|t_1^* - t_1| + |t_2^* - t_2|\right) \begin{cases} 0, & t_1 \ge t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \le t_2,\ t_1^* \le t_2^* \\ \\ 1, & t_1 \le t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \ge t_2,\ t_1^* \le t_2^* \end{cases} \le$$

$$\le L_3\left(|t_1^* - t_1| + |t_2^* - t_2|\right) \begin{cases} 1, & t_1 \ge t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \le t_2,\ t_1^* \le t_2^* \\ \\ 1, & t_1 \le t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \ge t_2,\ t_1^* \le t_2^* \end{cases} =$$

$$= L_3\left(|t_1^* - t_1| + |t_2^* - t_2|\right). \tag{2.54}$$

From (2.50), (2.54) we obtain (2.49). Let

$$t_1 = \frac{T-t}{2}x + \frac{T+t}{2}, \quad t_2 = \frac{T-t}{2}y + \frac{T+t}{2},$$

where $x, y \in [-1, 1]$. Then

$$K'(t_1, t_2) \equiv K''(x, y) = \begin{cases} \psi_2\left(h(x)\right)\psi_1\left(h(y)\right), & x \ge y \\ \\ \psi_1\left(h(x)\right)\psi_2\left(h(y)\right), & x \le y \end{cases},$$

where $x, y \in [-1, 1]$ and

$$h(x) = \frac{T-t}{2}x + \frac{T+t}{2}. \tag{2.55}$$

The inequality (2.49) can be rewritten in the form

$$|K''(x,y) - K''(x^*,y^*)| \le L^* \left(|x - x^*| + |y - y^*|\right), \tag{2.56}$$

where $L^* < \infty$ and (x,y), $(x^*,y^*) \in [-1,1]^2$.

Thus, the function $K''(x,y)$ satisfies the conditions of Lemma 2.1. Hence, for the function

$$K''(x,y)\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}$$

the inequality (2.43) is correct.

Due to the continuous differentiability of the functions $\psi_1(h(x))$ and $\psi_2(h(x))$ at the interval $[-1,1]$ we have $K''(x,y) \in L_2([-1,1]^2)$. In addition

$$\int\limits_{[-1,1]^2} \frac{K''(x,y)dxdy}{(1-x^2)^{1/4}(1-y^2)^{1/4}} \le C\left(\int\limits_{-1}^{1} \frac{1}{(1-x^2)^{1/4}}\int\limits_{-1}^{x}\frac{1}{(1-y^2)^{1/4}}dydx + \right.$$

$$\left. + \int\limits_{-1}^{1} \frac{1}{(1-x^2)^{1/4}}\int\limits_{x}^{1}\frac{1}{(1-y^2)^{1/4}}dydx\right) < \infty, \quad C < \infty.$$

Thus, the conditions of Proposition 2.1 are fulfilled for the function $K''(x,y)$. Note that the mentioned properties of the function $K''(x,y)$, $x,y \in [-1,1]$ also correct for the function $K'(t_1,t_2)$, $t_1,t_2 \in [t,T]$.

Remark 2.1. *On the basis of* (2.49) *it can be argued that the function* $K'(t_1,t_2)$ *belongs to the Hölder class with parameter* 1 *in* $[t,T]^2$*. Hence by Proposition* 2.2 *this function can be expanded into the uniformly convergent double trigonometric Fourier series in the square* $[t,T]^2$*, which summarized by Pringsheim method. However, the expansions of iterated stochastic integrals obtained by using the system of Legendre polynomials are essentially simpler than their analogues obtained by using the trigonometric system of functions* (*see Chapter* 5 *for details*)*.*

Let us expand the function $K'(t_1,t_2)$ into a multiple (double) Fourier–Legendre series or trigonometric Fourier series in the square $[t,T]^2$. This series is summable by the method of rectangular sums (Pringsheim method), i.e.

$$K'(t_1,t_2) = \lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2) =$$

$$
= \lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} \left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+ \right.
$$

$$
\left. + \int\limits_t^T \psi_1(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^T \psi_2(t_1)\phi_{j_1}(t_1)dt_1 \right) dt_2\phi_{j_1}(t_1)\phi_{j_2}(t_2) =
$$

$$
= \lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} \left(C_{j_2j_1} + C_{j_1j_2}\right) \phi_{j_1}(t_1)\phi_{j_2}(t_2), \qquad (2.57)
$$

where $(t_1, t_2) \in (t, T)^2$. At that, the convergence of the series (2.57) is uniform on the rectangle

$[t+\varepsilon, T-\varepsilon] \times [t+\delta, T-\delta]$ for any $\varepsilon, \delta > 0$ (in particular, we can choose $\varepsilon = \delta$).

In addition, the series (2.57) converges to $K'(t_1, t_2)$ at any inner point of the square $[t, T]^2$.

Note that Proposition 2.1 does not answer the question of convergence of the series (2.57) on the boundary of the square $[t, T]^2$.

In obtaining (2.57) we replaced the order of integration in the second iterated integral.

Let us substitute $t_1 = t_2$ in (2.57). After that, let us rewrite the limit on the right-hand side of (2.57) as two limits. Let us replace j_1 with j_2, j_2 with j_1, n_1 with n_2, and n_2 with n_1 in the second limit. Thus, we get

$$
\lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1), \quad t_1 \in (t, T). \qquad (2.58)
$$

According to the above reasoning, the convergence in (2.58) is uniform on the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon > 0$. Additionally, (2.58) holds at each interior point of the interval $[t, T]$.

Let us fix $\varepsilon > 0$ and integrate the equality (2.58) at the interval $[t+\varepsilon, T-\varepsilon]$. Due to the uniform convergence of the series (2.58) we can swap the series and the integral

$$
\lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} C_{j_2j_1} \int\limits_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 = \frac{1}{2}\int\limits_{t+\varepsilon}^{T-\varepsilon} \psi_1(t_1)\psi_2(t_1)dt_1. \qquad (2.59)
$$

Lemma 2.2. *Under the conditions of Theorem 2.2 the following limit*

$$\lim_{n\to\infty}\sum_{j_1=0}^{n} C_{j_1 j_1}$$

exists and is finite, where $C_{j_1 j_1}$ is defined by (2.36) if $j_1 = j_2$, i.e.

$$C_{j_1 j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2.$$

Lemma 2.2 has already been proved in Sect. 2.1.1 under stronger conditions. Further, in this section, another proof of Lemma 2.2 is given. This will allow us to obtain useful estimates that will be used later in Chapter 2.

Applying the equality (2.59) for $n_1 = n_2 = n$ and Lemma 2.2, we get

$$\frac{1}{2}\int\limits_{t+\varepsilon}^{T-\varepsilon}\psi_1(t_1)\psi_2(t_1)dt_1 = \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2 j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$= \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2 j_1}\left(\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \int\limits_t^{t+\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \right.$$

$$\left. - \int\limits_{T-\varepsilon}^{T}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1\right) =$$

$$= \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2 j_1}\left(\mathbf{1}_{\{j_1=j_2\}} - \Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)\varepsilon\right) =$$

$$= \lim_{n\to\infty}\sum_{j_1=0}^{n} C_{j_1 j_1} - \varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2 j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr), \quad (2.60)$$

where $\theta \in [t, t+\varepsilon]$, $\lambda \in [T-\varepsilon, T]$. In obtaining (2.60) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j = 0, 1, 2\ldots$

Applying (2.60), we obtain

$$\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2 j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr) =$$

$$= \lim_{n\to\infty}\sum_{j_1=0}^{n} C_{j_1j_1} - \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1,$$

where the limits

$$\lim_{n\to\infty}\sum_{j_1=0}^{n} C_{j_1j_1}, \quad \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1$$

exist and are finite (see Lemma 2.2 and the equality (2.59)). This means that the limit

$$\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)$$

also exists and is finite.

Suppose that the following relations

$$\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(T)\phi_{j_1}(T)\right| \le K<\infty, \quad \left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(t)\phi_{j_1}(t)\right| \le K<\infty \tag{2.61}$$

are satisfied for $n\in\mathbf{N}$ (the relations (2.61) will be proved further in this section); constant K does not depend on n.

Note that

$$\left|\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)\right| =$$

$$= \lim_{n\to\infty}\varepsilon\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right|. \tag{2.62}$$

Using (2.58) ($n_1=n_2=n$) and (2.61), we obtain

$$\varepsilon\lim_{n\to\infty}\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right| \le$$

$$\le \varepsilon\lim_{n\to\infty}\left(\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)\right|+\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right|\right) \le 2\varepsilon K_1 \ \to\ 0 \tag{2.63}$$

if $\varepsilon \to +0$, where $\theta \in [t, t+\varepsilon]$, $\lambda \in [T-\varepsilon, T]$, constant K_1 is independent on n.

Performing the passage to the limit $\lim\limits_{\varepsilon \to +0}$ in the equality (2.60) and taking into account (2.62), (2.63), we get

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{2.64}$$

Thus, to complete the proof of Theorem 2.2, it is necessary to prove (2.61). To prove (2.61), as well as for further consideration, we need some well known properties of the Legendre polynomials [115], [121].

The complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \tag{2.65}$$

where $P_j(x)$ is the Legendre polynomial.

It is known that the Legendre polynomial $P_j(x)$ is represented as

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

At the boundary points of the orthogonality interval the Legendre polynomials satisfy the following relations

$$P_j(1) = 1, \quad P_j(-1) = (-1)^j,$$

$$P_{j+1}(1) - P_j(1) = 0, \quad P_{j+1}(-1) + P_j(-1) = 0,$$

where $j = 0, 1, 2, \ldots$

Relation of the Legendre polynomial $P_j(x)$ with derivatives of the Legendre polynomials $P_{j+1}(x)$ and $P_{j-1}(x)$ is expressed by the following equality

$$P_j(x) = \frac{1}{2j+1}\left(P'_{j+1}(x) - P'_{j-1}(x)\right), \quad j = 1, 2, \ldots \tag{2.66}$$

The recurrent relation has the form

$$xP_j(x) = \frac{(j+1)P_{j+1}(x) + jP_{j-1}(x)}{2j+1}, \quad j = 1, 2, \ldots$$

Orthogonality of the Legendre polynomial $P_j(x)$ to any polynomial $Q_k(x)$ of lesser degree k we write in the following form

$$\int_{-1}^{1} Q_k(x)P_j(x)dx = 0, \quad k = 0, 1, 2, \ldots, j-1.$$

From the property

$$\int_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if } k \neq j \\ 2/(2j+1) & \text{if } k = j \end{cases}$$

it follows that the orthonormal on the interval $[-1, 1]$ Legendre polynomials determined by the relation

$$P_j^*(x) = \sqrt{\frac{2j+1}{2}}P_j(x), \quad j = 0, 1, 2, \ldots$$

Remind that there is the following estimate [115]

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j = 1, 2, \ldots, \tag{2.67}$$

where constant K does not depend on y and j.

Moreover,

$$|P_j(x)| \le 1, \quad x \in [-1, 1], \quad j = 0,\ 1, \ldots \tag{2.68}$$

The Christoffel–Darboux formula has the form

$$\sum_{j=0}^{n}(2j+1)P_j(x)P_j(y) = (n+1)\frac{P_n(x)P_{n+1}(y) - P_{n+1}(x)P_n(y)}{y-x}. \tag{2.69}$$

Let us prove (2.61) (see [28])). From (2.69) for $x = \pm 1$ we obtain

$$\sum_{j=0}^{n}(2j+1)P_j(y) = (n+1)\frac{P_{n+1}(y) - P_n(y)}{y-1}, \tag{2.70}$$

$$\sum_{j=0}^{n}(2j+1)(-1)^j P_j(y)=(n+1)(-1)^n\frac{P_{n+1}(y)+P_n(y)}{y+1}. \tag{2.71}$$

From the other hand (see (2.66))

$$\sum_{j=0}^{n}(2j+1)P_j(y)=1+\sum_{j=1}^{n}(2j+1)P_j(y)=$$

$$=1+\sum_{j=1}^{n}(P_{j+1}^{'}(y)-P_{j-1}^{'}(y))=1+\left(\sum_{j=1}^{n}(P_{j+1}(y)-P_{j-1}(y))\right)^{'}=$$

$$=1+(P_{n+1}(x)+P_n(x)-x-1)'=(P_n(x)+P_{n+1}(x))' \tag{2.72}$$

and

$$\sum_{j=0}^{n}(2j+1)(-1)^j P_j(y)=1+\sum_{j=1}^{n}(-1)^j(2j+1)P_j(y)=$$

$$=1+\sum_{j=1}^{n}(-1)^j(P_{j+1}^{'}(y)-P_{j-1}^{'}(y))=1+\left(\sum_{j=1}^{n}(-1)^j(P_{j+1}(y)-P_{j-1}(y))\right)^{'}=$$

$$=1+((-1)^n(P_{n+1}(x)-P_n(x))-x+1)'=(-1)^n(P_{n+1}(x)-P_n(x))'. \tag{2.73}$$

Applying (2.70)–(2.73), we get

$$(n+1)\frac{P_{n+1}(y)-P_n(y)}{y-1}=(P_n(x)+P_{n+1}(x))', \tag{2.74}$$

$$(n+1)\frac{P_{n+1}(y)+P_n(y)}{y+1}=(P_{n+1}(x)-P_n(x))'. \tag{2.75}$$

Let us prove the boundedness of the first sum in (2.61). We have

$$\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\phi_{j_2}(T)\phi_{j_1}(T)=$$

$$=\frac{1}{4}\sum_{j_2=0}^{n}\sum_{j_1=0}^{n}(2j_2+1)(2j_1+1)\int\limits_{-1}^{1}\psi_2(h(y))P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))P_{j_1}(y_1)dy_1dy=$$

$$
= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\sum_{j_2=0}^{n}(2j_2+1)P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))\sum_{j_1=0}^{n}(2j_1+1)P_{j_1}(y_1)dy_1dy =
$$

$$
= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)) =
$$

$$
= \frac{1}{4}\int\limits_{-1}^{1}\psi_1(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)) +
$$

$$
+ \frac{1}{4}\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)) =
$$

$$
= \frac{1}{4}I_1 + \frac{1}{4}I_2,
$$

where

$$
\Delta(h(y)) = \psi_2(h(y)) - \psi_1(h(y)), \quad h(y) = \frac{T-t}{2}y + \frac{T+t}{2}. \tag{2.76}
$$

Further,

$$
I_1 = \frac{1}{2}\Biggl(\int\limits_{-1}^{1}\psi_1(h(y))d(P_{n+1}(y)+P_n(y))\Biggr)^2 =
$$

$$
= \frac{1}{2}\Biggl(2\psi_1(T) - \int\limits_{-1}^{1}(P_{n+1}(y)+P_n(y))\psi_1'(h(y))\frac{T-t}{2}dy\Biggr)^2 < C_1 < \infty,
$$

where ψ_1' is a derivative of the function ψ_1 with respect to the variable y, constant C_1 does not depend on n.

By the Lagrange formula we obtain

$$
\Delta(h(y)) = \psi_2\Biggl(\frac{1}{2}(T-t)(y-1)+T\Biggr) - \psi_1\Biggl(\frac{1}{2}(T-t)(y-1)+T\Biggr) =
$$

$$
= \psi_2(T) - \psi_1(T) + (y-1)\Biggl(\psi_2'(\xi_y) - \psi_1'(\theta_y)\Biggr)\frac{1}{2}(T-t) =
$$

$$
= C_1 + \alpha_y(y-1), \tag{2.77}
$$

where $|\alpha_y| < \infty$ and $C_1 = \psi_2(T) - \psi_1(T)$.

Let us substitute (2.77) into the integral I_2

$$I_2 = I_3 + I_4,$$

where

$$I_3 = \int\limits_{-1}^{1} \alpha_y(y-1)\left(\int\limits_{-1}^{y} \psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)),$$

$$I_4 = C_1\int\limits_{-1}^{1} \left(\int\limits_{-1}^{y} \psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)).$$

Integrating by parts and using (2.74), we obtain

$$I_3 = \int\limits_{-1}^{1} \frac{\alpha_y(y-1)(n+1)(P_{n+1}(y)-P_n(y))}{y-1}\Biggl(\psi_1(h(y))(P_{n+1}(y)+P_n(y))-$$

$$-\int\limits_{-1}^{y}(P_{n+1}(y_1)+P_n(y_1))\psi_1'(h(y_1))\frac{1}{2}(T-t)dy_1\Biggr)dy.$$

Applying the etimate (2.67) and taking into account the boundedness of α_y and $\psi_1'(h(y_1))$, we have that $|I_3| < \infty$.

Using the integration order replacement in I_4, we get

$$I_4 = C_1\int\limits_{-1}^{1} \psi_1(h(y_1))\left(\int\limits_{y_1}^{1} d(P_{n+1}(y)+P_n(y))\right)d(P_{n+1}(y_1)+P_n(y_1)) =$$

$$= C_1\int\limits_{-1}^{1} \psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\int\limits_{-1}^{1} d(P_{n+1}(y)+P_n(y))-$$

$$-C_1\int\limits_{-1}^{1} \psi_1(h(y_1))\left(\int\limits_{-1}^{y_1} d(P_{n+1}(y)+P_n(y))\right)d(P_{n+1}(y_1)+P_n(y_1)) =$$

$$= I_5 - I_6.$$

Consider I_5

$$I_5 = 2C_1 \int\limits_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) =$$

$$= 2C_1 \left(2\psi_1(T) - \int\limits_{-1}^{1} (P_{n+1}(y_1) + P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t) dy_1 \right).$$

Applying the estimate (2.68) and using the boundedness of $\psi_1'(h(y_1))$, we obtain that $|I_5| < \infty$.

Since (see (2.77))

$$\psi_1(h(y)) = \psi_1\left(\frac{1}{2}(T-t)(y-1) + T\right) =$$

$$= \psi_1(T) + (y-1)\psi_1'(\theta_y)\frac{1}{2}(T-t) = C_2 + \beta_y(y-1),$$

where $|\beta_y| < \infty$ and $C_2 = \psi_1(T)$, then

$$I_6 = C_3 \int\limits_{-1}^{1} \left(\int\limits_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) +$$

$$+ C_1 \int\limits_{-1}^{1} \beta_{y_1}(y_1 - 1) \left(\int\limits_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) =$$

$$= \frac{C_3}{2} \left(\int\limits_{-1}^{1} d(P_{n+1}(y) + P_n(y)) \right)^2 +$$

$$+ C_1 \int\limits_{-1}^{1} \frac{\beta_{y_1}(y_1-1)(n+1)(P_{n+1}(y_1) - P_n(y_1))}{y_1 - 1} \left(\int\limits_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right) dy_1 =$$

$$= 2C_3 + C_1 \int\limits_{-1}^{1} \beta_{y_1}(n+1)(P_{n+1}(y_1) - P_n(y_1))(P_{n+1}(y_1) + P_n(y_1)) dy_1.$$

Using the estimate (2.67) and taking into account the bounedness of β_{y_1}, we obtain that $|I_6| < \infty$. Thus, the boundedness of the first sum in (2.61) is proved.

Let us prove the boundedness of the second sum in (2.61). We have

$$\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(t)\phi_{j_1}(t) =$$

$$= \frac{1}{4}\sum_{j_2=0}^{n}\sum_{j_1=0}^{n}(2j_2+1)(2j_1+1)(-1)^{j_1+j_2}\int\limits_{-1}^{1}\psi_2(h(y))P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))P_{j_1}(y_1)\times$$

$$\times dy_1dy =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\sum_{j_2=0}^{n}(2j_2+1)P_{j_2}(y)(-1)^{j_2}\int\limits_{-1}^{y}\psi_1(h(y_1))\times$$

$$\times\sum_{j_1=0}^{n}(2j_1+1)P_{j_1}(y_1)(-1)^{j_1}dy_1dy =$$

$$= \frac{(-1)^{2n}}{4}\int\limits_{-1}^{1}\psi_2(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)\times$$

$$\times d(P_{n+1}(y)-P_n(y)) =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_1(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y))+$$

$$+\frac{1}{4}\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y)) =$$

$$= \frac{1}{4}J_1 + \frac{1}{4}J_2,$$

where $\Delta(h(y))$, $h(y)$ are defined by (2.76).

Further,

$$J_1 = \frac{1}{2}\left(\int\limits_{-1}^{1}\psi_1(h(y))d(P_{n+1}(y)-P_n(y))\right)^2 =$$

$$=\frac{1}{2}\left(2(-1)^n\psi_1(t)-\int\limits_{-1}^{1}(P_{n+1}(y)-P_n(y))\psi_1'(h(y))\frac{T-t}{2}dy\right)^2<K_1<\infty, \tag{2.78}$$

where ψ_1' is a derivative of the function ψ_1 with respect to the variable y, constant K_1 is independent of n.

By the Lagrange formula we obtain

$$\Delta(h(y))=\psi_2\left(\frac{1}{2}(T-t)(y+1)+t\right)-\psi_1\left(\frac{1}{2}(T-t)(y+1)+t\right)=$$

$$=\psi_2(t)-\psi_1(t)+(y+1)\left(\psi_2'(\mu_y)-\psi_1'(\rho_y)\right)\frac{1}{2}(T-t)=$$

$$=K_2+\gamma_y(y+1), \tag{2.79}$$

where $|\gamma_y|<\infty$ and $K_2=\psi_2(t)-\psi_1(t)$.

Consider J_2

$$J_2=\int\limits_{-1}^{1}\Delta(h(y))d(P_{n+1}(y)-P_n(y))\int\limits_{-1}^{1}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))-$$

$$-\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{y}^{1}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y))=$$

$$=J_3J_4-J_5.$$

The integral J_4 was considered earlier (see J_1 and (2.78)), i.e. it has already been shown that $|J_4|<\infty$. Analogously, we have that $|J_3|<\infty$.

Let us substitute (2.79) into the integral J_5

$$J_5=J_6+J_7,$$

where

$$J_6=\int\limits_{-1}^{1}\gamma_y(y+1)\left(\int\limits_{y}^{1}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y)),$$

$$
J_7 = K_2 \int\limits_{-1}^{1} \left(\int\limits_{y}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)).
$$

Integrating by parts and using (2.75), we get

$$
J_6 = \int\limits_{-1}^{1} \frac{\gamma_y (y+1)(n+1)(P_{n+1}(y) + P_n(y))}{y+1} \Biggl(-\psi_1(h(y))(P_{n+1}(y) - P_n(y)) -
$$

$$
- \int\limits_{y}^{1} (P_{n+1}(y_1) - P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t) dy_1 \Biggr) dy.
$$

Applying the etimate (2.67) and taking into account the boundedness of γ_y and $\psi_1'(h(y_1))$, we have that $|J_6| < \infty$.

Using the integration order replacement in J_7, we obtain

$$
J_7 = K_2 \int\limits_{-1}^{1} \psi_1(h(y_1)) \left(\int\limits_{-1}^{y_1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) =
$$

$$
= K_2 \int\limits_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \int\limits_{-1}^{1} d(P_{n+1}(y) - P_n(y)) - K_2 J_8 =
$$

$$
= K_2 J_4 2(-1)^n - K_2 J_8,
$$

where

$$
J_8 = \int\limits_{-1}^{1} \psi_1(h(y_1)) \left(\int\limits_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)).
$$

Since (see (2.79))

$$
\psi_1(h(y)) = \psi_1\left(\frac{1}{2}(T-t)(y+1) + t \right) =
$$

$$
= \psi_1(t) + (y+1)\psi_1'(\rho_y) \frac{1}{2}(T-t) = K_3 + \varepsilon_y (y+1), \tag{2.80}
$$

where $|\varepsilon_y| < \infty$ and $K_3 = \psi_1(t)$, then

$$J_8 = K_3 \int\limits_{-1}^{1} \left(\int\limits_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) +$$

$$+ \int\limits_{-1}^{1} \varepsilon_y (y+1) \left(\int\limits_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) =$$

$$= \frac{K_3}{2} \left(\int\limits_{-1}^{1} d(P_{n+1}(y) - P_n(y)) \right)^2 +$$

$$+ \int\limits_{-1}^{1} \frac{\varepsilon_{y_1}(y_1+1)(n+1)(P_{n+1}(y_1) + P_n(y_1))}{y_1+1} (P_n(y_1) - P_{n+1}(y_1)) dy =$$

$$= 2K_3 + \int\limits_{-1}^{1} \varepsilon_{y_1}(n+1)(P_{n+1}(y_1) + P_n(y_1))(P_n(y_1) - P_{n+1}(y_1)) dy. \quad (2.81)$$

When obtaining the equality (2.81), we used (2.75). Applying the estimate (2.67) and taking into account the bounedness of ε_{y_1}, we obtain that $|J_8| < \infty$. Thus, the boundedness of the second sum in (2.61) is proved. The relations (2.61) are proved. Theorem 2.2 is proved.

Let us consider the proof of Lemma 2.2 under the conditions of Theorem 2.2. We will prove that

$$\sum_{j_1=0}^{n} C_{j_1 j_1}$$

is the Cauchy sequence for the cases of Legendre polynomials and trigonometric functions.

Consider the case of Legendre polynomials. Below in this section we write $\lim\limits_{n,m\to\infty}$ instead of $\lim\limits_{\substack{n,m\to\infty\\ n>m}}$. Fix $n > m$ ($n, m \in \mathbf{N}$). We have

$$\sum_{j_1=m+1}^{n} C_{j_1 j_1} = \sum_{j_1=m+1}^{n} \int\limits_{t}^{T} \psi_2(s)\phi_{j_1}(s) \int\limits_{t}^{s} \psi_1(\tau)\phi_{j_1}(\tau) d\tau ds =$$

$$
= \frac{T-t}{4}\sum_{j_1=m+1}^{n}(2j_1+1)\int\limits_{-1}^{1}\psi_2(h(x))P_{j_1}(x)\int\limits_{-1}^{x}\psi_1(h(y))P_{j_1}(y)dydx =
$$

$$
= \frac{T-t}{4}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-
$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_2(h(x))P_{j_1}(x)\int\limits_{-1}^{x}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))dydx =
$$

$$
= \frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\sum_{j_1=m+1}^{n}\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-
$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\int\limits_{y}^{1}P_{j_1}(x)\psi_2(h(x))dxdy =
$$

$$
= \frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx+
$$

$$
+\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\frac{1}{2j_1+1}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\times
$$

$$
\times\Biggl(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_2(h(y))+
$$

$$
+\frac{T-t}{2}\int\limits_{y}^{1}\left(P_{j_1+1}(x)-P_{j_1-1}(x)\right)\psi_2'(h(x))dx\Biggr)dy, \tag{2.82}
$$

where ψ_1', ψ_2' are derivatives of the functions ψ_1, ψ_2 with respect to the variable $h(y)$ (see (2.55)).

Applying the estimate (2.67) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we finally obtain

$$
\left|\sum_{j_1=m+1}^{n}C_{j_1j_1}\right| \le C_1\left(\frac{1}{n}+\frac{1}{m}\right)\int\limits_{-1}^{1}\frac{dx}{\left(1-x^2\right)^{1/2}}+
$$

$$
+C_2\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/2}}+\int\limits_{-1}^{1}\frac{1}{(1-y^2)^{1/4}}\int\limits_{y}^{1}\frac{dx}{(1-x^2)^{1/4}}dy\right)\le
$$

$$
\le C_3\left(\frac{1}{n}+\frac{1}{m}+\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\right)\to 0 \qquad (2.83)
$$

if $n,m\to\infty$ $(n>m)$, where constants C_1,C_2,C_3 do not depend on n and m.

Now consider the trigonometric case. Fix $n>m$ $(n,m\in\mathbf{N})$. Denote

$$
S_{n,m}\stackrel{\sf def}{=}\sum_{j_1=m+1}^{n}C_{j_1j_1}=\sum_{j_1=m+1}^{n}\int\limits_{t}^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.
$$

By analogy with (2.82) we obtain

$$
S_{2n,2m}=\sum_{j_1=2m+1}^{2n}\int\limits_{t}^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\frac{2}{T-t}\sum_{j_1=m+1}^{n}\left(\int\limits_{t}^{T}\psi_2(t_2){\rm sin}\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2+\right.
$$

$$
\left.+\int\limits_{t}^{T}\psi_2(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2\right)=
$$

$$
=\frac{T-t}{2\pi^2}\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\psi_1(t)\left(\psi_2(t)-\psi_2(T)+\int\limits_{t}^{T}\psi_2'(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}dt_2\right)-\right.
$$

$$
-\int\limits_{t}^{T}\psi_1'(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}\left(\psi_2(T)-\psi_2(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}-\right.
$$

$$
\left.-\int\limits_{t_1}^{T}\psi_2'(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}dt_2\right)dt_1+
$$

$$
+\int\limits_{t}^{T}\psi_1'(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}\left(\psi_2(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}+\right.
$$

$$
+\int\limits_{t_1}^{T}\psi_2'(t_2)\sin\frac{2\pi j_1(t_2-t)}{T-t}dt_2\Biggr)dt_1\Biggr), \qquad (2.84)
$$

where $\psi_1'(\tau)$, $\psi_2'(\tau)$ are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable τ.

From (2.84) we get

$$
|S_{2n,2m}|\le C\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\to 0 \qquad (2.85)
$$

if $n,m\to\infty$ $(n>m)$, where constant C does not depend on n and m.

Further,

$$
S_{2n-1,2m}=S_{2n,2m}-
$$

$$
-\frac{2}{T-t}\int\limits_{t}^{T}\psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2, \qquad (2.86)
$$

$$
S_{2n,2m-1}=S_{2n,2m}+
$$

$$
+\frac{2}{T-t}\int\limits_{t}^{T}\psi_2(t_2)\cos\frac{2\pi m(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1)\cos\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2, \qquad (2.87)
$$

$$
S_{2n-1,2m-1}=S_{2n,2m-1}-
$$

$$
-\frac{2}{T-t}\int\limits_{t}^{T}\psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2=
$$

$$
=S_{2n,2m}+\frac{2}{T-t}\int\limits_{t}^{T}\psi_2(t_2)\cos\frac{2\pi m(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1)\cos\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2-
$$

$$
-\frac{2}{T-t}\int\limits_{t}^{T}\psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2. \qquad (2.88)
$$

Integrating by parts in (2.86)–(2.88), we obtain

$$
|S_{2n-1,2m}|\le|S_{2n,2m}|+\frac{C_1}{n}, \qquad (2.89)
$$

$$|S_{2n,2m-1}| \le |S_{2n,2m}| + \frac{C_1}{m}, \tag{2.90}$$

$$|S_{2n-1,2m-1}| \le |S_{2n,2m}| + C_1\left(\frac{1}{m} + \frac{1}{n}\right), \tag{2.91}$$

where constant C_1 does not depend on n and m.

The relations (2.85), (2.89)–(2.91) imply that

$$\lim_{n,m\to\infty} |S_{2n,2m}| = \lim_{n,m\to\infty} |S_{2n-1,2m}| = \lim_{n,m\to\infty} |S_{2n,2m-1}| = \lim_{n,m\to\infty} |S_{2n-1,2m-1}| = 0. \tag{2.92}$$

From (2.92) we get

$$\lim_{n,m\to\infty} |S_{n,m}| = 0. \tag{2.93}$$

The relation (2.93) completes the proof.

2.1.3 Approach Based on Generalized Double Multiple and Iterated Fourier Series

This section is devoted to the proof of Theorem 2.1 using a slightly different method than the method proposed in Sect. 2.1.1. We will consider two different parts of the expansion of iterated Stratonovich stochastic integrals of second multiplicity. The mean-square convergence of the first part will be proved on the base of generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^2)$. The mean-square convergence of the second part will be proved on the base of generalized iterated (double) Fourier series converging pointwise.

Proof. Let us consider Lemma 1.1, definition of the multiple stochastic integral (1.16) together with the formula (1.19) when the function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary as well as Lemma 1.3 for the case $k=2$ (see Sect. 1.1.3).

In accordance to the standard relation between Stratonovich and Itô stochastic integrals (see (2.8)) we have w. p. 1

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1. \tag{2.94}$$

Let us consider the function $K^*(t_1,t_2)$ defined by (2.11)

$$K^*(t_1,t_2) = K(t_1,t_2) + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_2), \tag{2.95}$$

where

$$K(t_1,t_2) = \mathbf{1}_{\{t_1<t_2\}}\psi_1(t_1)\psi_2(t_2), \quad t_1,t_2\in[t,T]. \tag{2.96}$$

Lemma 2.3. *Under the conditions of Theorem* 2.2 *the following relation*

$$J[K^*]_{T,t}^{(2)} = J^*[\psi^{(2)}]_{T,t} \tag{2.97}$$

is valid w. p. 1, *where* $J[K^*]_{T,t}^{(2)}$ *is defined by the equality* (1.16).

Proof. Substituting (2.95) into (1.16) (the case $k=2$) and using Lemma 1.1 together with (1.19) (the case $k=2$) it is easy to see that w. p. 1

$$J[K^*]_{T,t}^{(2)} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 =$$

$$= J^*[\psi^{(2)}]_{T,t}. \tag{2.98}$$

Let us consider the following generalized double Fourier sum

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),$$

where $C_{j_2j_1}$ is the Fourier coefficient defined as follows

$$C_{j_2j_1} = \int\limits_{[t,T]^2} K^*(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2. \tag{2.99}$$

Further, subsitute the relation

$$K^*(t_1,t_2) = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2) + K^*(t_1,t_2) -$$

$$-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)$$

into $J[K^*]_{T,t}^{(2)}$. At that we suppose that $p_1, p_2 < \infty$.

Then using Lemma 1.3 (the case $k=2$), we obtain

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + J[R_{p_1p_2}]_{T,t}^{(2)} \quad \text{w. p. 1}, \tag{2.100}$$

where the stochastic integral $J[R_{p_1p_2}]_{T,t}^{(2)}$ is defined in accordance with (1.16) and

$$R_{p_1p_2}(t_1,t_2) = K^*(t_1,t_2) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{2.101}$$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)},$$

$$J[R_{p_1p_2}]_{T,t}^{(2)} = \int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1.$$

Using standard moment properties of stochastic integrals [100] (see (1.26), (1.27)), we get

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)}\right)^2\right\} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2 \le$$

$$
\le 2\left(\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2+\int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_2dt_1\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2=
$$

$$
=2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \qquad (2.102)
$$

We have

$$
\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2=
$$

$$
=\int\limits_{[t,T]^2}\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2 dt_1dt_2=
$$

$$
=\int\limits_{[t,T]^2}\left(K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2 dt_1dt_2. \qquad (2.103)
$$

The function $K(t_1,t_2)$ is piecewise continuous in the square $[t,T]^2$. At this situation it is well known that the generalized multiple Fourier series of the function $K(t_1,t_2)\in L_2([t,T]^2)$ is converging to this function in the square $[t,T]^2$ in the mean-square sense, i.e.

$$
\lim_{p_1,p_2\to\infty}\left\|K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^2)}=0,
$$

where notations are the same as in (1.7).

So, we obtain

$$
\lim_{p_1,p_2\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2=0. \qquad (2.104)
$$

Note that

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 =$$

$$= \int\limits_t^T \left(\frac{1}{2}\psi_1(t_1)\psi_2(t_1) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right) dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}} =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1}. \qquad (2.105)$$

From (2.105) and Lemma 2.2 we get

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \lim_{p_1\to\infty}\sum_{j_1=0}^{p_1} C_{j_1j_1} =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{\infty} C_{j_1j_1} =$$

$$= \lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1. \qquad (2.106)$$

If we prove the following relation

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0, \qquad (2.107)$$

then from (2.106) we obtain

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}, \tag{2.108}$$

$$\lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0. \tag{2.109}$$

From (2.102), (2.104), and (2.109) we get

$$\lim_{p_1,p_2\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} = 0$$

and Theorem 2.1 will be proved (see (2.100)).

The proof of the equality (2.107) can be carried out in the same way as in the proof of Theorem 2.1 or, under weaker conditions, as in the proof of Theorem 2.2.

2.1.4 Approach Based on Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$

Let us prove the equality (2.10) under weaker restrictions. Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau)\equiv\psi_2(\tau)$ or

$$\psi_1(\tau) = \psi_2(\tau)\int\limits_t^{\tau} g(\theta)d\theta, \tag{2.110}$$

where $\tau\in[t,T]$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $g(\tau)\in L_1([t,T])$.

Thus, we will prove the equality

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2 = \frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau \tag{2.111}$$

under the above conditions.

Using Fubini's Theorem, Lebesgue's Dominated Convergence Theorem and Parseval's equality, we have (see (2.110))

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2 =$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_2(t_1)\phi_j(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1dt_2 = \tag{2.112}$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\int\limits_\tau^{T}\psi_2(t_1)\phi_j(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_j(t_2)dt_2dt_1d\tau =$$

$$=\frac{1}{2}\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\left(\int\limits_\tau^{T}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 d\tau = \tag{2.113}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{\tau<t_1\}}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 d\tau = \tag{2.114}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\int\limits_t^{T}\mathbf{1}_{\{\tau<t_1\}}\psi_2^2(t_1)dt_1d\tau = \frac{1}{2}\int\limits_t^T g(\tau)\int\limits_\tau^{T}\psi_2^2(t_1)dt_1d\tau =$$

$$=\frac{1}{2}\int\limits_t^T \psi_2^2(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1 = \tag{2.115}$$

$$=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1, \tag{2.116}$$

where the transition from (2.113) to (2.114) is based on Lebesgue's Dominated Convergence Theorem. The integrable majorant exists due to Parseval's equality

$$|g(\tau)|\sum_{j=0}^{q}\left(\int\limits_\tau^{T}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 \le |g(\tau)|\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{\tau<t_1\}}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 =$$

$$
= |g(\tau)| \int\limits_t^T \left(\mathbf{1}_{\{\tau<t_1\}}\right)^2 \psi_2^2(t_1)dt_1 \le |g(\tau)| \, \|\psi_2\|^2_{L_2([t,T])} = C\,|g(\tau)|\,,
$$

where constant C does not depend on p.

From the other hand, using Fubini's Theorem and the generalized Parseval equality as well as the transition from (2.112) to (2.115), we get

$$
\sum_{j=0}^{\infty} \int\limits_t^T \psi_1(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_2(t_1)\phi_j(t_1)dt_1dt_2 =
$$

$$
= \sum_{j=0}^{\infty} \int\limits_t^T \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} g(\tau)d\tau \int\limits_t^{t_2} \psi_2(t_1)\phi_j(t_1)dt_1dt_2 =
$$

$$
= \sum_{j=0}^{\infty} \int\limits_t^T \psi_2(t_1)\phi_j(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} g(\tau)d\tau dt_2dt_1 =
$$

$$
= \sum_{j=0}^{\infty} \int\limits_t^T \psi_2(t_1)\phi_j(t_1)dt_1 \int\limits_{t}^{T} \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} g(\tau)d\tau dt_2 -
$$

$$
- \sum_{j=0}^{\infty} \int\limits_t^T \psi_2(t_1)\phi_j(t_1) \int\limits_{t}^{t_1} \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} g(\tau)d\tau dt_2dt_1 =
$$

$$
= \int\limits_t^T \psi_2(t_1)\cdot\psi_2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 - \frac{1}{2}\int\limits_t^T \psi_2^2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 =
$$

$$
= \frac{1}{2}\int\limits_t^T \psi_2^2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 = \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1. \qquad (2.117)
$$

In addition, for the case $\psi_1(\tau) \equiv \psi_2(\tau)$, using the Parseval equality, we obtain

$$
\sum_{j=0}^{\infty} \int\limits_t^T \psi_1(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1)dt_1dt_2 =
$$

$$
= \frac{1}{2}\sum_{j=0}^{\infty}\left(\int_t^T \psi_1(t_1)\phi_j(t_1)dt_1\right)^2 = \frac{1}{2}\int_t^T \psi_1^2(t_1)dt_1 = \frac{1}{2}\int_t^T \psi_1(t_1)\psi_1(t_2)dt_1. \tag{2.118}
$$

The equality (2.111) is proved for $\psi_1(\tau) \equiv \psi_2(\tau)$ or when the equality (2.110) is satisfied.

Further, let us suppose that $\psi_2(\tau) = (\tau - t)^l$, $g(\tau) = k(\tau - t)^{k-1}$, where $l = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$ Note that this case is important for applications (see Sect. 4.7 and 4.11).

From (2.110) we obtain

$$
\psi_1(\tau) = \psi_2(\tau)\int_t^{\tau} g(\theta)d\theta = k(\tau - t)^l \int_t^{\tau}(\theta - t)^{k-1}d\theta = (\tau - t)^{l+k}.
$$

Taking into account (2.116)–(2.118), we get

$$
\sum_{j=0}^{\infty}\int_t^T (t_2 - t)^l \phi_j(t_2)\int_t^{t_2}(t_1 - t)^{l+k}\phi_j(t_1)dt_1dt_2 =
$$

$$
= \sum_{j=0}^{\infty}\int_t^T (t_2 - t)^{l+k}\phi_j(t_2)\int_t^{t_2}(t_1 - t)^{l}\phi_j(t_1)dt_1dt_2 =
$$

$$
= \frac{1}{2}\int_t^T (\tau - t)^{2l+k}d\tau, \tag{2.119}
$$

where $k, l = 0, 1, 2, \ldots$

Let us rewrite the equality (2.119) in the following form

$$
\sum_{j=0}^{\infty}\int_t^T (t_2 - t)^l \phi_j(t_2)\int_t^{t_2}(t_1 - t)^{m}\phi_j(t_1)dt_1dt_2 = \frac{1}{2}\int_t^T (\tau - t)^l(\tau - t)^m d\tau, \tag{2.120}
$$

where $l, m = 0, 1, 2, \ldots$

The equality similar to (2.120) was obtained in [117], [118] using other arguments. These arguments are based on trace class operators and the equality of matrix and integral traces for such operators (see Sect. 2.27 for details).

In addition, the formula similar to (2.120) was used in [117], [118] to generalize the equality (2.111) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. This means that Theorems 2.1, 2.2 can be generalized to the case of continuous functions $\psi_1(\tau), \psi_2(\tau)$ (this condition is related to the definition (2.3) of the Stratonovich stochastic integral (see Sect. 2.1.1 for details)) and an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$.

Consider the mentioned approach [117], [118] in our interpretation (after this, we will consider an approach that is slightly different from the approach in [117], [118]). Since the equality (2.120) is valid for monomials with respect to $\tau - t$ ($\tau \in [t,T]$), it will obviously also be valid for Legendre polynomials that form a complete orthonormal system of functions in the space $L_2([t,T])$ and finite linear combinations of Legendre polynomials.

Let $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$ and $\psi_1^{(p)}(\tau), \psi_2^{(q)}(\tau)$ be approximations of the functions $\psi_1(\tau), \psi_2(\tau)$, respectively, which are partial sums of the corresponding Fourier–Legendre series. Then we have (see (2.120))

$$\sum_{j=0}^{\infty} \int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1^{(p)}(t_1)\phi_j(t_1)dt_1dt_2 = \frac{1}{2}\int\limits_t^T \psi_1^{(p)}(\tau)\psi_2^{(q)}(\tau)d\tau, \quad (2.121)$$

where $p, q \in \mathbf{N}$, the series converges absolutly and its sum does not depend on a basis system $\{\phi_j(x)\}_{j=0}^{\infty}$ (we mean permutation of the terms of the series on the left-hand side of (2.121) (any permutation of basis functions $\phi_j(x)$ forms a basis in $L_2([t,T])$ [127])).

Using Fubini's Theorem, we rewrite (2.121) in the form

$$\sum_{j=0}^{\infty} \left(\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1^{(p)}(t_1)\phi_j(t_1)dt_1dt_2 + \right.$$

$$\left. + \int\limits_t^T \psi_1^{(p)}(t_2)\phi_j(t_2) \int\limits_{t_2}^{T} \psi_2^{(q)}(t_1)\phi_j(t_1)dt_1dt_2 \right) = \int\limits_t^T \psi_1^{(p)}(\tau)\psi_2^{(q)}(\tau)d\tau. \quad (2.122)$$

Let us fix q in (2.122). The right-hand side of (2.122) for a fixed q defines (as a scalar product in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\psi_2^{(q)}$. The integral operator (which corresponds to the matrix trace on the left-hand side of (2.122)) is a trace class operator (see [118]). The matrix trace of the mentioned operator (on the left-hand side of (2.122)) is also a linear bounded (and therefore continuous) functional (in the space of trace class operators [127], [128]) which can be extended to the space $L_2([t,T])$ by continuity [147].

Let us implement the passage to the limit $\lim\limits_{p\to\infty}$ in (2.122)

$$\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_{t_2}^{T}\psi_2^{(q)}(t_1)\phi_j(t_1)dt_1dt_2\right)=\int\limits_t^T\psi_1(\tau)\psi_2^{(q)}(\tau)d\tau, \quad (2.123)$$

where $q\in\mathbf{N}$. Recall that $\psi_2^{(q)}(\tau)$ is a partial sum of the Fourier–Legendre series of any function $\psi_2(\tau)\in L_2([t,T])$, i.e. the equality (2.123) holds on a dense subset in $L_2([t,T])$. The right-hand side of (2.123) defines (as a scalar product in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function ψ_1. On the left-hand side of (2.123) (by virtue of the equality (2.123)) there is a linear continuous functional on a dense subset in $L_2([t,T])$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T])$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.123)

$$\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_{t_2}^{T}\psi_2(t_1)\phi_j(t_1)dt_1dt_2\right)=\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau. \quad (2.124)$$

Applying Fubini's Theorem to the left-hand side of (2.124), we obtain

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau, \quad (2.125)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

However, the equality (2.125) can be obtained somewhat more simply. Now let us consider an approach that is slightly different form the approach in [117], [118].

Consider the equality (2.121) and fix q in it. The right-hand side of (2.121) for a fixed q defines (as a scalar product of $\psi_1^{(p)}$ and $\frac{1}{2}\psi_2^{(q)}$ in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\frac{1}{2}\psi_2^{(q)}$.

On the left-hand side of (2.121) (by virtue of the equality (2.121)) there is a linear continuous functional on a dense subset in $L_2([t,T])$ (recall that $\psi_1^{(p)}(\tau)$ is a partial sum of the Fourier–Legendre series of any function $\psi_1(\tau)\in L_2([t,T])$). This functional can be uniquely extended to a linear continuous functional in $L_2([t,T])$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{p\to\infty}$ in the equality (2.121)

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2^{(q)}(\tau)d\tau, \quad (2.126)$$

where $q\in\mathbf{N}$.

Recall that $\psi_2^{(q)}(\tau)$ is a partial sum of the Fourier–Legendre series of any function $\psi_2(\tau)\in L_2([t,T])$, i.e. the equality (2.126) holds on a dense subset in $L_2([t,T])$. The right-hand side of (2.126) defines (as a scalar product of $\psi_2^{(q)}$ and $\frac{1}{2}\psi_1$ in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\frac{1}{2}\psi_1$. On the left-hand side of (2.126) (by virtue of the equality (2.126)) there is a linear continuous functional on a dense subset in $L_2([t,T])$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T])$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.126)

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. As a result, we obtained the equality (2.125).

Thus, we have the following theorem.

Theorem 2.3. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Moreover,* $\psi_1(\tau), \psi_2(\tau)$ *are continuous functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \tag{2.127}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 2.1, 2.2.

The condition of continuity of the functions $\psi_1(\tau), \psi_2(\tau)$ is related to the definition (2.3) of the Stratonovich stochastic integral that we use.

Theorem 2.3 can be generalized to the case $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$ if instead of the definition (2.3) we use another definition of the Stratonovich stochastic integral (see (2.960) and Theorem 2.44 in Sect. 2.18 for details).

2.1.5 Approach Based on Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and Symmetrized Kernel $K'(t_1, t_2)$

Let us list some useful facts that we will need further in this section.

Theorem A ([128], Theorem 8.1). *Let* $\mathbb{K} : L_2([t,T]) \to L_2([t,T])$ *be an integral operator defined by*

$$(\mathbb{K}f)(\tau) = \int\limits_t^T K(\tau, s) f(s) ds,$$

where $K(\tau, s)$ *is a continuous function on* $[t,T]\times[t,T]$. *If, in addition,* $\mathbb{K}$ *is a trace class operator then*

$$tr\mathbb{K} = \int\limits_t^T K(s,s) ds, \tag{2.128}$$

where trace $tr\mathbb{K}$ *is defined as a series of singular values* $s_j(\mathbb{K})$ *of* $\mathbb{K}$.

Theorem B ([128], P. 71). *Let*

$$(\mathbb{K}f)(\tau) = \int\limits_t^T K(\tau,s)f(s)ds,$$

the kernel $K(\tau,s)$ *is continuous on* $[t,T]\times[t,T]$ *and satisfies the condition*

$$|K(\tau,s_2) - K(\tau,s_1)| \le C\,|s_2 - s_1|^{\alpha}\,, \tag{2.129}$$

where $0<\alpha\le 1$. *If, in addition,* $\mathbb{K}$ *is a Hermitian operator and* $\alpha>1/2$, *then*

$$\sum_{j=0}^{\infty} s_j(\mathbb{K}) < \infty$$

i.e., $\mathbb{K}$ *is a trace class operator.*

Suppose that $\mathbb{A}: H\to H$ is a linear bounded operator. Recall [127] that $\mathbb{A}$ has a finite matrix trace if for any orthonormal basis $\{\phi_j(x)\}_{j=0}^{\infty}$ of the space H the series

$$\sum_{j=0}^{\infty} \langle \mathbb{A}\phi_j,\phi_j\rangle_H \tag{2.130}$$

converges, where $\langle\cdot,\cdot\rangle_H$ is a scalar probuct in H.

Note that the series (2.130) converges absolutely since its sum does not depend on the permutation of the terms of the series (2.130) (any permutation of basis functions $\phi_j(x)$ forms a basis in H) [127].

Theorem C ([128], Theorem 5.6). *Let* $\mathbb{K}: H\to H$ *be a trace class operator. Then*

$$tr\mathbb{A} = \sum_{j=0}^{\infty} \langle \mathbb{A}\phi_j,\phi_j\rangle_H \tag{2.131}$$

for any orthonormal basis $\{\phi_j(x)\}_{j=0}^{\infty}$ *of* H.

Consider an integral operator $\mathbb{K}': L_2([t,T])\to L_2([t,T])$ defined by the equality

$$(\mathbb{K}'f)(\tau) = \int\limits_t^T K'(\tau,s)f(s)ds,$$

where the continuous kernel $K'(\tau,s)$ has the form (2.48), i.e.

$$
K'(t_1,t_2)=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}\qquad (t_1,t_2\in[t,T])
$$

and $\psi_1(\tau),\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$.

Recall that (see (2.49))

$$
|K'(t_2,s_2)-K'(t_1,s_1)|\le L\left(|t_2-t_1|+|s_2-s_1|\right), \tag{2.132}
$$

where $L<\infty$ and (t_1,s_1), $(t_2,s_2)\in[t,T]^2$.

Let us substitute $t_1=t_2=\tau$ into (2.132)

$$
|K'(\tau,s_2)-K'(\tau,s_1)|\le L|s_2-s_1|. \tag{2.133}
$$

Thus, the condition (2.129) is fulfilled ($\alpha=1$). Further, using Fubini's Theorem, we have

$$
\langle\mathbb{K}'x,y\rangle_{L_2([t,T])}=\int\limits_t^T\psi_2(t_2)y(t_2)\int\limits_t^{t_2}\psi_1(t_1)x(t_1)dt_1dt_2+
$$

$$
+\int\limits_t^T\psi_1(t_2)y(t_2)\int\limits_{t_2}^T\psi_2(t_1)x(t_1)dt_1dt_2=\int\limits_t^T\psi_1(t_1)x(t_1)\int\limits_{t_1}^T\psi_2(t_2)y(t_2)dt_2dt_1+
$$

$$
+\int\limits_t^T\psi_2(t_1)x(t_1)\int\limits_t^{t_2}\psi_1(t_2)y(t_2)dt_2dt_1=\langle\mathbb{K}'y,x\rangle_{L_2([t,T])}. \tag{2.134}
$$

The conditions of Theorem B are fulfilled. Then, $\mathbb{K}'$ is a trace class operator. Since the kernel $K'(t_1,t_2)$ is continuous, then by Theorems A and C (see (2.128) and (2.131)) we obtain

$$
\sum_{j_1=0}^{\infty}\langle\mathbb{K}'\phi_{j_1},\phi_{j_1}\rangle_{L_2([t,T])}=\int\limits_t^T K'(s,s)ds=\int\limits_t^T\psi_1(s)\psi_2(s)ds. \tag{2.135}
$$

Combining (2.134), (2.135) and applying Fubini's Theorem, we get

$$
\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+\right.
$$

$$
+\int_t^T \psi_1(t_2)\phi_{j_1}(t_2)\int_{t_2}^T \psi_2(t_1)\phi_{j_1}(t_1)dt_1dt_2\Biggr) =
$$

$$
=\sum_{j_1=0}^{\infty}\Biggl(\int_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+
$$

$$
+\int_t^T \psi_2(t_1)\phi_{j_1}(t_1)\int_t^{t_2} \psi_1(t_2)\phi_{j_1}(t_2)dt_2dt_1\Biggr) =
$$

$$
=2\sum_{j_1=0}^{\infty}\int_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 = \int_t^T \psi_1(s)\psi_2(s)ds. \quad (2.136)
$$

From (2.136) we obtain

$$
\sum_{j_1=0}^{\infty}\int_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 = \frac{1}{2}\int_t^T \psi_1(s)\psi_2(s)ds, \quad (2.137)
$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$.

To further generalize of the equality (2.137) to the case when $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ it is necessary to set $\psi_2(\tau)=(\tau-t)^l$, $\psi_1(\tau)=(\tau-t)^m$ $(l,m=0,1,2,\ldots)$ and apply the reasoning of the previous section after the formula (2.120).

2.2 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3 Based on Theorem 1.1

This section is devoted to the development of the method of expansion and mean-square approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series converging in the mean (Theorem 1.1). We adapt this method for the iterated Stratonovich stochastic integrals of multiplicity 3. The main results of this section have been derived with using triple Fourier–Legendre series as well as triple trigonometric Fourier series for different cases of series summation and different cases of weight functions of iterated Stratonovich stochastic integrals.

2.2.1 The Case $p_1, p_2, p_3 \to \infty$ and Constant Weight Functions (The Case of Legendre Polynomials)

Theorem 2.4 [6]-[17], [35]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{2.138}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(s)\int\limits_t^s \phi_{j_2}(s_1)\int\limits_t^{s_1} \phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. If we prove w. p. 1 the following equalities

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right), \tag{2.139}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{2.140}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0, \tag{2.141}$$

then in accordance with the formulas (2.139)–(2.141), Theorem 1.1 (see (1.47)), standard relations between iterated Itô and Stratonovich stochastic integrals as well as in accordance with the formulas (they also follow from Theorem 1.1)

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} dsd\mathbf{f}_{\tau}^{(i_3)}=\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\quad \text{w. p. 1},$$

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau=\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\quad \text{w. p. 1}$$

we will have

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} dsd\mathbf{f}_{\tau}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau\quad \text{w. p. 1}.$$

It means that the expansion (2.138) will be proved.

Let us at first prove that

$$\sum_{j_1=0}^{\infty}C_{0j_1j_1}=\frac{1}{4}(T-t)^{3/2},\tag{2.142}$$

$$\sum_{j_1=0}^{\infty}C_{1j_1j_1}=\frac{1}{4\sqrt{3}}(T-t)^{3/2}.\tag{2.143}$$

We have

$$C_{000}=\frac{(T-t)^{3/2}}{6},$$

$$C_{0j_1j_1}=\int\limits_t^T\phi_0(s)\int\limits_t^s\phi_{j_1}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\frac{1}{2}\int\limits_t^T\phi_0(s)\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds,\quad j_1\ge 1,\tag{2.144}$$

where $\phi_j(s)$ looks as follows

$$\phi_j(s)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j\ge 0,\tag{2.145}$$

where $P_j(x)$ is the Legendre polynomial.

Let us substitute (2.145) into (2.144) and calculate $C_{0j_1j_1}$ $(j_1\ge 1)$

$$C_{0j_1j_1}=\frac{2j_1+1}{2(T-t)^{3/2}}\int\limits_t^T\left(\int\limits_{-1}^{z(s)}P_{j_1}(y)\frac{T-t}{2}dy\right)^2 ds=$$

$$=\frac{(2j_1+1)\sqrt{T-t}}{8}\int\limits_t^T\left(\int\limits_{-1}^{z(s)}\frac{1}{2j_1+1}\left(P^{'}_{j_1+1}(y)-P^{'}_{j_1-1}(y)\right)dy\right)^2 ds=$$

$$=\frac{\sqrt{T-t}}{8(2j_1+1)}\int\limits_t^T\left(P_{j_1+1}(z(s))-P_{j_1-1}(z(s))\right)^2 ds, \qquad (2.146)$$

where here and further

$$z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

In (2.146) we used the following well known properties of the Legendre polynomials

$$P_j(y)=\frac{1}{2j+1}\left(P^{'}_{j+1}(y)-P^{'}_{j-1}(y)\right),\quad P_j(-1)=(-1)^j,\quad j\ge 1.$$

Also, we denote

$$\frac{dP_j}{dy}(y)\stackrel{\rm def}{=}P^{'}_j(y).$$

From (2.146) using the property of orthogonality of the Legendre polynomials, we get the following relation

$$C_{0j_1j_1}=\frac{(T-t)^{3/2}}{16(2j_1+1)}\int\limits_{-1}^{1}\left(P^2_{j_1+1}(y)+P^2_{j_1-1}(y)\right)dy=$$

$$=\frac{(T-t)^{3/2}}{8(2j_1+1)}\left(\frac{1}{2j_1+3}+\frac{1}{2j_1-1}\right),$$

where we used the property

$$\int\limits_{-1}^{1}P_j^2(y)dy=\frac{2}{2j+1},\quad j\ge 0.$$

Then

$$\sum_{j_1=0}^{\infty} C_{0j_1j_1} = \frac{(T-t)^{3/2}}{6} +$$

$$+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{(2j_1+1)(2j_1+3)}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right)=$$

$$=\frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}-\frac{1}{3}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right)=$$

$$=\frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right)=\frac{(T-t)^{3/2}}{4}.$$

The relation (2.142) is proved.

Let us check the correctness of (2.143). Let us represent $C_{1j_1j_1}$ in the form

$$C_{1j_1j_1} = \frac{1}{2}\int\limits_t^T \phi_1(s)\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 ds =$$

$$=\frac{(T-t)^{3/2}(2j_1+1)\sqrt{3}}{16}\int\limits_{-1}^{1} P_1(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1\ge 1.$$

Since the functions

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2, \quad j_1\ge 1$$

are even, then the functions

$$P_1(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1\ge 1$$

are uneven. It means that $C_{1j_1j_1}=0$ $(j_1\ge 1)$. From the other side

$$C_{100}=\frac{\sqrt{3}(T-t)^{3/2}}{16}\int\limits_{-1}^{1} y(y+1)^2 dy = \frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_1=0}^{\infty} C_{1j_1j_1} = C_{100} + \sum_{j_1=1}^{\infty} C_{1j_1j_1} = \frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (2.143) is proved.

Let us prove the equality (2.139). Using (2.143), we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1}\sum_{j_3=2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1} \sum_{j_3=2, j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \quad (2.147)$$

Since

$$C_{j_3j_1j_1} = \frac{(T-t)^{3/2}(2j_1+1)\sqrt{2j_3+1}}{16}\int\limits_{-1}^{1} P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy$$

and degree of the polynomial

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

equals to $2j_1+2$, then $C_{j_3j_1j_1} = 0$ for $j_3 > 2j_1+2$. It explains that we put $2j_1+2$ instead of p_3 on the right-hand side of the formula (2.147).

Moreover, the function

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is uneven for uneven j_3. It means that $C_{j_3j_1j_1} = 0$ for uneven j_3. That is why we summarize using even j_3 on the right-hand side of the formula (2.147).

Then we have

$$\sum_{j_1=0}^{p_1}\ \sum_{j_3=2, j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=(j_3-2)/2}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.148)$$

We replaced $(j_3-2)/2$ by zero on the right-hand side of the formula (2.148), since $C_{j_3j_1j_1} = 0$ for $0 \le j_1 < (j_3-2)/2$.

Let us substitute (2.148) into (2.147)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)} +$$

$$+ \sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.149)$$

It is easy to see that the right-hand side of the formula (2.149) does not depend on p_3.

If we prove that

$$\lim_{p_1\to\infty} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\right)^2\right\} = 0, \qquad (2.150)$$

then the relaion (2.139) will be proved.

Using (2.149) and (2.142), we can rewrite the left-hand side of (2.150) in the following form

$$\lim_{p_1\to\infty} \mathsf{M}\left\{\left(\left(\sum_{j_1=0}^{p_1} C_{0j_1j_1} - \frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_3)} + \sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1} C_{0j_1j_1} - \frac{(T-t)^{3/2}}{4}\right)^2 + \lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 =$$

$$= \lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2. \qquad (2.151)$$

If we prove that

$$\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=0, \tag{2.152}$$

then the relation (2.139) will be proved.

We have

$$\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\left((s-t)-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2\right)ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2\le$$

$$\le\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2. \tag{2.153}$$

Obtaining (2.153), we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2=\int\limits_t^T\left(\mathbf{1}_{\{s_1<s\}}\right)^2ds_1=s-t \tag{2.154}$$

and the orthogonality property of the Legendre polynomials

$$\int\limits_t^T\phi_{j_3}(s)(s-t)ds=0,\quad j_3\ge 2. \tag{2.155}$$

Then we have for $j_1 \in \mathbf{N}$

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 = \frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)dy\right)^2 =
$$

$$
= \frac{T-t}{4(2j_1+1)}\left(\int\limits_{-1}^{z(s)} \left(P_{j_1+1}^{'}(y) - P_{j_1-1}^{'}(y)\right)dy\right)^2 =
$$

$$
= \frac{T-t}{4(2j_1+1)}\left(P_{j_1+1}\left(z(s)\right) - P_{j_1-1}\left(z(s)\right)\right)^2 \le
$$

$$
\le \frac{T-t}{2(2j_1+1)}\left(P_{j_1+1}^2\left(z(s)\right) + P_{j_1-1}^2\left(z(s)\right)\right). \tag{2.156}
$$

Remind that for the Legendre polynomials the following estimate is correct

$$
|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1,1), \quad j \in \mathbf{N}, \tag{2.157}
$$

where constant K does not depend on y and j.

The estimate (2.157) can be rewritten for the function $\phi_j(s)$ in the following form

$$
|\phi_j(s)| < \sqrt{\frac{2j+1}{j+1}}\frac{K}{\sqrt{T-t}}\frac{1}{(1-z^2(s))^{1/4}} <
$$

$$
< \frac{K_1}{\sqrt{T-t}}\frac{1}{(1-z^2(s))^{1/4}}, \tag{2.158}
$$

where $K_1 = K\sqrt{2}$, $s \in (t,T)$.

Let us estimate the right-hand side of (2.156) using the estimate (2.157)

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 < \frac{T-t}{2(2j_1+1)}\left(\frac{K^2}{j_1+2} + \frac{K^2}{j_1}\right)\frac{1}{(1-(z(s))^2)^{1/2}} <
$$

$$
< \frac{(T-t)K^2}{2j_1^2}\frac{1}{(1-(z(s))^2)^{1/2}}, \tag{2.159}
$$

where $s \in (t,T)$, $j_1 \in \mathbf{N}$.

Substituting the estimate (2.159) into the relation (2.153) and using in (2.153) the estimate (2.158) for $|\phi_{j_3}(s)|$, we obtain

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 <$$

$$< \frac{(T-t)K^4K_1^2}{16}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T \frac{ds}{\left(1-(z(s))^2\right)^{3/4}}\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2 =$$

$$= \frac{(T-t)^3K^4K_1^2(p_1+1)}{64}\left(\int\limits_{-1}^1 \frac{dy}{(1-y^2)^{3/4}}\right)^2\left(\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2. \qquad (2.160)$$

Since

$$\int\limits_{-1}^1 \frac{dy}{(1-y^2)^{3/4}} < \infty \qquad (2.161)$$

and

$$\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2} \le \int\limits_{p_1}^{\infty}\frac{dx}{x^2} = \frac{1}{p_1}, \qquad (2.162)$$

then from (2.160) we find

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 < \frac{C(T-t)^3(p_1+1)}{p_1^2} \quad \to\ 0 \quad \text{if} \quad p_1\to\infty, \qquad (2.163)$$

where constant C does not depend on p_1 and $T-t$. The relation (2.163) implies (2.152), and the relation (2.152) implies the correctness of the formula (2.139).

Let us prove the equaity (2.140). Let us at first prove that

$$\sum_{j_3=0}^{\infty} C_{j_3j_30} = \frac{1}{4}(T-t)^{3/2}, \qquad (2.164)$$

$$\sum_{j_3=0}^{\infty} C_{j_3j_31} = -\frac{1}{4\sqrt{3}}(T-t)^{3/2}. \qquad (2.165)$$

We have

$$\sum_{j_3=0}^{\infty} C_{j_3j_30} = C_{000} + \sum_{j_3=1}^{\infty} C_{j_3j_30},$$

$$C_{000}=\frac{(T-t)^{3/2}}{6},$$

$$C_{j_3j_30}=\frac{(T-t)^{3/2}}{16(2j_3+1)}\int\limits_{-1}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy=$$

$$=\frac{(T-t)^{3/2}}{8(2j_3+1)}\left(\frac{1}{2j_3+3}+\frac{1}{2j_3-1}\right),\quad j_3\ge 1.$$

Then

$$\sum_{j_3=0}^{\infty}C_{j_3j_30}=\frac{(T-t)^{3/2}}{6}+$$

$$+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_3=1}^{\infty}\frac{1}{(2j_3+1)(2j_3+3)}+\sum_{j_3=1}^{\infty}\frac{1}{4j_3^2-1}\right)=$$

$$=\frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_3=1}^{\infty}\frac{1}{4j_3^2-1}-\frac{1}{3}+\sum_{j_3=1}^{\infty}\frac{1}{4j_3^2-1}\right)=$$

$$=\frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right)=\frac{(T-t)^{3/2}}{4}.$$

The relation (2.164) is proved. Let us check the equality (2.165). We have

$$C_{j_3j_3j_1}=\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\int\limits_t^T\phi_{j_1}(s_2)ds_2\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\int\limits_{s_1}^T\phi_{j_3}(s)ds=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2ds_2=$$

$$=\frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1}P_{j_1}(y)\left(\int\limits_y^1P_{j_3}(y_1)dy_1\right)^2dy,\quad j_3\ge 1. \tag{2.166}$$

Since the functions

$$\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2, \quad j_3 \ge 1$$

are even, then the functions

$$P_1(y)\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2 dy, \quad j_3 \ge 1$$

are uneven. It means that $C_{j_3 j_3 1} = 0$ $(j_3 \ge 1)$.

Moreover,

$$C_{001} = \frac{\sqrt{3}(T-t)^{3/2}}{16}\int\limits_{-1}^{1} y(1-y)^2 dy = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 1} = C_{001} + \sum_{j_3=1}^{\infty} C_{j_3 j_3 1} = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (2.165) is proved. Using the obtained results, we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)} = \sum_{j_3=0}^{p_3} C_{j_3 j_3 0}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\sum_{j_1=2}^{p_1} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)} =$$

$$= \sum_{j_3=0}^{p_3} C_{j_3 j_3 0}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\ \sum_{j_1=2, j_1-\text{even}}^{2j_3+2} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.167)$$

Since

$$C_{j_3 j_3 j_1} = \frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1} P_{j_1}(y)\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2 dy, \quad j_3 \ge 1,$$

and degree of the polynomial

$$\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2$$

equals to $2j_3+2$, then $C_{j_3j_3j_1}=0$ for $j_1>2j_3+2$. It explains that we put $2j_3+2$ instead of p_1 on the right-hand side of the formula (2.167).

Moreover, the function

$$\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_1}(y)\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$$

is uneven for uneven j_1. It means that $C_{j_3j_3j_1}=0$ for uneven j_1. It explains the summation with respect to even j_1 on the right-hand side of (2.167).

Then we have

$$\sum_{j_3=0}^{p_3}\ \sum_{j_1=2,j_1-\mathrm{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2,j_1-\mathrm{even}}^{2p_3+2}\ \sum_{j_3=(j_1-2)/2}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_1=2,j_1-\mathrm{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{2.168}$$

We replaced $(j_1-2)/2$ by zero on the right-hand side of (2.168), since $C_{j_3j_3j_1}=0$ for $0\le j_3<(j_1-2)/2$.

Let us substitute (2.168) into (2.167)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_{0}^{(i_1)}-\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_{1}^{(i_1)}+$$

$$+\sum_{j_1=2,j_1-\mathrm{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{2.169}$$

It is easy to see that the right-hand side of the formula (2.169) does not depend on p_1.

If we prove that

$$\lim_{p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\right)^2\right\}=0, \tag{2.170}$$

then (2.140) will be proved.

Using (2.169) and (2.164), (2.165), we can rewrite the left-hand side of the formula (2.170) in the following form

$$
\lim_{p_3\to\infty} \mathsf{M}\left\{\left(\left(\sum_{j_3=0}^{p_3} C_{j_3j_30}-\frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_1)}+\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=
$$

$$
=\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_1} C_{j_3j_30}-\frac{(T-t)^{3/2}}{4}\right)^2+\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2=
$$

$$
=\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2.
$$

If we prove that

$$
\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2=0, \tag{2.171}
$$

then the relation (2.140) will be proved.

From (2.166) we obtain

$$
\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\left((T-s_2)-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2\right)ds_2\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2\le
$$

$$
\le\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\left|\phi_{j_1}(s_2)\right|\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2. \tag{2.172}
$$

In order to get (2.172) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int_s^T \phi_{j_1}(s_1)ds_1\right)^2=\int_t^T \left(\mathbf{1}_{\{s<s_1\}}\right)^2 ds_1=T-s \tag{2.173}$$

and the orthogonality property of the Legendre polynomials

$$\int_t^T \phi_{j_3}(s)(T-s)ds=0,\quad j_3\ge 2. \tag{2.174}$$

Then we have for $j_3\in\mathbf{N}$

$$\left(\int_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2=\frac{(T-t)}{4(2j_3+1)}\left(P_{j_3+1}\left(z(s_2)\right)-P_{j_3-1}\left(z(s_2)\right)\right)^2\le$$

$$\le\frac{T-t}{2(2j_3+1)}\left(P_{j_3+1}^2\left(z(s_2)\right)+P_{j_3-1}^2\left(z(s_2)\right)\right)<$$

$$<\frac{T-t}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\frac{1}{(1-(z(s_2))^2)^{1/2}}<$$

$$<\frac{(T-t)K^2}{2j_3^2}\frac{1}{(1-(z(s_2))^2)^{1/2}},\quad s_2\in(t,T). \tag{2.175}$$

In order to get (2.175) we used the estimate (2.157).

Substituting the estimate (2.175) into the relation (2.172) and using in (2.172) the estimate (2.158) for $|\phi_{j_1}(s_2)|$, we obtain

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2<$$

$$<\frac{(T-t)K^4K_1^2}{16}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int_t^T\frac{ds_2}{(1-z^2(s_2))^{3/4}}\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2=$$

$$=\frac{(T-t)^3K^4K_1^2(p_3+1)}{64}\left(\int_{-1}^1\frac{dy}{(1-y^2)^{3/4}}\right)^2\left(\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2. \tag{2.176}$$

Using (2.161) and (2.162) in (2.176), we get

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2<\frac{C(T-t)^3(p_3+1)}{p_3^2}\ \to 0\quad\text{with}\ \ p_3\to\infty,\tag{2.177}$$

where constant C does not depend on p_3 and $T-t$.

The relation (2.177) implies (2.171), and the relation (2.171) implies the correctness of the formula (2.140). The relation (2.140) is proved.

Let us prove the equality (2.141). Since $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)\equiv 1$, then the following relation for the Fourier coefficients is correct

$$C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3},$$

where $C_j=0$ for $j\ge 1$ and $C_0=\sqrt{T-t}$. Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}.\tag{2.178}$$

Therefore, considering (2.139) and (2.140), we can write w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=\frac{1}{2}C_0^3\zeta_0^{(i_2)}-$$

$$-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)-$$

$$-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)=0.\tag{2.179}$$

The relation (2.141) is proved. Theorem 2.4 is proved.

It is easy to see that the formula (2.138) can be proved for the case $i_1=i_2=i_3$ using the Itô formula

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}=\frac{1}{6}\left(\int\limits_t^{T}d\mathbf{f}_s^{(i_1)}\right)^3=\frac{1}{6}\left(C_0\zeta_0^{(i_1)}\right)^3=C_{000}\zeta_0^{(i_1)}\zeta_0^{(i_1)}\zeta_0^{(i_1)},$$

where the equality is fulfilled w. p. 1.

2.2.2 The Case $p_1, p_2, p_3 \to \infty$, Binomial Weight Functions, and Additional Restrictive Conditions (The Case of Legendre Polynomials)

Let us consider the following generalization of Theorem 2.4.

Theorem 2.5 [6]-[17], [35]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \int\limits_t^{*T} (t-t_3)^{l_3} \int\limits_t^{*t_3} (t-t_2)^{l_2} \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}$$

the following expansion

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \qquad (2.180)$$

that converges in the mean-square sense is valid for each of the following cases

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
2. $i_1 = i_2 \ne i_3$ *and* $l_1 = l_2 \ne l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
3. $i_1 \ne i_2 = i_3$ *and* $l_1 \ne l_2 = l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
4. $i_1, i_2, i_3 = 1, \ldots, m;\ l_1 = l_2 = l_3 = l$ *and* $l = 0, 1, 2, \ldots,$

where $i_1, i_2, i_3 = 1, \ldots, m,$

$$C_{j_3 j_2 j_1} = \int\limits_t^T (t-s)^{l_3} \phi_{j_3}(s) \int\limits_t^s (t-s_1)^{l_2} \phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^{l_1} \phi_{j_1}(s_2) ds_2 ds_1 ds,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. Case 1 directly follows from (1.47). Let us consider Case 2, i.e. $i_1 = i_2 \ne i_3$, $l_1 = l_2 = l \ne l_3$, and $l_1, l_3 = 0, 1, 2, \ldots$ So, we prove the following expansion

$$I_{l_1 l_1 l_3 T,t}^{*(i_1 i_1 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_1)} \zeta_{j_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m), \qquad (2.181)$$

where $l_1, l_3 = 0, 1, 2, \ldots$ $(l_1 = l)$ and

$$C_{j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3} \int\limits_t^s (t-s_1)^l \phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^l \phi_{j_1}(s_2) ds_2 ds_1 ds. \quad (2.182)$$

If we prove w. p. 1 the formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)}, \quad (2.183)$$

where coefficients $C_{j_3 j_1 j_1}$ are defined by (2.182), then using Theorem 1.1 and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.181).

Using Theorem 1.1, we obtain

$$\frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\sum_{j_3=0}^{2l+l_3+1} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)} \quad \text{w. p. 1},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 ds.$$

Then

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)} + \sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}.$$

Therefore,

$$\lim_{p_1,p_3\to\infty} \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)}\right)^2\right\} =$$

$$= \lim_{p_1\to\infty}\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 +$$

$$
+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \qquad (2.184)
$$

Let us prove that

$$
\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=0. \qquad (2.185)
$$

We have

$$
\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=
$$

$$
=\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds-\right.
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_t^s(t-s_1)^{2l}ds_1\right)ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s(t-s_1)^{2l}ds_1-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_t^s(t-s_1)^{2l}ds_1\right)ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \qquad (2.186)
$$

In order to get (2.186) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1, \tag{2.187}$$

where

$$K(s,s_1)=(t-s_1)^l\ \mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

Taking into account the nondecreasing of the functional sequence

$$u_n(s)=\sum_{j_1=0}^{n}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2,$$

continuity of its members and continuity of the limit function

$$u(s)=\int\limits_t^s (t-s_1)^{2l}ds_1$$

at the interval $[t,T]$ in accordance with the Dini Theorem we have uniform convergence of the functional sequences $u_n(s)$ to the limit function $u(s)$ at the interval $[t,T]$.

From (2.186) using the inequality of Cauchy–Bunyakovsky, we obtain

$$\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2 \le$$

$$\le \frac{1}{4}\int\limits_t^T \phi_{j_3}^2(s)(t-s)^{2l_3}ds\int\limits_t^T\left(\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right)^2 ds \le$$

$$\le \frac{1}{4}\varepsilon^2(T-t)^{2l_3}\int\limits_t^T \phi_{j_3}^2(s)ds(T-t)=\frac{1}{4}(T-t)^{2l_3+1}\varepsilon^2 \tag{2.188}$$

when $p_1>N(\varepsilon)$, where $N(\varepsilon)\in\mathbf{N}$ exists for any $\varepsilon>0$. The relation (2.188) implies (2.185).

Further,

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{2.189}$$

We put $2(j_1+l+1)+l_3$ instead of p_3, since $C_{j_3j_1j_1}=0$ for $j_3>2(j_1+l+1)+l_3$. This conclusion follows from the relation

$$C_{j_3j_1j_1}=\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)Q_{2(j_1+l+1)+l_3}(s)ds,$$

where $Q_{2(j_1+l+1)+l_3}(s)$ is a polynomial of degree $2(j_1+l+1)+l_3$.

It is easy to see that

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.190)$$

Note that we included some zero coefficients $C_{j_3j_1j_1}$ into the sum $\sum\limits_{j_1=0}^{p_1}$. From (2.189) and (2.190) we have

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=$$

$$=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s(t-s_1)^{2l}ds_1-\right.\right.$$

$$
-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\Biggr) ds\Biggr)^2 =
$$

$$
=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \tag{2.191}
$$

In order to get (2.191) we used the Parseval equality (2.187) and the following relation

$$
\int\limits_t^T \phi_{j_3}(s)Q_{2l+1+l_3}(s)ds=0,\quad j_3>2l+1+l_3,
$$

where $Q_{2l+1+l_3}(s)$ is a polynomial of degree $2l+1+l_3$.

Further, we have

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}(2j_1+1)}{2^{2l+2}}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)(1+y)^l dy\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}}{2^{2l+2}(2j_1+1)}\times
$$

$$
\times\left((1+z(s))^l R_{j_1}(s) - l\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2 \le
$$

$$
\le \frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_1+1)}\times
$$

$$
\times\left(\left(\frac{2(s-t)}{T-t}\right)^{2l} R_{j_1}^2(s) + l^2\left(\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2\right) \le
$$

$$
\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\times
$$

$$\times\left(2^{2l+1}Z_{j_1}(s)+l^2\int\limits_{-1}^{z(s)}(1+y)^{2l-2}dy\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)^2dy\right)\le$$

$$\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\times$$

$$\times\left(2^{2l+1}Z_{j_1}(s)+\frac{2l^2}{2l-1}\left(\frac{2(s-t)}{T-t}\right)^{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right)\le$$

$$\le\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(2Z_{j_1}(s)+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right),\qquad(2.192)$$

where $j_1\in\mathbf{N}$,

$$R_{j_1}(s)=P_{j_1+1}(z(s))-P_{j_1-1}(z(s)),$$
$$Z_{j_1}(s)=P_{j_1+1}^2(z(s))+P_{j_1-1}^2(z(s)).$$

Let us estimate the right-hand side of (2.192) using (2.157) ($j_1\in\mathbf{N}$)

$$\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2<$$

$$<\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\left(\frac{2}{(1-(z(s))^2)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/2}}\right)<$$

$$<\frac{(T-t)^{2l+1}K^2}{2j_1^2}\left(\frac{2}{(1-(z(s))^2)^{1/2}}+\frac{l^2\pi}{2l-1}\right),\quad s\in(t,T).\qquad(2.193)$$

From (2.191) and (2.193) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le$$

$$\le\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2\le$$

$$
\le \frac{1}{4}(T-t)^{2l_3}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2<
$$

$$
<\frac{(T-t)^{4l+2l_3+1}K^4K_1^2}{16}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\left(\int\limits_t^T\frac{2ds}{\left(1-(z(s))^2\right)^{3/4}}+\right.\right.
$$

$$
\left.\left.+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds}{\left(1-(z(s))^2\right)^{1/4}}\right)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2\le
$$

$$
\le\frac{(T-t)^{4l+2l_3+3}K^4K_1^2}{64}\ \frac{2p_1+1}{p_1^2}\left(\int\limits_{-1}^1\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^1\frac{dy}{(1-y^2)^{1/4}}\right)^2\le
$$

$$
\le C(T-t)^{4l+2l_3+3}\ \frac{2p_1+1}{p_1^2}\to 0\quad\text{when}\ \ p_1\ \to\ \infty, \tag{2.194}
$$

where constant C does not depend on p_1 and $T-t$.

The relations (2.184), (2.185), and (2.194) imply (2.183), and the relation (2.183) implies the correctness of the formula (2.181).

Let us consider Case 3, i.e. $i_2=i_3\ne i_1$, $l_2=l_3=l\ne l_1$, and $l_1,l_3=0,1,2,\ldots$ So, we prove the following expansion

$$
I_{l_1l_3l_3T,t}^{*(i_1i_3i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_3)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{2.195}
$$

where $l_1,l_3=0,1,2,\ldots$ $(l_3=l)$ and

$$
C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.196}
$$

If we prove w. p. 1 the formula

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds, \tag{2.197}
$$

where the coefficients $C_{j_3j_3j_1}$ are defined by (2.196), then using Theorem 1.1 and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.195).

Using Theorem 1.1 and the Itô formula, we have

$$\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsd\mathbf{f}_{s_1}^{(i_1)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)}\quad \text{w. p. 1},$$

where

$$\tilde{C}_{j_1}=\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsds_1.$$

Then

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)\zeta_{j_1}^{(i_1)}+\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}.$$

Therefore,

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds\right)^2\right\}=$$

$$=\lim_{p_3\to\infty}\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2+$$

$$+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}. \tag{2.198}$$

Let us prove that

$$\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2=0. \tag{2.199}$$

We have

$$\left(\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)^2 =$$

$$= \left(\sum_{j_3=0}^{p_3} \int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1} ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1 \int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds - \right.$$

$$\left. - \frac{1}{2}\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \int\limits_{s_1}^T (t-s)^{2l} ds ds_1\right)^2 =$$

$$= \left(\frac{1}{2}\sum_{j_3=0}^{p_3} \int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1} \left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2 - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \int\limits_{s_1}^T (t-s)^{2l} ds ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \left(\sum_{j_3=0}^{p_3}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2 - \right.\right.$$

$$\left.\left. - \int\limits_{s_1}^T (t-s)^{2l} ds\right) ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \left(\int\limits_{s_1}^T (t-s)^{2l} ds - \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2 - \right.\right.$$

$$\left.\left. - \int\limits_{s_1}^T (t-s)^{2l} ds\right) ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2 ds_1\right)^2. \qquad (2.200)$$

In order to get (2.200) we used the Parseval equality

$$\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2=\int\limits_t^T K^2(s,s_1)ds, \qquad (2.201)$$

where

$$K(s,s_1)=(t-s)^l\ \mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

Taking into account the nondecreasing of the functional sequence

$$u_n(s_1)=\sum_{j_3=0}^{n}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2,$$

continuity of its members and continuity of the limit function

$$u(s_1)=\int\limits_{s_1}^{T}(t-s)^{2l}ds$$

at the interval $[t,T]$ in accordance with the Dini Theorem we have uniform convergence of the functional sequence $u_n(s_1)$ to the limit function $u(s_1)$ at the interval $[t,T]$.

From (2.200) using the inequality of Cauchy–Bunyakovsky, we obtain

$$\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2\le$$

$$\le\frac{1}{4}\int\limits_t^T\phi_{j_1}^2(s_1)(t-s_1)^{2l_1}ds_1\int\limits_t^T\left(\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2\right)^2 ds_1\le$$

$$\le\frac{1}{4}\varepsilon^2(T-t)^{2l_1}\int\limits_t^T\phi_{j_1}^2(s_1)ds_1(T-t)=\frac{1}{4}(T-t)^{2l_1+1}\varepsilon^2 \qquad (2.202)$$

when $p_3>N(\varepsilon)$, where $N(\varepsilon)\in\mathbf{N}$ exists for any $\varepsilon>0$. The relation (2.199) follows from (2.202).

We have

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.203)$$

We put $2(j_3+l+1)+l_1$ instead of p_1, since $C_{j_3j_3j_1}=0$ when $j_1>2(j_3+l+1)+l_1$. This conclusion follows from the relation

$$C_{j_3j_3j_1}=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2=$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_2)Q_{2(j_3+l+1)+l_1}(s_2)ds_2,$$

where $Q_{2(j_3+l+1)+l_1}(s)$ is a polynomial of degree $2(j_3+l+1)+l_1$.

It is easy to see that

$$\sum_{j_3=0}^{p_3}\ \sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.204)$$

Note that we included some zero coefficients $C_{j_3j_3j_1}$ into the sum $\sum\limits_{j_3=0}^{p_3}$.

From (2.203) and (2.204) we have

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\ \sum_{j_1=2l+l_1+2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2=$$

$$=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T (t-s_1)^{2l}ds_1-\right.\right.$$

$$
- \sum_{j_3=p_3+1}^{\infty} \left(\int\limits_{s_2}^{T} \phi_{j_3}(s_1)(t-s_1)^l ds_1 \right)^2 \Biggr) ds_2 \Biggr)^2 =
$$

$$
= \frac{1}{4} \sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1} \left(\int\limits_{t}^{T} \phi_{j_1}(s_2)(t-s_2)^{l_1} \sum_{j_3=p_3+1}^{\infty} \left(\int\limits_{s_2}^{T} \phi_{j_3}(s_1)(t-s_1)^l ds_1 \right)^2 ds_2 \right)^2 . \tag{2.205}
$$

In order to get (2.205) we used the Parseval equality (2.201) and the following relation

$$
\int\limits_{t}^{T} \phi_{j_1}(s) Q_{2l+1+l_1}(s) ds = 0, \quad j_1 > 2l+1+l_1,
$$

where $Q_{2l+1+l_1}(s)$ is a polynomial of degree $2l+1+l_1$.

Further, we have

$$
\left(\int\limits_{s_2}^{T} \phi_{j_3}(s_1)(t-s_1)^l ds_1 \right)^2 =
$$

$$
= \frac{(T-t)^{2l+1}(2j_3+1)}{2^{2l+2}} \left(\int\limits_{z(s_2)}^{1} P_{j_3}(y)(1+y)^l dy \right)^2 =
$$

$$
= \frac{(T-t)^{2l+1}}{2^{2l+2}(2j_3+1)} \times
$$

$$
\times \left((1+z(s_2))^l Q_{j_3}(s_2) - l \int\limits_{z(s_2)}^{1} \left(P_{j_3+1}(y) - P_{j_3-1}(y) \right) (1+y)^{l-1} dy \right)^2 \le
$$

$$
\le \frac{(T-t)^{2l+1} 2}{2^{2l+2}(2j_3+1)} \times
$$

$$
\times \left(\left(\frac{2(s_2-t)}{T-t} \right)^{2l} Q_{j_3}^2(s_2) + l^2 \left(\int\limits_{z(s_2)}^{1} \left(P_{j_3+1}(y) - P_{j_3-1}(y) \right) (1+y)^{l-1} dy \right)^2 \right) \le
$$

$$
\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)} \times
$$

$$
\times\left(2^{2l+1}H_{j_3}(s_2)+l^2\int\limits_{z(s_2)}^{1}(1+y)^{2l-2}dy\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)^2dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\times
$$

$$
\times\left(2^{2l+1}H_{j_3}(s_2)+\frac{2^{2l}l^2}{2l-1}\left(1-\left(\frac{s_2-t}{T-t}\right)^{2l-1}\right)\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(2H_{j_3}(s_2)+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right),\qquad(2.206)
$$

where $j_3\in\mathbf{N}$,

$$
Q_{j_3}(s_2)=P_{j_3-1}(z(s_2))-P_{j_3+1}(z(s_2)),
$$

$$
H_{j_3}(s_2)=P_{j_3-1}^2(z(s_2))+P_{j_3+1}^2(z(s_2)).
$$

Let us estimate the right-hand side of (2.206) using (2.157) ($j_3\in\mathbf{N}$)

$$
\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2<
$$

$$
<\frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\frac{dy}{(1-y^2)^{1/2}}\right)<
$$

$$
<\frac{(T-t)^{2l+1}K^2}{2j_3^2}\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2\pi}{2l-1}\right),\quad s_2\in(t,T).\qquad(2.207)
$$

From (2.205) and (2.207) we obtain

$$
\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\le
$$

$$
\le\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_{t}^{T}|\phi_{j_1}(s_2)|(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2ds_2\right)^2\le
$$

$$\le \frac{1}{4}(T-t)^{2l_1}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T|\phi_{j_1}(s_2)|\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2<$$

$$<\frac{(T-t)^{4l+2l_1+1}K^4K_1^2}{16}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\left(\int\limits_t^T\frac{2ds_2}{\left(1-(z(s_2))^2\right)^{3/4}}+\right.\right.$$

$$\left.\left.+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds_2}{\left(1-(z(s_2))^2\right)^{1/4}}\right)\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2\le$$

$$\le\frac{(T-t)^{4l+2l_1+3}K^4K_1^2}{64}\,\frac{2p_3+1}{p_3^2}\left(\int\limits_{-1}^1\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^1\frac{dy}{(1-y^2)^{1/4}}\right)^2\le$$

$$\le C(T-t)^{4l+2l_1+3}\,\frac{2p_3+1}{p_3^2}\to 0\quad\text{when}\quad p_3\ \to\ \infty, \tag{2.208}$$

where constant C does not depend on p_3 and $T-t$.

The relations (2.198), (2.199), and (2.208) imply (2.197), and the relation (2.197) implies the correctness of the expansion (2.195).

Let us consider Case 4, i.e. $l_1=l_2=l_3=l=0,1,2,\ldots$ and $i_1,i_2,i_3=1,\ldots,m$. So, we will prove the following expansion for iterated Stratonovich stochastic integral of third multiplicity

$$I_{lll_{T,t}}^{*(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{2.209}$$

where the series converges in the mean-square sense, $l=0,1,2,\ldots,$ and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^l\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.210}$$

If we prove w. p. 1 the following formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{2.211}$$

where the coefficients $C_{j_3j_2j_1}$ are defined by (2.210), then using the formulas (2.183), (2.197) when $l_1=l_3=l$, Theorem 1.1, and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.209).

Since $\psi_1(s),\ \psi_2(s),\ \psi_3(s)\ \equiv\ (t-s)^l$, then the following equality for the Fourier coefficients takes place

$$C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3},$$

where the coefficients $C_{j_3j_2j_1}$ are defined by (2.210) and

$$C_{j_1}=\int\limits_t^T\phi_{j_1}(s)(t-s)^l ds.$$

Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \qquad (2.212)$$

Taking into account (2.183) and (2.197) when $l_3=l_1=l$ as well as the Itô formula, we have w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\sum_{j_3=0}^{l}C_{j_3}\zeta_{j_3}^{(i_2)}-$$

$$-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_t^T(t-s)^l\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_2)}-$$

$$-\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^l d\mathbf{f}_{s_1}^{(i_2)}ds=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}+\frac{1}{2(2l+1)}\int\limits_t^T(t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}-$$

$$
-\frac{1}{2}\int\limits_t^T (t-s_1)^l \int\limits_{s_1}^T (t-s)^{2l} ds d\mathbf{f}_{s_1}^{(i_2)} =
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} + \frac{1}{2(2l+1)}\int\limits_t^T (t-s)^{3l+1} d\mathbf{f}_s^{(i_2)} -
$$

$$
-\frac{1}{2(2l+1)}\left((T-t)^{2l+1}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} + \int\limits_t^T (t-s)^{3l+1} d\mathbf{f}_s^{(i_2)}\right) =
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} - \frac{(T-t)^{2l+1}}{2(2l+1)}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} =
$$

$$
=\frac{1}{2}\left(\sum_{j_1=0}^{l} C_{j_1}^2 - \int\limits_t^T (t-s)^{2l} ds\right)\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} = 0.
$$

Here the Parseval equality looks as follows

$$
\sum_{j_1=0}^{\infty} C_{j_1}^2 = \sum_{j_1=0}^{l} C_{j_1}^2 = \int\limits_t^T (t-s)^{2l} ds = \frac{(T-t)^{2l+1}}{2l+1}
$$

and

$$
\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} = \sum_{j_3=0}^{l} C_{j_3}\zeta_{j_3}^{(i_2)} \quad \text{w. p. 1.}
$$

The expansion (2.209) is proved. Theorem 2.5 is proved.

It is easy to see that using the Itô formula if $i_1 = i_2 = i_3$ we obtain (see (1.61))

$$
\int\limits_t^{*T} (t-s)^l \int\limits_t^{*s} (t-s_1)^l \int\limits_t^{*s_1} (t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)} d\mathbf{f}_{s_1}^{(i_1)} d\mathbf{f}_s^{(i_1)} =
$$

$$
=\frac{1}{6}\left(\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_1)}\right)^3 = \frac{1}{6}\left(\sum_{j_1=0}^{l} C_{j_1}\zeta_{j_1}^{(i_1)}\right)^3 =
$$

$$
=\sum_{j_1,j_2,j_3=0}^{l} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)} \quad \text{w. p. 1.} \tag{2.213}
$$

2.2.3 The Case $p_1, p_2, p_3 \to \infty$ and Constant Weight Functions (The Case of Trigonometric Functions)

In this section, we will prove the following theorem.

Theorem 2.6 [6]-[17], [35]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{2.214}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. If we prove w. p. 1 the following formulas

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}, \tag{2.215}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau, \tag{2.216}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{2.217}$$

then from the equalities (2.215)–(2.217), Theorem 1.1, and standard relations between iterated Itô and Stratonovich stochastic integrals we will obtain the expansion (2.214).

We have

$$S_{p_1,p_3} \stackrel{\text{def}}{=} \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \frac{(T-t)^{3/2}}{6}\zeta_0^{(i_3)}+$$

$$+\sum_{j_1=1}^{p_1} C_{0,2j_1,2j_1}\zeta_0^{(i_3)}+\sum_{j_1=1}^{p_1} C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{p_1} C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3} C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)}, \qquad (2.218)$$

where the summation is stopped, when $2j_1,\ 2j_1-1>p_1$ or $2j_3,\ 2j_3-1>p_3$ and

$$C_{0,2l,2l}=\frac{(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{0,2l-1,2l-1}=\frac{3(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{2l,0,0}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 l^2}, \qquad (2.219)$$

$$C_{2r-1,2l,2l}=0,\quad C_{2l-1,0,0}=-\frac{\sqrt{2}(T-t)^{3/2}}{4\pi l},\quad C_{2r-1,2l-1,2l-1}=0, \qquad (2.220)$$

$$C_{2r,2l,2l}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ 0, & r\ne 2l\end{cases}, \qquad (2.221)$$

$$C_{2r,2l-1,2l-1}=\begin{cases}\sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ -\sqrt{2}(T-t)^{3/2}/(4\pi^2 l^2), & r=l\\ 0, & r\ne l,\ r\ne 2l\end{cases}. \qquad (2.222)$$

Let us show that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3}. \qquad (2.223)$$

We have

$$S_{2p_1,2p_3} = S_{2p_1,2p_3-1} + \sum_{j_1=0}^{2p_1} C_{2p_3,j_1,j_1}\zeta_{2p_3}^{(i_3)}. \qquad (2.224)$$

Using the relations (2.219), (2.221), and (2.222), we obtain

$$\sum_{j_1=0}^{2p_1} C_{2p_3,j_1,j_1} = C_{2p_3,0,0} + \sum_{j_1=1}^{2p_1} C_{2p_3,j_1,j_1} =$$

$$= C_{2p_3,0,0} + \sum_{j_1=1}^{p_1}\Bigl(C_{2p_3,2j_1-1,2j_1-1} + C_{2p_3,2j_1,2j_1}\Bigr) =$$

$$= \frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 p_3^2}\bigl(1 - \mathbf{1}_{\{p_1\ge p_3\}}\bigr). \qquad (2.225)$$

From (2.224), (2.225) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1}. \qquad (2.226)$$

Further, we get (see (2.219)–(2.221))

$$S_{2p_1,2p_3-1} = S_{2p_1-1,2p_3-1} + \sum_{j_3=0}^{2p_3-1} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}, \qquad (2.227)$$

$$\sum_{j_3=0}^{2p_3-1} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} = C_{0,2p_1,2p_1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} - C_{2p_3,2p_1,2p_1}\zeta_{2p_3}^{(i_3)} =$$

$$= C_{0,2p_1,2p_1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{p_3}\Bigl(C_{2j_3-1,2p_1,2p_1}\zeta_{2j_3-1}^{(i_3)} + C_{2j_3,2p_1,2p_1}\zeta_{2j_3}^{(i_3)}\Bigr) - C_{2p_3,2p_1,2p_1}\zeta_{2p_3}^{(i_3)} =$$

$$= \frac{(T-t)^{3/2}}{8\pi^2 p_1^2}\zeta_0^{(i_3)} + \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_1^2}\bigl(\mathbf{1}_{\{p_3=2p_1\}} - \mathbf{1}_{\{p_3\ge 2p_1\}}\bigr)\zeta_{4p_1}^{(i_3)}. \qquad (2.228)$$

From (2.227), (2.228) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3-1}. \qquad (2.229)$$

Further, we have

$$S_{2p_1,2p_3} = S_{2p_1-1,2p_3} + \sum_{j_3=0}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}, \qquad (2.230)$$

$$\sum_{j_3=0}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} = C_{0,2p_1,2p_1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} =$$

$$= C_{0,2p_1,2p_1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{p_3}\left(C_{2j_3-1,2p_1,2p_1}\zeta_{2j_3-1}^{(i_3)} + C_{2j_3,2p_1,2p_1}\zeta_{2j_3}^{(i_3)}\right). \qquad (2.231)$$

From (2.231), (2.219)–(2.221) we obtain

$$\sum_{j_3=0}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} = \frac{(T-t)^{3/2}}{8\pi^2 p_1^2}\zeta_0^{(i_3)} - \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_1^2}\mathbf{1}_{\{p_3\ge 2p_1\}}\zeta_{4p_1}^{(i_3)}. \qquad (2.232)$$

The relations (2.230), (2.232) mean that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\; S_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\; S_{2p_1-1,2p_3}. \qquad (2.233)$$

The equalities (2.226), (2.229), and (2.233) imply (2.223). This means that instead of (2.215) it is enough to prove the following equality

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int_t^T\int_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} \quad \text{w. p. 1.} \qquad (2.234)$$

We have

$$S_{2p_1,2p_3} = \sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{(T-t)^{3/2}}{6}\zeta_0^{(i_3)} +$$

$$+\sum_{j_1=1}^{p_1} C_{0,2j_1,2j_1}\zeta_0^{(i_3)} + \sum_{j_1=1}^{p_1} C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{p_3} C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)} +$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)} + \sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)} + \sum_{j_3=1}^{p_3} C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)} +$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)} + \sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)}. \qquad (2.235)$$

After substituting (2.219)–(2.222) into (2.235), we obtain

$$\sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = (T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_3)} + \frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\;\zeta_0^{(i_3)} - \right.$$

$$
-\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}-\frac{\sqrt{2}}{4\pi^2}\sum_{j_3=1}^{\min\{p_1,p_3\}}\frac{1}{j_3^2}\zeta_{2j_3}^{(i_3)}+\frac{\sqrt{2}}{4\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\zeta_{2j_3}^{(i_3)}\Biggr). \tag{2.236}
$$

From (2.236) we have w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\pi^2}\sum_{j_1=1}^{\infty}\frac{1}{j_1^2}\ \zeta_0^{(i_3)}-
$$

$$
-\underset{p_3\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\Biggr). \tag{2.237}
$$

Using Theorem 1.1 and the system of trigonometric functions, we get w. p. 1

$$
\frac{1}{2}\int\limits_t^T\int\limits_t^s d\tau d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\int\limits_t^T(s-t)d\mathbf{f}_s^{(i_3)}=
$$

$$
=\frac{(T-t)^{3/2}}{4}\underset{p_3\to\infty}{\text{l.i.m.}}\Biggl(\zeta_0^{(i_3)}-\frac{\sqrt{2}}{\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\Biggr). \tag{2.238}
$$

From (2.237) and (2.238) it follows that

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=
$$

$$
=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{12}\zeta_0^{(i_3)}-\underset{p_3\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\Biggr)=
$$

$$
=(T-t)^{3/2}\Biggl(\frac{1}{4}\zeta_0^{(i_3)}-\underset{p_3\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\Biggr)=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^s d\tau d\mathbf{f}_s^{(i_3)},
$$

where the equality is fulfilled w. p. 1.

So, the relations (2.234) and (2.215) are proved for the case of trigonometric system of functions.

Let us prove the relation (2.216). We have

$$S'_{p_1,p_3} \stackrel{\sf def}{=} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)} = \frac{(T-t)^{3/2}}{6}\zeta_0^{(i_1)} +$$

$$+\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,0}\zeta_0^{(i_1)} + \sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} +$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + \sum_{j_1=1}^{p_1} C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + \sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)} +$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)} + \sum_{j_1=1}^{p_1} C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}, \tag{2.239}$$

where the summation is stopped, when $2j_3,\ 2j_3-1>p_3$ or $2j_1,\ 2j_1-1>p_1$ and

$$C_{2l,2l,0} = \frac{(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{2l-1,2l-1,0} = \frac{3(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{0,0,2r} = \frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 r^2}, \tag{2.240}$$

$$C_{2l-1,2l-1,2r-1} = 0,\quad C_{0,0,2r-1} = \frac{\sqrt{2}(T-t)^{3/2}}{4\pi r},\quad C_{2l,2l,2r-1} = 0, \tag{2.241}$$

$$C_{2l,2l,2r} = \begin{cases} -\sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ \\ 0, & r\ne 2l\end{cases}, \tag{2.242}$$

$$C_{2l-1,2l-1,2r} = \begin{cases} \sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ \\ -\sqrt{2}(T-t)^{3/2}/(4\pi^2 l^2), & r=l\\ \\ 0, & r\ne l,\ r\ne 2l\end{cases}. \tag{2.243}$$

Let us show that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3-1} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3-1} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3}. \tag{2.244}$$

We have

$$S'_{2p_1,2p_3} = S'_{2p_1-1,2p_3} + \sum_{j_3=0}^{2p_3} C_{j_3,j_3,2p_1}\zeta_{2p_1}^{(i_1)}. \tag{2.245}$$

Using the relations (2.240), (2.242), and (2.243), we obtain

$$\sum_{j_1=0}^{2p_3} C_{j_3,j_3,2p_1} = C_{0,0,2p_1} + \sum_{j_3=1}^{2p_3} C_{j_3,j_3,2p_1} =$$

$$= C_{0,0,2p_1} + \sum_{j_3=1}^{p_3}\Bigl(C_{2j_3-1,2j_3-1,2p_1} + C_{2j_3,2j_3,2p_1}\Bigr) =$$

$$= \frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 p_1^2}\bigl(1 - \mathbf{1}_{\{p_3\ge p_1\}}\bigr). \qquad (2.246)$$

From (2.245), (2.246) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3}. \qquad (2.247)$$

Further, we get (see (2.240)–(2.242))

$$S'_{2p_1-1,2p_3} = S'_{2p_1-1,2p_3-1} + \sum_{j_1=0}^{2p_1-1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)}, \qquad (2.248)$$

$$\sum_{j_1=0}^{2p_1-1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} - C_{2p_3,2p_3,2p_1}\zeta_{2p_1}^{(i_1)} =$$

$$= C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\Bigl(C_{2p_3,2p_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + C_{2p_3,2p_3,2j_1}\zeta_{2j_1}^{(i_1)}\Bigr) - C_{2p_3,2p_3,2p_1}\zeta_{2p_1}^{(i_1)} =$$

$$= \frac{(T-t)^{3/2}}{8\pi^2 p_3^2}\zeta_0^{(i_1)} + \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_3^2}\bigl(\mathbf{1}_{\{p_1=2p_3\}} - \mathbf{1}_{\{p_1\ge 2p_3\}}\bigr)\zeta_{4p_3}^{(i_1)}. \qquad (2.249)$$

From (2.248), (2.249) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3-1}. \qquad (2.250)$$

Further, we have

$$S'_{2p_1,2p_3} = S'_{2p_1,2p_3-1} + \sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)}, \qquad (2.251)$$

$$\sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} =$$

$$
= C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\left(C_{2p_3,2p_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + C_{2p_3,2p_3,2j_1}\zeta_{2j_1}^{(i_1)}\right). \qquad (2.252)
$$

From (2.252), (2.240)–(2.242) we obtain

$$
\sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = \frac{(T-t)^{3/2}}{8\pi^2 p_3^2}\zeta_0^{(i_1)} - \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_3^2}\mathbf{1}_{\{p_1\ge 2p_3\}}\zeta_{4p_3}^{(i_1)}. \qquad (2.253)
$$

The relations (2.251), (2.253) mean that

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3-1}. \qquad (2.254)
$$

The equalities (2.247), (2.250), and (2.254) imply (2.244). This means that instead of (2.216) it is enough to prove the following equality

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau \quad \text{w. p. 1.} \qquad (2.255)
$$

We have

$$
S'_{2p_1,2p_3} = \sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{(T-t)^{3/2}}{6}\zeta_0^{(i_1)} +
$$

$$
+\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,0}\zeta_0^{(i_1)} + \sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} +
$$

$$
+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + \sum_{j_1=1}^{p_1} C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + \sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)} +
$$

$$
+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)} + \sum_{j_1=1}^{p_1} C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}. \qquad (2.256)
$$

After substituting (2.240)–(2.243) into (2.256), we obtain

$$
\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = (T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)} + \frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\ \zeta_0^{(i_1)} +\right.
$$

$$
+\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}-\frac{\sqrt{2}}{4\pi^2}\sum_{j_1=1}^{\min\{p_1,p_3\}}\frac{1}{j_1^2}\zeta_{2j_1}^{(i_1)}+\frac{\sqrt{2}}{4\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\zeta_{2j_1}^{(i_1)}\Biggr). \tag{2.257}
$$

From (2.257) we have w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\pi^2}\sum_{j_3=1}^{\infty}\frac{1}{j_3^2}\ \zeta_0^{(i_1)}+
$$

$$
+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\Biggr). \tag{2.258}
$$

Using the Itô formula and Theorem 1.1 for the case of trigonometric system of functions, we obtain w. p. 1

$$
\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau=\frac{1}{2}\left((T-t)\int\limits_t^T d\mathbf{f}_s^{(i_1)}+\int\limits_t^T(t-s)d\mathbf{f}_s^{(i_1)}\right)=
$$

$$
=\frac{1}{4}(T-t)^{3/2}\Biggl(\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\Biggr). \tag{2.259}
$$

From (2.258) and (2.259) it follows that

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=
$$

$$
=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{12}\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\Biggr)=
$$

$$
=(T-t)^{3/2}\Biggl(\frac{1}{4}\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\Biggr)=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau,
$$

where the equality is fulfilled w. p. 1.

So, the relations (2.255) and (2.216) are proved for the case of trigonometric system of functions.

Let us prove the equality (2.217). Since $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)\equiv 1$, then the following relation for the Fourier coefficients is correct

$$
C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3}.
$$

Then w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \qquad (2.260)
$$

Taking into account (2.215) and (2.216), we can write w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\frac{1}{2}C_0^3\zeta_0^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-
$$

$$
-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_2)}\right)-
$$

$$
-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_2)}\right)=0.
$$

From Theorem 1.1 and (2.215)–(2.217) we obtain the expansion (2.214). Theorem 2.6 is proved.

2.2.4 The Case $p_1=p_2=p_3\to\infty$, Smooth Weight Functions, and Additional Restrictive Conditions (The Cases of Legendre Polynomials and Trigonometric Functions)

Let us consider the following modification of Theorem 2.5.

Theorem 2.7 [10]-[17], [35]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ *are continuously differentiable functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3)\int\limits_t^{*t_3} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{2.261}$$

that converges in the mean-square sense is valid for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$,
2. $i_1=i_2\ne i_3$ *and* $\psi_1(s)\equiv\psi_2(s)$,
3. $i_1\ne i_2=i_3$ *and* $\psi_2(s)\equiv\psi_3(s)$,
4. $i_1,i_2,i_3=1,\ldots,m$ *and* $\psi_1(s)\equiv\psi_2(s)\equiv\psi_3(s)$,

where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_2(s_1)\phi_{j_2}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. Let us consider at first the polynomial case. Case 1 directly follows from Theorem 1.1. Further, consider Case 2. We will prove the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1d\mathbf{f}_s^{(i_3)} \quad \text{w. p. 1},$$

where

$$C_{j_3j_1j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1} \psi(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.$$

Using Theorem 1.1, we can write w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\ \underset{p_3\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{p_3} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 ds.$$

We have

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} = \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\frac{1}{2}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\sum_{j_1=0}^{p}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 - \int\limits_t^s \psi^2(s_1)ds_1\right)ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds\right)^2. \qquad (2.262)$$

In order to get (2.262) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1 = \int\limits_t^s \psi^2(s_1)ds_1,$$

where

$$K(s,s_1) = \psi(s_1)\mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

We have for $j_1 \in \mathbf{N}$

$$\left(\int\limits_t^s \psi(s_1)\phi_{j_1}(s_1)ds_1\right)^2 =$$

$$= \frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)\psi\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2 =$$

$$= \frac{T-t}{4(2j_1+1)}\Biggl(\left(P_{j_1+1}(z(s))-P_{j_1-1}(z(s))\right)\psi(s)-$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(s)}\left(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi'\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2, \qquad (2.263)$$

where

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ' is a derivative of the function $\psi(s)$ with respect to the variable

$$\frac{T-t}{2}y+\frac{T+t}{2}.$$

Further consideration is similar to the proof of Case 2 from Theorem 2.5. Finally, from (2.262) and (2.263) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} <$$

$$< K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2 \le$$

$$\le \frac{K_1}{p} \to 0 \quad \text{if} \quad p\to\infty,$$

where constants K, K_1 do not depend on p. Case 2 is proved.

Let us consider Case 3. In this case we will prove the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T \psi^2(s)\int\limits_t^s \psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds \quad \text{w. p. 1},$$

where

$$C_{j_3j_3j_1}=\int\limits_t^T\psi(s)\phi_{j_3}(s)\int\limits_t^s\psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.$$

Using the Itô formula, we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi^2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)}. \tag{2.264}$$

Moreover, using Theorem 1.1, we have w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)}=\frac{1}{2}\ \underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}^{*}\zeta_{j_1}^{(i_1)}, \tag{2.265}$$

where

$$C_{j_1}^{*}=\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsds_1.$$

Further,

$$C_{j_3j_3j_1}=\int\limits_t^T\psi(s)\phi_{j_3}(s)\int\limits_t^s\psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\psi(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi(s)\phi_{j_3}(s)dsds_1ds_2=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\left(\int\limits_{s_2}^T\psi(s_1)\phi_{j_3}(s_1)ds_1\right)^2ds_2. \tag{2.266}$$

From (2.264)–(2.266) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^{*}\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}=\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^{*}\right)^2=$$

$$
=\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\left(\sum_{j_3=0}^{p}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_{s_1}^T\psi^2(s)ds\right)ds_1\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds\right)^2ds_1\right)^2. \qquad (2.267)
$$

In order to get (2.267) we used the Parseval equality

$$
\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds\right)^2=\int\limits_t^T K^2(s,s_1)ds=\int\limits_{s_1}^T\psi^2(s)ds,
$$

where

$$
K(s,s_1)=\psi(s)\mathbf{1}_{\{s>s_1\}},\quad s,s_1\in[t,T].
$$

Further consideration is similar to the proof of Case 3 from Theorem 2.5. Finally, from (2.267) we get

$$
\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^*\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}<
$$

$$
<K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2\le
$$

$$
\le\frac{K_1}{p}\to 0\quad\text{if}\quad p\to\infty,
$$

where constants K, K_1 do not depend on $p.$ Case 3 is proved.

Let us consider Case 4. We will prove w. p. 1 the following relation

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0\quad(\psi_1(s),\psi_2(s),\psi_3(s)\equiv\psi(s)).
$$

In Case 4 we obtain w. p. 1

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)} =
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p}\left(\frac{1}{2}C_{j_1}^2 C_{j_3} - C_{j_1 j_1 j_3} - C_{j_3 j_1 j_1}\right)\zeta_{j_3}^{(i_2)} =
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2\sum_{j_3=0}^{p} C_{j_3}\zeta_{j_3}^{(i_2)} - \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p} C_{j_1 j_1 j_3}\zeta_{j_3}^{(i_2)} -
$$

$$
-\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_2)} =
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{\infty} C_{j_1}^2\int\limits_t^T \psi(s)d\mathbf{f}_s^{(i_2)} - \frac{1}{2}\int\limits_t^T \psi^2(s)\int\limits_t^s \psi(s_1)d\mathbf{f}_{s_1}^{(i_2)}ds -
$$

$$
-\frac{1}{2}\int\limits_t^T \psi(s)\int\limits_t^s \psi^2(s_1)ds_1 d\mathbf{f}_s^{(i_2)} = \frac{1}{2}\int\limits_t^T \psi^2(s)ds\int\limits_t^T \psi(s)d\mathbf{f}_s^{(i_2)} -
$$

$$
-\frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_{s_1}^T \psi^2(s)ds d\mathbf{f}_{s_1}^{(i_2)} - \frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_t^{s_1} \psi^2(s)ds d\mathbf{f}_{s_1}^{(i_2)} =
$$

$$
=\frac{1}{2}\int\limits_t^T \psi^2(s)ds\int\limits_t^T \psi(s)d\mathbf{f}_s^{(i_2)} - \frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_t^T \psi^2(s)ds d\mathbf{f}_{s_1}^{(i_2)} = 0,
$$

where we used the Parseval equality

$$
\sum_{j_1=0}^{\infty} C_j^2 = \sum_{j=0}^{\infty}\left(\int\limits_t^T \psi(s)\phi_j(s)ds\right)^2 = \int\limits_t^T \psi^2(s)ds.
$$

Case 4 and Theorem 2.7 are proved for the case of Legendre polynomials.

Let us consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$
\phi_j(\theta)=\frac{1}{\sqrt{T-t}}\begin{cases} 1, & j=0\\ \\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r\end{cases}
$$

where $r=1,2,\ldots$

Integrating by parts, we have

$$
\int\limits_t^s \phi_{2r-1}(\theta)\psi(\theta)d\theta=\frac{\sqrt{2}}{\sqrt{T-t}}\int\limits_t^s \psi(\theta)\ \sin\frac{2\pi r(\theta-t)}{T-t}d\theta=
$$

$$
=\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(-\psi(s)\ \cos\frac{2\pi r(s-t)}{T-t}+\psi(t)+
$$

$$
+\int\limits_t^s \psi'(\theta)\ \cos\frac{2\pi r(\theta-t)}{T-t}d\theta\Biggr),
$$

$$
\int\limits_t^s \phi_{2r}(\theta)\psi(\theta)d\theta=\frac{\sqrt{2}}{\sqrt{T-t}}\int\limits_t^s \psi(\theta)\ \cos\frac{2\pi r(\theta-t)}{T-t}d\theta=
$$

$$
=\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(\psi(s)\ \sin\frac{2\pi r(s-t)}{T-t}-
$$

$$
-\int\limits_t^s \psi'(\theta)\ \sin\frac{2\pi r(\theta-t)}{T-t}d\theta\Biggr),
$$

where $r=1,2,\ldots$ and $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Then

$$
\left|\int\limits_t^s \phi_{2r-1}(\theta)\psi(\theta)d\theta\right|\le\frac{C}{r}=\frac{2C}{2r}<\frac{2C}{2r-1}, \tag{2.268}
$$

$$\left|\int\limits_t^s \phi_{2r}(\theta)\psi(\theta)d\theta\right| \le \frac{C}{r} = \frac{2C}{2r}, \tag{2.269}$$

where constant C does not depend on r $(r = 1, 2, \ldots)$.

From (2.268), (2.269) we get

$$\left|\int\limits_t^s \phi_{j_1}(\theta)\psi(\theta)d\theta\right| \le \frac{K}{j_1}, \tag{2.270}$$

where constant K is independent of j_1 $(j_1 = 1, 2, \ldots)$.

Analogously, we obtain

$$\left|\int\limits_s^T \phi_{j_1}(\theta)\psi(\theta)d\theta\right| \le \frac{K}{j_1}, \tag{2.271}$$

where constant K does not depend on j_1 $(j_1 = 1, 2, \ldots)$.

Using (2.262), (2.267), (2.270), and (2.271), we get

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{K_1}{p} \to 0 \quad \text{if} \quad p \to \infty,$$

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2}C^*_{j_1}\right)\zeta_{j_1}^{(i_1)}\right)^2\right\} \le \frac{K_1}{p} \to 0 \quad \text{if} \quad p \to \infty,$$

where constant K_1 is independent of p.

The consideration of Case 4 is similar to the case of Legendre polynomials. Theorem 2.7 is proved.

In the next section, an analogue of Theorem 2.7 will be proved without the restrictions 1–4 (see the formulation of Theorem 2.7).

2.2.5 The Case $p_1 = p_2 = p_3 \to \infty$, Smooth Weight Functions, and without Additional Restrictive Conditions (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 2.8 [10]-[17], [22], [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the* *space* $L_2([t,T])$. *At the same time* $\psi_2(s)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(s)$, $\psi_3(s)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{2.272}$$

that converges in the mean-square sense is valid, where

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s) \int\limits_t^s \psi_2(s_1)\phi_{j_2}(s_1) \int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. Let us consider the case of Legendre polynomials. From (1.47) for the case $p_1 = p_2 = p_3 = p$ and standard relations between Itô and Stratonovich stochastic integrals we conclude that Theorem 2.8 will be proved if w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} = \frac{1}{2} \int\limits_t^T \psi_3(s) \int\limits_t^s \psi_2(s_1)\psi_1(s_1) ds_1 d\mathbf{f}_s^{(i_3)}, \tag{2.273}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} = \frac{1}{2} \int\limits_t^T \psi_3(s)\psi_2(s) \int\limits_t^s \psi_1(s_1) d\mathbf{f}_{s_1}^{(i_1)} ds, \tag{2.274}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_1 j_3 j_1} \zeta_{j_3}^{(i_2)} = 0. \tag{2.275}$$

Let us prove (2.273). Using Theorem 1.1 for $k=1$ (also see (1.45)), we can write w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi_2(s_1)\psi_1(s_1)ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\ \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{p} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi_2(s_1)\psi_1(s_1)ds_1 ds.$$

We have

$$E_p \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\sum_{j_3=0}^{p}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 ds - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_1(s_1)\psi_2(s_1)ds_1 ds\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s\left(\sum_{j_1=0}^{p}\psi_2(s_1)\phi_{j_1}(s_1)\times\right.\right.$$

$$\left.\left.\times\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 - \frac{1}{2}\psi_1(s_1)\psi_2(s_1)\right)ds_1 ds\right)^2. \qquad (2.276)$$

Let us substitute $t_1 = t_2 = s_1$ into (2.12). Then for all $s_1\in(t,T)$

$$\sum_{j_1=0}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 = \frac{1}{2}\psi_1(s_1)\psi_2(s_1). \qquad (2.277)$$

From (2.276) and (2.277) it follows that

$$E_p=\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\sum_{j_1=p+1}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds\right)^2. \tag{2.278}$$

Applying (2.278) and (2.24), we obtain

$$E_p<C_1\sum_{j_3=0}^{p}\left(\int\limits_t^T|\phi_{j_3}(s)|\frac{1}{p}\left(\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/2}}+\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/4}}\right)ds\right)^2\le$$

$$\le\frac{C_2}{p^2}\sum_{j_3=0}^{p}\left(\int\limits_t^T|\phi_{j_3}(s)|ds\right)^2\le\frac{C_2(T-t)}{p^2}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}^2(s)ds=\frac{C_3p}{p^2}\to 0$$

if $p\to\infty$, where constants C_1,C_2,C_3 do not depend on p. The equality (2.273) is proved.

Let us prove (2.274). Using the Itô formula, we have

$$\frac{1}{2}\int\limits_t^T\psi_3(s)\psi_2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi_3(s)\psi_2(s)dsd\mathbf{f}_{s_1}^{(i_1)}\quad \text{w. p. 1.}$$

Moreover, using Theorem 1.1 for $k=1$ (also see (1.45)), we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi_1(s)\int\limits_s^T\psi_3(s_1)\psi_2(s_1)ds_1d\mathbf{f}_s^{(i_1)}=\frac{1}{2}\ \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}C_{j_1}^*\zeta_{j_1}^{(i_1)},$$

where

$$C_{j_1}^*=\int\limits_t^T\psi_1(s)\phi_{j_1}(s)\int\limits_s^T\psi_3(s_1)\psi_2(s_1)ds_1ds. \tag{2.279}$$

We have

$$E_p'\stackrel{\text{def}}{=}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^*\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^*\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$
=\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^{*}\right)^2, \tag{2.280}
$$

$$
C_{j_3j_3j_1}=\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds=
$$

$$
=\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)dsds_1ds_2. \tag{2.281}
$$

From (2.279)–(2.281) we obtain

$$
E_p'=\sum_{j_1=0}^{p}\left(\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\left(\sum_{j_3=0}^{p}\psi_2(s_1)\phi_{j_3}(s_1)\times\right.\right.
$$

$$
\left.\left.\times\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)ds-\frac{1}{2}\psi_3(s_1)\psi_2(s_1)\right)ds_1ds_2\right)^2. \tag{2.282}
$$

We will prove the following equality for all $s_1\in(t,T)$

$$
\sum_{j_3=0}^{\infty}\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)ds=\frac{1}{2}\psi_2(s_1)\psi_3(s_1). \tag{2.283}
$$

Let us denote

$$
K_1^*(t_1,t_2)=K_1(t_1,t_2)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_2(t_1)\psi_3(t_1), \tag{2.284}
$$

where

$$
K_1(t_1,t_2)=\psi_2(t_1)\psi_3(t_2)\mathbf{1}_{\{t_1<t_2\}},\quad t_1,t_2\in[t,T].
$$

Let us expand the function $K_1^*(t_1,t_2)$ using the variable t_2, when t_1 is fixed, into the Fourier–Legendre series at the interval (t,T)

$$
K_1^*(t_1,t_2)=\sum_{j_3=0}^{\infty}\psi_2(t_1)\int\limits_{t_1}^T\psi_3(t_2)\phi_{j_3}(t_2)dt_2\cdot\phi_{j_3}(t_2)\quad(t_2\neq t,T). \tag{2.285}
$$

The equality (2.285) is fulfilled in each point of the interval (t, T) with respect to the variable t_2, when $t_1 \in [t, T]$ is fixed, due to piecewise smoothness of the function $K_1^*(t_1, t_2)$ with respect to the variable $t_2 \in [t, T]$ (t_1 is fixed).

Obtaining (2.285), we also used the fact that the right-hand side of (2.285) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K_1(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K_1(t_1, t_1 - 0) + K_1(t_1, t_1 + 0)\right) = \frac{1}{2}\psi_2(t_1)\psi_3(t_1) = K_1^*(t_1, t_1).$$

Let us substitute $t_1 = t_2$ into (2.285). Then we have (2.283). From (2.282) and (2.283) we get

$$E_p' = \sum_{j_1=0}^{p}\left(\int_t^T \psi_1(s_2)\phi_{j_1}(s_2)\int_{s_2}^T \sum_{j_3=p+1}^{\infty} \psi_2(s_1)\phi_{j_3}(s_1)\int_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds ds_1 ds_2\right)^2. \tag{2.286}$$

Analogously with (2.24) we obtain for the twice continuously differentiable function $\psi_3(s)$ the following estimate

$$\left|\sum_{j_3=p+1}^{\infty} \phi_{j_3}(s_1)\int_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds\right| <$$

$$< \frac{C}{p}\left(\frac{1}{(1-(z(s_1))^2)^{1/2}} + \frac{1}{(1-(z(s_1))^2)^{1/4}}\right), \tag{2.287}$$

where $s_1 \in (t, T)$, $z(s_1)$ is defined by (2.20), and constant C does not depend on p.

Further consideration is analogously to the proof of (2.273). The relation (2.274) is proved.

Let us prove (2.275). We have

$$E_p'' \stackrel{\sf def}{=} {\sf M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\} = \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1 j_3 j_1}\right)^2, \tag{2.288}$$

$$C_{j_1 j_3 j_1} = \int_t^T \psi_3(s)\phi_{j_1}(s)\int_t^s \psi_2(s_1)\phi_{j_3}(s_1)\int_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 ds =$$

$$= \int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2\int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)dsds_1. \qquad (2.289)$$

After substituting (2.289) into (2.288), we obtain

$$E_p'' = \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1)\sum_{j_1=0}^{p}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)dsds_1\right)^2. \qquad (2.290)$$

The generalized Parseval equality gives

$$\sum_{j_1=0}^{\infty}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)ds =$$

$$= \sum_{j_1=0}^{\infty}\int\limits_t^T \mathbf{1}_{\{\theta<s_1\}}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_t^T \mathbf{1}_{\{s>s_1\}}\psi_3(s)\phi_{j_1}(s)ds =$$

$$= \int\limits_t^T \mathbf{1}_{\{\tau<s_1\}}\psi_1(\tau)\mathbf{1}_{\{\tau>s_1\}}\psi_3(\tau)d\tau = 0. \qquad (2.291)$$

Using (2.290) and (2.291), we get

$$E_p'' = \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)dsds_1\right)^2. \qquad (2.292)$$

Let us write the following relation

$$\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds = \frac{\sqrt{T-t}\sqrt{2j_1+1}}{2}\int\limits_{-1}^{z(x)} P_{j_1}(y)\psi_1(u(y))dy =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\Biggl((P_{j_1+1}(z(x)) - P_{j_1-1}(z(x)))\psi_1(x) -$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(x)}((P_{j_1+1}(y) - P_{j_1-1}(y))\psi_1{}'(u(y))dy\Biggr), \qquad (2.293)$$

where $x \in (t,T)$, $j_1 \ge p+1$, $z(x)$ and $u(y)$ are defined by (2.20), ${\psi_1}'$ is a derivative of the function $\psi_1(s)$ with respect to the variable $u(y)$.

Note that in (2.293) we used the following well known property of the Legendre polynomials [121]

$$P_{j+1}(-1) = -P_j(-1), \quad j = 0, 1, 2, \ldots$$

and (2.21).

From (2.157) and (2.293) we obtain

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds\right| < \frac{C}{j_1}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + C_1\right), \tag{2.294}$$

where $j_1 \in \mathbf{N}$, $x \in (t,T)$, constants C, C_1 do not depend on j_1.

Similarly to (2.294) and due to

$$P_j(1) = 1, \quad j = 0, 1, 2, \ldots$$

we obtain an analogue of (2.294) for the integral, which is similar to the integral on the left-hand side of (2.294), but with integration limits x and T.

From the formula (2.294) and its analogue for the integral with integration limits x and T we obtain

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds \int\limits_x^T \psi_3(s)\phi_{j_1}(s)ds\right| < \frac{K}{j_1^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + K_1\right), \tag{2.295}$$

where $j_1 \in \mathbf{N}$, $x \in (t,T)$, and constants K, K_1 do not depend on j_1.

Let us estimate the right-hand side of (2.292) using (2.295)

$$E_p'' \le$$

$$\le L\sum_{j_3=0}^{p}\left(\int\limits_t^T |\phi_{j_3}(s_1)| \sum_{j_1=p+1}^{\infty}\left|\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta \int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)ds\right| ds_1\right)^2 <$$

$$< L_1\sum_{j_3=0}^{p}\left(\int\limits_t^T |\phi_{j_3}(s_1)| \sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(s_1))^2)^{1/2}} + K_1\right) ds_1\right)^2 <$$

$$< \frac{L_2}{p^2}\sum_{j_3=0}^{p}\left(\int\limits_t^T \frac{ds_1}{(1-(z(s_1))^2)^{3/4}} + K_1\int\limits_t^T \frac{ds_1}{(1-(z(s_1))^2)^{1/4}}\right)^2 =$$

$$
= \frac{L_2(T-t)^2}{4p^2}\sum_{j_3=0}^{p}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+K_1\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2 \le
$$

$$
\le \frac{L_3 p}{p^2} = \frac{L_3}{p} \to 0 \tag{2.296}
$$

if $p\to\infty$, where constants L, L_1, L_2, L_3 do not depend on p and we used (2.25), (2.158) in (2.296). The relation (2.275) is proved. Theorem 2.8 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. Analogously to (2.34) we obtain

$$
\left|\int\limits_{s_2}^{T}\sum_{j_3=p+1}^{\infty}\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^{T}\psi_3(s)\phi_{j_3}(s)dsds_1\right| \le \frac{K_1}{p}, \tag{2.297}
$$

where $s_2\in(t,T)$ and constant K_1 does not depend on p.

Using (2.34) for $T=s$ and (2.278), we obtain

$$
E_p \le K\sum_{j_3=0}^{p}\left(\int\limits_{t}^{T}\left|\int\limits_{t}^{s}\sum_{j_1=p+1}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_{t}^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1\right|ds\right)^2 \le
$$

$$
\le K\sum_{j_3=0}^{p}\left((T-t)\frac{K_1}{p}\right)^2 \le \frac{K_2}{p^2}\sum_{j_3=0}^{p}(T-t)^2 \le \frac{L}{p}\to 0 \tag{2.298}
$$

if $p\to\infty$, where constants K, K_1, K_2, L do not depend on p.

Analogously, using (2.297) and (2.286), we obtain that $E_p'\to 0$ if $p\to\infty$. It is not difficult to see that in our case we have (see (2.270), (2.271))

$$
\left|\int\limits_{t}^{x}\psi_1(s)\phi_{j_1}(s)ds\int\limits_{x}^{T}\psi_3(s)\phi_{j_1}(s)ds\right| < \frac{C_1}{j_1^2}, \tag{2.299}
$$

where $j_1\in\mathbf{N}$, constant C_1 does not depend on j_1.

Using (2.292) and (2.299), we obtain

$$
E_p'' \le
$$

$$
\leq L \sum_{j_3=0}^{p} \left(\int\limits_t^T |\phi_{j_3}(s_1)| \sum_{j_1=p+1}^{\infty} \left| \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \int\limits_{s_1}^{T} \psi_3(s)\phi_{j_1}(s)ds \right| ds_1 \right)^2 \leq
$$

$$
\leq L_1 \sum_{j_3=0}^{p} \left((T-t) \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \right)^2 \leq \frac{L_1}{p^2} \sum_{j_3=0}^{p} (T-t)^2 \leq
$$

$$
\leq \frac{L_2}{p} \to 0 \qquad (2.300)
$$

if $p \to \infty$, where constants L, L_1, L_2 do not depend on p.

Theorem 2.8 is proved for the trigonometric case. Theorem 2.8 is proved.

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4 Based on Theorem 1.1. The Case $p_1 = \ldots = p_4 \to \infty,\ \psi_1(\tau), \ldots, \psi_4(\tau) \equiv 1$ (Cases of Legendre Polynomials and Trigonometric Functions)

In this section, we will develop the approach to expansion of iterated Stratonovich stochatic integrals based on Theorem 1.1 for the stochastic integrals of multiplicity 4.

Theorem 2.9 [8]-[17], [22], [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1, i_2, i_3, i_4 = 0, 1, \ldots, m)
$$

the following expansion

$$
J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \qquad (2.301)
$$

that converges in the mean-square sense is valid, where

$$
C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(s_4) \int\limits_t^{s_4} \phi_{j_3}(s_3) \int\limits_t^{s_3} \phi_{j_2}(s_2) \int\limits_t^{s_2} \phi_{j_1}(s_1) ds_1 ds_2 ds_3 ds_4
$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. The relation (1.48) (in the case when $p_1 = \ldots = p_4 = p \to \infty$) implies that

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{T,t}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}A_1^{(i_3i_4)} + \mathbf{1}_{\{i_1=i_3\neq 0\}}A_2^{(i_2i_4)} + \mathbf{1}_{\{i_1=i_4\neq 0\}}A_3^{(i_2i_3)} + \mathbf{1}_{\{i_2=i_3\neq 0\}}A_4^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_2=i_4\neq 0\}}A_5^{(i_1i_3)} + \mathbf{1}_{\{i_3=i_4\neq 0\}}A_6^{(i_1i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}B_1-$$

$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}B_2 - \mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}B_3, \tag{2.302}$$

where $J[\psi^{(4)}]_{T,t}$ has the form (2.7) for $\psi_1(s), \ldots, \psi_4(s) \equiv 1$ and $i_1, \ldots, i_4 = 0, 1, \ldots, m$,

$$A_1^{(i_3i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$A_2^{(i_2i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_3}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$A_3^{(i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$A_4^{(i_1i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_3j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)},$$

$$A_5^{(i_1i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$A_6^{(i_1i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$B_1 = \lim_{p\to\infty}\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_1},\quad B_2 = \lim_{p\to\infty}\sum_{j_4,j_3=0}^{p} C_{j_3j_4j_3j_4},$$

$$B_3 = \lim_{p\to\infty}\sum_{j_4,j_3=0}^{p} C_{j_4j_3j_3j_4}.$$

Using the integration order replacement in Riemann integrals, Theorem 1.1 for $k=2$ (see (1.46)) and (2.10), Parseval's equality and the integration order replacement technique for Itô stochastic integrals (see Chapter 3) [1]-[17], [77], [123], [124] or Itô's formula, we obtain

$$A_1^{(i_3i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_3}(s_1)\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1ds\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_3}(s_1)\sum_{j_1=0}^{p}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1ds\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_3}(s_1)\left((s_1-t)-\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2\right) ds_1ds\times$$

$$\times\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_3}(s_1)(s_1-t)ds_1ds\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \Delta_1^{(i_3i_4)} =$$

$$= \frac{1}{2}\int\limits_t^T\int\limits_t^s(s_1-t)d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_3}(s_1)(s_1-t)ds_1ds - \Delta_1^{(i_3i_4)} =$$

$$= \frac{1}{2}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(s_1-t)ds_1-\Delta_1^{(i_3i_4)}\quad \text{w. p. 1},\quad (2.303)$$

where

$$\Delta_1^{(i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$a_{j_4 j_3}^{p} = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1 ds. \qquad (2.304)$$

Let us consider $A_2^{(i_2 i_4)}$

$$A_2^{(i_2 i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_3}(s_3)ds_3\int\limits_{s_2}^{s}\phi_{j_3}(s_1)ds_1 ds_2 ds\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\left(\int\limits_t^s \phi_{j_3}(s_3)ds_3\right)^2\int\limits_t^s \phi_{j_2}(s_2)ds_2 ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)\left(\int\limits_t^{s_2}\phi_{j_3}(s_3)ds_3\right)^2 ds_2 ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)\left(\int\limits_{s_2}^{s}\phi_{j_3}(s_1)ds_1\right)^2 ds_2 ds\right)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)(s-t)\int\limits_t^s \phi_{j_2}(s_2)ds_2 ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)(s_2-t)ds_2 ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)(s-t+t-s_2)ds_2 ds\right)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\Delta_2^{(i_2 i_4)}+\Delta_1^{(i_2 i_4)}+\Delta_3^{(i_2 i_4)} = -\Delta_2^{(i_2 i_4)}+\Delta_1^{(i_2 i_4)}+\Delta_3^{(i_2 i_4)} \quad \text{w. p. 1}, \qquad (2.305)$$

where

$$\Delta_2^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} b_{j_4 j_2}^{p} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} c_{j_4 j_2}^{p} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$b_{j_4 j_2}^{p} = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s) \sum_{j_3=p+1}^{\infty} \left(\int\limits_t^s \phi_{j_3}(s_1)ds_1\right)^2 \int\limits_t^s \phi_{j_2}(s_1)ds_1 ds, \tag{2.306}$$

$$c_{j_4 j_2}^{p} = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_2}(s_3) \sum_{j_3=p+1}^{\infty} \left(\int\limits_{s_3}^s \phi_{j_3}(s_1)ds_1\right)^2 ds_3 ds. \tag{2.307}$$

Let us consider $A_5^{(i_1 i_3)}$

$$A_5^{(i_1 i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_4}(s_2) \int\limits_{s_2}^T \phi_{j_3}(s_1) \int\limits_{s_1}^T \phi_{j_4}(s) ds ds_1 ds_2 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s_1) \int\limits_{s_1}^T \phi_{j_4}(s) ds \int\limits_{s_3}^{s_1} \phi_{j_4}(s_2) ds_2 ds_1 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \left(\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \left(\int\limits_{s_3}^T \phi_{j_4}(s) ds\right)^2 \int\limits_{s_3}^T \phi_{j_3}(s_1) ds_1 ds_3 - \right.$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s_1) \left(\int\limits_{s_3}^{s_1} \phi_{j_4}(s_2) ds_2\right)^2 ds_1 ds_3 -$$

$$\left. -\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s_1) \left(\int\limits_{s_1}^T \phi_{j_4}(s) ds\right)^2 ds_1 ds_3 \right) \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} \left(\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)(T-s_3) \int\limits_{s_3}^T \phi_{j_3}(s_1) ds_1 ds_3 - \right.$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_1)(s_1-s_3)ds_1ds_3-
$$

$$
\left.-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_1)(T-s_1)ds_1ds_3\right)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}=-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}\quad \text{w. p. 1}, \qquad (2.308)
$$

where

$$
\Delta_4^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} d_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_5^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} e_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_6^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} f_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
d_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^T \phi_{j_4}(s)ds\right)^2\int\limits_{s_3}^T \phi_{j_3}(s)dsds_3, \qquad (2.309)
$$

$$
e_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^s \phi_{j_4}(s_1)ds_1\right)^2 dsds_3, \qquad (2.310)
$$

$$
f_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_4}(s_1)ds_1\right)^2 ds_2ds_3=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_4}(s_1)ds_1\right)^2\int\limits_t^{s_2} \phi_{j_1}(s_3)ds_3ds_2. \qquad (2.311)
$$

Moreover,

$$
A_3^{(i_2i_3)}+A_5^{(i_2i_3)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\left(C_{j_4j_3j_2j_4}+C_{j_4j_3j_4j_2}\right)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 ds_1 ds \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 \int\limits_{s_1}^{T} \phi_{j_4}(s) ds ds_1 \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \left(\int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{T} \phi_{j_4}(s_3) ds_3 \int\limits_{s_1}^{T} \phi_{j_4}(s) ds ds_2 ds_1 - \right.
$$

$$
\left. - \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left(\int\limits_{s_1}^{T} \phi_{j_4}(s) ds \right)^2 ds_2 ds_1 \right) \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left((T-s_1) - \sum_{j_4=0}^{p} \left(\int\limits_{s_1}^{T} \phi_{j_4}(s_3) ds_3 \right)^2 \right) ds_2 ds_1 \times
$$

$$
\times \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = 2\Delta_6^{(i_2 i_3)} \quad \text{w. p. 1.} \tag{2.312}
$$

Then

$$
A_3^{(i_2 i_3)} = 2\Delta_6^{(i_2 i_3)} - A_5^{(i_2 i_3)} = \Delta_4^{(i_2 i_3)} - \Delta_5^{(i_2 i_3)} + \Delta_6^{(i_2 i_3)} \quad \text{w. p. 1.} \tag{2.313}
$$

Let us consider $A_4^{(i_1 i_4)}$

$$
A_4^{(i_1 i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(s_2) \int\limits_{s_2}^{s} \phi_{j_3}(s_1) ds_1 ds_2 ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3) \sum_{j_3=0}^{p} \left(\int\limits_{s_3}^{s} \phi_{j_3}(s_2) ds_2 \right)^2 ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3)(s-s_3) ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \Delta_3^{(i_1 i_4)} =
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^s (s-s_3)d\mathbf{w}_{s_3}^{(i_1)}d\mathbf{w}_{s}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\ne 0\}}\lim\limits_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)(s-s_3)ds_3ds-\Delta_3^{(i_1i_4)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\sum_{j_4=0}^{\infty}\int\limits_t^T(s-t)\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)ds_3ds-\right.
$$

$$
\left.-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s(s_3-t)\phi_{j_4}(s_3)ds_3ds\right)-\Delta_3^{(i_1i_4)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-\Delta_3^{(i_1i_4)}\quad \hbox{w. p. 1.} \qquad (2.314)
$$

Let us consider $A_6^{(i_1i_2)}$

$$
A_6^{(i_1i_2)}=
$$

$$
=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_3,j_2,j_1=0}^{p}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\int\limits_{s_2}^T\phi_{j_3}(s_1)\int\limits_{s_1}^T\phi_{j_3}(s)dsds_1ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\sum_{j_3=0}^{p}\left(\int\limits_{s_2}^T\phi_{j_3}(s)ds\right)^2ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)(T-s_2)ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=
$$

$$
=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_1}(s_3)ds_3ds_2\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T (T-s_2)\int\limits_t^{s_2} d\mathbf{w}_{s_3}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_2=0}^{\infty}\int\limits_t^T \phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_2}(s_3)ds_3ds_2-\Delta_6^{(i_1i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T (T-s_2)ds_2-\Delta_6^{(i_1i_2)}\quad \text{w. p. 1.} \tag{2.315}
$$

Let us consider B_1, B_2, B_3

$$
B_1=\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1ds=
$$

$$
=\lim_{p\to\infty}\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)(s_1-t)ds_1ds-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^{p}=
$$

$$
=\frac{1}{4}\int\limits_t^T (s_1-t)ds_1-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^{p}, \tag{2.316}
$$

$$
B_2=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T \phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\int\limits_t^{s_2}\phi_{j_4}(s_3)ds_3\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1ds_2ds=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\left(\int\limits_t^{s}\phi_{j_4}(s_3)ds_3\right)^2\int\limits_t^{s}\phi_{j_3}(s_2)ds_2ds-\right.
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\left(\int\limits_t^{s_2}\phi_{j_4}(s_3)ds_3\right)^2 ds_2ds-
$$

$$
\left.-\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\left(\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1\right)^2 ds_2ds\right)=
$$

$$
=\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(s-t)\int\limits_t^s\phi_{j_3}(s_2)ds_2ds-\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p-
$$

$$
-\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s(s_2-t)\phi_{j_3}(s_2)ds_2ds+\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p-
$$

$$
-\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_3}(s_2)(s-t+t-s_2)ds_2ds+\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p=
$$

$$
=\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p+\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p-\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p. \tag{2.317}
$$

Moreover,

$$
B_2+B_3=\lim\limits_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(C_{j_3j_4j_3j_4}+C_{j_3j_4j_4j_3}\right)=
$$

$$
=\lim\limits_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2ds_1ds=
$$

$$
=\lim\limits_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2\int\limits_{s_1}^T\phi_{j_3}(s)dsds_1=
$$

$$
=\lim\limits_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\int\limits_t^T\phi_{j_3}(s_2)ds_2\int\limits_{s_1}^T\phi_{j_3}(s)dsds_3ds_1-\right.
$$

$$
\left.-\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\left(\int\limits_{s_1}^T\phi_{j_3}(s)ds\right)^2ds_3ds_1\right)=
$$

$$
=\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1-
$$

$$
-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1+2\lim\limits_{p\to\infty}\sum_{j_4=0}^{p}f_{j_4j_4}^p=
$$

$$
=2\lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}. \qquad (2.318)
$$

Therefore,

$$
B_3=2\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3}+\lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}. \quad (2.319)
$$

After substituting the relations (2.303)–(2.319) into (2.302), we obtain

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=
$$

$$
=J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_{s}^{(i_1)}ds_1 d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2 ds_1+R=J^*[\psi^{(4)}]_{T,t}+R\quad \text{w. p. 1}, \quad (2.320)
$$

where

$$
R=-\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2i_4)}+\Delta_1^{(i_2i_4)}+\Delta_3^{(i_2i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_4^{(i_2i_3)}-\Delta_5^{(i_2i_3)}+\Delta_6^{(i_2i_3)}\right)-\mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1i_4)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}\right)-\mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1i_2)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}+\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(2\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}-
$$

$$
-\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3}+\lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}\Biggr)+
$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p}a^p_{j_3j_3}. \tag{2.321}$$

From (2.320) and (2.321) it follows that Theorem 2.9 will be proved if

$$\Delta_k^{(ij)}=0\quad \text{w. p. 1}, \tag{2.322}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}a^p_{j_3j_3}=\lim_{p\to\infty}\sum_{j_3=0}^{p}b^p_{j_3j_3}=\lim_{p\to\infty}\sum_{j_3=0}^{p}c^p_{j_3j_3}=\lim_{p\to\infty}\sum_{j_3=0}^{p}f^p_{j_3j_3}=0, \tag{2.323}$$

where $k=1,2,\ldots,6,\ \ i,j=0,1,\ldots,m.$

Consider the case of Legendre polynomials. Let us prove that $\Delta_1^{(i_3i_4)}=0$ w. p. 1. We have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a^p_{j_4j_3}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(2a^p_{j_3j_3}a^p_{j_3'j_3'}+\left(a^p_{j_3j_3'}\right)^2+2a^p_{j_3j_3'}a^p_{j_3'j_3}+\left(a^p_{j_3'j_3}\right)^2\right)+3\sum_{j_3'=0}^{p}\left(a^p_{j_3'j_3'}\right)^2=$$

$$=\left(\sum_{j_3=0}^{p}a^p_{j_3j_3}\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(a^p_{j_3j_3'}+a^p_{j_3'j_3}\right)^2+2\sum_{j_3'=0}^{p}\left(a^p_{j_3'j_3'}\right)^2\quad (i_3=i_4\neq 0), \tag{2.324}$$

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a^p_{j_4j_3}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\sum_{j_3,j_4=0}^{p}\left(a^p_{j_4j_3}\right)^2\quad (i_3\neq i_4,\ i_3\neq 0,\ i_4\neq 0), \tag{2.325}$$

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a^p_{j_4j_3}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\begin{cases}(T-t)\sum\limits_{j_4=0}^{p}\left(a^p_{j_4,0}\right)^2 & \text{if}\quad i_3=0,\ i_4\neq 0\\ (T-t)\sum\limits_{j_3=0}^{p}\left(a^p_{0,j_3}\right)^2 & \text{if}\quad i_4=0,\ i_3\neq 0\\ (T-t)^2\left(a^p_{00}\right)^2 & \text{if}\quad i_3=i_4=0\end{cases}\cdot \tag{2.326}$$

Let us consider the case $i_3 = i_4 \neq 0$

$$a^p_{j_4 j_3} = \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_4}(y) \int\limits_{-1}^{y} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} (2j_1+1) \left(\int\limits_{-1}^{y_1} P_{j_1}(y_2)dy_2 \right)^2 dy_1 dy =$$

$$= \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 \int\limits_{y_1}^{1} P_{j_4}(y)dydy_1 =$$

$$= \frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1) \left(P_{j_4-1}(y_1) - P_{j_4+1}(y_1)\right) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 \neq 0$ and

$$a^p_{j_4 j_3} = \frac{(T-t)^2\sqrt{2j_3+1}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1)(1-y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 = 0$.

From (2.157) and the estimate $|P_j(y)| \le 1$, $y \in [-1,1]$ we obtain

$$|P_j(y)| = \sqrt{|P_j(y)|}\cdot\sqrt{|P_j(y)|} \le \frac{C}{j^{1/4}(1-y^2)^{1/8}}, \quad y \in (-1,1), \quad j \in \mathbf{N}. \quad (2.327)$$

Using (2.157) and (2.327), we get

$$\left|a^p_{j_4 j_3}\right| \le \frac{C_0}{(j_4)^{3/4}} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{7/8}} \le \frac{C_1}{p\,(j_4)^{3/4}} \quad (j_3 \neq 0,\ j_4 \ge 2), \quad (2.328)$$

$$\left|a^p_{0 j_3}\right| + \left|a^p_{1 j_3}\right| \le C_0 \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} \le \frac{C_1}{p} \quad (j_3 \neq 0), \quad (2.329)$$

$$
\left|a^p_{j_4 0}\right|+\left|a^p_{00}\right|\le C_0\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/2}}\le\frac{C_1}{p}\quad (j_4\ge 1), \tag{2.330}
$$

where constants C_0, C_1 do not depend on p.

Taking into account (2.324), (2.328)–(2.330), we have

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a^p_{j_4j_3}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\right)^2\right\}=\left(a^p_{00}+\sum_{j_3=1}^{p}a^p_{j_3j_3}\right)^2+\sum_{j_3'=1}^{p}\left(a^p_{0j_3'}+a^p_{j_3'0}\right)^2+
$$

$$
+\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\left(a^p_{j_3j_3'}+a^p_{j_3'j_3}\right)^2+2\left(\sum_{j_3'=1}^{p}\left(a^p_{j_3'j_3'}\right)^2+(a_{00})^2\right)\le
$$

$$
\le K_0\left(\frac{1}{p}+\frac{1}{p}\sum_{j_3=1}^{p}\frac{1}{(j_3)^{3/4}}\right)^2+\frac{K_1}{p}+K_2\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\frac{1}{p^2}\left(\frac{1}{(j_3')^{3/4}}+\frac{1}{(j_3)^{3/4}}\right)^2\le
$$

$$
\le K_0\left(\frac{1}{p}+\frac{1}{p}\int\limits_{0}^{p}\frac{dx}{x^{3/4}}\right)^2+\frac{K_1}{p}+\frac{K_3}{p}\sum_{j_3=1}^{p}\frac{1}{(j_3)^{3/2}}\le
$$

$$
\le K_0\left(\frac{1}{p}+\frac{4}{p^{3/4}}\right)^2+\frac{K_1}{p}+\frac{K_3}{p}\left(1+\int\limits_{1}^{p}\frac{dx}{x^{3/2}}\right)\le
$$

$$
\le\frac{K_4}{p}+\frac{K_3}{p}\left(3-\frac{2}{\sqrt{p}}\right)\le\frac{K_5}{p}\ \to 0
$$

if $p\to\infty$ $(i_3=i_4\ne 0)$.

The same result for the cases (2.325), (2.326) also follows from the estimates (2.328)–(2.330). Therefore,

$$
\Delta_1^{(i_3i_4)}=0\quad \hbox{w. p. 1.} \tag{2.331}
$$

It is not difficult to see that the formulas

$$
\Delta_2^{(i_2i_4)}=0,\quad \Delta_4^{(i_1i_3)}=0,\quad \Delta_6^{(i_1i_3)}=0\quad \hbox{w. p. 1} \tag{2.332}
$$

can be proved similarly with the proof of (2.331).

Moreover, from the estimates (2.328)–(2.330) we obtain

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}a^p_{j_3j_3}=0. \tag{2.333}
$$

The relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3j_3}^{p}=0 \quad \text{and} \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3j_3}^{p}=0 \tag{2.334}$$

can also be proved analogously with (2.333).

Let us consider $\Delta_3^{(i_2i_4)}$

$$\Delta_3^{(i_2i_4)}=\Delta_4^{(i_2i_4)}+\Delta_6^{(i_2i_4)}-\Delta_7^{(i_2i_4)}=-\Delta_7^{(i_2i_4)} \quad \text{w. p. 1}, \tag{2.335}$$

where

$$\Delta_7^{(i_2i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_2,j_4=0}^{p} g_{j_4j_2}^{p}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$g_{j_4j_2}^{p}=\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_1}(s_2)ds_2\int\limits_s^T\phi_{j_1}(s_2)ds_2\right)ds_1ds=$$

$$=\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2\int\limits_t^s\phi_{j_2}(s_1)\int\limits_{s_1}^T\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.336}$$

The last step in (2.336) follows from the estimate

$$\left|g_{j_4j_2}^{p}\right|\le K\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{1}{(1-y^2)^{1/2}}\int\limits_{-1}^{y}\frac{1}{(1-x^2)^{1/2}}dxdy\ \le\frac{K_1}{p}.$$

Note that

$$g_{j_4j_4}^{p}=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\right)^2, \tag{2.337}$$

$$g_{j_4j_2}^{p}+g_{j_2j_4}^{p}=\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds, \tag{2.338}$$

and

$$g_{j_4j_2}^{p}=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_2+1)}}{16}\times$$

$$\times \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \int\limits_{-1}^{1} P_{j_4}(y_1) \left(P_{j_1-1}(y_1) - P_{j_1+1}(y_1)\right) \times$$

$$\times \int\limits_{-1}^{y_1} P_{j_2}(y) \left(P_{j_1-1}(y) - P_{j_1+1}(y)\right) dy dy_1, \quad j_4, j_2 \le p.$$

Due to orthogonality of the Legendre polynomials we obtain

$$g_{j_4 j_2}^{p} + g_{j_2 j_4}^{p} = \frac{(T-t)^2 \sqrt{(2j_4+1)(2j_2+1)}}{16} \times$$

$$\times \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \int\limits_{-1}^{1} P_{j_4}(y_1) \left(P_{j_1-1}(y_1) - P_{j_1+1}(y_1)\right) dy_1 \times$$

$$\times \int\limits_{-1}^{1} P_{j_2}(y) \left(P_{j_1-1}(y) - P_{j_1+1}(y)\right) dy =$$

$$= \frac{(T-t)^2 (2p+1)}{16} \frac{1}{2p+3} \left(\int\limits_{-1}^{1} P_p^2(y_1) dy_1 \right)^2 \cdot \begin{cases} 1 & \text{if } j_2 = j_4 = p \\ \\ 0 & \text{otherwise} \end{cases} =$$

$$= \frac{(T-t)^2}{4(2p+3)(2p+1)} \cdot \begin{cases} 1 & \text{if } j_2 = j_4 = p \\ \\ 0 & \text{otherwise} \end{cases}, \tag{2.339}$$

$$g_{j_4 j_4}^{p} = \frac{(T-t)^2 (2j_4+1)}{16} \times$$

$$\times \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \cdot \frac{1}{2} \left(\int\limits_{-1}^{1} P_{j_4}(y_1) \left(P_{j_1-1}(y_1) - P_{j_1+1}(y_1)\right) dy_1 \right)^2 =$$

$$= \frac{(T-t)^2 (2p+1)}{32} \frac{1}{2p+3} \left(\int\limits_{-1}^{1} P_p^2(y_1) dy_1 \right)^2 \cdot \begin{cases} 1 & \text{if } j_4 = p \\ \\ 0 & \text{otherwise} \end{cases} =$$

$$= \frac{(T-t)^2}{8(2p+3)(2p+1)} \cdot \begin{cases} 1 & \text{if } j_4 = p \\ \\ 0 & \text{otherwise} \end{cases}. \tag{2.340}$$

From (2.324), (2.339), and (2.340) it follows that

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p} g_{j_4j_2}^{p}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\left(\sum_{j_3=0}^{p} g_{j_3j_3}^{p}\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(g_{j_3j_3'}^{p}+g_{j_3'j_3}^{p}\right)^2+2\sum_{j_3'=0}^{p}\left(g_{j_3'j_3'}^{p}\right)^2=
$$

$$
=\left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2+\ 0\ +2\left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2\to 0
$$

if $p\to\infty$ $(i_2=i_4\ne 0)$.

Let us consider the case $i_2\ne i_4,\ i_2\ne 0,\ i_4\ne 0$ (see (2.325)). It is not difficult to see that

$$
g_{j_4j_2}^{p}=\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_1)F_p(s,s_1)ds_1ds=\int\limits_{[t,T]^2}K_p(s,s_1)\phi_{j_4}(s)\phi_{j_2}(s_1)ds_1ds
$$

is a coefficient of the double Fourier–Legendre series of the function

$$
K_p(s,s_1)=\mathbf{1}_{\{s_1<s\}}F_p(s,s_1), \tag{2.341}
$$

where

$$
\sum_{j_1=p+1}^{\infty}\int\limits_{s_1}^{T}\phi_{j_1}(s_2)ds_2\int\limits_{s}^{T}\phi_{j_1}(s_2)ds_2\stackrel{\mathsf{def}}{=}F_p(s,s_1).
$$

The Parseval equality in this case looks as follows

$$
\lim_{p_1\to\infty}\sum_{j_4,j_2=0}^{p_1}\left(g_{j_4j_2}^{p}\right)^2=\int\limits_{[t,T]^2}\left(K_p(s,s_1)\right)^2ds_1ds=\int\limits_t^T\int\limits_t^s\left(F_p(s,s_1)\right)^2ds_1ds. \tag{2.342}
$$

From (2.157) we obtain

$$
\left|\int\limits_{s_1}^{T}\phi_{j_1}(\theta)d\theta\right|=\frac{1}{2}\sqrt{2j_1+1}\sqrt{T-t}\left|\int\limits_{z(s_1)}^{1}P_{j_1}(y)dy\right|=
$$

$$
=\frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\left|P_{j_1-1}(z(s_1))-P_{j_1+1}(z(s_1))\right|\le\frac{K}{j_1}\frac{1}{\left(1-z^2(s_1)\right)^{1/4}}, \qquad (2.343)
$$

where $z(s_1)$ is defined by (2.20), $s_1\in(t,T)$.

From (2.343) we have

$$
\left(F_p(s,s_1)\right)^2\le\frac{C^2}{p^2}\frac{1}{\left(1-z^2(s)\right)^{1/2}}\frac{1}{\left(1-z^2(s_1)\right)^{1/2}},\quad s,s_1\in(t,T). \qquad (2.344)
$$

From (2.344) it follows that $|F_p(s,s_1)|\le M_{\varepsilon}/p$ in the domain

$$
D_{\varepsilon}=\{(s,s_1):\ s\in[t+\varepsilon,T-\varepsilon],\ s_1\in[t+\varepsilon,s]\}\quad \text{for some small}\ \ \varepsilon>0,
$$

where constant M_{ε} does not depend on s,s_1. Then we have the uniform convergence

$$
\sum_{j_1=0}^{p}\int\limits_s^T\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T\phi_{j_1}(\theta)d\theta\to\sum_{j_1=0}^{\infty}\int\limits_s^T\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T\phi_{j_1}(\theta)d\theta \qquad (2.345)
$$

at the set D_{ε} if $p\to\infty$.

Because of continuity of the function on the left-hand side of (2.345) we obtain continuity of the limit function on the right-hand side of (2.345) at the set D_{ε}.

Using this fact and (2.344), we obtain

$$
\int\limits_t^T\int\limits_t^s\left(F_p(s,s_1)\right)^2ds_1ds=\lim_{\varepsilon\to+0}\int\limits_{t+\varepsilon}^{T-\varepsilon}\int\limits_{t+\varepsilon}^{s}\left(F_p(s,s_1)\right)^2ds_1ds\le
$$

$$
\le\frac{C^2}{p^2}\lim_{\varepsilon\to+0}\int\limits_{t+\varepsilon}^{T-\varepsilon}\int\limits_{t+\varepsilon}^{s}\frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}}\frac{ds}{\left(1-z^2(s)\right)^{1/2}}=
$$

$$
=\frac{C^2}{p^2}\int\limits_t^T\int\limits_t^s\frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}}\frac{ds}{\left(1-z^2(s)\right)^{1/2}}=
$$

$$
=\frac{K}{p^2}\int\limits_{-1}^{1}\int\limits_{-1}^{y}\frac{dy_1}{\left(1-y_1^2\right)^{1/2}}\frac{dy}{\left(1-y^2\right)^{1/2}}\le\frac{K_1}{p^2}, \qquad (2.346)
$$

where constant K_1 does not depend on p.

From (2.346) and (2.342) we get

$$0 \le \sum_{j_2,j_4=0}^{p} \left(g_{j_4j_2}^p\right)^2 \le \lim_{p_1\to\infty} \sum_{j_2,j_4=0}^{p_1} \left(g_{j_4j_2}^p\right)^2 = \sum_{j_2,j_4=0}^{\infty} \left(g_{j_4j_2}^p\right)^2 \le \frac{K_1}{p^2} \to 0 \quad (2.347)$$

if $p\to\infty$. The case $i_2\ne i_4,\ i_2\ne 0,\ i_4\ne 0$ is proved.

The same result for the cases
1) $i_2=0,\ i_4\ne 0$,
2) $i_4=0,\ i_2\ne 0$,
3) $i_2=0,\ i_4=0$
can also be obtained. Then $\Delta_7^{(i_2i_4)}=0$ and $\Delta_3^{(i_2i_4)}=0$ w. p. 1.

Let us consider $\Delta_5^{(i_1i_3)}$

$$\Delta_5^{(i_1i_3)}=\Delta_4^{(i_1i_3)}+\Delta_6^{(i_1i_3)}-\Delta_8^{(i_1i_3)} \quad \text{w. p. 1},$$

where

$$\Delta_8^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} h_{j_3j_1}^p \zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$h_{j_3j_1}^p=\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s)F_p(s_3,s)dsds_3.$$

Analogously, we obtain that $\Delta_8^{(i_1i_3)}=0$ w. p. 1. Here we consider the function

$$K_p(s_3,s)=\mathbf{1}_{\{s_3<s\}}F_p(s_3,s)$$

and the relation

$$h_{j_3j_1}^p=\int\limits_{[t,T]^2} K_p(s_3,s)\phi_{j_1}(s_3)\phi_{j_3}(s)dsds_3$$

for the case $i_1\ne i_3,\ i_1\ne 0,\ i_3\ne 0$.

For the case $i_1=i_3\ne 0$ we use (see (2.337), (2.338))

$$h_{j_1j_1}^p=\sum_{j_4=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^T \phi_{j_1}(s)\int\limits_s^T \phi_{j_4}(s_1)ds_1ds\right)^2,$$

$$
h^p_{j_3j_1}+h^p_{j_1j_3}=\sum_{j_4=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(s)\int\limits_s^T\phi_{j_4}(s_2)ds_2ds\int\limits_t^T\phi_{j_3}(s)\int\limits_s^T\phi_{j_4}(s_2)ds_2ds.
$$

Let us prove that

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}c^p_{j_3j_3}=0. \tag{2.348}
$$

We have

$$
c^p_{j_3j_3}=f^p_{j_3j_3}+d^p_{j_3j_3}-g^p_{j_3j_3}. \tag{2.349}
$$

Moreover,

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}f^p_{j_3j_3}=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}d^p_{j_3j_3}=0, \tag{2.350}
$$

where the first equality in (2.350) has been proved earlier. Analogously, we can prove the second equality in (2.350).

From (2.340) we obtain

$$
0\le\lim_{p\to\infty}\sum_{j_3=0}^{p}g^p_{j_3j_3}\le\lim_{p\to\infty}\frac{(T-t)^2}{8(2p+3)(2p+1)}=0.
$$

So, (2.348) is proved. The relations (2.322), (2.323) are proved for the polynomial case. Theorem 2.9 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. According to (2.304), we have

$$
a^p_{j_4j_3}=\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2\int\limits_{s_1}^T\phi_{j_4}(s)dsds_1. \tag{2.351}
$$

Moreover (see (2.270), (2.271)),

$$
\left|\int\limits_t^{s_1}\phi_j(s_2)ds_2\right|\le\frac{K}{j},\quad \left|\int\limits_{s_1}^{T}\phi_j(s_2)ds_2\right|\le\frac{K}{j}, \tag{2.352}
$$

where constant K does not depend on j ($j=1,2,\ldots$).

Note that

$$
\int\limits_{s_1}^T\phi_0(s)ds=\frac{T-s_1}{\sqrt{T-t}}.
$$

Using (2.351) and (2.352), we obtain

$$\left|a^p_{j_4 j_3}\right| \le \frac{C_1}{j_4}\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2} \le \frac{C_1}{pj_4} \quad (j_4\ne 0), \qquad \left|a^p_{0 j_3}\right| \le \frac{C_1}{p}, \tag{2.353}$$

where constant C_1 does not depend on p.

Taking into account (2.324)–(2.326) and (2.353), we obtain that $\Delta_1^{(i_3 i_4)}=0$ w. p. 1. Analogously, we get $\Delta_2^{(i_2 i_4)}=0$, $\Delta_4^{(i_1 i_3)}=0$, $\Delta_6^{(i_1 i_3)}=0$ w. p. 1 and

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3 j_3}=0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3 j_3}=0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3 j_3}=0.$$

Let us consider $\Delta_3^{(i_2 i_4)}$ for the case $i_2=i_4\ne 0$. For the values $g^{2m}_{j_4 j_2}+g^{2m}_{j_2 j_4}$ and $g^{2m-1}_{j_4 j_2}+g^{2m-1}_{j_2 j_4}$ $(m\in\mathbf{N})$ we have (see (2.338))

$$g^{2m}_{j_4 j_2}+g^{2m}_{j_2 j_4}=$$

$$=\sum_{j_1=2m+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds=$$

$$=\sum_{r=m+1}^{\infty}\left(\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r-1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2r-1}(s_2)ds_2ds+\right.$$

$$\left.+\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2r}(s_2)ds_2ds\right), \tag{2.354}$$

$$g^{2m-1}_{j_4 j_2}+g^{2m-1}_{j_2 j_4}=$$

$$=\sum_{j_1=2m}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds=$$

$$=g^{2m}_{j_4 j_2}+g^{2m}_{j_2 j_4}+$$

$$+\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2m}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2m}(s_2)ds_2ds, \tag{2.355}$$

where

$$\int_t^T \phi_{j_4}(s)\int_s^T \phi_{2r-1}(s_2)ds_2 ds = \sqrt{\frac{2}{T-t}}\int_t^T \phi_{j_4}(s)\int_s^T \sin\frac{2\pi r(s_2-t)}{T-t}ds_2 ds =$$

$$= \frac{\sqrt{2}\sqrt{T-t}}{2\pi r}\int_t^T \phi_{j_4}(s)\left(\cos\frac{2\pi r(s-t)}{T-t}-1\right)ds,$$

$$\int_t^T \phi_{j_4}(s)\int_s^T \phi_{2r}(s_2)ds_2 ds = \sqrt{\frac{2}{T-t}}\int_t^T \phi_{j_4}(s)\int_s^T \cos\frac{2\pi r(s_2-t)}{T-t}ds_2 ds =$$

$$= \frac{\sqrt{2}\sqrt{T-t}}{2\pi r}\int_t^T \phi_{j_4}(s)\left(-\sin\frac{2\pi r(s-t)}{T-t}\right)ds,$$

where $2r-1,\ 2r\ge p+1$, and $j_2, j_4 = 0, 1, \ldots, p$.

Due to orthogonality of the trigonometric functions we have

$$\int_t^T \phi_{j_4}(s)\int_s^T \phi_{2r-1}(s_2)ds_2 ds = \frac{\sqrt{2}(T-t)}{2\pi r}\cdot\begin{cases}-1 & \text{if } j_4=0\\ \\ 0 & \text{otherwise}\end{cases}, \tag{2.356}$$

$$\int_t^T \phi_{j_4}(s)\int_s^T \phi_{2r}(s_2)ds_2 ds = 0, \tag{2.357}$$

where $2r-1,\ 2r\ge p+1$, and $j_4 = 0, 1, \ldots, p$.

From (2.354), (2.356), and (2.357) we obtain

$$g_{j_4 j_2}^{2m} + g_{j_2 j_4}^{2m} =$$

$$= \sum_{j_1=m+1}^{\infty}\frac{(T-t)^2}{2\pi^2 j_1^2}\cdot\begin{cases}1 & \text{if } j_2=j_4=0\\ \\ 0 & \text{otherwise}\end{cases},$$

$$g_{j_4j_4}^{2m}=\frac{1}{2}\left(g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}\right)\Bigg|_{j_2=j_4}=$$

$$=\sum_{j_1=m+1}^{\infty}\frac{(T-t)^2}{4\pi^2 j_1^2}\cdot\begin{cases}1 & \text{if}\ \ j_4=0\\ & \\ 0 & \text{otherwise}\end{cases}.$$

Therefore (see (2.25)),

$$\begin{cases}\left|g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}\right|\le K_1/(2m) & \text{if}\ \ j_2=j_4=0\\ & \\ g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}=0 & \text{otherwise}\end{cases}, \tag{2.358}$$

$$\begin{cases}\left|g_{j_4j_4}^{2m}\right|\le K_1/(2m) & \text{if}\ \ j_4=0\\ & \\ g_{j_4j_4}^{2m}=0 & \text{otherwise}\end{cases}, \tag{2.359}$$

where constant K_1 does not depend on $p=2m$.

For $p=2m-1$ from (2.355) and (2.357) we have

$$g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}=$$

$$=\sum_{j_1=m+1}^{\infty}\frac{(T-t)^2}{2\pi^2 j_1^2}\cdot\begin{cases}1\ \text{ or }\ 0 & \text{if}\ j_2=j_4=0\\ & \\ 0 & \text{otherwise}\end{cases}. \tag{2.360}$$

The relation (2.360) implies that

$$g_{j_4j_4}^{2m-1}=\frac{1}{2}\left(g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}\right)\Bigg|_{j_2=j_4}=$$

$$=\sum_{j_1=m+1}^{\infty}\frac{(T-t)^2}{4\pi^2 j_1^2}\cdot\begin{cases}1\ \text{ or }\ 0 & \text{if}\ j_4=0\\ & \\ 0 & \text{otherwise}\end{cases}. \tag{2.361}$$

Using (2.360) and (2.361), we obtain

$$\begin{cases} \left|g_{j_4j_2}^{2m-1} + g_{j_2j_4}^{2m-1}\right| \le \ K_2/(2m-1) & \text{if} \ \ j_2 = j_4 = 0 \\ \\ g_{j_4j_2}^{2m-1} + g_{j_2j_4}^{2m-1} = 0 & \text{otherwise} \end{cases}, \tag{2.362}$$

$$\begin{cases} \left|g_{j_4j_4}^{2m-1}\right| \le \ K_2/(2m-1) & \text{if} \ \ j_4 = 0 \\ \\ g_{j_4j_4}^{2m-1} = 0 & \text{otherwise} \end{cases}, \tag{2.363}$$

where constant K_2 does not depend on $p = 2m-1$.

The relations (2.358), (2.359), (2.362), and (2.363) imply the following formulas

$$\begin{cases} \left|g_{j_4j_2}^{p} + g_{j_2j_4}^{p}\right| \le \ K_3/p & \text{if} \ \ j_2 = j_4 = 0 \\ \\ g_{j_4j_2}^{p} + g_{j_2j_4}^{p} = 0 & \text{otherwise} \end{cases}, \tag{2.364}$$

$$\begin{cases} \left|g_{j_4j_4}^{p}\right| \le \ K_3/p & \text{if} \ \ j_4 = 0 \\ \\ g_{j_4j_4}^{p} = 0 & \text{otherwise} \end{cases}, \tag{2.365}$$

where constant K_3 does not depend on p ($p \in \mathbf{N}$). Moreover, $g_{j_4j_4}^p \ge 0$ (see (2.337)).

From (2.324), (2.364), and (2.365) it follows that $\Delta_7^{(i_2i_4)} = 0$ and $\Delta_3^{(i_2i_4)} = 0$ w. p. 1 for $i_2 = i_4 \ne 0$. Analogously to the polynomial case, we obtain $\Delta_7^{(i_2i_4)} = 0$ and $\Delta_3^{(i_2i_4)} = 0$ w. p. 1 for $i_2 \ne i_4$, $i_2 \ne 0$, $i_4 \ne 0$. The similar arguments prove that $\Delta_5^{(i_1i_3)} = 0$ w. p. 1.

Taking into account (2.349), (2.364), (2.365) and the relations

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} f_{j_3j_3}^p = \lim_{p\to\infty} \sum_{j_3=0}^{p} d_{j_3j_3}^p = 0,$$

which follow from the estimates

$$|f_{jj}^p| \le \frac{C_1}{pj}, \quad |d_{jj}^p| \le \frac{C_1}{pj} \quad (j \ne 0), \quad |f_{00}^p| \le \frac{C_1}{p}, \quad |d_{00}^p| \le \frac{C_1}{p}, \tag{2.366}$$

we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p} = -\lim_{p\to\infty}\sum_{j_3=0}^{p} g_{j_3j_3}^{p},$$

$$0\le \lim_{p\to\infty}\sum_{j_3=0}^{p} g_{j_3j_3}^{p} \le \lim_{p\to\infty}\frac{K_3}{p}=0.$$

Note that the estimates (2.366) can be obtained by analogy with (2.353); constant C_1 in (2.366) has the same meaning as constant C_1 in (2.353).

Finally, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p} = 0.$$

The relations (2.322), (2.323) are proved for the trigonometric case. Theorem 2.9 is proved for the trigonometric case. Theorem 2.9 is proved.

Remark 2.2. *It should be noted that the proof of Theorem* 2.9 *can be somewhat simplified. More precisely, instead of* (2.324)–(2.326)*, we can use only one and rather simple estimate.*

We have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)+\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_4=0}^{p} a_{j_4j_4}^{p}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\sum_{j_4=0}^{p} a_{j_4j_4}^{p}\right)^2. \qquad (2.367)$$

The expression

$$\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)$$

can be interpreted as the multiple Wiener stochastic integral (1.258) (*also see* (1.23)) *of multiplicity* 2 *with nonrandom integrand function*

$$\sum_{j_3,j_4=0}^{p} a^p_{j_4 j_3}\phi_{j_3}(t_3)\phi_{j_4}(t_4).$$

From (1.25) *we obtain*

$$\mathsf{M}\left\{\left(J'[\Phi]^{(k)}_{T,t}\right)^2\right\} \le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k, \tag{2.368}$$

where $J'[\Phi]^{(k)}_{T,t}$ *is defined by* (1.23) *and* C_k *is a constant.*

Then

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a^p_{j_4 j_3}\left(\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} \le$$

$$\le C_2 \int\limits_{[t,T]^2} \left(\sum_{j_3,j_4=0}^{p} a^p_{j_4 j_3}\phi_{j_3}(t_3)\phi_{j_4}(t_4)\right)^2 dt_3 dt_4 = C_2 \sum_{j_3,j_4=0}^{p} \left(a^p_{j_4 j_3}\right)^2. \tag{2.369}$$

From (2.367) *and* (2.369) *we get*

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a^p_{j_4 j_3}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\right)^2\right\} \le C_2 \sum_{j_3,j_4=0}^{p} \left(a^p_{j_4 j_3}\right)^2 + \mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\sum_{j_4=0}^{p} a^p_{j_4 j_4}\right)^2. \tag{2.370}$$

Obviously, the estimate (2.370) *can be used in the proof of Theorem* 2.9 *instead of* (2.324)–(2.326).

The estimate (2.370) *can be refined. Using* (1.88), *we obtain*

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a^p_{j_4 j_3}\left(\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} =$$

$$
= \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^p\right)^2 + \mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^p a_{j_3 j_4}^p \le
$$

$$
\le \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^p\right)^2 + \mathbf{1}_{\{i_3=i_4\ne 0\}} \frac{1}{2} \sum_{j_3,j_4=0}^{p} \left(\left(a_{j_4 j_3}^p\right)^2 + \left(a_{j_3 j_4}^p\right)^2\right) =
$$

$$
= \left(1 + \mathbf{1}_{\{i_3=i_4\ne 0\}}\right) \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^p\right)^2. \tag{2.371}
$$

Combining (2.367) *and* (2.371), *we have*

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \left(1 + \mathbf{1}_{\{i_3=i_4\ne 0\}}\right) \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^p\right)^2 +
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}} \left(\sum_{j_4=0}^{p} a_{j_4 j_4}^p\right)^2. \tag{2.372}
$$

2.4 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$ Based on Generalized Iterated Fourier Series Converging Pointwise

This section is devoted to the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k $(k \in \mathbf{N})$ based on generalized iterated Fourier series. The case of trigonometric Fourier series are considered in detail. The obtained expansion provides a possibility to represent the iterated Stratonovich stochastic integral in the form of iterated series of products of standard Gaussian random variables. Convergence in the mean of degree $q = 2n$ $(n \in \mathbf{N})$ of the expansion is proved. The case of iterated Fourier–Legendre series for $k = 2$ and $q = 2$ is also considered.

The idea of representing of iterated Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using generalized iterated Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of author's publications [76] (1997), [77] (1998) (also see [5]-[17], [34]). The results of this section convincingly testify that there is a doubtless relation between the multiplier factor 1/2, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Itô stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its trigonometric Fourier series and Fourier–Legendre series converge to the value $(f(x+0)+f(x-0))/2$.

2.4.1 Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$)

Consider the following iterated Stratonovich and Itô stochastic integrals

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.373}$$

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^{T} \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.374}$$

where $\psi_l(\tau)$ $(l=1,\ldots,k)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m.$

Let us denote as $\{\phi_j(x)\}_{j=0}^\infty$ the complete orthonormal systems of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

In this section, we will pay attention on the well known facts about Fourier series with respect to these two systems of functions [115] (also see Sect. 2.1.1).

Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k \\ & \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^k \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l<t_{l+1}\}} \tag{2.375}$$

for $t_1,\ldots,t_k \in [t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, where $\mathbf{1}_A$ denotes the indicator of the set A.

Let us formulate the following theorem.

Theorem 2.10 [76] (1997), [77] (1998) (also see [5]-[17], [34]). *Suppose that every function $\psi_l(\tau)$ $(l=1,\ldots,k)$ is twice continuously differentiable at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, the iterated Stratonovich stochastic* *integral $J^*[\psi^{(k)}]_{T,t}$ defined by (2.373) is expanded into the converging in the mean of degree $2n$ ($n \in \mathbf{N}$) iterated series*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{2.376}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$) and

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \tag{2.377}$$

is the Fourier coefficient.

Note that (2.376) means the following

$$\lim_{p_1 \to \infty} \overline{\lim_{p_2 \to \infty}} \ldots \overline{\lim_{p_k \to \infty}} \mathsf{M} \left\{ \left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right)^{2n} \right\} = 0, \tag{2.378}$$

where $\overline{\lim}$ means lim sup.

Proof. The proof of Theorem 2.10 is based on Lemmas 1.1, 1.3 (see Sect. 1.1.3) and Theorems 2.11–2.13 (see below).

Define the function $K^*(t_1, \ldots, t_k)$ on the hypercube $[t, T]^k$ as follows

$$K^*(t_1, \ldots, t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \left(\mathbf{1}_{\{t_l < t_{l+1}\}} + \frac{1}{2} \mathbf{1}_{\{t_l = t_{l+1}\}} \right) =$$

$$= \prod_{l=1}^{k} \psi_l(t_l) \left(\prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} + \sum_{r=1}^{k-1} \frac{1}{2^r} \sum_{\substack{s_r, \ldots, s_1 = 1 \\ s_r > \ldots > s_1}}^{k-1} \prod_{l=1}^{r} \mathbf{1}_{\{t_{s_l} = t_{s_l+1}\}} \prod_{\substack{l=1 \\ l \neq s_1, \ldots, s_r}}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} \right) \tag{2.379}$$

for $t_1, \ldots, t_k \in [t, T]$ ($k \geq 2$) and $K^*(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$, where $\mathbf{1}_A$ is the indicator of the set A.

Theorem 2.11 [76] (1997). *Suppose that every function $\psi_l(\tau)$ $(l=1,\ldots,k)$ is continuously differentiable at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, the function $K^*(t_1,\ldots,t_k)$ is represented in any internal point of the hypercube $[t,T]^k$ by the generalized iterated Fourier series*

$$K^*(t_1,\ldots,t_k)=\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{2.380}$$

where $(t_1,\ldots,t_k)\in(t,T)^k$ and $C_{j_k\ldots j_1}$ is defined by (2.377). At that, the iterated series (2.380) converges at the boundary of the hypercube $[t,T]^k$ (not necessarily to the function $K^*(t_1,\ldots,t_k)$).

Proof. We will perform the proof using induction. Consider the case $k=2$. Let us expand the function $K^*(t_1,t_2)$ using the variable t_1, when t_2 is fixed, into the generalized Fourier series with respect to the system $\{\phi_j(x)\}_{j=0}^{\infty}$ at the interval (t,T)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2)\phi_{j_1}(t_1)\quad (t_1\neq t,T), \tag{2.381}$$

where

$$C_{j_1}(t_2)=\int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The equality (2.381) is satisfied pointwise at each point of the interval (t,T) with respect to the variable t_1, when $t_2\in[t,T]$ is fixed, due to a piecewise smoothness of the function $K^*(t_1,t_2)$ with respect to the variable $t_1\in[t,T]$ (t_2 is fixed).

Note also that due to the well known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (2.381) converges when $t_1=t,T$ (not necessarily to the function $K^*(t_1,t_2)$).

Obtaining (2.381), we also used the fact that the right-hand side of (2.381) converges when $t_1=t_2$ (point of a finite discontinuity of the function $K(t_1,t_2)$ defined by (2.375)) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

The function $C_{j_1}(t_2)$ is continuously differentiable at the interval $[t,T]$. Let us expand it into the generalized Fourier series at the interval (t,T)

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\ne t,T), \tag{2.382}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

and the equality (2.382) is satisfied pointwise at any point of the interval (t,T). Moreover, the right-hand side of (2.382) converges when $t_2=t,T$ (not necessarily to $C_{j_1}(t_2)$).

Let us substitute (2.382) into (2.381)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2. \tag{2.383}$$

Note that the series on the right-hand side of (2.383) converges at the boundary of the square $[t,T]^2$ (not necessarily to $K^*(t_1,t_2)$). Theorem 2.11 is proved for the case $k=2$.

Note that proving Theorem 2.11 for the case $k=2$ we obtained the following equality (see (2.381))

$$\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)=\sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\phi_{j_1}(t_1), \tag{2.384}$$

which is satisfied pointwise at the interval (t,T), besides the series on the right-hand side of (2.384) converges when $t_1=t,T$.

Let us introduce the induction assumption

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{2.385}$$

Then

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\psi_k(t_k)\times$$

$$\times\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}\prod_{l=1}^{k-1}\phi_{j_l}(t_l)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.386)$$

On the other hand, the left-hand side of (2.386) can be represented in the following form

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_k(t_k)\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}$$

into the generalized Fourier series at the interval (t,T) using the variable t_k. Theorem 2.11 is proved.

Let us introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.387)$$

where

$$\mathrm{A}_{k,l} = \bigl\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1,\ s_l,\ldots,s_1=1,\ldots,k-1\bigr\}, \qquad (2.388)$$

$$(s_l,\ldots,s_1)\in \mathrm{A}_{k,l},\quad l=1,\ldots,[k/2],\quad i_s=0,1,\ldots,m,\quad s=1,\ldots,k,$$

$[x]$ is an integer part of a real number x, and $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals $J^*[\psi^{(k)}]_{T,t}$, $J[\psi^{(k)}]_{T,t}$ of fixed multiplicity k, $k\in\mathbf{N}$ (see (2.373), (2.374)).

Theorem 2.12 [76] (1997). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous function at the interval $[t,T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals*

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1} \qquad (2.389)$$

is correct, where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Proof. Let us prove the equality (2.389) using induction. The case $k=1$ is obvious. If $k=2$, then from (2.389) we get

$$J^*[\psi^{(2)}]_{T,t}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}J[\psi^{(2)}]_{T,t}^{1}\quad \text{w. p. 1.} \tag{2.390}$$

Let us demonstrate that the equality (2.390) is correct w. p. 1. In order to do it let us consider the function $F(x,\tau)=x\psi_2(\tau)$ and the process $F(\eta_{\tau,t},\tau)$, where $\eta_{\tau,t}=J[\psi^{(1)}]_{\tau,t}$, $\tau\in[t,T]$. Then

$$\frac{\partial F}{\partial x}(x,\tau)=\psi_2(\tau),\quad d\eta_{\tau,t}=\psi_1(\tau)d\mathbf{w}_{\tau}^{(i_1)}. \tag{2.391}$$

From (2.391) we obtain that the diffusion coefficient of the process $\eta_{\tau,t}$, $\tau\in[t,T]$ equals to $\mathbf{1}_{\{i_1\ne 0\}}\psi_1(\tau)$. Further, using the standard relations between Stratonovich and Itô stochastic integrals (see (2.4), (2.5)), we obtain the relation (2.390). Thus, the statement of Theorem 2.12 is proved for $k=1$ and $k=2$.

Assume that the statement of Theorem 2.12 is correct for some integer k ($k>2$). Let us prove its correctness when the value k is greater per unit. Using the induction assumption, we have w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t}=$$

$$=\int\limits_t^{*T}\psi_{k+1}(\tau)\left(J[\psi^{(k)}]_{\tau,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1}\right)d\mathbf{w}_{\tau}^{(i_{k+1})}=$$

$$=\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}+$$

$$+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1}d\mathbf{w}_{\tau}^{(i_{k+1})}. \tag{2.392}$$

Using the standard relations between Stratonovich and Itô stochastic integrals (see (2.4), (2.5)), similarly to (2.390), we get w. p. 1

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}=J[\psi^{(k+1)}]_{T,t}+\frac{1}{2}J[\psi^{(k+1)}]_{T,t}^{k}, \tag{2.393}$$

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1}d\mathbf{w}_\tau^{(i_{k+1})}=$$

$$=\begin{cases}J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} & \text{if}\ \ s_r=k-1\\ \\ J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}+J[\psi^{(k+1)}]_{T,t}^{k,s_r,\ldots,s_1}/2 & \text{if}\ \ s_r<k-1\end{cases}. \qquad (2.394)$$

After substituting (2.393) and (2.394) into (2.392) and regrouping of summands, we pass to the following relations, which are valid w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t}=J[\psi^{(k+1)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k+1,r}}J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} \qquad (2.395)$$

when k is even and

$$J^*[\psi^{(k'+1)}]_{T,t}=J[\psi^{(k'+1)}]_{T,t}+\sum_{r=1}^{[k'/2]+1}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k'+1,r}}J[\psi^{(k'+1)}]_{T,t}^{s_r,\ldots,s_1} \qquad (2.396)$$

when $k'=k+1$ is uneven.

From (2.395) and (2.396) we have w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t}=J[\psi^{(k+1)}]_{T,t}+\sum_{r=1}^{[(k+1)/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k+1,r}}J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}. \qquad (2.397)$$

Theorem 2.12 is proved.

For example, from Theorem 2.12 for $k=1,2,3,4$ we obtain the following well known equalities [84], which are fulfilled w. p. 1

$$\int\limits_t^{*T}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}=\int\limits_t^{T}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)},$$

$$\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}=\int\limits_t^{T}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T \psi_2(t_2)\psi_1(t_2)dt_2, \qquad (2.398)
$$

$$
{\int\limits_t^{*}}^T \psi_3(t_3)\ldots {\int\limits_t^{*}}^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_3}^{(i_3)} = \int\limits_t^T \psi_3(t_3)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_3}^{(i_3)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3} \psi_2(t_2)\psi_1(t_2)dt_2 d\mathbf{w}_{t_3}^{(i_3)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3, \qquad (2.399)
$$

$$
{\int\limits_t^{*}}^T \psi_4(t_4)\ldots {\int\limits_t^{*}}^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)} = \int\limits_t^T \psi_4(t_4)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T \psi_4(t_4)\int\limits_t^{t_4} \psi_3(t_3)\int\limits_t^{t_3} \psi_1(t_2)\psi_2(t_2)dt_2 d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T \psi_4(t_4)\int\limits_t^{t_4} \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3 d\mathbf{w}_{t_4}^{(i_4)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4} \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_4 +
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2 dt_4. \qquad (2.400)
$$

Let us consider Lemma 1.1, definition of the multiple stochastic integral (1.16) together with the formula (1.19) when the function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary as well as Lemma 1.3 (see Sect. 1.1.3). Substituting (2.379) into (1.16) and using Lemma 1.1, (1.19), and Theorem 2.12 it is easy to see that w. p. 1

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J[K^*]_{T,t}^{(k)}, \quad (2.401)$$

where $J[K^*]_{T,t}^{(k)}$ is defined by (1.16) and $K^*(t_1,\ldots,t_k)$ has the form (2.379).

Let us subsitute the relation

$$K^*(t_1,\ldots,t_k) = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) + K^*(t_1,\ldots,t_k) -$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

into the right-hand side of (2.401) (here we suppose that $p_1,\ldots,p_k<\infty$). Then using Lemma 1.3 (see Sect. 1.1.3), we obtain

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + J[R_{p_1\ldots p_k}]_{T,t}^{(k)} \quad \text{w. p. 1}, \quad (2.402)$$

where the stochastic integral $J[R_{p_1\ldots p_k}]_{T,t}^{(k)}$ is defined by (1.16) and

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) = K^*(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \quad (2.403)$$

$$\zeta_{j_l}^{(i_l)} = \int\limits_t^T \phi_{j_l}(s)d\mathbf{w}_s^{(i_l)}.$$

According to Theorem 2.11, we have

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty} R_{p_1\ldots p_k}(t_1,\ldots,t_k) = 0 \quad \text{when} \quad (t_1,\ldots,t_k)\in(t,T)^k, \quad (2.404)$$

where the left-hand side of (2.404) is bounded on the boundary of $[t,T]^k$.

Theorem 2.13. *Under the conditions of Theorem* 2.10 *we have*

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}} \mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} = 0, \quad n\in\mathbf{N}.$$

Proof. At first let us analize in detail the cases $k=2,3,4$. Using (2.442) (see below) and (1.19), we have w. p. 1

$$
J[R_{p_1p_2}]_{T,t}^{(2)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\left(\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\right)R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}R_{p_1p_2}(\tau_{l_1},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}=
$$

$$
=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1, \qquad (2.405)
$$

where we used the same notations as in the formulas (1.16), (1.19) and Lemma 1.1 (see Sect. 1.1.3). Moreover,

$$
R_{p_1p_2}(t_1,t_2)=K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad p_1,p_2<\infty. \qquad (2.406)
$$

Let us consider the following well known estimates for moments of stochastic integrals [100]

$$
\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau df_\tau\right|^{2n}\right\}\le (T-t)^{n-1}\left(n(2n-1)\right)^n\int\limits_t^T\mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau, \qquad (2.407)
$$

$$
\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau d\tau\right|^{2n}\right\}\le (T-t)^{2n-1}\int\limits_t^T\mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau, \qquad (2.408)
$$

where the process ξ_τ such that $(\xi_\tau)^n\in \text{M}_2([t,T])$ and f_τ is a scalar standard Wiener process, $n=1,2,\ldots$ (definition of the class $\text{M}_2([t,T])$ see in Sect. 1.1.2).

Using (2.407) and (2.408), we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} \le C_n \left(\int\limits_t^T \int\limits_t^{t_2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2 + \right.$$

$$\left. + \int\limits_t^T \int\limits_t^{t_1} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_2 dt_1 + \mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \left(R_{p_1p_2}(t_1,t_1)\right)^{2n} dt_1 \right), \quad (2.409)$$

where constant $C_n < \infty$ depends on n and $T-t$ $(n=1,2,\ldots)$.

Further, we have

$$\int\limits_t^T \int\limits_t^{t_2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2 + \int\limits_t^T \int\limits_t^{t_1} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_2 dt_1 =$$

$$= \int\limits_t^T \int\limits_t^{t_2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2 + \int\limits_t^T \int\limits_{t_2}^{T} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2 =$$

$$= \int\limits_{[t,T]^2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2. \quad (2.410)$$

Combining (2.409) and (2.410), we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} \le$$

$$\le C_n \left(\int\limits_{[t,T]^2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1 dt_2 + \mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \left(R_{p_1p_2}(t_1,t_1)\right)^{2n} dt_1 \right), \quad (2.411)$$

where constant $C_n < \infty$ depends on n and $T-t$ $(n=1,2,\ldots)$.

Since the integrals on the right-hand side of (2.411) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty} \lim_{p_2\to\infty} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} = 0 \quad \text{when} \quad (t_1,t_2) \in (t,T)^2, \quad (2.412)$$

where $n \in \mathbf{N}$ and the left-hand side is bounded on the boundary of $[t,T]^2$.

According to (2.406), we have

$$R_{p_1p_2}(t_1,t_2)=\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2)-\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right). \qquad (2.413)$$

Then, applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}$) the Lebesgue's Dominated Convergence Theorem and taking into account (2.381), (2.382), and (2.413), we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2=0, \qquad (2.414)$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T\left(R_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1=0. \qquad (2.415)$$

We will discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem when we consider the case of arbitrary $k\in\mathbf{N}$ later in this section.

From (2.411), (2.414), and (2.415) we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\}=0, \quad n\in\mathbf{N}.$$

Recall that (2.415) for $2n=1$ has also been proved in Sect. 2.1.1, 2.1.2.

Let us consider the case $k=3$. Using (2.443) (see below) and (1.19), we have w. p. 1

$$J[R_{p_1p_2p_3}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+$$

$$\left.+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\left(R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+\right.$$

$$+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+$$

$$\left.+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}\left(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+\right.$$

$$+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$\left.+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_1)}+$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_2,t_2,t_3)dt_2d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_2,t_3,t_2)dt_2d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_2,t_2)dt_2d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_1,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}dt_3, \qquad (2.416)
$$

where we used the same notations as in the formulas (1.16), (1.19) and Lemma 1.1 (see Sect. 1.1.3). Using (2.407) and (2.408), we obtain from (2.416)

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}\le
$$

$$
\le C_n\Biggl(\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+
$$

$$
+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Biggr)dt_1dt_2dt_3+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\Biggr)dt_2dt_3\Biggr),\quad C_n<\infty. \tag{2.417}
$$

Due to (2.403) and Theorem 2.11 the function $R_{p_1p_2p_3}(t_1,t_2,t_3)$ is continuous in the open domains of integration of iterated integrals on the right-hand side of (2.417) and it is bounded at the boundaries of these domains. Moreover, everywhere in $(t,T)^3$ the following formula takes place

$$
\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}R_{p_1p_2p_3}(t_1,t_2,t_3)=0. \tag{2.418}
$$

Further, we have

$$
\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Biggr)dt_1dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3, \tag{2.419}
$$

$$
\int\limits_t^T\int\limits_t^{t_3}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Biggr)dt_2dt_3=
$$

$$
=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3, \tag{2.420}
$$

$$\int\limits_t^T\int\limits_t^{t_3}\left(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\right)dt_2dt_3=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3=$$

$$=\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3, \tag{2.421}$$

$$\int\limits_t^T\int\limits_t^{t_3}\left(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\right)dt_2dt_3=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3=$$

$$=\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3. \tag{2.422}$$

Combining (2.417) and (2.419)–(2.422), we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}\le C_n\left(\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3+\right.$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3+$$

$$\left.+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3\right). \tag{2.423}$$

Since the integrals on the right-hand side of (2.423) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty} R_{p_1p_2p_3}(t_1,t_2,t_3)=0 \quad \text{when} \quad (t_1,t_2,t_3)\in(t,T)^3,$$

where the left-hand side is bounded on the boundary of $[t,T]^3$.

According to the proof of Theorem 2.11 and (2.403) for $k=3$, we have

$$R_{p_1p_2p_3}(t_1,t_2,t_3)=\left(K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,t_3)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,t_3)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}(t_3)-\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\phi_{j_3}(t_3)\right)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right), \qquad (2.424)$$

where

$$C_{j_1}(t_2,t_3)=\int\limits_t^T K^*(t_1,t_2,t_3)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3)=\int\limits_{[t,T]^2} K^*(t_1,t_2,t_3)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2.$$

Then, applying three times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3=0, \qquad (2.425)$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=0, \qquad (2.426)$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3=0, \qquad (2.427)$$

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3=0. \qquad (2.428)$$

From (2.423) and (2.425)–(2.428) we get

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}.$$

Let us consider the case $k=4$. Using (2.444) (see below) and (1.19), we have w. p. 1

$$J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\times$$

$$\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{(l_2,l_2,l_3,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_1=0}^{l_3-1}\sum_{(l_1,l_3,l_3,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_4,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{(l_3,l_3,l_3,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_3},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{(l_2,l_2,l_4,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{l_4-1}\sum_{(l_1,l_4,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_4},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_4}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\int_t^T\int_t^{t_4}\int_t^{t_3}\int_t^{t_2}\sum_{(t_1,t_2,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\int_t^T\int_t^{t_4}\int_t^{t_3}\sum_{(t_1,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_1,t_3,t_4)dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\int_t^T\int_t^{t_4}\int_t^{t_2}\sum_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_1,t_4)dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\int_t^T\int_t^{t_3}\int_t^{t_2}\sum_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_1)dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\int_t^T\int_t^{t_4}\int_t^{t_2}\sum_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_2,t_4)d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_4\neq 0\}}\int_t^T\int_t^{t_3}\int_t^{t_2}\sum_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_3=i_4\neq 0\}}\int_t^T\int_t^{t_3}\int_t^{t_2}\sum_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\left(\int_t^T\int_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)dt_2dt_4+\right.$$

$$
+\int\limits_t^T\int\limits_t^{t_4} R_{p_1p_2p_3p_4}(t_4,t_4,t_2,t_2)dt_2dt_4\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\Biggl(\int\limits_t^T\int\limits_t^{t_4} R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)dt_2dt_4+
$$

$$
+\int\limits_t^T\int\limits_t^{t_4} R_{p_1p_2p_3p_4}(t_4,t_2,t_4,t_2)dt_2dt_4\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(\int\limits_t^T\int\limits_t^{t_4} R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)dt_2dt_4+
$$

$$
+\int\limits_t^T\int\limits_t^{t_4} R_{p_1p_2p_3p_4}(t_4,t_2,t_2,t_4)dt_2dt_4\Biggr), \tag{2.429}
$$

where the expression

$$
\sum_{(a_1,\ldots,a_k)}
$$

means the sum with respect to all possible permutations $(a_1,\ldots,a_k)$. Moreover, we used in (2.429) the same notations as in the proof of Theorem 1.1 (see Sect. 1.1.3). Note that an analogue of (2.429) will be obtained in Sect. 2.6 (also see [10]-[17], [36]) with using the another approach.

By analogy with (2.423) we obtain

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\}\le
$$

$$
\le C_n\Biggl(\int\limits_{[t,T]^4}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n}dt_1dt_2dt_3dt_4+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_3,t_4)\right)^{2n}dt_2dt_3dt_4+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_2,t_4)\right)^{2n}dt_2dt_3dt_4+
$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n}dt_2dt_3dt_4+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n}dt_2dt_3dt_4+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n}dt_2dt_3dt_4+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n}dt_2dt_3dt_4+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n}dt_2dt_4+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n}dt_2dt_4+$$

$$\left.+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n}dt_2dt_4\right),\quad C_n<\infty. \tag{2.430}$$

Since the integrals on the right-hand side of (2.430) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}\lim\limits_{p_4\to\infty}R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)=0\quad\text{when}\quad (t_1,t_2,t_3,t_4)\in(t,T)^4,$$

where the left-hand side is bounded on the boundary of $[t,T]^4$.

According to the proof of Theorem 2.11 and (2.403) for $k=4$, we have

$$R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)=$$

$$=\left(K^*(t_1,t_2,t_3,t_4)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,t_3,t_4)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,t_3,t_4)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,t_4)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}(t_3,t_4)-\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(t_4)\phi_{j_3}(t_3)\right)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(C_{j_3j_2j_1}(t_4)-\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\phi_{j_4}(t_4)\right)\phi_{j_3}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right),$$

where

$$C_{j_1}(t_2,t_3,t_4)=\int\limits_t^T K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3,t_4)=\int\limits_{[t,T]^2} K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$C_{j_3j_2j_1}(t_4)=\int\limits_{[t,T]^3} K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)dt_1dt_2dt_3.$$

Then, applying four times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}\int\limits_{[t,T]^4}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n}dt_1dt_2dt_3dt_4=0, \quad (2.431)$$

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_3,t_4)\right)^{2n}dt_2dt_3dt_4=0, \quad (2.432)$$

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_2,t_4)\right)^{2n}dt_2dt_3dt_4=0, \quad (2.433)$$

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \quad (2.434)$$

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n}dt_2dt_3dt_4=0, \quad (2.435)$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \tag{2.436}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n}dt_2dt_3dt_4=0, \tag{2.437}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n}dt_2dt_4=0, \tag{2.438}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n}dt_2dt_4=0, \tag{2.439}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n}dt_2dt_4=0. \tag{2.440}$$

Combaining (2.430) with (2.431)–(2.440), we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}.$$

Theorem 2.13 is proved for $k=4$.

Let us consider the case of arbitrary k, $k\in\mathbf{N}$. Let us analyze the stochastic integral defined by (1.16) and find its representation convenient for the following consideration. In order to do it we introduce several notations. Suppose that

$$S_N^{(k)}(a)=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{(j_1,\ldots,j_k)}a_{(j_1,\ldots,j_k)},$$

$$\mathrm{C}_{s_r}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=$$

$$=\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\ \sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\ \sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}a_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)},$$

where

$$\prod_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)\ \stackrel{\rm def}{=}\ \mathbf{I}_{j_{s_r},j_{s_r+1}}\ldots\mathbf{I}_{j_{s_1},j_{s_1+1}}(j_1,\ldots,j_k),$$

$$\mathrm{C}_{s_0}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=S_N^{(k)}(a),\quad \prod_{l=1}^{0}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)=(j_1,\ldots,j_k),$$

$$\mathbf{I}_{j_l,j_{l+1}}(j_{q_1},\ldots,j_{q_2},j_l,j_{q_3},\ldots,j_{q_{k-2}},j_l,j_{q_{k-1}},\ldots,j_{q_k})\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}(j_{q_1},\ldots,j_{q_2},j_{l+1},j_{q_3},\ldots,j_{q_{k-2}},j_{l+1},j_{q_{k-1}},\ldots,j_{g_k}),$$

where $l\ne q_1,\ldots,q_2,q_3,\ldots,q_{k-2},q_{k-1},\ldots,q_k$, $l\in\mathbf{N}$, $a_{(j_{q_1},\ldots,j_{q_k})}$ is a scalar value, $s_1,\ldots,s_r=1,\ldots,k-1$, $s_r>\ldots>s_1$, $q_1,\ldots,q_k=1,\ldots,k$, the expression

$$\sum_{(j_{q_1},\ldots,j_{q_k})}$$

means the sum with respect to all possible permutations $(j_{q_1},\ldots,j_{q_k})$.

Using induction it is possible to prove the following equality

$$\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{N-1}a_{(j_1,\ldots,j_k)}=\sum_{r=0}^{k-1}\ \sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\mathrm{C}_{s_r}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a),\qquad (2.441)$$

where $k=2,3,\ldots$

Hereinafter in this section, we will identify the following records

$$a_{(j_1,\ldots,j_k)}=a_{(j_1\ldots j_k)}=a_{j_1\ldots j_k}.$$

In particular, from (2.441) for $k=2,3,4$ we get the following formulas

$$\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1}a_{(j_1,j_2)}=S_N^{(2)}(a)+\mathrm{C}_1S_N^{(2)}(a)=$$

$$=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2)}a_{(j_1j_2)}+\sum_{j_2=0}^{N-1}a_{(j_2j_2)}=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2}+a_{j_2j_1})+$$

$$+\sum_{j_2=0}^{N-1}a_{j_2j_2},\qquad (2.442)$$

$$
\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2,j_3)} = S_N^{(3)}(a) + \mathrm{C}_1 S_N^{(3)}(a) + \mathrm{C}_2 S_N^{(3)}(a) + \mathrm{C}_2\mathrm{C}_1 S_N^{(3)}(a) =
$$

$$
= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3)} a_{(j_1j_2j_3)} + \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3)} a_{(j_2j_2j_3)} +
$$

$$
+ \sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3)} a_{(j_1j_3j_3)} + \sum_{j_3=0}^{N-1} a_{(j_3j_3j_3)} =
$$

$$
= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \left(a_{j_1j_2j_3} + a_{j_1j_3j_2} + a_{j_2j_1j_3} + a_{j_2j_3j_1} + a_{j_3j_2j_1} + a_{j_3j_1j_2}\right) +
$$

$$
+ \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \left(a_{j_2j_2j_3} + a_{j_2j_3j_2} + a_{j_3j_2j_2}\right) + \sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1} \left(a_{j_1j_3j_3} + a_{j_3j_1j_3} + a_{j_3j_3j_1}\right) +
$$

$$
+ \sum_{j_3=0}^{N-1} a_{j_3j_3j_3}, \tag{2.443}
$$

$$
\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2,j_3,j_4)} = S_N^{(4)}(a) + \mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_2 S_N^{(4)}(a) +
$$

$$
+\mathrm{C}_3 S_N^{(4)}(a) + \mathrm{C}_2\mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_2 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_2\mathrm{C}_1 S_N^{(4)}(a) =
$$

$$
= \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3,j_4)} a_{(j_1j_2j_3j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3,j_4)} a_{(j_2j_2j_3j_4)}
$$

$$
+ \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3,j_4)} a_{(j_1j_3j_3j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_4,j_4)} a_{(j_1j_2j_4j_4)} +
$$

$$
+ \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{(j_3,j_3,j_3,j_4)} a_{(j_3j_3j_3j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)} a_{(j_2j_2j_4j_4)} +
$$

$$
+ \sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_4,j_4,j_4)} a_{(j_1j_4j_4j_4)} + \sum_{j_4=0}^{N-1} a_{j_4j_4j_4j_4} =
$$

$$
= \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \left(a_{j_1j_2j_3j_4} + a_{j_1j_2j_4j_3} + a_{j_1j_3j_2j_4} + a_{j_1j_3j_4j_2} + \right.
$$

$$
+a_{j_1j_4j_3j_2}+a_{j_1j_4j_2j_3}+a_{j_2j_1j_3j_4}+a_{j_2j_1j_4j_3}+a_{j_2j_4j_1j_3}+a_{j_2j_4j_3j_1}+a_{j_2j_3j_1j_4}+
$$

$$
+a_{j_2j_3j_4j_1}+a_{j_3j_1j_2j_4}+a_{j_3j_1j_4j_2}+a_{j_3j_2j_1j_4}+a_{j_3j_2j_4j_1}+a_{j_3j_4j_1j_2}+a_{j_3j_4j_2j_1}+
$$

$$
+a_{j_4j_1j_2j_3}+a_{j_4j_1j_3j_2}+a_{j_4j_2j_1j_3}+a_{j_4j_2j_3j_1}+a_{j_4j_3j_1j_2}+a_{j_4j_3j_2j_1}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\Bigl(a_{j_2j_2j_3j_4}+a_{j_2j_2j_4j_3}+a_{j_2j_3j_2j_4}+a_{j_2j_4j_2j_3}+a_{j_2j_3j_4j_2}+a_{j_2j_4j_3j_2}+
$$

$$
+a_{j_3j_2j_2j_4}+a_{j_4j_2j_2j_3}+a_{j_3j_2j_4j_2}+a_{j_4j_2j_3j_2}+a_{j_4j_3j_2j_2}+a_{j_3j_4j_2j_2}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\Bigl(a_{j_3j_3j_1j_4}+a_{j_3j_3j_4j_1}+a_{j_3j_1j_3j_4}+a_{j_3j_4j_3j_1}+a_{j_3j_4j_1j_3}+a_{j_3j_1j_4j_3}+
$$

$$
+a_{j_1j_3j_3j_4}+a_{j_4j_3j_3j_1}+a_{j_4j_3j_1j_3}+a_{j_1j_3j_4j_3}+a_{j_1j_4j_3j_3}+a_{j_4j_1j_3j_3}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\Bigl(a_{j_4j_4j_1j_2}+a_{j_4j_4j_2j_1}+a_{j_4j_1j_4j_2}+a_{j_4j_2j_4j_1}+a_{j_4j_2j_1j_4}+a_{j_4j_1j_2j_4}+
$$

$$
+a_{j_1j_4j_4j_2}+a_{j_2j_4j_4j_1}+a_{j_2j_4j_1j_4}+a_{j_1j_4j_2j_4}+a_{j_1j_2j_4j_4}+a_{j_2j_1j_4j_4}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\Bigl(a_{j_3j_3j_3j_4}+a_{j_3j_3j_4j_3}+a_{j_3j_4j_3j_3}+a_{j_4j_3j_3j_3}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\Bigl(a_{j_2j_2j_4j_4}+a_{j_2j_4j_2j_4}+a_{j_2j_4j_4j_2}+a_{j_4j_2j_2j_4}+a_{j_4j_2j_4j_2}+a_{j_4j_4j_2j_2}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\Bigl(a_{j_1j_4j_4j_4}+a_{j_4j_1j_4j_4}+a_{j_4j_4j_1j_4}+a_{j_4j_4j_4j_1}\Bigr)+
$$

$$
+\sum_{j_4=0}^{N-1}a_{j_4j_4j_4j_4}. \qquad (2.444)
$$

Perhaps, the formula (2.441) for any k ($k\in\mathbf{N}$) was found by the author for the first time [76] (1997).

Assume that

$$
a_{(j_1,\ldots,j_k)}=\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)},
$$

where $\Phi(t_1,\ldots,t_k)$ is a nonrandom function of k variables. Then from (1.16) and (2.441) we have

$$J[\Phi]_{T,t}^{(k)}=\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\times$$

$$\times\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}\times$$

$$\times\Biggl[\Phi\Biggl(\tau_{j_1},\ldots,\tau_{j_{s_1-1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+2}},\ldots,\tau_{j_{s_r-1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+2}},\ldots,\tau_{j_k}\Biggr)\times$$

$$\times\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_{s_1-1}}}^{(i_{s_1-1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1+1})}\Delta\mathbf{w}_{\tau_{j_{s_1+2}}}^{(i_{s_1+2})}\ldots$$

$$\ldots\Delta\mathbf{w}_{\tau_{j_{s_r-1}}}^{(i_{s_r-1})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r+1})}\Delta\mathbf{w}_{\tau_{j_{s_r+2}}}^{(i_{s_r+2})}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_k)}\Biggr]=$$

$$=\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\quad\text{w. p. 1,}\tag{2.445}$$

where

$$I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}=\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times$$

$$\times\Biggl[\Phi\Biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Biggr)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1+1})}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r+1})}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Biggr],\tag{2.446}$$

where $k\ge 2$, the set $\mathrm{A}_{k,r}$ is defined by the relation (2.388). We suppose that the right-hand side of (2.446) exists as the Itô stochastic integral.

Remark 2.3. *The summands on the right-hand side of (2.446) should be understood as follows: for each permutation from the set*

$$
\prod_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)=
$$

$$
=\Biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Biggr)
$$

it is necessary to perform replacement on the right-hand side of (2.446) of all pairs (their number is equal to r) of differentials $d\mathbf{w}_{t_p}^{(i)}d\mathbf{w}_{t_p}^{(j)}$ with similar lower indices by the values $\mathbf{1}_{\{i=j\ne 0\}}dt_p$.

Note that the term in (2.445) for $r=0$ should be understood as follows

$$
\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Biggl(\Phi\left(t_1,\ldots,t_k\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Biggr),
$$

where notations are the same as in (1.24).

Using (2.407), (2.408), (2.445), and (2.446), we get

$$
\mathsf{M}\left\{\left|J[\Phi]_{T,t}^{(k)}\right|^{2n}\right\}\le
$$

$$
\le C_{nk}\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}, \tag{2.447}
$$

where

$$
\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}\le
$$

$$
\le C_{nk}^{s_1\ldots s_r}\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times
$$

$$
\times\Phi^{2n}\Biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Biggr)\times
$$

$$
\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k, \tag{2.448}
$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing are performed in (2.448) only in the values

$$\Phi^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr).$$

Consider (2.447) and (2.448) for $\Phi(t_1,\ldots,t_k)\equiv R_{p_1\ldots p_k}(t_1,\ldots,t_k)$

$$\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}\le$$

$$\le C_{nk}\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\},\tag{2.449}$$

where

$$\mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}\le$$

$$\le C_{nk}^{s_1\ldots s_r}\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times$$

$$\times R_{p_1\ldots p_k}^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr)\times$$

$$\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k,\tag{2.450}$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing are performed in (2.450) only in the values

$$R_{p_1\ldots p_k}^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr).$$

From the other hand, we can consider the generalization of the formulas (2.411), (2.423), (2.430) for the case of arbitrary k ($k\in\mathbf{N}$). In order to do this, let us consider the sum with respect to all possible partitions defined by (1.53)

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}.$$

Now we can generalize the formulas (2.411), (2.423), (2.430) for the case of arbitrary k ($k\in\mathbf{N}$)

$$
\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}\le C_{nk}\left(\int\limits_{[t,T]^k}\left(R_{p_1\ldots p_k}(t_1,\ldots,t_k)\right)^{2n}dt_1\ldots dt_k+\right.
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\ldots\mathbf{1}_{\{i_{g_{2r-1}}=i_{g_{2r}}\ne 0\}}\times
$$

$$
\times\int\limits_{[t,T]^{k-r}}\left(R_{p_1\ldots p_k}\left(t_1,\ldots,t_k\right)\Bigl|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}\right)^{2n}\times
$$

$$
\left.\times\left(dt_1\ldots dt_k\right)\Bigl|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}\right),\qquad(2.451)
$$

where C_{nk} is a constant,

$$
\left(t_1,\ldots,t_k\right)\Bigl|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}
$$

means the ordered set $(t_1,\ldots,t_k)$, where we put $t_{g_1}=t_{g_2},\ \ldots,\ t_{g_{2r-1}}=t_{g_{2r}}$.

Moreover,

$$
\left(dt_1\ldots dt_k\right)\Bigl|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}
$$

means the product $dt_1\ldots dt_k$, where we replace all pairs $dt_{g_1}dt_{g_2},\ \ldots,\ dt_{g_{2r-1}}dt_{g_{2r}}$ by $dt_{g_1},\ \ldots,\ dt_{g_{2r-1}}$ correspondingly.

Note that the estimate like (2.451), where all indicators $\mathbf{1}_{\{\cdot\}}$ must be replaced with 1, can be obtained from the estimates (2.449), (2.450).

The comparison of (2.451) with the formula (1.54) (see Theorem 1.2) shows their similar structure.

Let us consider the particular case of (2.451) for $k=4$

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]^{(4)}_{T,t}\right|^{2n}\right\} \le C_{n4}\left(\int\limits_{[t,T]^4}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n} dt_1dt_2dt_3dt_4+\right.
$$

$$
+\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} \mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigl|_{t_{g_1}=t_{g_2}}\right)^{2n}\times
$$

$$
\times\left(dt_1dt_2dt_3dt_4\right)\Bigl|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1}}+
$$

$$
+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} \mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\mathbf{1}_{\{i_{g_3}=i_{g_4}\ne 0\}}\times
$$

$$
\times\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigl|_{t_{g_1}=t_{g_2},t_{g_3}=t_{g_4}}\right)^{2n}\times
$$

$$
\left.\times\left(dt_1dt_2dt_3dt_4\right)\Bigl|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\left(dt_{g_3}dt_{g_4}\right)\curvearrowright dt_{g_3}}\right). \qquad (2.452)
$$

It is not difficult to notice that (2.452) is consistent with (2.430). According to (2.379) and (2.403), we have the following expression

$$
R_{p_1\ldots p_k}(t_1,\ldots,t_k)=
$$

$$
=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right)-
$$

$$
-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l). \qquad (2.453)
$$

Due to (2.453) the function $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ is continuous in the open domains of integration of integrals on the right-hand side of (2.450) and it is bounded at the boundaries of these domains for $p_1,\ldots,p_k<\infty$.

Let us perform the iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ldots\overline{\lim\limits_{p_k\to\infty}}$ under the integral signs on the right-hand side of the estimate (2.451) (it was similarly performed for the 2-dimensional, 3-dimentional, and 4-dimensional cases (see above)). Then, taking into account (2.404), we obtain the required result. More precisely, since the integrals on the right-hand side of (2.451) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}R_{p_1\ldots p_k}(t_1,\ldots,t_k)=0\quad\text{when}\quad(t_1,\ldots,t_k)\in(t,T)^k,$$

where the left-hand side is bounded on the boundary of $[t,T]^k$.

According to the proof of Theorem 2.11 and (2.403), we have

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)=$$

$$=\left(K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,\ldots,t_k)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$\ldots$$

$$+\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}}\left(C_{j_{k-1}\ldots j_1}(t_k)-\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)\right)\phi_{j_{k-1}}(t_{k-1})\ldots\phi_{j_1}(t_1)\right),\tag{2.454}$$

where

$$C_{j_1}(t_2,\ldots,t_k)=\int\limits_t^T K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3,\ldots,t_k)=\int\limits_{[t,T]^2}K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$\ldots$$

$$C_{j_{k-1}\ldots j_1}(t_k) = \int\limits_{[t,T]^{k-1}} K^*(t_1,\ldots,t_k)\prod_{l=1}^{k-1}\phi_{j_l}(t_l)dt_1\ldots dt_{k-1}.$$

Then, applying k times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ldots\overline{\lim\limits_{p_k\to\infty}}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (2.451), we obtain

$$\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ldots\overline{\lim\limits_{p_k\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}.$$

Let us discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in (2.451).

It is well known that [125]

$$\left|\sum_{k=1}^N\frac{\sin kx}{k}\right|\le C \qquad (2.455)$$

for all N and x, where constant C does not depend on N and x.

Moreover,

$$\sum_{j=1}^N\frac{1}{j^2}\le\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}. \qquad (2.456)$$

Applying double integration by parts (as in (2.28)), we estimate the partial sums of one-dimensional trigonometric Fourier series

$$\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1),$$

$$\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_2}(t_2),$$

$$\ldots$$

$$\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)$$

in (2.454) using (2.456) and (see (2.455))

$$\left|\sum_{k=1}^{N}\frac{1}{k}\sin\frac{2\pi k(x-y)}{T-t}\right|\le C,$$

$$\left|\sum_{k=1}^{N}\frac{1}{k}\sin\frac{2\pi k(x-t)}{T-t}\right|\le C$$

(here $N\in\mathbf{N}$ and $x,y\in\mathbf{R}$, constant C does not depend on N and x,y) as follows

$$\left|\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right|\le C_1,$$

$$\left|\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right|\le C_2,$$

$$\ldots$$

$$\left|\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)\right|\le C_k,$$

where constant C_1 does not depend on p_1, constant C_2 does not depend on p_2, etc.

Moreover,

$$|K^*(t_1,\ldots,t_k)|\le\tilde C_1,\quad |C_{j_1}(t_2,\ldots,t_k)|\le\tilde C_2,\quad \ldots\quad \left|C_{j_{k-1}\ldots j_1}(t_k)\right|\le\tilde C_k,$$

where constant $\tilde C_1$ does not depend on p_1, constant $\tilde C_2$ does not depend on p_2, etc.

Further, the construction of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in (2.451) is obvious.

For example, to pass to the limit $\overline{\lim\limits_{p_k\to\infty}}$, the integrable majorant has the form (it is constructed on the base of (2.454))

$$\left(R_{p_1\ldots p_k}(t_1,\ldots,t_k)\right)^{2n}\le$$

$$\le\Biggl(\left(\tilde C_1+C_1\right)+$$

$$
+\sum_{j_1=0}^{p_1}\left(\tilde{C}_2+C_2\right)\left|\phi_{j_1}(t_1)\right|+\ldots
$$

$$
\left.\ldots+\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}}\left(\tilde{C}_k+C_k\right)\left|\phi_{j_{k-1}}(t_{k-1})\ldots\phi_{j_1}(t_1)\right|\right)^{2n}\le
$$

$$
\le\left(\left(\tilde{C}_1+C_1\right)+\right.
$$

$$
+\sqrt{\frac{2}{T-t}}\left(p_1+1\right)\left(\tilde{C}_2+C_2\right)+\ldots
$$

$$
\left.\ldots+\left(\sqrt{\frac{2}{T-t}}\right)^{k-1}\left(p_1+1\right)\ldots\left(p_{k-1}+1\right)\left(\tilde{C}_k+C_k\right)\right)^{2n}, \qquad (2.457)
$$

where $n\in\mathbf{N}$, the numbers $p_1,\ldots,p_{k-1}$ are fixed and the right-hand side of (2.457) is independent of p_k.

Theorems 2.13 and 2.10 are proved.

It is easy to notice that if we expand the function $K^*(t_1,\ldots,t_k)$ into the generalized Fourier series at the interval (t,T) at first with respect to the variable t_k, after that with respect to the variable t_{k-1}, etc., then we will have the expansion

$$
K^*(t_1,\ldots,t_k)=\lim_{p_k\to\infty}\ldots\lim_{p_1\to\infty}\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) \qquad (2.458)
$$

instead of the expansion (2.380).

Let us prove the expansion (2.458). Similarly with (2.384) we have the following equality

$$
\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)=\sum_{j_k=0}^{\infty}\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\phi_{j_k}(t_k), \quad (2.459)
$$

which is satisfied pointwise at the interval (t,T), besides the series on the right-hand side of (2.459) converges when $t_1=t,T$.

Let us introduce the induction assumption

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.460)$$

Then

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\sum_{j_2=0}^{\infty}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2\prod_{l=2}^{k}\phi_{j_l}(t_l)=$$

$$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.461)$$

From the other hand, the left-hand side of (2.461) can be represented in the following form

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2$$

into the generalized Fourier series at the interval (t,T) using the variable t_1. Here we applied the following replacement of integration order

$$\int\limits_{t}^{T}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2dt_1=$$

$$=\int\limits_{t}^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\ldots dt_k=$$

$$C_{j_k\ldots j_1}.$$

The expansion (2.458) is proved. So, we can formulate the following theorem.

Theorem 2.14 [10] (2013) (also see [11]-[17], [34]). *Suppose that the conditions of Theorem* 2.10 *are fulfilled. Then*

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},\qquad(2.462)$$

where notations are the same as in Theorem 2.10.

Note that (2.462) means the following

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\,\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^{2n}\right\}=0,$$

where $n \in \mathbf{N}$.

Let us make a remark about how one can obtain an analogue of Theorem 2.10 for the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $n=1$ (the case of mean-square convergence), $k=2$.

From (2.102) we have

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} \le$$

$$\le 2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2 + \mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \quad (2.463)$$

From Remark 1.6 and (1.73), (2.103) we obtain for the case of Legendre polynomials

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2 = 0.$$

Further, we have (see (2.413))

$$R_{p_1p_2}(t_1,t_1) = \left(K^*(t_1,t_1) - \sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1)\right) +$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_1) - \sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_2}(t_1)\right)\phi_{j_1}(t_1)\right). \quad (2.464)$$

Then, taking into account (2.412), (2.464) and applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0.$$

Let us discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in our case.

Using double integration by parts (as in (2.22)), we estimate the partial sums of one-dimensional Fourier–Legendre series

$$\sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1), \quad \sum_{j_2=0}^{p_2} C_{j_2 j_1}\phi_{j_2}(t_1)$$

in (2.464) using (2.23) and (2.456) as follows

$$\left|\sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1)\right| \le K_1\left(1+\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}}+\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \quad (2.465)$$

$$\left|\sum_{j_2=0}^{p_2} C_{j_2 j_1}\phi_{j_2}(t_1)\right| \le K_2\left(1+\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \quad (2.466)$$

where $z(t_1)$ is defined by (2.20), constant K_1 does not depend on p_1, and constant K_2 does not depend on p_2.

Thus, integrable majorants in our case can be easily constracted using (2.464), (2.465) and (2.466) (see the proof of Theorem 2.10 for details).

An analogue of Theorem 2.10 for the case of Legendre polynomials and $n = 1$ (the case of mean-square convergence), $k = 2$ is obtained.

2.4.2 Further Remarks

In this section, we consider some approaches on the base of Theorems 2.10 and 1.1 for the case $k = 2$. Moreover, we explain the potential difficulties associated with the use of generalized multiple Fourier series converging almost everywhere (with respect to Lebesgue's measure (here and further in this section)) on the hypercube $[t, T]^k$ in the proof of Theorem 2.10.

First, we show how iterated series can be replaced by multiple one in Theorem 2.10 (the case $k = 2$ and $n = 1$) and in analogue of Theorem 2.10 for the case of Legendre polynomials (the case $k = 2$ and $n = 1$).

We have

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \lim_{p\to\infty}\overline{\lim_{q\to\infty}} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le$$

$$
\le \lim_{p\to\infty}\overline{\lim_{q\to\infty}}\left(2\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}+\right.
$$

$$
\left.+2\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}\right)=
$$

$$
=2\lim_{p\to\infty}\overline{\lim_{q\to\infty}}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}=
$$

$$
=2\lim_{p\to\infty}\overline{\lim_{q\to\infty}}\sum_{j_1=0}^{p}\sum_{j_1'=0}^{p}\sum_{j_2=p+1}^{q}\sum_{j_2'=p+1}^{q}C_{j_2j_1}C_{j_2'j_1'}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_1'}^{(i_1)}\right\}\mathsf{M}\left\{\zeta_{j_2}^{(i_2)}\zeta_{j_2'}^{(i_2)}\right\}=
$$

$$
=2\lim_{p\to\infty}\lim_{q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q}C_{j_2j_1}^2=
$$

$$
=2\lim_{p\to\infty}\lim_{q\to\infty}\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}^2\right)= \tag{2.467}
$$

$$
=2\left(\lim_{p,q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}^2-\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}^2\right)= \tag{2.468}
$$

$$
=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2=0, \tag{2.469}
$$

where the function $K(t_1,t_2)$ is defined by (1.6) for $k=2$.

Note that the transition from (2.467) to (2.468) is based on the theorem on reducing of a limit to iterated one. Moreover, the transition from (2.468) to (2.469) is based on the Parseval equality.

Thus, we obtain the following Theorem.

Theorem 2.15 [14]-[17], [34]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$ is twice continuously differentiable nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

(2.373) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense is valid, where

$$C_{j_2j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{2.470}$$

is the Fourier coefficient and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i=1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Note that Theorem 2.15 is a modification (for the case $p_1=p_2=p$ of series summation) of Theorem 2.1.

Using Theorem 2.10, we get

$$0\le\left|\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\mathsf{M}\left\{\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-J^*[\psi^{(k)}]_{T,t}\right\}\right|\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left|\mathsf{M}\left\{\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-J^*[\psi^{(k)}]_{T,t}\right\}\right|\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\mathsf{M}\left\{\left|J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right|\right\}\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\right)^{1/2}=0. \tag{2.471}$$

From the other hand,

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}\right)=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}. \quad (2.472)$$

Combining (2.471) and (2.472), we obtain

$$\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}. \quad (2.473)$$

Note that the relations (2.471)–(2.473) are also valid for the case of Legendre polynomials and $k=2$.

The formula (2.473) with $k=2$ implies the following

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(s)\psi_2(s)ds=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}, \quad (2.474)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Since

$$\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}},$$

then from (2.474) we obtain

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}, \quad (2.475)$$

where $C_{j_1 j_1}$ is defined by (2.470) for $j_1 = j_2$, i.e.

$$C_{j_1 j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2.$$

From (2.474) and (2.475) we obtain the following relation

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2}\int\limits_t^T \psi_1(s)\psi_2(s)ds. \tag{2.476}$$

Combining (1.46) and (2.476), we have

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(s)\psi_2(s)ds. \tag{2.477}$$

Since

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(s)\psi_2(s)ds \quad \text{w. p. 1}, \tag{2.478}$$

then from (2.477) we finally get the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}.$$

Thus, we obtain the statement of Theorem 2.1.

We have

$$J^*[\psi^{(2)}]_{T,t}^{p_1,p_2} \stackrel{\text{def}}{=} J[\psi^{(2)}]_{T,t}^{p_1,p_2} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(s)\psi_2(s)ds =$$

$$
=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(s)\psi_2(s)ds=
$$

$$
=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}\right),
\tag{2.479}
$$

where

$$
J[\psi^{(2)}]_{T,t}^{p_1,p_2}=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right)
$$

is the approximation of iterated Itô stochastic integral (2.374) ($k=2$) based on Theorem 1.1 (see (1.46)).

Moreover, from (1.74), (1.75), (1.329) and (2.8) we obtain

$$
\mathsf{M}\left\{\left(J^{*}[\psi^{(2)}]_{T,t}-J^{*}[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\}=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\}\ \to\ 0
\tag{2.480}
$$

if $p_1,p_2\to\infty$ $(n\in\mathbf{N})$.

Further,

$$
\mathsf{M}\left\{\left(J^{*}[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^{2n}\right\}=
$$

$$
=\mathsf{M}\Biggl\{\Biggl(J^{*}[\psi^{(2)}]_{T,t}-J^{*}[\psi^{(2)}]_{T,t}^{p_1,p_2}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}\right)\Biggr)^{2n}\Biggr\}\le
$$

$$
\le K_n\Biggl(\mathsf{M}\left\{\left(J^{*}[\psi^{(2)}]_{T,t}-J^{*}[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}\right)^{2n}\Biggr),
\tag{2.481}
$$

where constant $K_n < \infty$ depends on n.

Taking into account (2.480), (2.481) and also that the equality (2.476) is true under the conditions of Theorem 2.3, we get

$$\lim_{p_1,p_2\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^{2n}\right\} = 0. \qquad (2.482)$$

Thus, we obtain the following theorem.

Theorem 2.16. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Moreover,* $\psi_1(\tau), \psi_2(\tau)$ *are continuous nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral* (2.373) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1, i_2 = 0, 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean of degree $2n$*,* $n \in \mathbf{N}$ (see (2.482)) *is valid, where the Fourier coefficient* $C_{j_2j_1}$ *is defined by* (2.470) *and*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$)*; another notations are the same as in Theorem* 2.15.

Let us consider some other approaches close to the approaches outlined in this section.

Now we turn to multiple trigonometric Fourier series converging almost everywhere. Let us formulate the well known result from the theory of multiple trigonometric Fourier series.

Proposition 2.3 [126]. *Suppose that*

$$\int\limits_{[0,2\pi]^k} |f(x_1,\ldots,x_k)| \left(\log^+|f(x_1,\ldots,x_k)|\right)^k \log^+\log^+|f(x_1,\ldots,x_k)| \times$$

$$\times dx_1 \ldots dx_k < \infty. \tag{2.483}$$

Then, for the square partial sums

$$\sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(x_l)$$

of the multiple trigonometric Fourier series we have

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(x_l) = f(x_1, \ldots, x_k)$$

almost everywhere on $[0, 2\pi]^k$, *where* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of trigonometric functions in the space* $L_2([0, 2\pi])$, $\log^+ x = \log \max\{1,\ x\}$,

$$C_{j_k \ldots j_1} = \int\limits_{[0,2\pi]^k} f(x_1, \ldots, x_k) \prod_{l=1}^{k} \phi_{j_l}(x_l) dx_1 \ldots dx_k$$

is the Fourier coefficient of the function $f(x_1, \ldots, x_k)$.

Note that Proposition 2.3 can be reformulated for $[t, T]^k$ instead of $[0, 2\pi]^k$. If we tried to apply Proposition 2.3 in the proof of Theorem 2.10, then we would encounter the following difficulties. The right-hand side of (2.451) contains multiple integrals over hypercubes of various dimensions, namely over hypercubes $[t, T]^k$, $[t, T]^{k-1}$, etc. Obviously, the convergence almost everywhere on $[t, T]^k$ does not mean the convergence almost everywhere on $[t, T]^{k-1}$, $[t, T]^{k-2}$, etc. This means that we could not apply the Lebesgue's Dominated Convergence Theorem in the proof of Theorem 2.13 and thus we could not complete the proof of Theorem 2.10. Although multiple series are more convenient in terms of approximation than iterated series as in Theorem 2.10.

Suppose that the conditions of Theorem 2.16 are fulfilled. In the proof of Theorem 2.2 (see (2.58)) we deduced in particular that

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) = \frac{1}{2} \psi_1(t_1) \psi_2(t_1) = K^*(t_1, t_1), \tag{2.484}$$

where $t_1 \in (t, T)$, $C_{j_2 j_1}$ is defined by (2.470). This means that we can repeat the proof of Theorem 2.10 for the case $k = 2$ and apply the Lebesgue's Dominated Convergence Theorem in the formula (2.451), since Proposition 2.3 and (2.484) imply the convergence almost everywhere on $[t,T]^2$ and $[t,T]$ $(t_1=t_2\in[t,T])$ of the multiple trigonometric Fourier series

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad t_1,t_2\in[t,T]^2 \tag{2.485}$$

to the function $K^*(t_1,t_2)$ (the question of finding an integrable majorant for Lebesgue's Dominated Convergence Theorem is omitted here). So, we could obtain the particular case of Theorem 2.16.

Consider another possible way of the proof of Theorem 2.16, which is based on the function (2.48) and Theorem 2.10. The case $i_1\ne i_2$ follows from (2.479) and (2.480). Consider the case $i_1=i_2\ne 0$. We have $K^*(t_1,t_2)+K^*(t_2,t_1)=K'(t_1,t_2)$, where the functions $K'(t_1,t_2)$ and $K^*(t_1,t_2)$ are defined by (2.48) and (2.95) correspondingly. Note that the function $K'(t_1,t_2)$ is symmetric, i.e. $K'(t_1,t_2)=K'(t_2,t_1)$.

By analogy with (2.405) we get w. p. 1

$$J[K'/2]_{T,t}^{(2)}=\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}K'(\tau_{l_1},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}=$$

$$=\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\left(\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\right)K'(\tau_{l_1},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+$$

$$+\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}K'(\tau_{l_1},\tau_{l_1})\left(\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\right)^2=$$

$$=\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}\left(K'(\tau_{l_1},\tau_{l_2})+K'(\tau_{l_2},\tau_{l_1})\right)\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+$$

$$+\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}K'(\tau_{l_1},\tau_{l_1})\left(\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\right)^2=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}K'(\tau_{l_1},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+\frac{1}{2}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}K'(\tau_{l_1},\tau_{l_1})\left(\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\right)^2=$$

$$=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}+\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=$$

$$
=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}\stackrel{\sf def}{=}J^{*}[\psi^{(2)}]_{T,t},\tag{2.486}
$$

where we used the same notations as in (2.405).

Let us expand the function $K'(t_1,t_2)/2$ into a multiple (double) Fourier–Legendre series or trigonometric Fourier series in the square $[t,T]^2$ (see (2.57))

$$
\frac{1}{2}K'(t_1,t_2)=
$$

$$
=\frac{1}{2}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2)=
$$

$$
=\frac{1}{2}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+\right.
$$

$$
\left.+\int\limits_t^T\psi_1(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^T\psi_2(t_1)\phi_{j_1}(t_1)dt_1\right)dt_2\phi_{j_1}(t_1)\phi_{j_2}(t_2)=
$$

$$
=\frac{1}{2}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}+C_{j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2),\tag{2.487}
$$

where the series (2.487) converges to $K'(t_1,t_2)/2$ at any inner point of the square $[t,T]^2$ (see the proof of Theorem 2.2 for details).

In obtaining (2.487) we replaced the order of integration in the second iterated integral.

Using (2.486), (2.487), and the scheme of the proof of Theorem 2.10 ($k=2$), we can obtain the following relation (the question of finding an integrable majorant for Lebesgue's Dominated Convergence Theorem is omitted here)

$$
\lim_{p_1,p_2\to\infty}\mathsf{M}\left\{\left(J^{*}[\psi^{(2)}]_{T,t}-\frac{1}{2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}+C_{j_1j_2}\right)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^{2n}\right\}=0.\tag{2.488}
$$

Let us rewrite the sum on the left-hand side of (2.488) as two sums. Let us replace j_1 with j_2, j_2 with j_1, p_1 with p_2, and p_2 with p_1 in the second sum.

Thus, we get the statement of Theorem 2.16

$$\lim_{p_1,p_2\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^{2n}\right\} = 0.$$

Let us consider another approach. The following fact is well known [122].

Proposition 2.4. *Let* $\left\{x_{n_1,\ldots,n_k}\right\}_{n_1,\ldots,n_k=1}^{\infty}$ *be a multi-index sequence and let there exists the limit*

$$\lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k} < \infty.$$

Moreover, let there exists the limit

$$\lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = y_{n_1,\ldots,n_{k-1}} < \infty \quad \textit{for any} \quad n_1,\ldots,n_{k-1}.$$

Then there exists the iterated limit

$$\lim_{n_1,\ldots,n_{k-1}\to\infty}\ \lim_{n_k\to\infty} x_{n_1,\ldots,n_k}$$

and moreover,

$$\lim_{n_1,\ldots,n_{k-1}\to\infty}\ \lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = \lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k}.$$

Denote

$$C_{j_s\ldots j_1}(t_{s+1},\ldots,t_k) = \int\limits_{[t,T]^s} K(t_1,\ldots,t_k)\prod_{l=1}^{s}\phi_{j_l}(t_l)dt_1\ldots dt_s,$$

where $s = 1,\ldots,k-1$ and $K(t_1,\ldots,t_k)$ is defined by (1.6). For $s = k$ we suppose that $C_{j_k\ldots j_1}$ is defined by (1.8).

Consider the following Fourier series

$$\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{2.489}$$

$$\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3), \tag{2.490}$$

$$\ldots$$

$$\lim_{p_1,\ldots,p_{k-1}\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}} C_{j_{k-1}\ldots j_1}(t_k)\phi_{j_1}(t_1)\ldots\phi_{j_{k-1}}(t_{k-1}), \tag{2.491}$$

$$
\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k), \tag{2.492}
$$

where $t_1,\ldots,t_k\in[t,T]$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

The author does not know the answer to the question on the existence of limits (2.489)–(2.492) even for the case $p_1=\ldots=p_k$ and trigonometric Fourier series. Obviously, at least for the case $k=2$ and $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ the answere to the above question is positive for the Fourier–Legendre series as well as for the trigonometric Fourier series.

If we suppose that the limits (2.489)–(2.492) exist, then combining Proposition 2.4 and the proof of Theorem 2.11, we obtain

$$
K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)=
$$

$$
=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)= \tag{2.493}
$$

$$
=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)=
$$

$$
=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)=
$$

$$
=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{2.494}
$$

$$
=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{2.495}
$$

$$
=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{\infty}C_{j_4\ldots j_1}(t_5,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_3}(t_4)= \tag{2.496}
$$

$$
=\ldots=
$$

$$
= \lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \phi_{j_1}(t_1)\ldots \phi_{j_k}(t_k). \tag{2.497}
$$

Note that the transition from (2.494) to (2.495) is based on (2.493) and the proof of Theorem 2.11. The transition from (2.495) to (2.496) is based on (2.494) and the proof of Theorem 2.11.

Using (2.497), we could get the version of Theorem 2.10 with multiple series instead of iterated ones (see Hypothesis 2.3, Sect. 2.5).

2.4.3 Refinement of Theorems 2.10 and 2.14 for Iterated Stratonovich Stochastic Integrals of Multiplicities 2 and 3 ($i_1, i_2, i_3 = 1,\ldots,m$). The Case of Mean-Square Convergence

In this section, it will be shown that the upper limits in Theorems 2.10 and 2.14 (the cases $k = 2$, $k = 3$ and $n = 1$) can be replaced by the usual limits.

Theorem 2.17 [34]. *Suppose that every $\psi_l(\tau)$ $(l = 1, 2, 3)$ is twice continuously differentiable function at the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t, T])$. Then, the iterated Stratonovich stochastic integrals $J^*[\psi^{(2)}]_{T,t}$ and $J^*[\psi^{(3)}]_{T,t}$ $(i_1, i_2, i_3 = 1,\ldots,m)$ defined by (2.373) are expanded into the converging in the mean-square sense iterated series*

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} = 0, \tag{2.498}
$$

$$
\lim_{p_2\to\infty}\lim_{p_1\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_2=0}^{p_2}\sum_{j_1=0}^{p_1} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} = 0, \tag{2.499}
$$

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} = 0, \tag{2.500}
$$

$$
\lim_{p_3\to\infty}\lim_{p_2\to\infty}\lim_{p_1\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_3=0}^{p_3}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} = 0, \tag{2.501}
$$

where

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d{\bf f}_s^{(i)}\quad (i=1,\ldots,m,\quad j=0,1,\ldots)$$

are independent standard Gaussian random variables for various i *or* j *and* $C_{j_2j_1}$, $C_{j_3j_2j_1}$ *are defined by* (2.377) *and* (2.375).

Proof. We will prove the equalities (2.498) and (2.500) (the equalities (2.499) and (2.501) can be proved similarly using the expansion (2.458) instead of the expansion (2.380)).

From (2.405) we have w. p. 1

$$J^{*}[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=J[R_{p_1p_2}]_{T,t}^{(2)}=$$

$$=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}R_{p_1p_2}(t_1,t_2)d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_1}^{(i_1)}+$$

$$+{\bf 1}_{\{i_1=i_2\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1, \tag{2.502}$$

where we used the same notations as in (2.405).

Uning (2.502), we obtain

$${\sf M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\}=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+\int\limits_t^T\int\limits_t^{t_1}R_{p_1p_2}^2(t_1,t_2)dt_2dt_1+$$

$$+{\bf 1}_{\{i_1=i_2\}}\left(2\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\right)=$$

$$=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+\int\limits_t^T\int\limits_{t_2}^{T}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+$$

$$+{\bf 1}_{\{i_1=i_2\}}\left(\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\right.$$

$$
+\int\limits_t^T\int\limits_{t_1}^T R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_2dt_1\Biggr) + \mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2 =
$$

$$
=\int\limits_{[t,T]^2} R^2_{p_1p_2}(t_1,t_2)dt_1dt_2+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\Biggl(\int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+
$$

$$
+\int\limits_t^T\int\limits_{t_2}^T R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2\Biggr) + \mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2 =
$$

$$
=\int\limits_{[t,T]^2} R^2_{p_1p_2}(t_1,t_2)dt_1dt_2+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_{[t,T]^2} R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2 + \left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\right). \tag{2.503}
$$

Since the integrals on the right-hand side of (2.503) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$
\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty} R_{p_1p_2}(t_1,t_2)=0 \quad \text{when} \quad (t_1,t_2)\in(t,T)^2,
$$

where the left-hand side is bounded on the boundary of $[t,T]^2$ (see (2.404)).

Then, applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem (see the choice of integrable majorants in the proof of Theorem 2.10) and taking into account (2.381), (2.382), and (2.413), we obtain

$$
\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\int\limits_{[t,T]^2} R^2_{p_1p_2}(t_1,t_2)dt_1dt_2=0, \tag{2.504}
$$

$$
\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\int\limits_{[t,T]^2} R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2=0, \tag{2.505}
$$

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0. \qquad (2.506)
$$

The relations (2.503)–(2.506) imply the following equality

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} = 0.
$$

The formula (2.498) is proved.

Let us prove the relation (2.500). After replacement of the integration order in the iterated Itô stochastic integrals from (2.416) (see Chapter 3) [1]-[17], [77], [123], [124] we get w. p. 1

$$
J^*[\psi^{(3)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J[R_{p_1p_2p_3}]_{T,t}^{(3)} =
$$

$$
= \int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_2,t_2,t_3)dt_2\right)d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_1,t_2,t_2)dt_2\right)d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_3,t_2,t_3)dt_3\right)d\mathbf{f}_{t_2}^{(i_2)}. \qquad (2.507)
$$

Let us calculate the second moment of $J[R_{p_1p_2p_3}]_{T,t}^{(3)}$ using (2.507). We have

$$
\mathsf{M}\left\{\left(J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right)^2\right\}=
$$

$$
=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\left(\sum\limits_{(t_1,t_2,t_3)}R_{p_1p_2p_3}^2(t_1,t_2,t_3)\right)dt_1dt_2dt_3+ \qquad (2.508)
$$

$$
+2\left(\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(1)}(t_1,t_2,t_3)dt_1dt_2dt_3+\right.
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(2)}(t_1,t_2,t_3)dt_1dt_2dt_3+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(3)}(t_1,t_2,t_3)dt_1dt_2dt_3+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(4)}(t_1,t_2,t_3)dt_1dt_2dt_3\right)+
$$

$$
+\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{i_1=i_2\}}R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_2,t_3)+\right.
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_3,t_2,t_2)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}R_{p_1p_2p_3}(t_1,t_3,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)+
$$

$$
+2\cdot\mathbf{1}_{\{i_1=i_2=i_3\}}\Bigl(R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_2)+
$$

$$
+R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_2)+
$$

$$+R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)\Biggr)\Biggr)dt_1dt_2dt_3, \qquad (2.509)$$

where permutation (t_1,t_2,t_3) when summing in (2.508) are performed only in the value $R^2_{p_1p_2p_3}(t_1,t_2,t_3)$ and the functions $G^{(i)}_{p_1p_2p_3}(t_1,t_2,t_3)$ $(i=1,\ldots,4)$ are defined by the following relations

$$G^{(1)}_{p_1p_2p_3}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_2,t_1,t_3)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_3,t_1,t_2)+$$

$$+R_{p_1p_2p_3}(t_2,t_3,t_1)R_{p_1p_2p_3}(t_3,t_2,t_1),$$

$$G^{(2)}_{p_1p_2p_3}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_3,t_1,t_2),$$

$$G^{(3)}_{p_1p_2p_3}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_1,t_3,t_2)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_3,t_2,t_1)R_{p_1p_2p_3}(t_3,t_1,t_2),$$

$$G^{(4)}_{p_1p_2p_3}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_3,t_1,t_2)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_2,t_1,t_3)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_3,t_1)R_{p_1p_2p_3}(t_3,t_1,t_2).$$

Further (see (1.38)),

$$
\int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \left(\sum_{(t_1,t_2,t_3)} R^2_{p_1p_2p_3}(t_1,t_2,t_3) \right) dt_1 dt_2 dt_3 =
$$

$$
= \int\limits_{[t,T]^3} R^2_{p_1p_2p_3}(t_1,t_2,t_3) dt_1 dt_2 dt_3. \qquad (2.510)
$$

We will say that the function $\Phi(t_1,t_2,t_3)$ is symmetric if

$$
\Phi(t_1,t_2,t_3) = \Phi(t_1,t_3,t_2) = \Phi(t_2,t_1,t_3) = \Phi(t_2,t_3,t_1) =
$$

$$
= \Phi(t_3,t_1,t_2) = \Phi(t_3,t_2,t_1).
$$

For the symmetric function $\Phi(t_1,t_2,t_3)$, we have

$$
\int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \left(\sum_{(t_1,t_2,t_3)} \Phi(t_1,t_2,t_3) \right) dt_1 dt_2 dt_3 =
$$

$$
= 6 \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) dt_1 dt_2 dt_3 =
$$

$$
= \int\limits_{[t,T]^3} \Phi(t_1,t_2,t_3) dt_1 dt_2 dt_3. \qquad (2.511)
$$

The relation (2.511) implies that

$$
\int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) dt_1 dt_2 dt_3 = \frac{1}{6} \int\limits_{[t,T]^3} \Phi(t_1,t_2,t_3) dt_1 dt_2 dt_3. \qquad (2.512)
$$

It is easy to check that the functions $G^{(i)}_{p_1p_2p_3}(t_1,t_2,t_3)$ $(i = 1,\ldots,4)$ are symmetric. Using this property as well as (2.509), (2.510), and (2.512), we obtain

$$
\mathsf{M}\left\{ \left(J[R_{p_1p_2p_3}]^{(3)}_{T,t} \right)^2 \right\} = \int\limits_{[t,T]^3} R^2_{p_1p_2p_3}(t_1,t_2,t_3) dt_1 dt_2 dt_3 +
$$

$$
+\frac{1}{3}\int\limits_{[t,T]^3}\Biggl(\mathbf{1}_{\{i_1=i_2\}}G^{(1)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}G^{(2)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}G^{(3)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}}G^{(4)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3\Biggr)dt_1dt_2dt_3+
$$

$$
+\int\limits_{[t,T]^3}\Biggl(\mathbf{1}_{\{i_1=i_2\}}R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_2,t_3)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_3,t_2,t_2)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}R_{p_1p_2p_3}(t_1,t_3,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)+
$$

$$
+2\cdot\mathbf{1}_{\{i_1=i_2=i_3\}}\Biggl(R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_2)+
$$

$$
+R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_2)+
$$

$$
+R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)\Biggr)\Biggr)dt_1dt_2dt_3. \qquad (2.513)
$$

Since the integrals on the right-hand side of (2.513) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}R_{p_1p_2p_3}(t_1,t_2,t_3)=0\quad\text{when}\quad(t_1,t_2,t_3)\in(t,T)^3,
$$

where the left-hand side is bounded on the boundary of $[t,T]^3$ (see (2.404)).

Using (2.424) and applying three times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}$) the Lebesgue's Dominated Convergence Theorem (see the choice of integrable majorants in the proof of Theorem 2.10) in the equality (2.513), we obtain

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2p_3}]^{(3)}_{T,t}\right)^2\right\}=0.
$$

The relation (2.500) is proved. Theorem 2.17 is proved.

Developing the approach used in the proof of Theorem 2.17, we can in principle prove the following formulas

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}=0,$$

$$\lim_{p_k\to\infty}\ldots\lim_{p_1\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}=0,$$

which are correct under the conditions of Theorem 2.10 for $i_1,\ldots,i_k=1,\ldots,m.$

2.5 The Hypotheses on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k\in\mathbf{N}$) Based on Theorem 1.1

In this section, on the base of the presented theorems (see Sect. 1.1.3, 2.1–2.4) we formulate 3 hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k\in\mathbf{N}$) based on generalized multiple Fourier series converging in $L_2([t,T]^k)$. The considered expansions contain only one operation of the limit transition and substantially simpler than their analogues for iterated Itô stochastic integrals (Theorem 1.1).

Taking into account (1.44) and Theorems 2.1–2.10, 2.14, and 2.17, let us formulate the following hypotheses on expansions of iterated Stratonovich stochastic integrals of multiplicity k ($k\in\mathbf{N}$).

Hypothesis 2.1 [8]-[17], [39]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of multiplicity k*

$$I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{2.514}$$

the following expansion

$$I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{2.515}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau,$ $\lambda_l=0$ *if* $i_l=0$ *and* $\lambda_l=1$ *if* $i_l=1,\ldots,m$ $(l=1,\ldots,k)$*.*

Hypothesis 2.1 allows to approximate the iterated Stratonovich stochastic integral $I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}$ by the sum

$$I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)p}=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},\tag{2.516}$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}-I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)p}\right)^2\right\}=0.$$

The integrals (2.514) will be used in the Taylor–Stratonovich expansion (see Chapter 4). It means that the approximations (2.516) may be very useful for the construction of high-order strong numerical methods for Itô SDEs (see Chapter 4 for details).

The expansion (2.515) contains only one operation of the limit transition and by this reason is convenient for approximation of iterated Stratonovich stochastic integrals.

Let us consider the more general hypothesis than Hypothesis 2.1.

Hypothesis 2.2 [14]-[17], [39]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral* *of multiplicity* k

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \tag{2.517}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Hypothesis 2.2 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{2.518}$$

where

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\} = 0.$$

Let us consider the more general hypothesis than Hypotheses 2.1 and 2.2.

Hypothesis 2.3 [14]-[17], [39]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the* *space* $L_2([t,T])$. *Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth non-random function on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of multiplicity* k

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{2.519}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Let us consider the idea of the proof of Hypotheses 2.1–2.3.

According to (1.10), we have

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)}=J[\psi^{(k)}]_{T,t}+$$

$$+\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}\quad \text{w. p. 1}, \tag{2.520}$$

where notations are the same as in (1.10).

From (2.520) and Theorem 2.12 it follows that

$$J^*[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)} \qquad (2.521)$$

if

$$\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =$$

$$= \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)} \quad \text{w. p. 1},$$

where notations are the same as in Theorems 1.1 and 2.12.

Note that from Theorem 1.1 for pairwise different $i_1,\ldots,i_k$ $(i_1,\ldots,i_k = 0,1,\ldots,m)$ we obtain (2.521) (compare (1.44) and (2.521)).

In the case $p_1 = \ldots = p_k = p$ and $\psi_l(s)\equiv 1$ $(l = 1,\ldots,k)$ we obtain from (2.521) the statement of Hypothesis 2.1 (see (2.515)).

If $p_1 = \ldots = p_k = p$ and every $\psi_l(s)$ $(l = 1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$, then we obtain from (2.521) the statement of Hypothesis 2.2 (see (2.517)).

In the case when every $\psi_l(s)$ $(l = 1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$ we obtain from (2.521) the statement of Hypothesis 2.3 (see (2.519)).

2.6 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 and 4. Combained Approach Based on Generalized Multiple and Iterated Fourier series. Another Proof of Theorems 2.8 and 2.9

In this section, we develop the approach from Sect. 2.1.3 for iterated Stratonovich stochastic integrals of multiplicities 3 and 4. We call this approach the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series converging in the sense of norm in $L_2([t,T]^k)$,

$k=3,4$. The mean-square convergence of the second part is proved on the base of (1.46), (2.10), Parseval's equality, and generalized Fourier series converging pointwise. At that, we do not use iterated Itô stochastic integrals as a tool of the proof and directly consider iterated Stratonovich stochastic integrals.

2.6.1 Another Proof of Theorem 2.8

Let us consider (2.402) for $k=3$, $p_1=p_2=p_3=p$, and $i_1,i_2,i_3=1,\ldots,m$

$$J^*[\psi^{(3)}]_{T,t}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+J[R_{ppp}]_{T,t}^{(3)}\quad \text{w. p. 1},\qquad (2.522)$$

where

$$J[R_{ppp}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{ppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)},$$

$$R_{ppp}(t_1,t_2,t_3)\stackrel{\text{def}}{=}K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3),$$

$$K^*(t_1,t_2,t_3)=\prod_{l=1}^{3}\psi_l(t_l)\Biggl(\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2<t_3\}}+$$
$$+\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2=t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2=t_3\}}\Biggr).$$

Using (2.416), we obtain w. p. 1

$$J[R_{ppp}]_{T,t}^{(3)}=R_{T,t}^{(1)ppp}+R_{T,t}^{(2)ppp},$$

where

$$R_{T,t}^{(1)ppp}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_2,t_3)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_3,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_2)}+$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_2,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_2,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_3,t_2,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_3,t_1,t_2)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_1)},
$$

$$
R_{T,t}^{(2)ppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_2,t_2,t_3)dt_2d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_2,t_3,t_2)dt_2d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_2,t_2)dt_2d\mathbf{f}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3.
$$

We have

$$
\mathsf{M}\left\{\left(J[R_{ppp}]_{T,t}^{(3)}\right)^2\right\}\le 2\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}+2\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\}. \quad (2.523)
$$

Now, using standard estimates for moments of stochastic integrals [100], we obtain the following inequality

$$
\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\} \le
$$

$$
\le 6\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^2+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^2+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^2+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^2\Biggr)dt_1dt_2dt_3 =
$$

$$
= 6\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2 dt_1dt_2dt_3.
$$

We have

$$
\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2 dt_1dt_2dt_3 =
$$

$$
=\int\limits_{[t,T]^3}\left(K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1dt_2dt_3 =
$$

$$
=\int\limits_{[t,T]^3}\left(K(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1dt_2dt_3,
$$

where

$$
K(t_1,t_2,t_3)=\begin{cases}\psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1<t_2<t_3\\ & \\ 0, & \text{otherwise}\end{cases},\qquad t_1,t_2,t_3\in[t,T].
$$

So, we get

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\} \le
$$

$$\le 6\lim_{p\to\infty}\int\limits_{[t,T]^3}\left(K(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1dt_2dt_3=$$

$$=0, \qquad (2.524)$$

where $K(t_1,t_2,t_3)\in L_2([t,T]^3)$.

After replacement of the integration order in the iterated Itô stochastic integrals from $R_{T,t}^{(2)ppp}$ [1]-[17], [77], [123], [124] (see Chapter 3) we obtain w. p. 1

$$R_{T,t}^{(2)ppp}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_2,t_3)dt_2d\mathbf{f}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_2,t_2)dt_2d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_3,t_2)dt_2d\mathbf{f}_{t_3}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3\right)=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_2,t_2,t_1)dt_2d\mathbf{f}_{t_1}^{(i_3)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_2,t_2,t_1)dt_2d\mathbf{f}_{t_1}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_1,t_2,t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_1,t_2,t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_2,t_1,t_2)dt_2d\mathbf{f}_{t_1}^{(i_2)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_2,t_1,t_2)dt_2d\mathbf{f}_{t_1}^{(i_2)}\right)=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_2,t_2,t_3)dt_2\right)d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_1,t_2,t_2)dt_2\right)d\mathbf{f}_{t_1}^{(i_1)}+$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_3,t_2,t_3)dt_3\right)d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^T\left(\left(\frac{1}{2}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\right)\psi_1(t_2)\psi_2(t_2)\psi_3(t_3)-\right.
$$

$$
\left.-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_2)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)dt_2d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\left(\left(\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_1(t_1)\psi_2(t_2)\psi_3(t_2)-\right.
$$

$$
\left.-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_2)\right)dt_2d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\left(\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\psi_1(t_3)\psi_2(t_2)\psi_3(t_3)-\right.
$$

$$
\left.-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)dt_3d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\phi_{j_3}(t_3)\right)d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\phi_{j_1}(t_1)\right)d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T(-1)\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_1j_2j_1}\phi_{j_2}(t_2)d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}.
$$

From the proof of Theorem 2.8 we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\}\le 3\left(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\right.\right.\right.
$$

$$
\left.\left.-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}-\right.\right.
$$

$$
\left.\left.\left.-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\right)\to 0 \qquad (2.525)
$$

if $p\to\infty$. From (2.522)–(2.525) we obtain the expansion (2.272). Theorem 2.8 is proved.

2.6.2 Another Proof of Theorem 2.9

Let us consider (2.402) for $k=4$, $p_1=\ldots=p_4=p$, and $\psi_1(s),\ldots,\psi_4(s)\equiv 1$

$$
\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+J[R_{pppp}]_{T,t}^{(4)}\quad \text{w. p. 1},\quad (2.526)
$$

where

$$
J[R_{pppp}]_{T,t}^{(4)}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},
$$

$$R_{pppp}(t_1,t_2,t_3,t_4) \stackrel{\sf def}{=} K^*(t_1,t_2,t_3,t_4)-$$

$$-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\phi_{j_4}(t_4), \tag{2.527}$$

$$K^*(t_1,t_2,t_3,t_4) \stackrel{\sf def}{=} \prod_{l=1}^{3}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right) =$$

$$= \mathbf{1}_{\{t_1<t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2=t_3<t_4\}}+$$

$$+\frac{1}{4}\mathbf{1}_{\{t_1=t_2=t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2<t_3=t_4\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2<t_3=t_4\}}+$$

$$+\frac{1}{4}\mathbf{1}_{\{t_1<t_2=t_3=t_4\}}+\frac{1}{8}\mathbf{1}_{\{t_1=t_2=t_3=t_4\}}.$$

We have

$$J[R_{pppp}]_{T,t}^{(4)} = \sum_{i=0}^{7} R_{T,t}^{(i)pppp} \quad \text{w. p. 1}, \tag{2.528}$$

where

$$R_{T,t}^{(0)pppp} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\Biggl(R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\times$$

$$\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr),$$

where permutations (l_1,l_2,l_3,l_4) when summing are performed only in the expression, which is enclosed in parentheses,

$$R_{T,t}^{(1)pppp} = \mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_3,l_1=0\\ l_1\ne l_3,l_1\ne l_4,l_3\ne l_4}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(2)pppp} = \mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(3)pppp} = \mathbf{1}_{\{i_1=i_4\neq 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_2, l_1\neq l_3, l_2\neq l_3}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(4)pppp} = \mathbf{1}_{\{i_2=i_3\neq 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_2,l_1=0\\ l_1\neq l_2, l_1\neq l_4, l_2\neq l_4}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(5)pppp} = \mathbf{1}_{\{i_2=i_4\neq 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_2, l_1\neq l_3, l_2\neq l_3}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(6)pppp} = \mathbf{1}_{\{i_3=i_4\neq 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_2, l_1\neq l_3, l_2\neq l_3}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3},$$

$$R_{T,t}^{(7)pppp} = \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}} \lim_{N\to\infty} \sum_{\substack{l_4,l_2=0\\ l_2\neq l_4}}^{N-1} R_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}} \lim_{N\to\infty} \sum_{\substack{l_4,l_2=0\\ l_2\neq l_4}}^{N-1} R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}} \lim_{N\to\infty} \sum_{\substack{l_4,l_2=0\\ l_2\neq l_4}}^{N-1} R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}.$$

From (2.526) and (2.528) it follows that Theorem 2.9 will be proved if

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(i)pppp}\right)^2\right\} = 0, \quad i=0,1,\ldots,7.$$

We have (see (1.19), (1.24))

$$R_{T,t}^{(0)pppp} = \int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2} \sum_{(t_1,t_2,t_3,t_4)} \left(R_{pppp}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right),$$

where permutations (t_1,t_2,t_3,t_4) when summing are performed only in the expression, which is enclosed in parentheses.

From the other hand (see (1.24), (1.25))

$$R_{T,t}^{(0)pppp}=\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}R_{pppp}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)},$$

where permutations (t_1,t_2,t_3,t_4) when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation (t_1,t_2,t_3,t_4), then i_r swapped with i_q in the permutation (i_1,i_2,i_3,i_4).

So, we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{(0)pppp}\right)^2\right\}\le 24\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\left(R_{pppp}(t_1,t_2,t_3,t_4)\right)^2dt_1dt_2dt_3dt_4=$$

$$=24\int\limits_{[t,T]^4}\left(R_{pppp}(t_1,t_2,t_3,t_4)\right))^2dt_1dt_2dt_3dt_4\ \to 0$$

if $p\to\infty$, $K^*(t_1,t_2,t_3,t_4)\in L_2([t,T]^4)$ (see (2.527)).

Let us consider $R_{T,t}^{(1)pppp}$

$$R_{T,t}^{(1)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_1\ne l_3,l_1\ne l_4,l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}\left(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}}+\right.$$

$$+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_3}<\tau_{l_4}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}=\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_3}=\tau_{l_4}\}}-$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\right)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}} -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4}) \Biggr) \Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}} -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4}) \Biggr) \Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1} \Biggl(0 -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4}) \Biggr) \Delta\tau_{l_1}\Delta\tau_{l_4} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} - \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \right) +
$$

$$
+ \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} \quad \text{w. p. 1.}
$$

When proving Theorem 2.9 we have proved that

$$
\lim_{p\to\infty} \sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} = \frac{1}{4}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} +
$$

$$
+ \mathbf{1}_{\{i_3=i_4\ne 0\}} \frac{1}{4}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2 \quad \text{w. p. 1.}
$$

Then

$$\lim_{p\to\infty}\ \mathsf{M}\left\{\left(R_{T,t}^{(1)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(2)pppp}$

$$R_{T,t}^{(2)pppp}=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}\left(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-\right.$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\right)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\tau_{l_4}=$$

$$=-\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p}C_{j_4j_1j_4j_1}\quad \text{w. p. 1}.$$

When proving Theorem 2.9 we have proved that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}=0\quad \text{w. p. 1},$$

$$\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p}C_{j_4j_1j_4j_1}=0.$$

Then

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(2)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(3)pppp}$

$$R_{T,t}^{(3)pppp}=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-$$

$$-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_3})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\tau_{l_3}=$$

$$=-\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_4,j_3,j_2=0}^{p}C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_4,j_2=0}^{p}C_{j_4j_2j_2j_4}\quad \text{w. p. 1.}$$

When proving Theorem 2.9 we have proved that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=0\quad \text{w. p. 1},$$

$$\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0.$$

Then

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(3)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(4)pppp}$

$$R_{T,t}^{(4)pppp}=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}+$$

$$+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-$$

$$-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-$$

$$-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-$$

$$
\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\right)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_4})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}\quad \text{w. p. 1.}
$$

When proving Theorem 2.9 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(4)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(5)pppp}$

$$
R_{T,t}^{(5)pppp}=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1}\left(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-\right.
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=-\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_3})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_3}=
$$

$$
=-\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}\quad \text{w. p. 1.}
$$

When proving Theorem 2.9 we have proved that

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=0\quad \text{w. p. 1,}
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0.
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(5)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(6)pppp}$

$$
R_{T,t}^{(6)pppp}=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_3}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \left(\frac{1}{2}\int_t^T\int_t^{t_3}\int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 - \sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right) -
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1) \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Delta\tau_{l_1}\Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \left(\frac{1}{2}\int_t^T\int_t^{t_3}\int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 - \sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right) +
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}} \Biggl(\frac{1}{2}\int_t^T\int_t^{t_3}\int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int_t^T\int_t^{t_3} dt_1 dt_3 -
$$

$$
-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\Biggr) +
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\right)\quad \text{w. p. 1.}
$$

When proving Theorem 2.9 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(6)pppp}\right)^2\right\}=0.
$$

Finally, let us consider $R_{T,t}^{(7)pppp}$

$$
R_{T,t}^{(7)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1} G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{4}\int_t^T\int_t^{t_4} dt_2dt_4-\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}.
$$

When proving Theorem 2.9 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} = \frac{1}{4}\int_t^T\int_t^{t_4} dt_2dt_4, \tag{2.529}
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1} = 0, \quad \lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4} = 0. \tag{2.530}
$$

Then

$$\lim_{p\to\infty} R_{T,t}^{(7)pppp} = 0.$$

Theorem 2.9 is proved.

2.7 Modification of Theorems 2.1, 2.8, and 2.9 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4 and Wong–Zakai Type Theorems

2.7.1 Modification of Theorem 2.1 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 2

Let us prove the following theorem.

Theorem 2.18. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{s,t} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \qquad (2.531)$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ (s is fixed),

$$C_{j_2j_1}(s) = \int\limits_t^{s} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2, \qquad (2.532)$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

The condition of continuity of the functions $\psi_1(\tau), \psi_2(\tau)$ is related to the definition (2.3) of the Stratonovich stochastic integral that we use.

Proof. The case $s = T$ is considered in Theorems 2.1–2.3. Below we consider the case $s \in (t, T)$. In accordance to the standard relations between Stratonovich and Itô stochastic integrals (see (2.4) and (2.5)) we have w. p. 1

$$J^*[\psi^{(2)}]_{s,t} = J[\psi^{(2)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1, \qquad (2.533)$$

where $\psi_1(\tau), \psi_2(\tau)$ are continuous functions on $[t, T]$, $s \in (t, T)$ (s is fixed), $\mathbf{1}_A$ is the indicator of the set A.

From the other side according to (1.255), we obtain

$$J[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}} \lim_{p_1,p_2\to\infty} \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1}(s). \qquad (2.534)$$

From (2.533) and (2.534) it follows that Theorem 2.18 will be proved if

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}(s), \qquad (2.535)$$

where $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$.

We have (see Sect. 2.1.4)

$$\frac{1}{2}\int\limits_t^T \bar{\psi}_1(\tau)\bar{\psi}_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_t^T \bar{\psi}_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} \bar{\psi}_1(t_1)\phi_j(t_1)dt_1dt_2, \qquad (2.536)$$

where $\bar{\psi}_1(\tau), \bar{\psi}_2(\tau) \in L_2([t, T])$.

Suppose that

$$\bar{\psi}_1(\tau) = \psi_1(\tau)\mathbf{1}_{\{\tau<s\}}, \quad \bar{\psi}_2(\tau) = \psi_2(\tau)\mathbf{1}_{\{\tau<s\}}, \qquad (2.537)$$

where $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, $s \in (t,T)$ (s is fixed).

Combining (2.536) and (2.537), we get

$$\frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)\mathbf{1}_{\{\tau<s\}}d\tau = \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\mathbf{1}_{\{t_2<s\}}\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\mathbf{1}_{\{t_1<s\}}\phi_j(t_1)dt_1dt_2,$$

i.e.

$$\frac{1}{2}\int\limits_t^s \psi_1(\tau)\psi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_t^s \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2.$$

The equality (2.535) is proved. Theorem 2.18 is proved.

Let us reformulate Theorem 2.18 in terms on the convergence of the solution of system of ordinary differential equations (ODEs) to the solution of system of Stratonovich SDEs (the so-called Wong–Zakai type theorem).

By analogy with (2.1477) for $k = 2$, $i_1, i_2 = 1, \ldots, m$, and $s \in (t,T]$ (s is fixed) we obtain

$$\int\limits_t^s \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \qquad (2.538)$$

where $p_1, p_2 \in \mathbf{N}$ and $d\mathbf{f}_{\tau}^{(i)p}$ is defined by (2.1474); another notations are the same as in Theorem 2.18.

The iterated Riemann–Stiltjes integrals

$$Y_{s,t}^{(i_1i_2)p_1p_2} = \int\limits_t^s \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2},$$

$$X_{s,t}^{(i_1)p_1} = \int\limits_t^s \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}$$

are the solution of the following system of ODEs

$$\begin{cases} dY_{s,t}^{(i_1i_2)p_1p_2} = \psi_2(s)X_{s,t}^{(i_1)p_1}d\mathbf{f}_s^{(i_2)p_2}, & Y_{t,t}^{(i_1i_2)p_1p_2} = 0 \\ \\ dX_{s,t}^{(i_1)p_1} = \psi_1(s)d\mathbf{f}_s^{(i_1)p_1}, & X_{t,t}^{(i_1)p_1} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$Y_{s,t}^{(i_1 i_2)} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)},$$

$$X_{s,t}^{(i_1)} = \int\limits_t^{*s} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ \\ dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* \, d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorem 2.18 and (1.254) we obtain the following theorem.

Theorem 2.19 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t, T])$. *Moreover,* $\psi_1(\tau), \psi_2(\tau)$ *are continuous functions on* $[t, T]$. *Then for any fixed* s $(s \in (t, T])$

$$\underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \; Y_{s,t}^{(i_1 i_2) p_1 p_2} = Y_{s,t}^{(i_1 i_2)}, \quad \underset{p_1 \to \infty}{\text{l.i.m.}} \; X_{s,t}^{(i_1) p_1} = X_{s,t}^{(i_1)}.$$

2.7.2 Modification of Theorem 2.8 for the Case of Integration Interval $[t, s]$ $(s \in (t, T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 3

Let us prove the following theorem.

Theorem 2.20 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t, T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{s,t} = \int\limits_t^{*s} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{s,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $s \in (t,T]$ *(*s *is fixed),*

$$C_{j_3j_2j_1}(s) = \int\limits_t^s \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

Proof. The case $s=T$ is considered in Theorem 2.8. Below we consider the case $s\in(t,T)$. First, let us consider the case of Legendre polynomials. From (1.256) for the case $p_1=p_2=p_3=p$ and standard relations between Itô and Stratonovich stochastic integrals we conclude that Theorem 2.20 will be proved if w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_1j_1}(s)\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^s \psi_3(\tau)\int\limits_t^{\tau}\psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_\tau^{(i_3)}, \tag{2.539}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^s \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau, \tag{2.540}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1j_3j_1}(s)\zeta_{j_3}^{(i_2)} = 0. \tag{2.541}$$

The proof of the formulas (2.539), (2.541) is absolutely similar to the proof of the formulas (2.273), (2.275). It is only necessary to replace the interval of integration $[t,T]$ by $[t,s]$ in the proof of the formulas (2.273), (2.275) and use Theorem 1.11 instead of Theorem 1.1. Also in the case (2.541) it is necessary to use the estimate (1.211).

Let us prove (2.540). Using Theorem 1.11 for $k=2$ (see (1.255) for $i_1 = 1,\ldots,m,\ i_2=0$), we obtain w. p. 1 (also see (2.699), (2.700))

$$\frac{1}{2}\int\limits_t^s \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau = \frac{1}{2}\ \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} C_{j_1}^*(s)\zeta_{j_1}^{(i_1)},$$

where

$$C^*_{j_1}(s)=\int\limits_t^s\psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)\phi_{j_1}(s_1)ds_1d\tau=$$

$$=\int\limits_t^s\psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau ds_1. \qquad (2.542)$$

We have

$$E_p'(s)\stackrel{\sf def}{=}{\sf M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{p}C^*_{j_1}(s)\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$={\sf M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s)-\frac{1}{2}C^*_{j_1}(s)\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s)-\frac{1}{2}C^*_{j_1}(s)\right)^2, \qquad (2.543)$$

$$C_{j_3j_3j_1}(s)=\int\limits_t^s\psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_t^{\tau}\psi_1(s_1)\phi_{j_1}(s_1)ds_1d\tau d\theta=$$

$$=\int\limits_t^s\psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1. \qquad (2.544)$$

From (2.542)–(2.544) we obtain

$$E_p'(s)=\sum_{j_1=0}^{p}\left(\int\limits_t^s\psi_1(s_1)\phi_{j_1}(s_1)\left(\sum_{j_3=0}^{p}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau-\right.\right.$$

$$\left.\left.-\frac{1}{2}\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau\right)ds_1\right)^2. \qquad (2.545)$$

Let us show that

$$\sum_{j_3=0}^{\infty}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau=\frac{1}{2}\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau. \qquad (2.546)$$

Using (2.536) and Fubini's Theorem, we have

$$\frac{1}{2}\int\limits_t^T \bar{\psi}_1(\tau)\bar{\psi}_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_t^T \bar{\psi}_1(t_1)\phi_j(t_1)\int\limits_{t_1}^T \bar{\psi}_2(t_2)\phi_j(t_2)dt_2dt_1, \quad (2.547)$$

where $\bar{\psi}_1(\tau), \bar{\psi}_2(\tau) \in L_2([t,T])$.

Suppose that

$$\bar{\psi}_1(\tau) = \psi_2(\tau)\mathbf{1}_{\{s_1<\tau<s\}}, \quad \bar{\psi}_2(\tau) = \psi_3(\tau)\mathbf{1}_{\{\tau<s\}}. \quad (2.548)$$

Using (2.547) and (2.548), we get (2.546). Combining (2.545) and (2.546), we obtain

$$E_p'(s) = \sum_{j_1=0}^{p}\left(\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1\right)^2 \le$$

$$\le K\sum_{j_1=0}^{p}\left(\int\limits_t^s |\phi_{j_1}(s_1)|\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| ds_1\right)^2, \quad (2.549)$$

where constant K does not depend on p.

Let us estimate the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right|.$$

Note that, by virtue of the additivity property of the integral, we obtain

$$\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau =$$

$$= \int\limits_t^s \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta -$$

$$- \int\limits_t^{s_1} \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta -$$

$$
-\int\limits_{s_1}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta\int\limits_{t}^{s_1}\psi_2(\tau)\phi_{j_3}(\tau)d\tau.
$$

Further, we have

$$
\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| \le
$$

$$
\le\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t}^{s}\psi_3(\theta)\phi_{j_3}(\theta)\int\limits_{t}^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta\right|+
$$

$$
+\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t}^{s_1}\psi_3(\theta)\phi_{j_3}(\theta)\int\limits_{t}^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta\right|+
$$

$$
+\sum_{j_3=p+1}^{\infty}\left|\int\limits_{s_1}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta\int\limits_{t}^{s_1}\psi_2(\tau)\phi_{j_3}(\tau)d\tau\right|. \qquad (2.550)
$$

Applying the estimate (2.648) (see Sect. 2.9), we can write

$$
\left|\sum_{j_1=p+1}^{\infty}C_{j_1j_1}(s)\right|\le\frac{C}{p}\left(1+\frac{1}{\left(1-(z(s))^2\right)^{1/4}}\right), \qquad (2.551)
$$

where $s\in(t,T)$, constant C does not depend on p, $z(s)$ has the form (2.20), and $C_{j_1j_1}(s)$ is defined by (2.532) for the case $j_1=j_2$.

Applying the estimates (1.211), (2.294), (2.551) to the right-hand side of (2.550) gives

$$
\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right|\le\frac{L}{p}\left(1+\frac{1}{\left(1-(z(s_1))^2\right)^{1/4}}\right)\times
$$

$$
\times\left(1+\frac{1}{\left(1-(z(s))^2\right)^{1/4}}+\frac{1}{\left(1-(z(s_1))^2\right)^{1/4}}\right), \qquad (2.552)
$$

where $s,s_1\in(t,T)$ and constant L is independent of p.

Combining the estimates (2.158), (2.549), and (2.552), we finally obtain

$$E_p'(s) \le \frac{L(s)p}{p^2} = \frac{L(s)}{p}$$

if $p \to \infty$, where constant $L(s)$ (s is fixed, $s \in (t,T)$) does not depend on p. The relation (2.540) is proved for the polynomial case. Theorem 2.20 is proved for the case of Legendre polynomials.

For the trigonometric case, by analogy with the proof of Lemma 2.2 (Sect. 2.1.2), we obtain the following analog of (2.551)

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_1}(s)\right| \le \frac{C}{p}, \tag{2.553}$$

where $s \in [t,T]$, constant C does not depend on p, and $C_{j_1j_1}(s)$ is defined by (2.532) for the case $j_1 = j_2$.

Note the following obvious estimates for the trigonometric case

$$\left|\int\limits_{s_1}^{s} \psi_3(\theta)\phi_j(\theta)d\theta\right| \le \frac{C}{j}, \quad \left|\int\limits_{t}^{s_1} \psi_2(\tau)\phi_j(\tau)d\tau\right| \le \frac{C}{j} \quad (j \ne 0), \tag{2.554}$$

where $s, s_1 \in [t,T]$, constant C does not depend on p.

Applying (2.549), (2.550), (2.553), and (2.554), we obtain the assertion of Theorem 2.20 for the trigonometric case. Theorem 2.20 is proved.

Let us reformulate Theorem 2.20 in terms on the convergence of the solution of system of ODEs to the solution of system of Stratonovich SDEs (the so-called Wong–Zakai type theorem).

By analogy with (2.1477) for the case $k = 3$, $p_1 = p_2 = p_3 = p$, $i_1, i_2, i_3 = 1, \ldots, m$, and $s \in (t,T]$ (s is fixed) we obtain

$$\int\limits_t^s \psi_3(t_3)\int\limits_t^{t_3} \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p}d\mathbf{f}_{t_2}^{(i_2)p}d\mathbf{f}_{t_3}^{(i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{2.555}$$

where $p \in \mathbf{N}$ and $d\mathbf{f}_\tau^{(i)p}$ is defined by (2.1474); another notations are the same as in Theorem 2.20.

The iterated Riemann–Stiltjes integrals

$$Z_{s,t}^{(i_1i_2i_3)p} = \int\limits_t^s \psi_3(t_3)\int\limits_t^{t_3} \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p}d\mathbf{f}_{t_2}^{(i_2)p}d\mathbf{f}_{t_3}^{(i_3)p},$$

$$
Y_{s,t}^{(i_1 i_2)p} = \int\limits_t^s \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p},
$$

$$
X_{s,t}^{(i_1)p} = \int\limits_t^s \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p}
$$

are the solution of the following system of ODEs

$$
\begin{cases}
dZ_{s,t}^{(i_1 i_2 i_3)p} = \psi_3(s) Y_{s,t}^{(i_1 i_2)p} d\mathbf{f}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\
dY_{s,t}^{(i_1 i_2)p} = \psi_2(s) X_{s,t}^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\
dX_{s,t}^{(i_1)p} = \psi_1(s) d\mathbf{f}_s^{(i_1)p}, & X_{t,t}^{(i_1)p} = 0
\end{cases}.
$$

From the other hand, the iterated Stratonovich stochastic integrals

$$
Z_{s,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*s} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)},
$$

$$
Y_{s,t}^{(i_1 i_2)} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)},
$$

$$
X_{s,t}^{(i_1)} = \int\limits_t^{*s} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}
$$

are the solution of the following system of Stratonovich SDEs

$$
\begin{cases}
dZ_{s,t}^{(i_1 i_2 i_3)} = \psi_3(s) Y_{s,t}^{(i_1 i_2)} * d\mathbf{f}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\
dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\
dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0
\end{cases},
$$

where $* d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 2.19 and 2.20 we obtain the following theorem.

Theorem 2.21 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$. *Then for any fixed* s $(s\in(t,T])$

$$\underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1i_2i_3)p}=Z_{s,t}^{(i_1i_2i_3)},\quad \underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1i_2)p}=Y_{s,t}^{(i_1i_2)},$$

$$\underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)p}=X_{s,t}^{(i_1)}.$$

2.7.3 Modification of Theorem 2.9 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 4

Let us prove the following theorem.

Theorem 2.22 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{s,t}=\int\limits_t^{*s}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(4)}]_{s,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ *(*s *is fixed),*

$$C_{j_4j_3j_2j_1}(s)=\int\limits_t^{s}\phi_{j_4}(s_4)\int\limits_t^{s_4}\phi_{j_3}(s_3)\int\limits_t^{s_3}\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_1}(s_1)ds_1ds_2ds_3ds_4$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Proof. The case $s = T$ is considered in Theorem 2.9. Below we consider the case $s \in (t, T)$. The relation (1.257) (in the case when $p_1 = \ldots = p_4 = p \to \infty$) implies that

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{s,t}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}A_1^{(i_3i_4)}(s)+\mathbf{1}_{\{i_1=i_3\neq 0\}}A_2^{(i_2i_4)}(s)+\mathbf{1}_{\{i_1=i_4\neq 0\}}A_3^{(i_2i_3)}(s)+\mathbf{1}_{\{i_2=i_3\neq 0\}}A_4^{(i_1i_4)}(s)+
$$

$$
+\mathbf{1}_{\{i_2=i_4\neq 0\}}A_5^{(i_1i_3)}(s)+\mathbf{1}_{\{i_3=i_4\neq 0\}}A_6^{(i_1i_2)}(s)-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}B_1(s)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}B_2(s)-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}B_3(s), \qquad (2.556)
$$

where $J[\psi^{(4)}]_{s,t}$ has the form (1.238) for $\psi_1(\tau), \ldots, \psi_4(\tau) \equiv 1$ and $i_1, \ldots, i_4 = 0, 1, \ldots, m$,

$$
A_1^{(i_3i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},
$$

$$
A_2^{(i_2i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_3}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},
$$

$$
A_3^{(i_2i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},
$$

$$
A_4^{(i_1i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)},
$$

$$
A_5^{(i_1i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
A_6^{(i_1i_2)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
B_1(s) = \lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_1}(s), \quad B_2(s) = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_3j_4j_3j_4}(s),
$$

$$B_3(s) = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_4j_3j_3j_4}(s).$$

Using the integration order replacement in Riemann integrals, Theorem 1.11 for $k = 2$ (see (1.255)) and (2.535), Parseval's equality and the integration order replacement technique for Itô stochastic integrals (see Chapter 3) [1]-[17], [77], [123], [124] or Itô's formula, we obtain (see the derivation of the formula (2.303))

$$A_1^{(i_3i_4)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{\tau} \int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_{\tau}^{(i_4)} +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\neq 0\}} \int\limits_t^s (s_1 - t) ds_1 - \Delta_1^{(i_3i_4)}(s) \quad \text{w. p. 1}, \tag{2.557}$$

where

$$\Delta_1^{(i_3i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_4=0}^{p} a_{j_4j_3}^p(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$a_{j_4j_3}^p(s) = \frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau) \int\limits_t^{\tau} \phi_{j_3}(s_1) \sum_{j_1=p+1}^{\infty} \left(\int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2\right)^2 ds_1 d\tau. \tag{2.558}$$

Let us consider $A_2^{(i_2i_4)}(s)$ (see the derivation of the formula (2.305))

$$A_2^{(i_2i_4)}(s) = -\Delta_2^{(i_2i_4)}(s) + \Delta_1^{(i_2i_4)}(s) + \Delta_3^{(i_2i_4)}(s) \text{ w. p. 1}, \tag{2.559}$$

where

$$\Delta_2^{(i_2i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} b_{j_4j_2}^p(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} c_{j_4j_2}^p(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$b_{j_4j_2}^p(s) = \frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau) \sum_{j_3=p+1}^{\infty} \left(\int\limits_t^{\tau} \phi_{j_3}(s_1) ds_1\right)^2 \int\limits_t^{\tau} \phi_{j_2}(s_1) ds_1 d\tau,$$

$$c_{j_4j_2}^p(s) = \frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau) \int\limits_t^{\tau} \phi_{j_2}(s_3) \sum_{j_3=p+1}^{\infty} \left(\int\limits_{s_3}^{\tau} \phi_{j_3}(s_1) ds_1\right)^2 ds_3 d\tau.$$

Further, we have w. p. 1 (see the derivation of the formula (2.308))

$$A_5^{(i_1 i_3)}(s) = -\Delta_4^{(i_1 i_3)}(s) + \Delta_5^{(i_1 i_3)}(s) + \Delta_6^{(i_1 i_3)}(s) \quad \text{w. p. 1}, \tag{2.560}$$

where

$$\Delta_4^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} d_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$\Delta_5^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} e_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$\Delta_6^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} f_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$d_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_3}^{s} \phi_{j_4}(\tau)d\tau\right)^2 \int\limits_{s_3}^{s} \phi_{j_3}(\tau)d\tau ds_3,$$

$$e_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(\tau) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_3}^{\tau} \phi_{j_4}(s_1)ds_1\right)^2 d\tau ds_3,$$

$$f_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_2}^{s} \phi_{j_4}(s_1)ds_1\right)^2 ds_2 ds_3 =$$

$$= \frac{1}{2}\int\limits_t^s \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_2}^{s} \phi_{j_4}(s_1)ds_1\right)^2 \int\limits_t^{s_2} \phi_{j_1}(s_3) ds_3 ds_2.$$

Let us consider $A_4^{(i_1 i_4)}(s)$ (see the derivation of the formula (2.314))

$$A_4^{(i_1 i_4)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{s_2} \int\limits_t^{s_1} d\mathbf{w}_{\tau}^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} - \Delta_3^{(i_1 i_4)}(s) \quad \text{w. p. 1}. \tag{2.561}$$

Moreover (see the derivation of the formula (2.315)),

$$A_6^{(i_1 i_2)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{s_1} \int\limits_t^{s_2} d\mathbf{w}_{\tau}^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1 +$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^s (s-s_2)ds_2-\Delta_6^{(i_1i_2)}(s)\quad \text{w. p. 1.} \tag{2.562}
$$

Further, we have w. p. 1 (see the derivation of the formula (2.312))

$$
A_3^{(i_2i_3)}(s)+A_5^{(i_2i_3)}(s)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{2.563}
$$

Using (2.563) and the generalized Parseval equality, we obtain w. p. 1

$$
A_3^{(i_2i_3)}(s)+A_5^{(i_2i_3)}(s)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\sum_{j_4=0}^{p}\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=
$$

$$
=-\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\sum_{j_4=p+1}^{\infty}\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1\times
$$

$$
\times\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=
$$

$$
=\Delta_6^{(i_2i_3)}(s)+\Delta_2^{(i_2i_3)}(s)-\Delta_9^{(i_2i_3)}(s), \tag{2.564}
$$

where

$$
\Delta_9^{(i_2i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}q_{j_2j_3}^{p}(s)\zeta_{j_2}^{(i_3)}\zeta_{j_3}^{(i_3)},
$$

$$
q_{j_2j_3}^{p}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2ds_1\sum_{j_4=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_4}(\tau)d\tau\right)^2.
$$

From (2.560) and (2.564) we get

$$
A_3^{(i_2i_3)}(s)=\Delta_2^{(i_2i_3)}(s)+\Delta_4^{(i_2i_3)}(s)-\Delta_5^{(i_2i_3)}(s)-\Delta_9^{(i_2i_3)}(s)\quad \text{w. p. 1.} \tag{2.565}
$$

Let us consider $B_1(s), B_2(s), B_3(s)$. We have (see the derivation of the formulas (2.316), (2.317))

$$B_1(s) = \frac{1}{4}\int\limits_t^s (s_1 - t)ds_1 - \lim_{p\to\infty}\sum_{j_4=0}^{p} a^p_{j_4j_4}(s), \tag{2.566}$$

$$B_2(s) = \lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}(s) + \lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3}(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}(s). \tag{2.567}$$

Moreover (see the derivation of the formula (2.318)),

$$B_2(s) + B_3(s) =$$

$$= \lim_{p\to\infty}\ \sum_{j_4=0}^{p}\int\limits_t^s \phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)ds_2\sum_{j_3=0}^{p}\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_3}(\tau)d\tau ds_1. \tag{2.568}$$

Using (2.568) and the generalized Parseval equality, we obtain

$$B_2(s) + B_3(s) =$$

$$= -\lim_{p\to\infty}\ \sum_{j_4=0}^{p}\int\limits_t^s \phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)ds_2\ \sum_{j_3=p+1}^{\infty}\ \int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_3}(\tau)d\tau ds_1 =$$

$$= \lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}(s) + \lim_{p\to\infty}\sum_{j_4=0}^{p} b^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} q^p_{j_4j_4}(s). \tag{2.569}$$

Combining (2.567) and (2.569), we have

$$B_3(s) = 2\lim_{p\to\infty}\sum_{j_4=0}^{p} b^p_{j_4j_4}(s) + \lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} c^p_{j_4j_4}(s) -$$

$$- \lim_{p\to\infty}\sum_{j_4=0}^{p} a^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} q^p_{j_4j_4}(s). \tag{2.570}$$

After substituting the relations (2.557), (2.559)–(2.562), (2.565)–(2.567), (2.570) into (2.556), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}(s)\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4} =$$

$$= J[\psi^{(4)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^s\int\limits_t^{\tau}\int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_{\tau}^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^s\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_{\tau}^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} + \frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^s\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{\tau}^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2 ds_1 + R(s) = J^*[\psi^{(4)}]_{s,t}+$$

$$+R(s)\quad \text{w. p. 1}, \tag{2.571}$$

where

$$R(s) = -\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3 i_4)}(s) + \mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2 i_4)}(s) + \Delta_1^{(i_2 i_4)}(s) + \Delta_3^{(i_2 i_4)}(s)\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_2^{(i_2 i_3)}(s) + \Delta_4^{(i_2 i_3)}(s) - \Delta_5^{(i_2 i_3)}(s) - \Delta_9^{(i_2 i_3)}(s)\right) - \mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1 i_4)}(s)+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1 i_3)}(s) + \Delta_5^{(i_1 i_3)}(s) + \Delta_6^{(i_1 i_3)}(s)\right) - \mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1 i_2)}(s)-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3 j_3}^p(s) + \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3 j_3}^p(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3 j_3}^p(s)\right)-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(2\lim_{p\to\infty}\sum_{j_4=0}^{p} b_{j_4 j_4}^p(s) + \lim_{p\to\infty}\sum_{j_4=0}^{p} f_{j_4 j_4}^p(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} c_{j_4 j_4}^p(s)-\right.$$

$$\left.-\lim_{p\to\infty}\sum_{j_4=0}^{p} a_{j_4 j_4}^p(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} q_{j_4 j_4}^p(s)\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3 j_3}^p(s). \tag{2.572}$$

Let us prove that

$$R(s) = 0\quad \text{w. p. 1}. \tag{2.573}$$

Consider the case of Legendre polynomials. First, we prove that

$$\Delta_1^{(i_3 i_4)}(s) = 0\quad \text{w. p. 1}. \tag{2.574}$$

We have

$$a^p_{j_4j_3}(s)=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times\int\limits_{-1}^{z(s)}P_{j_4}(y)\int\limits_{-1}^{y}P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}(2j_1+1)\left(\int\limits_{-1}^{y_1}P_{j_1}(y_2)dy_2\right)^2dy_1dy=$$

$$=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times\int\limits_{-1}^{z(s)}P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2\int\limits_{y_1}^{z(s)}P_{j_4}(y)dydy_1=$$

$$=\frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times$$

$$\times\int\limits_{-1}^{z(s)}P_{j_3}(y_1)\left((P_{j_4+1}(z(s))-P_{j_4-1}(z(s)))-(P_{j_4+1}(y_1)-P_{j_4-1}(y_1))\right)\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2dy_1$$

if $j_4\neq 0$ and

$$a^p_{j_4j_3}(s)=\frac{(T-t)^2\sqrt{2j_3+1}}{32}\times$$

$$\times\int\limits_{-1}^{z(s)}P_{j_3}(y_1)(z(s)-y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2dy_1$$

if $j_4=0$, where $z(s)$ is defined by (2.20).

We assume that $s\in(t,T)$ $(z(s)\neq\pm 1)$ since the case $s=T$ has already been considered in Theorem 2.9. Now the further proof of the equality (2.574) is completely analogous to the proof of the equality (2.331).

It is not difficult to see that the formulas

$$\Delta_2^{(i_2i_4)}(s)=0,\quad \Delta_4^{(i_1i_3)}(s)=0,\quad \Delta_6^{(i_1i_3)}(s)=0,\quad \Delta_9^{(i_2i_3)}(s)=0\quad \text{w. p. 1}\quad (2.575)$$

can be proved similarly with the proof of (2.574).

Moreover, the relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}(s)=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}(s)=0, \tag{2.576}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3}(s)=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p} q^p_{j_3j_3}(s)=0 \tag{2.577}$$

can also be proved analogously with (2.333), (2.334).

Let us consider $\Delta_3^{(i_2i_4)}(s)$ and prove that

$$\Delta_3^{(i_2i_4)}(s)=0\quad \text{w. p. 1}. \tag{2.578}$$

We have

$$\Delta_3^{(i_2i_4)}(s)=\Delta_4^{(i_2i_4)}(s)+\Delta_6^{(i_2i_4)}(s)-\Delta_7^{(i_2i_4)}(s)=-\Delta_7^{(i_2i_4)}(s) \tag{2.579}$$

w. p. 1, where

$$\Delta_7^{(i_2i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_2,j_4=0}^{p} g^p_{j_4j_2}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$g^p_{j_4j_2}(s)=\int\limits_t^s \phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_2}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_{s_1}^{s}\phi_{j_1}(s_2)ds_2\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2\right)ds_1d\tau.$$

Note that (see (2.337))

$$g^p_{j_4j_4}(s)=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2d\tau\right)^2. \tag{2.580}$$

The proof of (2.578) for the case $i_2=i_4\ne 0$ differs from the proof of the equality

$$\Delta_3^{(i_2i_4)}=0\quad \text{w. p. 1}$$

for the case $i_2=i_4\ne 0$ (see the proof of Theorem 2.9). In our case we will use Parseval's equality instead of the orthogonality property of the Legendre polynomials.

Using the Parseval equality, we obtain

$$
\sum_{j_4=0}^{p} g_{j_4j_4}^{p}(s)=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s \phi_{j_4}(\tau)\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s \phi_{j_4}(\tau)\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2-\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2\right)d\tau\right)^2\le
$$

$$
\le\sum_{j_4=0}^{p}\left(\int\limits_t^s\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2d\tau\right)^2\le
$$

$$
\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2d\tau\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2\left(\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2\right)^2d\tau=
$$

$$
=(s-t)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+\sum_{j_1=p+1}^{\infty}\int\limits_t^s\left(\int\limits_t^{\tau}\phi_{j_1}(s_2)ds_2\right)^2d\tau. \qquad (2.581)
$$

We assume that $s \in (t, T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 2.9. Then from (2.581) and (2.159) we obtain

$$0 \le \sum_{j_4=0}^{p} g_{j_4 j_4}^{p}(s) \le \frac{C(s)}{p}, \tag{2.582}$$

where constant $C(s)$ (s is fixed) is independent of p.

Combining (2.23) and (2.327) with (1.210), we obtain for $j \in \mathbf{N}$

$$\left| \int\limits_{s_1}^{s} \phi_j(\theta) d\theta \right| < \frac{K}{j^{1/2+m/4}} \left(\frac{1}{(1-z^2(s))^{m/8}} + \frac{1}{(1-z^2(s_1))^{m/8}} \right), \tag{2.583}$$

where $s, s_1 \in (t, T)$, $m = 1$ or $m = 2$, $z(s)$ is defined by (2.20), constant K does not depend on j.

Using the Parseval equality, we get

$$\lim_{p_1 \to \infty} \sum_{j_4, j_2 = 0}^{p_1} \left(g_{j_4 j_2}^{p}(s) \right)^2 = \int\limits_{[t,T]^2} \left(K_p(\tau, s_1, s) \right)^2 ds_1 d\tau = \int\limits_{t}^{s} \int\limits_{t}^{\tau} \left(F_p(\tau, s_1, s) \right)^2 ds_1 d\tau, \tag{2.584}$$

where

$$g_{j_4 j_2}^{p}(s) = \int\limits_{t}^{T} \mathbf{1}_{\{\tau < s\}} \phi_{j_4}(\tau) \int\limits_{t}^{\tau} \phi_{j_2}(s_1) F_p(\tau, s_1, s) ds_1 d\tau =$$

$$= \int\limits_{[t,T]^2} K_p(\tau, s_1, s) \phi_{j_4}(\tau) \phi_{j_2}(s_1) ds_1 d\tau$$

is a coefficient of the double Fourier–Legendre series of the function

$$K_p(\tau, s_1, s) = \mathbf{1}_{\{\tau < s\}} \mathbf{1}_{\{s_1 < \tau < s\}} F_p(\tau, s_1, s),$$

where

$$\sum_{j_1=p+1}^{\infty} \int\limits_{s_1}^{s} \phi_{j_1}(s_2) ds_2 \int\limits_{\tau}^{s} \phi_{j_1}(s_2) ds_2 \stackrel{\mathsf{def}}{=} F_p(\tau, s_1, s). \tag{2.585}$$

From (2.583) for $m = 1$ and $m = 2$ we have

$$|F_p(\tau, s_1, s)| < \sum_{j_1=p+1}^{\infty} \frac{K_1}{(j_1)^{7/4}} \left(\frac{1}{(1-z^2(s))^{1/8}} + \frac{1}{(1-z^2(s_1))^{1/8}} \right) \times$$

$$
\times\left(\frac{1}{(1-z^2(s))^{1/4}}+\frac{1}{(1-z^2(\tau))^{1/4}}\right)\le
$$

$$
\le\frac{K_2}{p^{3/4}}\left(\frac{1}{(1-z^2(s))^{1/8}}+\frac{1}{(1-z^2(s_1))^{1/8}}\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+\frac{1}{(1-z^2(\tau))^{1/4}}\right),
\tag{2.586}
$$

where $s,s_1,\tau\in(t,T)$, constant K_2 is independent of p and we used the estimate (2.743) in (2.586).

The relations (2.584) and (2.586) imply the estimate

$$
\sum_{j_2,j_4=0}^{p}\left(g^p_{j_4j_2}(s)\right)^2\le\frac{C_1(s)}{p^{3/2}}
\tag{2.587}
$$

for the case $s\in(t,T)$ or $z(s)\in(-1,1)$ (the case $s=T$ has already been considered in Theorem 2.9), where constant $C_1(s)$ (s is fixed) does not depend on p.

Then from analogue of (2.372) for $s\in(t,T)$ (s is fixed), (2.582), and (2.587) we have

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p}g^p_{j_4j_2}(s)\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\right)^2\right\}\le\left(1+\mathbf{1}_{\{i_2=i_4\ne 0\}}\right)\sum_{j_2,j_4=0}^{p}\left(g^p_{j_4j_2}(s)\right)^2+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\sum_{j_4=0}^{p}g^p_{j_4j_4}(s)\right)^2\le\frac{C_2(s)}{p^{3/2}}\ \to 0
$$

if $p\to\infty$, where constant $C_2(s)$ (s is fixed) does not depend on p. The equality (2.578) is proved.

Let us consider $\Delta_5^{(i_1i_3)}(s)$

$$
\Delta_5^{(i_1i_3)}(s)=\Delta_4^{(i_1i_3)}(s)+\Delta_6^{(i_1i_3)}(s)-\Delta_8^{(i_1i_3)}(s)\quad\text{w. p. 1},
$$

where

$$
\Delta_8^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}h^p_{j_3j_1}(s)\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3},
$$

$$
h^p_{j_3j_1}(s)=\int\limits_t^s\phi_{j_1}(s_3)\int\limits_{s_3}^s\phi_{j_3}(\tau)F_p(s_3,\tau,s)d\tau ds_3,
$$

where $F_p(s_3,\tau,s)$ is defined by (2.585).

Analogously to (2.578), we obtain that $\Delta_8^{(i_1 i_3)}(s)=0$ w. p. 1. In this case we consider the function

$$K_p(s_3,\tau,s)=\mathbf{1}_{\{s_3<s\}}\mathbf{1}_{\{s_3<\tau<s\}}F_p(s_3,\tau,s)$$

and the relations (see (2.580))

$$h^p_{j_3 j_1}(s)=\int\limits_{[t,T]^2} K_p(s_3,\tau,s)\phi_{j_1}(s_3)\phi_{j_3}(\tau)d\tau ds_3,$$

$$h^p_{j_1 j_1}(s)=\sum_{j_4=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s \phi_{j_1}(\tau)\int\limits_\tau^s \phi_{j_4}(s_1)ds_1 d\tau\right)^2.$$

Let us prove that

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3 j_3}(s)=0. \tag{2.588}$$

We have

$$c^p_{j_3 j_3}(s)=f^p_{j_3 j_3}(s)+d^p_{j_3 j_3}(s)-g^p_{j_3 j_3}(s). \tag{2.589}$$

Moreover,

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3 j_3}(s)=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p} d^p_{j_3 j_3}(s)=0, \tag{2.590}$$

where the first equality in (2.590) has been proved earlier. Analogously, we can prove the second equality in (2.590).

From (2.582) we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} g^p_{j_3 j_3}(s)=0.$$

So, (2.588) is proved. The relation (2.573) is proved for the polynomial case. Theorem 2.22 is proved for the case of Legendre polynomials.

It is easy to see that the trigonometric case is considered by analogy with the case of Legendre polynomials using the estimates (2.554). Theorem 2.22 is proved.

Let us reformulate Theorem 2.22 in terms on the convergence of the solution of system of ODEs to the solution of system of Stratonovich SDEs.

By analogy with (2.1477) for the case $k = 4$, $p_1 = \ldots = p_4 = p$, $i_1, \ldots, i_4 = 0, 1, \ldots, m$, and $s \in (t, T]$ (s is fixed) we obtain

$$\int\limits_t^s \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p} = \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

where $p \in \mathbf{N}$ and $d\mathbf{w}_\tau^{(i)p}$ is defined by (2.1476); another notations are the same as in Theorem 2.22.

The iterated Riemann–Stiltjes integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)p} = \int\limits_t^s \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p}, \tag{2.591}$$

$$Z_{s,t}^{(i_1 i_2 i_3)p} = \int\limits_t^s \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p}, \tag{2.592}$$

$$Y_{s,t}^{(i_1 i_2)p} = \int\limits_t^s \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p}, \tag{2.593}$$

$$X_{s,t}^{(i_1)p} = \int\limits_t^s d\mathbf{w}_{t_1}^{(i_1)p} \tag{2.594}$$

are the solution of the following system of ODEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)p} = Z_{s,t}^{(i_1 i_2 i_3)p} d\mathbf{w}_s^{(i_4)p}, & V_{t,t}^{(i_1 i_2 i_3 i_4)p} = 0 \\ dZ_{s,t}^{(i_1 i_2 i_3)p} = Y_{s,t}^{(i_1 i_2)p} d\mathbf{w}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\ dY_{s,t}^{(i_1 i_2)p} = X_{s,t}^{(i_1)p} d\mathbf{w}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\ dX_{s,t}^{(i_1)p} = 1 \cdot d\mathbf{w}_s^{(i_1)p}, & X_{t,t}^{(i_1)p} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_4} {\int\limits_t^{*}}^{t_3} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}, \tag{2.595}$$

$$Z_{s,t}^{(i_1 i_2 i_3)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_3} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}, \tag{2.596}$$

$$Y_{s,t}^{(i_1 i_2)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \tag{2.597}$$

$$X_{s,t}^{(i_1)} = {\int\limits_t^{*}}^{s} d\mathbf{w}_{t_1}^{(i_1)} \tag{2.598}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)} = Z_{s,t}^{(i_1 i_2 i_3)} * d\mathbf{w}_s^{(i_4)}, & V_{t,t}^{(i_1 i_2 i_3 i_4)} = 0 \\ dZ_{s,t}^{(i_1 i_2 i_3)} = Y_{s,t}^{(i_1 i_2)} * d\mathbf{w}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\ dY_{s,t}^{(i_1 i_2)} = X_{s,t}^{(i_1)} * d\mathbf{w}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ dX_{s,t}^{(i_1)} = 1 * d\mathbf{w}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* \, d\mathbf{w}_s^{(i)}$, $i = 0, 1, \ldots, m$ is the Stratonovich differential, $* \, d\mathbf{w}_s^{(0)} = ds$.

Then from Theorems 2.19, 2.21, and 2.22 we obtain the following theorem.

Theorem 2.23 [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then for any fixed s $(s \in (t,T])$*

$$\underset{p\to\infty}{\text{l.i.m.}}\ V_{s,t}^{(i_1 i_2 i_3 i_4)p} = V_{s,t}^{(i_1 i_2 i_3 i_4)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1 i_2 i_3)p} = Z_{s,t}^{(i_1 i_2 i_3)},$$

$$\underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1 i_2)p} = Y_{s,t}^{(i_1 i_2)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)p} = X_{s,t}^{(i_1)}.$$

2.8 Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4 in Theorems 2.2, 2.8, and 2.9

2.8.1 Rate of the Mean-Square Convergence of Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2

This section is devoted to the proof of the following theorem.

Theorem 2.24 [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C}{p} \tag{2.599}$$

is valid, where constant C is independent of p,

$$C_{j_2 j_1} = \int\limits_t^T \psi_2(s_2)\phi_{j_2}(s_2) \int\limits_t^{s_2} \psi_1(s_1)\phi_{j_1}(s_1) ds_1 ds_2,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. Applying (2.8), we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$
= \mathsf{M}\Biggl\{\Biggl(J[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr)+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{p} C_{j_1j_1}\Biggr)^2\Biggr\} =
$$

$$
= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+
$$

$$
+\left(\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{p} C_{j_1j_1}\right)^2 =
$$

$$
= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{p,p}\right)^2\right\}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1j_1}\right)^2. \tag{2.600}
$$

From Remark 1.7 (see (1.225)) we have

$$
\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{p,p}\right)^2\right\} \le \frac{C_1}{p}, \tag{2.601}
$$

where constant C_1 is independent of p.

From Theorem 2.2 (see (2.37)) we get

$$
\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1j_1} = \sum_{j_1=p+1}^{\infty} C_{j_1j_1}. \tag{2.602}
$$

Let us consider the case of Legendre polynomials. The estimate (2.83) implies that

$$
\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_1}\right| \le C_2\left(\frac{1}{p} + \sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\right), \tag{2.603}
$$

where constant C_2 does not depend on p.

Using (2.25) and (2.603), we have

$$S_p \stackrel{\text{def}}{=} \left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}\right| \le \frac{C_3}{p}, \qquad (2.604)$$

where constant C_3 is independent of p.

Applying the ideas that we used to obtain the relations (2.85), (2.89)–(2.91), we can prove the following estimates for the trigonometric case

$$S_{2p} = \left|\sum_{j_1=2p+1}^{\infty} C_{j_1 j_1}\right| \le \frac{K_1}{p}, \qquad (2.605)$$

$$S_{2p-1} = \left|\sum_{j_1=2p}^{\infty} C_{j_1 j_1}\right| \le S_{2p} + \frac{K_2}{p}, \qquad (2.606)$$

where constants K_1, K_2 do not depend on p.

Using (2.605) and (2.606), we get the estimate (2.604) for the trigonometric case. Combining (2.600)–(2.602), (2.604), we obtain (2.599). Theorem 2.24 is proved.

2.8.2 Rate of the Mean-Square Convergence of Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3

In this section, we consider the following theorem.

Theorem 2.25 [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \qquad (2.607)$$

is valid, where constant C is independent of p,

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(s_3)\phi_{j_3}(s_3)\int\limits_t^{s_3}\psi_2(s_2)\phi_{j_2}(s_2)\int\limits_t^{s_2}\psi_1(s_1)\phi_{j_1}(s_1)ds_1ds_2ds_3,$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. We have (see (2.399))

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}+\right.\right.$$

$$\left.\left.+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}+\right.\right.$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)-$$

$$\left.\left.-\mathbf{1}_{\{i_1=i_3\}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\},\qquad(2.608)$$

where (see (1.47))

$$J[\psi^{(3)}]_{T,t}^{p,p,p}=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\right.$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr).
$$

Using (2.608) and the elementary inequality

$$
(a+b+c+d)^2\le 4\left(a^2+b^2+c^2+d^2\right),
$$

we obtain

$$
\mathsf{M}\left\{\left(J^{*}[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le 4\Biggl(\mathsf{M}\left\{\left(J[\psi^{(3)}]_{s,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}\right)^2\right\}+\mathbf{1}_{\{i_1=i_2\}}E_p^{(1)}+\mathbf{1}_{\{i_2=i_3\}}E_p^{(2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}E_p^{(3)}\Biggr), \tag{2.609}
$$

where

$$
E_p^{(1)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},
$$

$$
E_p^{(2)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\},
$$

$$
E_p^{(3)}=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\}.
$$

From Remark 1.7 (see (1.225)) we have

$$
\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}\right)^2\right\}\le\frac{C_1}{p}, \tag{2.610}
$$

where constant C_1 is independent of p.

Moreover, from (2.296) and (2.300) we have the following estimate

$$
E_p^{(3)}\le\frac{C_2}{p} \tag{2.611}
$$

for the polynomial and trigonometric cases, where constant C_2 does not depend on p.

Using Theorem 1.1 for $k=1$ (also see (1.45)), we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi_2(s_1)\psi_1(s_1)ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde C_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi_2(s_1)\psi_1(s_1)ds_1 ds.$$

Applying the Itô formula, we have

$$\int\limits_t^T \psi_3(s)\psi_2(s)\int\limits_t^s \psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds = \int\limits_t^T \psi_1(s_1)\int\limits_{s_1}^T \psi_3(s)\psi_2(s)ds d\mathbf{f}_{s_1}^{(i_1)} \quad \text{w. p. 1.}$$

Using Theorem 1.1 for $k=1$ (also see (1.45)), we have w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi_1(s)\int\limits_s^T \psi_3(s_1)\psi_2(s_1)ds_1 d\mathbf{f}_s^{(i_1)} = \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}C^*_{j_1}\zeta_{j_1}^{(i_1)},$$

where

$$C^*_{j_1} = \int\limits_t^T \psi_1(s)\phi_{j_1}(s)\int\limits_s^T \psi_3(s_1)\psi_2(s_1)ds_1 ds.$$

Further, we get

$$E_p^{(1)} \le 2G_p^{(1)} + 2G_p^{(2)}, \tag{2.612}$$

$$E_p^{(2)} \le 2H_p^{(1)} + 2H_p^{(2)}, \tag{2.613}$$

where

$$G_p^{(1)} = \mathsf{M}\left\{\frac{1}{4}\left(\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2 d\mathbf{f}_{t_3}^{(i_3)} - \sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$G_p^{(2)} = \mathsf{M}\left\{\left(\frac{1}{2}\sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)} - \sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$H_p^{(1)} = \mathsf{M}\left\{\frac{1}{4}\left(\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3 - \sum_{j_1=0}^{p} C_{j_1}^{*}\zeta_{j_1}^{(i_1)}\right)^2\right\},$$

$$H_p^{(2)} = \mathsf{M}\left\{\left(\frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^{*}\zeta_{j_1}^{(i_1)} - \sum_{j_1,j_3=0}^{p} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}.$$

From Remark 1.7 (see (1.225)) we have

$$G_p^{(1)} \le \frac{C_2}{p}, \quad H_p^{(1)} \le \frac{C_2}{p}, \tag{2.614}$$

where constant C_2 is independent of p.

The estimates

$$G_p^{(2)} \le \frac{C_3}{p}, \quad H_p^{(2)} \le \frac{C_3}{p} \tag{2.615}$$

are proved in Sect. 2.2.5 (see the proof of Theorem 2.8) for the polynomial and trigonometric cases; constant C_3 does not depend on p.

Combining the estimates (2.609)–(2.615), we obtain the inequality (2.607). Theorem 2.25 is proved.

2.8.3 Rate of the Mean-Square Convergence of Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4

This section is devoted to the proof of the following theorem.

Theorem 2.26 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4 = 0,1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p} \tag{2.616}$$

is valid, where constant C is independent of p,

$$C_{j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(s_4) \int\limits_t^{s_4} \phi_{j_3}(s_3) \int\limits_t^{s_3} \phi_{j_2}(s_2) \int\limits_t^{s_2} \phi_{j_1}(s_1) ds_1 ds_2 ds_3 ds_4,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. First, let us prove that Theorem 2.8 is valid for the case $i_1, i_2, i_3 = 0, 1, \ldots, m$. The case $i_1, i_2, i_3 = 1, \ldots, m$ has been proved in Theorem 2.8. From (1.47) and the standard relation (2.399) between Stratonovich and Itô stochastic integrals of third multiplicity we have that Theorem 2.8 is valid for the following cases

$$i_1 = i_2 = 0, \quad i_3 = 1, \ldots, m,$$

$$i_1 = i_3 = 0, \quad i_2 = 1, \ldots, m,$$

$$i_2 = i_3 = 0, \quad i_1 = 1, \ldots, m.$$

Thus, it remains to consider the following three cases

$$i_1,\ i_2 = 1, \ldots, m, \quad i_3 = 0, \tag{2.617}$$

$$i_2,\ i_3 = 1, \ldots, m, \quad i_1 = 0, \tag{2.618}$$

$$i_1,\ i_3 = 1, \ldots, m, \quad i_2 = 0. \tag{2.619}$$

The relations (1.47) and (2.399) imply that for the case (2.617) we need to prove the following equality

$$\sum_{j_1=0}^{\infty} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1) dt_1 dt_2 dt_3 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_1(t_1)\psi_2(t_1) dt_1 dt_3. \tag{2.620}$$

Using the relation (2.10), we get

$$
\sum_{j_1=0}^{\infty}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{T}\phi_{j_1}(t_2)\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)dt_3dt_2dt_1=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{T}\phi_{j_1}(t_2)\tilde\psi_2(t_2)dt_2dt_1=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_2)\tilde\psi_2(t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi_1(t_2)\tilde\psi_2(t_2)dt_2, \tag{2.621}
$$

where

$$
\tilde\psi_2(t_2)=\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)dt_3. \tag{2.622}
$$

From (2.621) and (2.622) we obtain

$$
\sum_{j_1=0}^{\infty}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi_1(t_2)\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)dt_3dt_2=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2dt_3. \tag{2.623}
$$

The relation (2.620) is proved.

From (1.47) and (2.399) it follows that for the case (2.618) we need to prove the following equality

$$\sum_{j_2=0}^{\infty}\int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)dt_1dt_3. \qquad (2.624)$$

Using the relation (2.10), we have

$$\sum_{j_2=0}^{\infty}\int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \sum_{j_2=0}^{\infty}\int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\bar{\psi}_2(t_2)dt_2dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_3(t_3)\bar{\psi}_2(t_3)dt_3,$$

where

$$\bar{\psi}_2(t_2) = \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1. \qquad (2.625)$$

The relation (2.624) is proved.

The relations (1.47) and (2.399) imply that for the case (2.619) we need to prove the following equality

$$\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3 = 0. \qquad (2.626)$$

We have

$$\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3 =$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{t_3}\psi_2(t_2)dt_2dt_1dt_3=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\left(\int\limits_{t_1}^{T}\psi_2(t_2)dt_2-\int\limits_{t_3}^{T}\psi_2(t_2)dt_2\right)dt_1dt_3=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)dt_2dt_1dt_3-
$$

$$
-\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_3}^{T}\psi_2(t_2)dt_2dt_1dt_3=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3-
$$

$$
-\sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi_3(t_1)\tilde{\psi}_1(t_1)dt_1-\frac{1}{2}\int\limits_t^T\tilde{\psi}_3(t_1)\psi_1(t_1)dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi_3(t_1)\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)dt_2dt_1-\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_3(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)dt_2dt_1=0,
$$

where

$$
\tilde{\psi}_1(t_1)=\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)dt_2, \tag{2.627}
$$

$$
\tilde{\psi}_3(t_3)=\psi_3(t_3)\int\limits_{t_3}^{T}\psi_2(t_2)dt_2. \tag{2.628}
$$

The relation (2.626) is proved. Theorem 2.8 is proved for the case $i_1,i_2,i_3=0,1,\ldots,m$.

Using (2.399) and (2.400), we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+\right.\right.
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+
$$

$$
\left.\left.+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}-\right.\right.
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1+
$$

$$
\left.\left.+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+\right.\right.
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\Biggr)^2\Biggr\}=
$$

$$
=\mathsf{M}\Biggl\{\Biggl(J[\psi^{(4)}]_{T,t}-J[\psi^{(4)}]_{T,t}^{p,p,p,p}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\Biggl(\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)}-S_1^{(i_3i_4)p}\Biggr)+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-S_2^{(i_1i_4)p}\Biggr)+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\Biggl(\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-S_3^{(i_1i_2)p}\Biggr)-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\Biggl(\frac{1}{4}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-
$$

$$
-\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)(s_1-t)ds_1ds\Biggr)-R_p\Biggr)^2\Biggr\}, \qquad (2.629)
$$

where $S_1^{(i_3i_4)p}$, $S_2^{(i_1i_4)p}$, $S_3^{(i_1i_2)p}$ are the approximations of the iterated Stratonovich stochastic integrals

$$
\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)},\quad \int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)},
$$

$$
\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1,
$$

respectively (these approximations are obtained by the version of Theorem 2.8 for the case $i_1,i_2,i_3=0,1,\ldots,m$); $J[\psi^{(4)}]_{T,t}^{p,p,p,p}$ is the approximation of the iterated Itô stochastic integral $J[\psi^{(4)}]_{T,t}$ obtained by Theorem 1.1 (see (1.48))

$$J[\psi^{(4)}]_{T,t}^{p,p,p,p}=\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}A_1^{(i_3i_4)p}-\mathbf{1}_{\{i_1=i_3\ne 0\}}A_2^{(i_2i_4)p}-\mathbf{1}_{\{i_1=i_4\ne 0\}}A_3^{(i_2i_3)p}-\mathbf{1}_{\{i_2=i_3\ne 0\}}A_4^{(i_1i_4)p}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}A_5^{(i_1i_3)p}-\mathbf{1}_{\{i_3=i_4\ne 0\}}A_6^{(i_1i_2)p}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}B_1^p+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}B_2^p+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}B_3^p,$$

where

$$A_1^{(i_3i_4)p}=\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},\quad A_2^{(i_2i_4)p}=\sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_3}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$A_3^{(i_2i_3)p}=\sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},\quad A_4^{(i_1i_4)p}=\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_3j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)},$$

$$A_5^{(i_1i_3)p}=\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},\quad A_6^{(i_1i_2)p}=\sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$B_1^p=\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_1},\quad B_2^p=\sum_{j_4,j_3=0}^{p} C_{j_3j_4j_3j_4},$$

$$B_3^p=\sum_{j_4,j_3=0}^{p} C_{j_4j_3j_3j_4};$$

R_p is the expression on the right-hand side of (2.321) before passing to the limits, i.e.

$$R_p=-\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3i_4)p}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2i_4)p}+\Delta_1^{(i_2i_4)p}+\Delta_3^{(i_2i_4)p}\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_4^{(i_2i_3)p}-\Delta_5^{(i_2i_3)p}+\Delta_6^{(i_2i_3)p}\right)-\mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1i_4)p}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1i_3)p}+\Delta_5^{(i_1i_3)p}+\Delta_6^{(i_1i_3)p}\right)-\mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1i_2)p}-$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\sum_{j_3=0}^{p} a_{j_3j_3}^p+\sum_{j_3=0}^{p} c_{j_3j_3}^p-\sum_{j_3=0}^{p} b_{j_3j_3}^p\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(2\sum_{j_3=0}^{p} f_{j_3j_3}^p-\sum_{j_3=0}^{p} a_{j_3j_3}^p-\sum_{j_3=0}^{p} c_{j_3j_3}^p+\sum_{j_3=0}^{p} b_{j_3j_3}^p\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_3=0}^{p} a_{j_3j_3}^p,
$$

where

$$
\Delta_1^{(i_3i_4)p}=\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^p\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},\quad \Delta_2^{(i_2i_4)p}=\sum_{j_4,j_2=0}^{p} b_{j_4j_2}^p\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},
$$

$$
\Delta_3^{(i_2i_4)p}=\sum_{j_4,j_2=0}^{p} c_{j_4j_2}^p\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},\quad \Delta_4^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p} d_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_5^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p} e_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},\quad \Delta_6^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p} f_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

where

$$
a_{j_4j_3}^p,\quad b_{j_4j_2}^p,\quad c_{j_4j_2}^p,\quad d_{j_3j_1}^p,\quad e_{j_3j_1}^p,\quad f_{j_3j_1}^p
$$

are defined by the relations (2.304), (2.306), (2.307), (2.309)–(2.311).

Using (2.629) and the elementary inequality

$$
(a_1+\ldots+a_6)^2\le 6\left(a_1^2+\ldots+a_6^2\right),
$$

we get

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le
$$

$$
\le 6\left(Q_p^{(1)}+Q_p^{(2)}+Q_p^{(3)}+Q_p^{(4)}+Q_p^{(5)}+Q_p^{(6)}\right), \tag{2.630}
$$

where

$$
Q_p^{(1)}=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}-J[\psi^{(4)}]_{T,t}^{p,p,p,p}\right)^2\right\},
$$

$$Q_p^{(2)}=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathsf{M}\left\{\left(\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)}-S_1^{(i_3i_4)p}\right)^2\right\},$$

$$Q_p^{(3)}=\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathsf{M}\left\{\left(\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-S_2^{(i_1i_4)p}\right)^2\right\},$$

$$Q_p^{(4)}=\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathsf{M}\left\{\left(\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-S_3^{(i_1i_2)p}\right)^2\right\},$$

$$Q_p^{(5)}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times$$
$$\times\left(\frac{1}{4}\int\limits_t^T(s_1-t)ds_1-\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_1)(s_1-t)ds_1ds\right)^2,$$

$$Q_p^{(6)}=\mathsf{M}\left\{(R_p)^2\right\}.$$

From Remark 1.7 (see (1.225)) we have

$$Q_p^{(1)}\le\frac{C_1}{p},\tag{2.631}$$

where constant C_1 is independent of p.

Let us prove the version of Theorem 2.25 for the case $i_1,i_2,i_3=0,1,\ldots,m$. The case $i_1,i_2,i_3=1,\ldots,m$ has been proved in Theorem 2.25. It is easy to see that, in addition to the proof of Theorem 2.25, we need to prove the following inequalities

$$\Biggl|\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_1)\psi_2(t_1)dt_1dt_3-$$
$$-\sum_{j_1=0}^{p}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3\Biggr|\le\frac{C}{p},\tag{2.632}$$

$$
\left|\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)dt_1dt_3-\right.
$$

$$
\left.-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\psi_3(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1dt_2dt_3\right|\le\frac{C}{p},\qquad(2.633)
$$

$$
\left|\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3\right|\le\frac{C}{p},\qquad(2.634)
$$

where constant C is independent of p.

The inequalities (2.632) and (2.633) are equivalent to the following inequalities (see the proof of the cases (2.617), (2.618))

$$
\left|\frac{1}{2}\int\limits_t^T\psi_1(t_2)\tilde\psi_2(t_2)dt_2-\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_2)\tilde\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2\right|\le\frac{C}{p},\qquad(2.635)
$$

$$
\left|\frac{1}{2}\int\limits_t^T\psi_3(t_3)\bar\psi_2(t_3)dt_3-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\bar\psi_2(t_2)dt_2dt_3\right|\le\frac{C}{p},\qquad(2.636)
$$

where $\tilde\psi_2(t_2)$, $\bar\psi_2(t_2)$ are defined by (2.622) and (2.625), respectively. The inequalities (2.635), (2.636) follow from (2.602), (2.604)–(2.606).

Let us prove (2.634). By analogy with the proof of (2.626) we have

$$
\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3=
$$

$$
=\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde\psi_1(t_1)dt_1dt_3-
$$

$$
-\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\tilde\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3=
$$

$$
=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3-
$$

$$
-\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3-
$$

$$
-\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3+
$$

$$
+\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3=
$$

$$
=-\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3+
$$

$$
+\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3, \tag{2.637}
$$

where $\tilde{\psi}_1(t_1)$, $\tilde{\psi}_3(t_3)$ are defined by (2.627), (2.628), respectively.

Now the estimate (2.634) follows from (2.637) and (2.604)–(2.606). Theorem 2.25 is proved for the case $i_1,i_2,i_3=0,1,\ldots,m.$

Using the version of Theorem 2.25 for the case $i_1,i_2,i_3=0,1,\ldots,m,$ we obtain the following estimates

$$
Q_p^{(2)}\le\frac{C_2}{p},\quad Q_p^{(3)}\le\frac{C_2}{p},\quad Q_p^{(4)}\le\frac{C_2}{p}, \tag{2.638}
$$

where constant C_2 does not depend on p.

From Theorem 2.2 (see (2.37)) we get

$$
\frac{1}{2}\int\limits_t^T(s_1-t)ds_1-\sum_{j_4=0}^{p}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)(s_1-t)ds_1ds=
$$

$$
=\sum_{j_4=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)(s_1-t)ds_1ds. \tag{2.639}
$$

Let us consider the case of Legendre polynomials. From (2.604) and (2.639) we have

$$\left|\sum_{j_4=p+1}^{\infty}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_4}(s_1)(s_1-t)ds_1 ds\right| \le \frac{C_3}{p}, \tag{2.640}$$

where constant C_3 is independent of p.

By analogy with (2.605) and (2.606) we have the estimate (2.640) for the trigonometric case. Then

$$Q_p^{(5)} \le \frac{C_4}{p^2}, \tag{2.641}$$

where constant C_4 does not depend on p.

Analyzing the proof of Theorem 2.9, we conclude that

$$Q_p^{(6)} \le \frac{C_5}{p} \tag{2.642}$$

for the polynomial and trigonometric cases; constant C_5 is independent of p.

Combining (2.630), (2.631), (2.638), (2.641), (2.642), we get (2.616). Theorem 2.26 is proved.

2.9 Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4 in Theorems 2.18, 2.20, and 2.22 (The Case of Integration Interval $[t, s]$ $(s \in (t, T])$)

Let us prove the following theorem.

Theorem 2.27 [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Moreover, $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{s,t} = \int\limits_t^{*s} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{s,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C(s)}{p} \tag{2.643}$$

is valid, where $s\in (t,T]$ (s is fixed), constant $C(s)$ is independent of p,

$$C_{j_2j_1}(s)=\int\limits_t^s \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. The case $s=T$ has already been considered in Theorem 2.24. Below we consider the case $s\in (t,T)$. By analogy with (2.600) we obtain

$$\mathsf{M}\left\{\left(J^{*}[\psi^{(2)}]_{s,t}-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{s,t}-J[\psi^{(2)}]_{s,t}^{p,p}\right)^2\right\}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^s\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p}C_{j_1j_1}(s)\right)^2, \qquad (2.644)$$

where (see (1.255))

$$J[\psi^{(2)}]_{s,t}^{p,p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right).$$

From Remark 1.12 (see (1.252)) we have

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{s,t}-J[\psi^{(2)}]_{s,t}^{p,p}\right)^2\right\}\le\frac{C_1(s)}{p}, \qquad (2.645)$$

where constant $C_1(s)$ is independent of p.

Using (2.535), we obtain (the existence of a limit on the right-hand side of (2.535) and a useful estimate for this limit will be proved further in this section)

$$\frac{1}{2}\int\limits_t^s\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p}C_{j_1j_1}(s)=\sum_{j_1=p+1}^{\infty}C_{j_1j_1}(s). \qquad (2.646)$$

Consider the case of Legendre polynomials. By analogy with (2.82) we get for $n > m$ $(n, m \in \mathbf{N})$

$$\sum_{j_1=m+1}^{n} C_{j_1j_1}(s) = \sum_{j_1=m+1}^{n} \int\limits_{t}^{s} \psi_2(\theta)\phi_{j_1}(\theta) \int\limits_{t}^{\theta} \psi_1(\tau)\phi_{j_1}(\tau)d\tau d\theta =$$

$$= \frac{T-t}{4} \int\limits_{-1}^{z(s)} \psi_1(h(x))\psi_2(h(x)) \left(P_{n+1}(x)P_n(x) - P_{m+1}(x)P_m(x)\right) dx -$$

$$-\frac{(T-t)^2}{8} \sum_{j_1=m+1}^{n} \frac{1}{2j_1+1} \int\limits_{-1}^{z(s)} \left(P_{j_1+1}(y) - P_{j_1-1}(y)\right) \psi_1'(h(y)) \times$$

$$\times \Biggl(\left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s))\right) \psi_2(s) - \left(P_{j_1+1}(y) - P_{j_1-1}(y)\right) \psi_2(h(y)) -$$

$$- \frac{T-t}{2} \int\limits_{y}^{z(s)} \left(P_{j_1+1}(x) - P_{j_1-1}(x)\right) \psi_2'(h(x)) dx \Biggr) dy, \tag{2.647}$$

where

$$h(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $h(y)$ (see (2.55)).

Applying the estimate (2.67) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we finally obtain

$$\left|\sum_{j_1=m+1}^{n} C_{j_1j_1}(s)\right| \le C_1 \left(\frac{1}{n} + \frac{1}{m}\right) \int\limits_{-1}^{z(s)} \frac{dx}{\left(1-x^2\right)^{1/2}} +$$

$$+ C_2 \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \left(\int\limits_{-1}^{z(s)} \frac{dy}{\left(1-y^2\right)^{1/2}} + \frac{1}{\left(1-z^2(s)\right)^{1/4}} \int\limits_{-1}^{z(s)} \frac{dy}{\left(1-y^2\right)^{1/4}} + \right.$$

$$\left. + \int\limits_{-1}^{z(s)} \frac{1}{\left(1-y^2\right)^{1/4}} \int\limits_{y}^{z(s)} \frac{dx}{\left(1-x^2\right)^{1/4}} dy \right), \tag{2.648}$$

where constants C_1, C_2 do not depend on n and m.

We assume that $s \in (t,T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 2.24. Then

$$\left|\sum_{j_1=m+1}^{n} C_{j_1 j_1}(s)\right| \le C_3(s)\left(\frac{1}{n}+\frac{1}{m}+\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\right), \tag{2.649}$$

where constant $C_3(s)$ does not depend on n and m. Thus, the limit

$$\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1 j_1}(s) \tag{2.650}$$

exists for the polynomial case. For the trigonometric case, the existence of the limit (2.650) can be proved by analogy with the proof of Lemma 2.2 (Sect. 2.1.2). We also note that the existence of these limits follows from Sect. 2.1.4.

The relations (2.649) and (2.25) imply that

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le C_3(s)\left(\frac{1}{p}+\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\right) \le \frac{C_4(s)}{p}, \tag{2.651}$$

where constant $C_4(s)$ is independent of p.

For the trigonometric case, the analog of the inequality (2.651) can be obtained by analogy with (2.605) and (2.606) (see the proof of Lemma 2.2).

Combining (2.644)–(2.646), (2.651), we obtain the estimate (2.643). Theorem 2.27 is proved.

The arguments given earlier in Chapters 1 and 2 of this book allow us to formulate the following two theorems.

Theorem 2.28 [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{s,t} = \int\limits_t^{*s}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)},$$

where $i_1, i_2, i_3 = 0, 1, \ldots, m,$ *the following estimate*

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{s,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C(s)}{p}$$

is valid, where $s \in (t, T]$ *(*s *is fixed), constant* $C(s)$ *is independent of* p*,*

$$C_{j_3 j_2 j_1}(s) = \int\limits_t^s \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

Theorem 2.29 [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{s,t} = \int\limits_t^{*s} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1, i_2, i_3, i_4 = 0, 1, \ldots, m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{s,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C(s)}{p}$$

is valid, where $s \in (t, T]$ *(*s *is fixed), constant* $C(s)$ *is independent of* p*,*

$$C_{j_4 j_3 j_2 j_1}(s) = \int\limits_t^s \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

2.10 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbf{N}$). Proof of Hypotheses 2.2 and 2.3 Under the Condition of Convergence of Trace Series

In this section, we prove the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) under the condition of convergence of trace series. Let us introduce some notations.

Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{2.652}
$$

where

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

Consider the sum (1.53) with respect to all possible partitions (2.652)

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}}
$$

and the Fourier coefficient

$$
C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k \tag{2.653}
$$

corresponding to the function (1.6), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. At that we suppose $\phi_0(x) = 1/\sqrt{T-t}$.

Denote

$$
C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots j_1}\Bigg|_{(j_l j_l) \curvearrowright (\cdot)} \stackrel{\mathsf{def}}{=}
$$

$$\stackrel{\mathsf{def}}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k = \qquad (2.654)$$

$$= \sqrt{T-t}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k =$$

$$= \sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau),\ \psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Let

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Biggl|_{(j_lj_l)\curvearrowright j_m} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k = \qquad (2.655)$$

$$= \bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1},$$

i.e. $\bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,$ $\psi_{l-2}(\tau),\ \psi_{l-1}(\tau)\psi_l(\tau),\ \psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (2.656)$$

Introduce the following notation

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\sf def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (2.657)$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \qquad (2.658)$$

as follows: S_l multiplies (2.658) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (2.659)$$

Note that we write

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$
C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_2})\curvearrowright j_m,j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_1})\curvearrowright j_m,j_{g_1}=j_{g_2}},
$$

$$
C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=
$$

$$
=C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\quad \text{etc.}
$$

Since (2.659) is again the Fourier coefficient, then the action of superposition S_lS_m on (2.659) is obvious. For example, for $r=3$

$$
S_3S_2S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=
$$

$$
=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},
$$

$$
S_3S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=
$$

$$
=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},
$$

$$
S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\biggr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.
$$

Theorem 2.30 [33], [38], [39], [64]. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j=0}^{\infty}\int\limits_t^s \Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (2.660)$$

holds for all $s\in(t,T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t,T]$ *and the series on the right-hand side of* (2.660) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \qquad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s\in(t,T)$ *and for some* $\alpha,\beta>0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$, $j,p\in\mathbf{N}$, *and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau<\infty, \qquad \int\limits_t^T \left|\Psi_2(\tau)\right|d\tau<\infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (*see* (2.652)) *and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\sf def}{=} \bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \qquad (2.661)$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \qquad (2.662)$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (2.663)$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. First note that (2.660) is true (see the proof of Theorem 2.18). The proof of Theorem 2.30 ($k\ge 2$) will consist of several steps (the case $k=1$ is obvious (see (1.45)).

Step 1. Let us find a representation of the quantity

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

Recall the equality (1.272) (also see (1.316), (1.390))

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+$$

$$+\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}$$

$$(2.664)$$

w. p. 1, where notations are the same as in Theorem 1.2 and $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (1.23) (also see (1.258)).

From (2.664) we obtain

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}-$$

$$-\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{2.665}$$

w. p. 1.

By iteratively applying the formula (2.665) (also see (1.46)–(1.50)), we obtain the following representation of the product

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1}, \tag{2.666}$$

where

$$J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\mathsf{def}}{=}1$$

for $k=2r$.

Multiplying both sides of the equality (2.666) by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get w. p. 1

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \tag{2.667}
$$

Denote

$$
K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{2.668}
$$

$$
K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}), \tag{2.669}
$$

where $C_{j_k\ldots j_1}$ is defined by (2.663) and

$$
\prod_{l=1}^{0}\phi_{j_{q_l}}(t_{q_l}) \stackrel{\text{def}}{=} 1, \tag{2.670}
$$

i.e. $k=2r$ in (2.670).

The equality (2.667) can be written as

$$
J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{2.671}
$$

w. p. 1, where $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$ are defined by the equalities (2.668), (2.669), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral (1.16) (also see (2.1493)) and $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$, $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined by (1.23) (also see (1.258)).

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (2.667) or (2.671), we get w. p. 1 (see Theorems 1.1, 1.2 and (1.43))

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}= \tag{2.672}
$$

$$
=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\underset{p\to\infty}{\text{l.i.m.}}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{2.673}
$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral

$$
J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{2.674}
$$

If we prove that w. p. 1

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}, \qquad (2.675)$$

then (see (2.672), (2.675), and Theorem 2.12)

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \qquad (2.676)$$

w. p. 1, where notations in (2.676) are the same as in Theorem 2.12. Thus Theorem 2.30 will be proved.

From (2.671) we have that the multiple Stratonovich stochastic integral $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k is expressed as a sum of some constant value and multiple Wiener stochastic integrals

$$J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$$

and

$$J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$$

of multiplicities $k,\ k-2,\ k-4,\ \ldots,\ k-2[k/2]$ $(r=1,2,\ldots,[k/2])$.

The formulas (2.667), (2.671) can be considered as new representations of the Hu-Meyer formula for the case of a multidimensional Wiener process [142] (also see [139], [141]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (2.668)).

Further, we will use the representation (2.667) for $p_1 = \ldots = p_k = p$, i.e.

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+ \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} \times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{2.677}$$

For example, for $k = 2, 3, 4, 5, 6$ from (2.667) we have w. p. 1

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} = J'[K_{p_1p_2}]_{T,t}^{(i_1i_2)} + \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}, \tag{2.678}$$

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)} +$$

$$+ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}]_{T,t}^{(i_3)} + \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}]_{T,t}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}\Biggr), \tag{2.679}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J'[K_{p_1p_2p_3p_4}]_{T,t}^{(i_1i_2i_3i_4)} +$$

$$+ \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)} +$$

$$
+\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\right), \tag{2.680}
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=J'[K_{p_1p_2p_3p_4p_5}]_{T,t}^{(i_1i_2i_3i_4i_5)}+
$$

$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_3i_4i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_4i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_2i_3i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2i_3i_4)}+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_4i_5)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_3i_5)}+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_1i_2i_5)}+\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1i_2i_4)}+
$$

$$
+\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_2i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}\Biggr), \qquad (2.681)
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}=J'[K_{p_1p_2p_3p_4p_5p_6}]_{T,t}^{(i_1i_2i_3i_4i_5i_6)}+
$$

$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_3i_4i_5)}+
$$

$$
+\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_3i_4i_5)}+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_2i_4i_5)}+
$$

$$
+\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_2i_3i_5)}+\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_3i_4i_5i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_4i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_3i_5i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_2i_3i_4i_6)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_4i_5i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_3i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_3i_4i_6)}+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_2i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_2i_4i_6)}+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_1i_2i_3i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+
$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$

$$+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$

$$
+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\right). \tag{2.682}
$$

Note that the relation (2.680) can be written in the following form

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{p_4}\left(\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_4j_3j_1j_1}\right) J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_2=0}^{p_2}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_1,p_3\}} C_{j_4j_3j_2j_3}\right) J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_1,p_4\}} C_{j_4j_3j_2j_4}\right) J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_1=0}^{p_1}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_4j_3j_3j_1}\right) J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_2,p_4\}} C_{j_4j_3j_4j_1}\right) J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\neq 0\}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(\sum_{j_4=0}^{\min\{p_3,p_4\}} C_{j_4j_4j_2j_1}\right) J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\sum_{j_2=0}^{\min\{p_2,p_3\}}\sum_{j_4=0}^{\min\{p_1,p_4\}} C_{j_4j_2j_2j_4}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_3=0}^{\min\{p_1,p_3\}}\sum_{j_4=0}^{\min\{p_2,p_4\}} C_{j_4j_3j_4j_3}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_2=0}^{\min\{p_1,p_2\}}\sum_{j_4=0}^{\min\{p_3,p_4\}} C_{j_4j_4j_2j_2}\quad \text{w. p. 1.}
$$

Step 2. Let us prove that

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = 0 \tag{2.683}$$

or

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}, \tag{2.684}$$

where $l-1 \ge s+1$.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume for simplicity that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

Using the integration order replacement, we have

$$C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s \times$$

$$\times \Biggl(\int\limits_{t_{s+1}}^T \phi_{j_{s+2}}(t_{s+2})\ldots \int\limits_{t_{l-2}}^T \phi_{j_{l-1}}(t_{l-1}) \int\limits_{t_{l-1}}^T \phi_{j_l}(t_l) \int\limits_{t_l}^T \phi_{j_{l+1}}(t_{l+1})\ldots$$

$$\ldots \int\limits_{t_{k-1}}^T \phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_l dt_{l-1}\ldots dt_{s+2} \Biggr) dt_{s+1} =$$

$$
= \int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \underbrace{\int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)} dt_s \times
$$

$$
\times \int\limits_{t_{s+1}}^T \phi_{j_l}(t_l) \underbrace{\int\limits_{t_l}^T \phi_{j_{l+1}}(t_{l+1}) \ldots \int\limits_{t_{k-1}}^T \phi_{j_k}(t_k) dt_k \ldots dt_{l+1}}_{H_{j_k\ldots j_{l+1}}(t_l)} \times
$$

$$
\times \left(\underbrace{\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1}}_{Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1})} dt_l \right) dt_{s+1} =
$$

$$
= \int\limits_t^T \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1}\ldots j_1}(t_s) dt_s \times
$$

$$
\times \int\limits_{t_{s+1}}^T \phi_{j_l}(t_l) H_{j_k\ldots j_{l+1}}(t_l) Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1}) dt_l dt_{s+1}. \qquad (2.685)
$$

Applying the additive property of the integral, we obtain

$$
Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1}) =
$$

$$
= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1} =
$$

$$
= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3}) \int\limits_t^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} dt_{s+3} \ldots dt_{l-1} -
$$

$$-\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}}\phi_{j_{s+3}}(t_{s+3})dt_{s+3}\ldots dt_{l-1}\int\limits_{t}^{t_{s+1}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}=$$

$$\ldots$$

$$=\sum_{m=1}^{d}h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}),\qquad (2.686)$$

where $d<\infty$.

Combining (2.685) and (2.686), we have

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=\sum_{m=1}^{d}\left(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=0}^{p}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right).\qquad (2.687)$$

Using the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l=$$

$$=\int\limits_{t}^{T}\mathbf{1}_{\{\tau<t_{s+1}\}}G_{j_{s-1}\ldots j_1}(\tau)\cdot\mathbf{1}_{\{\tau>t_{s+1}\}}H_{j_k\ldots j_{l+1}}(\tau)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(\tau)d\tau=0.\qquad (2.688)$$

From (2.687) and (2.688) we get

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$
= -\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.
$$

$$
\left.\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\right). \qquad (2.689)
$$

Combining Condition 2 of Theorem 2.30 and (2.685)–(2.687), (2.689), we have

$$
\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =
$$

$$
= -\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.
$$

$$
\left.\times\int\limits_{t_{s+1}}^T \phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\right) =
$$

$$
= -\sum_{j_l=p+1}^{\infty}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =
$$

$$
= -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}. \qquad (2.690)
$$

The equality (2.690) implies (2.683), (2.684).

Step 3. Under the conditions of Theorem 2.30 we prove that

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} =$$

$$= \frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_l j_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} \qquad (2.691)$$

or

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} = \frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_l j_l)\curvearrowright(\cdot)}. \qquad (2.692)$$

Denote

$$C_{j_{l-2}\ldots j_1}(t_{l-1}) = \int\limits_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}.$$

Using the integration order replacement and Condition 1 of Theorem 2.30, we obtain

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} =$$

$$= \sum_{j_l=0}^{\infty} \int\limits_t^{T} \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times$$

$$\times \int\limits_t^{t_{l+1}} \psi_l(t_l)\phi_{j_l}(t_l) \int\limits_t^{t_l} \psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})C_{j_{l-2}\ldots j_1}(t_{l-1})dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \sum_{j_l=0}^{\infty} \int\limits_t^{T} \psi_l(t_l)\phi_{j_l}(t_l) \int\limits_t^{t_l} \psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})C_{j_{l-2}\ldots j_1}(t_{l-1})dt_{l-1}\times$$

$$\times \int\limits_{t_l}^{T} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\ldots \int\limits_{t_{k-1}}^{T} \psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_l =$$

$$
=\frac{1}{2}\sum_{j_l=0}^{\infty}\int\limits_t^T \psi_l(t_l)\psi_{l-1}(t_l)C_{j_{l-2}\ldots j_1}(t_l)\times
$$

$$
\times\int\limits_{t_l}^T \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\ldots\int\limits_{t_{k-1}}^T \psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_l=
$$

$$
=\frac{1}{2}\sum_{j_l=0}^{\infty}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times\int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)C_{j_{l-2}\ldots j_1}(t_l)dt_l dt_{l+1}\ldots dt_k=
$$

$$
=\frac{1}{2}C_{j_k\ldots j_1}\Biggr|_{(j_l j_l)\curvearrowright(\cdot)}. \qquad (2.693)
$$

The equalities (2.691) and (2.692) are proved.

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (2.677), we have (see (1.43))

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \qquad (2.694)
$$

w. p. 1.

Taking into account (2.684) and (2.691), we obtain for $r=1$

$$
\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
= -\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2>g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+
$$

$$
+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \frac{1}{2} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowleft(\cdot),j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}-
$$

$$
-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =
$$

$$
= -\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2}} \times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+
$$

$$
+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \frac{1}{2} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowleft(\cdot),j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} = \qquad (2.695)
$$

$$
= \frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}} J[\psi^{(k)}]_{T,t}^{g_1} + \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)1,g_1,g_2} \quad \text{w. p. 1}, \qquad (2.696)
$$

where $J[\psi^{(k)}]_{T,t}^{g_1}$ $(g_1 = 1, 2, \ldots, k-1)$ is defined by (2.387),

$$
R_{T,t}^{(p)1,g_1,g_2} = -\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}.
$$

Let us explain the transition from (2.695) to (2.696). We have for $g_2 = g_1+1$

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0,j_{g_1}=\ j_{g_2}}\times$$

$$\times\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times$$

$$\times\zeta_{j_{m_1}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(0 i_{q_1}\ldots i_{q_{k-2}})}= \qquad (2.697)$$

$$=\frac{1}{2}J[\psi^{(k)}]_{T,t}^{g_1}\quad \text{w. p. 1}, \qquad (2.698)$$

where

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2},g_2=g_1+1}=$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{g_1+3}}\psi_l(t_{g_1+2})\phi_{j_{g_1+2}}(t_{g_1+2})\int\limits_t^{t_{g_1+2}}\psi_{g_1+1}(t_{g_1})\psi_{g_1}(t_{g_1})\phi_{j_{m_1}}(t_{g_1})\times$$

$$\times \int\limits_{t}^{t_{g_1}} \psi_l(t_{g_1-1})\phi_{j_{g_1-1}}(t_{g_1-1}) \ldots \int\limits_{t}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_{g_1-1}dt_{g_1}dt_{g_1+2} \ldots dt_k,$$

$$\zeta_{j_{m_1}}^{(0)} = \int\limits_{t}^{T} \phi_{j_{m_1}}(\tau)d\mathbf{w}_{\tau}^{(0)} = \int\limits_{t}^{T} \phi_{j_{m_1}}(\tau)d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_{m_1} = 0 \\ & , \\ 0 & \text{if } j_{m_1} \neq 0 \end{cases} \tag{2.699}$$

$$\phi_0(\tau) = \frac{1}{\sqrt{T-t}}. \tag{2.700}$$

The transition from (2.697) to (2.698) is based on (1.43) or (1.319).

By Condition 3 of Theorem 2.30 we have (also see the property (2.368) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)1,g_1,g_2}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\neq g_1,g_2}}^{p} \left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2}\right)^2 = 0,$$

where constant K does not depend on p.

Thus

$$\mathbf{1}_{\{i_{g_1} = i_{g_2} \neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \mathbf{1}_{\{j_{g_1} = j_{g_2}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =$$

$$= \frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}} J[\psi^{(k)}]_{T,t}^{g_1} \quad \text{w. p. 1.}$$

Involving into consideration the second pair $\{g_3, g_4\}$ (the first pair is $\{g_1, g_2\}$), we obtain from (2.695) for $r = 2$

$$\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =$$

$$= \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \left(\frac{1}{4} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} - \right.$$

$$-\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \mathbf{1}_{\{g_4=g_3+1\}} -$$

$$-\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \mathbf{1}_{\{g_2=g_1+1\}} +$$

$$\left. + \sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\right) J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} = \qquad (2.701)$$

$$= \frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_2,s_1} + \prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)2,g_1,g_2,g_3,g_4} \qquad (2.702)$$

w. p. 1, where $g_3 \stackrel{\text{def}}{=} s_2$, $g_1 \stackrel{\text{def}}{=} s_1$, $(s_2,s_1)\in \mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,2}$ is defined by (2.388),

$$R_{T,t}^{(p)2,g_1,g_2,g_3,g_4} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4} - \right.$$

$$\left. -S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\} - S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}.$$

Let us explain the transition from (2.701) to (2.702). We have for $g_2=g_1+1$, $g_4=g_3+1$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \frac{1}{4} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times$$

$$\times \prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =$$

$$
= \frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright 0(j_{g_4}j_{g_3})\curvearrowright 0,j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times
$$

$$
\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \zeta_0^{(0)}\zeta_0^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
= \frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times
$$

$$
\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
= \frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times
$$

$$
\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(00i_{q_1}\ldots i_{q_{k-4}})} = \qquad (2.703)
$$

$$
= \frac{1}{4} J[\psi^{(k)}]_{T,t}^{s_2,s_1} \quad \text{w. p. 1.} \qquad (2.704)
$$

The transition from (2.703) to (2.704) is based on (1.43) or (1.319).

Note that

$$
C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}
$$

is the Fourier coefficient, where $g_2 = g_1 + 1$. Therefore, the value

$$
C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} =
$$

$$
= C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}(j_{g_3}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}
$$

is determined recursively using (2.655) in an obvious way for $g_2 = g_1 + 1$ and $g_4 = g_3 + 1$.

By Condition 3 of Theorem 2.30 we have (also see the property (2.368) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}\right)^2\right\} \le$$

$$\le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right)^2 + \right.$$

$$\left. + \left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2 + \left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2\right) = 0,$$

where constant K is independent of p.

Thus

$$\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} = \frac{1}{4}\prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_2,s_1} \quad \text{w. p. 1},$$

where $g_3 \stackrel{\mathsf{def}}{=} s_2$, $g_1 \stackrel{\mathsf{def}}{=} s_1$, $(s_2, s_1) \in \mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,2}$ is defined by (2.388).

Involving into consideration the third pair $\{g_6, g_5\}$ ($\{g_1, g_2\}$ is the first pair and $\{g_4, g_3\}$ is the second pair), we obtain from (2.701) for $r = 3$

$$\prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{3} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} = \prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4,g_5,g_6}}^{p} \Biggl(\frac{1}{2^3} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \times$$

$$\times \prod_{s=1}^{3} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} -$$

$$-\frac{1}{2^2}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_4=g_3+1\}}\mathbf{1}_{\{g_6=g_5+1\}} -$$

$$-\frac{1}{2^2}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_6=g_5+1\}} -$$

$$-\frac{1}{2^2}\sum_{j_{g_5}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}} +$$

$$+\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_6=g_5+1\}} +$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_4=g_3+1\}} +$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot), j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}} -$$

$$-\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=\ j_{g_2}, j_{g_3}=\ j_{g_4}, j_{g_5}=\ j_{g_6}}\Biggr) \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} =$$

$$= \frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1} + \prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}$$

w. p. 1, where $g_{2i-1} \stackrel{\text{def}}{=} s_i$; $i = 1, 2, 3$, $(s_3, s_2, s_1) \in \mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,3}$ is defined by (2.388),

$$
R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p} \left(-\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6} + \right.
$$

$$
+S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} + S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} +
$$

$$
+S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} -
$$

$$
-S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} - S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} -
$$

$$
\left. -S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right) J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}.
$$

By Condition 3 of Theorem 2.30 we have (also see the property (2.368) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p} \left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right)^2 + \right.
$$

$$
+\left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +
$$

$$
+\left(S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +
$$

$$
+\left(S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +
$$

$$
\left. +\left(S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2\right) = 0,
$$

where constant K does not depend on p.

Thus

$$
\underset{p\to\infty}{\text{l.i.m.}}\ \prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{3}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}\quad \text{w. p. 1},
$$

where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,3,\ (s_3,s_2,s_1)\in \mathrm{A}_{k,3},\ J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,3}$ is defined by (2.388).

Repeating the previous steps, we obtain for an arbitrary r ($r=1,2,\ \ldots,$ $[k/2]$)

$$
\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$
+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}= \tag{2.705}
$$

$$
=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{2.706}
$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2]\,,\ (s_r,\ldots,s_1)\in\mathrm{A}_{k,r},$ $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,r}$ is defined by (2.388),

$$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl((-1)^r\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r}S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \qquad (2.707)$$

Let us explain the transition from (2.705) to (2.706). We have for $g_2=g_1+1,\ldots,g_{2r}=g_{2r-1}+1$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$
=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}\ldots\zeta_{j_{m_{2r-1}}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots0i_{q_1}\ldots i_{q_{k-2r}})}= \tag{2.708}
$$

$$
=\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\text{w. p. 1.} \tag{2.709}
$$

The transition from (2.708) to (2.709) is based on (1.43) or (1.319).

Note that

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}
$$

is the Fourier coefficient, where $g_2=g_1+1$. Therefore, the value

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}=
$$

$$
=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d-1}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}
$$

is determined recursively using (2.655) in an obvious way for $g_2=g_1+1,\ldots,$ $g_{2d}=g_{2d-1}+1$ and $d=2,\ldots,r$.

By Condition 3 of Theorem 2.30 we have (also see the property (2.368) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\le$$

$$\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\left(\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2+\right.$$

$$+\sum_{l_1=1}^{r}\left(S_{l_1}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$+\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$\ldots$$

$$\left.+\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right)=0,$$

where constant K does not depend on p.

So we have

$$\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\text{w. p. 1},\tag{2.710}$$

where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2],\ (s_r,\ldots,s_1)\in\mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (2.387) and $\mathrm{A}_{k,r}$ is defined by (2.388).

Note that

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \Biggl|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1} A_{g_1,g_3,\ldots,g_{2r-1}} =
$$

$$
= \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} A_{s_1,s_2,\ldots,s_r}, \tag{2.711}
$$

where $A_{g_1,g_3,\ldots,g_{2r-1}}$, $A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1}=s_i$; $i=1,2,\ldots,r$; $r=1,2,\ldots,[k/2]$, $\mathrm{A}_{k,r}$ is defined by (2.388):

$$
\mathrm{A}_{k,r}=\Bigl\{(s_r,\ldots,s_1):\ s_r>s_{r-1}+1,\ldots,s_2>s_1+1,\ s_r,\ldots,s_1=1,\ldots,k-1\Bigr\}.
$$

Using (2.694), (2.710), (2.711), and Theorem 2.12, we finally get

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =
$$

$$
= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{2.712}
$$

w. p. 1, where (see (2.387))

$$
J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\text{def}}{=} \prod_{q=1}^{r}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\neq 0\}}\ \times
$$

$$
\times \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_{s_r+3}}\psi_{s_r+2}(t_{s_r+2})\int\limits_t^{t_{s_r+2}}\psi_{s_r}(t_{s_r+1})\psi_{s_r+1}(t_{s_r+1})\times
$$

$$
\times \int\limits_t^{t_{s_r+1}}\psi_{s_r-1}(t_{s_r-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times
$$

$$
\times \int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots
$$

$$\ldots \ d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})} dt_{s_r+1} d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})} \ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{2.713}$$

Theorem 2.30 is proved.

Remark 2.4. *Let us make a number of remarks about Theorem* 2.30. *An expansion similar to* (2.662) *was obtained in* [142] *for the multiple Stratonovich stochastic integral* (2.978). *The proof from* [142] *is somewhat simpler than the proof proposed in this section. However, the results from* [142] *were obtained under the condition of convergence of trace series. The verification of this condition for the kernel* (1.6) *is a separate problem. In our proof we essentially use the structure of the Fourier coefficients* (2.663) *corresponding to the kernel* (1.6). *This circumstance actually made it possible to prove Theorem* 2.30 *using not the condition of finiteness of trace series, but using the condition of convergence to zero of explicit expressions for the remainders of the mentioned series. This leaves hope that it is possible to prove an analogue of Theorems* 2.24–2.26, 2.37–2.39 *for the case of an arbitrary* k $(k \in \mathbf{N})$.

Note that under the conditions of Theorem 2.30 (*also see* (2.684), (2.691)) *the sequential order of the series*

$$\sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty}$$

in (2.656) *is not important. We also note that Conditions* 1, 2 *of Theorem* 2.30 *are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space* $L_2([t,T])$ (*see the proofs of Theorems* 2.1, 2.2, 2.8, 2.9, 2.27). *Moreover, the equality* (2.660) *is true for an arbitrary basis in* $L_2([t,T])$ (*see* (2.535)). *It is easy to see that in the proofs of Theorems* 2.1–2.9 *the conditions of Theorem* 2.30 *are verified for various special cases of iterated Stratonovich stochastic integrals of multiplicities* 2–4.

Taking into account Theorem 1.11, we can formulate an analogue of Theorem 2.30 for the case of integration interval $[t,s]$ $(s \in (t,T)$; the case $s = T$ is considered in Theorem 2.30) of iterated Stratonovich stochastic integrals of multiplicity k $(k \in \mathbf{N})$.

Denote

$$\bar{C}_{j_k \ldots j_q \ldots j_1}^{(p)}(s) \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}(s) \Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}$$

and introduce the following notation

$$S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}{\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}\sum\limits_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum\limits_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum\limits_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum\limits_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum\limits_{j_{g_{3}}=p+1}^{\infty}\ \sum\limits_{j_{g_{1}}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

where $l=1,2,\ldots,r,$

$$C_{j_k\ldots j_1}(s)\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot)}$$

is defined by analogy with (2.654),

$$C_{j_k\ldots j_1}(s)=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k. \qquad (2.714)$$

Theorem 2.31 [33], [38], [39], [64]. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum\limits_{j=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (2.715)$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (2.715) converges absolutely.

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\qquad\left|\int\limits_{\tau}^s\phi_j(\theta)\Phi_2(\theta)d\theta\right|\le\frac{\Psi_2(s,\tau)}{j^{1/2+\alpha}},$$

$$\left|\sum\limits_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_3(s)}{p^{\beta}}$$

hold for all s, τ *such that* $t < \tau < s < T$ *and for some* $\alpha, \beta > 0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$, $j, p \in \mathbf{N}$, *and*

$$\int\limits_t^s |\Psi_1(\tau)\Psi_2(s,\tau)|\, d\tau < \infty, \qquad \int\limits_t^s |\Psi_3(\tau)|\, d\tau < \infty$$

for all $s \in (t, T)$.

3. *The condition*

$$\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (*see* (2.652)) *and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\textsf{def}}{=} \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} = \int\limits_t^{*s} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{2.716}$$

the following expansion

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (2.714), l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m$, $s \in (t, T)$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

It is easy to see that the estimates (1.211), (1.217), (2.270), (2.294), and the results of Sect. 2.9 imply the fulfillment of Condition 2 of Theorem 2.31 for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$.

Also the equality (2.535) guarantees the fulfillment of Condition 1 of Theorem 2.31 for these two systems of functions.

It should be noted that (see (2.707))

$$(-1)^r \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Biggr|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}} +$$

$$+(-1)^{r-1} \sum_{l_1=1}^{r} S_{l_1} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Biggr|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} +$$

$$+(-1)^{r-2} \sum_{\substack{l_1, l_2=1 \\ l_1 > l_2}}^{r} S_{l_1} S_{l_2} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Biggr|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} +$$

$$\ldots$$

$$+(-1)^{1} \sum_{\substack{l_1, l_2, \ldots, l_{r-1}=1 \\ l_1 > l_2 > \ldots > l_{r-1}}}^{r} S_{l_1} S_{l_2} \ldots S_{l_{r-1}} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Biggr|_{q \neq g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} =$$

$$= \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggr|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggr|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot),\, j_{g_1}=\, j_{g_2}, \ldots, j_{g_{2r-1}}=\, j_{g_{2r}}}, \quad (2.717)$$

where the meaning of the notations used in (2.707) is preserved.

For example, from (2.717) for the case $r = 2$ we get

$$\sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Biggr|_{j_{g_1}=\, j_{g_2}, j_{g_3}=\, j_{g_4}} -$$

$$
-\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} -
$$

$$
-\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} =
$$

$$
=\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} -
$$

$$
-\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}.
$$

As a result, Condition 3 of Theorem 2.30 can be replaced by a weaker condition

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -\right.
$$

$$
\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2=0, \tag{2.718}
$$

where $r=1,2,\ldots,[k/2]$.

However, Condition 3 of Theorem 2.30 itself contains a way of proving of the condition (2.718), which is partially realized in the proof of Theorems 2.33–2.36 (see below).

In fact, when proving Theorem 2.35 (the case $r=3$ is proved in Theorem 2.36 for $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$), we proved the following equality

$$
\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} =
$$

$$
=\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}.
$$

On the other hand, iterative application of (2.684), (2.691) gives

$$
\sum_{j_{g_1}=0}^{\infty}\sum_{j_{g_3}=0}^{\infty}\ldots\sum_{j_{g_{2r-1}}=0}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$
=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}},
$$

where $r=1,2,\ldots,[k/2]$.

Moreover, we have (see (2.709))

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}-\right.
$$

$$
\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$
+\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} - \right.
$$

$$
\left. -\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+
$$

$$
+\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1.} \tag{2.719}
$$

Using (2.719) and the condition (2.718), we obtain (2.710). This means that we get (2.712). Thus the expansion (2.662) is proved.

Analyzing the proof of Theorems 2.30 and 2.12 and taking into account the above arguments, it is easy to see that the following theorem is true.

Theorem 2.32 [38], [39]. *Assume that the continuous functions $\psi_1(\tau),\ldots,$ $\psi_k(\tau)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following condition*

$$
\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times
$$

$$
\times\left(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\sum_{j_{g_3}=0}^{\min\{p_{g_3},p_{g_4}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.
$$

$$
\left. -\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2=0 \tag{2.720}
$$

is satisfied for all $r=1,2,\ldots,[k/2]$ and for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652)). Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

2.11 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case $p_1=p_2=p_3\to\infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

In this section, we present a simple proof of Theorem 2.8 based on Theorem 2.30. In this case, the conditions of Theorem 2.8 will be weakened.

First, consider the following equalities

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int\limits_{t_1}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau, \tag{2.721}$$

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_1(\theta)\phi_j(\theta)\int\limits_{\theta}^{t_2}\Phi_2(\tau)\phi_j(\tau)d\tau d\theta \tag{2.722}$$

that will be used further, where $t \le t_1 < t_2 \le T$, $\Phi_1(\tau), \Phi_2(\tau) \in L_2([t,T])$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is the same as in the conditions of Theorem 2.8.

The equality (2.722) is proved in Sect. 2.7.2 (see (2.546)–(2.548)). Using (2.722) and Fubini's Theorem, we get (2.721).

Theorem 2.33 [33], [38], [39], [64]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \qquad (2.723)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$,

$$C_{j_3 j_2 j_1} = \int\limits_t^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As noted in Remark 2.4, Conditions 1 and 2 of Theorem 2.30 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 2.30 for the iterated Stratonovich stochastic integral (2.723). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_3 j_1 j_1} \right)^2 = 0, \qquad (2.724)$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1}\right)^2=0, \tag{2.725}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=0. \tag{2.726}$$

We have

$$\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_1j_1}\right)^2=$$

$$=\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{2.727}$$

$$=\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2\le \tag{2.728}$$

$$\le\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{2.729}$$

$$=\int\limits_t^T\psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3\le \tag{2.730}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{2.731}$$

if $p\to\infty$, where constant K does not depend on p.

Note that the transition from (2.727) to (2.728) is based on the estimate (2.651) for the polynomial case and its analogue for the trigonometric case, the transition from (2.729) to (2.730) is based on the Parseval equality, and the transition from (2.730) to (2.731) is also based on the estimate (2.651) and its analogue for the trigonometric case.

By analogy with the previous case we have

$$\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1}\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2= \tag{2.732}$$

$$=\sum_{j_1=0}^{p}\left(\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2\le \tag{2.733}$$

$$\le\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2=$$

$$=\int\limits_t^T\psi_1^2(t_1)\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_1(t_3)\phi_{j_3}(t_3)dt_3dt_2\right)^2dt_1\le \tag{2.734}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{2.735}$$

if $p\to\infty$, where constant K is independent of p.

The transition from (2.732) to (2.733) is based on an analogue of the estimate (2.651) for the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2\right|$$

for the polynomial and trigonometric cases, the transition from (2.734) to (2.735) is also based on the mentioned analogue of the estimate (2.651).

Further, we have

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=$$

$$=\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2=$$

$$
=\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2= \tag{2.736}
$$

$$
=\sum_{j_2=0}^{p}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2\le \tag{2.737}
$$

$$
\le\sum_{j_2=0}^{\infty}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2=
$$

$$
=\int\limits_t^T\psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2dt_2. \tag{2.738}
$$

The transition from (2.736) to (2.737) is based on the estimate (2.295) and its obvious analogue for the trigonometric case. However, the estimate (2.295) cannot be used to estimate the right-hand side of (2.738), since we get the divergent integral. For this reason, we will obtain a new estimate based on the relation (2.293).

From (2.157) and the estimate $|P_j(y)|\le 1,\ y\in[-1,1]$ we obtain

$$
|P_j(y)|=|P_j(y)|^{\varepsilon}\cdot|P_j(y)|^{1-\varepsilon}\le|P_j(y)|^{1-\varepsilon}<\frac{C}{j^{1/2-\varepsilon/2}(1-y^2)^{1/4-\varepsilon/4}}, \tag{2.739}
$$

where $y\in(-1,1)$, $j\in\mathbf{N}$, $\varepsilon\in(0,1)$ is an arbitrary small positive real number.

Combining (2.293) and (2.739), we have the following estimate

$$
\left|\int\limits_t^s\psi_1(\tau)\phi_{j_1}(\tau)d\tau\right|<\frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right), \tag{2.740}
$$

where $j_1\in\mathbf{N}$, $s\in(t,T)$, $z(s)$ is defined by (2.20), constant C does not depend on j_1.

Similarly to (2.740) we obtain

$$
\left|\int\limits_s^T\psi_3(\tau)\phi_{j_1}(\tau)d\tau\right|<\frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right), \tag{2.741}
$$

where $j_1 \in \mathbf{N}$, $s \in (t, T)$, constant C does not depend on j_1.

Combining (2.294) and (2.741), we have

$$\left| \int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau \int\limits_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau \right| <$$

$$< \frac{L}{(j_1)^{2-\varepsilon/2}} \left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} + 1 \right) \left(\frac{1}{(1-z^2(s))^{1/4}} + 1 \right), \qquad (2.742)$$

where $j_1 \in \mathbf{N}$, $s \in (t, T)$, $z(s)$ is defined by (2.20), constant L does not depend on j_1.

Observe that

$$\sum_{j_1=p+1}^{\infty} \frac{1}{(j_1)^{2-\varepsilon/2}} \le \int\limits_p^{\infty} \frac{dx}{x^{2-\varepsilon/2}} = \frac{1}{(1-\varepsilon/2)p^{1-\varepsilon/2}}. \qquad (2.743)$$

Applying (2.742) and (2.743) to estimate the right-hand side of (2.738) gives

$$\sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1} \right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \qquad (2.744)$$

if $p \to \infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

The estimation of the right-hand side of (2.738) for the trigonometric case is carried out using the estimates (2.270), (2.271). At that we obtain the estimate (2.744) with $\varepsilon = 0$. Theorem 2.33 is proved.

2.12 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case $p_1 = \ldots = p_4 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau), \ldots, \psi_4(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 2.34 [33], [38], [39], [64]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in* *the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{2.745}$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3,i_4=0,1,\ldots,m,$

$$C_{j_4j_3j_2j_1}=\int\limits_t^{T}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\times$$
$$\times dt_2dt_3dt_4$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. As noted in Remark 2.4, Conditions 1 and 2 of Theorem 2.30 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 2.30 for the iterated Stratonovich stochastic integral (2.745). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=0, \tag{2.746}$$

$$\lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=0, \tag{2.747}$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=0, \tag{2.748}$$

$$\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=0, \tag{2.749}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=0, \tag{2.750}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=0, \tag{2.751}$$

$$\lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2=0, \tag{2.752}$$

$$\lim_{p\to\infty}\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2=0, \tag{2.753}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=0, \tag{2.754}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0, \tag{2.755}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{2.756}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{2.757}$$

where in (2.755)–(2.757) we use the notation (2.654).

Applying arguments similar to those we used in the proof of Theorem 2.33, we obtain for (2.746)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{2.758}$$

$$
= \sum_{j_3,j_4=0}^{p} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3)\times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \right)^2 \le \qquad (2.759)
$$

$$
\le \sum_{j_3,j_4=0}^{\infty} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3)\times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \right)^2 = \qquad (2.760)
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}}\psi_4^2(t_4)\psi_3^2(t_3)\times
$$

$$
\times \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \right)^2 dt_3dt_4 \le \qquad (2.761)
$$

$$
\le \frac{K}{p^2} \ \to \ 0 \qquad (2.762)
$$

if $p \to \infty$, where constant K is independent of p.

Note that the transition from (2.758) to (2.759) is based on the estimate (2.651) for the polynomial case and its analogue for the trigonometric case, the transition from (2.760) to (2.761) is based on the Parseval equality, and the transition from (2.761) to (2.762) is also based on the estimate (2.651) and its analogue for the trigonometric case.

Further, we have for (2.747)

$$
\sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4j_1j_2j_1} \right)^2 = \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)\times \right.
$$

$$
\left. \times \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 \right)^2 = \qquad (2.763)
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.
$$

$$
\left. \times \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4 \right)^2 = \qquad (2.764)
$$

$$
= \sum_{j_2,j_4=0}^{p} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4 \right)^2 \le
$$

$$
\le \sum_{j_2,j_4=0}^{\infty} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.
$$

$$
\left. \times \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<t_4\}} \psi_4^2(t_4)\psi_2^2(t_2) \times
$$

$$
\times \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_1}(t_3)dt_3 \right)^2 dt_2dt_4 \le
$$

$$
\le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \qquad (2.765)
$$

if $p \to \infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

The relation (2.765) was obtained by the same method as (2.762). Note that in obtaining (2.765) we used the estimates (1.211) and (2.740) for the polynomial case and (1.217) and (2.270) for the trigonometric case. We also used the integration order replacement in the iterated Riemann integrals (see (2.763), (2.764)).

Repeating the previous steps for (2.748) and (2.749), we get

$$
\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.
$$

$$
\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.
$$

$$
\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2=
$$

$$
=\sum_{j_2,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.
$$

$$
\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2\le
$$

$$
\le\sum_{j_2,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.
$$

$$
\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\psi_3^2(t_3)\psi_2^2(t_2)\times
$$

$$
\times\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4\right)^2dt_2dt_3\le
$$

$$
\le\frac{K}{p^2}\ \to\ 0 \qquad (2.766)
$$

if $p\to\infty$, where constant K does not depend on p;

$$
\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\times\right.
$$

$$
\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\int\limits_{t_1}^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\right)^2=
$$

$$
=\sum_{j_1,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\right)^2\le
$$

$$
\le\sum_{j_1,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_4\}}\psi_4^2(t_4)\psi_1^2(t_1)\times
$$

$$
\times\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2\right)^2dt_1dt_4. \qquad (2.767)
$$

Note that, by virtue of the additivity property of the integral, we have

$$\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2= \tag{2.768}$$

$$=\sum_{j_2=p+1}^{\infty}\int\limits_{t}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3-$$

$$-\sum_{j_2=p+1}^{\infty}\int\limits_{t}^{t_1}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3-$$

$$-\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3\int\limits_{t}^{t_1}\psi_2(t_2)\phi_{j_2}(t_2)dt_2. \tag{2.769}$$

However, all three series on the right-hand side of (2.769) have already been evaluated in (2.762) and (2.765). From (2.767) and (2.769) we finally obtain

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{2.770}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

In complete analogy with (2.765), we have for (2.750)

$$\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t}^{T}\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_{t}^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2\le
$$

$$
\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_3\}}\psi_3^2(t_3)\psi_1^2(t_1)\times
$$

$$
\times\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4\right)^2dt_1dt_3\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \quad (2.771)
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

We have for (2.751)

$$
\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.
$$

$$
\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
= \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} \int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1 \right)^2 =
$$

$$
= \sum_{j_1,j_2=0}^{p} \left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1 \right)^2 \le
$$

$$
\le \sum_{j_1,j_2=0}^{\infty} \left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_2}(t_2)\times \right.
$$

$$
\left. \times \sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2\}}\psi_1^2(t_1)\psi_2^2(t_2)\times
$$

$$
\times \left(\sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3 \right)^2 dt_2dt_1. \qquad (2.772)
$$

It is easy to see that the integral (see (2.772))

$$
\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^T \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3
$$

is similar to the integral from the formula (2.768) if in the last integral we substitute $t_4 = T$. Therefore, by analogy with (2.770), we obtain

$$
\sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1} \right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \qquad (2.773)
$$

if $p \to \infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

Now consider (2.752)–(2.754). We have for (2.752) (see **Step 2** in the proof of Theorem 2.30)

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2. \qquad (2.774)$$

Consider (2.750) and (2.771). We have

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2\Biggl|_{j_1=j_3} \le$$

$$\le \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \qquad (2.775)$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (2.774) and (2.775), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \to 0$$

if $p \to \infty$, where constant K_1 does not depend on p.

Similarly for (2.753) we have (see (2.749), (2.770))

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2, \qquad (2.776)$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2\Biggl|_{j_1=j_4} \le$$

$$
\le \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{2.777}
$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. Combining (2.776) and (2.777), we obtain

$$
\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0
$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider (2.754). Using (2.691), we obtain

$$
\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{\infty} C_{j_3j_3j_1j_1} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1} =
$$

$$
= \frac{1}{2}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1}, \tag{2.778}
$$

where (see (2.654))

$$
C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)} = \int\limits_t^T \psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2dt_3dt_4.
$$

From the estimate (2.83) (polynomial case) and its analogue for the trigonometric case (see the proof of Lemma 2.2, Sect. 2.1.2) we get

$$
\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{2.779}
$$

where constant C is independent of p.

Further, we have (see (2.773))

$$
\left(\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 =
$$

$$
= (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2\Bigg|_{j_1=j_2} \le
$$

$$
\le (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}}, \qquad (2.780)
$$

where constant K_1 does not depend on p.

Combining (2.778)–(2.780), we obtain

$$
\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le \frac{K_2}{p^{1-\varepsilon}} \ \to\ 0
$$

if $p\to\infty$, where constant K_2 does not depend on p.

Let us prove (2.755)–(2.757). It is not difficult to see that the estimate (2.779) proves (2.755).

Using the integration order replacement, we obtain

$$
\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} =
$$

$$
= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_4 =
$$

$$
= \sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\psi_2(t_2)\int\limits_{t_2}^T\psi_4(t_4)\psi_3(t_4)dt_4\right)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2, \quad (2.781)
$$

$$
\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} =
$$

$$
= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_3dt_4 =
$$

$$
= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =
$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \left(\int\limits_t^{t_4} - \int\limits_t^{t_1} \right) \psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int\limits_t^T \left(\psi_4(t_4) \int\limits_t^{t_4} \psi_3(t_3)\psi_2(t_3)dt_3 \right) \phi_{j_1}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_4 - \quad (2.782)$$

$$- \sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int\limits_t^{t_4} \left(\psi_1(t_1) \int\limits_t^{t_1} \psi_3(t_3)\psi_2(t_3)dt_3 \right) \phi_{j_1}(t_1)dt_1dt_4. \quad (2.783)$$

Applying the estimate (2.83) (polynomial case) and its analogue for the trigonometric case (see the proof of Lemma 2.2, Sect. 2.1.2) to the right-hand sides of (2.781)–(2.783), we get

$$\left| \sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)} \right| \le \frac{C}{p}, \quad (2.784)$$

$$\left| \sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)} \right| \le \frac{C}{p}, \quad (2.785)$$

where constant C is independent of p. The estimates (2.784), (2.785) prove (2.756), (2.757).

The relations (2.746)–(2.757) are proved. Theorem 2.34 is proved.

2.13 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case $p_1 = \ldots = p_5 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau), \ldots, \psi_5(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 2.35 [33], [38], [39], [64]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \ldots, \psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich* *stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \qquad (2.786)$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5 = 0,1,\ldots,m,$

$$C_{j_5\ldots j_1} = \int\limits_t^{T} \psi_5(t_5)\phi_{j_5}(t_5)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau.$

Proof. Note that in this proof we write k instead of 5 when this is true for an arbitrary k ($k\in\mathbf{N}$). As noted in Remark 2.4, Conditions 1 and 2 of Theorem 2.30 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 2.30 for the iterated Stratonovich stochastic integral (2.786). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2 = 0, \qquad (2.787)$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 = 0, \qquad (2.788)$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 = 0, \qquad (2.789)$$

where $(\{g_1,g_2\},\{g_3,g_4\},\{q_1\})$ and $(\{g_1,g_2\},\{q_1,q_2,q_3\})$ are partitions of the set $\{1,2,\ldots,5\}$ that is $\{g_1,g_2,g_3,g_4,q_1\}=\{g_1,g_2,q_1,q_2,q_3\}=\{1,2,\ldots,5\}$; braces mean an unordered set, and parentheses mean an ordered set.

Let us find a representation for $C_{j_k\ldots j_1}\Bigl|_{j_{g_1}=j_{g_2},\ g_2>g_1+1}$ that will be convenient for further consideration.

Using the integration order replacement in Riemann integrals, we obtain

$$\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l)dt_l\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k-$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots
$$

$$
\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (2.790)
$$

where $2 < l < k-1$ and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$. The case $l=1$ is obvious. By analogy with (2.790) we have for $l=k$

$$
\int\limits_t^T h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_l =
$$

$$
=\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^T h_l(t_l)dt_l dt_{l-1}\ldots dt_2dt_1 =
$$

$$
=\left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1}\ldots dt_2dt_1-
$$

$$
-\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)dt_{l-1}\ldots dt_2dt_1 =
$$

$$
=\left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}-
$$

$$
-\int\limits_t^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}. \qquad (2.791)
$$

The formulas (2.790), (2.791) will be used further.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume for simplicity that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

Let us continue the proof. Applying (2.790) to $C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}$ (more precisely to $h_s(t_s)=\psi_s(t_s)\phi_{j_l}(t_s)$), we obtain for $l+1\le k$, $s-1\ge 1$, $l-1\ge s+1$

$$
\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}= \tag{2.792}
$$

$$
=\sum_{j_l=p+1}^{\infty}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k=
$$

$$
=\sum_{j_l=p+1}^{\infty}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\left(\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)dt_s\right)\int\limits_t^{t_{s+1}}\phi_{j_{s-1}}(t_{s-1})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k-
$$

$$
-\sum_{j_l=p+1}^{\infty}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_{s-1}}(t_{s-1})\left(\int\limits_t^{t_{s-1}}\phi_{j_l}(t_s)dt_s\right)\int\limits_t^{t_{s-1}}\phi_{j_{s-2}}(t_{s-2})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-2}dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k=
$$

$$
=\sum_{j_l=p+1}^{\infty}A_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}-\sum_{j_l=p+1}^{\infty}B_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}.
$$

Now we apply the formula (2.790) to the quantities $A_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}$ and $B_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}$ (more precisely to $h_l(t_l) = \psi_l(t_l)\phi_{j_l}(t_l)$). Then we have for $l+1 \le k,\ s-1 \ge 1,\ l-1 \ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=1\\ g\ne l,s}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_{l+1}\ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_{l+1}j_{l-1}\ldots j_{s+1}j_{s-1}\ldots j_1} = \sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne l,s}, \qquad (2.793)$$

where

$$F_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \qquad (2.794)$$

$$F_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \qquad (2.795)$$

$$F_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \qquad (2.796)$$

$$F_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$
= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau. \tag{2.797}
$$

By analogy with (2.793) we can consider the expressions

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l}, \tag{2.798}
$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1} j_l j_{l-1}\ldots j_2 j_l} \quad (l+1\le k), \tag{2.799}
$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1} j_l j_{s-1}\ldots j_1} \quad (s-1\ge 1). \tag{2.800}
$$

Then we have for (2.798)–(2.800) (see (2.790), (2.791))

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} G_p^{(d)}(t_2,\ldots,t_{k-1}) \prod_{g=2}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{k-1}, \tag{2.801}
$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1} j_l j_{l-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} E_p^{(d)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times
$$

$$
\times \prod_{\substack{g=2\\ g\ne l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{l-1}dt_{l+1}\ldots dt_k, \tag{2.802}
$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1} j_l j_{s-1}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} D_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})\times
$$

$$
\times \prod_{\substack{g=1\\ g\ne s}}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{k-1}, \tag{2.803}
$$

where

$$
G_p^{(1)}(t_2,\ldots,t_{k-1}) = \mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{T} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
G_p^{(2)}(t_2,\ldots,t_{k-1})=-\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
E_p^{(1)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)=
$$

$$
=\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
E_p^{(2)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)=
$$

$$
=-\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau.
$$

Now let us consider the value $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2=g_1+1}$. To do this, we will make the following transformations

$$
\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots
$$

$$
\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}}\right) h_l(t_{l-1}) \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-1}}\right) h_l(t_l)dt_l dt_{l-1}dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l \int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times
$$

$$
\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k -
$$

$$
- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots
$$

$$
\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right) dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k -
$$

$$
- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times
$$

$$
\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k +
$$

$$
+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k =
$$

$$
=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k -
$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k -
$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)\times
$$

$$
\times\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k +
$$

$$
+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)\times
$$

$$
\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k, \qquad (2.804)
$$

where $l+1\le k$, $l-2\ge 1$, and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$. The case $l=k$ follows from (2.804) with $t_{l+1}=T$, and the case $l=2$ is obvious.

Applying (2.804) to $C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots\ldots j_1}$, we obtain for $l+1\le k,\ l-2\ge 1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots\ldots j_1}=$$

$$=\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)\prod_{\substack{g=1\\ g\ne l-1,l}}^{k}\psi_g(t_g)\phi_{j_g}(t_g)\times$$

$$\times dt_1\ldots dt_{l-2}dt_{l+1}\ldots dt_k=$$

$$=\sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}=\sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne l-1,l}, \tag{2.805}$$

where

$$H_p^{(1)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l+1}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{2.806}$$

$$H_p^{(2)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-2}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{2.807}$$

$$H_p^{(3)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau, \tag{2.808}$$

$$H_p^{(4)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-2}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau. \tag{2.809}$$

By analogy with (2.805) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l}, \tag{2.810}$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_l j_{k-2}\ldots j_1}. \qquad (2.811)
$$

Then we have for (2.810), (2.811) (see (2.804) and its analogue for $t_{l+1}=T$)

$$
\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l} = \int\limits_{[t,T]^{k-2}} L_p(t_3,\ldots,t_k)\prod_{g=3}^{k}\psi_g(t_g)\phi_{j_g}(t_g)dt_3\ldots dt_k, \qquad (2.812)
$$

$$
\sum_{j_l=p+1}^{\infty} C_{j_l j_l j_{k-2}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} M_p^{(d)}(t_1,\ldots,t_{k-2})\prod_{g=1}^{k-2}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{k-2}, \qquad (2.813)
$$

where

$$
L_p(t_3,\ldots,t_k) = \mathbf{1}_{\{t_3<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_l}(\theta)d\theta d\tau,
$$

$$
M_p^{(1)}(t_1,\ldots,t_{k-2}) =
$$

$$
= \mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
M_p^{(2)}(t_1,\ldots,t_{k-2}) =
$$

$$
= -\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
M_p^{(3)}(t_1,\ldots,t_{k-2}) =
$$

$$
= -\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau,
$$

$$
M_p^{(4)}(t_1,\ldots,t_{k-2}) =
$$

$$
= \mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau.
$$

It is important to note that $C^{*(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$, $C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$ $(d=1,\ldots,4)$ are Fourier coefficients (see (2.793), (2.805)), that is, we can use Parseval's equality in the further proof.

Combining the equalities (2.793)–(2.797) (the case $g_2>g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 2.33, 2.34, we obtain for (2.793)

$$
\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\times\right.
$$

$$
\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\right)^2=
$$

$$
=\int\limits_{[t,T]^{k-2}}\left(\sum_{d=1}^{4}F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\right)^2\times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\le
$$

$$
\le 4\sum_{d=1}^{4}\int\limits_{[t,T]^{k-2}}\left(F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\right)^2\times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \qquad (2.814)
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (2.798)–(2.800) are considered analogously.

Absolutely similarly (see (2.814)) combining the equalities (2.805)–(2.809) (the case $g_2=g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 2.33, 2.34, we get for (2.805)

$$
\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggl|_{q\ne g_1,g_2}\right)^2\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggl|_{q\ne g_1,g_2}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\times\right.
$$

$$
\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_1+1}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\right)^2=
$$

$$
=\int\limits_{[t,T]^{k-2}}\left(\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_1+1}}^{k}\psi_q(t_q)\right)^2\times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots\ldots dt_k\le
$$

$$
\le 4\sum_{d=1}^{4}\int\limits_{[t,T]^{k-2}}\left(H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_1+1}}^{k}\psi_q(t_q)\right)^2\times
$$

$$
\times dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \qquad (2.815)
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (2.810), (2.811) are considered analogously.

From (2.814), (2.815) and their analogues for the cases (2.798)–(2.800), (2.810), (2.811) we obtain

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2\le\frac{K}{p^{2-\varepsilon}}, \tag{2.816}$$

where constant K is independent of p. Thus the equality (2.787) is proved.

Let us prove the equality (2.788). Consider the following cases

1. $g_2>g_1+1,\ g_4=g_3+1,$ 2. $g_2=g_1+1,\ g_4>g_3+1,$
3. $g_2>g_1+1,\ g_4>g_3+1,$ 4. $g_2=g_1+1,\ g_4=g_3+1.$

The proof for Cases 1–3 will be similar. Consider, for example, Case 2. Using (2.690), we obtain

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=0}^{p}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2\le \tag{2.817}$$

$$\le(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2\Bigg|_{j_{g_3}=j_{g_4}}\le$$

$$\le(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2. \tag{2.818}$$

It is easy to see that the expression (2.818) (without the multiplier $p+1$) is a particular case ($k=5, g_4>g_3+1, g_2=g_1+1$) of the left-hand side of (2.816). Combining (2.816) and (2.818), we have

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2\le\frac{(p+1)K}{p^{2-\varepsilon}}\le\frac{K_1}{p^{1-\varepsilon}}\to 0 \qquad (2.819)
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K_1 does not depend on p.

Consider Case 4 ($g_2=g_1+1,\ g_4=g_3+1$). We have (see (2.691))

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=
$$

$$
=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\left(\sum_{j_{g_3}=0}^{\infty}-\sum_{j_{g_3}=0}^{p}\right)C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=
$$

$$
=\sum_{j_{q_1}=0}^{p}\left(\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}-\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2\le
$$

$$
\le\frac{1}{2}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2+ \qquad (2.820)
$$

$$
+2\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2. \qquad (2.821)
$$

An expression similar to (2.821) was estimated (see (2.817)–(2.819)). Let us estimate (2.820). We have

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2=
$$

$$
=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright 0}\right)^2\le
$$

$$
\le (T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright j_{g_3}}\right)^2, \qquad (2.822)
$$

where the notations are the same as in the proof of Theorem 2.30.

The expression (2.822) without the multiplier $T-t$ is an expression of type (2.746)–(2.751) before passing to the limit $\lim\limits_{p\to\infty}$ (the only difference is the replacement of one of the weight functions $\psi_1(\tau),\ldots,\psi_4(\tau)$ in (2.746)–(2.751) by the product $\psi_{l+1}(\tau)\psi_l(\tau)$ $(l=1,\ldots,4)$. Therefore, for Case 4 ($g_2=g_1+1$, $g_4=g_3+1$), we obtain the estimate

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1,g_2=g_1+1}\right)^2 \le \frac{K}{p^{1-\varepsilon}}, \qquad (2.823)
$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K is independent of p.

The estimates (2.819), (2.823) prove (2.788).

Let us prove (2.789). By analogy with (2.822) we have

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =
$$

$$
=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =
$$

$$
=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright 0,j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \le
$$

$$
\le (T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{g_1},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2. \qquad (2.824)
$$

Thus, we obtain the estimate (see (2.822) and the proof of Theorem 2.34)

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \qquad (2.825)
$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

The estimate (2.825) proves (2.789). Theorem 2.35 is proved.

2.14 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case $p_1 = \ldots = p_6 \to \infty$ and $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 2.36 [33], [38], [39], [65]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = {\int\limits_t^{*}}^{T} \ldots {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{2.826}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1, \ldots, i_6 = 0, 1, \ldots, m$,

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_6$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As noted in Remark 2.4, Conditions 1 and 2 of Theorem 2.30 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 2.30 for the iterated Stratonovich stochastic integral (2.826). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3},j_{q_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2}}\right)^2=0, \tag{2.827}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0, \tag{2.828}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1}\right)^2=0, \tag{2.829}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right)^2=0, \tag{2.830}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_6=g_5+1}\right)^2=0, \tag{2.831}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_4=g_3+1,g_6=g_5+1}\right)^2=0, \tag{2.832}$$

where the expressions $(\{g_1,g_2\},\{g_3,g_4\},\{g_5,g_6\})$, $(\{g_1,g_2\},\{g_3,g_4\},\{q_1,q_2\})$, $(\{g_1,g_2\},\{q_1,q_2,q_3,q_4\})$ are partitions of the set $\{1,2,\ldots,6\}$ that is $\{g_1,g_2,g_3,g_4,g_5,g_6\}=\{g_1,g_2,g_3,g_4,q_1,q_2\}=\{g_1,g_2,q_1,q_2,q_3,q_4\}=\{1,2,\ldots,6\}$; braces mean an unordered set, and parentheses mean an ordered set.

The equalities (2.827), (2.829) were proved earlier (see the proof of equalities (2.816), (2.822)). The relation (2.832) follows from the estimate (2.83) for the polynomial case and its analogue for the trigonometric case. It is easy to see that the equalities (2.828) and (2.831) are proved in complete analogy with the proof of (2.788), (2.822).

Thus, we have to prove the relation (2.830). The equality (2.830) is equivalent to the following equalities

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=0, \tag{2.833}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_3j_2j_3j_2j_1}=0, \tag{2.834}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=0, \tag{2.835}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=0, \tag{2.836}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_2j_3j_3j_1}=0, \tag{2.837}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=0, \tag{2.838}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=0, \tag{2.839}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=0, \tag{2.840}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=0, \tag{2.841}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0, \tag{2.842}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_1j_3j_3j_2j_1}=0, \tag{2.843}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0, \tag{2.844}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0, \tag{2.845}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0, \tag{2.846}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0. \tag{2.847}$$

Consider in detail the case of Legendre polynomials (the case of trigonometric functions is considered in complete analogy).

First, we prove the following equality for the Fourier coefficients for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

$$C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$
$$+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}. \tag{2.848}$$

Using the integration order replacement, we have

$$C_{j_6j_5j_4j_3j_2j_1}=$$
$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5dt_6=$$
$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6-$$
$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6=$$
$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$
$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6+$$
$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$
$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$
$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)dt_5dt_6\ C_{j_4j_3j_2j_1}+$$

$$
+\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\int\limits_{t_5}^T \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6 =
$$

$$
= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} +
$$

$$
+\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\int\limits_{t_5}^T \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6 =
$$

$$
\ldots
$$

$$
= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} - C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} -
$$

$$
-\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\ldots\int\limits_{t_2}^T \phi_{j_1}(t_1)dt_1\ldots dt_5dt_6 =
$$

$$
= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} -
$$

$$
-C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} - C_{j_1j_2j_3j_4j_5j_6}. \qquad (2.849)
$$

The equality (2.849) completes the proof of the relation (2.848).

Let us consider (2.833). From (2.684) we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}. \qquad (2.850)
$$

Applying (2.848), we get

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_1j_2j_3} = 2\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1} - C_{j_2j_3}C_{j_1j_3j_2j_1} + C_{j_1j_2j_3}C_{j_3j_2j_1} -
$$

$$
-C_{j_3j_1j_2j_3}C_{j_2j_1} + C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr). \qquad (2.851)
$$

Note that

$$
C_{j_2 j_1} = \int\limits_t^T \phi_{j_2}(\tau) \int\limits_t^{\tau} \phi_{j_1}(\theta) d\theta d\tau =
$$

$$
= \frac{T-t}{2} \begin{cases} 1/\sqrt{(2j_1+1)(2j_1+3)} & \text{if } j_2 = j_1 + 1,\ j_1 = 0, 1, 2, \ldots \\ \\ -1/\sqrt{4j_1^2 - 1} & \text{if } j_2 = j_1 - 1,\ j_1 = 1, 2, \ldots \\ \\ 1 & \text{if } j_1 = j_2 = 0 \\ \\ 0 & \text{otherwise} \end{cases}, \tag{2.852}
$$

$$
C_{j_1} = \int\limits_t^T \phi_{j_1}(\tau) d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_1 = 0 \\ \\ 0 & \text{if } j_1 \neq 0 \end{cases}. \tag{2.853}
$$

Moreover, the generalized Parseval equality gives

$$
\lim_{p\to\infty} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_1 j_2 j_3} C_{j_3 j_2 j_1} =
$$

$$
= \lim_{p\to\infty} \sum_{j_1, j_2, j_3 = 0}^{p} \int\limits_t^T \phi_{j_1}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_3}(t_1) dt_1 dt_2 dt_3 \times
$$

$$
\times \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 =
$$

$$
= \lim_{p\to\infty} \sum_{j_1, j_2, j_3 = 0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \times
$$

$$
\times \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 =
$$

$$
= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 \times
$$

$$
\times \int\limits_{[t,T]^3} \mathbf{1}_{\{t_1<t_2<t_3\}} \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 =
$$

$$
= \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \mathbf{1}_{\{t_1<t_2<t_3\}} dt_1 dt_2 dt_3 = 0. \qquad (2.854)
$$

Using the above arguments and also (2.684), (2.850), and (2.851), we get

$$
-\lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_2 j_1 j_3 j_2 j_1} = \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1 j_3 j_2 j_1} =
$$

$$
= \frac{1}{2} \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_3} C_{j_2 j_1 j_3 j_2 j_1} - C_{j_2 j_3} C_{j_1 j_3 j_2 j_1} -
$$

$$
- C_{j_3 j_1 j_2 j_3} C_{j_2 j_1} + C_{j_2 j_3 j_1 j_2 j_3} C_{j_1} \Biggr) =
$$

$$
= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \Bigl(C_{j_3} C_{j_2 j_1 j_3 j_2 j_1} - C_{j_3 j_1 j_2 j_3} C_{j_2 j_1} \Bigr) =
$$

$$
= \sqrt{T-t} \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1 0 j_2 j_1} - \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_1 j_2 j_3} C_{j_2 j_1} =
$$

$$
= \sqrt{T-t} \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1 0 j_2 j_1} + \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_2 j_3} C_{j_2 j_1}. \qquad (2.855)
$$

By analogy with the proof of (2.752) (see the proof of Theorem 2.34) we obtain

$$
\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1 0 j_2 j_1} = \lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} C_{j_2 j_1 0 j_2 j_1} = 0, \qquad (2.856)
$$

where we used the following representation

$$
C_{j_2 j_1 0 j_2 j_1} =
$$

$$
=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5=
$$

$$
=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}dt_3dt_2dt_4dt_5=
$$

$$
=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)(t_4-t)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5+
$$

$$
+\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\stackrel{\sf def}{=}
$$

$$
\stackrel{\sf def}{=}\bar C_{j_2j_1j_2j_1}+\tilde C_{j_2j_1j_2j_1}.
$$

Further, we have (see (2.852))

$$
\lim\limits_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}=\lim\limits_{p\to\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{00}C_{j_300j_3}+
$$

$$
+\sum_{j_1=1}^{p}C_{j_1-1,j_1}C_{j_3j_1,j_1-1,j_3}+\sum_{j_1=1}^{p-1}C_{j_1+1,j_1}C_{j_3j_1,j_1+1,j_3}+C_{1,0}C_{j_301j_3}\Biggr). \quad (2.857)
$$

Observe that

$$
|C_{j_1-1,j_1}|+|C_{j_1+1,j_1}|\le\frac{K}{j_1}\quad (j_1=1,\ldots,p), \quad (2.858)
$$

$$
|C_{j_300j_3}|+|C_{j_3j_1,j_1-1,j_3}|+|C_{j_3j_1,j_1+1,j_3}|+|C_{j_301j_3}|\le
$$

$$
\le\frac{K_1}{j_3^2}\quad (j_3\ge p+1), \quad (2.859)
$$

where constants K, K_1 do not depend on j_1, j_3.

The estimate (2.858) follows from (2.852). At the same time, the estimate (2.859) can be obtained using the following reasoning. First note that the integration order replacement gives

$$
C_{j_3j_1j_2j_3}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3dt_4=
$$

$$
= \int\limits_t^T \phi_{j_1}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \left(\int\limits_t^{t_2} \phi_{j_3}(t_1) dt_1 \right) dt_2 \left(\int\limits_{t_3}^T \phi_{j_3}(t_4) dt_4 \right) dt_3. \quad (2.860)
$$

Applying the estimates (2.158), (2.159), and (2.175) to (2.860) gives the estimate (2.859).

Using (2.857), (2.858), and (2.859), we obtain

$$
\left| \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_2 j_3} C_{j_2 j_1} \right| \le K \sum_{j_3=p+1}^{\infty} \frac{1}{j_3^2} \left(1 + \sum_{j_1=1}^{p} \frac{1}{j_1} \right) \le
$$

$$
\le K \int\limits_p^{\infty} \frac{dx}{x^2} \left(2 + \int\limits_1^p \frac{dx}{x} \right) = \frac{K(2 + lnp)}{p} \to 0 \qquad (2.861)
$$

if $p \to \infty$, where constant K is independent of p. Thus, the equality (2.833) is proved (see (2.855), (2.856), (2.861)).

The relation (2.834) is proved in complete analogy with the proof of equality (2.833). For (2.834) we have (see (2.848))

$$
\lim_{p\to\infty} \left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_1 j_3 j_2 j_3 j_2 j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1 j_2 j_3 j_2 j_3 j_1} \right) = 2 \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_1 j_3 j_2 j_3 j_2 j_1} =
$$

$$
= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \Bigl(C_{j_1} C_{j_3 j_2 j_3 j_2 j_1} - C_{j_3 j_1} C_{j_2 j_3 j_2 j_1} + C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} -
$$

$$
- C_{j_3 j_2 j_3 j_1} C_{j_2 j_1} + C_{j_2 j_3 j_2 j_3 j_1} C_{j_1} \Bigr) =
$$

$$
= 2 \lim_{p\to\infty} \left(\sqrt{T-t} \sum_{j_2,j_3=0}^{p} C_{j_3 j_2 j_3 j_2 0} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1} C_{j_3 j_2 j_3 j_1} \right) =
$$

$$
= -2 \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1} C_{j_3 j_2 j_3 j_1}.
$$

To estimate the Fourier coefficient $C_{j_3 j_2 j_3 j_1}$, we use the following (see the proof of (2.833) for more details)

$$
C_{j_3 j_2 j_3 j_1} = \int\limits_t^T \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_3}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
=\int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_3}\phi_{j_3}(t_2)dt_2dt_1dt_3dt_4=
$$

$$
=\int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_4-
$$

$$
-\int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1dt_3dt_4=
$$

$$
=\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\left(\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4\right)dt_3-
$$

$$
-\int\limits_t^T \phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1\left(\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4\right)dt_3.
$$

Let us prove (2.835). From (2.684) we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_3j_1j_2j_1}. \tag{2.862}
$$

Applying (2.848) and (2.862), we get (we replaced j_3 by j_4)

$$
\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}+\sum_{j_1,j_2,j_4=0}^{p}C_{j_1j_2j_1j_4j_2j_4}=2\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}=
$$

$$
=\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_4}C_{j_2j_4j_1j_2j_1}-C_{j_2j_4}C_{j_4j_1j_2j_1}+C_{j_4j_2j_4}C_{j_1j_2j_1}-
$$

$$
-C_{j_1j_4j_2j_4}C_{j_2j_1}+C_{j_2j_1j_4j_2j_4}C_{j_1}\Biggr)=
$$

$$
=2\sum_{j_1,j_2,j_4=0}^{p}\Bigl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Bigr)+
$$

$$
+\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4}C_{j_1j_2j_1}. \tag{2.863}
$$

Further, we have (see (2.684))

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p} C_{j_4j_2j_4}C_{j_1j_2j_1} = \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}\right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_1}\right)^2 = 0, \qquad (2.864)$$

where we applied the equality (2.726).

Furthermore, by analogy with the proof of (2.833), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\left(C_{j_2j_1j_4j_2j_4}C_{j_1} - C_{j_1j_4j_2j_4}C_{j_2j_1}\right) = 0. \qquad (2.865)$$

To estimate the Fourier coefficient $C_{j_1j_4j_2j_4}$ in (2.865), we use the following (see the proof of (2.833) for more details)

$$C_{j_1j_4j_2j_4} = \int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_3dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4 -$$

$$- \int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_3)dt_3\right)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4.$$

The relations (2.862)–(2.865) complete the proof of equality (2.835).

Let us prove (2.836). Using (2.684), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_2j_3j_3j_2j_1} = \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_1j_2j_3j_3j_2j_1}. \qquad (2.866)$$

Applying (2.848) and (2.866), we obtain

$$2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}+\left(C_{j_3j_2j_1}\right)^2-$$

$$-C_{j_3j_3j_2j_1}C_{j_2j_1}+C_{j_2j_3j_3j_2j_1}C_{j_1}\Biggr)=$$

$$=2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}\Biggr)+$$

$$+\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2. \tag{2.867}$$

Using the estimate (1.219), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2=0. \tag{2.868}$$

By analogy with the proof of (2.833), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}\Biggr)=0, \tag{2.869}$$

where we applied the equality (2.753). To estimate the Fourier coefficient $C_{j_3j_3j_2j_1}$ in (2.869), we used the following (see the proof of (2.833) for more details)

$$C_{j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1=$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\left(\int\limits_{t_2}^T\phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \tag{2.870}
$$

Combining the equalities (2.866)–(2.869), we obtain (2.836).

Let us prove (2.837) (we replace j_2 by j_4 and j_3 by j_2 in (2.837)). As noted in Remark 2.4, the sequential order of the series

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}
$$

is not important. This follows directly from the formulas (2.691) and (2.684).

Applying the mentioned property and (2.684), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=-\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}. \tag{2.871}
$$

Observe that (see the above reasoning)

$$
\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=\sum_{j_4=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}. \tag{2.872}
$$

Using (2.848) and (2.872), we obtain

$$
\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\left(C_{j_1j_4j_4j_2j_2j_1}+C_{j_1j_2j_2j_4j_4j_1}\right)=2\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}+C_{j_4j_4j_1}C_{j_2j_2j_1}-
$$

$$
-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Biggr)=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\left(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\right)+
$$

$$
+\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2. \tag{2.873}
$$

The equality

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2=0 \qquad (2.874)$$

follows from the relation (2.725).

By analogy with the proof of equality (2.833) we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Biggr)=0, \qquad (2.875)$$

where we applied the equality (2.754). To estimate the Fourier coefficient $C_{j_2j_4j_4j_1}$ in (2.875), we used the following (see the proof of (2.833) for more details)

$$C_{j_2j_4j_4j_1}=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4+$$

$$+\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4-$$

$$-\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)dt_1dt_4.$$

The relation (2.837) follows from (2.871), (2.873)–(2.875).

Consider (2.838). Using the integration order replacement, we obtain

$$C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{4}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3. \quad (2.876)$$

Applying the estimates (2.158), (2.159), and (2.175) to (2.876) gives the following estimate

$$\left|C_{j_3j_3j_2j_2j_1j_1}\right|\le\frac{K}{j_1^2j_3^2}\quad (j_1,j_3>0,\ j_2\ge 0), \quad (2.877)$$

where constant K does not depend on j_1,j_2,j_3.

Further, we get (see (2.691))

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}, \quad (2.878)$$

where

$$C_{j_3j_3j_2j_2j_1j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\int\limits_t^{t_4}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_5}dt_4dt_2dt_5dt_6=$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5)(t_5 - t) \int\limits_t^{t_5} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_5 dt_6 +
$$

$$
+ \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_2)(t - t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_5 dt_6 \stackrel{\sf def}{=}
$$

$$
\stackrel{\sf def}{=} C'_{j_3 j_3 j_1 j_1} + C''_{j_3 j_3 j_1 j_1}. \tag{2.879}
$$

Let us substitute (2.879) into (2.878)

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1} = \frac{1}{2} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C'_{j_3 j_3 j_1 j_1} +
$$

$$
+ \frac{1}{2} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C''_{j_3 j_3 j_1 j_1} - \sum_{j_2=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1}. \tag{2.880}
$$

The relation (2.754) implies that

$$
\lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C'_{j_3 j_3 j_1 j_1} = 0, \quad \lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C''_{j_3 j_3 j_1 j_1} = 0. \tag{2.881}
$$

From the estimate (2.877) we get

$$
\left| \sum_{j_2=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1} \right| \le K(p+1) \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \sum_{j_3=p+1}^{\infty} \frac{1}{j_3^2} \le
$$

$$
\le K(p+1) \left(\int\limits_p^{\infty} \frac{dx}{x^2} \right)^2 \le \frac{K(p+1)}{p^2} \ \to \ 0 \tag{2.882}
$$

if $p \to \infty$, where constant K is independent of p.

The relations (2.880)–(2.882) complete the proof of (2.838).

Let us prove (2.839). Using the integration order replacement, we get

$$
C_{j_2 j_3 j_3 j_2 j_1 j_1} =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_2}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 dt_3 dt_4 dt_5 dt_6 =
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6dt_5dt_4dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\times
$$

$$
\times dt_5dt_3-
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)\times
$$

$$
\times dt_5dt_3. \qquad (2.883)
$$

Applying (2.684) and (2.691), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=-\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}-
$$

$$
-\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}-\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}-
$$

$$
-\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}. \qquad (2.884)
$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=0 \tag{2.885}$$

follows from the inequality similar to (2.780) (see the proof of Theorem 2.34), where we used the following representation

$$C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{t_6}dt_4dt_3dt_6=$$

$$+\int\limits_t^T\phi_{j_2}(t_6)(t_6-t)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6+$$

$$+\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)(t-t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}C^{*}_{j_2j_2j_1j_1}+C^{**}_{j_2j_2j_1j_1}. \tag{2.886}$$

Applying the estimates (2.158), (2.159), (2.175), and (2.740) ($\varepsilon=1/2$) to (2.883) gives the following estimates

$$|C_{j_2j_3j_3j_2j_1j_1}|\le\frac{K}{j_1^2j_2j_3^{3/4}}\quad(j_1,j_2,j_3>0), \tag{2.887}$$

$$|C_{j_200j_2j_1j_1}|\le\frac{K}{j_1^2j_2}\quad(j_1,j_2>0), \tag{2.888}$$

$$|C_{0j_3j_30j_1j_1}|\le\frac{K}{j_1^2j_3}\quad(j_1,j_3>0), \tag{2.889}$$

$$|C_{0000j_1j_1}|\le\frac{K}{j_1^2}\quad(j_1>0). \tag{2.890}$$

Using the estimate (2.887), we have

$$\left|\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_2j_3j_3j_2j_1j_1}\right| \le K\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_2=1}^{p}\frac{1}{j_2}\sum_{j_3=1}^{p}\frac{1}{j_3^{3/4}} \le$$

$$\le K\int\limits_p^{\infty}\frac{dx}{x^2}\left(1+\int\limits_1^p\frac{dx}{x}\right)\left(1+\int\limits_1^p\frac{dx}{x^{3/4}}\right) \le K_1\frac{1+lnp}{p^{3/4}} \to 0 \qquad (2.891)$$

if $p\to\infty$, where constants K, K_1 do not depend on p.

Similarly we get (see (2.888)–(2.890))

$$\left|\sum_{j_1=p+1}^{\infty} C_{0000j_1j_1}\right| + \left|\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{0j_3j_30j_1j_1}\right| + \left|\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_200j_2j_1j_1}\right| \to 0 \quad (2.892)$$

if $p\to\infty$.

The relations (2.884), (2.885), (2.891), (2.892) prove (2.839).

Consider (2.840). Using the integration order replacement, we get

$$C_{j_3j_2j_3j_2j_1j_1} =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_2}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\left(\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\right)\times$$

$$\times dt_5dt_3 -$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_2}(t_5)\left(\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\right)\times$$

$$
\times dt_5 dt_3. \tag{2.893}
$$

Applying (2.684), we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1}=
$$

$$
=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1}. \tag{2.894}
$$

Further proof of the equality (2.840) is based on the relations (2.893), (2.894) and is similar to the proof of the formula (2.839).

Let us prove (2.841). Applying the integration order replacement, we obtain

$$
C_{j_3j_3j_2j_1j_2j_1}=
$$

$$
=\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=
$$

$$
=\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_4)\int\limits_{t_4}^{T}\phi_{j_3}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_1}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\times
$$

$$
\times dt_2dt_4-
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_2)\left(\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\right)^2 \times
$$

$$
\times dt_2 dt_4. \tag{2.895}
$$

Using (2.684), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1} = \sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1} =
$$

$$
= -\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1}. \tag{2.896}
$$

Further proof of the equality (2.841) is based on the relations (2.895), (2.896) and is similar to the proof of the relations (2.839), (2.840).

Consider (2.842). Using the integration order replacement, we have

$$
C_{j_3j_3j_1j_2j_2j_1} =
$$

$$
= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_3}(t_5)\int\limits_t^{t_5} \phi_{j_1}(t_4)\int\limits_t^{t_4} \phi_{j_2}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$
= \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_5)\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4} \phi_{j_2}(t_3)dt_3dt_2dt_4 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \left(\int\limits_t^{t_4} \phi_{j_2}(t_3)dt_3\right)\int\limits_t^{t_4} \phi_{j_2}(t_2)\left(\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\right) \times
$$

$$\times dt_2 dt_4-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_2)\left(\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_2} \phi_{j_2}(t_3)dt_3\right)\times$$

$$\times dt_2 dt_4. \tag{2.897}$$

Applying (2.684) and (2.691), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_2 j_2 j_1} = -\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2 j_3 j_1 j_2 j_2 j_1} =$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2 j_3 j_1 j_2 j_2 j_1} = \sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_2 j_3 j_1 j_2 j_2 j_1} =$$

$$=\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_2 j_2 j_1}\Bigg|_{(j_2 j_2)\curvearrowright(\cdot)} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_2 j_2 j_1}. \tag{2.898}$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_2 j_2 j_1}\Bigg|_{(j_2 j_2)\curvearrowright(\cdot)} = 0 \tag{2.899}$$

follows from the inequality (2.780), where we proceed similarly to the proof of equality (2.885) (see (2.886)).

The relation

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_2 j_2 j_1} = 0 \tag{2.900}$$

is proved on the basis of (2.897) and similarly with the proof of (2.839). The equalities (2.898)–(2.900) prove (2.842).

Let us prove (2.843). Using (2.684) and (2.691), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2 j_1 j_3 j_3 j_2 j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1} =$$

$$=\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1}. \tag{2.901}$$

Using the equality (2.752) we have

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=0, \tag{2.902}$$

where we proceed similarly to the proof of equality (2.885) (see (2.886)).

Further, we will prove the following relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1j_3j_3j_2j_1}=0 \tag{2.903}$$

using the equality (2.848). From (2.848) we have

$$\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1j_3j_3j_2j_1}=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_1j_3j_3j_2j_1}+C_{j_1j_2j_3j_3j_1j_2}\Biggr)=$$

$$=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2}C_{j_1j_3j_3j_2j_1}-C_{j_1j_2}C_{j_3j_3j_2j_1}+C_{j_3j_1j_2}C_{j_3j_2j_1}-$$

$$-C_{j_3j_3j_1j_2}C_{j_2j_1}+C_{j_2j_3j_3j_1j_2}C_{j_1}\Biggr)=$$

$$=\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr)+$$

$$+\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}. \tag{2.904}$$

The generalized Parseval equality gives (by analogy with (2.854))

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}=0. \tag{2.905}$$

Let us prove the following equality

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr)=0. \tag{2.906}$$

The relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2}C_{j_1}=0 \qquad (2.907)$$

is proved by the same methods as in the proof of equality (2.833) and also using Theorem 2.34 and (2.691).

Further, we have (see (2.691))

$$\sum_{j_3=0}^{p} C_{j_3j_3j_1j_2}=\frac{1}{2}C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}. \qquad (2.908)$$

Moreover,

$$C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3=$$

$$=\int\limits_t^T\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1\int\limits_{t_2}^{T}dt_3dt_2=$$

$$=\int\limits_t^T(T-t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2=$$

$$=\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^{T}(T-t_2)\phi_{j_1}(t_2)dt_2dt_1=$$

$$=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^{T}(T-t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}(T-t_1)\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\tilde C_{j_2j_1}. \qquad (2.909)$$

Using (2.908), (2.909), and the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1}=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1}\tilde C_{j_2j_1}-$$

$$-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1}=-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1}. \quad (2.910)$$

We have (see (2.870))

$$C_{j_3j_3j_1j_2}=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^T\phi_{j_1}(t_2)\left(\int\limits_{t_2}^T\phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \quad (2.911)$$

By analogy with (2.861) and also using (2.911), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1}=0. \quad (2.912)$$

Combining (2.910) and (2.912), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1}=0. \quad (2.913)$$

The relation (2.906) follows from (2.907) and (2.913). From (2.904)–(2.906) we get (2.903). The equalities (2.901)–(2.903) complete the proof of (2.843).

For the proof of (2.844)–(2.847) we will use a new idea. More precisely, we will consider the sums of expressions (2.844)–(2.847) with the expressions already studied throughout this proof.

Let us begin from (2.844). Applying the integration order replacement, we obtain

$$C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) \int\limits_{t_3}^{t_5} \phi_{j_2}(t_4) dt_4 \times
$$

$$
\times dt_3 dt_5 dt_6 =
$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \left(\int\limits_t^{t_5} \phi_{j_2}(t_4) dt_4 \right) \int\limits_t^{t_5} \phi_{j_3}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \times
$$

$$
\times \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 dt_5 dt_6 -
$$

$$
- \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right)^2 \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) \times
$$

$$
\times dt_3 dt_5 dt_6 =
$$

$$
= \int\limits_t^T \phi_{j_1}(t_5) \left(\int\limits_t^{t_5} \phi_{j_2}(t_4) dt_4 \right) \int\limits_t^{t_5} \phi_{j_3}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \times
$$

$$
\times \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6) dt_6 \right) dt_5 -
$$

$$
- \int\limits_t^T \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right)^2 \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 \times
$$

$$
\times \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6) dt_6 \right) dt_5. \qquad (2.914)
$$

Using (2.684), we get

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_2 j_3 j_2 j_1} + C_{j_3 j_1 j_2 j_3 j_1 j_2} \Bigr) =
$$

$$
= \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_2 j_3 j_2 j_1} + C_{j_3 j_1 j_2 j_3 j_1 j_2} \Bigr). \qquad (2.915)
$$

Further, by analogy with the proof of equality (2.839) and using (2.914), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0. \tag{2.916}$$

From (2.915) and (2.916) we get

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0. \tag{2.917}$$

Moreover (see (2.833)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_1j_2}=0. \tag{2.918}$$

Combining (2.917) and (2.918), we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0.$$

The equality (2.844) is proved.

Consider (2.845). Using the integration order replacement, we have

$$C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-
$$

$$
-\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\times
$$

$$
\times dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times
$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-
$$

$$
-\int\limits_t^T\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3\times
$$

$$
\times\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5. \qquad (2.919)
$$

Using (2.684), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right)=
$$

$$
=\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right). \qquad (2.920)
$$

By analogy with the proof of (2.839) and applying (2.919), we get

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right)=0. \qquad (2.921)
$$

From (2.920) and (2.921) we have

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right)=0. \qquad (2.922)
$$

Moreover (see (2.834)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_1j_2}=0. \qquad (2.923)$$

Combining (2.922) and (2.923), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0.$$

The equality (2.845) is proved.

Now consider (2.846). Using the integration order replacement, we obtain

$$C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=$$

$$
=\int\limits_t^T \phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times
$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5-
$$

$$
-\int\limits_t^T \phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times
$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5. \qquad (2.924)
$$

Applying (2.684) and (2.691), we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=
$$

$$
=-\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)-
$$

$$
-\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}. \qquad (2.925)
$$

The equality

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}=0 \qquad (2.926)
$$

follows from the equality (2.752), where we proceed similarly to the proof of equality (2.885) (see (2.886)).

By analogy with the proof of (2.839) and applying (2.924), we get

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \qquad (2.927)
$$

From (2.925)–(2.927) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \qquad (2.928)$$

Moreover (see (2.835)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_1j_2}=0. \qquad (2.929)$$

Combining (2.928) and (2.929), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0.$$

The equality (2.846) is proved.

Finally consider (2.847). Using the integration order replacement, we have

$$C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times
$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-
$$

$$
-\int\limits_t^T\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times
$$

$$
\times\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5. \qquad (2.930)
$$

Using (2.684) and (2.691), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=
$$

$$
=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)-
$$

$$
-\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=
$$

$$
=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)+
$$

$$
+\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)-
$$

$$
-\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}. \qquad (2.931)
$$

The equalities

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)=0, \quad (2.932)$$

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\lim_{p\to\infty}\frac{1}{4}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)(j_3j_3)\curvearrowright(\cdot)}-$$

$$-\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}=0 \qquad (2.933)$$

follows from the equalities (2.752), (2.753), where we used the same technique as in (2.886). When proving (2.933), we also applied (2.691) and (2.83).

By analogy with the proof of (2.839) and applying (2.930), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=0. \qquad (2.934)$$

From (2.931)–(2.934) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=0. \qquad (2.935)$$

Furthermore (see (2.837)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}=0. \qquad (2.936)$$

Combining (2.935) and (2.936), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0.$$

The equality (2.847) is proved. Theorem 2.36 is proved.

2.15 Estimates for the Mean-Square Approximation Error of Iterated Stratonovich Stochastic Integrals of Multiplicity k in Theorems 2.30, 2.31

In this section, we estimate the mean-square approximation error for iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbf{N}$) in Theorems 2.30, 2.31.

Theorem 2.37 [33], [38], [39], [64]. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuously differentiable nonrandom function at the interval $[t, T]$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then the following estimates*

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le$$

$$\le K_1\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right), \tag{2.937}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le$$

$$\le K_2(s)\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right) \tag{2.938}$$

hold, where $s \in (t, T]$ (s is fixed), $i_1, \ldots, i_k = 1, \ldots, m$,

$$R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} = R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Bigg|_{T=s},$$

$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ *is defined by* (2.707), $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *and* $J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}$ *are iterated Stratonovich stochastic integrals* (2.661) *and* (2.716), $C_{j_k\ldots j_1}$ *and* $C_{j_k\ldots j_1}(s)$ *are Fourier coefficients* (2.653) *and* (2.714), *constants* K_1 *and* $K_2(s)$ *are independent of* p; *another notations are the same as in Theorems* 1.1, 2.30, 2.31.

Proof. Note that Conditions 1 and 2 of Theorems 2.30, 2.31 are satisfied under the conditions of Theorem 2.37 (see Remark 2.4). From the proof of Theorem 2.30 it follows that the expression (2.712) ($i_1,\ldots,i_k=1,\ldots,m$) before passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ has the form

$$
\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}+
$$

$$
+\sum_{r=1}^{[k/2]}\left(\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}+\right.
$$

$$
\left.+\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right) \tag{2.939}
$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the approximation (1.224) of the iterated Itô stochastic integral (2.674), $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}$ is the approximation obtained using (1.224) for the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ (see (2.713)).

Using (2.939) and Theorem 2.12, we have

$$
\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}+
$$

$$
+\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-\right.
$$

$$
\left.-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=
$$

$$
=J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-\right.
$$

$$
\left.-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \qquad (2.940)
$$

w. p. 1, where we denote $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$.

Applying (1.228) (see Remark 1.7), we obtain the following estimates

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le\frac{C}{p}, \qquad (2.941)
$$

$$
\mathsf{M}\left\{\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)^2\right\}\le
$$

$$
\le\frac{C}{p}, \qquad (2.942)
$$

where constant C does not depend on p.

From (2.940)–(2.942) and the elementary inequality

$$
(a_1+a_2+\ldots+a_n)^2\le n\left(a_1^2+a_2^2+\ldots+a_n^2\right),\quad n\in\mathbf{N},
$$

we obtain (2.937). The estimate (2.938) is obtained similarly to the estimate (2.937) using Theorems 1.11, 2.31 and (1.252) (see Remark 1.12). Theorem 2.37 is proved.

2.16 Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3–5 in Theorems 2.33–2.35

In this section, we consider the rate of convergence of approximations of iterated Stratonovich stochastic integrals in Theorems 2.33–2.35. It is easy to see that in Theorems 2.33–2.35 the second term in parentheses on the right-hand side of (2.937) is estimated for $k = 3, 4, 5$. Combining these results with Theorem 2.37, we obtain the following theorems.

Theorem 2.38 [33], [38], [39], [64]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = {\int\limits_t^{*}}^{T} \psi_3(t_3) {\int\limits_t^{*}}^{t_3} \psi_2(t_2) {\int\limits_t^{*}}^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p}$$

is fulfilled, where $i_1, i_2, i_3 = 1, \ldots, m$, *constant* C *is independent of* p,

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Theorem 2.39 [33], [38], [39], [64]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}d\mathbf{f}_{t_4}^{(i_4)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

holds, where $i_1,i_2,i_3,i_4=1,\ldots,m$, *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=\int\limits_t^{T}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\times$$
$$\times dt_2dt_3dt_4;$$

another notations are the same as in Theorem 2.38.

Note that Theorem 2.26 is an analog of Theorem 2.39. At that $\varepsilon=0$, $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$, and $i_1,\ldots,i_k=0,1,\ldots,m$ in Theorem 2.26.

Theorem 2.40 [33], [38], [39], [64]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_5}^{(i_5)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

is valid, where $i_1, \ldots, i_5 = 1, \ldots, m$*, constant* C *is independent of* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t, T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t, T])$*,*

$$C_{j_5 \ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_5;$$

another notations are the same as in Theorem 2.38, 2.39*.*

2.17 Generalization of Theorems 2.4–2.8. The Case p_1, p_2, $p_3 \to \infty$ and Continuously Differetiable Weight Functions (The Cases of Legendre Polynomials and Trigonometric Functions). Proof of Hypothesis 2.3 for the Case $k = 3$

This section is devoted to the following theorem.

Theorem 2.41 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \qquad (2.943)$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$*,*

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau.$

Proof. Let us consider the case of Legendre polynomials (the trigonometric case is simpler and can be considered similarly). Applying (2.679), we obtain

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)}\qquad(2.944)$$

w. p. 1, where notations are the same as in (2.679).

Using (2.399), Theorem 1.1 (see (1.43)), Theorem 2.12 (see (2.389)) as well as (2.698) (see the derivation of (2.698)) and (2.691), we get

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3=$$

$$=J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}+\frac{1}{2}J[\psi^{(3)}]_{T,t}^{1}+\frac{1}{2}J[\psi^{(3)}]_{T,t}^{2}=$$

$$=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl|_{(j_2j_1)\curvearrowright(\cdot),j_1=j_2} J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\Biggl|_{(j_3j_2)\curvearrowright(\cdot),j_2=j_3} J'[\phi_{j_1}]_{T,t}^{(i_1)}=
$$

$$
=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty} C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)} \tag{2.945}
$$

w. p. 1.

Using (2.944), (2.945) and the elementary inequality

$$
(a+b+c+d)^2\le 4\left(a^2+b^2+c^2+d^2\right),
$$

we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le 4\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}-J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathsf{M}\left\{\left(\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)}\right)^2\right\}=
$$

$$
=4A_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}B_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}C_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}D_{p_1p_2p_3}. \tag{2.946}
$$

Theorem 1.1 gives (see (1.43))

$$
\lim_{p_1,p_2,p_3\to\infty}A_{p_1p_2p_3}=0. \tag{2.947}
$$

Further, in complete analogy with (2.744) and using (2.684), we obtain

$$
D_{p_1p_2p_3}=
$$

$$
=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}\right)^2=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le
$$

$$
\le\sum_{j_2=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le\frac{K}{\left(\min\{p_1,p_3\}\right)^{2-\varepsilon}}\ \to\ 0 \tag{2.948}
$$

if $p_1,p_2,p_3\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

We have

$$
B_{p_1p_2p_3}=
$$

$$
=\mathsf{M}\left\{\left(\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)+\right.\right.
$$

$$
\left.\left.+\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)\right)^2\right\}\le
$$

$$
\le 2E_{p_3}+2F_{p_1p_2p_3}, \tag{2.949}
$$

where

$$
E_{p_3}=
$$

$$
= \mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)} - \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\},
$$

$$
F_{p_1p_2p_3} =
$$

$$
= \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)} - \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\} =
$$

$$
= \sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2. \tag{2.950}
$$

By analogy with (2.731) we get

$$
\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2 \le
$$

$$
\le \sum_{j_3=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2 \le
$$

$$
\le \frac{K}{\left(\min\{p_1,p_2\}\right)^2}\ \to\ 0 \tag{2.951}
$$

if $p_1, p_2, p_3 \to \infty$, where constant K does not depend on p.

Moreover,

$$
\lim_{p_3\to\infty} E_{p_3} = \lim_{p_1,p_2,p_3\to\infty} E_{p_3} = 0. \tag{2.952}
$$

Combining (2.949)–(2.952), we obtain

$$
\lim_{p_1,p_2,p_3\to\infty} B_{p_1p_2p_3} = 0. \tag{2.953}
$$

Consider $C_{p_1p_2p_3}$. We have

$$
C_{p_1p_2p_3} =
$$

$$
= \mathsf{M}\left\{\left(\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)+\right.\right.
$$

$$
\left.\left.+\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)\right)^2\right\} \le
$$

$$
\le 2G_{p_1} + 2H_{p_1p_2p_3}, \tag{2.954}
$$

where

$$
G_{p_1} =
$$

$$
= \mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\},
$$

$$
H_{p_1p_2p_3} =
$$

$$
= \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\ \sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\} =
$$

$$
= \sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1}\right)^2. \tag{2.955}
$$

By analogy with (2.735) we get

$$
\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1}\right)^2 \le
$$

$$
\le \sum_{j_1=0}^{\infty}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1}\right)^2 \le
$$

$$
\le \frac{K}{\left(\min\{p_2,p_3\}\right)^2}\ \to\ 0 \tag{2.956}
$$

if $p_1, p_2, p_3 \to \infty$, where constant K does not depend on p.

Moreover,

$$\lim_{p_1\to\infty} G_{p_1} = \lim_{p_1,p_2,p_3\to\infty} G_{p_1} = 0. \tag{2.957}$$

Combining (2.954)–(2.957), we obtain

$$\lim_{p_1,p_2,p_3\to\infty} C_{p_1p_2p_3} = 0. \tag{2.958}$$

The relations (2.946)–(2.948), (2.953), (2.958) complete the proof of Theorem 2.41. Theorem 2.41 is proved.

2.18 Generalization of Theorem 2.30 for Complete Orthonormal Systems of Functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ such that the Condition (2.720) is Satisfied

In this section, we generalize Theorem 2.30 to the case of complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau) \in L_2([t,T])$ such that the condition (2.720) is satisfied.

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) \stackrel{\mathsf{def}}{=} f_t : [0,T]\times\Omega \rightarrow \mathbf{R}$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let $\xi(\tau,\omega) \stackrel{\mathsf{def}}{=} \xi_\tau : [0,T]\times\Omega\rightarrow\mathbf{R}$ be some random process, which is measurable with respect to the pair of variables (τ,ω) and satisfies to the following condition

$$\int\limits_t^T |\xi_\tau| d\tau < \infty \quad \text{w. p. 1} \quad (t\ge 0).$$

Let $\tau_j^{(N)}$, $j=0,1,\ldots,N$ be a partition of the interval $[t,T]$, $t\ge 0$ such that

$$t=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty. \tag{2.959}$$

Further, for simplicity, we write τ_j instead of $\tau_j^{(N)}$.

Consider the definition of the Stratonovich stochastic integral, which differs from the definition given in Sect. 2.1.1.

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{j=0}^{N-1}\frac{1}{\tau_{j+1}-\tau_j}\int\limits_{\tau_j}^{\tau_{j+1}}\xi_s ds\ \left(f_{\tau_{j+1}}-f_{\tau_j}\right)\stackrel{\text{def}}{=}\int\limits_t^T\xi_\tau\circ df_\tau \tag{2.960}$$

is called [143], [144] the Stratonovich stochastic integral of the process ξ_τ, $\tau\in[t,T]$, where τ_j, $j=0,1,\ldots,N$ is a partition of the interval $[t,T]$ satisfying the condition (2.959).

We also denote by

$$\int\limits_t^\tau\xi_s\circ df_s$$

the Stratonovich stochastic integral like (2.960) (if it exists) of $\xi_s\mathbf{1}_{\{s\in[t,\tau]\}}$ for $\tau\in[t,T]$, $t\ge 0$.

It is known [144] (Lemma A.2) that the following iterated Stratonovich stochastic integral

$$J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)}=\int\limits_t^\tau\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\ldots\circ d\mathbf{w}_{t_k}^{(i_k)} \tag{2.961}$$

exists for the case $i_1=\ldots=i_k\ne 0$, where $\tau\in[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

Note that in [145] (2021) an analogue of Theorem 2.12 (1997) is proved for the integral $J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)}$ $(i_1=\ldots=i_k\ne 0,\ \psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T]))$.

Let us denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\stackrel{\text{def}}{=}\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}, \tag{2.962}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.963)$$

$\sum\limits_{\emptyset}$ is supposed to be equal to zero; another notations as in Theorem 2.12.

We will also notice that

$$J^S[\psi^{(1)}]_{T,t}^{(i_1)}=J[\psi^{(1)}]_{T,t}^{(i_1)}\quad \text{w. p. 1.} \qquad (2.964)$$

Further, by analogy with (2.667), (2.671) and using (1.316) (also see Theorem 1.23) instead of (1.272) we obtain the following generalization of (2.667) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1,} \qquad (2.965)$$

where $k\ge 2$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$, $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals (see (1.304)) and $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\mathsf{def}}{=}1$ for $k=2r$.

Using the equalities (1.316) and (1.321), we can reformulate Theorem 1.16 as follows

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \qquad (2.966)$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral defined by (1.304); another notations are the same as in Theorem 1.16.

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in (2.965) and using the equality (2.966), we get w. p. 1

$$
\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})},\tag{2.967}
$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ is the multiple Wiener stochastic integral defined by (1.304), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2.963).

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$. Then we have

$$
\sum_{j=0}^{\infty}\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le
$$

$$
\le\frac{1}{2}\sum_{j=0}^{\infty}\left(\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_j(\tau)\Phi_1(\tau)d\tau\right)^2+\left(\int\limits_t^T\mathbf{1}_{\{\tau>s\}}\phi_j(\tau)\Phi_2(\tau)d\tau\right)^2\right)=
$$

$$
=\frac{1}{2}\left(\int\limits_t^s\Phi_1^2(\tau)d\tau+\int\limits_s^T\Phi_2^2(\tau)d\tau\right)\le\frac{1}{2}\left(\|\Phi_1\|_{L_2([t,T])}^2+\|\Phi_2\|_{L_2([t,T])}^2\right)<\infty,\tag{2.968}
$$

i.e.

$$
\left|\sum_{j=0}^{p}\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le C<\infty,\tag{2.969}
$$

where $p\in\mathbf{N}$.

By interpreting the integrals in (2.685)–(2.688) as Lebesgue integrals, using Fubini's Theorem in (2.685) and Lebesgue's Dominated Convergence Theorem in (2.689), we obtain (2.683) (see (2.969)) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Using the equality (2.535) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ as well as Fubini's Theorem when deriving (2.693), we obtain the generalization of (2.691) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Repeating the steps of the proof of Theorem 2.30 below the formula (2.694) using (2.962), (2.967) or steps of the proof of Theorem 2.32 using (2.962), (2.967), we obtain for complete orthonormal systems $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ (for which the condition (2.720) is satisfied) the following equality

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=$$

$$=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=\bar J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\qquad(2.970)$$

w. p. 1, where notations in (2.970) are the same as in Theorem 2.12 and $\bar J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (2.962).

Thus the following two theorems are proved (compare with Theorem 1.16 (Sect. 1.11) on the expansion of iterated Itô stochastic integrals).

Theorem 2.42 [33], [38], [39]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ *are such that*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\sum_{j_{g_3}=0}^{\min\{p_{g_3},p_{g_4}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=0$$

(2.971)

for all $r = 1, 2, \ldots, [k/2]$ *and for all possible* $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (2.652)*). Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (2.962) *the following expansion*

$$
\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{2.972}
$$

that converges in the mean-square sense is valid, where

$$
C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k
$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$
\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau.$

Theorem 2.43 [33], [38], [39]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x) = 1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2, 3, \ldots, k)$ *are such that the condition*

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0
$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (2.652)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$
S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\text{def}}{=} \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}
$$

for $d = 0$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals* *defined by* (2.962) *the following expansion*

$$
\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}
$$

that converges in the mean-square sense is valid, where

$$
C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k
$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Note that in Theorems 2.42, 2.43 (the case $k=2$) the condition $\psi_1(\tau)\psi_2(\tau)\in L_2([t,T])$ can be omitted.

Using Theorem 2.12 together with Proposition 3.1 [145] and the proof of Lemma A.2 [144], we can write $\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ w. p. 1 and reformulate Theorems 2.42, 2.43 for $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ defined by (2.961).

Let us consider the special case $k=2$ of Theorem 2.42 in more detail. In this case, the condition (2.971) takes the following form (compare with (2.10))

$$
\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1. \tag{2.973}
$$

As follows from Sect. 2.1.4, the equality (2.973) is valid for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

From Proposition 3.1 [145] for the case $k=2$ we obtain

$$
\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i)}\circ d\mathbf{w}_{t_2}^{(i)}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i)}d\mathbf{w}_{t_2}^{(i)}+
$$

$$
+\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 \qquad (2.974)
$$

w. p. 1, where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $i=1,\ldots,m$,

$$
\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i)}\circ d\mathbf{w}_{t_2}^{(i)}
$$

is defined by (2.960), (2.961) and

$$
\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i)}d\mathbf{w}_{t_2}^{(i)}
$$

is the iterated Itô stochastic integral of the form (2.7) ($k=2$).

On the other hand, it is not difficult to show that

$$
\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i)}\circ d\mathbf{w}_{t_2}^{(j)}=\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i)}d\mathbf{w}_{t_2}^{(j)} \qquad (2.975)
$$

w. p. 1, where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $i\ne j$ ($i,j=1,\ldots,m$), another notations are the same as in (2.974).

Combining (2.974) and (2.975), we get (see (2.962))

$$
\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\circ d\mathbf{w}_{t_2}^{(i_2)}=\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 \stackrel{\sf def}{=} \bar J^{*}[\psi^{(2)}]_{T,t}^{(i_1i_2)} \qquad (2.976)
$$

w. p. 1, where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $i_1,i_2=1,\ldots,m$.

It is easy to see that the condition $\phi_0(x)=1/\sqrt{T-t}$ can be omitted in Theorems 2.42, 2.43 for the case $k=2$ (see the proof of Theorem 2.30).

Summing up the above arguments, we obtain the following generalization of Theorem 2.3 to the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

Theorem 2.44 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral*

$$J^S[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{f}_{t_1}^{(i_1)}\circ d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^S[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{2.977}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 2.1–2.3 and $J^S[\psi^{(2)}]_{T,t}^{(i_1i_2)}$ is defined by (2.961).

Note that the analog of (2.977) for $k=1$ is also true (see (1.45) and (2.964)).

In this section, it is also appropriate to mention the so-called multiple Stratonovich stochastic integral [143], [144] (also see [139]).

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}\ldots\sum_{l_k=0}^{N-1}\frac{1}{\Delta\tau_{l_1}\ldots\Delta\tau_{l_k}}\int\limits_{[\tau_{l_1},\tau_{l_1+1}]\times\ldots\times[\tau_{l_k},\tau_{l_k+1}]}K(t_1,\ldots,t_k)dt_1\ldots dt_k\times$$

$$\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\stackrel{\text{def}}{=}\bar{J}^S[K]_{T,t}^{(i_1\ldots i_k)} \tag{2.978}$$

is called [143], [144] the multiple Stratonovich stochastic integral of the function $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$, where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\Delta\tau_j=\tau_{j+1}-\tau_j$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (2.959), $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$, $\mathbf{f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

Note that in [144] the case $i_1=\ldots=i_k\neq 0$ was considered. We also denote by $\bar{J}^S[K]_{s,t}^{(i_1\ldots i_k)}$ the stochastic integral (2.978) (if it exists) of the function $K(t_1,\ldots,t_k)\mathbf{1}_{\{(t_1,\ldots,t_k)\in[t,s]^k\}}$, where $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$, $s\in[t,T]$, $t\ge 0$.

Let the function $K(t_1,\ldots,t_k)$ be chosen as follows

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1\le\ldots\le t_k\\ 0, & \text{otherwise}\end{cases}, \tag{2.979}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T]),\ t_1,\ldots,t_k\in[t,T]\ (k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

We will denote the multiple Stratonovich stochastic integral (2.978) of the function (2.979) as $\bar J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$.

It is known [144] (Lemma A.2) that the Stratonovich stochastic integrals $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ and $\bar J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ exist for the case $i_1=\ldots=i_k\ne 0$. Moreover, $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\bar J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ w. p. 1 for this case [144] (Lemma A.2).

Recall that an expansion similar to (2.972) for $p_1=\ldots=p_k=p$ was obtained in [142] for the multiple Stratonovich stochastic integral (2.978) under the condition of convergence of trace series (see Remarks 2.4, 2.7 for details).

Recently, another approach to the expansion of integral (2.978) has been proposed (assuming that the integral (2.978) exists), where multiple Fourier–Walsh and Fourier–Haar series ($k\in\mathbf{N}$) have been applied [221]. The convergence was proved with respect to the special subsequence ($p_1=\ldots=p_k=p=2^m$, $m\to\infty$ in a formula similar to (2.972) [221]).

2.19 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case of an Arbitrary Complete Orthonormal System of Functions ($\phi_0(x)=1/\sqrt{T-t}$) in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$

In this section, we will prove the following theorem.

Theorem 2.45 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\qquad(2.980)$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. First, note that under the conditions of Theorem 2.45 the equality

$$\bar J^{*}[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

is true w. p. 1 (see Theorem 2.12), where $\bar J^{*}[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}$ is defined by (2.962).

According to Theorem 2.42, we come to the conclusion that Theorem 2.45 will be proved if we prove the following equalities

$$\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_2j_1}\Biggl|_{j_1=j_2}-\frac{1}{2}C_{j_3j_2j_1}\Biggl|_{(j_1j_2)\curvearrowright(\cdot),j_1=j_2}\right)^2=0, \qquad (2.981)$$

$$\lim\limits_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_2j_1}\Biggl|_{j_2=j_3}-\frac{1}{2}C_{j_3j_2j_1}\Biggl|_{(j_2j_3)\curvearrowright(\cdot),j_2=j_3}\right)^2=0, \qquad (2.982)$$

$$\lim\limits_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_2j_1}\Biggl|_{j_1=j_3}\right)^2=0. \qquad (2.983)$$

Note that using Theorem 2.43 (also see (2.125)), we can rewrite the relations (2.981)–(2.983)) in the form (compare with (2.724)–(2.726))

$$\lim\limits_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_2j_1}\Biggl|_{j_1=j_2}\right)^2=0,\quad \lim\limits_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1}\Biggl|_{j_2=j_3}\right)^2=0,$$

$$\lim\limits_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_2j_1}\Biggl|_{j_1=j_3}\right)^2=0.$$

Let us prove (2.981). Using Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_2j_1}\Bigg|_{j_1=j_2}-\frac{1}{2}C_{j_3j_2j_1}\Bigg|_{(j_1j_2)\curvearrowright(\cdot),j_1=j_2}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_2j_1}\Bigg|_{(j_1j_2)\curvearrowright(\cdot),j_1=j_2}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\int\limits_t^T\phi_{j_3}(\tau)\left(\frac{1}{2}\int\limits_t^{\tau}ds-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2\right)d\tau\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\phi_{j_3}(\tau)\left(\frac{1}{2}(\tau-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2\right)d\tau\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(\frac{1}{2}(\tau-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2\right)^2d\tau. \qquad (2.984)$$

Applying the Parseval equality, we have

$$\sum_{j_1=0}^{\infty}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2=\sum_{j_1=0}^{\infty}\frac{1}{2}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_{j_1}(s)ds\right)^2=$$

$$=\frac{1}{2}\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2ds=\frac{1}{2}(\tau-t). \qquad (2.985)$$

Moreover,

$$\left|\frac{1}{2}(\tau-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2\right|\le\frac{1}{2}(\tau-t)\le\frac{1}{2}(T-t)<\infty. \qquad (2.986)$$

Using (2.985), (2.986) and applying Lebesgue's Dominated Convergence Theorem in (2.984), we obtain the equality (2.981).

Note that we could use Dini's Theorem instead of Lebesgue's Dominated Convergence Theorem. Using the continuity of the functions $u_p(\tau)$ (see below), the nondecreasing property of the functional sequence

$$u_p(\tau)=\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{\tau}\phi_{j_1}(s)ds\right)^2,$$

and the continuity of the limit function $u(\tau)=(\tau-t)/2$ according to Dini's Theorem, we have the uniform convergence $u_p(\tau)$ to $u(\tau)$ at the interval $[t,T]$. Then we can swap the limit and integral in (2.984) and get (2.981).

Let us prove (2.982). Using Fubini's Theorem and Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_2j_1}\Biggl|_{j_2=j_3}-\frac{1}{2}C_{j_3j_2j_1}\Biggl|_{(j_2j_3)\curvearrowright(\cdot),j_2=j_3}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_2j_1}\Biggl|_{(j_2j_3)\curvearrowright(\cdot),j_2=j_3}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(\theta)\int\limits_t^{\theta}\phi_{j_3}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau d\theta\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_1}(s)(T-s)ds-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_1}(s)\int\limits_s^T\phi_{j_3}(\tau)\int\limits_{\tau}^T\phi_{j_3}(\theta)d\theta d\tau ds\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s)\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)ds\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(s)\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)ds\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)^2ds. \qquad (2.987)$$

Using the Parseval equality, we get

$$
\sum_{j_3=0}^{\infty}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2=\sum_{j_3=0}^{\infty}\frac{1}{2}\left(\int\limits_t^T\mathbf{1}_{\{s<\tau\}}\phi_{j_3}(\tau)d\tau\right)^2=
$$

$$
=\frac{1}{2}\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2d\tau=\frac{1}{2}(T-s). \tag{2.988}
$$

Moreover,

$$
\left|\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right|\le\frac{1}{2}(T-s)\le\frac{1}{2}(T-t)<\infty. \tag{2.989}
$$

Combining (2.987)–(2.989) and using the same reasoning as in the proof of (2.981), we obtain

$$
\lim_{p\to\infty}\int\limits_t^T\left(\frac{1}{2}(T-s)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_s^T\phi_{j_3}(\tau)d\tau\right)^2\right)^2ds=0.
$$

The equality (2.982) is proved.

Let us prove (2.983). Applying Fubini's Theorem and Parseval's equality, we have

$$
\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=
$$

$$
=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(\theta)\int\limits_t^{\theta}\phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)dsd\tau d\theta\right)^2=
$$

$$
=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta d\tau\right)^2\le
$$

$$
\le\lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(\int\limits_t^T\phi_{j_2}(\tau)\sum_{j_1=0}^{p}\int\limits_t^{\tau}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}\phi_{j_1}(\theta)d\theta d\tau\right)^2=
$$

$$
= \lim_{p\to\infty} \int\limits_t^T \left(\sum_{j_1=0}^{p} \int\limits_t^{\tau} \phi_{j_1}(s)ds \int\limits_{\tau}^{T} \phi_{j_1}(\theta)d\theta \right)^2 d\tau. \qquad (2.990)
$$

Applying (2.968), we obtain

$$
\left| \sum_{j_1=0}^{p} \int\limits_t^{\tau} \phi_{j_1}(s)ds \int\limits_{\tau}^{T} \phi_{j_1}(\theta)d\theta \right| \le \sum_{j_1=0}^{p} \left| \int\limits_t^{\tau} \phi_{j_1}(s)ds \int\limits_{\tau}^{T} \phi_{j_1}(\theta)d\theta \right| \le
$$

$$
\le \sum_{j_1=0}^{\infty} \left| \int\limits_t^{\tau} \phi_{j_1}(s)ds \int\limits_{\tau}^{T} \phi_{j_1}(\theta)d\theta \right| \le \frac{1}{2}(T-t) < \infty. \qquad (2.991)
$$

Using the generalized Parseval equality, we get

$$
\lim_{p\to\infty} \sum_{j_1=0}^{p} \int\limits_t^{\tau} \phi_{j_1}(s)ds \int\limits_{\tau}^{T} \phi_{j_1}(\theta)d\theta = \sum_{j_1=0}^{\infty} \int\limits_t^{T} \mathbf{1}_{\{s<\tau\}}\phi_{j_1}(s)ds \int\limits_t^{T} \mathbf{1}_{\{s>\tau\}}\phi_{j_1}(s)ds =
$$

$$
= \int\limits_t^{T} \mathbf{1}_{\{s<\tau\}}\mathbf{1}_{\{s>\tau\}}ds = 0. \qquad (2.992)
$$

Taking into account (2.991), (2.992) and applying Lebesgue's Dominated Convergence Theorem in (2.990), we obtain the equality (2.983). Theorem 2.45 is proved.

2.20 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case of an Arbitrary Complete Orthonormal System of Functions $(\phi_0(x) = 1/\sqrt{T-t})$ in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_4(\tau) \equiv 1$

In this section, we will prove the following theorem.

Theorem 2.46 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x) = 1/\sqrt{T-t})$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$.

Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where

$$C_{j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. First, note that under the conditions of Theorem 2.46 the equality

$$\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1i_2i_3i_4)} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

is valid w. p. 1 (see Theorem 2.12), where $\bar{J}^*[\psi^{(4)}]_{T,t}^{(i_1i_2i_3i_4)}$ is defined by (2.962).

It is easy to see that Theorem 2.46 will be proved if we prove the following equalities (see Theorem 2.42)

$$\lim_{p\to\infty}\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_4j_3j_1j_1} - \frac{1}{2}C_{j_4j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2 = 0, \qquad (2.993)$$

$$\lim_{p\to\infty}\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_4j_1j_2j_1}\right)^2 = 0, \qquad (2.994)$$

$$\lim_{p\to\infty}\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_3j_2j_1}\right)^2 = 0, \qquad (2.995)$$

$$\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_4j_2j_2j_1}-\frac{1}{2}C_{j_4j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{2.996}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}\right)^2=0, \tag{2.997}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_2j_1}-\frac{1}{2}C_{j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{2.998}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}=\frac{1}{4}C_{j_3j_3j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)(j_1j_1)\curvearrowright(\cdot)}=\frac{1}{8}(T-t)^2, \tag{2.999}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=0, \tag{2.1000}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=0. \tag{2.1001}$$

Let us prove the equalities (2.993)–(2.998). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (2.993)–(2.998)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4j_3j_1j_1}-\frac{1}{2}C_{j_4j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=$$

$$=\sum_{j_3,j_4=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)dt_3dt_4-\right.$$

$$\left.-\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}(t_3-t)-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\right)dt_3dt_4\right)^2=$$

$$= \sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}(t_3-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(s)ds\right)^2\right)dt_3dt_4\right)^2 \le$$

$$\le \sum_{j_3,j_4=0}^{\infty}\left(\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}(t_3-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(s)ds\right)^2\right)dt_3dt_4\right)^2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_4\}}\left(\frac{1}{2}(t_3-t)-\sum_{j_1=0}^{p}\frac{1}{2}\left(\int\limits_t^{t_3}\phi_{j_1}(s)ds\right)^2\right)^2 dt_3dt_4, \qquad (2.1002)$$

$$\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_4j_1j_2j_1}\right)^2 =$$

$$= \sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =$$

$$= \sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2 =$$

$$= \sum_{j_2,j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2 \le$$

$$\le \sum_{j_2,j_4=0}^{\infty}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\right)^2 dt_2dt_4, \qquad (2.1003)$$

$$\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_3j_2j_1}\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 \le$$

$$\le \sum_{j_2,j_3=0}^{\infty}\left(\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\phi_{j_1}(t_4)dt_4\right)^2 dt_2dt_3, \qquad (2.1004)$$

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_4j_2j_2j_1}-\frac{1}{2}C_{j_4j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2 =$$

$$= \sum_{j_1,j_4=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4-\right.$$

$$\left.-\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =$$

$$= \sum_{j_1,j_4=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}dt_2dt_1dt_4-\right.$$

$$\left.-\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\right)^2 =$$

$$= \sum_{j_1,j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{t_4-t_1}{2}-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)dt_1dt_4\right)^2 \le$$

$$
\le \sum_{j_1,j_4=0}^{\infty}\left(\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{t_4-t_1}{2}-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)dt_1dt_4\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_4\}}\left(\frac{1}{2}(t_4-t_1)-\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(s)ds\right)^2\right)^2 dt_1dt_4, \qquad (2.1005)
$$

$$
\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4dt_3\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2\le
$$

$$
\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_3\}}\left(\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)dt_4\right)^2dt_1dt_3, \qquad (2.1006)
$$

$$
\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_2j_1}-\frac{1}{2}C_{j_3j_3j_2j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=
$$

$$
=\sum_{j_1,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3-\right.
$$

$$
\left.-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T dt_3dt_2dt_1-\right.
$$

$$
\left.-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2=
$$

$$
=\sum_{j_1,j_2=0}^{p}\left(\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\left(\frac{T-t_2}{2}-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^T\phi_{j_3}(s)ds\right)^2\right)dt_2dt_1\right)^2\le
$$

$$
\le\sum_{j_1,j_2=0}^{\infty}\left(\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\left(\frac{T-t_2}{2}-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^T\phi_{j_3}(s)ds\right)^2\right)dt_2dt_1\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\left(\frac{1}{2}(T-t_2)-\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^T\phi_{j_3}(s)ds\right)^2\right)^2dt_2dt_1. \qquad (2.1007)
$$

Using Parseval's equality, generalized Parseval's equality and Lebesgue's Dominated Convergence Theorem, as well as applying the same reasoning as in the proof of Theorem 2.45, we obtain that the right-hand sides of (2.1002)–(2.1007) tend to zero when $p\to\infty$. The equalities (2.993)–(2.998) are proved.

Let us prove the equalities (2.999)–(2.1001). We will use our idea from Sect. 2.14. More precisely, we consider the following analogue of the equality (2.848)

$$
C_{j_4j_3j_2j_1}+C_{j_1j_2j_3j_4}=C_{j_4}C_{j_3j_2j_1}-C_{j_3j_4}C_{j_2j_1}+
$$

$$
+C_{j_2j_3j_4}C_{j_1}. \qquad (2.1008)
$$

Using Fubini's Theorem, we have

$$
C_{j_4j_3j_2j_1}=
$$

$$
= \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 -
$$

$$
- \int\limits_t^T \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= C_{j_4} C_{j_3 j_2 j_1} -
$$

$$
- \int\limits_t^T \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_3}(t_3) \int\limits_t^{T} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 +
$$

$$
+ \int\limits_t^T \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= C_{j_4} C_{j_3 j_2 j_1} - C_{j_3 j_4} C_{j_2 j_1} +
$$

$$
+ \int\limits_t^T \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_t^{T} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 -
$$

$$
- \int\limits_t^T \phi_{j_4}(t_4) \int\limits_{t_4}^{T} \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= C_{j_4} C_{j_3 j_2 j_1} - C_{j_3 j_4} C_{j_2 j_1} + C_{j_2 j_3 j_4} C_{j_1} - C_{j_1 j_2 j_3 j_4}. \tag{2.1009}
$$

The equality (2.1009) completes the proof of the relation (2.1008).

Let us prove (2.999). Substitute $j_4 = j_3$, $j_2 = j_1$ into (2.1008)

$$
C_{j_3 j_3 j_1 j_1} + C_{j_1 j_1 j_3 j_3} = C_{j_3} C_{j_3 j_1 j_1} - C_{j_3 j_3} C_{j_1 j_1} +
$$

$$
+ C_{j_1 j_3 j_3} C_{j_1}. \tag{2.1010}
$$

From (2.1010) we obtain

$$
\sum_{j_1, j_3 = 0}^{p} \left(C_{j_3 j_3 j_1 j_1} + C_{j_1 j_1 j_3 j_3} \right) = \sum_{j_1, j_3 = 0}^{p} C_{j_3} C_{j_3 j_1 j_1} - \sum_{j_1, j_3 = 0}^{p} C_{j_3 j_3} C_{j_1 j_1} +
$$

$$
+\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3}C_{j_1}.
$$

Then

$$
2\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}=2\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}-\left(\sum_{j_1=0}^{p} C_{j_1j_1}\right)^2. \qquad (2.1011)
$$

From (2.1011) we get

$$
\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}=\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}-\frac{1}{2}\left(\sum_{j_1=0}^{p} C_{j_1j_1}\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}-\frac{1}{2}\left(\sum_{j_1=0}^{p} \frac{1}{2}\left(C_{j_1}\right)^2\right)^2=\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}-\frac{1}{8}\left(\sum_{j_1=0}^{p}\left(C_{j_1}\right)^2\right)^2. \qquad (2.1012)
$$

Recall that $\phi_0(\tau)=1/\sqrt{T-t}$. Then

$$
C_j=\int\limits_t^T \phi_j(\tau)d\tau=\begin{cases}\sqrt{T-t} & \text{if } j=0\\ & \\ 0 & \text{if } j\ne 0\end{cases}. \qquad (2.1013)
$$

Combining (2.1012), (2.1013) and using Fubini's Theorem, we obtain

$$
\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}=\sqrt{T-t}\sum_{j_1=0}^{p} C_{0j_1j_1}-\frac{1}{8}(T-t)^2=
$$

$$
=\sum_{j_1=0}^{p}\int\limits_t^T\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3-\frac{1}{8}(T-t)^2=
$$

$$
=\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_1}(t_2)\int\limits_{t_2}^{T}dt_3dt_2dt_1-\frac{1}{8}(T-t)^2=
$$

$$
=\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_1}(t_2)(T-t_2)dt_2dt_1-\frac{1}{8}(T-t)^2=
$$

$$
=\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_2)(T-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2-\frac{1}{8}(T-t)^2. \tag{2.1014}
$$

Finally applying (2.125) and (2.1014), we have

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}=\frac{1}{2}\int\limits_t^T(T-t_2)dt_2-\frac{1}{8}(T-t)^2=\frac{1}{8}(T-t)^2.
$$

The equality (2.999) is proved.

Let us prove (2.1000). Substitute $j_4=j_1,\ j_2=j_3$ into (2.1008)

$$
C_{j_1j_3j_3j_1}+C_{j_1j_3j_3j_1}=C_{j_1}C_{j_3j_3j_1}-C_{j_3j_1}C_{j_3j_1}+
$$

$$
+C_{j_3j_3j_1}C_{j_1}. \tag{2.1015}
$$

Using (2.1015), we get

$$
2\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=2\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2. \tag{2.1016}
$$

Then applying (2.1016), (2.1013), Parseval's equality, and (2.125), we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1}C_{j_3j_3j_1}-\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2=
$$

$$
=\sqrt{T-t}\sum_{j_3=0}^{\infty}C_{j_3j_30}-\frac{1}{2}\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\phi_{j_3}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\right)^2=
$$

$$
=\sum_{j_3=0}^{\infty}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\int\limits_t^{t_2}dt_1dt_2dt_3-
$$

$$
-\frac{1}{2}\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\phi_{j_1}(t_1)\phi_{j_3}(t_2)dt_1dt_2\right)^2=
$$

$$
=\sum_{j_3=0}^{\infty}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)(t_2-t)dt_2dt_3-\frac{1}{2}\int\limits_{[t,T]^2}\left(\mathbf{1}_{\{t_1<t_2\}}\right)^2dt_1dt_2=
$$

$$
=\frac{1}{2}\int\limits_t^T (t_2-t)dt_2-\frac{1}{2}\int\limits_t^T\int\limits_t^{t_2} dt_1dt_2=0.
$$

The equality (2.1000) is proved.

Let us prove (2.1001). Substitute $j_3=j_1,\ j_4=j_2$ into (2.1008)

$$
C_{j_2j_1j_2j_1}+C_{j_1j_2j_1j_2}=C_{j_2}C_{j_1j_2j_1}-C_{j_1j_2}C_{j_2j_1}+
$$

$$
+C_{j_2j_1j_2}C_{j_1}. \tag{2.1017}
$$

Then

$$
\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1j_2j_1}+C_{j_1j_2j_1j_2}\right)=\sum_{j_1,j_2=0}^{p}\left(C_{j_2}C_{j_1j_2j_1}+C_{j_2j_1j_2}C_{j_1}\right)-
$$

$$
-\sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1}. \tag{2.1018}
$$

From (2.1018) we have

$$
2\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=2\sum_{j_1,j_2=0}^{p}C_{j_1}C_{j_2j_1j_2}-
$$

$$
-\sum_{j_1,j_2=0}^{p}\frac{1}{2}\left(\left(C_{j_1j_2}+C_{j_2j_1}\right)^2-\left(C_{j_1j_2}\right)^2-\left(C_{j_2j_1}\right)^2\right)=
$$

$$
=2\sum_{j_1,j_2=0}^{p}C_{j_1}C_{j_2j_1j_2}-\frac{1}{2}\sum_{j_1,j_2=0}^{p}\left(C_{j_1j_2}+C_{j_2j_1}\right)^2+
$$

$$
+\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1}\right)^2. \tag{2.1019}
$$

Using Fubini's Theorem, we obtain (also see (1.60))

$$
C_{j_1j_2}+C_{j_2j_1}=C_{j_1}C_{j_2}. \tag{2.1020}
$$

Applying (2.1019), (2.1020), (2.1013), Fubini's Theorem, Parseval's equality, and (2.125), we get

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1}C_{j_2j_1j_2}-\frac{1}{4}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(C_{j_1j_2}+C_{j_2j_1}\right)^2+
$$

$$+\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1}\right)^2=$$

$$=\sqrt{T-t}\sum_{j_2=0}^{\infty}C_{j_20j_2}-\frac{1}{4}\sum_{j_1,j_2=0}^{\infty}\left(C_{j_1}C_{j_2}\right)^2+\frac{1}{2}\sum_{j_1,j_2=0}^{\infty}\left(C_{j_2j_1}\right)^2=$$

$$=\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(t_3)\int\limits_t^{t_3}\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3-\frac{1}{4}(T-t)^2+\frac{1}{2}\int\limits_{[t,T]^2}\left(\mathbf{1}_{\{t_1<t_2\}}\right)^2dt_1dt_2=$$

$$=\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_1)\int\limits_{t_1}^{t_3}dt_2dt_1dt_3=$$

$$=\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(t_3)(t_3-t)\int\limits_t^{t_3}\phi_{j_2}(t_1)dt_1dt_3+\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_1)(t-t_1)dt_1dt_3=$$

$$=\frac{1}{2}\int\limits_t^T(t_3-t)dt_3+\frac{1}{2}\int\limits_t^T(t-t_3)dt_3=0.$$

The equality (2.1001) is proved. The equalities (2.993)–(2.1001) are proved. Theorem 2.46 is proved.

2.21 Condition $\phi_0(x)=1/\sqrt{T-t}$ in Theorems 2.45 and 2.46 can be Omitted

In this section, we will show that the condition $\phi_0(x)=1/\sqrt{T-t}$ in Theorems 2.45 and 2.46 can be omitted.

Theorem 2.47 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad(i_1,i_2,i_3=0,1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \qquad (2.1021)$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. Analyzing the proof of Theorems 2.42 and 2.45 (also see the derivation of (2.709) and (2.719)), we notice that Theorem 2.47 will be proved if we prove that

$$\int\limits_t^T\int\limits_t^{t_3} dt_2d\mathbf{w}_{t_3}^{(i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{p}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3} dt_2dt_3\ \zeta_{j_3}^{(i_3)}, \qquad (2.1022)$$

$$\int\limits_t^T\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2 = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\int\limits_t^T\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2\ \zeta_{j_1}^{(i_1)}. \qquad (2.1023)$$

The equality (2.1022) immediately follows from Theorem 1.16 (see (1.321) for $k = 1$).

Let us prove (2.1023). Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and (3.31)) or the Itô formula, Theorem 1.16 (see (1.321) for $k = 1$) and Fubini's Theorem, we obtain w. p. 1

$$\int\limits_t^T\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2 = \int\limits_t^T\int\limits_{t_1}^{T} dt_2d\mathbf{w}_{t_1}^{(i_1)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T} dt_2dt_1\ \zeta_{j_1}^{(i_1)} =$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p}\int\limits_t^T\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\ \zeta_{j_1}^{(i_1)}.
$$

The equality (2.1023) is proved. Theorem 2.47 is proved.

Let us develop this approach and prove the following generalization of Theorem 2.46.

Theorem 2.48 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)
$$

the following expansion

$$
J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}
$$

that converges in the mean-square sense is valid, where

$$
C_{j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. Considering the proof of Theorems 2.42 and 2.46 (also see the derivation of (2.709) and (2.719)), we conclude that Theorem 2.48 will be proved if we prove that under the conditions of Theorem 2.48 the following equalities

$$
\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_2,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}dt_1dt_2dt_3\times
$$

$$
\times J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}, \tag{2.1024}
$$

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_3}^{(i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\times$$

$$\times J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)}, \tag{2.1025}$$

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3 = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} \int\limits_t^T \int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\times$$

$$\times J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_2)}, \tag{2.1026}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = \frac{1}{4}C_{j_3j_3j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)(j_1j_1)\curvearrowright(\cdot)} = \frac{1}{8}(T-t)^2, \tag{2.1027}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1} = 0, \tag{2.1028}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = 0 \tag{2.1029}$$

holds, where we use (1.319), i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{2.1030}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral defined by (1.304).

Moreover, for $k=4, r=2, g_1=1, g_2=2, g_3=3, g_4=4$ we can write (see the derivation of (2.709))

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$
= \frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}C_{j_3j_3j_1j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)(j_1j_1)\curvearrowright(\cdot)} =
$$

$$
= \frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_2}dt_1dt_2 = \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{(T-t)^2}{8},
$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$.

The equality (2.1024) immediately follows from Theorem 1.16 (see (1.319) or (2.1030) for $k=2$).

Let us prove (2.1026). Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and (3.34)) or the Itô formula, Theorem 1.16 (see (1.319) or (2.1030) for $k=2$) and Fubini's Theorem, we get w. p. 1

$$
\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3 = \int\limits_t^T(T-t_2)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T(T-t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)(T-t_2)dt_2dt_1J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}dt_3dt_2dt_1J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}.
$$

The equality (2.1026) is proved. To prove (2.1025) we will use the above arguments ((2.1031) (see below) also directly follows from the Itô formula)

$$
\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_3}^{(i_3)} = [\text{by Theorems 3.1, 3.3}] = \int\limits_t^T\int\limits_t^{t_3}d\mathbf{w}_{t_1}^{(i_1)}\int\limits_{t_1}^{t_3}dt_2d\mathbf{w}_{t_3}^{(i_3)} =
$$

$$
= \int\limits_t^T \int\limits_t^{t_3} (t_3 - t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_3}^{(i_3)} =
$$

$$
= \int\limits_t^T (t_3 - t) \int\limits_t^{t_3} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_3}^{(i_3)} - \int\limits_t^T \int\limits_t^{t_3} (t_1 - t) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_3}^{(i_3)} = \tag{2.1031}
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T (t_3 - t)\phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 dt_3 J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)} -
$$

$$
- \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} (t_1 - t)\phi_{j_1}(t_1) dt_1 dt_3 J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \left(\int\limits_t^T (t_3 - t)\phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 dt_3 - \right.
$$

$$
\left. - \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} (t_1 - t)\phi_{j_1}(t_1) dt_1 dt_3 \right) J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} (t_3 - t + t - t_1)\phi_{j_1}(t_1) dt_1 dt_3 J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) \int\limits_{t_1}^{t_3} dt_2 dt_1 dt_3 J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_3)}.
$$

The equality (2.1025) is proved. Let us prove (2.1027)–(2.1029). Using (2.1012), we obtain

$$
\sum_{j_1,j_3=0}^{p} C_{j_3 j_3 j_1 j_1} = \sum_{j_1,j_3=0}^{p} C_{j_3} C_{j_3 j_1 j_1} - \frac{1}{8} \left(\sum_{j_1=0}^{p} \left(C_{j_1}\right)^2 \right)^2 . \tag{2.1032}
$$

Applying Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(C_{j_1}\right)^2=\int\limits_t^T 1^2 d\tau=T-t. \qquad (2.1033)$$

Combining (2.1032) and (2.1033), we get

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}=\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}-\frac{(T-t)^2}{8}. \qquad (2.1034)$$

Further, we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}=$$

$$=\frac{1}{2}\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \qquad (2.1035)$$

Applying the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T \phi_{j_3}(\tau)d\tau\int\limits_t^T \phi_{j_3}(\tau)\int\limits_t^{\tau} d\theta d\tau=$$

$$=\int\limits_t^T 1\cdot\int\limits_t^{\tau} d\theta d\tau=\frac{(T-t)^2}{2}. \qquad (2.1036)$$

From (2.1035) and (2.1036) we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3}C_{j_3j_1j_1}=$$

$$=\frac{(T-t)^2}{4}-\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \qquad (2.1037)$$

Combining (2.1034) and (2.1037), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}=\frac{(T-t)^2}{8}-\lim_{p\to\infty}\sum_{j_3=0}^{p} C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p} C_{j_3j_1j_1}\right). \qquad (2.1038)$$

Due to the inequality of Cauchy–Bunyakovsky and (2.981), (2.1033), we get

$$\lim_{p\to\infty}\left(\sum_{j_3=0}^{p}C_{j_3}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(C_{j_3}\right)^2\ \sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(C_{j_3}\right)^2\ \sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=(T-t)\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=0. \qquad (2.1039)$$

Taking into account (2.1038) and (2.1039), we obtain (2.1027). It is not difficult to see that by analogy with (2.1027) we get

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(s)=\frac{1}{8}(s-t)^2, \qquad (2.1040)$$

where $s\in(t,T]$ and

$$C_{j_4j_3j_2j_1}(s)=\int\limits_t^s\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4. \quad (2.1041)$$

Let us prove (2.1028). Using (2.1018), we have

$$\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=\sum_{j_1,j_2=0}^{p}C_{j_2}C_{j_1j_2j_1}-\frac{1}{2}\sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1}. \qquad (2.1042)$$

Fubini's Theorem and the generalized Parseval equality give

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1j_2}C_{j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_1)dt_1dt_2\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2=$$

$$= \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_1\}}\mathbf{1}_{\{t_1<t_2\}}dt_1dt_2 = 0. \qquad (2.1043)$$

The equalities (2.1042) and (2.1043) imply the relation

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1} = \lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2}C_{j_1j_2j_1}. \qquad (2.1044)$$

Further, we have (see the derivation of (2.1039))

$$\lim_{p\to\infty}\left(\sum_{j_2=0}^{p}C_{j_2}\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2 \le \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(C_{j_2}\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2 \le$$

$$\le \lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(C_{j_2}\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2 = (T-t)\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2 = 0, \qquad (2.1045)$$

where (2.1045) follows from (2.983).

The relations (2.1044) and (2.1045) complete the proof of (2.1028). By analogy with the above reasoning, we obviously get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s) = 0, \qquad (2.1046)$$

where $s\in(t,T]$ and $C_{j_2j_1j_2j_1}(s)$ is defined by (2.1041).

Let us prove (2.1029). Using (2.1016), we obtain

$$\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = \sum_{j_1,j_3=0}^{p} C_{j_1}C_{j_3j_3j_1} - \frac{1}{2}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2. \qquad (2.1047)$$

Parseval's equality gives

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}\right)^2 = \lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\phi_{j_1}(t_1)\phi_{j_3}(t_2)dt_1dt_2\right)^2 =$$

$$
= \int\limits_{[t,T]^2} \left(\mathbf{1}_{\{t_1<t_2\}}\right)^2 dt_1 dt_2 = \frac{(T-t)^2}{2}. \tag{2.1048}
$$

Combining (2.1047) and (2.1048), we have

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = \lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1}C_{j_3j_3j_1} - \frac{(T-t)^2}{4}. \tag{2.1049}
$$

Further, we have

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1}C_{j_3j_3j_1} =
$$

$$
= \frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} - \lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{2.1050}
$$

Applying Fubini's Theorem and the generalized Parseval equality, we obtain

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} = \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(\tau)d\tau \int\limits_t^T\int\limits_t^{t_2}\phi_{j_1}(\tau)d\tau dt_2 =
$$

$$
= \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(\tau)d\tau\int\limits_t^T\phi_{j_1}(\tau)\int\limits_\tau^T dt_2 d\tau = \int\limits_t^T 1\cdot\int\limits_\tau^T d\theta d\tau = \frac{(T-t)^2}{2}. \tag{2.1051}
$$

From (2.1050) and (2.1051) we have

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1}C_{j_3j_3j_1} =
$$

$$
= \frac{(T-t)^2}{4} - \lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{2.1052}
$$

Combining (2.1049) and (2.1052), we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = -\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=0}^{p} C_{j_3j_3j_1}\right). \tag{2.1053}
$$

Due to the inequality of Cauchy–Bunyakovsky and (2.982), (2.1033), we get

$$\lim_{p\to\infty}\left(\sum_{j_1=0}^{p}C_{j_1}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{p}(C_{j_1})^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}(C_{j_1})^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2=$$

$$=(T-t)\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2=0. \qquad (2.1054)$$

The relations (2.1053) and (2.1054) complete the proof of (2.1029). By analogy with the above reasoning, we obviously have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}(s)=0, \qquad (2.1055)$$

where $s\in(t,T]$ and $C_{j_1j_3j_3j_1}(s)$ is defined by (2.1041).

The equalities (2.1024)–(2.1029) are proved. Theorem 2.48 is proved.

Note that the equalities (2.1046) and (2.1055) can be proved by another way. Using Fubini's Theorem, we obtain

$$C_{j_2j_1j_2j_1}(s)=\frac{1}{2}\left(C_{j_2j_1}(s)\right)^2-2C_{j_2j_2j_1j_1}(s), \qquad (2.1056)$$

$$\sum_{(j_1,j_2,j_3,j_4)}C_{j_4j_3j_2j_1}(s)=C_{j_1}(s)C_{j_2}(s)C_{j_3}(s)C_{j_4}(s), \qquad (2.1057)$$

where $s\in(t,T]$,

$$\sum_{(j_1,j_2,j_3,j_4)}$$

means the sum with respect to all possible permutations (j_1,j_2,j_3,j_4) and

$$C_{j_k\ldots j_1}(s)=\int\limits_t^s\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k\quad(k=1,\ldots,4).$$

Taking into account (2.1040), (2.1048) (for s instead of T), (2.1056), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(s)=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1}(s)\right)^2-2\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1j_1}(s)=$$

$$=\frac{1}{2}\cdot\frac{(s-t)^2}{2}-2\cdot\frac{(s-t)^2}{8}=0.$$

The equality (2.1046) is proved. Let us substitute $j_2=j_1$ and $j_4=j_3$ into (2.1057). Then we obtain

$$4\Biggl(C_{j_3j_3j_1j_1}(s)+C_{j_1j_1j_3j_3}(s)+C_{j_3j_1j_1j_3}(s)+C_{j_1j_3j_3j_1}(s)+$$

$$+C_{j_3j_1j_3j_1}(s)+C_{j_1j_3j_1j_3}(s)\Biggr)=\left(C_{j_1}(s)\right)^2\left(C_{j_3}(s)\right)^2. \qquad (2.1058)$$

The equality (2.1058) implies that

$$8\sum_{j_1,j_3=0}^{p}\Biggl(C_{j_3j_3j_1j_1}(s)+C_{j_1j_3j_3j_1}(s)+C_{j_3j_1j_3j_1}(s)\Biggr)=\sum_{j_1=0}^{p}\left(C_{j_1}(s)\right)^2\sum_{j_3=0}^{p}\left(C_{j_3}(s)\right)^2. \qquad (2.1059)$$

Passing to the limit $\lim\limits_{p\to\infty}$ in (2.1059) and taking into account (2.1033) (for s instead of T), (2.1040), (2.1046), we get

$$8\left(\frac{(s-t)^2}{8}+\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}(s)+0\right)=(s-t)^2.$$

The equality (2.1055) is proved.

2.22 Generalization of Theorem 2.42 to the Case When the Conditions $\phi_0(x)=1/\sqrt{T-t}$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ are Omitted

In this section, we will consider the following generalization of Theorem 2.42.

Theorem 2.49 [33], [38], [39]. *Assume that the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ are such that*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\left(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\sum_{j_{g_3}=0}^{\min\{p_{g_3},p_{g_4}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2=0 \tag{2.1060}$$

for all $r=1,2,\ldots,[k/2]$ *and for all possible* $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (2.652)). Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by (2.962) the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{2.1061}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. To prove Theorem 2.49, we need to prove that under the conditions of Theorem 2.49 the following equality

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \tag{2.1062}$$

holds w. p. 1, where $g_2 = g_1 + 1, \ldots, g_{2r} = g_{2r-1} + 1,\ g_{2i-1} \stackrel{\text{def}}{=} s_i;\ i = 1, 2, \ldots, r;$ $r = 1, 2, \ldots, [k/2]$, $(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]^{s_r,\ldots,s_1}_{T,t}$ is defined by (2.387) and $\mathrm{A}_{k,r}$ is defined by (2.388); also we put $p_1 = \ldots = p_k = p$ in (2.1062) to simplify the notation; another notations in (2.1062) are the same as in Sect. 2.10.

Using the Itô formula, we obtain w. p. 1

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) \int\limits_t^{t_{l-1}} \psi_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} dt_{l-1} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) dt_{l-1} \right) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)} -$$

$$- \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) dt_{l-1} \right) \times$$

$$\times \int\limits_t^{t_{l-2}} \psi_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{l-3}}^{(i_{l-3})} d\mathbf{w}_{t_{l-2}}^{(i_{l-2})} d\mathbf{w}_{t_{l+1}}^{(i_{l+1})} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.1063}$$

where $l \ge 3$. Note that the formula (2.1063) will change in an obvious way for the case $t_{l+1} = T$. We will also assume that the transformation (2.1063) is not carried out for $l = 2$ since the integral

$$\int\limits_t^{t_3} \psi_2(t_1)\psi_1(t_1) dt_1$$

is an internal integral on the left-hand side of (2.1063) for this case.

It is important to note that the transformation (2.1063) fully complies with the classical rules for replacing the order of integration (Fubini's Theorem) if we replace all differentials of the form $d\mathbf{w}_{t_j}^{(i_j)}$ with dt_j in (2.1063).

Indeed, formally changing the order of integration on the left-hand side of (2.1063) according to the classical rules, we have

$$
\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1}) \int\limits_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\ldots \qquad (2.1064)
$$

$$
\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}dt_{l-1}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =
$$

$$
= \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2})d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}\times\right.
$$

$$
\left.\times \int\limits_{t_{l-2}}^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =
$$

$$
= \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2})d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}\times\right.
$$

$$
\left.\times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}}\right) \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =
$$

$$
= \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots
$$

$$
\ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2})d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} -
$$

$$
- \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots \int\limits_{t_{l-3}}^{t_{l+1}} \psi_{l-2}(t_{l-2})\times
$$

$$
\times \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} =
$$

$$
= \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2})\ldots
$$

$$
\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)} -
$$

$$
- \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} \psi_l(t_{l-1})\psi_{l-1}(t_{l-1})dt_{l-1}\right) \times
$$

$$
\times \int\limits_t^{t_{l-2}} \psi_{l-3}(t_{l-3})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{l-3}}^{(i_{l-3})}d\mathbf{w}_{t_{l-2}}^{(i_{l-2})}d\mathbf{w}_{t_{l+1}}^{(i_{l+1})}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{2.1065}
$$

Comparing the right-hand sides of (2.1063) and (2.1065) we come to the conclusion that we got the same result.

The strict mathematical meaning of the transformations leading to (2.1065) is explained in Chapter 3, at least for the case when $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous functions on the interval $[t,T]$.

Note that under the conditions of Theorem 2.49, the derivation of the formulas (2.1063) and (2.1065) will remain valid if in (2.1063) and (2.1065) we replace all differentials $d\mathbf{w}_{t_j}^{(i_j)}$ with dt_j (this follows from Fubini's Theorem).

Recall that

$$
J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\rm def}{=} \prod_{q=1}^{r} \mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}} \times
$$

$$
\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_r+3}} \psi_{s_r+2}(t_{s_r+2}) \int\limits_t^{t_{s_r+2}} \psi_{s_r}(t_{s_r+1})\psi_{s_r+1}(t_{s_r+1})\times
$$

$$
\times \int\limits_t^{t_{s_r+1}} \psi_{s_r-1}(t_{s_r-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times
$$

$$
\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots
$$

$$
\ldots\ d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}dt_{s_r+1}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.1066}
$$

where $\mathrm{A}_{k,r}$ is defined by (2.388):

$$\mathrm{A}_{k,r}=\Bigl\{(s_r,\ldots,s_1):\ s_r>s_{r-1}+1,\ldots,s_2>s_1+1,\ s_r,\ldots,s_1=1,\ldots,k-1\Bigr\}.$$

Temporarily denote

$$J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\stackrel{\mathsf{def}}{=}I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}.$$

Let us carry out the transformation (2.1063) for the iterated Itô stochastic integral $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ iteratively for $s_1,\ldots,s_r$. After this, apply (2.1030) to each of the obtained iterated Itô stochastic integrals. As a result, we obtain w. p. 1

$$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}=\prod_{q=1}^{r}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}}\times$$

$$\times\sum_{d=1}^{2^r}\left(\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}-\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)=$$

$$=\prod_{q=1}^{r}\mathbf{1}_{\{i_{s_q}=i_{s_q+1}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_{s_1-1},j_{s_1+2},\ldots,j_{s_r-1},j_{s_r+2},\ldots,j_k=0}^{p}\ \sum_{d=1}^{2^r}\left(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}-\right.$$

$$\left.-\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}\right)\times$$

$$\times J'[\phi_{j_1}\ldots\phi_{j_{s_1-1}}\phi_{j_{s_1+2}}\ldots\phi_{j_{s_r-1}}\phi_{j_{s_r+2}}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)},\quad (2.1067)$$

where some terms in the sum

$$\sum_{d=1}^{2^r}$$

can be identically equal to zero due to the remark to (2.1063).

Taking into account that the integrals $\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and the Fourier coefficients $\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$ are formed on the basis of the same kernels (the same applies to the integrals $\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and the Fourier coefficients $\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}$), as well as a remark about the relationship of the transformation (2.1063) based on the Itô formula and on the basis of classical rules for replacing the order of integration (see the derivation of (2.1065)), we obtain using Fubini's theorem (applying the inverse transformation from (2.1065) to (2.1064) in which all differentials of the form $d\mathbf{w}_{t_j}^{(i_j)}$ are replaced with dt_j)

$$\sum_{d=1}^{2^r}\left(\hat{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}-\bar{C}^{(d)}_{j_1\ldots j_{s_1-1}j_{s_1+2}\ldots j_{s_r-1}j_{s_r+2}\ldots j_k}\right)=$$

$$=C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}, \qquad (2.1068)$$

where $g_2=g_1+1,\ldots,g_{2r}=g_{2r-1}+1$. Combining (2.1067) and (2.1068), we get w. p. 1

$$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})},$$

where we use the notations from Sect. 2.10. The equality (2.1062) is proved for the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Thus, the condition $\phi_0(x)=1/\sqrt{T-t}$ in Theorem 2.42 can be omitted.

Let us separately explain why the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ in Theorem 2.42 can also be omitted. Recall that this condition appeared due to the direct application of (1.319) to the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ defined by (2.1066) (see the transition from (2.708) to (2.709)).

It is easy to see that the kernels $\hat{K}_d(t_1,\ldots,t_{k-2r})$ and $\bar{K}_d(t_1,\ldots,t_{k-2r})$ of the iterated Itô stochastic integrals $\hat{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ and $\bar{I}[\psi^{(k)}]_{T,t}^{d(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$ have the same structure as (1.310) but with new wight functions $\hat{\psi}_1(\tau),\ldots,\hat{\psi}_{k-2r}(\tau)$ and $\bar{\psi}_1(\tau),\ldots,\bar{\psi}_{k-2r}(\tau)$, some of which possibly coincide with $\psi_1(\tau),\ldots,\psi_k(\tau)$ (see (2.1063)). Moreover, the conditions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$ guarantee that $\hat{K}_d(t_1,\ldots,t_{k-2r}), \bar{K}_d(t_1,\ldots,t_{k-2r})\in L_2([t,T])$ (see (2.1063)). This means that the formula (2.1067) is true if $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$. Furthermore, the formula (2.1068) holds under the conditions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$.

Since the condition $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ implies the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$, then the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_1([t,T])$ $(l=2,3,\ldots,k)$ can be omitted in the above reasoning.

Thus, the equalities (2.1067) and (2.1068) are satisfied under the condition $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and the condition $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ can be omitted in Theorem 2.42. Theorem 2.49 is proved.

2.23 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv 1$

Theorem 2.50 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)}$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1, \ldots, i_5 = 0, 1, \ldots, m,$

$$C_{j_5 \ldots j_1} = \int\limits_t^T \phi_{j_5}(t_5) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_5$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau.$

Proof. Step 1. According to Theorem 2.49, we conclude that Theorem 2.50 will be proved if we prove the following equalities (see (2.1060) for $k = 5, r = 1$ and $k = 5, r = 2$ $(p_1 = \ldots = p_5 = p)$) under the conditions of Theorem 2.50

$$\lim_{p\to\infty} \sum_{j_3,j_4,j_5=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_5 j_4 j_3 j_1 j_1} - \frac{1}{2} C_{j_5 j_4 j_3 j_1 j_1} \Bigg|_{(j_1 j_1) \curvearrowright (\cdot)} \right)^2 = 0, \qquad (2.1069)$$

$$\lim_{p\to\infty} \sum_{j_2,j_4,j_5=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_5 j_4 j_1 j_2 j_1} \right)^2 = 0, \qquad (2.1070)$$

$$\lim_{p\to\infty} \sum_{j_2,j_3,j_5=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_5 j_1 j_3 j_2 j_1} \right)^2 = 0, \qquad (2.1071)$$

$$\lim_{p\to\infty} \sum_{j_2,j_3,j_4=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_4 j_3 j_2 j_1} \right)^2 = 0, \qquad (2.1072)$$

$$\lim_{p\to\infty} \sum_{j_1,j_4,j_5=0}^{p} \left(\sum_{j_2=0}^{p} C_{j_5 j_4 j_2 j_2 j_1} - \frac{1}{2} C_{j_5 j_4 j_2 j_2 j_1} \Bigg|_{(j_2 j_2) \curvearrowright (\cdot)} \right)^2 = 0, \qquad (2.1073)$$

$$\lim_{p\to\infty} \sum_{j_1,j_3,j_5=0}^{p} \left(\sum_{j_2=0}^{p} C_{j_5 j_2 j_3 j_2 j_1} \right)^2 = 0, \qquad (2.1074)$$

$$\lim_{p\to\infty} \sum_{j_1,j_3,j_4=0}^{p} \left(\sum_{j_2=0}^{p} C_{j_2 j_4 j_3 j_2 j_1} \right)^2 = 0, \qquad (2.1075)$$

$$\lim_{p\to\infty} \sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_5 j_3 j_3 j_2 j_1} - \frac{1}{2} C_{j_5 j_3 j_3 j_2 j_1} \Bigg|_{(j_3 j_3) \curvearrowright (\cdot)} \right)^2 = 0, \qquad (2.1076)$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_4j_3j_2j_1}\right)^2=0, \tag{2.1077}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\left(\sum_{j_4=0}^{p}C_{j_4j_4j_3j_2j_1}-\frac{1}{2}C_{j_4j_4j_3j_2j_1}\Big|_{(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{2.1078}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_5j_3j_3j_1j_1}-\frac{1}{4}C_{j_5j_3j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot),(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{2.1079}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_4j_3j_1j_1}\right)^2=0, \tag{2.1080}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_3j_1j_1}-\frac{1}{4}C_{j_4j_4j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{2.1081}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_2j_1j_2j_1}\right)^2=0, \tag{2.1082}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_4j_1j_2j_1}\right)^2=0, \tag{2.1083}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_1j_2j_1}\right)^2=0, \tag{2.1084}$$

$$\lim_{p\to\infty}\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_5j_1j_2j_2j_1}\right)^2=0, \tag{2.1085}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_2j_1}\right)^2=0, \tag{2.1086}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_3j_2j_1}\right)^2=0, \tag{2.1087}$$

$$\lim_{p\to\infty}\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_4j_2j_2j_1}\right)^2=0, \tag{2.1088}$$

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_3j_2j_1}\right)^2=0, \tag{2.1089}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_2j_1}\right)^2=0, \tag{2.1090}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_4=0}^{p}C_{j_4j_4j_2j_2j_1}-\frac{1}{4}C_{j_4j_4j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=0, \tag{2.1091}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_2j_1}\right)^2=0, \tag{2.1092}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p}C_{j_2j_3j_3j_2j_1}\right)^2=0. \tag{2.1093}$$

Step 2. Let us prove the equalities (2.1069)–(2.1078). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (2.1069)–(2.1078)

$$\sum_{j_3,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_5j_4j_3j_1j_1}-\frac{1}{2}C_{j_5j_4j_3j_1j_1}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=$$

$$=\sum_{j_3,j_4,j_5=0}^{p}\left(\int_t^T\phi_{j_5}(t_5)\int_t^{t_5}\phi_{j_4}(t_4)\int_t^{t_4}\phi_{j_3}(t_3)\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)\times\right.$$

$$\left.\times dt_3dt_4dt_5\right)^2\le$$

$$\le\sum_{j_3,j_4,j_5=0}^{\infty}\left(\int_t^T\phi_{j_5}(t_5)\int_t^{t_5}\phi_{j_4}(t_4)\int_t^{t_4}\phi_{j_3}(t_3)\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)\times\right.$$

$$\left.\times dt_3dt_4dt_5\right)^2=$$

$$=\int_{[t,T]^3}\left(\mathbf{1}_{\{t_3<t_4<t_5\}}\right)^2\left(\sum_{j_1=0}^{p}\frac{1}{2}\left(\int_t^{t_3}\phi_{j_1}(\tau)d\tau\right)^2-\frac{t_3-t}{2}\right)^2dt_3dt_4dt_5, \tag{2.1094}$$

$$\sum_{j_2,j_4,j_5=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_5 j_4 j_1 j_2 j_1}\right)^2 =$$

$$= \sum_{j_2,j_4,j_5=0}^{p}\Biggl(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\times$$

$$\times dt_2 dt_4 dt_5\Biggr)^2 \le$$

$$\le \sum_{j_2,j_4,j_5=0}^{\infty}\Biggl(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\times$$

$$\times dt_2 dt_4 dt_5\Biggr)^2 =$$

$$= \int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_2<t_4<t_5\}}\right)^2\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\right)^2 dt_2 dt_4 dt_5, \qquad (2.1095)$$

$$\sum_{j_2,j_3,j_5=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_5 j_1 j_3 j_2 j_1}\right)^2 =$$

$$= \sum_{j_2,j_3,j_5=0}^{p}\Biggl(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4\times$$

$$\times dt_2 dt_3 dt_5\Biggr)^2 \le$$

$$\le \sum_{j_2,j_3,j_5=0}^{\infty}\Biggl(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4\times$$

$$\times dt_2 dt_3 dt_5\Biggr)^2 =$$

$$
=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_2<t_3<t_5\}}\right)^2\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4\right)^2 dt_2dt_3dt_5, \qquad (2.1096)
$$

$$
\sum_{j_2,j_3,j_4=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_4j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_2,j_3,j_4=0}^{p}\Biggl(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\times
$$

$$
\times dt_2dt_3dt_4\Biggr)^2\le
$$

$$
\le\sum_{j_2,j_3,j_4=0}^{\infty}\Biggl(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\times
$$

$$
\times dt_2dt_3dt_4\Biggr)^2=
$$

$$
=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_2<t_3<t_4\}}\right)^2\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_2dt_3dt_4, \qquad (2.1097)
$$

$$
\sum_{j_1,j_4,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_4j_2j_2j_1}-\frac{1}{2}C_{j_5j_4j_2j_2j_1}\Bigr|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=
$$

$$
=\sum_{j_1,j_4,j_5=0}^{p}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2\times
$$

$$
\times dt_1dt_4dt_5-\frac{1}{2}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\Biggr)^2=
$$

$$
=\sum_{j_1,j_4,j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)\times\right.
$$

$$
\left.\times dt_1 dt_4 dt_5\right)^2 \le
$$

$$
\le \sum_{j_1,j_4,j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)\times\right.
$$

$$
\left.\times dt_1 dt_4 dt_5\right)^2 =
$$

$$
=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_1<t_4<t_5\}}\right)^2\left(\sum_{j_2=0}^{p}\frac{1}{2}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1dt_4dt_5, \quad (2.1098)
$$

$$
\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_5j_2j_3j_2j_1}\right)^2 =
$$

$$
=\sum_{j_1,j_3,j_5=0}^{p}\left(\sum_{j_2=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_3 dt_5\right)^2 =
$$

$$
=\sum_{j_1,j_3,j_5=0}^{p}\left(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_1 dt_3 dt_5\right)^2 \le
$$

$$
\le\sum_{j_1,j_3,j_5=0}^{\infty}\left(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\sum_{j_2=0}^{p}\int\limits_{t_1}^{t_3}\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4\times\right.
$$

$$
\left.\times dt_1 dt_3 dt_5\right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_3<t_5\}}\right)^2 \left(\sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2)dt_2 \int\limits_{t_3}^{t_5} \phi_{j_2}(t_4)dt_4\right)^2 dt_1dt_3dt_5, \quad (2.1099)
$$

$$
\sum_{j_1,j_3,j_4=0}^{p} \left(\sum_{j_2=0}^{p} C_{j_2j_4j_3j_2j_1}\right)^2 =
$$

$$
= \sum_{j_1,j_3,j_4=0}^{p} \left(\sum_{j_2=0}^{p} \int\limits_{t}^{T} \phi_{j_4}(t_4) \int\limits_{t}^{t_4} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3 \int\limits_{t_4}^{T} \phi_{j_2}(t_5)dt_5 \times \right.
$$

$$
\left. \times dt_4\right)^2 =
$$

$$
= \sum_{j_1,j_3,j_4=0}^{p} \left(\int\limits_{t}^{T} \phi_{j_4}(t_4) \int\limits_{t}^{t_4} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2)dt_2 \int\limits_{t_4}^{T} \phi_{j_2}(t_5)dt_5 \times \right.
$$

$$
\left. \times dt_1dt_3dt_4\right)^2 \le
$$

$$
\le \sum_{j_1,j_3,j_4=0}^{\infty} \left(\int\limits_{t}^{T} \phi_{j_4}(t_4) \int\limits_{t}^{t_4} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2)dt_2 \int\limits_{t_4}^{T} \phi_{j_2}(t_5)dt_5 \times \right.
$$

$$
\left. \times dt_1dt_3dt_4\right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_3<t_4\}}\right)^2 \left(\sum_{j_2=0}^{p} \int\limits_{t_1}^{t_3} \phi_{j_2}(t_2)dt_2 \int\limits_{t_4}^{T} \phi_{j_2}(t_5)dt_5\right)^2 dt_1dt_3dt_4, \quad (2.1100)
$$

$$
\sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_5j_3j_3j_2j_1} - \frac{1}{2} C_{j_5j_3j_3j_2j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2 =
$$

$$
= \sum_{j_1,j_2,j_5=0}^{p} \left(\sum_{j_3=0}^{p} \int\limits_{t}^{T} \phi_{j_5}(t_5) \int\limits_{t}^{t_5} \phi_{j_1}(t_1) \int\limits_{t_1}^{t_5} \phi_{j_2}(t_2) \int\limits_{t_2}^{t_5} \phi_{j_3}(t_3) \int\limits_{t_3}^{t_5} \phi_{j_3}(t_4)dt_4dt_3 \times \right.
$$

$$\times dt_2dt_1dt_5-\frac{1}{2}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_5\Biggr)^2=$$

$$=\sum_{j_1,j_2,j_5=0}^{p}\Biggl(\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3\times$$

$$\times dt_2dt_1dt_5-\frac{1}{2}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_5}dt_3dt_2dt_1dt_5\Biggr)^2=$$

$$=\sum_{j_1,j_2,j_5=0}^{p}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\left(\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2-\frac{t_5-t_2}{2}\right)\times$$

$$\times dt_2dt_1dt_5\Biggr)^2\le$$

$$\le\sum_{j_1,j_2,j_5=0}^{\infty}\Biggl(\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\left(\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2-\frac{t_5-t_2}{2}\right)\times$$

$$\times dt_2dt_1dt_5\Biggr)^2=$$

$$=\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{t_1<t_2<t_5\}}\right)^2\left(\sum_{j_3=0}^{p}\frac{1}{2}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2-\frac{t_5-t_2}{2}\right)^2dt_2dt_1dt_5,\quad(2.1101)$$

$$\sum_{j_1,j_2,j_4=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_4j_3j_2j_1}\right)^2=$$

$$=\sum_{j_1,j_2,j_4=0}^{p}\Biggl(\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3\times$$

$$\times dt_2dt_1\Biggr)^2=$$

$$
= \sum_{j_1,j_2,j_4=0}^{p} \left(\int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^T \phi_{j_2}(t_2) \int\limits_{t_2}^T \phi_{j_4}(t_4) \sum_{j_3=0}^{p} \int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \int\limits_{t_2}^{t_4} \phi_{j_3}(t_3) dt_3 dt_4 \times \right.
$$

$$
\left. \times dt_2 dt_1 \right)^2 \le
$$

$$
\le \sum_{j_1,j_2,j_4=0}^{\infty} \left(\int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^T \phi_{j_2}(t_2) \int\limits_{t_2}^T \phi_{j_4}(t_4) \sum_{j_3=0}^{p} \int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \int\limits_{t_2}^{t_4} \phi_{j_3}(t_3) dt_3 dt_4 \times \right.
$$

$$
\left. \times dt_2 dt_1 \right)^2 =
$$

$$
= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_2<t_4\}} \right)^2 \left(\sum_{j_3=0}^{p} \int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \int\limits_{t_2}^{t_4} \phi_{j_3}(t_3) dt_3 \right)^2 dt_4 dt_2 dt_1, \qquad (2.1102)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} \left(\sum_{j_4=0}^{p} C_{j_4 j_4 j_3 j_2 j_1} - \frac{1}{2} C_{j_4 j_4 j_3 j_2 j_1} \Bigr|_{(j_4 j_4) \curvearrowright (\cdot)} \right)^2 =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} \left(\int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \sum_{j_4=0}^{p} \int\limits_{t_3}^T \phi_{j_4}(t_4) \int\limits_{t_4}^T \phi_{j_4}(t_5) dt_5 dt_4 \times \right.
$$

$$
\left. \times dt_3 - \frac{1}{2} \int\limits_t^T \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} \left(\int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) \left(\sum_{j_4=0}^{p} \frac{1}{2} \left(\int\limits_{t_3}^T \phi_{j_4}(t_4) dt_4 \right)^2 - \frac{T-t_3}{2} \right) \times \right.
$$

$$
\left. \times dt_1 dt_2 dt_3 \right)^2 \le
$$

$$
\le \sum_{j_1,j_2,j_3=0}^{\infty} \left(\int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) \left(\sum_{j_4=0}^{p} \frac{1}{2} \left(\int\limits_{t_3}^T \phi_{j_4}(t_4) dt_4 \right)^2 - \frac{T-t_3}{2} \right) \times \right.
$$

$$\times dt_1 dt_2 dt_3\Biggr)^2 =$$

$$= \int\limits_{[t,T]^3} \left(\mathbf{1}_{\{t_1<t_2<t_3\}}\right)^2 \left(\sum_{j_4=0}^{p}\frac{1}{2}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2 - \frac{T-t_3}{2}\right)^2 dt_1 dt_2 dt_3. \quad (2.1103)$$

Further, applying the Parseval equality and the generalized Parseval equality as well as using the Cauchy–Bunyakovsky inequality, we have (see the proof of Theorem 2.45)

$$\sum_{j=0}^{\infty}\left(\int\limits_{t_1}^{t_2}\phi_j(s)ds\right)^2 = \int\limits_{t}^{T}\left(\mathbf{1}_{\{t_1<s<t_2\}}\right)^2 ds = t_2 - t_1, \quad (2.1104)$$

$$\sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\phi_j(s)ds\int\limits_{t_3}^{t_4}\phi_j(s)ds = \sum_{j=0}^{\infty}\int\limits_{t}^{T}\mathbf{1}_{\{t_1<s<t_2\}}\phi_j(s)ds\int\limits_{t}^{T}\mathbf{1}_{\{t_3<s<t_4\}}\phi_j(s)ds =$$

$$= \int\limits_{t}^{T}\mathbf{1}_{\{t_1<s<t_2\}}\mathbf{1}_{\{t_3<s<t_4\}}ds = 0, \quad (2.1105)$$

$$\left|(t_2-t_1) - \sum_{j=0}^{p}\left(\int\limits_{t_1}^{t_2}\phi_j(s)ds\right)^2\right| \le t_2 - t_1 \le T - t < \infty, \quad (2.1106)$$

$$\left(\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}\phi_j(s)ds\int\limits_{t_3}^{t_4}\phi_j(s)ds\right)^2 \le \sum_{j=0}^{p}\left(\int\limits_{t_1}^{t_2}\phi_j(s)ds\right)^2\sum_{j=0}^{p}\left(\int\limits_{t_3}^{t_4}\phi_j(s)ds\right)^2 \le$$

$$\le (t_2-t_1)(t_4-t_3) \le (T-t)^2 < \infty, \quad (2.1107)$$

where $t \le t_1 < t_2 \le t_3 < t_4 \le T$.

Using Lebesgue's Dominated Convergence Theorem and (2.1104)–(2.1107), we obtain that the right-hand sides of (2.1094)–(2.1103) tend to zero when $p \to \infty$. The equalities (2.1069)–(2.1078) are proved.

Step 3. Before proving the equalities (2.1079)–(2.1093), we show that

$$\left|\sum_{j_1,j_3=0}^{p} C_{j_3 j_3 j_1 j_1}(s,\tau)\right| \le K, \quad (2.1108)$$

$$\left|\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau)\right| \le K, \tag{2.1109}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau)\right| \le K, \tag{2.1110}$$

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(s,\tau)\right)^2 \le \int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{s}\phi_{j_1}(t_3)dt_3\right)^2 dt_2, \tag{2.1111}$$

where constant K does not depend on p, s, τ; here and further in this proof

$$C_{j_k\ldots j_1}(s,\tau)=\int\limits_{\tau}^{s}\phi_{j_k}(t_k)\ldots\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k \quad (k=1,\ldots,4,\ t\le\tau<s\le T).$$

Further, by K, K_1, K_2 we will denote contants that can change from line to line.

By analogy with (2.1032), (2.1042), (2.1047) and (2.1040), (2.1046), (2.1055) we get

$$\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau)=\sum_{j_1,j_3=0}^{p} C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau)-\frac{1}{8}\left(\sum_{j_1=0}^{p}\left(C_{j_1}(s,\tau)\right)^2\right)^2, \tag{2.1112}$$

$$\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau)=\sum_{j_1,j_2=0}^{p} C_{j_2}(s,\tau)C_{j_1j_2j_1}(s,\tau)-\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_1j_2}(s,\tau)C_{j_2j_1}(s,\tau), \tag{2.1113}$$

$$\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau)=\sum_{j_1,j_3=0}^{p} C_{j_1}(s,\tau)C_{j_3j_3j_1}(s,\tau)-\frac{1}{2}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}(s,\tau)\right)^2, \tag{2.1114}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau)=\frac{1}{8}(s-\tau)^2, \tag{2.1115}$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau)=0, \tag{2.1116}$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau)=0. \tag{2.1117}$$

Using (2.1112), Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2+2b^2$, we obtain

$$\left(\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau)\right)^2 \le 2\left(\sum_{j_1,j_3=0}^{p} C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau)\right)^2 +$$

$$+2\cdot\frac{1}{64}\left(\sum_{j_1=0}^{p}\bigl(C_{j_1}(s,\tau)\bigr)^2\right)^4 \le$$

$$\le 2\sum_{j_3=0}^{p}(C_{j_3}(s,\tau))^2\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}(s,\tau)\right)^2 + K_1 \le$$

$$\le K_2\sum_{j_3=0}^{\infty}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}(s,\tau)\right)^2 + K_1 =$$

$$= K_2\sum_{j_3=0}^{\infty}\left(\int_{\tau}^{s}\phi_{j_3}(t_3)\sum_{j_1=0}^{p}\int_{\tau}^{t_3}\phi_{j_1}(t_2)\int_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 + K_1 =$$

$$= K_2\int_{\tau}^{s}\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int_{\tau}^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\right)^2 dt_3 + K_1 \le$$

$$\le K_2\int_{\tau}^{s}\left(\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int_{\tau}^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\right)^2 dt_3 + K_1 =$$

$$= K_2\int_{\tau}^{s}\left(\frac{1}{2}(t_3-\tau)\right)^2 dt_3 + K_1 \le K < \infty,$$

where constants K, K_1, K_2 do not depend on p, s, τ. The equality (2.1108) is proved.

Let us prove (2.1109). Using (2.1114) and the above reasoning, we get

$$\left(\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau)\right)^2 \le 2\left(\sum_{j_1,j_3=0}^{p} C_{j_1}(s,\tau)C_{j_3j_3j_1}(s,\tau)\right)^2 +$$

$$+2\cdot\frac{1}{4}\left(\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1}(s,\tau)\right)^2\right)^2\le$$

$$\le 2\sum_{j_1=0}^{p}\left(C_{j_1}(s,\tau)\right)^2\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s,\tau)\right)^2+K_1\le$$

$$\le K_2\sum_{j_1=0}^{\infty}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s,\tau)\right)^2+K_1=$$

$$=K_2\sum_{j_1=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_1}(t_1)\sum_{j_3=0}^{p}\int\limits_{t_1}^{s}\phi_{j_3}(t_2)\int\limits_{t_2}^{s}\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2+K_1=$$

$$=K_2\int\limits_{\tau}^{s}\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_1}^{s}\phi_{j_3}(t_2)dt_2\right)^2\right)^2dt_1+K_1\le$$

$$\le K_2\int\limits_{\tau}^{s}\left(\frac{1}{2}\sum_{j_3=0}^{\infty}\left(\int\limits_{t_1}^{s}\phi_{j_3}(t_2)dt_2\right)^2\right)^2dt_1+K_1=$$

$$=K_2\int\limits_{\tau}^{s}\left(\frac{1}{2}(s-t_1)\right)^2dt_1+K_1\le K<\infty,$$

where constants K, K_1, K_2 do not depend on p, s, τ. The equality (2.1109) is proved.

Let us prove (2.1110), (2.1111). Applying (2.1113), (2.1107) and the above reasoning, we have

$$\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(s,\tau)\right)^2\le 2\left(\sum_{j_1,j_2=0}^{p}C_{j_2}(s,\tau)C_{j_1j_2j_1}(s,\tau)\right)^2+$$

$$+2\cdot\frac{1}{4}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2}(s,\tau)C_{j_2j_1}(s,\tau)\right)^2\le$$

$$\le 2\sum_{j_2=0}^{p}\left(C_{j_2}(s,\tau)\right)^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+$$

$$
+\frac{1}{2}\sum_{j_1,j_2=0}^{p}\left(C_{j_1j_2}(s,\tau)\right)^2\sum_{j_1,j_2=0}^{p}\left(C_{j_2j_1}(s,\tau)\right)^2\le
$$

$$
\le K_2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+K_1\le K_2\sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}(s,\tau)\right)^2+K_1= \tag{2.1118}
$$

$$
=K_2\sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_2}(t_2)\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{s}\phi_{j_1}(t_3)dt_3dt_2\right)^2+K_1=
$$

$$
=K_2\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{s}\phi_{j_1}(t_3)dt_3\right)^2dt_2+K_1\le \tag{2.1119}
$$

$$
\le K_2\int\limits_{\tau}^{s}\left((t_2-\tau)(s-t_2)\right)^2dt_2+K_1\le K<\infty,
$$

where constants K,K_1,K_2 do not depend on p,s,τ. The equalities (2.1110) and (2.1111) (see (2.1118), (2.1119)) are proved.

Step 4. Let us start proving the equalities (2.1079)–(2.1093). Using Fubini's Theorem and Parseval's equality, we obtain the following relations for the prelimit expressions on the left-hand sides of (2.1079), (2.1082), (2.1085), (2.1091)–(2.1093)

$$
\sum_{j_5=0}^{p}\left(\sum_{j_1,j_3=0}^{p}C_{j_5j_3j_3j_1j_1}-\frac{1}{4}C_{j_5j_3j_3j_1j_1}\Bigr|_{(j_1j_1)\curvearrowright(\cdot),(j_3j_3)\curvearrowright(\cdot)}\right)^2=
$$

$$
=\sum_{j_5=0}^{p}\left(\int\limits_{t}^{T}\phi_{j_5}(t_5)\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{4}\int\limits_{t}^{t_5}(\tau-t)d\tau\right)dt_5\right)^2\le
$$

$$
\le\sum_{j_5=0}^{\infty}\left(\int\limits_{t}^{T}\phi_{j_5}(t_5)\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{4}\int\limits_{t}^{t_5}(\tau-t)d\tau\right)dt_5\right)^2=
$$

$$
=\int\limits_{t}^{T}\left(\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}(t_5,t)-\frac{1}{8}(t_5-t)^2\right)^2dt_5, \tag{2.1120}
$$

$$
\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_5j_2j_1j_2j_1}\right)^2=\sum_{j_5=0}^{p}\left(\int\limits_t^T \phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(t_5,t)dt_5\right)^2\le
$$

$$
\le\sum_{j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(t_5,t)dt_5\right)^2=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(t_5,t)\right)^2 dt_5, \tag{2.1121}
$$

$$
\sum_{j_5=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_5j_1j_2j_2j_1}\right)^2=\sum_{j_5=0}^{p}\left(\int\limits_t^T \phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,t)dt_5\right)^2\le
$$

$$
\le\sum_{j_5=0}^{\infty}\left(\int\limits_t^T \phi_{j_5}(t_5)\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,t)dt_5\right)^2=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,t)\right)^2 dt_5, \tag{2.1122}
$$

$$
\sum_{j_1=0}^{p}\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2j_1}-\frac{1}{4}C_{j_4j_4j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=
$$

$$
=\sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_4=0}^{p}\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_4}(t_4)\int\limits_{t_4}^T \phi_{j_4}(t_5)dt_5dt_4dt_3dt_2dt_1-\right.
$$

$$
\left.-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_5\right)^2=
$$

$$
=\sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(t_1)\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{4}\int\limits_{t_1}^T (T-t_3)dt_3\right)dt_1\right)^2\le
$$

$$
\le\sum_{j_1=0}^{\infty}\left(\int\limits_t^T \phi_{j_1}(t_1)\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{8}(T-t_1)^2\right)dt_1\right)^2=
$$

$$
=\int\limits_t^T\left(\sum_{j_2,j_4=0}^{p} C_{j_4j_4j_2j_2}(T,t_1)-\frac{1}{8}(T-t_1)^2\right)^2 dt_1, \tag{2.1123}
$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1}\right)^2 =$$

$$= \sum_{j_1=0}^{p}\Biggl(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_2}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5dt_4\times$$

$$\times dt_3dt_2dt_1\Biggr)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_2}(T,t_1)dt_1\right)^2 \le$$

$$\le \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_2}(T,t_1)dt_1\right)^2 = \int\limits_t^T \left(\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_2}(T,t_1)\right)^2 dt_1, \tag{2.1124}$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1}\right)^2 =$$

$$= \sum_{j_1=0}^{p}\Biggl(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p}\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_3}(t_4)\int\limits_{t_4}^T \phi_{j_2}(t_5)dt_5dt_4\times$$

$$\times dt_3dt_2dt_1\Biggr)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)dt_1\right)^2 \le$$

$$\le \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \phi_{j_1}(t_1)\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)dt_1\right)^2 = \int\limits_t^T \left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3j_3j_2}(T,t_1)\right)^2 dt_1. \tag{2.1125}$$

Using Lebesgue's Dominated Convergence Theorem and (2.1108)–(2.1110), (2.1115)–(2.1117), we obtain that the right-hand sides of (2.1120)–(2.1125) tend to zero when $p \to \infty$. The equalities (2.1079), (2.1082), (2.1085), (2.1091)–(2.1093) are proved.

Further, let us prove the equalities (2.1081), (2.1083), (2.1086), (2.1087), (2.1089).

Using Fubini's Theorem, Parseval's equality and Cauchy–Bunyakovsky's inequality, we have the following relations for the prelimit expressions on the left-hand sides of (2.1081), (2.1083), (2.1086), (2.1087), (2.1089)

$$
\sum_{j_3=0}^{p}\left(\sum_{j_1,j_4=0}^{p} C_{j_4j_4j_3j_1j_1}-\frac{1}{4}C_{j_4j_4j_3j_1j_1}\Bigg|_{(j_1j_1)\curvearrowright(\cdot),(j_4j_4)\curvearrowright(\cdot)}\right)^2=
$$

$$
=\sum_{j_3=0}^{p}\left(\int\limits_t^T\phi_{j_3}(t_3)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_4}(t_4)\int\limits_{t_4}^{T}\phi_{j_4}(t_5)dt_5dt_4dt_3-\right.
$$

$$
\left.-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}dt_1dt_3dt_4\right)^2\le
$$

$$
\le\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\phi_{j_3}(t_3)\left(\sum_{j_1,j_4=0}^{p}\frac{1}{4}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\right.\right.
$$

$$
\left.\left.-\frac{1}{4}(t_3-t)\int\limits_{t_3}^{T}dt_4\right)dt_3\right)^2=
$$

$$
=\int\limits_t^T\left(\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{1}{4}(t_3-t)(T-t_3)\right)^2dt_3,
\tag{2.1126}
$$

$$
\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_4j_1j_2j_1}\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\left(\int\limits_t^T\phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_2}(t_5)dt_5dt_4\right)^2\le
$$

$$\le \sum_{j_4=0}^{\infty}\left(\int\limits_t^T \phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_4)dt_4\right)^2 =$$

$$= \int\limits_t^T \left(\sum_{j_2=0}^{p}\sum_{j_1=0}^{p} C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_4)\right)^2 dt_4 \le$$

$$\le \int\limits_t^T \sum_{j_2=0}^{p}(C_{j_2}(T,t_4))^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(t_4,t)\right)^2 dt_4 \le$$

$$\le \int\limits_t^T \sum_{j_2=0}^{\infty}(C_{j_2}(T,t_4))^2\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(t_4,t)\right)^2 dt_4 \le$$

$$\le K_1\int\limits_t^T \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(t_4,t)\right)^2 dt_4 \le \tag{2.1127}$$

$$\le K_1\int\limits_t^T\int\limits_t^{t_4}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\right)^2 dt_2dt_4 = \tag{2.1128}$$

$$= K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3\right)^2 dt_2dt_4, \tag{2.1129}$$

where constant K_1 does not depend on p and the transition from (2.1127) to (2.1128) is based on (2.1111);

$$\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_2j_1}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\int\limits_t^T\phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_1}(t_4)\int\limits_{t_4}^{T}\phi_{j_2}(t_5)dt_5dt_4dt_3\right)^2 \le$$

$$\le \sum_{j_3=0}^{\infty}\left(\int\limits_t^T\phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)dt_2dt_1dt_3\right)^2 =$$

$$
=\int\limits_t^T \left(\sum_{j_1,j_2=0}^{p} \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2 \int\limits_{t_3}^{T} \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2)dt_2dt_1\right)^2 dt_3 =
$$

$$
=\int\limits_t^T \left(\sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\times\right.
$$

$$
\left.\times \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2 dt_3, \qquad (2.1130)
$$

where, using the generalized Parseval equality and the Cauchy–Bunyakovsky inequality, we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_2>t_1>t_3\}}dt_1dt_2 = 0,
$$

$$
\left(\sum_{j_1,j_2=0}^{p} \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2 \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2 \le
$$

$$
\le \sum_{j_1,j_2=0}^{p} \left(\int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2 \times
$$

$$
\times \sum_{j_1,j_2=0}^{p} \left(\int\limits_{[t,T]^2} \mathbf{1}_{\{t_2>t_1>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2 \le K_1 < \infty,
$$

where constant K_1 does not depend on p;

$$
\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_3j_1j_3j_2j_1}\right)^2 =
$$

$$
=\sum_{j_2=0}^{p}\left(\int\limits_t^T\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3dt_2\right)^2\le
$$

$$
\le\sum_{j_2=0}^{\infty}\left(\int\limits_t^T\phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3dt_2\right)^2=
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5dt_4dt_3\right)^2dt_2=
$$

$$
=\int\limits_t^T\left(\sum_{j_1=0}^{p}C_{j_1}(t_2,t)\sum_{j_3=0}^{p}\int\limits_{t_2}^T\phi_{j_3}(t_5)\int\limits_{t_2}^{t_5}\phi_{j_1}(t_4)\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4dt_5\right)^2dt_2=
$$

$$
=\int\limits_t^T\left(\sum_{j_1=0}^{p}C_{j_1}(t_2,t)\sum_{j_3=0}^{p}C_{j_3j_1j_3}(T,t_2)\right)^2dt_2\le
$$

$$
\le\int\limits_t^T\sum_{j_1=0}^{p}\left(C_{j_1}(t_2,t)\right)^2\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_1j_3}(T,t_2)\right)^2dt_2\le
$$

$$
\le K_1\int\limits_t^T\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_1j_3}(T,t_2)\right)^2dt_2\le \tag{2.1131}
$$

$$
\le K_1\int\limits_t^T\int\limits_{t_2}^T\left(\sum_{j_3=0}^{p}\int\limits_{t_2}^{\theta}\phi_{j_3}(t_1)dt_1\int\limits_{\theta}^T\phi_{j_3}(t_3)dt_3\right)^2d\theta dt_2= \tag{2.1132}
$$

$$
=K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<\theta\}}\left(\sum_{j_3=0}^{p}\int\limits_{t_2}^{\theta}\phi_{j_3}(t_1)dt_1\int\limits_{\theta}^T\phi_{j_3}(t_3)dt_3\right)^2d\theta dt_2, \tag{2.1133}
$$

where constant K_1 does not depend on p and the transition from (2.1131) to (2.1132) is based on (2.1111);

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_3j_2j_1}\right)^2=
$$

$$
=\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_4)\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4dt_3\right)^2\le
$$

$$
\le\sum_{j_3=0}^{\infty}\left(\int\limits_t^T \phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2=
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3=
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\times\right.
$$

$$
\left.\times\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3, \qquad (2.1134)
$$

where, using the generalized Parseval equality and the Cauchy–Bunyakovsky inequality, we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_1>t_2>t_3\}}dt_1dt_2=0,
$$

$$
\left(\sum_{j_1,j_2=0}^{p}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le
$$

$$
\le\sum_{j_1,j_2=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\times
$$

$$
\times\sum_{j_1,j_2=0}^{p}\left(\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1>t_2>t_3\}}\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2\right)^2\le K_1<\infty,
$$

where constant K_1 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem, we obtain that the right-hand sides of (2.1126), (2.1129), (2.1130), (2.1133), (2.1134) tend to zero when $p \to \infty$. The equalities (2.1081), (2.1083), (2.1086), (2.1087), (2.1089) are proved.

Step 5. Finally, let us prove the equalities (2.1080), (2.1084), (2.1088), (2.1090).

Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain for the prelimit expression on the left-hand side of (2.1080)

$$\sum_{j_4=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_3 j_4 j_3 j_1 j_1}\right)^2 =$$

$$= \sum_{j_4=0}^{p}\left(\int_t^T \phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2 \le$$

$$\le \sum_{j_4=0}^{\infty}\left(\int_t^T \phi_{j_4}(t_4)\sum_{j_1,j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5dt_4\right)^2 =$$

$$= \int_t^T\left(\sum_{j_1,j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4 =$$

$$= \int_t^T\left(\sum_{j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2 \mp \frac{t_3-t}{2}\right)dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4 \le$$

$$\le 2\int_t^T\left(\sum_{j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2 - \frac{t_3-t}{2}\right)dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4 +$$

$$+2\int_t^T\left(\sum_{j_3=0}^{p}\int_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4 \le$$

$$
\le 2\int\limits_t^T \sum_{j_3=0}^{p} \left(C_{j_3}(T,t_4)\right)^2 \times
$$

$$
\times \sum_{j_3=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2-\frac{t_3-t}{2}\right)dt_3\right)^2 dt_4+\varepsilon_p \le
$$

$$
\le K_1\int\limits_t^T \sum_{j_3=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2-\frac{t_3-t}{2}\right)dt_3\right)^2 dt_4+\varepsilon_p \le
$$

$$
\le K_1\int\limits_t^T \sum_{j_3=0}^{\infty}\left(\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2-\frac{t_3-t}{2}\right)dt_3\right)^2 dt_4+\varepsilon_p =
$$

$$
= K_1\int\limits_t^T \int\limits_t^{t_4}\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2-\frac{t_3-t}{2}\right)^2 dt_3dt_4+\varepsilon_p =
$$

$$
= K_1\int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}}\left(\frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_2)dt_2\right)^2-\frac{t_3-t}{2}\right)^2 dt_3dt_4+\varepsilon_p, \quad (2.1135)
$$

where constant K_1 does not depend on p,

$$
\varepsilon_p = 2\int\limits_t^T\left(\sum_{j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4.
$$

By analogy with (2.1105), (2.1107) we get

$$
\left(\sum_{j_3=0}^{p}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 \le K_2<\infty, \quad (2.1136)
$$

$$
\sum_{j_3=0}^{\infty}\int\limits_t^{t_4}\phi_{j_3}(t_3)\frac{t_3-t}{2}dt_3\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5 = 0, \quad (2.1137)
$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (2.1104), (2.1106), (2.1136), (2.1137), we obtain that the right-hand side of (2.1135) tends to zero when $p \to \infty$. The equality (2.1080) is proved.

Let us prove the equality (2.1084). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain for the prelimit expression on the left-hand side of (2.1084)

$$
\sum_{j_2=0}^{p}\left(\sum_{j_1,j_4=0}^{p} C_{j_4 j_4 j_1 j_2 j_1}\right)^2 =
$$

$$
= \sum_{j_2=0}^{p}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_4}(t_5)dt_5dt_4dt_3dt_2\right)^2 \le
$$

$$
\le \sum_{j_2=0}^{\infty}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_4}(t_5)dt_5dt_4dt_3dt_2\right)^2 =
$$

$$
= \int\limits_t^T\left(\sum_{j_1,j_4=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_4}(t_5)dt_5dt_4dt_3\right)^2 dt_2 =
$$

$$
= \int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^T\phi_{j_4}(t_4)dt_4\right)^2 \mp \frac{T-t_3}{2}\right)dt_3\right)^2 dt_2 \le
$$

$$
\le 2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^T\phi_{j_4}(t_4)dt_4\right)^2 - \frac{T-t_3}{2}\right)dt_3\right)^2 dt_2 +
$$

$$
+2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3\right)^2 dt_2 \le
$$

$$
\le 2\int\limits_t^T\sum_{j_1=0}^{p}\left(C_{j_1}(t_2,t)\right)^2 \times
$$

$$
\times\sum_{j_1=0}^{p}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{T-t_3}{2}\right)dt_3\right)^2 dt_2+\mu_p\le
$$

$$
\le K_1\int\limits_{t}^{T}\sum_{j_1=0}^{p}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{T-t_3}{2}\right)dt_3\right)^2 dt_2+\mu_p\le
$$

$$
\le K_1\int\limits_{t}^{T}\sum_{j_1=0}^{\infty}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{T-t_3}{2}\right)dt_3\right)^2 dt_2+\mu_p=
$$

$$
=K_1\int\limits_{t}^{T}\int\limits_{t_2}^{T}\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{T-t_3}{2}\right)^2 dt_3dt_2+\mu_p=
$$

$$
=K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\left(\frac{1}{2}\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2-\frac{T-t_3}{2}\right)^2 dt_3dt_2+\mu_p, \quad (2.1138)
$$

where constant K_1 does not depend on p,

$$
\mu_p=2\int\limits_{t}^{T}\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3\right)^2 dt_2.
$$

By analogy with (2.1105), (2.1107) we get

$$
\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3\right)^2\le K_2<\infty, \qquad (2.1139)
$$

$$
\sum_{j_1=0}^{\infty}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\frac{T-t_3}{2}dt_3=0, \qquad (2.1140)
$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (2.1104), (2.1106), (2.1139), (2.1140), we obtain that the right-hand side of (2.1138) tends to zero when $p\to\infty$. The equality (2.1084) is proved.

Let us prove the equality (2.1088). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (2.1088)

$$
\sum_{j_4=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_4j_2j_2j_1}\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\left(\int\limits_t^T \phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4\right)^2\le
$$

$$
\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T \phi_{j_4}(t_4)\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5dt_4\right)^2=
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4=
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_2=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4=
$$

$$
=\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2\mp\frac{t_4-t_1}{2}\right)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4\le
$$

$$
\le 2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4+
$$

$$
+2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2dt_4\le
$$

$$
\le 2\int\limits_t^T\sum_{j_1=0}^{p}\left(C_{j_1}(T,t_4)\right)^2\times
$$

$$\times\sum_{j_1=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p\le$$

$$\le K_1\int\limits_t^T\sum_{j_1=0}^{p}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p\le$$

$$\le K_1\int\limits_t^T\sum_{j_1=0}^{\infty}\left(\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)dt_1\right)^2 dt_4+\rho_p=$$

$$=K_1\int\limits_t^T\int\limits_t^{t_4}\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1dt_4+\rho_p=$$

$$=K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_4\}}\left(\frac{1}{2}\sum_{j_2=0}^{p}\left(\int\limits_{t_1}^{t_4}\phi_{j_2}(t_2)dt_2\right)^2-\frac{t_4-t_1}{2}\right)^2 dt_1dt_4+\rho_p, \quad (2.1141)$$

where constant K_1 does not depend on p,

$$\rho_p=2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2 dt_4.$$

By analogy with (2.1105), (2.1107) we get $(t_4-t_1=(t_4-t)+(t-t_1))$

$$\left(\sum_{j_1=0}^{p}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5\right)^2\le K_2<\infty, \qquad (2.1142)$$

$$\sum_{j_1=0}^{\infty}\int\limits_t^{t_4}\phi_{j_1}(t_1)\frac{t_4-t_1}{2}dt_1\int\limits_{t_4}^{T}\phi_{j_1}(t_5)dt_5=0, \qquad (2.1143)$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (2.1104), (2.1106), (2.1142), (2.1143), we obtain that the right-hand side of (2.1141) tends to zero when $p\to\infty$. The equality (2.1088) is proved.

Let us prove the equality (2.1090). Using Parseval's equality, Cauchy–Bunyakovsky's inequality, as well as Fubini's Theorem and the elementary inequality $(a+b)^2 \le 2a^2+2b^2$, we obtain for the prelimit expression on the left-hand side of (2.1090)

$$
\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_2j_1}\right)^2 =
$$

$$
=\sum_{j_2=0}^{p}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_1}(t_5)dt_5dt_4dt_3dt_2\right)^2 \le
$$

$$
\le\sum_{j_2=0}^{\infty}\left(\int\limits_t^T \phi_{j_2}(t_2)\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_1}(t_5)dt_5dt_4dt_3dt_2\right)^2 =
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_1}(t_5)dt_5dt_4dt_3\right)^2 dt_2 =
$$

$$
=\int\limits_t^T\left(\sum_{j_1,j_3=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)\int\limits_{t_2}^{t_4}\phi_{j_3}(t_3)dt_3dt_4dt_5\right)^2 dt_2 =
$$

$$
=\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2\mp\frac{t_5-t_2}{2}\right)dt_5\right)^2 dt_2 \le
$$

$$
\le 2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2 dt_2+
$$

$$
+2\int\limits_t^T\left(\sum_{j_1=0}^{p}\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2 dt_2 \le
$$

$$
\le 2\int\limits_t^T\sum_{j_1=0}^{p}\left(C_{j_1}(t_2,t)\right)^2\times
$$

$$
\times \sum_{j_1=0}^{p}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2 dt_2+\chi_p \le
$$

$$
\le K_1\int\limits_{t}^{T}\sum_{j_1=0}^{p}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2 dt_2+\chi_p \le
$$

$$
\le K_1\int\limits_{t}^{T}\sum_{j_1=0}^{\infty}\left(\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)dt_5\right)^2 dt_2+\chi_p =
$$

$$
= K_1\int\limits_{t}^{T}\int\limits_{t_2}^{T}\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)^2 dt_5dt_2+\chi_p =
$$

$$
= K_1\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_5\}}\left(\frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_4)dt_4\right)^2-\frac{t_5-t_2}{2}\right)^2 dt_5dt_2+\chi_p, \quad (2.1144)
$$

where constant K_1 does not depend on p,

$$
\chi_p = 2\int\limits_{t}^{T}\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2 dt_2.
$$

By analogy with (2.1105), (2.1107) we get $(t_5-t_2=(t_5-t)+(t-t_2))$

$$
\left(\sum_{j_1=0}^{p}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5\right)^2 \le K_2<\infty, \quad (2.1145)
$$

$$
\sum_{j_1=0}^{\infty}\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{T}\phi_{j_1}(t_5)\frac{t_5-t_2}{2}dt_5=0, \quad (2.1146)
$$

where constant K_2 does not depend on p.

Using Lebesgue's Dominated Convergence Theorem and (2.1104), (2.1106), (2.1145), (2.1146), we obtain that the right-hand side of (2.1144) tends to zero when $p\to\infty$. The equality (2.1090) is proved. The equalities (2.1069)–(2.1093) are proved. Theorem 2.50 is proved.

2.24 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and Binomial Weight Functions

In this section, we will consider a generalization of Theorems 2.45, 2.47. Namely, we will prove the following theorem.

Theorem 2.51 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)}=\int\limits_t^{*T}(t_3-t)^{l_3}\int\limits_t^{*t_3}(t_2-t)^{l_2}\int\limits_t^{*t_2}(t_1-t)^{l_1}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \qquad (2.1147)$$

the following expansion

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \qquad (2.1148)$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m$; $l_1,l_2,l_3=0,1,2,\ldots,$

$$C_{j_3 j_2 j_1}=\int\limits_t^{T}(t_3-t)^{l_3}\phi_{j_3}(t_3)\int\limits_t^{t_3}(t_2-t)^{l_2}\phi_{j_2}(t_2)\int\limits_t^{t_2}(t_1-t)^{l_1}\phi_{j_1}(t_1)dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$.

Note that the iterated Stratonovich stochastic integrals (2.1147) are important for applications (see Chapter 4).

Proof. According to Theorems 2.49, 2.12, we come to the conclusion that Theorem 2.51 will be proved if we prove the following equalities

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3 j_1 j_1}\Biggl|_{(j_1 j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3 j_1 j_1}\right)^2=0, \qquad (2.1149)$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}\right)^2=0, \tag{2.1150}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=0. \tag{2.1151}$$

First, we prove that

$$\left|\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t_1}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right|\le K<\infty, \tag{2.1152}$$

where $l,m=0,1,2,\ldots,$ $t\le t_1<t_2\le T$, constant K does not depend on p,t_1,t_2.

Using Fubini's Theorem and Parseval's equality, we have for $m>l$ $(l,m=0,1,2,\ldots)$

$$\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^l(\tau-t)^{m-l}\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^l\phi_j(\tau)\int\limits_{t}^{\tau}(\theta-t)^{m-l-1}(m-l)d\theta d\tau ds=$$

$$=(m-l)\sum_{j=0}^{p}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\int\limits_{\theta}^{t_2}(\tau-t)^l\phi_j(\tau)\int\limits_{\tau}^{t_2}(s-t)^l\phi_j(s)dsd\tau d\theta=$$

$$=(m-l)\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\frac{1}{2}\sum_{j=0}^{p}\left(\int\limits_{\theta}^{t_2}(\tau-t)^l\phi_j(\tau)d\tau\right)^2d\theta\le$$

$$\le\frac{m-l}{2}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\sum_{j=0}^{\infty}\left(\int\limits_{\theta}^{t_2}(\tau-t)^l\phi_j(\tau)d\tau\right)^2d\theta=$$

$$=\frac{m-l}{2}\int\limits_{t}^{t_2}(\theta-t)^{m-l-1}\int\limits_{\theta}^{t_2}(\tau-t)^{2l}d\tau d\theta\le K_1<\infty, \tag{2.1153}$$

where constant K_1 does not depend on p, t_2.

For $l > m$ $(l, m = 0, 1, 2, \ldots)$ we get

$$\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau-$$

$$-\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{s}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau-$$

$$-\sum_{j=0}^{p}\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)\int\limits_{t}^{\tau}(s-t)^l\phi_j(s)dsd\tau. \qquad (2.1154)$$

Applying Cauchy–Bunyakovsky's inequality and Parseval's equality, we obtain

$$\left(\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau\right)^2\le$$

$$\le\sum_{j=0}^{p}\left(\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2\sum_{j=0}^{p}\left(\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau\right)^2\le$$

$$\le\sum_{j=0}^{\infty}\left(\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2\sum_{j=0}^{\infty}\left(\int\limits_{t}^{t_2}(\tau-t)^m\phi_j(\tau)d\tau\right)^2=$$

$$=\int\limits_{t}^{t_2}(s-t)^{2l}ds\int\limits_{t}^{t_2}(\tau-t)^{2m}d\tau\le K_2<\infty, \qquad (2.1155)$$

where constant K_2 does not depend on p, t_2.

Using (2.1153)–(2.1155), we obtain

$$\left|\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right|\le K_3<\infty,\qquad(2.1156)$$

where $l>m$ $(l,m=0,1,2,\ldots)$, constant K_3 does not depend on p,t_2.

For the case $l=m$ we get

$$\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^l\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\frac{1}{2}\left(\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2\le\sum_{j=0}^{\infty}\frac{1}{2}\left(\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)ds\right)^2=$$

$$=\frac{1}{2}\int\limits_{t}^{t_2}(s-t)^{2l}ds\le K_4<\infty,\qquad(2.1157)$$

where constant K_4 does not depend on p,t_2.

Combining (2.1153), (2.1156), (2.1157), we have

$$\left|\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds\right|\le K_5<\infty,\qquad(2.1158)$$

where $l,m=0,1,2,\ldots$, constant K_5 does not depend on p,t_2.

Note that

$$\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t_1}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds=$$

$$=\sum_{j=0}^{p}\int\limits_{t}^{t_2}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds-$$

$$-\sum_{j=0}^{p}\int\limits_{t}^{t_1}(s-t)^l\phi_j(s)\int\limits_{t}^{s}(\tau-t)^m\phi_j(\tau)d\tau ds-$$

$$-\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_1}(\tau-t)^m\phi_j(\tau)d\tau,\qquad(2.1159)$$

where $l,m=0,1,2,\ldots$ and $t\le t_1<t_2\le T$.

By analogy with (2.1155) we get

$$\left|\sum_{j=0}^{p}\int\limits_{t_1}^{t_2}(s-t)^l\phi_j(s)ds\int\limits_{t}^{t_1}(\tau-t)^m\phi_j(\tau)d\tau\right|\le K_6<\infty, \qquad (2.1160)$$

where $l,m=0,1,2,\ldots$, constant K_6 does not depend on p,t_2.

Combining (2.1159), (2.1158), and (2.1160), we obtain (2.1152).

Let us prove (2.1149). Using Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\int\limits_{t}^{T}(\tau-t)^{l_3}\phi_{j_3}(\tau)\left(\frac{1}{2}\int\limits_{t}^{\tau}(s-t)^{l_1+l_2}ds-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_{t}^{\tau}(s-t)^{l_2}\phi_{j_1}(s)\int\limits_{t}^{s}(\theta-t)^{l_1}\phi_{j_1}(\theta)d\theta ds\right)d\tau\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(\int\limits_{t}^{T}(\tau-t)^{l_3}\phi_{j_3}(\tau)\left(\frac{1}{2}\int\limits_{t}^{\tau}(s-t)^{l_1+l_2}ds-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_{t}^{\tau}(s-t)^{l_2}\phi_{j_1}(s)\int\limits_{t}^{s}(\theta-t)^{l_1}\phi_{j_1}(\theta)d\theta ds\right)d\tau\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_{t}^{T}(\tau-t)^{2l_3}\left(\frac{1}{2}\int\limits_{t}^{\tau}(s-t)^{l_1+l_2}ds-\right.$$

$$\left.-\sum_{j_1=0}^{p}\int\limits_{t}^{\tau}(s-t)^{l_2}\phi_{j_1}(s)\int\limits_{t}^{s}(\theta-t)^{l_1}\phi_{j_1}(\theta)d\theta ds\right)^2d\tau. \qquad (2.1161)$$

Using (2.721), (2.1152) and applying Lebesgue's Dominated Convergence Theorem in (2.1161), we obtain the equality (2.1149).

Let us prove (2.1150). Using Fubini's Theorem and Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}C_{j_2j_2j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}\int\limits_t^T(s-t)^{l_2+l_3}\int\limits_t^s(\theta-t)^{l_1}\phi_{j_1}(\theta)d\theta ds-\right.$$

$$\left.-\sum_{j_2=0}^{p}\int\limits_t^T(s-t)^{l_3}\phi_{j_2}(s)\int\limits_t^s(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_t^\tau(\theta-t)^{l_1}\phi_{j_1}(\theta)d\theta d\tau ds\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\int\limits_t^T(\theta-t)^{l_1}\phi_{j_1}(\theta)\left(\frac{1}{2}\int\limits_\theta^T(s-t)^{l_2+l_3}ds-\right.\right.$$

$$\left.\left.-\sum_{j_2=0}^{p}\int\limits_\theta^T(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_\tau^T(s-t)^{l_3}\phi_{j_2}(s)dsd\tau\right)d\theta\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(\int\limits_t^T(\theta-t)^{l_1}\phi_{j_1}(\theta)\left(\frac{1}{2}\int\limits_\theta^T(s-t)^{l_2+l_3}ds-\right.\right.$$

$$\left.\left.-\sum_{j_2=0}^{p}\int\limits_\theta^T(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_\tau^T(s-t)^{l_3}\phi_{j_2}(s)dsd\tau\right)d\theta\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T(\theta-t)^{2l_1}\left(\frac{1}{2}\int\limits_\theta^T(s-t)^{l_2+l_3}ds-\right.$$

$$\left.-\sum_{j_2=0}^{p}\int\limits_\theta^T(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_\tau^T(s-t)^{l_3}\phi_{j_2}(s)dsd\tau\right)^2d\theta=$$

$$=\lim_{p\to\infty}\int\limits_t^T(\theta-t)^{2l_1}\left(\frac{1}{2}\int\limits_\theta^T(s-t)^{l_2+l_3}ds-\right.$$

$$\left.-\sum_{j_2=0}^{p}\int\limits_\theta^T(s-t)^{l_3}\phi_{j_2}(s)\int\limits_\theta^s(\tau-t)^{l_2}\phi_{j_2}(\tau)d\tau ds\right)^2d\theta. \qquad (2.1162)$$

Applying (2.721), (2.1152) and using Lebesgue's Dominated Convergence Theorem in (2.1162), we get the equality (2.1150).

Let us prove (2.1151). Applying Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T(\theta-t)^{l_3}\phi_{j_1}(\theta)\int\limits_t^{\theta}(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)dsd\tau d\theta\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T(\tau-t)^{l_2}\phi_{j_2}(\tau)\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}(\theta-t)^{l_3}\phi_{j_1}(\theta)d\theta d\tau\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(\int\limits_t^T(\tau-t)^{l_2}\phi_{j_2}(\tau)\sum_{j_1=0}^{p}\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}(\theta-t)^{l_3}\phi_{j_1}(\theta)d\theta d\tau\right)^2\le$$

$$=\lim_{p\to\infty}\int\limits_t^T(\tau-t)^{2l_2}\left(\sum_{j_1=0}^{p}\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}(\theta-t)^{l_3}\phi_{j_1}(\theta)d\theta\right)^2d\tau. \quad (2.1163)$$

Applying (2.969), we obtain

$$\left|\sum_{j_1=0}^{p}\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}(\theta-t)^{l_3}\phi_{j_1}(\theta)d\theta\right|\le C<\infty, \quad (2.1164)$$

where constant C does not depend on p,τ.

Using the generalized Parseval equality, we get

$$\sum_{j_1=0}^{\infty}\int\limits_t^{\tau}(s-t)^{l_1}\phi_{j_1}(s)ds\int\limits_{\tau}^{T}(\theta-t)^{l_3}\phi_{j_1}(\theta)d\theta=$$

$$=\int\limits_t^T(s-t)^{l_1+l_3}\mathbf{1}_{\{s<\tau\}}\mathbf{1}_{\{s>\tau\}}ds=0. \quad (2.1165)$$

Taking into account (2.1164), (2.24) and applying Lebesgue's Dominated Convergence Theorem in (2.1163), we obtain the equality (2.1151). Theorem 2.51 is proved.

2.25 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3 Under Additional Assumptions. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\in L_2([t,T])$

In this section, we will prove the following two theorems.

Theorem 2.52 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\psi_2(\tau),\psi_3(\tau)\in L_2([t,T])$ are such that*

$$\left|\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right|^2\le K<\infty, \qquad (2.1166)$$

$$\left|\sum_{j_3=0}^{p}\int\limits_s^T\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right|^2\le K<\infty \qquad (2.1167)$$

$\forall p\in\mathbf{N}$, where constant K does not depend on p and s $(t\le s\le T)$. Then, for the sum $\bar J^{*}[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}$ $(i_1,i_2,i_3=0,1,\ldots,m)$ of iterated Itô stochastic integrals defined by (2.962) $(k=3)$ the following expansion

$$\bar J^{*}[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$.

Theorem 2.53 [33], [38], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\psi_2(\tau),\psi_3(\tau)$ are continuous functions on $[t,T]$. Furthermore, let the conditions (2.1166), (2.1167) are satisfied. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where notations are the same as in Theorem 2.52.

Note that Theorem 2.53 is a simple consequence of Theorem 2.52 and Theorem 2.12 ($k=3$). Let us prove Theorem 2.52.

Proof. First, let us note some facts that follows from Monotone Convergence Theorem ([147], Theorem 3.5.1) and Lebesgue's Dominated Convergence Theorem. Suppose that $\{g_j(x)\}_{j=0}^{\infty}$ is an arbitrary sequence of real-valued measurable functions such that

$$\sum_{j=0}^{\infty}|g_j(x)|\le K<\infty \tag{2.1168}$$

almost everywhere on X (with respect to Lebesgue's measure), where constant K does not depend on x.

It is easy to see that under the above conditions the following equality

$$\lim_{p\to\infty}\int\limits_X h^2(x)\left(\sum_{j=0}^{p}g_j(x)\right)^2dx=\int\limits_X h^2(x)\left(\sum_{j=0}^{\infty}g_j(x)\right)^2dx \tag{2.1169}$$

is true, where $h(x)\in L_2(X)$ (further, we put $h(x)\equiv 1$ for simplicity). Indeed, we have

$$|g_j(x)|=g_j^{+}(x)+g_j^{-}(x),\quad g_j(x)=g_j^{+}(x)-g_j^{-}(x),$$

where

$$g_j^{+}(x)=\max\{g_j(x),0\}\ge 0,\quad g_j^{-}(x)=-\min\{g_j(x),0\}\ge 0.$$

Moreover,

$$\sum_{j=0}^{\infty}|g_j(x)|=\sum_{j=0}^{\infty}g_j^{+}(x)+\sum_{j=0}^{\infty}g_j^{-}(x), \tag{2.1170}$$

$$\sum_{j=0}^{\infty}g_j(x)=\sum_{j=0}^{\infty}g_j^{+}(x)-\sum_{j=0}^{\infty}g_j^{-}(x).$$

Using (2.1168), we obtain that the series (with non-negative terms) on the right-hand side of (2.1170) satisfy the condition (2.1168). Further, using Monotone Convergence Theorem, we obtain

$$\lim_{p\to\infty}\int\limits_X\left(\sum_{j=0}^{p}g_j(x)\right)^2dx=\lim_{p\to\infty}\int\limits_X\left(\sum_{j=0}^{p}g_j^{+}(x)-\sum_{j=0}^{p}g_j^{-}(x)\right)^2dx=$$

$$=\lim_{p\to\infty}\int\limits_X\left(\sum_{j=0}^{p}g_j^{+}(x)\right)^2dx-\lim_{p\to\infty}2\int\limits_X\sum_{j=0}^{p}g_j^{+}(x)\sum_{j=0}^{p}g_j^{-}(x)dx+$$

$$+\lim_{p\to\infty}\int\limits_X\left(\sum_{j=0}^{p}g_j^{-}(x)\right)^2dx=$$

$$=\int\limits_X\lim_{p\to\infty}\left(\sum_{j=0}^{p}g_j^{+}(x)\right)^2dx-2\int\limits_X\lim_{p\to\infty}\sum_{j=0}^{p}g_j^{+}(x)\sum_{j=0}^{p}g_j^{-}(x)dx+$$

$$+\int\limits_X\lim_{p\to\infty}\left(\sum_{j=0}^{p}g_j^{-}(x)\right)^2dx=$$

$$=\int\limits_X\left(\sum_{j=0}^{\infty}g_j^{+}(x)\right)^2dx-2\int\limits_X\sum_{j=0}^{\infty}g_j^{+}(x)\sum_{j=0}^{\infty}g_j^{-}(x)dx+\int\limits_X\left(\sum_{j=0}^{\infty}g_j^{-}(x)\right)^2dx=$$

$$=\int\limits_X\left(\sum_{j=0}^{\infty}g_j^{+}(x)-\sum_{j=0}^{\infty}g_j^{-}(x)\right)^2dx=\int\limits_X\left(\sum_{j=0}^{\infty}g_j(x)\right)^2dx. \tag{2.1171}$$

The equality (2.1169) can be obtained under another conditions. If we replace the condition (2.1168) with

$$\left|\sum_{j=0}^{p}g_j(x)\right|\le K<\infty\quad\forall p\in\mathbf{N}\quad\text{and}\quad\lim_{p\to\infty}\sum_{j=0}^{p}g_j(x)\ \ \text{exists} \tag{2.1172}$$

almost everywhere on X (with respect to Lebesgue's measure), then by Lebesgue's Dominated Convergence Theorem we obtain (2.1169). Here constant K does not depend on x and p.

According to Theorem 2.49, we come to the conclusion that Theorem 2.52 will be proved if we prove the following equalities

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=0, \tag{2.1173}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2=0, \tag{2.1174}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=0. \tag{2.1175}$$

Let us prove (2.1173). Using Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^\tau\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)ds\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\int\limits_t^s\psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^\tau\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)ds\right)^2=$$

$$=\lim_{p\to\infty}\int\limits_t^T\psi_3^2(s)\left(\frac{1}{2}\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau-\right.$$

$$
-\sum_{j_1=0}^{p}\int\limits_t^s \psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\Biggr)^2 ds = \qquad (2.1176)
$$

$$
=\int\limits_t^T \psi_3^2(s)\lim_{p\to\infty}\Biggl(\frac{1}{2}\int\limits_t^s \psi_2(\tau)\psi_1(\tau)d\tau-
$$

$$
-\sum_{j_1=0}^{p}\int\limits_t^s \psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau\Biggr)^2 ds = 0, \qquad (2.1177)
$$

where (2.1177) follows from from (2.125) (also see (2.721)) and the transition from (2.1176) to (2.1177) is based on (2.1169), (2.1172) and Lebesgue's Dominated Convergence Theorem (see (2.1166)). The equality (2.1173) is proved.

Let us prove (2.1174). Using Fubini's Theorem and Parseval's equality, we obtain

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\Biggl(\frac{1}{2}C_{j_3j_3j_1}\Biggr|_{(j_3j_3)\frown(\cdot)}-\sum_{j_3=0}^{p}C_{j_3j_3j_1}\Biggr)^2 =
$$

$$
=\lim_{p\to\infty}\sum_{j_1=0}^{p}\Biggl(\frac{1}{2}\int\limits_t^T \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds d\tau-
$$

$$
-\sum_{j_3=0}^{p}\int\limits_t^T \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_t^{\tau}\psi_1(s)\phi_{j_1}(s)ds d\tau d\theta\Biggr)^2 =
$$

$$
=\lim_{p\to\infty}\sum_{j_1=0}^{p}\Biggl(\frac{1}{2}\int\limits_t^T \psi_1(s)\phi_{j_1}(s)\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau ds-
$$

$$
-\sum_{j_3=0}^{p}\int\limits_t^T \psi_1(s)\phi_{j_1}(s)\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds\Biggr)^2 =
$$

$$
=\lim_{p\to\infty}\sum_{j_1=0}^{p}\Biggl(\int\limits_t^T \psi_1(s)\phi_{j_1}(s)\Biggl(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-
$$

$$
-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\Biggr) ds\Biggr)^2 \le
$$

$$\le \lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_1(s)\phi_{j_1}(s)\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.\right.$$

$$\left.\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)ds\right)^2 =$$

$$= \lim_{p\to\infty}\int\limits_t^T \psi_1^2(s)\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.$$

$$\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)^2 ds = \qquad (2.1178)$$

$$= \int\limits_t^T \psi_1^2(s)\lim_{p\to\infty}\left(\frac{1}{2}\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau-\right.$$

$$\left.-\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_\tau^T \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right)^2 ds = 0, \qquad (2.1179)$$

where (2.1179) follows from (2.125) and the transition from (2.1178) to (2.1179) is based on (2.1169), (2.1172) and Lebesgue's Dominated Convergence Theorem (see (2.1167)). The equality (2.1174) is proved.

Let us prove (2.1175). Applying Fubini's Theorem and Parseval's equality, we have

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}\right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_3(\theta)\phi_{j_1}(\theta)\int\limits_t^\theta \psi_2(\tau)\phi_{j_2}(\tau)\int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds d\tau d\theta\right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_2(\tau)\phi_{j_2}(\tau)\int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds\int\limits_\tau^T \psi_3(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2 \le$$

$$\le \lim_{p\to\infty}\sum_{j_2=0}^{\infty}\left(\int\limits_t^T \psi_2(\tau)\phi_{j_2}(\tau)\sum_{j_1=0}^{p}\int\limits_t^\tau \psi_1(s)\phi_{j_1}(s)ds\int\limits_\tau^T \psi_3(\theta)\phi_{j_1}(\theta)d\theta d\tau\right)^2 =$$

$$= \lim_{p\to\infty} \int\limits_t^T \psi_2^2(\tau) \left(\sum_{j_1=0}^{p} \int\limits_t^{\tau} \psi_1(s)\phi_{j_1}(s)ds \int\limits_{\tau}^{T} \psi_3(\theta)\phi_{j_1}(\theta)d\theta \right)^2 d\tau = \quad (2.1180)$$

$$= \int\limits_t^T \psi_2^2(\tau) \lim_{p\to\infty} \left(\sum_{j_1=0}^{p} \int\limits_t^{\tau} \psi_1(s)\phi_{j_1}(s)ds \int\limits_{\tau}^{T} \psi_3(\theta)\phi_{j_1}(\theta)d\theta \right)^2 d\tau = 0, \quad (2.1181)$$

where (2.1181) follows from the equality

$$\sum_{j_1=0}^{\infty} \int\limits_t^{\tau} \psi_1(s)\phi_{j_1}(s)ds \int\limits_{\tau}^{T} \psi_3(\theta)\phi_{j_1}(\theta)d\theta = \int\limits_t^T \psi_1(s)\mathbf{1}_{\{s<\tau\}}\psi_3(s)\mathbf{1}_{\{s>\tau\}}ds = 0$$
(2.1182)

(the relation (2.1182) follows from the generalized Parseval equality) and the transition from (2.1180) to (2.1181) is based on (2.1169), (2.1172) and Lebesgue's Dominated Convergence Theorem (see (2.969)). The equality (2.1175) is proved. Theorem 2.52 is proved.

2.26 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 4 and 5 Under Additional Assumptions. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_5(\tau) \in L_2([t,T])$

Let us develop the approach discussed in the previous section. It is easy to see (according to Theorem 2.49) that analogues of Theorems 2.52 and 2.53 for the cases $k=4$ and $k=5$ ($\psi_1(\tau),\ldots,\psi_5(\tau)\in L_2([t,T])$) will be true if the relations (2.993)–(2.998), (2.1069)–(2.1093) as well as the equalities

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1} = \frac{1}{4} \int\limits_t^T \psi_4(t_3)\psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_1)\psi_1(t_1)dt_1dt_3, \quad (2.1183)$$

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1} = 0, \quad (2.1184)$$

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1} = 0, \quad (2.1185)$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau)=\frac{1}{4}\int\limits_{\tau}^{s}\psi_4(t_3)\psi_3(t_3)\int\limits_{\tau}^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1dt_3, \quad (2.1186)$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau)=0, \quad (2.1187)$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau)=0 \quad (2.1188)$$

are satisfied, provided that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, $\psi_1(\tau),\ldots,\psi_5(\tau)\in L_2([t,T])$, the series on the left-hand sides of (2.1183)–(2.1188) converge absolutely, and

$$C_{j_4\ldots j_1}=\int\limits_{t}^{T}\psi_4(t_4)\phi_{j_4}(t_4)\ldots\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_4,$$

$$C_{j_5\ldots j_1}=\int\limits_{t}^{T}\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5,$$

$$C_{j_4\ldots j_1}(s,\tau)=\int\limits_{\tau}^{s}\psi_4(t_4)\phi_{j_4}(t_4)\ldots\int\limits_{\tau}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_4 \quad (t\le\tau<s\le T)$$

in (2.993)–(2.998), (2.1069)–(2.1093), (2.1183)–(2.1188).

It is obvious that the equalities (2.1186)–(2.1188) follow from the equalities (2.1183)–(2.1185) if in (2.1183)–(2.1185) we replace $\psi_4(t_4),\psi_3(t_3),\psi_2(t_2),\psi_1(t_1)$ with $\mathbf{1}_{\{\tau<t_4<s\}}\psi_4(t_4)$, $\mathbf{1}_{\{\tau<t_3\}}\psi_3(t_3)$, $\mathbf{1}_{\{\tau<t_2\}}\psi_2(t_2)$, $\mathbf{1}_{\{\tau<t_1\}}\psi_1(t_1)$, respectively.

Further, the proofs of Theorems 2.46 and 2.50 must be modified and carried out by analogy with the proof of Theorem 2.52, i.e. using the equalities (2.1169), (2.1172) and Lebesgue's Dominated Convergence Theorem. At that, the derivation of formulas similar to (2.1002)–(2.1007), (2.1094)–(2.1103), (2.1120)–(2.1126), (2.1129), (2.1130), (2.1133), (2.1134), (2.1135), (2.1138), (2.1141), (2.1144) is carried out completely similarly to (2.1002)–(2.1007), (2.1094)–(2.1103), (2.1120)–(2.1126), (2.1129), (2.1130), (2.1133), (2.1134), (2.1135), (2.1138), (2.1141), (2.1144), adjusted for the fact that in (2.1002)–(2.1007), (2.1094)–(2.1103), (2.1120)–(2.1126), (2.1129), (2.1130), (2.1133), (2.1134), (2.1135), (2.1138), (2.1141), (2.1144) the functions $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv 1$ are replaced by $\psi_1(\tau),\ldots,\psi_5(\tau)\in L_2([t,T])$.

Furthermore, the following additional conditions

$$\left|\sum_{j=0}^{p}\int\limits_{\tau}^{s}\psi_{m+1}(t_2)\phi_j(t_2)\int\limits_{\tau}^{t_2}\psi_m(t_1)\phi_j(t_1)dt_1dt_2\right|^2\le K<\infty\quad (m=1,2,3,4), \tag{2.1189}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1j_1}^{\psi_{m+3}\psi_{m+2}\psi_{m+1}\psi_m}(s,\tau)\right|^2\le K<\infty\quad (m=1,2), \tag{2.1190}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}^{\psi_{m+3}\psi_{m+2}\psi_{m+1}\psi_m}(s,\tau)\right|^2\le K<\infty\quad (m=1,2), \tag{2.1191}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}^{\psi_{m+3}\psi_{m+2}\psi_{m+1}\psi_m}(s,\tau)\right|^2\le K<\infty\quad (m=1,2), \tag{2.1192}$$

must be satisfied $\forall p\in\mathbf{N}$, where constant K does not depend on p,τ,s,

$$C_{j_4j_3j_2j_1}^{\psi_{m+3}\psi_{m+2}\psi_{m+1}\psi_m}(s,\tau)=\int\limits_{\tau}^{s}\psi_{m+3}(t_4)\phi_{j_4}(t_4)\times$$

$$\times\int\limits_{\tau}^{t_4}\psi_{m+2}(t_3)\phi_{j_3}(t_3)\int\limits_{\tau}^{t_3}\psi_{m+1}(t_2)\phi_{j_2}(t_2)\int\limits_{\tau}^{t_2}\psi_m(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4,$$

where $m=1,2$ and $t\le\tau<s\le T$.

The conditions (2.1189)–(2.1192) are required to perform the passage to the limit using Lebesgue's Dominated Convergence Theorem (see the proofs of Theorems 2.50, 2.52 for details).

The equality (2.1183) is proved in [118] for the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_4(\tau)\in L_2([t,T])$. The equalities (2.1184), (2.1185) can also be obtained [119] using the approach from [118]. At that, the series on the left-hand sides of (2.1183)–(2.1185) converge absolutly. We will return to these issues in Sect. 2.27.3 and 2.27.4. Sect. 2.27.3 will be devoted to the method from [118] based on trace class operators. In Sect. 2.27.4 we will prove the equalities

(2.1183)–(2.1185) using an approach based on the generalized Parseval equality and (2.125) (the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\psi_4(\tau)\in L_2([t,T])$).

Taking into account everything said above in this section, we obtain the following four theorems.

Theorem 2.54 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ \ldots,\psi_4(\tau)\in L_2([t,T])$. *Furthermore, let the condition* (2.1189) $(m=1,2,3)$ *is satisfied. Then, for the sum* $\bar J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}$ $(i_1,\ldots,i_4=0,1,\ldots,m)$ *of iterated Itô stochastic integrals defined by* (2.962) $(k=4)$ *the following expansion*

$$\bar J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where

$$C_{j_4\ldots j_1}=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_4$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Theorem 2.55 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ \ldots,\psi_4(\tau)$ *are continuous functions on* $[t,T]$. *Furthermore, let the condition* (2.1189) $(m=1,2,3)$ *is satisfied. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$\int\limits_t^{*T}\psi_4(t_4)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}\quad(i_1,\ldots,i_4=0,1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\psi_4(t_4)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where notations are the same as in Theorem 2.54.

Theorem 2.56 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)\in L_2([t,T])$. *Furthermore, let the conditions* (2.1189)–(2.1192) *are satisfied. Then, for the sum* $\bar{J}^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}$ $(i_1,\ldots,i_5=0,1,\ldots,m)$ *of iterated Itô stochastic integrals defined by* (2.962) $(k=5)$ *the following expansion*

$$
\bar{J}^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where

$$
C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Theorem 2.57 [33], [38], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuous functions on* $[t,T]$. *Furthermore, let the conditions* (2.1189)–(2.1192) *are satisfied. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)}\quad(i_1,\ldots,i_5=0,1,\ldots,m)
$$

the following expansion

$$
\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where notations are the same as in Theorem 2.56.

Note that Theorems 2.55 and 2.57 are simple consequences of Theorems 2.54 and 2.56, respectively (see Theorem 2.12 ($k=4,\ 5$)).

2.27 On the Calculation of Matrix Traces of Volterra–Type Integral Operators

2.27.1 Introduction

It is easy to see that the function (1.310) for even $k=2r$ ($r\in\mathbf{N}$) forms a family of integral operators $\mathbb{K}: L_2([t,T]^r)\to L_2([t,T]^r)$ (with the kernel (1.310)) of the form

$$(\mathbb{K}f)\left(t_{g_1},\ldots,t_{g_r}\right)=\int\limits_{[t,T]^r}K(t_1,\ldots,t_k)f(t_{g_{r+1}},\ldots,t_{g_k})dt_{g_{r+1}}\ldots dt_{g_k}, \quad (2.1193)$$

where $\{g_1,\ldots,g_k\}=\{1,\ldots,k\}$, the kernel $K(t_1,\ldots,t_k)$ is defined by (1.310), i.e. has the form

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \quad (2.1194)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T]),\ t_1,\ldots,t_k\in[t,T]$ ($k\ge 2$) and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

For example,

$$(\mathbb{K}f)(t_2)=\int\limits_t^T K(t_1,t_2)f(t_1)dt_1=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)f(t_1)dt_1, \quad (2.1195)$$

$$(\mathbb{K}f)(t_3,t_4)=\int\limits_{[t,T]^2}K(t_1,\ldots,t_4)f(t_1,t_2)dt_1dt_2=$$

$$=\mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_4(t_4)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)f(t_1,t_2)dt_1dt_2,$$

$$
(\mathbb{K}f)(t_1,t_2)=\int\limits_{[t,T]^2} K(t_1,\ldots,t_4)f(t_3,t_4)dt_3dt_4=
$$

$$
=\psi_1(t_1)\psi_2(t_2)\mathbf{1}_{\{t_1<t_2\}}\int\limits_{t_2}^{T}\psi_3(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)f(t_3,t_4)dt_4dt_3.
$$

The simplest representative of the family (2.1193) has the form

$$
(\mathbb{V}f)(x)=\int\limits_{0}^{x} f(\tau)d\tau \qquad (2.1196)
$$

and is called the Volterra integral operator, where $\mathbb{V}: L_2([0,1])\to L_2([0,1])$, $f(\tau)\in L_2([0,1])$. The kernel of the Volterra integral operator has the following form

$$
K(\tau,x)=\begin{cases}1, & \tau<x\\ \\ 0, & \text{otherwise}\end{cases},\quad \tau,x\in[0,1].
$$

Suppose that $\mathbb{A}: H\to H$ is a linear bounded operator. Recall [127] that $\mathbb{A}$ has a finite matrix trace if for any orthonormal basis $\{\Psi_j(x)\}_{j=0}^{\infty}$ of the space H the series

$$
\sum_{j=0}^{\infty}\langle\mathbb{A}\Psi_j,\Psi_j\rangle_H \qquad (2.1197)
$$

converges, where $\langle\cdot,\cdot\rangle_H$ is a scalar probuct in H.

Note that the series (2.1197) converges absolutely since its sum does not depend on the permutation of the terms of the series (2.1197) (any permutation of basis functions $\Psi_j(x)$ forms a basis in H) [127].

It is well known that the Volterra integral operator (2.1196) is not a trace class operator since its singular values are equal to $s_j(\mathbb{V})=2\left(\pi(2j+1)\right)^{-1}$ [150]. On the other hand, it is known [150] that for trace class operators the equality of matrix and integral traces holds. It turns out that for the Volterra integral operator (2.1196) (although it is not a trace class operator), the equality of matrix and integral traces is also true [150]. Thus, one cannot count on the fact that operators of the more general form (2.1193) (from the same family of operators as the Volterra integral operator (2.1196)) are operators of the trace class. Nevertheless, the proof of the equalities of matrix and integral traces for Volterra–type integral operators (2.1193) (which is obviously a problem) provides a way to calculate the matrix traces of these operators.

Why do we talk so much in this section about matrix traces of operators from the family (2.1193)? The point is that matrix traces of operators of the form (2.1193) are of great importance for obtaining of expansions of iterated Stratonovich stochastic integrals.

Throughout the Chapter 2, we have already calculated the matrix traces mentioned above many times (see the formulas (2.10), (2.37), (2.125), (2.316), (2.317), (2.318), (2.529), (2.530), (2.535), (2.566), (2.567), (2.570), (2.683), (2.692), (2.833)–(2.847), (2.903), (2.999)–(2.1001), (2.1027)–(2.1029), (2.1183)–(2.1188)).

Let us consider some illustrative examples. We have

$$\sum_{j_1=0}^{\infty}\left\langle \mathbb{K}\phi_{j_1},\phi_{j_1}\right\rangle_{L_2([t,T])} = \qquad (2.1198)$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=\sum_{j_1=0}^{\infty}C_{j_1j_1}, \qquad (2.1199)$$

$$\sum_{j_1,j_2=0}^{\infty}\left\langle \mathbb{K}\Psi_{j_1j_2},\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)} = \qquad (2.1200)$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_2j_1j_1}, \qquad (2.1201)$$

where $\{\Psi_{j_1j_2}(x,y)\}_{j_1,j_2=0}^{\infty}=\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$, $(\mathbb{K}f)(t_2)$ in (2.1198) is defined by (2.1195), and $(\mathbb{K}f)(t_2,t_3)$ in (2.1200) has the following form

$$(\mathbb{K}f)(t_2,t_3)=\int\limits_{[t,T]^2}K(t_1,\ldots,t_4)f(t_1,t_4)dt_1dt_4=$$

$$=\psi_2(t_2)\psi_3(t_3)\mathbf{1}_{\{t_2<t_3\}}\int\limits_t^{t_2}\psi_1(t_1)\int\limits_{t_3}^{T}\psi_4(t_4)f(t_1,t_4)dt_4dt_1,$$

where $K(t_1,\ldots,t_4)$ is defined by (2.1194).

The expressions on the right-hand sides of (2.1199) and (2.1201) were considered earlier in Chapter 2 under various assumptions on $\{\phi_j(x)\}_{j=0}^{\infty}$ and $\psi_1(\tau),\ldots,\psi_4(\tau)$ (see the formulas (2.10), (2.37), (2.125), (2.316), (2.529), (2.535), (2.566), (2.999), (2.1027), (2.1183)).

2.27.2 Approach Based on Generalized Parseval's Equality and (2.125). Symmetrical Case When $\psi_1(\tau)=\psi_k(\tau),\ \psi_2(\tau)=\psi_{k-1}(\tau),\ldots,\psi_r(\tau)=\psi_{r+1}(\tau)$ $(k=2r, r=2,3,4,\ldots)$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$

Let us consider one of the possible ways to calculate matix traces of Volterra-type integral operators (2.1193) based Fubini's Theorem, Parseval's equality and generalized Parseval's equality.

Recall the equalities (2.848) and (2.1008)

$$C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$
$$+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}, \tag{2.1202}$$

$$C_{j_4j_3j_2j_1}+C_{j_1j_2j_3j_4}=C_{j_4}C_{j_3j_2j_1}-C_{j_3j_4}C_{j_2j_1}+C_{j_2j_3j_4}C_{j_1}, \tag{2.1203}$$

where $C_{j_k\cdots j_1}$ is defined by the formula

$$C_{j_k\cdots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k\quad (k\in\mathbf{N})$$

for the case $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

It is easy to see (see the derivation of (2.848) and (2.1008)) that analogues of the relations (2.1202), (2.1203) (with appropriate changes) hold for $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

By analogy with (2.1202), (2.1203) (see the derivation of (2.848) and (2.1008)) we obtain for $k=2r$ $(r=2,3,4,\ldots)$

$$C_{j_kj_{k-1}\cdots j_1}^{\psi_k\psi_{k-1}\cdots\psi_1}+C_{j_1j_2\cdots j_k}^{\psi_1\psi_2\cdots\psi_k}=C_{j_k}^{\psi_k}\cdot C_{j_{k-1}j_{k-2}\cdots j_1}^{\psi_{k-1}\psi_{k-2}\cdots\psi_1}-C_{j_{k-1}j_k}^{\psi_{k-1}\psi_k}\cdot C_{j_{k-2}j_{k-3}\cdots j_1}^{\psi_{k-2}\psi_{k-3}\cdots\psi_1}+$$
$$+C_{j_{k-2}j_{k-1}j_k}^{\psi_{k-2}\psi_{k-1}\psi_k}\cdot C_{j_{k-3}j_{k-4}\cdots j_1}^{\psi_{k-3}\psi_{k-4}\cdots\psi_1}-\ldots-C_{j_3j_4\cdots j_k}^{\psi_3\psi_4\cdots\psi_k}\cdot C_{j_2j_1}^{\psi_2\psi_1}+C_{j_2j_3\cdots j_k}^{\psi_2\psi_3\cdots\psi_k}\cdot C_{j_1}^{\psi_1}, \tag{2.1204}$$

where

$$C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k\quad (k\in\mathbf{N}).\quad (2.1205)$$

When proving Theorems 2.46 and 2.48, using (2.1204) (the case $k=4$, $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$), we obtained the following formulas

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}=\frac{1}{8}(T-t)^2,$$

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}=0,$$

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=0,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and we use the notation $C_{j_k\ldots j_1}$ instead of $C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1}$ for the case $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

In principle, using (2.1204), we can calculate any matrix traces for which the following symmetry condition

$$\psi_1(\tau)=\psi_k(\tau),\ \ \psi_2(\tau)=\psi_{k-1}(\tau),\ \ldots,\ \ \psi_r(\tau)=\psi_{r+1}(\tau)\ \ (k=2r, r=2,3,4,\ldots)\quad (2.1206)$$

is satisfied. Obviously, the case $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$ is possible since it is a special case of (2.1206). This case is important because it covers the mean-square approximation of iterated Stratonovich stochastic integrals from the classical Taylor–Stratonovich expansions (see Chapter 4).

Consider the case $k=4$ of (2.1204)

$$C_{j_4j_3j_2j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_1j_2j_3j_4}^{\psi_1\psi_2\psi_3\psi_4}=C_{j_4}^{\psi_4}C_{j_3j_2j_1}^{\psi_3\psi_2\psi_1}-C_{j_3j_4}^{\psi_3\psi_4}C_{j_2j_1}^{\psi_2\psi_1}+C_{j_2j_3j_4}^{\psi_2\psi_3\psi_4}C_{j_1}^{\psi_1},\quad (2.1207)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Substitute $j_4=j_3$, $j_2=j_1$ into (2.1207)

$$C_{j_3j_3j_1j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_1j_1j_3j_3}^{\psi_1\psi_2\psi_3\psi_4}=C_{j_3}^{\psi_4}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}-C_{j_3j_3}^{\psi_3\psi_4}C_{j_1j_1}^{\psi_2\psi_1}+C_{j_1j_3j_3}^{\psi_2\psi_3\psi_4}C_{j_1}^{\psi_1}.\quad (2.1208)$$

Applying (2.1208), we get

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_3j_1j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_1j_1j_3j_3}^{\psi_1\psi_2\psi_3\psi_4}\right)=\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3}^{\psi_4}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}-$$

$$-\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3}^{\psi_3\psi_4}C_{j_1j_1}^{\psi_2\psi_1}+\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3}^{\psi_2\psi_3\psi_4}C_{j_1}^{\psi_1}. \qquad (2.1209)$$

From (2.721) we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3j_3}^{\psi_3\psi_4}\sum_{j_1=0}^{p}C_{j_1j_1}^{\psi_2\psi_1}=\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3j_3}^{\psi_3\psi_4}\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1j_1}^{\psi_2\psi_1}=$$

$$=\frac{1}{4}\int\limits_t^T\psi_4(s)\psi_3(s)ds\int\limits_t^T\psi_2(s)\psi_1(s)ds. \qquad (2.1210)$$

Further, we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}=\frac{1}{2}\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-$$

$$-\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right). \qquad (2.1211)$$

Applying the generalized Parseval equality, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\psi_4(s)\phi_{j_3}(s)ds\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds=$$

$$=\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds. \qquad (2.1212)$$

From (2.1211) and (2.1212) we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}=\frac{1}{2}\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds-$$

$$-\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right). \qquad (2.1213)$$

Due to Cauchy–Bunyakovsky's inequality, Parseval's equality and (2.1173), we get

$$\lim_{p\to\infty}\left(\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right)\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(C_{j_3}^{\psi_4}\right)^2\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right)^2\le$$

$$\le\lim_{p\to\infty}\sum_{j_3=0}^{\infty}\left(C_{j_3}^{\psi_4}\right)^2\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right)^2=$$

$$=\int\limits_t^T\psi_4^2(s)ds\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\frac{1}{2}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\Bigl|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}\right)^2=0. \qquad (2.1214)$$

Combining (2.1213) and (2.1214), we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}C_{j_3}^{\psi_4}\sum_{j_1=0}^{p}C_{j_3j_1j_1}^{\psi_3\psi_2\psi_1}=\frac{1}{2}\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds. \qquad (2.1215)$$

Absolutely similarly to (2.1215) we get

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_1}\sum_{j_3=0}^{p}C_{j_1j_3j_3}^{\psi_2\psi_3\psi_4}=\frac{1}{2}\int\limits_t^T\psi_2(s)\psi_1(s)\int\limits_t^s\psi_3(\tau)\psi_4(\tau)d\tau ds. \qquad (2.1216)$$

Combining (2.1209), (2.1210), (2.1215), (2.1216) and applying Fubini's Theorem, we have

$$\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_3j_1j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_1j_1j_3j_3}^{\psi_1\psi_2\psi_3\psi_4}\right)=\frac{1}{2}\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds+$$

$$+\frac{1}{2}\int\limits_t^T\psi_2(s)\psi_1(s)\int\limits_t^s\psi_3(\tau)\psi_4(\tau)d\tau ds-\frac{1}{4}\int\limits_t^T\psi_4(s)\psi_3(s)ds\int\limits_t^T\psi_2(s)\psi_1(s)ds=$$

$$
=\frac{1}{4}\int\limits_t^T\psi_4(s)\psi_3(s)ds\int\limits_t^T\psi_2(s)\psi_1(s)ds=
$$

$$
=\frac{1}{4}\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds+\frac{1}{4}\int\limits_t^T\psi_2(s)\psi_1(s)\int\limits_t^s\psi_3(\tau)\psi_4(\tau)d\tau ds. \tag{2.1217}
$$

Let us rewrite (2.1217) in the form

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_3j_1j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_3j_3j_1j_1}^{\psi_1\psi_2\psi_3\psi_4}\right)=
$$

$$
=\frac{1}{4}\int\limits_t^T\psi_4(s)\psi_3(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds+\frac{1}{4}\int\limits_t^T\psi_2(s)\psi_1(s)\int\limits_t^s\psi_3(\tau)\psi_4(\tau)d\tau ds. \tag{2.1218}
$$

It is easy to see that the left-hand side of (2.1218) does not depend on the simultaneous rearrangement of ψ_4 with ψ_1 and ψ_3 with ψ_2.

Using the above arguments and using derivation method of (2.1028) and (2.1029), we get

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_3j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_3j_1j_3j_1}^{\psi_1\psi_2\psi_3\psi_4}\right)=0, \tag{2.1219}
$$

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}\left(C_{j_1j_3j_3j_1}^{\psi_4\psi_3\psi_2\psi_1}+C_{j_1j_3j_3j_1}^{\psi_1\psi_2\psi_3\psi_4}\right)=0. \tag{2.1220}
$$

Using (2.1218)–(2.1220) under the conditions $\psi_1(\tau)=\psi_4(\tau)$, $\psi_2(\tau)=\psi_3(\tau)$, we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1j_1}^{\psi_1\psi_2\psi_2\psi_1}=\frac{1}{4}\int\limits_t^T\psi_2(s)\psi_1(s)\int\limits_t^s\psi_2(\tau)\psi_1(\tau)d\tau ds,
$$

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_3j_1}^{\psi_1\psi_2\psi_2\psi_1}=0,
$$

$$
\lim_{p\to\infty}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_3j_1}^{\psi_1\psi_2\psi_2\psi_1}=0.
$$

2.27.3 Approach Based on Trace Class Operators

An efficient method for calculating of matrix traces of Volterra–type integral operators of the form (2.1193) was proposed in [118]. This method is based on Theorem 3.1 from [150]. Theorem 3.1 [150] implies the following statement.

Theorem D. *Let* $\mathbb{K}: L_2([t,T]^k)\to L_2([t,T]^k)$ $(k=2r,\ r=1,2,\ldots)$ *be a trace class operator. Then* $\tilde{K}(t_1,\ldots,t_r,t_1,\ldots,t_r)$ *exists almost everywhere* $[dt_1\ldots dt_r]$ *and*

$$tr\mathbb{K}=\int\limits_{[t,T]^r}\tilde{K}(t_1,\ldots,t_r,t_1,\ldots,t_r)dt_1\ldots dt_r, \qquad (2.1221)$$

where $K(t_1,\ldots,t_{2r})\in L_2([t,T]^{2r})$ *is defined by* (2.1194),

$$\tilde{F}(x_1,\ldots,x_m)\stackrel{\rm def}{=}\lim_{u\to 0}A_uF(x_1,\ldots,x_m),$$

$$A_uF(x_1,\ldots,x_m)\stackrel{\rm def}{=}\frac{1}{(2u)^m}\int\limits_{[-u,u]^m}F(x_1+\tau_1,\ldots,x_m+\tau_m)d\tau_1\ldots d\tau_m \quad (m\in\mathbf{N}).$$

Let us prove the equality (2.1183) using the method from [118] in our interpretation. Consider two symmetric functions of the form (2.48) which we introduced in Sect. 2.1.2

$$K'(t_1,t_2)=\psi_1(t_1)f_2(t_2)\mathbf{1}_{\{t_1\le t_2\}}+\psi_1(t_2)f_2(t_1)\mathbf{1}_{\{t_1\ge t_2\}}, \qquad (2.1222)$$

$$K''(t_3,t_4)=f_3(t_3)\psi_4(t_4)\mathbf{1}_{\{t_3\le t_4\}}+f_3(t_4)\psi_4(t_3)\mathbf{1}_{\{t_3\ge t_4\}}, \qquad (2.1223)$$

where we suppose that $\psi_1(\tau),\psi_4(\tau)$ are continuously differentiable functions on $[t,T]$ (the case $\psi_1(\tau),\psi_4(\tau)\in L_2([t,T])$ will be considered further) and $f_2(\tau),f_3(\tau)$ are polynomials of finite degrees.

By Theorem B (see Sect. 2.1.5) and (2.133) we have that the kernels $K'(t_1,t_2)$ and $K''(t_3,t_4)$ (see (2.1222), (2.1223)) correspond to the trace class integral operators.

It is known [150] that the integral operator $\mathbb{A}$ is a trace class operator if and only if the kernel $K(x,y)$ of $\mathbb{A}$ has the following representation

$$K(x,y)=\int\limits_{[t,T]^{2n}}K_1(x,\tau)K_2(\tau,y)d\tau \qquad (2.1224)$$

almost everywhere $[dxdy]$, where $K_1(x,y), K_2(x,y)$ are kernels of Hilbert–Schmidt operators, $x,y\in\mathbf{R}^n$ $(n\ge 1)$.

Since $K'(t_1,t_2)$ and $K''(t_3,t_4)$ are kernels of the trace class integral operators, then (see (2.1224))

$$K'(t_1,t_2)=\int\limits_{[t,T]}K_1'(t_1,\tau)K_2'(\tau,t_2)d\tau,\quad K''(t_1,t_2)=\int\limits_{[t,T]}K_1''(t_1,\tau)K_2''(\tau,t_2)d\tau \tag{2.1225}$$

almost everywhere $[dt_1dt_2]$, where $K_1',K_2',K_1'',K_2''\in L_2([t,T]^2)$. Then, we have

$$K'(t_1,t_2)K''(t_3,t_4)=\int\limits_{[t,T]}K_1'(t_1,\tau_1)K_2'(\tau_1,t_2)d\tau_1\int\limits_{[t,T]}K_1''(t_3,\tau_2)K_2''(\tau_2,t_4)d\tau_2=$$

$$=\int\limits_{[t,T]^2}K_1'(t_1,\tau_1)K_1''(t_3,\tau_2)K_2'(\tau_1,t_2)K_2''(\tau_2,t_4)d\tau_1d\tau_2. \tag{2.1226}$$

The equality (2.1226) can be written as follows

$$F(t_1,t_3,t_2,t_4)=\int\limits_{[t,T]^2}F_1(t_1,t_3,\tau_1,\tau_2)F_2(\tau_1,\tau_2,t_2,t_4)d\tau_1d\tau_2$$

almost everywhere $[dt_1dt_2dt_3dt_4]$, where $F(t_1,t_3,t_2,t_4)=K'(t_1,t_2)K''(t_3,t_4)$, $F_1(t_1,t_3,\tau_1,\tau_2)=K_1'(t_1,\tau_1)K_1''(t_3,\tau_2)$, and $F_2(\tau_1,\tau_2,t_2,t_4)=K_2'(\tau_1,t_2)K_2''(\tau_2,t_4)$.

As a result, the product $K'(t_1,t_2)K''(t_3,t_4)$ is also the kernel of the trace class operator (see (2.1224)). Let us denote it by $\mathbb{K}'$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$. Then $\{\Psi_{j_1j_2}(x,y)\}_{j_1,j_2=0}^{\infty}=\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$ is an orthonormal basis in $L_2([t,T]^2)$.

Consider matrix trace of $\mathbb{K}'$. Using Fubini's Theorem, we obtain

$$\sum_{j_1,j_2=0}^{\infty}\left\langle\Psi_{j_1j_2},\mathbb{K}'\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^2}\phi_{j_2}(t_4)\phi_{j_1}(t_1)\int\limits_{[t,T]^2}K'(t_1,t_2)K''(t_3,t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)dt_2dt_3dt_1dt_4=$$

$$=\sum_{j_1,j_2=0}^{\infty}\Biggl(\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_4}f_3(t_3)\phi_{j_2}(t_3)\int\limits_{t_1}^T f_2(t_2)\phi_{j_1}(t_2)\times$$

$$\times dt_2dt_3dt_1dt_4+$$

$$+\int\limits_t^T f_3(t_4)\phi_{j_2}(t_4)\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_4}^T \psi_4(t_3)\phi_{j_2}(t_3)\int\limits_{t_1}^T f_2(t_2)\phi_{j_1}(t_2)\times$$

$$\times dt_2dt_3dt_1dt_4+$$

$$+\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^T f_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_4} f_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_1} \psi_1(t_2)\phi_{j_1}(t_2)\times$$

$$\times dt_2dt_3dt_1dt_4+$$

$$+\int\limits_t^T f_2(t_1)\phi_{j_1}(t_1)\int\limits_t^T \psi_4(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3} f_3(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_1} \psi_1(t_2)\phi_{j_1}(t_2)\times$$

$$\left.\times dt_2dt_4dt_3dt_1\right)=$$

$$=\sum_{j_1,j_2=0}^{\infty}\Biggl(\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4} f_3(t_3)\phi_{j_2}(t_3)\int\limits_t^T f_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4+$$

$$+\int\limits_t^T \psi_4(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3} f_3(t_4)\phi_{j_2}(t_4)\int\limits_t^T f_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_4dt_3+$$

$$+\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4} f_3(t_3)\phi_{j_2}(t_3)\int\limits_t^T f_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1} \psi_1(t_2)\phi_{j_1}(t_2)\times$$

$$\times dt_2dt_1dt_3dt_4+$$

$$
+\int\limits_t^T \psi_4(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3} f_3(t_4)\phi_{j_2}(t_4)\int\limits_t^T f_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1} \psi_1(t_2)\phi_{j_1}(t_2)\times
$$

$$
\left.\times dt_2dt_1dt_4dt_3\right) =
$$

$$
=4\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4} f_3(t_3)\phi_{j_2}(t_3)\int\limits_t^T f_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4. \qquad (2.1227)
$$

According to (2.1227) and (2.1221), we get

$$
\sum_{j_1,j_2=0}^{\infty}\langle\Psi_{j_1j_2},\mathbb{K}'\Psi_{j_1j_2}\rangle_{L_2([t,T]^2)} =
$$

$$
=4\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4} f_3(t_3)\phi_{j_2}(t_3)\int\limits_t^T f_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4 = \int\limits_{[t,T]^2}\lim_{u\to 0}A_uK'(t_2,t_2)K''(t_4,t_4)dt_2dt_4 =
$$

$$
=\int\limits_{[t,T]^2}\lim_{u\to 0}A_uK'(t_2,t_2)\lim_{u\to 0}A_uK''(t_4,t_4)dt_2dt_4 = \int\limits_{[t,T]^2}K'(t_2,t_2)K''(t_4,t_4)dt_2dt_4 =
$$

$$
=\int\limits_{[t,T]^2}\psi_4(t_4)f_3(t_4)f_2(t_2)\psi_1(t_2)dt_2dt_4. \qquad (2.1228)
$$

Recall that $f_2(\tau)$ and $f_3(\tau)$ are polynomials of finite degrees. For example, $f_2(\tau)$ and $f_3(\tau)$ can be Legendre polynomials that form a complete orthonormal system of functions in $L_2([t,T])$.

Denote

$$
s_q(t_2,t_3)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3), \qquad (2.1229)
$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$ and $C_{l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3) = \psi_2(t_2)\psi_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\psi_2(\tau), \psi_3(\tau) \in L_2([t,T]))$, i.e.

$$C_{l_2 l_1} = \int\limits_t^T \psi_3(t_3)\bar{\phi}_{l_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\bar{\phi}_{l_1}(t_2) dt_2 dt_3.$$

Further, we have

$$\lim_{q\to\infty} \int\limits_{[t,T]^2} \left(s_q(t_2,t_3) - g(t_2,t_3)\right)^2 dt_2 dt_3 = 0 \quad \text{or} \quad \lim_{q\to\infty} \|s_q - g\|^2_{L_2([t,T]^2)} = 0.$$

From (2.1228) we obtain (the sum on the right-hand side of (2.1229) is finite)

$$\sum_{j_1,j_2=0}^{\infty} \langle \Psi_{j_1 j_2}, \mathbb{K}'_q \Psi_{j_1 j_2}\rangle_{L_2([t,T]^2)} =$$

$$= 4 \sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_3<t_4\}}\psi_4(t_4)\phi_{j_2}(t_4)s_q(t_2,t_3)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1 dt_2 dt_3 dt_4 = \int\limits_{[t,T]^2} \psi_4(t_4) s_q(t_2,t_4)\psi_1(t_2) dt_2 dt_4, \qquad (2.1230)$$

where the operator $\mathbb{K}'_q$ (more precisely, its kernel) is obtained from the operator $\mathbb{K}'$ (more precisely, from its kernel) by replacing $f_2 f_3$ with s_q.

Note that the equality (2.1230) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$, i.e. the equality holds on a dense subset in $L_2([t,T]^2)$.

Trace class operators form a linear space. Therefore, on the left-hand side of (2.1230) there is a matrix trace of a trace class operator $\mathbb{K}'_q$. The mentioned matrix trace is a linear bounded (and therefore continuous) functional in the space of trace class operators [127], [128] (this functional can be extended to the space $L_2([t,T]^2)$ by continuity [147]).

From the other hand, the right-hand side of (2.1230) defines (as a scalar product of $s_q(t_2,t_4)$ and $\psi_4(t_4)\psi_1(t_2)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\psi_4(t_4)\psi_1(t_2)$. On the left-hand side of (2.1230) (by virtue of the equality

(2.1230)) there is a linear continuous functional on a dense subset in $L_2([t,T]^2)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^2)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in the equality (2.1230) (at that we suppose that s_q is defined by (2.1229))

$$\sum_{j_1,j_2=0}^{\infty}\left\langle \Psi_{j_1j_2}, \mathbb{K}''\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)}=$$

$$=4\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\int\limits_t^T\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4, \qquad (2.1231)$$

where the operator $\mathbb{K}''$ (more precisely, its kernel) is obtained from the operator $\mathbb{K}'_q$ (more precisely, from its kernel) by replacing s_q with $\lim\limits_{q\to\infty}s_q=g\in L_2([t,T]^2)$, $\psi_2(\tau),\psi_3(\tau)\in L_2([t,T])$ and $\psi_1(\tau),\psi_4(\tau)$ are continuously differentiable functions on $[t,T]$.

Further, the formula (2.1231) will remain valid if we choose

$$\psi_1(\tau)=\bar{\psi}_1^{(p)}(\tau),\quad \psi_4(\tau)=\bar{\psi}_4^{(p)}(\tau),$$

where

$$\bar{\psi}_1^{(p)}(\tau)=\sum_{j=0}^{p}\bar{\phi}_j(\tau)\int\limits_t^T\bar{\psi}_1(s)\bar{\phi}_j(s)ds,\quad \bar{\psi}_4^{(p)}(\tau)=\sum_{j=0}^{p}\bar{\phi}_j(\tau)\int\limits_t^T\bar{\psi}_4(s)\bar{\phi}_j(s)ds, \qquad (2.1232)$$

where $p\in\mathbf{N}$, $\bar{\psi}_1(\tau),\bar{\psi}_4(\tau)\in L_2([t,T])$, and $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$.

Substitute (2.1232) into (2.1231)

$$\sum_{j_1,j_2=0}^{\infty}\left\langle \Psi_{j_1j_2}, \mathbb{K}''_p\Psi_{j_1j_2}\right\rangle_{L_2([t,T]^2)}=$$

$$=4\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\bar{\psi}_4^{(p)}(t_4)\psi_3(t_3)\psi_2(t_2)\bar{\psi}_1^{(p)}(t_1)\phi_{j_2}(t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\phi_{j_1}(t_1)\times$$

$$
\times dt_1 dt_2 dt_3 dt_4 = \int\limits_t^T \bar{\psi}_4^{(p)}(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\bar{\psi}_1^{(p)}(t_2)dt_2 dt_4. \qquad (2.1233)
$$

where the operator $\mathbb{K}''_p$ (more precisely, its kernel) is obtained from the operator $\mathbb{K}''$ (more precisely, from its kernel) by replacing ψ_4 and ψ_1 with $\bar{\psi}_4^{(p)}$ and $\bar{\psi}_1^{(p)}$, respectively.

Note that the equality (2.1233) will also remain true if $\bar{\psi}_4^{(p)}\bar{\psi}_1^{(p)}$ is replaced by s_p (s_p is the partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$), i.e. the modified equality (2.1233) is true on a dense subset of $L_2([t,T]^2)$. Next, we can apply the reasoning below the formula (2.1230) and obtain the equality of two linear continuous functionals in $L_2([t,T]^2)$. Let us implement the passage to the limit $\lim\limits_{p\to\infty}$ in the mentioned equality under the condition $s_p = \bar{\psi}_4^{(p)}\bar{\psi}_1^{(p)}$

$$
4\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\bar{\psi}_4(t_4)\psi_3(t_3)\psi_2(t_2)\bar{\psi}_1(t_1)\phi_{j_2}(t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1 dt_2 dt_3 dt_4 = \int\limits_t^T \bar{\psi}_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\bar{\psi}_1(t_2)dt_2 dt_4, \qquad (2.1234)
$$

where $\bar{\psi}_1(\tau),\psi_2(\tau),\psi_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T])$.

Rewrite the equality (2.1234) in the form

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1j_1} =
$$

$$
= \sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1 dt_2 dt_3 dt_4 = \frac{1}{4}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2 dt_4, \qquad (2.1235)
$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (2.1235) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (2.1183) is proved.

In [118], the equality (2.1235) is generalized as follows

$$
\lim_{p\to\infty}\sum_{j_k,j_{k-2},\ldots,j_2=0}^{p} C_{j_kj_kj_{k-2}j_{k-2}\ldots j_2j_2} =
$$

$$
=\frac{1}{2^r}\int\limits_t^T \psi_k(t_k)\psi_{k-1}(t_k)\int\limits_t^{t_k}\psi_{k-2}(t_{k-2})\psi_{k-3}(t_{k-2})\ldots
$$

$$
\ldots\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{k-2}dt_k, \qquad (2.1236)
$$

where $k=2r$ $(r=2,3,\ldots)$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

The equalities (2.1184), (2.1185) can also be obtained [119] using the approach from [118] and the series on the left-hand sides of (2.1184), (2.1185) converge absolutely.

In the notations of Theorem 2.49, the equality (2.1236) can be written in the form

$$
\lim_{p\to\infty}\sum_{j_1,j_3,\ldots,j_{2r-1}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_1=j_2,\ldots,j_{2r-1}=j_{2r}} =
$$

$$
=\frac{1}{2^r}C_{j_k\ldots j_1}\Biggr|_{(j_2j_1)\curvearrowright(\cdot)(j_4j_3)\curvearrowright(\cdot)\ldots(j_{2r}j_{2r-1})\curvearrowright(\cdot),j_1=j_2,j_3=j_4,\ldots,j_{2r-1}=j_{2r}}, \qquad (2.1237)
$$

where $k=2r$ $(r=2,3,\ldots)$ and $C_{j_k\ldots j_1}$ is defined by (2.1061).

In principle, using the method from [118] the following equality can be obtained [119]

$$
\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$
=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}
$$

for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652)), where $k=2r$ $(r=2,3,\ldots)$, $C_{j_k\ldots j_1}$ is defined by (2.1061), another notations are the same as in Theorem 2.49.

2.27.4 Approach Based on Generalized Parseval's Equality and (2.125). General Case When $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ $(k=2r, r=2,3,4,\ldots)$

Let us prove the equalities (2.1183)–(2.1185) using a method based on generalized Parseval's equality and (2.125).

Consider (2.1183). Using (2.125), we have

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4=
$$

$$
=\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_4\times
$$

$$
\times\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\lim_{p\to\infty}\sum_{j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_4\times
$$

$$
\times\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\frac{1}{4}\int\limits_t^T\psi_4(t_4)\psi_3(t_4)dt_4\int\limits_t^T\psi_2(t_2)\psi_1(t_2)dt_2=
$$

$$
=\frac{1}{4}\int\limits_{[t,T]^2}\psi_4(t_4)\psi_3(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4, \qquad (2.1238)
$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Suppose that $\psi_2(\tau)$ and $\psi_3(\tau)$ are polynomials of finite degrees. For example, $\psi_2(\tau)$ and $\psi_3(\tau)$ can be Legendre polynomials that form a complete orthonormal system of functions in $L_2([t,T])$.

Denote

$$s_q(t_2,t_3)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3), \tag{2.1239}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3)=\bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau)\in L_2([t,T]))$, i.e.

$$C_{l_2l_1}=\int\limits_t^T\bar{\psi}_3(t_3)\bar{\phi}_{l_2}(t_3)\int\limits_t^{t_3}\bar{\psi}_2(t_2)\bar{\phi}_{l_1}(t_2)dt_2dt_3$$

and $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^2)}=0$.

From (2.1238) we obtain (the sum on the right-hand side of (2.1239) is finite)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_3<t_4\}}\psi_4(t_4)\phi_{j_2}(t_4)s_q(t_2,t_3)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{4}\int\limits_{[t,T]^2}\psi_4(t_4)s_q(t_2,t_4)\psi_1(t_2)dt_2dt_4. \tag{2.1240}$$

Note that the equality (2.1240) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$, i.e. the equality holds on a dense subset in $L_2([t,T]^2)$.

The right-hand side of (2.1240) defines (as a scalar product of $s_q(t_2,t_4)$ and $\frac{1}{4}\psi_4(t_4)\psi_1(t_2)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\frac{1}{4}\psi_4(t_4)\psi_1(t_2)$. On the left-hand side of (2.1240) (by virtue of the equality (2.1240)) there is a linear continuous functional on a dense subset in $L_2([t,T]^2)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^2)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1240) (at that we suppose that s_q is defined by (2.1239))

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_2}(t_3)\phi_{j_1}(t_2)\phi_{j_1}(t_1)\times$$

$$
\times dt_1dt_2dt_3dt_4 = \frac{1}{4}\int\limits_t^T \psi_4(t_4)\bar{\psi}_3(t_4)\int\limits_t^{t_4}\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4, \qquad (2.1241)
$$

where $\psi_1(\tau), \bar{\psi}_2(\tau), \bar{\psi}_3(\tau), \psi_4(\tau) \in L_2([t,T])$.

Rewrite the equality (2.1241) in the form

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1j_1} =
$$

$$
= \sum_{j_1,j_2=0}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4 = \frac{1}{4}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4, \qquad (2.1242)
$$

where $\psi_1(\tau),\ldots,\psi_4(\tau) \in L_2([t,T])$.

Note that the series on the left-hand side of (2.1242) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (2.1183) is proved.

Let us prove (2.1185). Using the generalized Parseval equality, we obtain

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{T}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4 =
$$

$$
= \sum_{j_1,j_2=0}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_4\times
$$

$$
\times\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 =
$$

$$
= \sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_4(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_4)dt_3dt_4\times
$$

$$
\times \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}} \psi_1(t_3)\psi_2(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_4)dt_3dt_4 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_4\}} \psi_3(t_3)\psi_2(t_4)\psi_4(t_4)\psi_1(t_3)dt_3dt_4 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_2\}} \psi_3(t_3)\psi_2(t_2)\psi_4(t_2)\psi_1(t_3)dt_3dt_2, \qquad (2.1243)
$$

where $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau), \psi_4(\tau) \in L_2([t,T])$.

Suppose that $\psi_2(\tau)$ and $\psi_3(\tau)$ are Legendre polynomials of finite degrees. Denote

$$
s_q(t_2,t_3) = \sum_{l_1,l_2=0}^{q} C_{l_2 l_1} \bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3), \qquad (2.1244)
$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3) = \bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau), \bar{\psi}_3(\tau) \in L_2([t,T]))$, i.e.

$$
C_{l_2 l_1} = \int\limits_t^T \bar{\psi}_3(t_3)\bar{\phi}_{l_2}(t_3) \int\limits_t^{t_3} \bar{\psi}_2(t_2)\bar{\phi}_{l_1}(t_2)dt_2dt_3
$$

and $\lim\limits_{q\to\infty} \|s_q - g\|^2_{L_2([t,T]^2)} = 0$.

From (2.1243) we obtain (the sum on the right-hand side of (2.1244) is finite)

$$
\sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_3<t_4\}}\psi_4(t_4)s_q(t_2,t_3)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times
$$

$$
\times dt_1dt_2dt_3dt_4 = \int\limits_{[t,T]^2} \mathbf{1}_{\{t_3<t_2\}} s_q(t_2,t_3)\psi_1(t_3)\psi_4(t_2)dt_3dt_2. \qquad (2.1245)
$$

Note that the equality (2.1245) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$, i.e. the equality holds on a dense subset in $L_2([t,T]^2)$.

The right-hand side of (2.1245) defines (as a scalar product of $s_q(t_2,t_3)$ and $\mathbf{1}_{\{t_3<t_2\}}\psi_1(t_3)\psi_4(t_2)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous)

functional in $L_2([t,T]^2)$, which is given by the function $\mathbf{1}_{\{t_3<t_2\}}\psi_1(t_3)\psi_4(t_2)$. On the left-hand side of (2.1245) (by virtue of the equality (2.1245)) there is a linear continuous functional on a dense subset in $L_2([t,T]^2)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^2)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1245) (at that we suppose that s_q is defined by (2.1244))

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2>t_3\}}\mathbf{1}_{\{t_2<t_3\}}\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_3)\psi_4(t_2)dt_3dt_2=0. \tag{2.1246}$$

Rewrite the equality (2.1246) in the form

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0, \tag{2.1247}$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (2.1247) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (2.1185) is proved.

Let us prove (2.1184). Using Fubini's Theorem and generalized Parseval's equality, we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{T}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1}^{\psi_4}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}C_{j_1}^{\psi_4}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Bigg|_{(j_2j_2)\frown(\cdot)}-$$

$$
-\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)=
$$

$$
=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \psi_4(s)\phi_{j_1}(s)ds\int\limits_t^T \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\phi_{j_1}(s)\psi_1(s)dsd\tau-
$$

$$
-\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)=
$$

$$
=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \psi_4(s)\phi_{j_1}(s)ds\int\limits_t^T \phi_{j_1}(s)\psi_1(s)\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau ds-
$$

$$
-\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi_4(s)\psi_1(s)\int\limits_s^T \psi_3(\tau)\psi_2(\tau)d\tau ds-
$$

$$
-\lim_{p\to\infty}\sum_{j_1=0}^{p} C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right), \qquad (2.1248)
$$

where $C_{j_1}^{\psi_4}$ and $C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}$ are defined by (2.1205).

Due to Cauchy–Bunyakovsky's inequality, Parseval's equality and (2.1174), we get

$$
\lim_{p\to\infty}\left(\sum_{j_1=0}^{p} C_{j_1}^{\psi_4}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)\right)^2\le
$$

$$
\le\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(C_{j_1}^{\psi_4}\right)^2\ \sum_{j_1=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2\le
$$

$$
\le\lim_{p\to\infty}\sum_{j_1=0}^{\infty}\left(C_{j_1}^{\psi_4}\right)^2\ \sum_{j_2=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2=
$$

$$
=\int\limits_t^T \psi_4^2(s)ds\ \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\frac{1}{2}C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p} C_{j_2j_2j_1}^{\psi_3\psi_2\psi_1}\right)^2=0. \quad (2.1249)
$$

Combining (2.1248) and (2.1249), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int\limits_t^T\psi_4(s)\psi_1(s)\int\limits_s^T\psi_3(\tau)\psi_2(\tau)d\tau ds=$$

$$=\frac{1}{2}\int\limits_{[t,T]^2}\psi_3(t_3)\psi_4(t_4)\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3, \qquad (2.1250)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Suppose that $\psi_3(\tau)$ and $\psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4), \qquad (2.1251)$$

where $\{\bar{\phi}_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$ and $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4)=\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_3<t_4\}}$ $(\bar{\psi}_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T]))$, i.e.

$$C_{l_2l_1}=\int\limits_t^T\bar{\psi}_4(t_4)\bar{\phi}_{l_2}(t_4)\int\limits_t^{t_4}\bar{\psi}_3(t_3)\bar{\phi}_{l_1}(t_3)dt_3dt_4$$

and $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^2)}=0$.

From (2.1250) we obtain (the sum on the right-hand side of (2.1251) is finite)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}\phi_{j_1}(t_4)\phi_{j_2}(t_3)s_q(t_3,t_4)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int\limits_{[t,T]^2}s_q(t_3,t_4)\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3. \qquad (2.1252)$$

Note that the equality (2.1252) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^2)$, i.e. the equality holds on a dense subset in $L_2([t,T]^2)$.

The right-hand side of (2.1252) defines (as a scalar product of $s_q(t_3,t_4)$ and $\frac{1}{2}\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)$ in $L_2([t,T]^2)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^2)$, which is given by the function $\frac{1}{2}\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)$. On the left-hand side of (2.1252) (by virtue of the equality (2.1252)) there is a linear continuous functional on a dense subset in $L_2([t,T]^2)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^2)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1252) (at that we suppose that s_q is defined by (2.1251))

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}\bar{\psi}_4(t_4)\phi_{j_1}(t_4)\bar{\psi}_3(t_3)\phi_{j_2}(t_3)\psi_2(t_2)\phi_{j_2}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=\frac{1}{2}\int\limits_{[t,T]^2}\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_3<t_4\}}\mathbf{1}_{\{t_4<t_3\}}\psi_1(t_4)\psi_2(t_3)dt_4dt_3=0. \tag{2.1253}$$

Rewrite the equality (2.1253) in the form

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4=0, \tag{2.1254}$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Note that the series on the left-hand side of (2.1254) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^2)$ is $\{\phi_{j_1}(x)\phi_{j_2}(y)\}_{j_1,j_2=0}^{\infty}$). The equality (2.1184) is proved. The equalities (2.1183)–(2.1185) are proved.

By induction we prove the following equality (i.e. by a different method compared with [118])

$$\lim_{p\to\infty}\sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{p}C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2j_2}=\frac{1}{2^r}\int\limits_t^T\psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times$$

$$\times \int\limits_t^{t_{2r}} \psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2}) \ldots \int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2)dt_2 \ldots dt_{2r-2}dt_{2r}, \quad (2.1255)$$

where $r \in \mathbf{N}$, $C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2j_2}$ is defined by

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_k \quad (k \in \mathbf{N}),$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\psi_1(\tau), \ldots, \psi_{2r}(\tau) \in L_2([t,T])$.

Note that the equality (2.1183) is a particular case of (2.1255) for $r = 2$ and the equality (2.125) is a particular case of (2.1255) for $r = 1$. Thus, the equality (2.1255) is true for $r = 1, 2$. Suppose that the equality (2.1255) is true for some $r > 2$. Then, using (2.125), we get

$$\lim_{p\to\infty} \sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p} \int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2}) \int\limits_t^{t_{2r+2}} \psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1}) \times$$

$$\times \int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r}) \int\limits_t^{t_{2r}} \psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1}) \ldots$$

$$\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_2}(t_1)dt_1dt_2 \ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2} =$$

$$= \sum_{j_{2r+2}=0}^{\infty} \int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2}) \int\limits_t^{t_{2r+2}} \psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1})dt_{2r+1}dt_{2r+2} \times$$

$$\times \sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{\infty} \int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r}) \int\limits_t^{t_{2r}} \psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1}) \times$$

$$\times \int\limits_t^{t_{2r-1}} \psi_{2r-2}(t_{2r-2})\phi_{j_{2r-2}}(t_{2r-2}) \int\limits_t^{t_{2r-2}} \psi_{2r-3}(t_{2r-3})\phi_{j_{2r-2}}(t_{2r-3}) \ldots$$

$$\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_2}(t_1)dt_1dt_2 \ldots dt_{2r-3}dt_{2r-2}dt_{2r-1}dt_{2r} =$$

$$
=\frac{1}{2}\int\limits_t^T \psi_{2r+2}(t_{2r+2})\psi_{2r+1}(t_{2r+2})dt_{2r+2}\cdot\frac{1}{2^r}\int\limits_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times
$$

$$
\times\int\limits_t^{t_{2r}}\psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}. \quad (2.1256)
$$

Let us rewrite the equality (2.1256) in the form

$$
\lim_{p\to\infty}\sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p}\int\limits_t^T \psi_{2r+2}(t_{2r+2})\phi_{j_{2r+2}}(t_{2r+2})\int\limits_t^{t_{2r+2}}\psi_{2r+1}(t_{2r+1})\phi_{j_{2r+2}}(t_{2r+1})\times
$$

$$
\times\int\limits_t^T \psi_{2r}(t_{2r})\phi_{j_{2r}}(t_{2r})\int\limits_t^{t_{2r}}\psi_{2r-1}(t_{2r-1})\phi_{j_{2r}}(t_{2r-1})\ldots
$$

$$
\ldots\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_2}(t_1)dt_1dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2}=
$$

$$
=\frac{1}{2^{r+1}}\int\limits_t^T \psi_{2r+2}(t_{2r+2})\psi_{2r+1}(t_{2r+2})\int\limits_t^T \psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times
$$

$$
\times\int\limits_t^{t_{2r}}\psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}dt_{2r+2}, \quad (2.1257)
$$

where $\psi_1(\tau),\ldots,\psi_{2r+2}(\tau)\in L_2([t,T])$.

Suppose that $\psi_1(\tau),\psi_3(\tau),\ldots,\psi_{2r-3}(\tau),\psi_{2r}(\tau),\psi_{2r+1}(\tau)$ in (2.1257) are Legendre polynomials of finite degrees. Denote

$$
h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})=
$$

$$
=\psi_2(t_2)\psi_4(t_4)\ldots\psi_{2r-2}(t_{2r-2})\psi_{2r-1}(t_{2r-1})\psi_{2r+2}(t_{2r+2}),
$$

$$
g(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})=
$$

$$
=\bar{\psi}_1(t_1)\bar{\psi}_3(t_3)\ldots\bar{\psi}_{2r-3}(t_{2r-3})\bar{\psi}_{2r}(t_{2r})\bar{\psi}_{2r+1}(t_{2r+1})\mathbf{1}_{\{t_{2r}<t_{2r+1}\}}, \quad (2.1258)
$$

$$
s_q(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})=
$$

$$
= \sum_{l_1,\ldots,l_{r+1}=0}^{q} C_{l_{r+1}\ldots l_1}\bar{\phi}_{l_1}(t_1)\bar{\phi}_{l_2}(t_3)\ldots\bar{\phi}_{l_{r-1}}(t_{2r-3})\bar{\phi}_{l_r}(t_{2r})\bar{\phi}_{l_{r+1}}(t_{2r+1}), \quad (2.1259)
$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$, $C_{l_{r+1}\ldots l_1}$ are Fourier–Legendre coefficients for the function (2.1258), and $\bar{\psi}_1(\tau), \bar{\psi}_3(\tau), \ldots, \bar{\psi}_{2r-3}(\tau), \bar{\psi}_{2r}(\tau), \bar{\psi}_{2r+1}(\tau) \in L_2([t,T])$. Then we have

$$
\lim_{q\to\infty} \|s_q - g\|^2_{L_2([t,T]^{r+1})} = 0.
$$

From (2.1257) we obtain (the sum on the right-hand side of (2.1259) is finite)

$$
\lim_{p\to\infty} \sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p} \int\limits_{[t,T]^{2r+2}} \mathbf{1}_{\{t_1<t_2<\ldots<t_{2r}\}}\mathbf{1}_{\{t_{2r+1}<t_{2r+2}\}} s_q(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})\times
$$

$$
\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})\times
$$

$$
\times \prod_{d=1}^{r+1} \phi_{j_{2d}}(t_{2d-1})\phi_{j_{2d}}(t_{2d}) dt_1 dt_2 \ldots dt_{2r-1} dt_{2r} dt_{2r+1} dt_{2r+2} =
$$

$$
= \frac{1}{2^{r+1}} \int\limits_{[t,T]^{r+1}} \mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}} s_q(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})\times
$$

$$
\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2}) dt_2 dt_4 \ldots dt_{2r-2} dt_{2r} dt_{2r+2}. \quad (2.1260)
$$

The right-hand side of the equality (2.1260) defines (as a scalar product of $s_q(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})$ and $\frac{1}{2^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})$ in the space $L_2([t,T]^{r+1})$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^{r+1})$. The mentioned functional is given by the function $\frac{1}{2^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})$.

Note that the equality (2.1260) will also remain true if s_q is replaced by $\bar{s}_q$ ($\bar{s}_q$ is the partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^{r+1})$), i.e. the modified equality (2.1260) is true on a dense subset in $L_2([t,T]^{r+1})$. On the left-hand side of (2.1260) (by virtue of the equality (2.1260)) there is a linear continuous functional on a dense subset in $L_2([t,T]^{r+1})$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^{r+1})$ (see [116], Theorem I.7, P. 9). Thus, we have the equality of two linear continuous functionals in $L_2([t,T]^{r+1})$. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in the mentioned equality if instead of $\bar{s}_q$ we choose s_q of the form (2.1259) (i.e. passage to the limit $\lim\limits_{q\to\infty}$ in (2.1260))

$$\lim_{p\to\infty}\sum_{j_{2r+2},j_{2r},\ldots,j_2=0}^{p}\ \int\limits_{[t,T]^{2r+2}}\mathbf{1}_{\{t_1<t_2<\ldots<t_{2r}\}}\mathbf{1}_{\{t_{2r+1}<t_{2r+2}\}}g(t_1,t_3,\ldots,t_{2r-3},t_{2r},t_{2r+1})\times$$

$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r-1},t_{2r+2})\times$$

$$\times\prod_{d=1}^{r+1}\phi_{j_{2d}}(t_{2d-1})\phi_{j_{2d}}(t_{2d})dt_1dt_2\ldots dt_{2r-1}dt_{2r}dt_{2r+1}dt_{2r+2}=$$

$$=\frac{1}{2^{r+1}}\int\limits_{[t,T]^{r+1}}\mathbf{1}_{\{t_2<t_4<\ldots<t_{2r}\}}g(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})\times$$

$$\times h(t_2,t_4,\ldots,t_{2r-2},t_{2r},t_{2r+2})dt_2dt_4\ldots dt_{2r-2}dt_{2r}dt_{2r+2},\qquad (2.1261)$$

where $\bar{\psi}_1(\tau),\bar{\psi}_3(\tau),\ldots,\bar{\psi}_{2r-3}(\tau)\bar{\psi}_{2r}(\tau),\bar{\psi}_{2r+1}(\tau)\in L_2([t,T])$.

It is easy to see that the equality (2.1261) (up to notations) is the equality (2.1255) in which r is replaced by $r+1$. So, we proved the equality (2.1255) by induction.

Note that the series on the left-hand side of (2.1255) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$).

Further, let us show that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\qquad (2.1262)$$

for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652)), where $k=2r$ $(r=2,3,\ldots)$, $C_{j_k\ldots j_1}$ is defined by (2.1061), another notations are the same as in Theorem 2.49.

The case

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=1$$

corresponds to (2.1255).

Thus, it remains to prove that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=0 \qquad (2.1263)$$

for the case

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=0.$$

Below we consider two examples that clearly explain the algorithm for the proof of equality (2.1263). After this we will formulate the algorithm.

First, let us prove that

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p} C_{j_3j_4j_4j_3j_1j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{t_6}\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=0, \qquad (2.1264)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 1. Using (2.1255) ($r=1$) and generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{T}\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6= \qquad (2.1265)$$

$$=\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\psi_6(t_6)\phi_{j_3}(t_6)dt_6\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3\times$$

$$\times \lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T \psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)dt_4dt_5\times$$

$$\times \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 =$$

$$= \int\limits_t^T \psi_6(t_6)\psi_3(t_6)dt_6\cdot\frac{1}{2}\int\limits_t^T \psi_5(t_4)\psi_4(t_4)dt_4\cdot\frac{1}{2}\int\limits_t^T \psi_2(t_2)\psi_1(t_2)dt_2. \quad (2.1266)$$

Let us rewrite (2.1266) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_t^T \psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^T \psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int\limits_t^T \psi_6(t_6)\psi_3(t_6)\int\limits_t^T \psi_5(t_4)\psi_4(t_4)\int\limits_t^T \psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \quad (2.1267)$$

Step 2. Suppose that $\psi_2(\tau),\psi_3(\tau),\psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2,t_3,t_4) = \sum_{l_1,l_2.l_3=0}^{q} C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3)\bar{\phi}_{l_3}(t_4), \quad (2.1268)$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in $L_2([t,T])$ and $C_{l_3l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3,t_4) = \bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau),\bar{\psi}_4(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0$.

From (2.1267) we obtain (the sum on the right-hand side of (2.1268) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_4<t_5\}}s_q(t_2,t_3,t_4)\psi_6(t_6)\psi_5(t_5)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\times$$

$$
\times\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$
=\frac{1}{4}\int\limits_{[t,T]^3} s_q(t_2,t_6,t_4)\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)dt_2dt_4dt_6. \qquad (2.1269)
$$

Note that the equality (2.1269) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (2.1269) defines (as a scalar product of $s_q(t_2,t_6,t_4)$ and $\frac{1}{4}\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)$. On the left-hand side of (2.1269) (by virtue of the equality (2.1269)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1269) (at that we suppose that s_q is defined by (2.1268))

$$
\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times
$$

$$
\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$
=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\bar{\psi}_3(t_6)\psi_5(t_4)\bar{\psi}_4(t_4)\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \qquad (2.1270)
$$

Rewrite the equality (2.1270) in the form

$$
\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times
$$

$$
\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$
=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6, \qquad (2.1271)
$$

where $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 3. Suppose that $\psi_3(\tau),\psi_4(\tau),\psi_1(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4,t_1)=\sum_{l_1,l_2.l_3=0}^{q}C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4)\bar{\phi}_{l_3}(t_1), \tag{2.1272}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (2.1268) and $C_{l_3l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4,t_1)=\bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\bar{\psi}_1(t_1)\mathbf{1}_{\{t_3<t_4\}}$ $(\bar{\psi}_3(\tau),\bar{\psi}_4(\tau),\bar{\psi}_1(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0$.

From (2.1271) we obtain (the sum on the right-hand side of (2.1272) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3\}}\mathbf{1}_{\{t_4<t_5\}}s_q(t_3,t_4,t_1)\psi_6(t_6)\psi_5(t_5)\psi_2(t_2)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}s_q(t_6,t_4,t_2)\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)dt_2dt_4dt_6. \tag{2.1273}$$

Note that the equality (2.1273) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (2.1273) defines (as a scalar product of $s_q(t_6,t_4,t_2)$ and $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_5(t_4)\psi_2(t_2)$. On the left-hand side of (2.1273) (by virtue of the equality (2.1273)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1273) (at that we suppose that s_q is defined by (2.1272))

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\psi_2(t_2)\bar{\psi}_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int\limits_{[t,T]^3} \mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_6(t_6)\bar{\psi}_3(t_6)\psi_5(t_4)\bar{\psi}_4(t_4)\psi_2(t_2)\bar{\psi}_1(t_2)dt_2dt_4dt_6. \tag{2.1274}$$

Rewrite (2.1274) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6} \mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int\limits_{[t,T]^3} \mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6, \tag{2.1275}$$

where $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2([t,T])$.

Step 4. Suppose that $\psi_5(\tau),\psi_6(\tau),\psi_2(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_5,t_6,t_2) = \sum_{l_1,l_2.l_3=0}^{q} C_{l_3l_2l_1}\bar{\phi}_{l_1}(t_5)\bar{\phi}_{l_2}(t_6)\bar{\phi}_{l_3}(t_2), \tag{2.1276}$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (2.1268) and $C_{l_3l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_5,t_6,t_2) = \bar{\psi}_5(t_5)\bar{\psi}_6(t_6)\bar{\psi}_2(t_2)\mathbf{1}_{\{t_5<t_6\}}$ $(\bar{\psi}_5(\tau),\bar{\psi}_6(\tau),\bar{\psi}_2(\tau)\in L_2([t,T]))$, i.e. $\lim\limits_{q\to\infty}\|s_q-g\|^2_{L_2([t,T]^3)}=0$.

From (2.1275) we obtain (the sum on the right-hand side of (2.1276) is finite)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6} \mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5\}}s_q(t_5,t_6,t_2)\psi_4(t_4)\psi_3(t_3)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4}\int\limits_{[t,T]^3} \mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}s_q(t_4,t_6,t_2)\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)dt_2dt_4dt_6. \tag{2.1277}$$

Note that the equality (2.1277) remains true when s_q is a partial sum of the Fourier–Legendre series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

The right-hand side of (2.1277) defines (as a scalar product of $s_q(t_4,t_6,t_2)$ and $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)$ in $L_2([t,T]^3)$) a linear bounded (and therefore continuous) functional in $L_2([t,T]^3)$, which is given by the function $\frac{1}{4}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\psi_3(t_6)\psi_4(t_4)\psi_1(t_2)$. On the left-hand side of (2.1277) (by virtue of the equality (2.1277)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$ (see [116], Theorem I.7, P. 9).

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1277) (at that we suppose that s_q is defined by (2.1276))

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2<t_3<t_4<t_5<t_6\}}\bar{\psi}_6(t_6)\bar{\psi}_5(t_5)\psi_4(t_4)\psi_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\mathbf{1}_{\{t_6<t_4\}}\mathbf{1}_{\{t_4<t_6\}}\bar{\psi}_6(t_6)\psi_3(t_6)\bar{\psi}_5(t_4)\psi_4(t_4)\bar{\psi}_2(t_2)\psi_1(t_2)dt_2dt_4dt_6=0. \tag{2.1278}$$

It is obvious that the equality (2.1278) (up to notations) is (2.1264). The equality (2.1264) is proved.

As a second example, we will prove the equality (2.1185). In this case, we will use the same approach as in the proof of equality (2.1264). Thus, we prove that

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}=0. \tag{2.1279}$$

Step 1. Using generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^T\psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4= \tag{2.1280}$$

$$
= \lim_{p\to\infty}\sum_{j_2=0}^{p}\int\limits_t^T \psi_4(t_4)\phi_{j_2}(t_4)dt_4\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)dt_2\times
$$

$$
\times \lim_{p\to\infty}\sum_{j_1=0}^{p}\int\limits_t^T \psi_3(t_3)\phi_{j_1}(t_3)dt_3\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)dt_1 =
$$

$$
= \int\limits_t^T \psi_4(t_4)\psi_2(t_4)dt_4\int\limits_t^T \psi_3(t_3)\psi_1(t_3)dt_3. \qquad (2.1281)
$$

Rewrite the equality (2.1281) in the form

$$
\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4} \psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =
$$

$$
= \int\limits_{[t,T]^2} \psi_4(t_4)\psi_2(t_4)\psi_3(t_2)\psi_1(t_2)dt_2dt_4. \qquad (2.1282)
$$

Step 2. Suppose that $\psi_1(\tau),\psi_2(\tau)$ are Legendre polynomials of finite degrees. Denote

$$
s_q(t_1,t_2)=\sum_{l_1,l_2=0}^{q} C_{l_2l_1}\bar{\phi}_{l_1}(t_1)\bar{\phi}_{l_2}(t_2),
$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (2.1268), $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_1,t_2)=\bar{\psi}_1(t_1)\bar{\psi}_2(t_2)\mathbf{1}_{\{t_1<t_2\}}$ $(\bar{\psi}_1(\tau),\bar{\psi}_2(\tau)\in L_2([t,T]))$.

From (2.1282) we obtain

$$
\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4} s_q(t_1,t_2)\psi_4(t_4)\psi_3(t_3)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =
$$

$$
= \int\limits_{[t,T]^2} s_q(t_2,t_4)\psi_4(t_4)\psi_3(t_2)dt_2dt_4. \qquad (2.1283)
$$

The left-hand and right-hand sides of (2.1283) define linear continuous functionals in $L_2([t,T]^2)$ (see explanation earlier in this section). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1283)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\psi_4(t_4)\psi_3(t_3)\bar{\psi}_2(t_2)\bar{\psi}_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4(t_4)\bar{\psi}_2(t_4)\psi_3(t_2)\bar{\psi}_1(t_2)dt_2dt_4. \qquad (2.1284)$$

Rewrite the equality (2.1284) in the form

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4(t_4)\psi_2(t_4)\psi_3(t_2)\psi_1(t_2)dt_2dt_4, \qquad (2.1285)$$

where $\psi_1(\tau),\ldots,\psi_4(\tau)\in L_2([t,T])$.

Step 3. Suppose that $\psi_2(\tau),\psi_3(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_2,t_3)=\sum_{l_1,l_2=0}^{q}C_{l_2l_1}\bar{\phi}_{l_1}(t_2)\bar{\phi}_{l_2}(t_3),$$

where $\left\{\bar{\phi}_j(x)\right\}_{j=0}^{\infty}$ as in (2.1268), $C_{l_2l_1}$ are Fourier–Legendre coefficients for the function $g(t_2,t_3)=\bar{\psi}_2(t_2)\bar{\psi}_3(t_3)\mathbf{1}_{\{t_2<t_3\}}$ $(\bar{\psi}_2(\tau),\bar{\psi}_3(\tau)\in L_2([t,T]))$.

From (2.1285) we obtain

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2\}}s_q(t_2,t_3)\psi_4(t_4)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}s_q(t_4,t_2)\psi_4(t_4)\psi_1(t_2)dt_2dt_4. \qquad (2.1286)$$

The left-hand and right-hand sides of (2.1286) define linear continuous functionals in $L_2([t,T]^2)$. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1286)

$$\sum_{j_1,j_2=0}^{\infty}\int\limits_{[t,T]^4}\mathbf{1}_{\{t_1<t_2<t_3\}}\psi_4(t_4)\bar{\psi}_3(t_3)\bar{\psi}_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_2<t_4\}} \mathbf{1}_{\{t_4<t_2\}} \psi_4(t_4)\bar{\psi}_2(t_4)\bar{\psi}_3(t_2)\psi_1(t_2)dt_2 dt_4 = 0. \qquad (2.1287)$$

Rewrite the equality (2.1287) in the form

$$\sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2<t_3\}} \psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1 dt_2 dt_3 dt_4 = 0. \qquad (2.1288)$$

Step 4. Suppose that $\psi_3(\tau), \psi_4(\tau)$ are Legendre polynomials of finite degrees. Denote

$$s_q(t_3,t_4) = \sum_{l_1,l_2=0}^{q} C_{l_2 l_1} \bar{\phi}_{l_1}(t_3)\bar{\phi}_{l_2}(t_4),$$

where $\{\bar{\phi}_j(x)\}_{j=0}^{\infty}$ as in (2.1268), $C_{l_2 l_1}$ are Fourier–Legendre coefficients for the function $g(t_3,t_4) = \bar{\psi}_3(t_3)\bar{\psi}_4(t_4)\mathbf{1}_{\{t_3<t_4\}}$ $(\bar{\psi}_3(\tau), \bar{\psi}_4(\tau) \in L_2([t,T]))$.

From (2.1288) we obtain

$$\sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2<t_3\}} s_q(t_3,t_4)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1 dt_2 dt_3 dt_4 = 0. \qquad (2.1289)$$

The left-hand and right-hand sides of (2.1289) define linear continuous functionals in $L_2([t,T]^2)$ (we interpret the right-hand side of (2.1289) as the zero functional in $L_2([t,T]^2)$). Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (2.1289)

$$\sum_{j_1,j_2=0}^{\infty} \int\limits_{[t,T]^4} \mathbf{1}_{\{t_1<t_2<t_3<t_4\}} \bar{\psi}_4(t_4)\bar{\psi}_3(t_3)\psi_2(t_2)\psi_1(t_1)\phi_{j_2}(t_4)\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\times$$

$$\times dt_1 dt_2 dt_3 dt_4 = 0. \qquad (2.1290)$$

It is easy to see that the equality (2.1290) (up to notations) is the equality (2.1185). The equality (2.1185) is proved.

Let us formulate the ideas used when considering the two above examples in the form of an algorithm.

Step 1. Suppose $k = 2r$ $(r = 2, 3, 4, \ldots)$, where r is the number of pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (2.652)). Let us select blocks in the multi-index $j_k \ldots j_1$ that correspond to the fulfillment of the condition

$$\prod_{l=1}^{r_d} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1,$$

where r_d is the number of pairs (see (2.652)) in the block with number d.

Step 2. Let us write the Volterra–type kernel (2.1194) in the form

$$K(t_1, \ldots, t_k) = \psi_1(t_1) \ldots \psi_k(t_k) \mathbf{1}_{\{t_1<t_2\}} \mathbf{1}_{\{t_2<t_3\}} \ldots \mathbf{1}_{\{t_{k-1}<t_k\}}, \quad (2.1291)$$

where $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$, $t_1, \ldots, t_k \in [t, T]$, $k \geq 4$.

Let us save multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the expression (2.1291) that correspond to the above blocks. At that, we remove the remaining multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ from the expression (2.1291). As a result, we get a modified kernel $\bar{K}(t_1, \ldots, t_k)$. Let us write an analogue of the left-hand side of equality (2.1263) for the modified kernel $\bar{K}(t_1, \ldots, t_k)$ (see (2.1265) and (2.1280) as examples). For definiteness, let us denote this expression by $(^-)$.

Step 3. Using generalized Parseval's equality and (2.1255), we represent the expression $(^-)$ as an integral over the hypercube $[t, T]^r$ (see the right-hand sides of (2.1267) and (2.1282) as examples). For definiteness, let us denote the obtained equality by $(\bar{K})$ ((2.1267) and (2.1282) are examples of $(\bar{K})$).

Step 4. Further, transformations and passages to the limit in the equality $(\bar{K})$ are performed iteratively in such a way as to restore the removed multipliers $\mathbf{1}_{\{t_n<t_{n+1}\}}$ on the left-hand side of $(\bar{K})$ (for more details, see the proof of formulas (2.1264), (2.1279)). As a result, we obtain the equality (2.1263). More precisely, we can move from left to right along a multi-index corresponding to the left-hand side of $(\bar{K})$. Let us assume that at the n-th step we need to restore the multiplier $\mathbf{1}_{\{t_n<t_{n+1}\}}$. Then the function g (see the proof of formulas (2.1264), (2.1279)) will be the product of $\mathbf{1}_{\{t_n<t_{n+1}\}} \psi_n(t_n) \psi_{n+1}(t_{n+1})$ and $r - 2$ weight functions that are chosen so that on the right-hand side of the equality $(\bar{K})$ there is a scalar product in $L_2([t, T]^r)$ involving s_q (s_q is an approximation of g).

Using the above algorithm, we prove the equality (2.1262) for the case $k = 2r$ $(r = 2, 3, \ldots)$. The equality (2.1262) is proved.

Note that the series on the left-hand side of (2.1262) converges absolutly since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$).

2.28 Revision of Hypotheses on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k\in\mathbf{N}$)

In Sect. 2.5, we formulated three hypotheses on expansion of iterated Stratonovich stochastic integrals based on the results obtained by the author in the 2010s. In light of recent results (Theorems 2.3, 2.42–2.57), a new vision of the above problem has appeared. In particular, it became clear that it is possible to methodically obtain results related to the expansion of iterated Stratonovich stochastic integrals for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$.

Definition (2.3) of the Stratonovich stochastic integral, which we mainly use in this book, imposes its own limitations. In particular, this definition assumes that $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)$ are continuous functions at the interval $[t,T]$.

Based on Theorems 2.3, 2.42–2.57, we formulate the following hypothesis on expansion of the sum $\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ of iterated Itô stochastic integrals (see (2.962)).

Hypothesis 2.4. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then, for the sum $\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ of iterated Itô stochastic integrals*

$$
\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}
$$

the following expansion

$$
\bar{J}^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\qquad(2.1292)
$$

that converges in the mean-square sense is valid, where

$$
C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k
$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*; another notations are the same as in Theorem* 2.12*.*

Using Theorem 2.12, we obtain the following hypothesis.

Hypothesis 2.5. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity* k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

2.29 Proof of Hypotheses 2.4, 2.5 Under the Condition (2.1293) for the Case $k \ge 2r$, $p_1 = \ldots = p_k = p$ and Under Some Additional Assumptions

Suppose that the equality

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \qquad (2.1293)$$

is satisfied for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (2.652)) and for any fixed $j_1, \ldots, j_q, \ldots, j_k$ ($q \ne g_1, g_2, \ldots, g_{2r-1},\ g_{2r}$), where $k \ge 2r$, $r = 1, 2, \ldots, [k/2]$, $C_{j_k\ldots j_1}$ is defined by (2.1061), another notations are the same as in Theorem 2.49. Recall that the case $k = 2r$ is considered in Sect. 2.27.4.

Moreover, suppose that the series

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

converges absolutly for any fixed $j_1, \ldots, j_q, \ldots, j_k$, where $q \ne g_1, g_2, \ldots, g_{2r-1}$, g_{2r} and $k > 2r$.

It should be noted that the above assumptions will be proved further (see Sect. 2.30).

Hypotheses 2.4 and 2.5 will be proved for the case $p_1 = \ldots = p_k = p$ if we prove that (see Theorem 2.49 for the case $p_1 = \ldots = p_k = p$)

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2 = 0 \qquad (2.1294)$$

for all $r = 1, 2, \ldots, [k/2]$, where notations are the same as in (2.1293).

We have

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2 \le$$

$$\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2, \tag{2.1295}$$

where

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty} \stackrel{\mathsf{def}}{=} \lim_{q\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q}. \tag{2.1296}$$

Consider the following analogue of Monotone Convergence Theorem for infinite series.

Proposition 2.5. *Suppose that $x_{m,n}\ge 0$ for all $m,n\in\mathbf{N}$,*

$$\lim_{m\to\infty}x_{m,n}=y_n \quad (\textit{for any fixed } n\in\mathbf{N}),$$

and $x_{m,n}\le x_{m+1,n}$ for all $m\in\mathbf{N}$ and for any fixed $n\in\mathbf{N}$. Then

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}x_{m,n}=\sum_{n=1}^{\infty}\lim_{m\to\infty}x_{m,n}=\sum_{n=1}^{\infty}y_n. \tag{2.1297}$$

Proof. Proposition 2.5 can be easily proved using the following version of Fatou's Lemma for infinite series

$$\sum_{n=1}^{\infty}\liminf_{m\to\infty}x_{m,n}\le\liminf_{m\to\infty}\sum_{n=1}^{\infty}x_{m,n}, \tag{2.1298}$$

where it is assumed that the conditions of Proposition 2.5 are fulfilled. Indeed, we have

$$0\le x_{m,n}\le y_n.$$

Then

$$\sum_{n=1}^{\infty} x_{m,n} \le \sum_{n=1}^{\infty} y_n$$

and (see (2.1298))

$$\limsup_{m\to\infty} \sum_{n=1}^{\infty} x_{m,n} \le \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \liminf_{m\to\infty} x_{m,n} \le \liminf_{m\to\infty} \sum_{n=1}^{\infty} x_{m,n}. \qquad (2.1299)$$

From (2.1299) we get

$$\sum_{n=1}^{\infty} y_n = \liminf_{m\to\infty} \sum_{n=1}^{\infty} x_{m,n} = \limsup_{m\to\infty} \sum_{n=1}^{\infty} x_{m,n} = \lim_{m\to\infty} \sum_{n=1}^{\infty} x_{m,n},$$

i.e. the equality (2.1297) is proved.

To prove (2.1298) we note that

$$\inf_{j\ge m} x_{j,n} \le x_{k,n} \quad (\forall k \ge m).$$

Then

$$\sum_{n=1}^{N} \inf_{j\ge m} x_{j,n} \le \sum_{n=1}^{N} x_{k,n} \quad (\forall k \ge m)$$

and

$$\sum_{n=1}^{N} \inf_{j\ge m} x_{j,n} \le \inf_{k\ge m} \sum_{n=1}^{N} x_{k,n} \le \inf_{k\ge m} \sum_{n=1}^{\infty} x_{k,n}. \qquad (2.1300)$$

Passing to the limit $\lim\limits_{m\to\infty}$ in (2.1300), we obtain

$$\sum_{n=1}^{N} \lim_{m\to\infty} \inf_{j\ge m} x_{j,n} \le \lim_{m\to\infty} \inf_{k\ge m} \sum_{n=1}^{\infty} x_{k,n}. \qquad (2.1301)$$

Passing to the limit $\lim\limits_{N\to\infty}$ in (2.1301), we get

$$\sum_{n=1}^{\infty} \lim_{m\to\infty} \inf_{j\ge m} x_{j,n} \le \lim_{m\to\infty} \inf_{k\ge m} \sum_{n=1}^{\infty} x_{k,n},$$

i.e. the equality (2.1298) is satisfied. Proposition 2.5 is proved.

Proposition 2.6. *Suppose that*

$$\sum_{j=1}^{\infty} g_{j,n} = 0, \tag{2.1302}$$

the series (2.1302) converges absolutely for any fixed $n \in \mathbf{N}$ and

$$\sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} |g_{j,n}|\right)^2 < \infty.$$

Then

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} g_{j,n}\right)^2 = \sum_{n=1}^{\infty} \lim_{m\to\infty} \left(\sum_{j=1}^{m} g_{j,n}\right)^2 = 0. \tag{2.1303}$$

Proof. We have $g_{j,n} = g^+_{j,n} - g^-_{j,n}$, $|g_{j,n}| = g^+_{j,n} + g^-_{j,n}$, where

$$g^+_{j,n} = \max\{g_{j,n}, 0\} = \frac{1}{2}\left(|g_{j,n}| + g_{j,n}\right) \geq 0,$$

$$g^-_{j,n} = -\min\{g_{j,n}, 0\} = \frac{1}{2}\left(|g_{j,n}| - g_{j,n}\right) \geq 0.$$

Moreover,

$$\sum_{j=1}^{\infty} g_{j,n} = \sum_{j=1}^{\infty} g^+_{j,n} - \sum_{j=1}^{\infty} g^-_{j,n} = 0, \tag{2.1304}$$

$$\sum_{j=1}^{\infty} |g_{j,n}| = \sum_{j=1}^{\infty} g^+_{j,n} + \sum_{j=1}^{\infty} g^-_{j,n} = 2\sum_{j=1}^{\infty} g^+_{j,n} = 2\sum_{j=1}^{\infty} g^-_{j,n}. \tag{2.1305}$$

Since the series (2.1302) converges absolutely, then by virtue of the equality (2.1305) the series (with non-negative terms) on the right-hand side of (2.1305) and on the right-hand side of (2.1304) converge.

Further, using Proposition 2.5 and (2.1304), (2.1305), we obtain

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} g_{j,n}\right)^2 = \lim_{m\to\infty} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} g^+_{j,n} - \sum_{j=1}^{m} g^-_{j,n}\right)^2 =$$

$$= \lim_{m\to\infty} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} g^+_{j,n}\right)^2 - \lim_{m\to\infty} \sum_{n=1}^{\infty} \left(2\sum_{j=1}^{m} g^+_{j,n} \sum_{j=1}^{m} g^-_{j,n}\right) +$$

$$
+\lim_{m\to\infty}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{m}g_{j,n}^{-}\right)^2=
$$

$$
=\sum_{n=1}^{\infty}\lim_{m\to\infty}\left(\sum_{j=1}^{m}g_{j,n}^{+}\right)^2-\sum_{n=1}^{\infty}\lim_{m\to\infty}\left(2\sum_{j=1}^{m}g_{j,n}^{+}\sum_{j=1}^{m}g_{j,n}^{-}\right)+
$$

$$
+\sum_{n=1}^{\infty}\lim_{m\to\infty}\left(\sum_{j=1}^{m}g_{j,n}^{-}\right)^2=
$$

$$
=\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}g_{j,n}^{+}\right)^2-2\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}g_{j,n}^{+}\sum_{j=1}^{\infty}g_{j,n}^{-}\right)+
$$

$$
+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}g_{j,n}^{-}\right)^2=
$$

$$
=\frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}|g_{j,n}|\right)^2-\frac{1}{2}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}|g_{j,n}|\right)^2+
$$

$$
+\frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}|g_{j,n}|\right)^2=0.
$$

Proposition 2.6 is proved.

It is easy to see that by analogy with Propositions 2.5, 2.6 the following statements can be proved.

Proposition 2.7. *Suppose that $h_{p,k_1,\ldots,k_d}\ge 0$ for all $p\in\mathbf{N}$ and for any fixed $k_1,\ldots,k_d\in\mathbf{N}$,*

$$
\lim_{p\to\infty}h_{p,k_1,\ldots,k_d}=u_{k_1,\ldots,k_d}\quad (\textit{for any fixed } k_1,\ldots,k_d\in\mathbf{N}),
$$

and $h_{p,k_1,\ldots,k_d}\le h_{p+1,k_1,\ldots,k_d}$ for all $p\in\mathbf{N}$ and for any fixed $k_1,\ldots,k_d\in\mathbf{N}$. Then

$$
\lim_{p\to\infty}\sum_{k_1,\ldots,k_d=1}^{\infty}h_{p,k_1,\ldots,k_d}=\sum_{k_1,\ldots,k_d=1}^{\infty}\lim_{p\to\infty}h_{p,k_1,\ldots,k_d}=\sum_{k_1,\ldots,k_d=1}^{\infty}u_{k_1,\ldots,k_d},\quad (2.1306)
$$

where $h_{p,k_1,\ldots,k_d},u_{k_1,\ldots,k_d}\in\mathbf{R}$, $d\in\mathbf{N}$, the series on the left-hand side of (2.1306) is understood in the same sense as in (2.1296).

Proposition 2.8. *Suppose that*

$$\lim_{p\to\infty}\sum_{j_1,\ldots,j_q=1}^{p} h_{j_1,\ldots,j_q,k_1,\ldots,k_d} \stackrel{\sf def}{=} \sum_{j_1,\ldots,j_q=1}^{\infty} h_{j_1,\ldots,j_q,k_1,\ldots,k_d}=0, \qquad (2.1307)$$

the series (2.1307) *converges absolutely for any fixed* $k_1,\ldots,k_d\in\mathbf{N}$ *and*

$$\sum_{k_1,\ldots,k_d=1}^{\infty}\left(\sum_{j_1,\ldots,j_q=1}^{\infty}\left|h_{j_1,\ldots,j_q,k_1,\ldots,k_d}\right|\right)^2<\infty.$$

Then

$$\lim_{p\to\infty}\sum_{k_1,\ldots,k_d=1}^{\infty}\left(\sum_{j_1,\ldots,j_q=1}^{p} h_{j_1,\ldots,j_q,k_1,\ldots,k_d}\right)^2=$$

$$=\sum_{k_1,\ldots,k_d=1}^{\infty}\lim_{p\to\infty}\left(\sum_{j_1,\ldots,j_q=1}^{p} h_{j_1,\ldots,j_q,k_1,\ldots,k_d}\right)^2=0,$$

where

$$\lim_{n\to\infty}\sum_{k_1,\ldots,k_d=1}^{n} \stackrel{\sf def}{=} \sum_{k_1,\ldots,k_d=1}^{\infty},$$

$h_{j_1,\ldots,j_q,k_1,\ldots,k_d}\in\mathbf{R}$ *and* $d,q\in\mathbf{N}$.

Obviously, Proposition 2.8 follows from Proposition 2.7 in the same way as Proposition 2.6 follows from Proposition 2.5. Applying Proposition 2.8 to the right-hand side of (2.1295) (using (2.1293) and the absolute convergence of the series on the left-hand side of (2.1293)), we obtain (2.1294). At that, we used the conditions

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\left|C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right|\right)^2<\infty, \qquad (2.1308)$$

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2<\infty. \qquad (2.1309)$$

Note that (2.1309) follows from the Parseval equality since the expression

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\ \stackrel{\rm def}{=}\ H_{j_{q_1}\ldots j_{q_{k-2r}}}$$

is a finite linear combination of the Fourier coefficients of $L_2([t,T]^{k-2r})$–functions after iteratively applying transformations (2.1315), (2.1316) (see Sect. 2.30) to $H_{j_{q_1}\ldots j_{q_{k-2r}}}$ for integrations not involving the basis functions $\phi_{j_{q_1}},\ldots,\phi_{j_{q_{k-2r}}}$.

Let us consider another sufficient condition under which the equality (2.1294) is satisfied. Suppose that $k>2r$ and

$$\exists\ \lim\limits_{p,q\to\infty}\sum\limits_{\substack{j_1,\ldots,j_m,\ldots,j_k=0\\ m\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q}\Biggl(\sum\limits_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^r {\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2<\infty \tag{2.1310}$$

for all $r=1,2,\ldots,[k/2]$, where notations are the same as in (2.1293). Then, by Proposition 1.1 (see Sect. 1.7.2) and (2.1293) we obtain

$$\lim\limits_{p,q\to\infty}\sum\limits_{\substack{j_1,\ldots,j_m,\ldots,j_k=0\\ m\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q}\Biggl(\sum\limits_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^r {\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=$$

$$=\lim\limits_{q\to\infty}\sum\limits_{\substack{j_1,\ldots,j_m,\ldots,j_k=0\\ m\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q}\lim\limits_{p\to\infty}\Biggl(\sum\limits_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^r {\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=0.$$

Thus, we get

$$\lim_{p,q\to\infty}\sum_{\substack{j_1,\ldots,j_m,\ldots,j_k=0\\ m\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{q}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2=0. \qquad (2.1311)$$

Substituting $p=q$ in (2.1311), we obtain (2.1294) (recall that the case $k=2r$ is proved in Sect. 2.27.4).

As a result, Hypotheses 2.4 and 2.5 are proved under the conditions formulated above in this section.

2.30 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k\in\mathbf{N}$). The Case of an Arbitrary Complete Orthonormal System of Functions in $L_2([t,T])$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Proof of Hypotheses 2.4, 2.5 for the Case $p_1=\ldots=p_k=p$ and Under the Condition (2.1310)

This section is devoted to the following two theorems.

Theorem 2.58. *Suppose that the condition* (2.1310) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$$

the following expansion

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (2.1312)$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$; another notations are the same as in Theorem 2.12.

Using Theorem 2.12, we obtain the following corollary of Theorem 2.58.

Theorem 2.59. *Suppose that the condition (2.1310) is satisfied, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous functions at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of multiplicity k $(k\in\mathbf{N})$*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \qquad (2.1313)$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Proof of Theorem 2.58. According to the results of Sect. 2.29, Theorem 2.58 will be proved if we prove (see (2.1293)) that the equality

$$\lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}} \qquad (2.1314)$$

is satisfied for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (2.652)), where $k \ge 2r$, $r = 1, 2, \ldots, [k/2]$, $C_{j_k\ldots j_1}$ is defined by (2.1312), another notations are the same as in Theorem 2.49.

Moreover (assuming that (2.1314) is proved), the series on the left-hand side of (2.1314) converges absolutly (the case $k = 2r$ (see Sect. 2.27.4)) and converges absolutly for any fixed $j_1, \ldots, j_q, \ldots, j_k$ and $q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (the case $k > 2r$) since its sum does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$). Recall that any permutation of basis functions in a Hilbert space forms a basis in this Hilbert space [127].

The case $k = 2r$ of (2.1314) is considered in Sect. 2.27.4. Consider the case $k > 2r$ of (2.1314).

Using Fubini's Theorem, we obtain

$$\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots$$

$$\ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l) dt_l \times$$

$$\times dt_{l-1} \ldots dt_2 dt_1 dt_{l+1} \ldots dt_k =$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\times
$$

$$
\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k-
$$

$$
- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\times
$$

$$
\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots
$$

$$
\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k-
$$

$$
- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots
$$

$$
\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (2.1315)
$$

where $2 < l < k-1$ and $h_1(\tau),\ldots,h_k(\tau) \in L_2([t,T])$.

By analogy with (2.1315) we have for $l = k$

$$
\int\limits_t^T h_l(t_l) \int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_l =
$$

$$
= \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2)\ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^T h_l(t_l)dt_l dt_{l-1}\ldots dt_2 dt_1 =
$$

$$
= \left(\int\limits_t^T h_l(t_l)dt_l\right) \int\limits_t^T h_1(t_1) \int\limits_{t_1}^T h_2(t_2)\ldots \int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1}\ldots dt_2 dt_1-
$$

$$
-\int\limits_{t}^{T}h_1(t_1)\int\limits_{t_1}^{T}h_2(t_2)\ldots\int\limits_{t_{l-2}}^{T}h_{l-1}(t_{l-1})\left(\int\limits_{t}^{t_{l-1}}h_l(t_l)dt_l\right)dt_{l-1}\ldots dt_2dt_1=
$$

$$
=\left(\int\limits_{t}^{T}h_l(t_l)dt_l\right)\int\limits_{t}^{T}h_{l-1}(t_{l-1})\ldots\int\limits_{t}^{t_2}h_1(t_1)dt_1\ldots dt_{l-1}-
$$

$$
-\int\limits_{t}^{T}h_{l-1}(t_{l-1})\left(\int\limits_{t}^{t_{l-1}}h_l(t_l)dt_l\right)\int\limits_{t}^{t_{l-1}}h_{l-2}(t_{l-2})\ldots\int\limits_{t}^{t_2}h_1(t_1)dt_1\ldots dt_{l-1}. \qquad (2.1316)
$$

We will assume that for $l=1$ the transformation (2.1315) is not carried out since

$$
\int\limits_{t}^{t_2}h_1(t_1)dt_1
$$

is the innermost integral on the left-hand side of (2.1315). The formulas (2.1315), (2.1316) will be used further.

Let us carry out the transformations (2.1315), (2.1316) for

$$
C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}
$$

iteratively for $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$. As a result, we obtain

$$
C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=
$$

$$
=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\left(\hat C^{(d)}_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\bar C^{(d)}_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right), \qquad (2.1317)
$$

where some terms in the sum

$$
\sum_{d=1}^{2^{k-2r}}
$$

can be identically equal to zero due to the remark to (2.1315), (2.1316).

Using (2.1317), we get

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. - \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right) =$$

$$= \sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. - \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right). \tag{2.1318}$$

Further, consider 3 possible cases.

Case 1. The quantities

$$\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}, \quad \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{2.1319}$$

are such that

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1 \tag{2.1320}$$

for $d = 1, 2, \ldots, 2^{k-2r}$ and

$$C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \tag{2.1321}$$

is such that the condition (2.1320) is fulfilled for (2.1321).

Case 2. The quantities (2.1319) are such that the condition (2.1320) is satisfied for $d = 1, 2, \ldots, 2^{k-2r}$ and (2.1321) is such that the condition

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0 \tag{2.1322}$$

is fulfilled for (2.1321).

Case 3. The quantities (2.1319) are such that the condition (2.1322) is satisfied for $d = 1, 2, \ldots, 2^{k-2r}$ and (2.1321) is such that the condition (2.1322) is fulfilled for (2.1321).

For Case 1, applying (2.1314) for the case $k = 2r$ and (2.1318), we get for any fixed $j_1, \ldots, j_q, \ldots, j_k$ $(q \ne g_1, g_2, \ldots, g_{2r-1},\ g_{2r})$

$$
\lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$
= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \left(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.
$$

$$
\left. - \bar{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right) =
$$

$$
= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \times
$$

$$
\times \left(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} - \right.
$$

$$
\left. - \bar{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \right) = \tag{2.1323}
$$

$$
= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \frac{1}{2^r} \left(\hat{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} - \right.
$$

$$
\left. - \bar{C}^{(d)}_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \right), \tag{2.1324}
$$

where $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (2.652), $k > 2r$, $r = 1, 2, \ldots, [k/2]$.

It is not difficult to see that the left-hand side of (2.1320) is a constant for the quantities (2.1319) for all $d = 1, 2, \ldots, 2^{k-2r}$.

Using (2.1315), (2.1316), we obtain

$$\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\frac{1}{2^r}\left(\hat{C}_{j_k\ldots j_1}^{(d)}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-\right.$$

$$\left.-\ \bar{C}_{j_k\ldots j_1}^{(d)}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)=$$

$$=\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}. \qquad (2.1325)$$

Combining (2.1324) and (2.1325), we have for any fixed $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}, \qquad (2.1326)$$

where $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (2.652), $k>2r$, $r=1,2,\ldots,[k/2]$.

From (2.1314) for the case $k=2r$ and (2.1326) $(k>2r)$ we obtain (2.1314) for the case $k\ge 2r$. The equality (2.1314) is proved for Case 1.

For Case 2, applying (2.1314) for the case $k=2r$ and (2.1318), we get (2.1324) for any fixed $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$. Further, note that

$$\hat{C}_{j_k\ldots j_1}^{(d)}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}=$$

$$=\ \bar{C}_{j_k\ldots j_1}^{(d)}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} \qquad (2.1327)$$

for Case 2. Combining (2.1324) and (2.1327), we obtain (Case 2) for any fixed $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=0. \qquad (2.1328)$$

From (2.1314) for the case $k=2r$ and (2.1328) $(k>2r)$ we obtain (2.1328) for the case $k\ge 2r$. The equality (2.1314) is proved for Case 2.

For Case 3, applying (2.1314) for the case $k=2r$ and (2.1318), we get (2.1323) for any fixed $j_1,\ldots,j_q,\ldots,j_k$ $(q\ne g_1,g_2,\ldots,g_{2r-1},\ g_{2r})$.

Since

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=0 \tag{2.1329}$$

for Case 3, then from (2.1323) we get (2.1328) for $k>2r$ (recall that the left-hand side of (2.1329) is a constant for the quantities (2.1319) for all $d=1,2,\ldots,2^{k-2r}$).

From (2.1314) for $k=2r$ and (2.1328) for $k>2r$ (Case 3) we obtain (2.1328) for $k\ge 2r$ (Case 3). The equality (2.1314) is proved for Case 3. The equality (2.1314) is proved. Thus, Theorem 2.58 is proved. Theorem 2.59 is also proved.

2.31 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k $(k\in\mathbf{N})$. The Case of an Arbitrary Complete Orthonormal System of Functions in $L_2([t,T])$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Proof of Hypotheses 2.4, 2.5 for the Case $p_1=\ldots=p_k=p$ and Under the Condition (2.1341)

We will start this section with an example. Let us assume that $h_1(\tau),\ldots,$ $h_{12}(\tau)\in L_2([t,T])$ and consider the following integral

$$I\stackrel{\rm def}{=}\int\limits_t^T h_{12}(t_{12})\int\limits_t^{t_{12}} h_{11}(t_{11})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{11}dt_{12}.$$

We want to transform the integral I in such a way that

$$I=\int\limits_t^T h_{10}(t_{10})\int\limits_t^{t_{10}} h_6(t_6)\int\limits_t^{t_6} h_4(t_4)\int\limits_t^{t_4} h_3(t_3)\left(\ldots\right)dt_3dt_4dt_6dt_{10},$$

where $(\ldots)$ is some expression.

Using Fubini's Theorem, we obtain

$$I=\int\limits_t^T h_{12}(t_{12})\int\limits_t^{t_{12}} h_{11}(t_{11})\int\limits_t^{t_{11}} \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} h_9(t_9)\int\limits_t^{t_9} h_8(t_8)\int\limits_t^{t_8} h_7(t_7)\int\limits_t^{t_7} \underline{h_6(t_6)}\times$$

$$\times\int\limits_t^{t_6} h_5(t_5)\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)dt_1dt_2dt_3dt_4dt_5dt_6dt_7dt_8\times$$

$$\times dt_9dt_{10}dt_{11}dt_{12}=$$

$$=\int\limits_t^T \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} h_9(t_9)\int\limits_t^{t_9} h_8(t_8)\int\limits_t^{t_8} h_7(t_7)\int\limits_t^{t_7} \underline{h_6(t_6)}\int\limits_t^{t_6} h_5(t_5)\times$$

$$\times\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)dt_1dt_2dt_3dt_4dt_5dt_6dt_7dt_8dt_9\times$$

$$\times\left(\int\limits_{t_{10}}^T h_{11}(t_{11})\int\limits_{t_{11}}^T h_{12}(t_{12})dt_{12}dt_{11}\right)dt_{10}=$$

$$=\int\limits_t^T \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} \underline{h_6(t_6)}\int\limits_t^{t_6} h_5(t_5)\int\limits_t^{t_5} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)\times$$

$$\times dt_1dt_2dt_3dt_4dt_5\left(\int\limits_{t_6}^{t_{10}} h_7(t_7)\int\limits_{t_7}^{t_{10}} h_8(t_8)\int\limits_{t_8}^{t_{10}} h_9(t_9)dt_9dt_8dt_7\right)dt_6\times$$

$$\times\left(\int\limits_{t_{10}}^T h_{11}(t_{11})\int\limits_{t_{11}}^T h_{12}(t_{12})dt_{12}dt_{11}\right)dt_{10}=$$

$$=\int\limits_t^T \underline{h_{10}(t_{10})}\int\limits_t^{t_{10}} \underline{h_6(t_6)}\int\limits_t^{t_6} \underline{h_4(t_4)}\int\limits_t^{t_4} \underline{h_3(t_3)}\left(\int\limits_t^{t_3} h_2(t_2)\int\limits_t^{t_2} h_1(t_1)dt_1dt_2\right)dt_3\times$$

$$\times\left(\int\limits_{t_4}^{t_6} h_5(t_5)dt_5\right)dt_4\left(\int\limits_{t_6}^{t_{10}} h_7(t_7)\int\limits_{t_7}^{t_{10}} h_8(t_8)\int\limits_{t_8}^{t_{10}} h_9(t_9)dt_9dt_8dt_7\right)dt_6\times$$

$$
\times \left(\int\limits_{t_{10}}^{T} h_{11}(t_{11}) \int\limits_{t_{11}}^{T} h_{12}(t_{12}) dt_{12} dt_{11} \right) dt_{10} =
$$

$$
= \int\limits_{t}^{T} \underline{h_{10}(t_{10})} \int\limits_{t}^{t_{10}} \underline{h_6(t_6)} \int\limits_{t}^{t_6} \underline{h_4(t_4)} \int\limits_{t}^{t_4} \underline{h_3(t_3)} \left(\int\limits_{t}^{t_3} h_2(t_2) \int\limits_{t}^{t_2} h_1(t_1) dt_1 dt_2 \right) \times
$$

$$
\times \left(\int\limits_{t_4}^{t_6} h_5(t_5) dt_5 \right) \left(\int\limits_{t_6}^{t_{10}} h_9(t_9) \int\limits_{t_6}^{t_9} h_8(t_8) \int\limits_{t_6}^{t_8} h_7(t_7) dt_7 dt_8 dt_9 \right) \times
$$

$$
\times \left(\int\limits_{t_{10}}^{T} h_{12}(t_{12}) \int\limits_{t_{10}}^{t_{12}} h_{11}(t_{11}) dt_{11} dt_{12} \right) dt_3 dt_4 dt_6 dt_{10}. \qquad (2.1330)
$$

Further, suppose that $h_l(\tau) = \psi_l(\tau)\phi_{j_l}(\tau)$ $(l = 1, \ldots, 12)$ in (2.1330) (here $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_{12}(\tau) \in L_2([t,T])$). Thus, we get

$$
C_{j_{12}j_{11}\underline{j_{10}}j_9 j_8 j_7 \underline{j_6} j_5 \underline{j_4 j_3} j_2 j_1} = \int\limits_{t}^{T} \psi_{10}(t_{10})\phi_{j_{10}}(t_{10}) \int\limits_{t}^{t_{10}} \psi_6(t_6)\phi_{j_6}(t_6) \int\limits_{t}^{t_6} \psi_4(t_4)\phi_{j_4}(t_4) \times
$$

$$
\times \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) C_{j_{12}j_{11}}^{\psi_{12}\psi_{11}}(T, t_{10}) C_{j_9 j_8 j_7}^{\psi_9\psi_8\psi_7}(t_{10}, t_6) C_{j_5}^{\psi_5}(t_6, t_4) C_{j_2 j_1}^{\psi_2\psi_1}(t_3, t) \times
$$

$$
\times dt_3 dt_4 dt_6 dt_{10}, \qquad (2.1331)
$$

where

$$
C_{j_k \ldots j_1}^{\psi_k \ldots \psi_1}(s, \tau) = \int\limits_{\tau}^{s} \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_{\tau}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k \quad (t \le \tau < s \le T).
$$

Suppose that $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (2.652) and $k > 2r$, $r \ge 1$ (the case $k = 2r$ see in Sect. 2.27.4). Consider $d_1, e_1, \ldots, d_f, e_f, f \in \mathbf{N}$ such that

$$
1 \le d_1 - e_1 + 1 < \ldots < d_1 - 1 < d_1 < \ldots < d_f - e_f + 1 < \ldots < d_f - 1 < d_f \le k,
$$

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} =
$$

$$
= \{d_1 - e_1 + 1, \ldots, d_1 - 1, d_1\} \cup \ldots \cup \{d_f - e_f + 1, \ldots, d_f - 1, d_f\},
$$

$$e_1+e_2+\ldots+e_f=2r,\quad \{1,\ldots,k\}\setminus\{g_1,g_2,\ldots,g_{2r-1},g_{2r}\}=\{q_1,\ldots,q_{k-2r}\}.$$

We will say that the condition (A) *is satisfied if* $\forall\ \{g_{2l-1},g_{2l}\}$ $(l=1,\ldots,r)$ $\exists\ h\in\{1,\ldots,f\}$ *such that*

$$\{g_{2l-1},g_{2l}\}\subset\{d_h-e_h+1,\ldots,d_h-1,d_h\}. \qquad (2.1332)$$

Moreover, $\forall\ h\in\{1,\ldots,f\}$ $\exists\ \{g_{2l-1},g_{2l}\}$ $(l=1,\ldots,r)$ *such that* (2.1332) *is fulfilled.*

If the condition (A) is satisfied, then $e_1,\ldots,e_f$ are even and we can write

$$\{d_1-e_1+1,\ldots,d_1\}=\left\{g_1^{(1)},g_2^{(1)},\ldots,g_{2r_1-1}^{(1)},g_{2r_1}^{(1)}\right\},$$
$$\ldots$$
$$\{d_f-e_f+1,\ldots,d_f\}=\left\{g_1^{(f)},g_2^{(f)},\ldots,g_{2r_f-1}^{(f)},g_{2r_f}^{(f)}\right\},$$

$$\left\{g_1,g_2,\ldots,g_{2r-1},g_{2r}\right\}=$$
$$=\left\{g_1^{(1)},g_2^{(1)},\ldots,g_{2r_1-1}^{(1)},g_{2r_1}^{(1)},\ldots,g_1^{(f)},g_2^{(f)},\ldots,g_{2r_f-1}^{(f)},g_{2r_f}^{(f)}\right\}.$$

If the condition (A) is not fulfilled, then some of $e_1,\ldots,e_f$ can be uneven.

Using (2.1262) and a modification of the algorithm from Sect. 2.27.4 (see below for details) it can be proved that

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots$$
$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$
$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),\,j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}} \qquad (2.1333)$$

if the condition (A) is satisfied, and

$$
\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=0 \qquad (2.1334)
$$

if the condition (A) is not fulfilled, where $t_{k+1}\stackrel{\rm def}{=}T$, $t_0\stackrel{\rm def}{=}t$, $e_1+\ldots+e_f=2r$ in (2.1333), (2.1334) and $e_h=2r_h$ $(h=1,\ldots,f)$, $r_1+\ldots+r_f=r$ in (2.1333).

Note that the series on the left-hand sides of (2.1333) and (2.1334) converge absolutly since their sums do not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$). Recall that any permutation of basis functions in a Hilbert space forms a basis in this Hilbert space [127].

Let us prove the formulas (2.1333) and (2.1334).

1. Suppose that the condition (A) is satisfied and

$$
\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}=1 \qquad (2.1335)
$$

for all $h=1,\ldots,f$. In this case we can use the results from Sect. 2.27.4. We have (see (2.1262))

$$
\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=
$$

$$
=\lim_{p\to\infty}\sum_{j_{g_1^{(1)}},j_{g_3^{(1)}},\ldots,j_{g_{2r_1-1}^{(1)}}=0}^{p}C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr|_{j_{g_1^{(1)}}=j_{g_2^{(1)}},\ldots,j_{g_{2r_1-1}^{(1)}}=j_{g_{2r_1}^{(1)}}}\times
$$

$$
\ldots
$$

$$
\times\lim_{p\to\infty}\sum_{j_{g_1^{(f)}},j_{g_3^{(f)}},\ldots,j_{g_{2r_f-1}^{(f)}}=0}^{p}C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\Biggr|_{j_{g_1^{(f)}}=j_{g_2^{(f)}},\ldots,j_{g_{2r_f-1}^{(f)}}=j_{g_{2r_f}^{(f)}}}=
$$

$$
=\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggl|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}.
$$

Thus, we get the formula (2.1333).

2. Suppose that the condition (A) is satisfied and for some $h=1,\ldots,f$

$$
\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}=0. \tag{2.1336}
$$

In this case, we act the same as in the previous case. Applying (2.1262), we obtain

$$
\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=
$$

$$
=\lim_{p\to\infty}\sum_{j_{g_1^{(1)}},j_{g_3^{(1)}},\ldots,j_{g_{2r_1-1}^{(1)}}=0}^{p}C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggl|_{j_{g_1^{(1)}}=j_{g_2^{(1)}},\ldots,j_{g_{2r_1-1}^{(1)}}=j_{g_{2r_1}^{(1)}}}\times
$$

$$
\ldots
$$

$$
\times\lim_{p\to\infty}\sum_{j_{g_1^{(f)}},j_{g_3^{(f)}},\ldots,j_{g_{2r_f-1}^{(f)}}=0}^{p}C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\Biggl|_{j_{g_1^{(f)}}=j_{g_2^{(f)}},\ldots,j_{g_{2r_f-1}^{(f)}}=j_{g_{2r_f}^{(f)}}}=0 \tag{2.1337}
$$

(al least one of the multipliers is equal to zero on the right-hand side of (2.1337)).

The equality (2.1333) is proved in our case (the right-hand side of (2.1333) is equal to zero for the considered case (see (2.1336))).

3. Suppose that the condition (A) is not satisfied. In this case, we act according to the algorithm from Sect. 2.27.4. More precisely, let us select blocks in the multi-index $j_{d_h}\ldots j_{d_h-e_h+1}$ $(h=1,\ldots,f)$ that correspond to the fulfillment of the condition

$$\prod_{l=1}^{r_{m,h}}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}=1,$$

where $r_{m,h}$ is the number of pairs $\{g_{2l-1}^{(h)},g_{2l}^{(h)}\}$ (from the set $\{g_1,g_2,\ldots,$ $g_{2r-1},g_{2r}\}$) in the block with number m that corresponds to the multi-index $j_{d_h}\ldots j_{d_h-e_h+1}$.

Let us save multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the Volterra–type kernels corresponding to the Fourier coefficients

$$C_{j_{d_1}\cdots j_{d_1-e_1+1}}^{\psi_{d_1}\cdots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1}),\ldots,C_{j_{d_f}\cdots j_{d_f-e_f+1}}^{\psi_{d_f}\cdots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\tag{2.1338}$$

and corresponding to the above blocks.

At that, we remove the remaining multipliers of the form $\mathbf{1}_{\{t_n<t_{n+1}\}}$ in the Volterra–type kernels corresponding to the Fourier coefficients (2.1338).

As a result, we get a modified left-hand side of the equality (2.1334). For definiteness, let us denote this expression by $(^-)$.

Using generalized Parseval's equality (Parseval's equality for two functions) and (2.1255), we represent the expression $(^-)$ as an integral over the hypercube $[t,T]^r$.

It is not difficult to see that the indicated integral over the hypercube $[t,T]^r$ is represented as a product of integrals over hypercubes of smaller dimentions. At that, at least one of these integrals is equal to zero due to the generalized Parseval equality (Parseval's equality for two functions) and the fulfillment of the condition $t\le t_{d_1-e_1}\le t_{d_1+1}\le\ldots\le t_{d_f-e_f}\le t_{d_f+1}\le T$ (see the above example and (2.1330) and (2.1331)). For definiteness, let us denote the equality of $(^-)$ to zero by $(\bar K)$. We interpret the above zero as the zero functional in $L_2([t,T]^r)$. Further, transformations and passages to the limit in the equality $(\bar K)$ are performed iteratively in such a way as to restore the removed multipliers $\mathbf{1}_{\{t_n<t_{n+1}\}}$ on the left-hand side of $(\bar K)$ (for more details, see Sect. 2.27.4). As a result, we obtain the equality (2.1334). The equalities (2.1333) and (2.1334) are proved.

For definiteness, suppose that $q_1<\ldots<q_{k-2r}$, $k>2r$, $r\ge 1$ (recall that the case $k=2r$ is proved in Sect. 2.27.4). Using Fubini's Theorem (as in the above example (see (2.1330)), we obtain

$$\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \int\limits_t^T \psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots \int\limits_t^{t_{q_1+1}} \psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times$$

$$\times \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} \biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots$$

$$\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times dt_{q_1}\ldots dt_{q_{k-2r}}, \qquad (2.1339)$$

$$\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}} =$$

$$= \int\limits_t^T \psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots \int\limits_t^{t_{q_1+1}} \psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times$$

$$\times\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\times$$

$$\times dt_{q_1}\ldots dt_{q_{k-2r}}. \qquad (2.1340)$$

Suppose that

$$
\Biggl|\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\biggr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr|\le K<\infty, \quad (2.1341)
$$

where constant K does not depend on p and $t_{d_1+1},t_{d_1-e_1},\ldots,t_{d_f+1},t_{d_f-e_f}$ (here $d_1-e_1\ge 1$ and $d_f+1\le k$). In (2.1341): $t_{k+1}\stackrel{\sf def}{=}T$, $t_0\stackrel{\sf def}{=}t$, $e_1+\ldots+e_f=2r$; another notations as above in this section.

Applying (2.1333), (2.1334), (2.1339), (2.1340), we obtain ($k>2r$, $r\ge 1$)

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2\le
$$

$$
\le\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=
$$

$$
=\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\Biggl(\int\limits_t^T\psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times
$$

$$
\times\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\Biggr)\times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}}\Biggr)^2 = \tag{2.1342}
$$

$$
=\lim_{p\to\infty}\int\limits_t^T\psi_{q_{k-2r}}^2(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi_{q_1}^2(t_{q_1})\times
$$

$$
\times\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Biggl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Biggr)\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times
$$

$$
\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Biggr|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}}\Biggr)^2\times
$$

$$
\times dt_{q_1}\ldots dt_{q_{k-2r}} = \tag{2.1343}
$$

$$
= \int\limits_t^T \psi_{q_{k-2r}}^2(t_{q_{k-2r}}) \ldots \int\limits_t^{t_{q_1+1}} \psi_{q_1}^2(t_{q_1}) \times
$$

$$
\times \lim_{p\to\infty} \left(\sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \left(C_{j_{d_f}\cdots j_{d_f-e_f+1}}^{\psi_{d_f}\cdots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \ldots \right.\right.
$$

$$
\left.\left. \ldots C_{j_{d_1}\cdots j_{d_1-e_1+1}}^{\psi_{d_1}\cdots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \right) \right|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}} \prod_{h=1}^{f} \frac{1}{2^{r_h}} \prod_{l=1}^{r_h} \mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}} \times
$$

$$
\left. \times C_{j_{d_h}\cdots j_{d_h-e_h+1}}^{\psi_{d_h}\cdots\psi_{d_h-e_h+1}}(t_{d_h+1}, t_{d_h-e_h}) \right|_{(j_{g_2^{(h)}} j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}} j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot),\, j_{g_1^{(h)}}=j_{g_2^{(h)}},\ldots,j_{g_{2r_h-1}^{(h)}}=j_{g_{2r_h}^{(h)}}} \Biggr)^2 \times
$$

$$
\times dt_{q_1} \ldots dt_{q_{k-2r}} = 0, \tag{2.1344}
$$

where the transition from (2.1342) to (2.1343) is based on the Parseval equality and the transition from (2.1343) to (2.1344) is based on Lebesgue's Dominated Convergence Theorem (see (2.1169), (2.1172), (2.1333), (2.1334), (2.1341)) and also on convergence to zero (almost everywhere on

$$
X = \{(t_{q_1}, \ldots, t_{q_{k-2r}}) : t \le t_{q_1} \le \ldots \le t_{q_{k-2r}} \le T\}
$$

with respect to Lebesgue's measure) of the integrand function in (2.1343).

Thus, the equality (2.1060) and Hypotheses 2.4, 2.5 are proved for the case $p_1 = \ldots = p_k = p$ under the condition (2.1341) and we have the following theorem.

Theorem 2.60. *Suppose that the condition (2.1341) is fulfilled, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Then, for the sum $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ of iterated Itô* *stochastic integrals*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$$

the following expansion

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (2.1345)$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$*; another notations are the same as in Theorem* 2.12*.*

Using Theorem 2.12, we obtain the following corollary of Theorem 2.60.

Theorem 2.61. *Suppose that the condition* (2.1341) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of multiplicity* k $(k\in\mathbf{N})$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \qquad (2.1346)$$

that converges in the mean-square sense is valid; another notations are the same as in Theorem 2.60*.*

Note that the condition (2.1341) can be weakened. Namely, the constant K^2 can be replaced by the function F such that $\psi_{q_1}^2 \ldots \psi_{q_{k-2r}}^2 F \in L_1([t,T]^{k-2r})$ (integrable majorant). More precisely, the condition (2.1341) can be replaced by the following condition

$$\left(\sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \left(C_{j_{d_f} \ldots j_{d_f - e_f + 1}}^{\psi_{d_f} \ldots \psi_{d_f - e_f + 1}} (t_{d_f+1}, t_{d_f - e_f}) \ldots \right.\right.$$

$$\left.\left. \ldots C_{j_{d_1} \ldots j_{d_1 - e_1 + 1}}^{\psi_{d_1} \ldots \psi_{d_1 - e_1 + 1}} (t_{d_1+1}, t_{d_1 - e_1}) \right) \Biggr|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} \right)^2 \le F(t_{q_1}, \ldots, t_{q_{k-2r}}) \tag{2.1347}$$

almost everywhere on

$$X = \{(t_{q_1}, \ldots, t_{q_{k-2r}}) : t \le t_{q_1} \le \ldots \le t_{q_{k-2r}} \le T\}$$

with respect to Lebesgue's measure, where the function $F(t_{q_1}, \ldots, t_{q_{k-2r}})$ is such that

$$\psi_{q_1}^2(t_{q_1}) \ldots \psi_{q_{k-2r}}^2(t_{q_{k-2r}}) F(t_{q_1}, \ldots, t_{q_{k-2r}}) \in L_1([t,T]^{k-2r}),$$

where $F(t_{q_1}, \ldots, t_{q_{k-2r}})$ does not depend on p. In (2.1347): $t_{k+1} \stackrel{\text{def}}{=} T$, $t_0 \stackrel{\text{def}}{=} t$, $e_1 + \ldots + e_f = 2r$; another notations as above in this section.

2.32 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$

This section is devoted to the following theorem.

Theorem 2.62. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J^*[\psi^{(6)}]_{T,t} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J^*[\psi^{(6)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6 \tag{2.1348}$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. Our proof will be based on Theorem 2.61 and verification of the equality (2.1341) under the conditions of Theorem 2.62 (the case $k=6>2r$, where $r=1,2$). Recall that the case $k=2r$ is considered in Sect. 2.27.4 (see (2.1262)). Under the conditions of Theorem 2.62, this means that $k=6=2r$, where $r=3$.

Let throughout this proof

$$C_{j_k\ldots j_1}(s,\tau) = \int\limits_\tau^s \phi_{j_k}(t_k)\ldots\int\limits_\tau^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

where $k=1,\ldots,4$, $t\le\tau<s\le T$, and $C_{j_6\ldots j_1}$ is defined by (2.1348).

Using Fubini's Theorem and the technique that leads to the formulas (2.1330), (2.1331), we obtain (note that we find all possible combinations of pairs using the equality (2.682)):

1. $r=1$ (15 combinations)

$$C_{j_1j_5j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(T,t_5)dt_2dt_3dt_4dt_5,$$

$$C_{j_2j_5j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2}(T,t_5)dt_1dt_3dt_4dt_5,$$

$$C_{j_3 j_5 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_3}(t_4, t_2) C_{j_3}(T, t_5) dt_1 dt_2 dt_4 dt_5,$$

$$C_{j_4 j_5 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4}(t_5, t_3) C_{j_4}(T, t_5) dt_1 dt_2 dt_3 dt_5,$$

$$C_{j_5 j_5 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_5 j_5}(T, t_4) dt_1 dt_2 dt_3 dt_4,$$

$$C_{j_6 j_5 j_4 j_3 j_1 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) C_{j_1 j_1}(t_3, t) dt_3 dt_4 dt_5 dt_6,$$

$$C_{j_6 j_5 j_4 j_1 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_4, t_2) dt_2 dt_4 dt_5 dt_6,$$

$$C_{j_6 j_5 j_1 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_5, t_3) dt_2 dt_3 dt_5 dt_6,$$

$$C_{j_6 j_1 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_6, t_4) dt_2 dt_3 dt_4 dt_6,$$

$$C_{j_6 j_5 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) C_{j_2 j_2}(t_4, t_1) dt_1 dt_4 dt_5 dt_6,$$

$$C_{j_6 j_5 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(t_5, t_3) dt_1 dt_3 dt_5 dt_6,$$

$$C_{j_6 j_2 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(t_6, t_4) dt_1 dt_3 dt_4 dt_6,$$

$$C_{j_6 j_5 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_3 j_3}(t_5, t_2) dt_1 dt_2 dt_5 dt_6,$$

$$C_{j_6 j_3 j_4 j_3 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)C_{j_3}(t_4,t_2)C_{j_3}(t_6,t_4)dt_1dt_2dt_4dt_6,$$

$$C_{j_6 j_4 j_4 j_3 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_3}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)C_{j_4 j_4}(t_6,t_3)dt_1dt_2dt_3dt_6,$$

2. $r=2$ (45 combinations)

$$C_{j_6 j_5 j_3 j_3 j_1 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_5}(t_5)C_{j_3 j_3 j_1 j_1}(t_5,t)dt_5dt_6,$$

$$C_{j_6 j_3 j_4 j_3 j_1 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_4}(t_4)C_{j_3 j_1 j_1}(t_4,t)C_{j_3}(t_6,t_4)dt_4dt_6,$$

$$C_{j_6 j_4 j_4 j_3 j_1 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_3}(t_3)C_{j_1 j_1}(t_3,t)C_{j_4 j_4}(t_6,t_3)dt_3dt_6,$$

$$C_{j_6 j_5 j_2 j_1 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_5}(t_5)C_{j_2 j_1 j_2 j_1}(t_5,t)dt_5dt_6,$$

$$C_{j_6 j_2 j_4 j_1 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_4}(t_4)C_{j_1 j_2 j_1}(t_4,t)C_{j_2}(t_6,t_4)dt_4dt_6,$$

$$C_{j_6 j_4 j_4 j_1 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_4 j_4 j_1}(t_6,t_2)dt_2dt_6,$$

$$C_{j_6 j_5 j_1 j_2 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_5}(t_5)C_{j_1 j_2 j_2 j_1}(t_5,t)dt_5dt_6,$$

$$C_{j_6 j_2 j_1 j_3 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_3}(t_3)C_{j_2 j_1}(t_3,t)C_{j_2 j_1}(t_6,t_3)dt_3dt_6,$$

$$C_{j_6 j_3 j_1 j_3 j_2 j_1}=\int\limits_t^T \phi_{j_6}(t_6)\int\limits_t^{t_6} \phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_3 j_1 j_3}(t_6,t_2)dt_2dt_6,$$

$$C_{j_6j_1j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_4}(t_4)C_{j_2j_2j_1}(t_4,t)C_{j_1}(t_6,t_4)dt_4dt_6,$$

$$C_{j_6j_1j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_1j_2}(t_6,t_3)dt_3dt_6,$$

$$C_{j_6j_1j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1j_3j_3}(t_6,t_2)dt_2dt_6,$$

$$C_{j_6j_4j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_1)C_{j_4j_4j_2j_2}(t_6,t_1)dt_1dt_6,$$

$$C_{j_6j_3j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_1)C_{j_3j_2j_3j_2}(t_6,t_1)dt_1dt_6,$$

$$C_{j_6j_2j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_1)C_{j_2j_3j_3j_2}(t_6,t_1)dt_1dt_6,$$

$$C_{j_1j_5j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_3j_3}(t_5,t_2)C_{j_1}(T,t_5)dt_2dt_5,$$

$$C_{j_1j_3j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_3}(t_4,t_2)C_{j_1j_3}(T,t_4)dt_2dt_4,$$

$$C_{j_1j_2j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_1j_2}(T,t_4)dt_3dt_4,$$

$$C_{j_1j_5j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_2}(t_5,t_3)C_{j_1}(T,t_5)dt_3dt_5,$$

$$C_{j_1j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1j_4j_4}(T,t_3)dt_2dt_3,$$

$$
C_{j_1j_5j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)C_{j_2j_2j_1}(t_4,t)C_{j_1}(T,t_5)dt_4dt_5,
$$

$$
C_{j_2j_3j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)C_{j_2j_3}(t_4,t_1)C_{j_2j_3}(T,t_4)dt_1dt_4,
$$

$$
C_{j_2j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2j_4j_4}(T,t_3)dt_1dt_3,
$$

$$
C_{j_2j_5j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)C_{j_2j_3j_3}(t_5,t_1)C_{j_2}(T,t_5)dt_1dt_5,
$$

$$
C_{j_2j_1j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_2j_1}(T,t_4)dt_3dt_4,
$$

$$
C_{j_2j_5j_1j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_1}(t_5,t_3)C_{j_2}(T,t_5)dt_3dt_5,
$$

$$
C_{j_2j_5j_4j_1j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_5)dt_4dt_5,
$$

$$
C_{j_3j_2j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)C_{j_3j_2}(t_4,t_1)C_{j_3j_2}(T,t_4)dt_1dt_4,
$$

$$
C_{j_3j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)C_{j_3j_4j_4j_3}(T,t_2)dt_1dt_2,
$$

$$
C_{j_3j_5j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)C_{j_2j_3j_2}(t_5,t_1)C_{j_3}(T,t_5)dt_1dt_5,
$$

$$
C_{j_3j_1j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_3}(t_4,t_2)C_{j_3j_1}(T,t_4)dt_2dt_4,
$$

$$C_{j_3 j_5 j_1 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1 j_3}(t_5, t_2) C_{j_3}(T, t_5) dt_2 dt_5,$$

$$C_{j_3 j_5 j_4 j_3 j_1 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) C_{j_3 j_1 j_1}(t_4, t) C_{j_3}(T, t_5) dt_4 dt_5,$$

$$C_{j_4 j_3 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4 j_3 j_4 j_3}(T, t_2) dt_1 dt_2,$$

$$C_{j_4 j_2 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_4 j_2 j_4}(T, t_3) dt_1 dt_3,$$

$$C_{j_4 j_5 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_1) C_{j_4 j_2 j_2}(t_5, t_1) C_{j_4}(T, t_5) dt_1 dt_5,$$

$$C_{j_4 j_1 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_4 j_1 j_4}(T, t_3) dt_2 dt_3,$$

$$C_{j_4 j_5 j_4 j_1 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_4 j_1}(t_5, t_2) C_{j_4}(T, t_5) dt_2 dt_5,$$

$$C_{j_4 j_5 j_4 j_3 j_1 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) C_{j_1 j_1}(t_3, t) C_{j_4}(t_5, t_3) C_{j_4}(T, t_5) dt_3 dt_5,$$

$$C_{j_5 j_5 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_5 j_5 j_3 j_3}(T, t_2) dt_1 dt_2,$$

$$C_{j_5 j_5 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_5 j_5 j_2}(T, t_3) dt_1 dt_3,$$

$$C_{j_5 j_5 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) C_{j_2 j_2}(t_4, t_1) C_{j_5 j_5}(T, t_4) dt_1 dt_4,$$

$$C_{j_5j_5j_1j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_5j_5j_1}(T,t_3)dt_2dt_3,$$

$$C_{j_5j_5j_4j_1j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(t_4,t_2)C_{j_5j_5}(T,t_4)dt_2dt_4,$$

$$C_{j_5j_5j_4j_3j_1j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_1j_1}(t_3,t)C_{j_5j_5}(T,t_4)dt_3dt_4.$$

It is not difficult to see (based on the above equalities) that the condition (2.1341) will be satisfied under the conditions of Theorem 2.62 if

$$\left|\sum_{j_1=0}^{p}C_{j_1j_1}(s,\tau)\right|\le K, \tag{2.1349}$$

$$\left|\sum_{j_1=0}^{p}C_{j_1}(s,\tau)C_{j_1}(\theta,u)\right|\le K, \tag{2.1350}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1j_1}(s,\tau)\right|\le K, \tag{2.1351}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(s,\tau)\right|\le K, \tag{2.1352}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_2j_1}(s,\tau)\right|\le K, \tag{2.1353}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_1}(s,\tau)C_{j_2}(\theta,u)\right|\le K, \tag{2.1354}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}(s,\tau)C_{j_2}(\theta,u)\right|\le K, \tag{2.1355}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1}(s,\tau)C_{j_1}(\theta,u)\right|\le K, \tag{2.1356}$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_1j_1}(s,\tau)C_{j_2j_2}(\theta,u)\right|\le K, \tag{2.1357}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_2j_1}(\theta,u)\right| \le K, \tag{2.1358}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_1j_2}(\theta,u)\right| \le K, \tag{2.1359}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_1}(\rho,v)C_{j_2j_2}(\theta,u)\right| \le K, \tag{2.1360}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_2}(\rho,v)C_{j_1j_2}(\theta,u)\right| \le K, \tag{2.1361}$$

where $p\in\mathbf{N}$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, $t\le v<\rho\le T$, constant K does not depend on p,s,τ,u,θ,v,ρ (but only on t,T) and may differ from line to line.

The equalities (2.1351)–(2.1353) have been proved earlier (see (2.1108)–(2.1110)).

Using Fubini's Theorem and Parseval's equality, we get

$$\left|\sum_{j_1=0}^{p} C_{j_1j_1}(s,\tau)\right| = \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2(s,\tau) \le$$

$$\le \frac{1}{2}\sum_{j_1=0}^{\infty} C_{j_1}^2(s,\tau) = \frac{1}{2}(s-\tau) \le$$

$$\le \frac{1}{2}(T-t) \le K.$$

The equality (2.1349) is proved. Moreover, (2.1357) follows from (2.1349).

Using the inequality of Cauchy–Bunyakovsky and Parseval's equality, we obtain

$$\left(\sum_{j_1=0}^{p} C_{j_1}(s,\tau)C_{j_1}(\theta,u)\right)^2 \le$$

$$\le \sum_{j_1=0}^{p} C_{j_1}^2(s,\tau)\sum_{j_1=0}^{p} C_{j_1}^2(\theta,u) \le$$

$$\le \sum_{j_1=0}^{\infty} C_{j_1}^2(s,\tau)\sum_{j_1=0}^{\infty} C_{j_1}^2(\theta,u) =$$

$$= (s-\tau)(\theta-u) \le (T-t)^2 \le K^2,$$

$$\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_2j_1}(\theta,u)\right)^2 \le \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2(s,\tau)\sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2(\theta,u) \le$$

$$\le \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}^2(s,\tau)\sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}^2(\theta,u) =$$

$$= \int\limits_{\tau}^{s}\int\limits_{\tau}^{v} dxdv \int\limits_{u}^{\theta}\int\limits_{u}^{v} dxdv \le \frac{1}{4}(T-t)^4 \le K^2.$$

Thus, the inequalities (2.1350), (2.1358) are proved. The inequalities (2.1359), (2.1361) are proved similarly to (2.1358). Moreover, (2.1360) follows from (2.1349), (2.1350).

Further, let us prove the equalities (2.1354)–(2.1356). Applying the Cauchy–Bunyakovsky inequality as well as Parseval's equality and (2.1349), we have

$$\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_1}(s,\tau)C_{j_2}(\theta,u)\right)^2 \le \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_2j_1j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{p} C_{j_2}^2(\theta,u) \le$$

$$\le \sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p} C_{j_2j_1j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{\infty} C_{j_2}^2(\theta,u) =$$

$$= \sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s} \phi_{j_2}(v)\sum_{j_1=0}^{p} C_{j_1j_1}(v,\tau)dv\right)^2 \cdot (\theta-u) =$$

$$= (\theta-u)\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p} C_{j_1j_1}(v,\tau)\right)^2 dv \le$$

$$\le K^2(\theta-u)(s-\tau) \le K^2(T-t)^2 = K_1.$$

The equality (2.1354) is proved.

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (2.1350), we have

$$\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}(s,\tau)C_{j_2}(\theta,u)\right)^2 \le \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{p} C_{j_2}^2(\theta,u) \le$$

$$\le \sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{s}\phi_{j_1}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdydz\right)^2\sum_{j_2=0}^{\infty}C_{j_2}^2(\theta,u)=$$

$$=\sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dx\int\limits_{y}^{s}\phi_{j_1}(z)dzdy\right)^2\cdot(\theta-u)=$$

$$=(\theta-u)\sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_2}(y)\sum_{j_1=0}^{p}C_{j_1}(y,\tau)C_{j_1}(s,y)dy\right)^2=$$

$$=(\theta-u)\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p}C_{j_1}(y,\tau)C_{j_1}(s,y)\right)^2dy\le$$

$$\le K^2(\theta-u)(s-\tau)\le K^2(T-t)^2=K_1.$$

The equality (2.1355) is proved.

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (2.1349), we have

$$\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1}(s,\tau)C_{j_1}(\theta,u)\right)^2\le\sum_{j_1=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_2j_1}(s,\tau)\right)^2\sum_{j_1=0}^{p}C_{j_1}^2(\theta,u)\le$$

$$\le \sum_{j_1=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdydz\right)^2\sum_{j_1=0}^{\infty}C_{j_1}^2(\theta,u)=$$

$$=\sum_{j_1=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_1}(x)\int\limits_{x}^{s}\phi_{j_2}(y)\int\limits_{y}^{s}\phi_{j_2}(z)dzdydx\right)^2\cdot(\theta-u)=$$

$$=(\theta-u)\sum_{j_1=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_1}(x)\int\limits_{x}^{s}\phi_{j_2}(z)\int\limits_{x}^{z}\phi_{j_2}(y)dydzdx\right)^2=$$

$$=(\theta-u)\sum_{j_1=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_1}(x)\sum_{j_2=0}^{p}C_{j_2j_2}(s,x)dx\right)^2=$$

$$
= (\theta - u) \int\limits_{\tau}^{s} \left(\sum_{j_2=0}^{p} C_{j_2 j_2}(s, x) \right)^2 dx \leq
$$

$$
\leq K^2 (\theta - u)(s - \tau) \leq K^2 (T - t)^2 = K_1.
$$

The equality (2.1356) is proved. The equalities (2.1349)–(2.1361) are proved.

Thus, the condition (2.1341) of Theorem 2.61 is satisfied under the conditions of Theorem 2.62. The assertion of Theorem 2.62 now follows from Theorem 2.61. Theorem 2.62 is proved.

2.33 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t, T])$ and Binomial Weight Functions

Let us prove the following theorem.

Theorem 2.63. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
I_{l_1 l_2 l_3 l_4 T,t}^{*(i_1 i_2 i_3 i_4)} = \int\limits_{t}^{*T} (t_4 - t)^{l_4} \int\limits_{t}^{*t_4} (t_3 - t)^{l_3} \int\limits_{t}^{*t_3} (t_2 - t)^{l_2} \int\limits_{t}^{*t_2} (t_1 - t)^{l_1} \times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}
$$

the following expansion

$$
I_{l_1 l_2 l_3 l_4 T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2, j_3, j_4 = 0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}
$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3, i_4 = 0, 1, \ldots, m;$ $l_1, l_2, l_3, l_4 = 0, 1, 2, \ldots,$

$$
C_{j_4 j_3 j_2 j_1} = \int\limits_{t}^{T} (t_4 - t)^{l_4} \phi_{j_4}(t_4) \int\limits_{t}^{t_4} (t_3 - t)^{l_3} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} (t_2 - t)^{l_2} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} (t_1 - t)^{l_1} \phi_{j_1}(t_1) \times
$$

$$\times dt_1 dt_2 dt_3 dt_4 \tag{2.1362}$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau.$

Proof. The following proof will be based on Theorem 2.61 and verification of the equality (2.1341) under the conditions of Theorem 2.63 (the case $k = 4 > 2r$, where $r = 1$). Note that the case $k = 2r$ is proved in Sect. 2.27.4 (see (2.1262)). Under the conditions of Theorem 2.63, the equality $k = 2r$ means that $k = 4$ and $r = 2$.

Let throughout this proof

$$C_{j_1 j_1}^{\psi_{i+1}\psi_i}(s,\tau) = \int\limits_\tau^s \psi_{i+1}(y)\phi_{j_1}(y) \int\limits_\tau^y \psi_i(x)\phi_{j_1}(x) dx dy, \ \ C_{j_1}^{\psi_q}(s,\tau) = \int\limits_\tau^s \psi_q(x)\phi_{j_1}(x) dx,$$

where $i = 1, 2, 3$, $t \leq \tau < s \leq T$, $\psi_q(x) = (x - t)^{l_q}$, $l_q = 0, 1, 2, \ldots$, $q = 1, \ldots, 4$, $x \in [t, T]$, and $C_{j_4 j_3 j_2 j_1}$ is defined by (2.1362).

Using Fubini's Theorem and the technique that leads to the formulas (2.1330), (2.1331), we obtain (note that we find all possible combinations of pairs using the equality (2.680)):

$$C_{j_4 j_3 j_1 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) C_{j_1 j_1}^{\psi_2\psi_1}(t_3, t) dt_3 dt_4,$$

$$C_{j_4 j_1 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) C_{j_1}^{\psi_1}(t_2, t) C_{j_1}^{\psi_3}(t_4, t_2) dt_2 dt_4,$$

$$C_{j_1 j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) C_{j_1}^{\psi_1}(t_2, t) C_{j_1}^{\psi_4}(T, t_3) dt_2 dt_3,$$

$$C_{j_4 j_2 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) C_{j_2 j_2}^{\psi_3\psi_2}(t_4, t_1) dt_1 dt_4,$$

$$
C_{j_2 j_3 j_2 j_1}=\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)C_{j_2}^{\psi_2}(t_3,t_1)C_{j_2}^{\psi_4}(T,t_3)dt_1 dt_3,
$$

$$
C_{j_3 j_3 j_1 j_1}=\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)C_{j_3 j_3}^{\psi_4\psi_3}(T,t_2)dt_1 dt_2.
$$

It is easy to see (based on the above equalities) that the condition (2.1341) will be satisfied under the conditions of Theorem 2.63 if

$$
\left|\sum_{j_1=0}^{p} C_{j_1 j_1}^{\psi_{i+1}\psi_i}(s,\tau)\right|\le K, \tag{2.1363}
$$

$$
\left|\sum_{j_1=0}^{p} C_{j_1}^{\psi_k}(s,\tau)C_{j_1}^{\psi_q}(\theta,u)\right|\le K, \tag{2.1364}
$$

where $p\in\mathbf{N}$, $i=1,2,3$, $k,q=1,\ldots,4$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, constant K does not depend on p,s,τ,u,θ (but only on t,T).

The equality (2.1363) has been proved earlier (see (2.1152)). Obviously, the relation (2.1364) is proved in complete analogy with (2.1155).

Thus, the condition (2.1341) of Theorem 2.61 is fulfilled under the conditions of Theorem 2.63. Then Theorem 2.63 follows from Theorem 2.61. Theorem 2.63 is proved.

2.34 Another Proof of Theorem 2.50 Based on Theorem 2.61

The following proof will be based on Theorem 2.61 and verification of the equality (2.1341) under the conditions of Theorem 2.50 (the case $k=5>2r$, where $r=1$ or $r=2$).

Further, suppose that

$$
C_{j_k\ldots j_1}(s,\tau)=\int\limits_\tau^s \phi_{j_k}(t_k)\ldots\int\limits_\tau^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k,
$$

where $k=1,\ldots,4$, $t\le\tau<s\le T$, and

$$C_{j_5\ldots j_1}=\int\limits_t^T\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5.$$

Applying the technique that leads to (2.1330), we obtain (note that we find all possible combinations of pairs using the equality (2.681))

$$C_{j_5j_4j_3j_1j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_1j_1}(t_3,t)dt_3dt_4dt_5,$$

$$C_{j_5j_4j_1j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(t_4,t_2)dt_2dt_4dt_5,$$

$$C_{j_5j_1j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(t_5,t_3)dt_2dt_3dt_5,$$

$$C_{j_1j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(T,t_4)dt_2dt_3dt_4,$$

$$C_{j_5j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)C_{j_2j_2}(t_4,t_1)dt_1dt_4dt_5,$$

$$C_{j_5j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2}(t_5,t_3)dt_1dt_3dt_5,$$

$$C_{j_2j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2}(T,t_4)dt_1dt_3dt_4,$$

$$C_{j_5j_3j_3j_2j_1}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)C_{j_3j_3}(t_5,t_2)dt_1dt_2dt_5,$$

$$C_{j_3j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)C_{j_3}(t_4,t_2)C_{j_3}(T,t_4)dt_1dt_2dt_4,$$

$$C_{j_4 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4 j_4}(T, t_3) dt_1 dt_2 dt_3,$$

$$C_{j_5 j_3 j_3 j_1 j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_3 j_3 j_1 j_1}(t_5, t) dt_5,$$

$$C_{j_5 j_2 j_1 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_2 j_1 j_2 j_1}(t_5, t) dt_5,$$

$$C_{j_5 j_1 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_1 j_2 j_2 j_1}(t_5, t) dt_5,$$

$$C_{j_4 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_4 j_4 j_2 j_2}(T, t_1) dt_1,$$

$$C_{j_3 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_3 j_2 j_3 j_2}(T, t_1) dt_1,$$

$$C_{j_2 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_2 j_3 j_3 j_2}(T, t_1) dt_1,$$

$$C_{j_4 j_4 j_3 j_1 j_1} = \int\limits_t^T \phi_{j_3}(t_3) C_{j_1 j_1}(t_3, t) C_{j_4 j_4}(T, t_3) dt_3,$$

$$C_{j_2 j_4 j_1 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) C_{j_1 j_2 j_1}(t_4, t) C_{j_2}(T, t_4) dt_4,$$

$$C_{j_2 j_1 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_3) C_{j_2 j_1}(t_3, t) C_{j_2 j_1}(T, t_3) dt_3,$$

$$C_{j_3 j_1 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_3 j_1 j_3}(T, t_2) dt_2,$$

$$C_{j_1j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_1j_2}(T,t_3)dt_3,$$

$$C_{j_3j_4j_3j_1j_1}=\int\limits_t^T\phi_{j_4}(t_4)C_{j_3j_1j_1}(t_4,t)C_{j_3}(T,t_4)dt_4,$$

$$C_{j_4j_4j_1j_2j_1}=\int\limits_t^T\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_4j_4j_1}(T,t_2)dt_2,$$

$$C_{j_1j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)C_{j_2j_2j_1}(t_4,t)C_{j_1}(T,t_4)dt_4,$$

$$C_{j_1j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1j_3j_3}(T,t_2)dt_2.$$

It is easy to see (based on the above relations) that (2.1341) will be satisfied (under the conditions of Theorem 2.50) if (2.1349)–(2.1359) are fulfilled. The equalities (2.1349)–(2.1359) are proved in Sect. 2.32. The assertion of Theorem 2.50 now follows from Theorem 2.61. Theorem 2.50 is proved.

Recall that for the case $k=6$, together with (2.1349)–(2.1359), the conditions (2.1360), (2.1361) and the equality (2.1262) ($k=2r$, $k=6$, $r=3$) must be satisfied (see the proof of Theorem 2.62).

2.35 Partial Proof of the Condition (2.1341)

In this section, we will prove (2.1341) for the case when the condition (A) and the relation (2.1335) are satisfied (see Sect. 2.31).

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

It is easy to see that (2.1341) will be proved for the above case if we prove that

$$\left|\sum_{j_r,j_{r-2},\ldots,j_2=0}^{p}C_{j_rj_rj_{r-2}j_{r-2}\ldots j_2j_2}(s,\tau)\right|\le K<\infty, \tag{2.1365}$$

where $p\in\mathbf{N}$, $r=2,4,6,\ldots$, constant K does not depend on p,s,τ (but only on t,T),

$$C_{j_k\ldots j_1}(s,\tau)=\int\limits_{\tau}^{s}\phi_{j_k}(t_k)\ldots\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k, \tag{2.1366}$$

where $k\in\mathbf{N}$, $t\le\tau<s\le T$.

By analogy with (2.1204) we obtain

$$C_{j_rj_rj_{r-2}j_{r-2}\ldots j_2j_2}(s,\tau)+C_{j_2j_2\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)=$$

$$=C_{j_r}(s,\tau)\cdot C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)-C_{j_rj_r}(s,\tau)\cdot C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)+$$

$$+C_{j_{r-2}j_rj_r}(s,\tau)\cdot C_{j_{r-2}j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)-\ldots$$

$$-C_{j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\cdot C_{j_2j_2}(s,\tau)+C_{j_2j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\cdot C_{j_2}(s,\tau). \tag{2.1367}$$

Applying (2.1367), we get

$$2\sum_{j_r,j_{r-2},\ldots,j_4,j_2=0}^{p}C_{j_rj_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)=$$

$$=\sum_{j_r=0}^{p}C_{j_r}(s,\tau)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p}C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)-$$

$$-\sum_{j_r=0}^{p}C_{j_rj_r}(s,\tau)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p}C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)+$$

$$+\sum_{j_{r-2}=0}^{p}\sum_{j_r=0}^{p}C_{j_{r-2}j_rj_r}(s,\tau)\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p}C_{j_{r-2}j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)-\ldots$$

$$-\sum_{j_r,j_{r-2},\ldots,j_4=0}^{p}C_{j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\sum_{j_2=0}^{p}C_{j_2j_2}(s,\tau)+$$

$$+\sum_{j_2=0}^{p}\sum_{j_r,j_{r-2},\ldots,j_4=0}^{p}C_{j_2j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\cdot C_{j_2}(s,\tau). \tag{2.1368}$$

Let us prove (2.1365) by induction. The equality (2.1365) is proved for $r = 2, 4$ (see (1.60), (2.1106), (2.1108)). Suppose that

$$\left|\sum_{j_6,j_4,j_2=0}^{p} C_{j_6j_6j_4j_4j_2j_2}(s,\tau)\right| \le K < \infty, \tag{2.1369}$$

$$\left|\sum_{j_8,j_6,j_4,j_2=0}^{p} C_{j_8j_8j_6j_6j_4j_4j_2j_2}(s,\tau)\right| \le K < \infty, \tag{2.1370}$$

$$\ldots$$

$$\left|\sum_{j_{r-2},j_{r-4},\ldots,j_2=0}^{p} C_{j_{r-2}j_{r-2}j_{r-4}j_{r-4}\ldots j_2j_2}(s,\tau)\right| \le K < \infty \tag{2.1371}$$

and prove (2.1365).

Using the induction hypothesis (see (2.1369)–(2.1371)), we obtain

$$\left|\sum_{j_r=0}^{p} C_{j_rj_r}(s,\tau) \sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right| \le K^2 < \infty, \tag{2.1372}$$

$$\left|\sum_{j_r,j_{r-2}=0}^{p} C_{j_{r-2}j_{r-2}j_rj_r}(s,\tau) \sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)\right| \le K^2 < \infty, \tag{2.1373}$$

$$\ldots$$

$$\left|\sum_{j_r,j_{r-2},\ldots,j_4=0}^{p} C_{j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau) \sum_{j_2=0}^{p} C_{j_2j_2}(s,\tau)\right| \le K^2 < \infty. \tag{2.1374}$$

Applying the inequality of Cauchy–Bunyakovsky, Parseval's equality and the induction hypothesis, we obtain

$$\left(\sum_{j_r=0}^{p} C_{j_r}(s,\tau) \sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \le$$

$$\le \sum_{j_r=0}^{p} \left(C_{j_r}(s,\tau)\right)^2 \sum_{j_r=0}^{p} \left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \le$$

$$\leq \sum_{j_r=0}^{\infty}\left(C_{j_r}(s,\tau)\right)^2\sum_{j_r=0}^{\infty}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \leq$$

$$\leq K_1\sum_{j_r=0}^{\infty}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 =$$

$$= K_1\sum_{j_r=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_r}(u)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(u,\tau)du\right)^2 =$$

$$= K_1\int\limits_{\tau}^{s}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(u,\tau)\right)^2 du \leq$$

$$\leq K_1K^2\int\limits_{\tau}^{s}du \leq (T-t)K_1K^2 = K_2 < \infty, \tag{2.1375}$$

where constant K_2 does not depend on p, s, τ;

$$\left(\sum_{j_{r-2}=0}^{p}\sum_{j_r=0}^{p} C_{j_{r-2}j_rj_r}(s,\tau)\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \leq$$

$$\leq \sum_{j_{r-2}=0}^{p}\left(\sum_{j_r=0}^{p} C_{j_{r-2}j_rj_r}(s,\tau)\right)^2\sum_{j_{r-2}=0}^{p}\left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \leq$$

$$\leq \sum_{j_{r-2}=0}^{\infty}\left(\sum_{j_r=0}^{p} C_{j_{r-2}j_rj_r}(s,\tau)\right)^2\sum_{j_{r-2}=0}^{\infty}\left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 =$$

$$= \sum_{j_{r-2}=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_{r-2}}(u)\sum_{j_r=0}^{p} C_{j_rj_r}(u,\tau)du\right)^2 \times$$

$$\times\sum_{j_{r-2}=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_{r-2}}(u)\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(u,\tau)du\right)^2 =$$

$$= \int\limits_{\tau}^{s}\left(\sum_{j_r=0}^{p} C_{j_rj_r}(u,\tau)\right)^2 du\times$$

$$\times \int\limits_{\tau}^{s} \left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(u,\tau) \right)^2 du \le K^4(T-t)^2 = K_3 < \infty. \tag{2.1376}$$

Similarly, we get

$$\left(\sum_{j_{r-4}=0}^{p} \sum_{j_r,j_{r-2}=0}^{p} C_{j_{r-4}j_{r-2}j_{r-2}j_rj_r}(s,\tau) \sum_{j_{r-6},\ldots,j_4,j_2=0}^{p} C_{j_{r-4}j_{r-6}j_{r-6}\ldots j_4j_4j_2j_2}(s,\tau) \right)^2 \le$$

$$\le K_4 < \infty, \tag{2.1377}$$

$$\ldots$$

$$\left(\sum_{j_4=0}^{p} \sum_{j_r,j_{r-2},\ldots,j_6=0}^{p} C_{j_4j_6j_6\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau) \sum_{j_2=0}^{p} C_{j_4j_2j_2}(s,\tau) \right)^2 \le K_4 < \infty, \tag{2.1378}$$

$$\left(\sum_{j_2=0}^{p} \sum_{j_r,j_{r-2},\ldots,j_4=0}^{p} C_{j_2j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau) \cdot C_{j_2}(s,\tau) \right)^2 \le K_4 < \infty, \tag{2.1379}$$

where constant K_4 does not depend on p, s, τ.

Combining (2.1368), (2.1372)–(2.1374), (2.1375), (2.1376), (2.1377)–(2.1379), we obtain (2.1365). The equality (2.1341) is proved for the case when the condition (A) and the relation (2.1335) are satisfied.

2.36 Further Development of the Approach Based on Theorem 2.61 for the Case $\psi_1(\tau), \ldots, \psi_7(\tau) \equiv 1$. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 7 (The Cases of Legendre Polynomials and Trigonometric Functions)

Unfortunately, the approach from the previous section can be generalized only partially to the case when the condition (A) and the relation (2.1336) are satisfied (see Sect. 2.31). In particular, the mentioned approach is applicable to the proof of inequality

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}(s,\tau)\right| \le K < \infty,$$

but is not applicable to the proof of inequality

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right| \le K < \infty,$$

where $C_{j_k\ldots j_1}(s,\tau)$ is defined by (2.1366), constant K does not depend on p,s,τ $(p\in\mathbf{N},\ t\le\tau<s\le T)$.

In this section, we will restrict ourselves to the case $k=7$, $r=1,2,3$ and we will also assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the condition (2.1341) can be weakened. Namely, the constant K^2 can be replaced by the function F such that $\psi_{q_1}^2\ldots\psi_{q_{k-2r}}^2 F\in L_1([t,T]^{k-2r})$ (see (2.1347)). For the trigonometric case, we will prove (2.1341) for $k=7$, $r=1,2,3$. For the polynomial case, we will prove a weakened version of (2.1341) for $k=7$, $r=1,2,3$ (the constant K and the above function F will be used in the weakened version of (2.1341)).

Obviously, that the conditions (2.1349)–(2.1361) together with the following condition

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_2}(\rho,v)C_{j_1}(\theta,u)C_{j_2}(\mu,w)\right| \le K \qquad (2.1380)$$

cover the case $k=7$, $r=1,2$ (see (2.1341)), where $p\in\mathbf{N}$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, $t\le v<\rho\le T$, $t\le w<\mu\le T$, constant K does not depend on $p,s,\tau,u,\theta,v,\rho,w,\mu$ (but only on t,T). The inequality (2.1380) is easily verified using (2.969).

Now let us focus on the proof of (2.1341) for the case $k=7$ and $r=3$. So, we need to prove that

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p} C_{j_{d_1}j_{d_1-1}j_{d_1-2}j_{d_1-3}j_{d_1-4}j_{d_1-5}}(s,\tau)\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right| \le K < \infty, \qquad (2.1381)$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}j_{d_2-3}j_{d_2-4}}(s,\tau)C_{j_{d_1}}(\theta,u)\right)\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le K<\infty, \tag{2.1382}$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}j_{d_2-3}}(s,\tau)C_{j_{d_1}j_{d_1-1}}(\theta,u)\right)\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le K<\infty, \tag{2.1383}$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}}(s,\tau)C_{j_{d_1}j_{d_1-1}j_{d_1-2}}(\theta,u)\right)\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le K<\infty, \tag{2.1384}$$

where $p\in\mathbf{N}$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, constant K does not depend on p,s,τ,u,θ (but only on t,T) and may differ from line to line; another notations are the same as in Sect. 2.31.

The inequalities (2.1382)–(2.1384) are proved using the same technique as inequalities (2.1349)–(2.1361) (see Sect. 2.32). Here we will only prove as an example the following special case of the inequality (2.1383)

$$\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)\right|\le K<\infty. \tag{2.1385}$$

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (2.1350), we have

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)\right)^2\le$$

$$\le\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=0}^{p}C_{j_2j_3j_2j_1}(s,\tau)\right)^2\sum_{j_1,j_3=0}^{p}C_{j_3j_1}^2(\theta,u)\le$$

$$\le\sum_{j_1,j_3=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(u)\int\limits_{\tau}^{u}\phi_{j_3}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdydzdu\right)^2\times$$

$$\times\sum_{j_1,j_3=0}^{\infty}C_{j_3j_1}^2(\theta,u)=$$

$$=\sum_{j_1,j_3=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_3}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdy\int\limits_{z}^{s}\phi_{j_2}(u)dudz\right)^2\cdot\frac{(\theta-u)^2}{2}=$$

$$
=\frac{(\theta-u)^2}{2}\sum_{j_1,j_3=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_3}(z)\int\limits_{\tau}^{z}\phi_{j_1}(x)\int\limits_{x}^{z}\phi_{j_2}(y)dydx\int\limits_{z}^{s}\phi_{j_2}(u)dudz\right)^2=
$$

$$
=\frac{(\theta-u)^2}{2}\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_3}(z)\int\limits_{\tau}^{z}\phi_{j_1}(x)\sum_{j_2=0}^{p}C_{j_2}(z,x)C_{j_2}(s,z)dxdz\right)^2=
$$

$$
=\frac{(\theta-u)^2}{2}\int\limits_{\tau}^{s}\int\limits_{\tau}^{z}\left(\sum_{j_2=0}^{p}C_{j_2}(z,x)C_{j_2}(s,z)\right)^2dxdz\le
$$

$$
\le K^2\frac{(\theta-u)^2}{2}\frac{(s-\tau)^2}{2}\le K^2\frac{(T-t)^4}{4}=K_1. \qquad (2.1386)
$$

The equality (2.1385) is proved.

The main difficulty is related to the proof of the inequality (2.1381). Further, we prove (2.1381) for all 15 possible cases under the assumption that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. As we noted above, in some situations we will need a function $F\in L_1([t,T])$ instead of a constant K^2 for the polynomial case.

It is easy to see that (2.1381) reduces to the following 15 inequalities

$$
\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1j_3j_2j_1}(s,\tau)\right|\le K<\infty, \qquad (2.1387)
$$

$$
\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_3j_2j_3j_2j_1}(s,\tau)\right|\le K<\infty, \qquad (2.1388)
$$

$$
\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_3j_1j_2j_1}(s,\tau)\right|\le K<\infty, \qquad (2.1389)
$$

$$
\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_2j_3j_3j_2j_1}(s,\tau)\right|\le K<\infty, \qquad (2.1390)
$$

$$
\left|\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_2j_2j_3j_3j_1}(s,\tau)\right|\le K<\infty, \qquad (2.1391)
$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_2j_1j_1}(s,\tau)\right| \le K < \infty, \tag{2.1392}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1j_1}(s,\tau)\right| \le K < \infty, \tag{2.1393}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1j_1}(s,\tau)\right| \le K < \infty, \tag{2.1394}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1395}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1396}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1397}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2j_3j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1398}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1399}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1400}$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right| \le K < \infty, \tag{2.1401}$$

where $p \in \mathbf{N}$, $t \le \tau < s \le T$, constant K does not depend on p, s, τ (but only on t, T) and may differ from line to line.

More precisely, the conditions (2.1387)–(2.1401) need to be proved in two cases: 1. $\tau = t$, 2. $s = T$. Further, we will not carry out such a refinement if some estimate from (2.1387)–(2.1401) is true for all $\tau, s \in [t, T]$ $(\tau < s)$. Looking ahead, we note that consideration of Cases 1 and 2 will be required only for some inequalities from (2.1387)–(2.1401) for the polynomial case.

The relation (2.1392) is a particular case of (2.1365). Let us prove (2.1387)–(2.1391), (2.1393)–(2.1401) using ideas from Sect. 2.11, 2.14, 2.32.

Step 1. First, we prove (2.1387)–(2.1391), (2.1397) using special symmetry properties of the Fourier coefficients.

By analogy with (2.848) we obtain

$$
\begin{gathered}
C_{j_6 j_5 j_4 j_3 j_2 j_1}(s,\tau)+C_{j_1 j_2 j_3 j_4 j_5 j_6}(s,\tau)= \\
=C_{j_6}(s,\tau)C_{j_5 j_4 j_3 j_2 j_1}(s,\tau)-C_{j_5 j_6}(s,\tau)C_{j_4 j_3 j_2 j_1}(s,\tau)+ \\
+C_{j_4 j_5 j_6}(s,\tau)C_{j_3 j_2 j_1}(s,\tau)-C_{j_3 j_4 j_5 j_6}(s,\tau)C_{j_2 j_1}(s,\tau)+ \\
+C_{j_2 j_3 j_4 j_5 j_6}(s,\tau)C_{j_1}(s,\tau). \qquad (2.1402)
\end{gathered}
$$

Using (2.1402), we get

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1 j_3 j_2 j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}(s,\tau)C_{j_2 j_1 j_3 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_2 j_3}(s,\tau)C_{j_1 j_3 j_2 j_1}(s,\tau)+C_{j_1 j_2 j_3}(s,\tau)C_{j_3 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_3 j_1 j_2 j_3}(s,\tau)C_{j_2 j_1}(s,\tau)+C_{j_2 j_3 j_1 j_2 j_3}(s,\tau)C_{j_1}(s,\tau)\Biggr), \qquad (2.1403)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_1 j_3 j_2 j_3 j_2 j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_1}(s,\tau)C_{j_3 j_2 j_3 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_3 j_1}(s,\tau)C_{j_2 j_3 j_2 j_1}(s,\tau)+C_{j_2 j_3 j_1}(s,\tau)C_{j_3 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_3 j_2 j_3 j_1}(s,\tau)C_{j_2 j_1}(s,\tau)+C_{j_2 j_3 j_2 j_3 j_1}(s,\tau)C_{j_1}(s,\tau)\Biggr), \qquad (2.1404)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_3 j_1 j_2 j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}(s,\tau)C_{j_2 j_3 j_1 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_2 j_3}(s,\tau)C_{j_3 j_1 j_2 j_1}(s,\tau)+C_{j_3 j_2 j_3}(s,\tau)C_{j_1 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_1 j_3 j_2 j_3}(s,\tau)C_{j_2 j_1}(s,\tau)+C_{j_2 j_1 j_3 j_2 j_3}(s,\tau)C_{j_1}(s,\tau)\Biggr), \qquad (2.1405)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_1 j_2 j_3 j_3 j_2 j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_1}(s,\tau)C_{j_2 j_3 j_3 j_2 j_1}(s,\tau)-
$$
$$
-C_{j_2 j_1}(s,\tau)C_{j_3 j_3 j_2 j_1}(s,\tau)+\left(C_{j_3 j_2 j_1}(s,\tau)\right)^2-
$$

$$
-C_{j_3j_3j_2j_1}(s,\tau)C_{j_2j_1}(s,\tau)+C_{j_2j_3j_3j_2j_1}(s,\tau)C_{j_1}(s,\tau)\Biggr), \qquad (2.1406)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_3j_2j_2j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_1}(s,\tau)C_{j_3j_3j_2j_2j_1}(s,\tau)-
$$

$$
-C_{j_3j_1}(s,\tau)C_{j_3j_2j_2j_1}(s,\tau)+C_{j_3j_3j_1}(s,\tau)C_{j_2j_2j_1}(s,\tau)-
$$

$$
-C_{j_2j_3j_3j_1}(s,\tau)C_{j_2j_1}(s,\tau)+C_{j_2j_2j_3j_3j_1}(s,\tau)C_{j_1}(s,\tau)\Biggr), \qquad (2.1407)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2}(s,\tau)C_{j_1j_3j_3j_2j_1}(s,\tau)-
$$

$$
-C_{j_1j_2}(s,\tau)C_{j_3j_3j_2j_1}(s,\tau)+C_{j_3j_1j_2}(s,\tau)C_{j_3j_2j_1}(s,\tau)-
$$

$$
-C_{j_3j_3j_1j_2}(s,\tau)C_{j_2j_1}(s,\tau)+C_{j_2j_3j_3j_1j_2}(s,\tau)C_{j_1}(s,\tau)\Biggr). \qquad (2.1408)
$$

Applying to the right-hand sides of (2.1403)–(2.1408) the technique that led to the estimate (2.1386), we obtain the inequalities (2.1387)–(2.1391), (2.1397).

Step 2. It is not difficult to see that

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)=\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_1j_2j_3j_3j_2}(s,\tau), \qquad (2.1409)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)=\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_1j_2j_3j_2j_3}(s,\tau), \qquad (2.1410)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)=\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_2j_3j_1j_3}(s,\tau). \qquad (2.1411)
$$

Further, using (2.1409)–(2.1411) and (2.1402), we get

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1j_1}(s,\tau)+\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)=
$$

$$
=\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1j_1}(s,\tau)+\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_1j_2j_3j_3j_2}(s,\tau)=
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_2}(s,\tau) C_{j_3j_3j_2j_1j_1}(s,\tau) -
$$

$$
- C_{j_3j_2}(s,\tau) C_{j_3j_2j_1j_1}(s,\tau) + C_{j_3j_3j_2}(s,\tau) C_{j_2j_1j_1}(s,\tau) -
$$

$$
- C_{j_2j_3j_3j_2}(s,\tau) C_{j_1j_1}(s,\tau) + C_{j_1j_2j_3j_3j_2}(s,\tau) C_{j_1}(s,\tau) \Biggr), \qquad (2.1412)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1j_1}(s,\tau) + \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau) =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1j_1}(s,\tau) + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_1j_2j_3j_2j_3}(s,\tau) =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_3}(s,\tau) C_{j_2j_3j_2j_1j_1}(s,\tau) -
$$

$$
- C_{j_2j_3}(s,\tau) C_{j_3j_2j_1j_1}(s,\tau) + C_{j_3j_2j_3}(s,\tau) C_{j_2j_1j_1}(s,\tau) -
$$

$$
- C_{j_2j_3j_2j_3}(s,\tau) C_{j_1j_1}(s,\tau) + C_{j_1j_2j_3j_2j_3}(s,\tau) C_{j_1}(s,\tau) \Biggr), \qquad (2.1413)
$$

$$
\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_2j_1}(s,\tau) + \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau) =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_2j_1}(s,\tau) + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_2j_3j_1j_3}(s,\tau) =
$$

$$
= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_3}(s,\tau) C_{j_1j_3j_2j_2j_1}(s,\tau) -
$$

$$
- C_{j_1j_3}(s,\tau) C_{j_3j_2j_2j_1}(s,\tau) + C_{j_3j_1j_3}(s,\tau) C_{j_2j_2j_1}(s,\tau) -
$$

$$
- C_{j_2j_3j_1j_3}(s,\tau) C_{j_2j_1}(s,\tau) + C_{j_2j_2j_3j_1j_3}(s,\tau) C_{j_1}(s,\tau) \Biggr). \qquad (2.1414)
$$

Applying to the right-hand sides of (2.1412)–(2.1414) the technique that led to the estimate (2.1386), we obtain the inequalities

$$
\left| \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1j_1}(s,\tau) + \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau) \right| \le K < \infty, \quad (2.1415)
$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1j_1}(s,\tau)+\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)\right|\le K<\infty, \quad (2.1416)$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_2j_1}(s,\tau)+\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right|\le K<\infty, \quad (2.1417)$$

where $p\in\mathbf{N}$, $t\le\tau<s\le T$, constant K does not depend on p,s,τ (but only on t,T) and may differ from line to line.

Note that $|a|\le K_1+K$ follows from $|b|\le K$ and $|a+b|\le K_1$, where $a,b,K,K_1\in\mathbf{R}$. Indeed, we have $|a|=|a+b-b|\le|a+b|+|b|\le K_1+K$. Then from (2.1415)–(2.1417) it follows that if we prove (2.1395), (2.1396), (2.1401), then (2.1394), (2.1393), (2.1400) will be proved. Thus, it remains to prove (2.1395), (2.1396), (2.1398), (2.1399), (2.1401).

Step 3. Let us prove (2.1395), (2.1396), (2.1398), (2.1399), (2.1401). Consider (2.1399). Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality, (2.684), (2.969) and Lebesgue's Dominated Convergence Theorem, we have

$$\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2=\left(\sum_{j_2=0}^{p}1\cdot\sum_{j_1,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2\le$$

$$\le\sum_{j_2=0}^{p}1^2\cdot\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2=$$

$$=(p+1)\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2=$$

$$=(p+1)\sum_{j_2=0}^{p}\left(\sum_{j_1,j_3=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(t_6)\int\limits_{\tau}^{t_6}\phi_{j_2}(t_2)C_{j_1}(t_2,\tau)C_{j_3j_1j_3}(t_6,t_2)dt_2dt_6\right)^2\le$$

$$\le(p+1)\sum_{j_2,j_2'=0}^{p}\left(\sum_{j_1,j_3=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(t_6)\int\limits_{\tau}^{t_6}\phi_{j_2'}(t_2)C_{j_1}(t_2,\tau)C_{j_3j_1j_3}(t_6,t_2)dt_2dt_6\right)^2\le$$

$$\le(p+1)\sum_{j_2,j_2'=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_2}(t_6)\int\limits_{\tau}^{t_6}\phi_{j_2'}(t_2)\sum_{j_1,j_3=0}^{p}C_{j_1}(t_2,\tau)C_{j_3j_1j_3}(t_6,t_2)dt_2dt_6\right)^2=$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\left(\sum_{j_1=0}^{p} C_{j_1}(t_2,\tau)\sum_{j_3=0}^{p} C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 =
$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\left(\sum_{j_1=0}^{p} C_{j_1}(t_2,\tau)\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 \le
$$

$$
\le (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_1=0}^{p} C_{j_1}^2(t_2,\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 \le
$$

$$
\le (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_1=0}^{\infty} C_{j_1}^2(t_2,\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 =
$$

$$
\le (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 =
$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^{t_6}\phi_{j_1}(\theta)C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)d\theta\right)^2 dt_2dt_6 =
$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{p}\left(\int\limits_{t_2}^{t_6}\phi_{j_1}(\theta)\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)d\theta\right)^2 dt_2dt_6 \le
$$

$$
\le (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{\infty}\left(\int\limits_{t_2}^{t_6}\phi_{j_1}(\theta)\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)d\theta\right)^2 dt_2dt_6 =
$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}(t_2-\tau)\int\limits_{t_2}^{t_6}\left(\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)\right)^2 d\theta dt_2dt_6. \quad (2.1418)
$$

For the trigonometric case (see (1.69)), we have the following obvious estimate

$$
|C_j(x,v)| = \left|\int\limits_{v}^{x}\phi_j(\tau)d\tau\right| < \frac{C}{j} \quad (j>0), \qquad (2.1419)
$$

where constant C does not depend on j, x, v.

Recall that (see (2.25))

$$\sum_{j=p+1}^{\infty}\frac{1}{j^2}\le\int\limits_{p}^{\infty}\frac{dx}{x^2}=\frac{1}{p}. \tag{2.1420}$$

Combining (2.1418)–(2.1420), we get

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2\le\frac{K_1(p+1)}{p^2}\le K^2,$$

where constants K, K_1 depend only on t, T. The inequality (2.1399) is proved for the trigonometric case.

For the polynomial case (see (2.65)), by analogy with (1.211) and (2.740) we have

$$|C_j(x,v)|=\left|\int\limits_{v}^{x}\phi_j(\tau)d\tau\right|<\frac{C}{j^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(x))^{1/4-\varepsilon/4}}+\frac{1}{(1-z^2(v))^{1/4-\varepsilon/4}}\right), \tag{2.1421}$$

where $j\in\mathbf{N}$, $z(x),z(v)\in(-1,1)$ ($z(x)$ is defined by (2.20)), $x,v\in(t,T)$, $\varepsilon\in(0,1)$ is an arbitrary small positive real number, constant C does not depend on j.

Recall that (see (2.743))

$$\sum_{j=p+1}^{\infty}\frac{1}{j^{2-\varepsilon}}\le\int\limits_{p}^{\infty}\frac{dx}{x^{2-\varepsilon}}=\frac{1}{(1-\varepsilon)p^{1-\varepsilon}}. \tag{2.1422}$$

Combining (2.1418), (2.1421), (2.1422) ($\varepsilon=1/4$), we obtain

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right)^2\le\frac{K_1(p+1)}{p^{3/2}}\le K^2,$$

where constants K, K_1 depend only on t, T. The inequality (2.1399) is proved for the polynomial case.

Let us prove (2.1398). In complete analogy with the proof of (2.1399) we have

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2j_3j_2j_1}(s,\tau)\right)^2\le$$

$$\le (p+1)\int\limits_{\tau}^{s}(s-t_5)\int\limits_{\tau}^{t_5}\int\limits_{t_1}^{t_5}\left(\sum_{j_2=p+1}^{\infty}C_{j_2}(\theta,t_1)C_{j_2}(t_5,\theta)\right)^2 d\theta dt_1 dt_5.$$

The further proof is the same as in the case of (2.1399). The inequality (2.1398) is proved.

Let us prove (2.1401). By analogy with the proof of (2.1399) (see (2.1418)) we get

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right)^2 \le$$

$$\le (p+1)\int\limits_{\tau}^{s}(s-t_5)\int\limits_{\tau}^{t_5}\int\limits_{\tau}^{t_4}\left(\sum_{j_1=p+1}^{\infty}C_{j_1}(\theta,\tau)C_{j_1}(t_4,\theta)\right)^2 d\theta dt_4 dt_5. \quad (2.1423)$$

The further proof for the trigonometric case is the same as for the inequality (2.1399).

Consider the polynomial case. In this case, we note that it is actually necessary to consider the following two cases of (2.1423)

$$1.\ \tau=t, \quad 2.\ s=T. \quad (2.1424)$$

For Case 1, the estimate (2.1421) is simplified as follows (see (2.293), (2.739) and (2.740))

$$|C_j(x,t)|=\left|\int\limits_{t}^{x}\phi_j(\tau)d\tau\right|<\frac{C}{j^{1-\varepsilon/2}}\frac{1}{(1-z^2(x))^{1/4-\varepsilon/4}}, \quad (2.1425)$$

where notations are the same as in (2.1421).

Combining (2.1423), (2.1421), (2.1422), (2.1425) ($\varepsilon=1/4$), we obtain

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_3j_1j_2j_1}(s,t)\right)^2 \le \frac{K_1(p+1)}{p^{3/2}} \le K^2, \quad (2.1426)$$

where constants K, K_1 depend only on t, T. The inequality (2.1401) is proved for the polynomial case (Case 1).

Consider Case 2. Combining (2.1423), (2.1421), (2.1422) ($\varepsilon=1/4$), we obtain

$$\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_3j_1j_2j_1}(T,\tau)\right)^2 \le \frac{K_1(p+1)}{p^{3/2}}\frac{1}{(1-z^2(\tau))^{3/8}} \le$$

$$
\le \frac{K^2}{(1-z^2(\tau))^{3/8}} \stackrel{\mathsf{def}}{=} F(\tau),
$$

where constants K, K_1 depend only on t, T and $F(\tau) \in L_1([t,T])$ (integrable majorant (see above in this section)). The following weakened version of the inequality (2.1401)

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(T,\tau)\right)^2 \le F(\tau) \tag{2.1427}
$$

is proved for the polynomial case (Case 2), where

$$
F(\tau) = \frac{K^2}{(1-z^2(\tau))^{3/8}}.
$$

Let us prove (2.1396). Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem and Parseval's equality, we have

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 = \left(\sum_{j_3=0}^{p} 1 \cdot \sum_{j_1,j_2=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 \le
$$

$$
\le \sum_{j_3=0}^{p} 1^2 \cdot \sum_{j_3=0}^{p} \left(\sum_{j_1,j_2=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 =
$$

$$
= (p+1) \sum_{j_3=0}^{p} \left(\sum_{j_1,j_2=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 =
$$

$$
= (p+1) \sum_{j_3=0}^{p} \left(\sum_{j_1,j_2=0}^{p} \int\limits_{\tau}^{s} \phi_{j_3}(t_6) \int\limits_{\tau}^{t_6} \phi_{j_3}(t_5) C_{j_1j_2j_2j_1}(t_5,\tau) dt_5 dt_6\right)^2 \le
$$

$$
\le (p+1) \sum_{j_3,j_3'=0}^{p} \left(\int\limits_{\tau}^{s} \phi_{j_3}(t_6) \int\limits_{\tau}^{t_6} \phi_{j_3'}(t_5) \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,\tau) dt_5 dt_6\right)^2 \le
$$

$$
\le (p+1) \sum_{j_3,j_3'=0}^{\infty} \left(\int\limits_{\tau}^{s} \phi_{j_3}(t_6) \int\limits_{\tau}^{t_6} \phi_{j_3'}(t_5) \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,\tau) dt_5 dt_6\right)^2 =
$$

$$
= (p+1) \int\limits_{\tau}^{s} \int\limits_{\tau}^{t_6} \left(\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(t_5,\tau)\right)^2 dt_5 dt_6 =
$$

$$
= (p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\left(\sum_{j_2=0}^{p} 1\cdot\sum_{j_1=0}^{p} C_{j_1j_2j_2j_1}(t_5,\tau)\right)^2 dt_5dt_6 \le
$$

$$
\le (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_2j_1}(t_5,\tau)\right)^2 dt_5dt_6 =
$$

$$
= (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_5}\phi_{j_2}(t_3)\int\limits_{\tau}^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,\tau)C_{j_1}(t_5,t_3)dt_2dt_3\right)^2 dt_5dt_6 \le
$$

$$
\le (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2,j_2'=0}^{p}\left(\int\limits_{\tau}^{t_5}\phi_{j_2}(t_3)\times\right.
$$

$$
\left.\times\int\limits_{\tau}^{t_3}\phi_{j_2'}(t_2)\sum_{j_1=0}^{p} C_{j_1}(t_2,\tau)C_{j_1}(t_5,t_3)dt_2dt_3\right)^2 dt_5dt_6 \le
$$

$$
\le (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2,j_2'=0}^{\infty}\left(\int\limits_{\tau}^{t_5}\phi_{j_2}(t_3)\times\right.
$$

$$
\left.\times\int\limits_{\tau}^{t_3}\phi_{j_2'}(t_2)\left(\sum_{j_1=0}^{\infty}-\sum_{j_1=p+1}^{\infty}\right)C_{j_1}(t_2,\tau)C_{j_1}(t_5,t_3)dt_2dt_3\right)^2 dt_5dt_6 =
$$

$$
= (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\int\limits_{\tau}^{t_5}\int\limits_{\tau}^{t_3}\left(\sum_{j_1=p+1}^{\infty} C_{j_1}(t_2,\tau)C_{j_1}(t_5,t_3)\right)^2 dt_2dt_3dt_5dt_6. \quad (2.1428)
$$

Consider the trigonometric case. Combining (2.1428), (2.1419), (2.1420), we obtain

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 \le \frac{K_1(p+1)^2}{p^2} \le K^2,
$$

where constants K, K_1 depend only on t, T. The inequality (2.1396) is proved for the trigonometric case.

Consider the polynomial case for two cases (2.1424). Let $\tau = t$. The modification of the estimate (2.1421) for $\varepsilon = 0$ is as follows (see also (1.210), (1.211))

$$
|C_j(x,v)| = \left|\int\limits_{v}^{x}\phi_j(\tau)d\tau\right| < \frac{C}{j}\left(\frac{1}{(1-z^2(x))^{1/4}}+\frac{1}{(1-z^2(v))^{1/4}}\right), \quad (2.1429)
$$

where $j \in \mathbf{N}$, $z(x), z(v) \in (-1, 1)$ ($z(x)$ is defined by (2.20)), $x, v \in (t, T)$, constant C does not depend on j.

For $v = t$, the estimate (2.1429) is simplified as follows (see (2.293), (2.294))

$$|C_j(x, t)| = \left| \int\limits_t^x \phi_j(\tau) d\tau \right| < \frac{C}{j(1 - z^2(x))^{1/4}}, \tag{2.1430}$$

where notations are the same as in (2.1429).

Combining (2.1428), (2.1429), (2.1430), we get

$$\left(\sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_3 j_1 j_2 j_2 j_1}(s, t) \right)^2 \le \frac{K_1 (p+1)^2}{p^2} \le K^2,$$

where constants K, K_1 depend only on t, T. The inequality (2.1396) is proved for the polynomial case ($\tau = t$).

Now let $s = T$. Combining (2.1428) and (2.1429), we obtain

$$\left(\sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_3 j_1 j_2 j_2 j_1}(T, \tau) \right)^2 \le \frac{K_1 (p+1)^2}{p^2} \frac{1}{(1 - z^2(\tau))^{1/2}} \le$$

$$\le \frac{K^2}{(1 - z^2(\tau))^{1/2}} \stackrel{\sf def}{=} F(\tau),$$

where constants K, K_1 depend only on t, T and $F(\tau) \in L_1([t, T])$ (integrable majorant (see above in this section)). The following weakened version of the inequality (2.1396)

$$\left(\sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_3 j_1 j_2 j_2 j_1}(T, \tau) \right)^2 \le F(\tau) \tag{2.1431}$$

is proved for the polynomial case ($s = T$), where

$$F(\tau) = \frac{K^2}{(1 - z^2(\tau))^{1/2}}.$$

Finally, we prove the inequality (2.1395). By analogy with (2.1428) we get

$$\left(\sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_3 j_2 j_1 j_2 j_1}(s, \tau) \right)^2 \le$$

$$
\le (p+1)\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)\right)^2 =
$$

$$
=(p+1)\sum_{j_3=0}^{p}\left(\sum_{j_1,j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_3}(t_6)\int\limits_{\tau}^{t_6}\phi_{j_3}(t_5)C_{j_2j_1j_2j_1}(t_5,\tau)dt_5dt_6\right)^2 \le
$$

$$
\le (p+1)\sum_{j_3,j_3'=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_3}(t_6)\int\limits_{\tau}^{t_6}\phi_{j_3'}(t_5)\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(t_5,\tau)dt_5dt_6\right)^2 =
$$

$$
=(p+1)\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_2j_1}(t_5,\tau)\right)^2 dt_5dt_6 =
$$

$$
\le (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_2j_1j_2j_1}(t_5,\tau)\right)^2 dt_5dt_6 =
$$

$$
=(p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_5}\phi_{j_2}(t_4)\int\limits_{\tau}^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,\tau)C_{j_1}(t_4,t_2)dt_2dt_4\right)^2 dt_5dt_6 \le
$$

$$
\le (p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\sum_{j_2,j_2'=0}^{\infty}\left(\int\limits_{\tau}^{t_5}\phi_{j_2}(t_4)\times\right.
$$

$$
\left.\times\int\limits_{\tau}^{t_4}\phi_{j_2'}(t_2)\left(\sum_{j_1=0}^{\infty}-\sum_{j_1=p+1}^{\infty}\right)C_{j_1}(t_2,\tau)C_{j_1}(t_4,t_2)dt_2dt_4\right)^2 dt_5dt_6 =
$$

$$
=(p+1)^2\int\limits_{\tau}^{s}\int\limits_{\tau}^{t_6}\int\limits_{\tau}^{t_5}\int\limits_{\tau}^{t_4}\left(\sum_{j_1=p+1}^{\infty}C_{j_1}(t_2,\tau)C_{j_1}(t_4,t_2)\right)^2 dt_2dt_4dt_5dt_6. \quad (2.1432)
$$

The further proof of inequality (2.1395) for the trigonometric case and the weakened analogue of inequality (2.1395) for the polynomial case is completely analogous to the proof of (2.1401) and its weakened analogue (see (2.1423), (2.1426), (2.1427)).

Thus, the following theorem is proved.

Theorem 2.64. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then,* *for the iterated Stratonovich stochastic integral of seventh multiplicity*

$$J^*[\psi^{(7)}]_{T,t} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_7}^{(i_7)}$$

the following expansion

$$J^*[\psi^{(7)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_7=0}^{p} C_{j_7\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_7}^{(i_7)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_7 = 0,1,\ldots,m,$

$$C_{j_7\ldots j_1} = \int\limits_t^T \phi_{j_7}(t_7) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_7$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau.$

2.37 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 8 for the Case $\psi_1(\tau),\ldots,\psi_8(\tau) \equiv 1$ (The Cases of Legendre Polynomials and Trigonometric Functions)

This section is devoted to the following theorem.

Theorem 2.65. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of eighth multiplicity*

$$J^*[\psi^{(8)}]_{T,t} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_8}^{(i_8)}$$

the following expansion

$$J^*[\psi^{(8)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_8=0}^{p} C_{j_8\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_8}^{(i_8)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_8=0,1,\ldots,m,$

$$
C_{j_8\ldots j_1}=\int\limits_t^T\phi_{j_8}(t_8)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_8
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_{\tau}^{(0)}=\tau.$

Proof. To prove the theorem, we need to check the condition (2.1341) (or the condition (2.1347)) for the case $k=8>2r$, where $r=1,2,3$ (see Theorem 2.61). Recall that the case $k=2r$ is considered in Sect. 2.27.4 (see (2.1262)). Under the conditions of Theorem 2.65, this means that $k=8=2r$, where $r=4$.

The relations (2.1349)–(2.1361), (2.1380) cover the case $k=8$, $r=1,2$ (see (2.1341)).

Thus, it remains to consider the case $k=8$, $r=3$. The case $k=7$, $r=3$ was considered in the previous section. Here we will focus on the differences between these two cases.

Since now $k=8$, then along with inequalities (2.1381)–(2.1384), it is necessary to prove the following inequalities

$$
\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_3}j_{d_3-1}j_{d_3-2}j_{d_3-3}}(s,\tau)C_{j_{d_2}}(\theta,u)C_{j_{d_1}}(\rho,v)\right)\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le
$$

$$
\le K<\infty, \tag{2.1433}
$$

$$
\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_3}j_{d_3-1}j_{d_3-2}}(s,\tau)C_{j_{d_2}j_{d_2-1}}(\theta,u)C_{j_{d_1}}(\rho,v)\right)\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le
$$

$$
\le K<\infty, \tag{2.1434}
$$

$$
\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p}\left(C_{j_{d_3}j_{d_3-1}}(s,\tau)C_{j_{d_2}j_{d_2-1}}(\theta,u)C_{j_{d_1}j_{d_1-1}}(\rho,v)\right)\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right|\le
$$

$$
\le K<\infty, \tag{2.1435}
$$

where $p\in \mathbf{N}$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, $t\le v<\rho\le T$, constant K does not depend on p,s,τ,θ,u,ρ,v (but only on t,T) and may differ from line to line; another notations are the same as in Sect. 2.31.

The inequalities (2.1433)–(2.1435) are proved using the same technique as inequalities (2.1349)–(2.1361) (see Sect. 2.32). Here we will only prove as an example the following special case of the inequality (2.1435)

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)C_{j_2j_3}(\rho,v)\right|\le K<\infty. \tag{2.1436}$$

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (2.1350), we have

$$\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)C_{j_2j_3}(\rho,v)\right)^2=$$

$$=\left(\sum_{j_2,j_3=0}^{p} C_{j_2j_3}(\rho,v)\sum_{j_1=0}^{p} C_{j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)\right)^2\le$$

$$\le\sum_{j_2,j_3=0}^{p} C_{j_2j_3}^2(\rho,v)\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)\right)^2\le$$

$$\le\sum_{j_2,j_3=0}^{\infty} C_{j_2j_3}^2(\rho,v)\sum_{j_2,j_3=0}^{\infty}\left(\sum_{j_1=0}^{p} C_{j_2j_1}(s,\tau)C_{j_3j_1}(\theta,u)\right)^2=$$

$$=\frac{(\rho-v)^2}{2}\sum_{j_2,j_3=0}^{\infty}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(t_2)\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{u}^{\theta}\phi_{j_3}(t_4)\int\limits_{u}^{t_4}\phi_{j_1}(t_3)dt_3dt_4\right)^2=$$

$$=\frac{(\rho-v)^2}{2}\sum_{j_2,j_3=0}^{\infty}\left(\int\limits_{\tau}^{s}\int\limits_{u}^{\theta}\phi_{j_2}(t_2)\phi_{j_3}(t_4)\times\right.$$

$$\left.\times\sum_{j_1=0}^{p}\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{u}^{t_4}\phi_{j_1}(t_3)dt_1dt_3dt_4dt_2\right)^2=$$

$$=\frac{(\rho-v)^2}{2}\int\limits_{\tau}^{s}\int\limits_{u}^{\theta}\left(\sum_{j_1=0}^{p} C_{j_1}(t_2,\tau)C_{j_1}(t_4,u)\right)^2dt_4dt_2\le$$

$$
\le K_1^2 \frac{(\rho - v)^2}{2}(s-\tau)(\theta - u) \le K_1^2 \frac{(T-t)^4}{2} = K.
$$

The inequality (2.1436) is proved.

The inequalities (2.1381)–(2.1384) for the case $k=8$ are proved similarly to the inequalities (2.1381)–(2.1384) for the case $k=7$ (see Sect. 2.36). There will be minor differences only when proving (2.1381) for the case $k=8$ (polynomial case). The above differences will be due to the fact that along with the two cases (2.1424) the following third case

$$
\tau, s \in (t, T)
$$

will now appear when proving (2.1395), (2.1396), (2.1401).

Using the technique that led to the estimates (2.1427), (2.1431), we obtain for Case 3

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)\right)^2 \le \frac{K^2}{(1-z^2(\tau))^{3/8}} \stackrel{\sf def}{=} F(\tau) \quad (\text{for } (2.1395)),
$$

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right)^2 \le \frac{K^2}{(1-z^2(\tau))^{1/2}} \stackrel{\sf def}{=} F(\tau) \quad (\text{for } (2.1396)),
$$

$$
\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right)^2 \le \frac{K^2}{(1-z^2(\tau))^{3/8}} \stackrel{\sf def}{=} F(\tau) \quad (\text{for } (2.1401)),
$$

where constant K depends only on t, T and $F(\tau) \in L_1([t,T])$ (integrable majorant (see (2.1347)). Theorem 2.65 is proved.

2.38 Verification of the Conditions of Theorems 2.52–2.57 for the Case $\psi_1(\tau),\ldots,\psi_5(\tau) \equiv \psi(\tau)$ and Generalization of Theorems 2.62, 2.64, 2.65 to the Case $\psi_1(\tau),\ldots,\psi_8(\tau) \equiv \psi(\tau)$

It is easy to see that Theorems 2.52–2.57 will be true if $\psi_1(\tau),\ldots,\psi_5(\tau) \equiv \psi(\tau)$, where $\psi(\tau) \in L_2([t,T])$ or $\psi(\tau)$ is a continuous function on $[t,T]$. Furthermore, Theorems 2.62, 2.64, 2.65 can be generalized to the case $\psi_1(\tau),\ldots,\psi_8(\tau) \equiv$

$\psi(\tau)$, where $\psi(\tau)$ is a continuous or continuously differentiable function on $[t,T]$. Let us provide the corresponding explanations.

Using Fubini's Theorem and Parseval's equality, we obtain for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv\psi(\tau)\in L_2([t,T])$

$$\sum_{j_1=0}^{p}\int\limits_t^s \psi_2(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_1}(\theta)d\theta d\tau =$$

$$=\sum_{j_1=0}^{p}\int\limits_t^s \psi(\tau)\phi_{j_1}(\tau)\int\limits_t^{\tau}\psi(\theta)\phi_{j_1}(\theta)d\theta d\tau = \frac{1}{2}\sum_{j_1=0}^{p}\left(\int\limits_t^s \psi(\tau)\phi_{j_1}(\tau)d\tau\right)^2 \le$$

$$\le \frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \psi(\tau)\phi_{j_1}(\tau)d\tau\right)^2 = \int\limits_t^s \psi^2(\tau)d\tau \le$$

$$\le \int\limits_t^T \psi^2(\tau)d\tau = K<\infty, \tag{2.1437}$$

$$\sum_{j_3=0}^{p}\int\limits_s^T \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{T}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau =$$

$$=\sum_{j_3=0}^{p}\int\limits_s^T \psi(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{T}\psi(\theta)\phi_{j_3}(\theta)d\theta d\tau = \frac{1}{2}\sum_{j_3=0}^{p}\left(\int\limits_s^T \psi(\tau)\phi_{j_3}(\tau)d\tau\right)^2 \le$$

$$\le \frac{1}{2}\sum_{j_3=0}^{\infty}\left(\int\limits_s^T \psi(\tau)\phi_{j_3}(\tau)d\tau\right)^2 = \int\limits_s^T \psi^2(\tau)d\tau \le$$

$$\le \int\limits_t^T \psi^2(\tau)d\tau = K<\infty$$

$\forall p\in\mathbf{N}$, where constant K does not depend on p and s $(t\le s\le T)$.

Thus, the conditions (2.1166), (2.1167) are fulfilled and Theorem 2.52 is true for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv\psi(\tau)\in L_2([t,T])$. Therefore, Theorem 2.53 is also true for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv\psi(\tau)$, where $\psi(\tau)$ is a continuous function on $[t,T]$.

By analogy with (2.1437) and (2.1108)–(2.1110) we obtain the inequalities (2.1189)–(2.1192) for the case $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv\psi(\tau)\in L_2([t,T])$. Thus, Theorems 2.54, 2.56 are true for the above case. Moreover, Theorems 2.55, 2.57 are true for the case $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv\psi(\tau)$, where $\psi(\tau)$ is a continuous function on $[t,T]$.

Generalizations (for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv\psi(\tau)\in L_2([t,T])$) of the relations (2.1349)–(2.1361) (those that were not mentioned earlier in this section) are proved similarly to (2.1349)–(2.1361) (see Sect. 2.32). This means that Theorem 2.62 is generalized to the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv\psi(\tau)$, where $\psi(\tau)$ is a continuous function on $[t,T]$.

In addition to all that has been said, we note that the proofs of Theorems 2.64, 2.65 can be easily modified to the case $\psi_1(\tau),\ldots,\psi_8(\tau)\equiv\psi(\tau)$, where $\psi(\tau)$ is a continuously differentiable function on $[t,T]$. In this case, it is necessary to use (1.211), (2.294), (2.740) as well as the following estimate for the case of Legendre polynomials

$$\left|\int\limits_v^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(x))^{1/4-\varepsilon/4}}+\frac{1}{(1-z^2(v))^{1/4-\varepsilon/4}}\right),$$

where $j\in\mathbf{N}$, $z(x),z(v)\in(-1,1)$ ($z(x)$ is defined by (2.20)), $x,v\in(t,T)$, $\varepsilon\in(0,1)$ is an arbitrary small positive real number, $\psi(\tau)$ is a continuously differentiable function on $[t,T]$, constant C does not depend on j.

2.39 Convergence of the Expansion (2.1346) to the Iterated Stratonovich Stochastic Integrals in the Sense of Mathematical Expectation

In the previous sections, we actually proved that the value

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

converges if $p\to\infty$ (under suitable conditions) to the iterated Stratonovich stochastic integrals (2.6) in the sense of mathematical expectation. Let us explain this fact in more detail.

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)$ $(k\in\mathbf{N})$ are continuous functions on $[t,T]$ and consider Theorem 2.12.

First, let $k=2q+1$, $q\in\mathbf{N}$. We represent (w. p. 1) each stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ from the right-hand side of (2.389) using the transformation (2.1063) as a finite linear combination of the iterated Itô stochastic integrals. Thus, we have (see (2.389))

$$\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}=0, \tag{2.1438}$$

where $J^*[\psi^{(k)}]_{T,t}$ is defined by (2.6). On the other hand,

$$\mathsf{M}\left\{\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}=0, \tag{2.1439}$$

since $\zeta_{j_l}^{(i_l)}$ has Gaussian distribution and $k=2q+1$, $q\in\mathbf{N}$.

Combining (2.1438) and (2.1439), we obtain

$$\lim_{p\to\infty}\left|\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}\right|=0. \tag{2.1440}$$

Now let $k=2q$, $q\in\mathbf{N}$. In this case, using the above reasoning, we get (see (2.389))

$$\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}=$$

$$=\frac{1}{2^q}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\ldots\mathbf{1}_{\{i_{2q-1}=i_{2q}\ne 0\}}\times$$

$$\times\int\limits_t^T\psi_{2q}(t_{2q})\psi_{2q-1}(t_{2q})\ldots\int\limits_t^{t_6}\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4\ldots dt_{2q}. \tag{2.1441}$$

Recall that the multiple Wiener stochastic integral (1.304) has zero expectation (see (1.305)). Then, using (2.965), (2.1255) and (2.1441), we have

$$\lim_{p\to\infty}\mathsf{M}\left\{\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\ldots\mathbf{1}_{\{i_{2q-1}=i_{2q}\ne 0\}}\times$$

$$\times \lim_{p\to\infty} \sum_{j_q,j_{q-2},\ldots,j_2=0}^{p} C_{j_q j_q j_{q-2} j_{q-2}\ldots j_2 j_2} =$$

$$= \frac{1}{2^q} \mathbf{1}_{\{i_1=i_2\ne 0\}} \mathbf{1}_{\{i_3=i_4\ne 0\}} \ldots \mathbf{1}_{\{i_{2q-1}=i_{2q}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_{2q}(t_{2q})\psi_{2q-1}(t_{2q}) \ldots \int\limits_t^{t_6} \psi_4(t_4)\psi_3(t_4) \int\limits_t^{t_4} \psi_2(t_2)\psi_1(t_2) dt_2 dt_4 \ldots dt_{2q} =$$

$$= \mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}. \qquad (2.1442)$$

Applying (2.1442), we obtain

$$\lim_{p\to\infty} \left| \mathsf{M}\left\{ J^*[\psi^{(k)}]_{T,t} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right\}\right| =$$

$$= \left| \mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\} - \lim_{p\to\infty} \mathsf{M}\left\{ \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right\}\right| =$$

$$= 0.$$

The equality (2.1440) is proved.

2.40 Algorithm of the Proof of Hypothesis 2.2

Sect. 2.27–2.38 were written recently, namely in 2024-2025. At the same time, this section (Sect. 2.40) reflects the author's vision of the problem under consideration in 2021-2022.

Let us make some remarks about the development of the approach based on Theorem 2.30 and describe the algorithm of the proof of Hypothesis 2.2 (see Sect. 2.5). First, consider the case $k = 2n+1$, $n = 3, 4, \ldots$ (k is the multiplicity of the iterated Stratonovich stochastic integral (2.661)). Let Condition 2 of Theorem 2.30 be satisfied (Condition 1 of this theorem is satisfied automatically (see the proof of Theorem 2.18)). Consider the equality (2.717). The right-hand side of (2.717) has the form

$$
\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}.
$$

Iterated application of the formulas (2.790), (2.791), (2.804) separately to the values

$$
\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}
$$

and

$$
\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}
$$

($g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (2.652), $r = 1, 2, \ldots, [k/2]$, $2r < k$) gives the following representation (see (2.718))

$$
\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2 \le
$$

$$
\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2 =
$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\Biggl(\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times
$$

$$
\times \prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\Biggr)^2, \tag{2.1443}
$$

where

$$
\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times
$$

$$
\times \prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k
$$

is the Fourier coefficient of

$$
\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=
$$

$$
=R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q),
$$

where

$$
R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=
$$

$$
=\sum_{d=1}^{4^r}\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)-
$$

$$
-\sum_{d=1}^{2^r}\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\in L_2([t,T]^{k-2r})
$$

and some of the functions $\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$ and $\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$ can be identically equal to zero. Obviously, we could use another representation for the function

$$
R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) \tag{2.1444}
$$

based on the left-hand side of the equality (2.717) and (2.790), (2.791), (2.804) (see Sect. 2.13 for details). In Sect. 2.13, we considered the function (2.1444) in detail for the case $k\ge 5$, $r=1$.

Parseval's equality gives

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\right)^2=$$

$$=\int\limits_{[t,T]^{k-2r}}\left(\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\right)^2 dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots$$

$$\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k=\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{2.1445}$$

Combining (2.1443) and (2.1445), we obtain

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2\le$$

$$\le\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{2.1446}$$

Assume that we have succeeded in proving the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0. \tag{2.1447}$$

Applying (2.1446) and (2.1447), we get (compare with (2.718))

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=0. \tag{2.1448}
$$

As noted in Sect. 2.10, Condition 3 of Theorem 2.30 can be replaced by a weaker condition (2.718) (or (2.1448)). Also Condition 3 of Theorem 2.30 can be replaced by (2.1447). From (2.1448) we obviously obtain for $2r < k$

$$
\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=
$$

$$
=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}. \tag{2.1449}
$$

According to (2.717), the equality (2.1449) will be satisfied if

$$
\lim_{p\to\infty} S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=0, \tag{2.1450}
$$

where $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (2.652), $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, $r=1,2,\ldots,[k/2]$,

$$
S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\text{def}}{=}\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}
$$

for $d=0$, where

$$
\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}},\qquad S_l\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}
$$

are defined by (2.656), (2.657), $l=1,2,\ldots,r$ (see Sect. 2.10 for details).

Let us make some remarks about the function (2.1444) for the case $k>5$, $r=2$. In this case, using the left-hand side of the equality (2.717) and (2.790), (2.791), (2.804), we represent the function (2.1444) as the sum of several functions. In particular, among these functions will be the following functions

$$
Q_p(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_{q-1},t_{q+1},\ldots,t_{g-1},t_{g+1},\ldots,t_k)=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_{q-1}<t_{q+1}<\ldots<t_{g-1}<t_{g+1}<\ldots<t_k\}}\times
$$

$$
\times\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times
$$

$$
\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{q+1}}\psi_q(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{g-1}}\psi_g(\tau)\phi_{j_q}(\tau)d\tau, \qquad (2.1451)
$$

$$
\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times
$$

$$
\times\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times
$$

$$
\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right), \qquad (2.1452)
$$

$$
\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times
$$

$$
\times\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times
$$

$$
\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau, \qquad (2.1453)
$$

$$
\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)=
$$

$$
=\mathbf{1}_{\{t_1<\ldots<t_{l-1}<t_{l+2}<\ldots<t_{q-1}<t_{q+2}<\ldots<t_k\}}\times
$$

$$
\times \sum_{j_l=p+1}^{\infty} \sum_{j_{l+1}=p+1}^{\infty} \left(\int\limits_{t}^{t_{l+2}} \psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_{t}^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta \right) \times
$$

$$
\times \left(\int\limits_{t}^{t_{q+2}} \psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_{t}^{\theta} \psi_q(u)\phi_{j_l}(u)dud\theta \right). \qquad (2.1454)
$$

Note that the pairs (g_1, g_2), (g_3, g_4) for the functions (2.1452) and (2.1453) have the property: $g_2 = g_1 + 1$, $g_4 = g_3 + 1$, $g_3 = g_2 + 1$. At the same time, the pairs (g_1, g_2), (g_3, g_4) for the function (2.1451) have the following property: $g_2 > g_1 + 1$, $g_4 > g_3 + 1$, $g_3 \geq g_2 + 1$. For the function (2.1454), the pairs (g_1, g_2), (g_3, g_4) chosen as follows: $g_2 > g_1 + 1$, $g_4 > g_3 + 1$, $g_4 = g_2 + 1$, $g_3 = g_1 + 1$. Generally speaking, all possible pairs (g_1, g_2), (g_3, g_4) must be considered. We consider the functions (2.1451)–(2.1454) only as an example.

Suppose that $s + 1 = l - 1$, $l + 1 = q - 1$, $q + 1 = g - 1$ in (2.1451). Let us show that (we consider the case of Legendre polynomials; the trigonometric case is simpler and can be considered similarly)

$$
\lim_{p\to\infty} \left\|Q_p\right\|^2_{L_2([t,T]^{k-4})} = 0, \qquad (2.1455)
$$

$$
\lim_{p\to\infty} \left\|\bar{Q}_p\right\|^2_{L_2([t,T]^{k-4})} = 0, \qquad (2.1456)
$$

$$
\lim_{p\to\infty} \left\|\tilde{Q}_p\right\|^2_{L_2([t,T]^{k-4})} = 0, \qquad (2.1457)
$$

$$
\lim_{p\to\infty} \left\|\hat{Q}_p\right\|^2_{L_2([t,T]^{k-4})} = 0. \qquad (2.1458)
$$

First consider the proof of (2.1455). We have ($s + 1 = l - 1$, $l + 1 = q - 1$, $q + 1 = g - 1$)

$$
\left(Q_p(t_1, \ldots, t_{l-3}, t_{l-1}, t_{l+1}, t_{l+3}, t_{l+5}, \ldots, t_k)\right)^2 =
$$

$$
= \mathbf{1}_{\{t_1<\ldots<t_{l-3}<t_{l-1}<t_{l+1}<t_{l+3}<t_{l+5}<\ldots<t_k\}} \times
$$

$$
\times \left(\sum_{j_l=p+1}^{\infty} \int\limits_{t}^{t_{l-1}} \psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau \int\limits_{t}^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \times \right.
$$

$$
\times \sum_{j_q=p+1}^{\infty} \int_t^{t_{l+3}} \psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau \int_t^{t_{l+3}} \psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau \Biggr)^2. \qquad (2.1459)
$$

Using the estimate (2.740), we obtain

$$
\left| \int_t^s \psi(\tau)\phi_j(\tau)d\tau \right| < \frac{K}{j^{1-\varepsilon/2}(1-z^2(s))^{1/4-\varepsilon/4}}, \qquad (2.1460)
$$

where $j \in \mathbf{N}$, $s \in (t,T)$, $z(s)$ is defined by (2.20), $\varepsilon \in (0,1)$, constant K does not depend on j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, $\psi(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$.

Applying (2.1460) and (2.743) (we take ε instead of $\varepsilon/2$ in (2.743)), we get

$$
\Biggl(\sum_{j_l=p+1}^{\infty} \int_t^{t_{l-1}} \psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau \int_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \times
$$

$$
\times \sum_{j_q=p+1}^{\infty} \int_t^{t_{l+3}} \psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau \int_t^{t_{l+3}} \psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau \Biggr)^2 \le
$$

$$
\le \frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-1}))^{1-\varepsilon}(1-z^2(t_{l+3}))^{1-\varepsilon}}, \qquad (2.1461)
$$

where $t_{l-1}, t_{l+3} \in (t,T)$, constant K_1 is independent of p. Combining (2.1459) and (2.1461), we have (2.1455).

Let us prove (2.1456). Applying the estimate (2.739) in (2.647) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we obtain

$$
\left| \sum_{j=m+1}^{n} C_{jj}(s) \right| \le C_1 \left(\frac{1}{n^{1-\varepsilon}} + \frac{1}{m^{1-\varepsilon}} \right) \int_{-1}^{z(s)} \frac{dx}{(1-x^2)^{1/2-\varepsilon/2}} +
$$

$$
+C_2 \sum_{j=m+1}^{n} \frac{1}{j^{2-\varepsilon}} \Biggl(\int_{-1}^{z(s)} \frac{dy}{(1-y^2)^{1/2-\varepsilon/2}} + \frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} \int_{-1}^{z(s)} \frac{dy}{(1-y^2)^{1/4-\varepsilon/4}} +
$$

$$
\left.+\int\limits_{-1}^{z(s)}\frac{1}{\left(1-y^2\right)^{1/4-\varepsilon/4}}\int\limits_{y}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/4-\varepsilon/4}}dy\right), \qquad (2.1462)
$$

where

$$
C_{jj}(s)=\int\limits_t^s\psi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_j(\theta)d\theta d\tau,
$$

$s\in(t,T)$, constants C_1, C_2 do not depend on n and m.

From (2.1462) we have

$$
\left|\sum_{j=m+1}^{\infty}C_{jj}(s)\right|\le\frac{K_1}{m^{1-\varepsilon}}+K_2\sum_{j=m+1}^{\infty}\frac{1}{j^{2-\varepsilon}}\left(1+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\right), \qquad (2.1463)
$$

where $s\in(t,T)$, constants K_1, K_2 do not depend on m.

Applying (2.743) (we take ε instead of $\varepsilon/2$ in (2.743)) in (2.1463), we get

$$
\left|\sum_{j=m+1}^{\infty}C_{jj}(s)\right|\le\frac{K}{m^{1-\varepsilon}\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}, \qquad (2.1464)
$$

where $s\in(t,T)$, constant K is independent of m.

Using the estimate (2.1464), we obtain (see (2.1452))

$$
\left(\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times
$$

$$
\times\left(\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.
$$

$$
\left.\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right)\right)^2\le
$$

$$
\le\frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-2}))^{1-\varepsilon}}, \qquad (2.1465)
$$

where $t_{l-2}\in(t,T)$, constant K_1 is independent of p. The inequality (2.1465) completes the proof of (2.1456).

Let us prove (2.1457). Applying (2.646) in (2.1453), we get

$$
\left(\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2 \le
$$

$$
\le \left(\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.
$$

$$
\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 =
$$

$$
= \left(\frac{1}{2}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\psi_{l+2}(\tau)d\tau-\right.
$$

$$
-\sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times
$$

$$
\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 = (a-b)^2 \le 2(|a|^2+|b|^2). \tag{2.1466}
$$

Further, we have

$$
|a| \le \frac{1}{2}\int\limits_t^{t_{l+3}}|\psi_{l+1}(\tau)|\left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right||\psi_{l+2}(\tau)|\,d\tau, \tag{2.1467}
$$

$$
|b| \le \sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}\left|\psi_{l+1}(\tau)\phi_{j_q}(\tau)\right|\left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right|\times
$$

$$
\times\left|\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)du\right|d\tau. \tag{2.1468}
$$

Combining (2.1464) and (2.1467), we obtain

$$
|a| \le \frac{C}{p^{1-\varepsilon}}, \tag{2.1469}
$$

where constant C is independent of p.

Separating in (2.1468) the term with the number $j_q=0$ and then applying (2.158), (2.294), (2.1464), we obtain

$$|b| \le \frac{K}{p^{1-\varepsilon}}\left(\int\limits_t^{t_{l+3}}\frac{d\tau}{\left(1-z^2(\tau)\right)^{1/2-\varepsilon/4}}+\sum_{j_q=1}^{p}\frac{1}{j_q}\int\limits_t^{t_{l+3}}\frac{d\tau}{\left(1-z^2(\tau)\right)^{3/4-\varepsilon/4}}\right)\le$$

$$\le \frac{K_1}{p^{1-\varepsilon}}\left(1+\sum_{j_q=1}^{p}\frac{1}{j_q}\right)\le \frac{K_1}{p^{1-\varepsilon}}\left(2+\int\limits_1^p\frac{dx}{x}\right)=$$

$$=\frac{K_1\left(2+lnp\right)}{p^{1-\varepsilon}}\ \to\ 0 \qquad (2.1470)$$

if $p\to\infty$. The estimates (2.1466), (2.1469), (2.1470) complete the proof of (2.1457).

Finally, consider the proof of (2.1458). Using the elementary inequality $|ab|\le (a^2+b^2)/2$ and Parseval's equality, we have

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2\le$$

$$\le\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left|\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right|\times\right.$$

$$\left.\times\left|\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right|\right)^2\le$$

$$\le\frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\le$$

$$
\le \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.
$$

$$
\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\le
$$

$$
\le \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+2}}\psi_{l+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)du\right)^2 d\theta+\right.
$$

$$
\left.+\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{q+2}}\psi_{q+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)du\right)^2 d\theta\right)^2. \qquad (2.1471)
$$

From (2.1471) and (2.25), (2.294) we obtain

$$
\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2\le\frac{K}{p^2}\ \to\ 0
$$

if $p\to\infty$, where constant K does not depend on p. Thus the equalities (2.1455)–(2.1458) are proved.

Recall that the function (2.1444) (this function is defined using the left-hand side of the equality (2.717)) for the case $k>5$, $r=2$ is represented as the sum of several functions. Four of them, namely Q_p, $\bar{Q}_p$, $\tilde{Q}_p$, $\hat{Q}_p$ (these functions correspond to the particular case of choosing the pairs (g_1,g_2), (g_3,g_4); generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered), have been studied above. Absolutely similarly, we can consider the remaining functions (for all possible pairs (g_1,g_2), (g_3,g_4)) whose sum is the function (2.1444) for the case $k>5$, $r=2$. As a result, we will have

$$
\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=2).
$$

After that, we can go to the function (2.1444) for the case $k>5$, $r=3$, $2r<k$ (this function is defined using the left-hand side of the equality (2.717)) and follow the same steps as above. This will lead us to the following equality

$$
\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=3,\ 2r<k).
$$

Then we can move on to the next step and so on. As a result, we get the equality (2.1447) ($r = 1, 2, \ldots, [k/2]$) and thus prove Hypothesis 2.2 for the case $k = 2n + 1$, $n = 3, 4, \ldots$ (see Sect. 2.5).

For the case $k = 2n$, $n = 3, 4, \ldots$ we follow the above steps for $r = 1, 2, \ldots, [k/2] - 1$ $(2r \le k - 2)$. For $2r = k$ we use the same technique as in the proof of the equalities (2.752)–(2.754). Recall that we used (2.684), (2.691) and Parseval's equality in the proof of (2.752)–(2.754). For $2r = k$ we can also use the equality (2.1237).

The obvious disadvantage of the proposed algorithm is the drastic increase of complexity of the proof when moving from $r = 1$ to $r = 2$, $r = 2$ to $r = 3$ and so on.

The proofs of Theorems 2.34 and 2.35 contain a rather simple trick of passing from $r = 1$ to $r = 2$. Unfortunately, this procedure cannot be applied already at the transition from $r = 2$ to $r = 3$. Note that the case $k = 6$, $r = 3$ was successfully considered in Theorem 2.36 under the following simplifying assumption: $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$.

Nevertheless, the results obtained in the previous sections of Chapter 2 are quite sufficient for practical needs (see Chapters 4 and 5 for details).

2.41 Theorems 2.1–2.9, 2.33–2.36, 2.41, 2.45–2.48, 2.50, 2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.65 on Expansion of Iterated Stratonovich Stochastic Integrals from Point of View of the Wong–Zakai Approximation

The iterated Itô stochastic integrals and solutions of Itô SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s \in [0, T]$. Let $\mathbf{f}_s^{(i)p}$, $p \in \mathbf{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i = 1, \ldots, m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimentional Wiener process $\mathbf{f}_s$?

The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [73], [74], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to the iterated Itô stochastic integrals and solutions of Itô SDEs.

The piecewise linear approximation as well as the regularization by convolution [73]-[75] relate to the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s \in [0, T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [129], [130] (also see Sect. 6.1 of this book for detail)

$$\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^\tau \phi_j(s) ds\ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}, \tag{2.1472}$$

where $\tau \in [t, T]$, $t \geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (2.1472) converges for any $\tau \in [t, T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p} = \sum_{j=0}^{p} \int\limits_t^\tau \phi_j(s) ds\ \zeta_j^{(i)}. \tag{2.1473}$$

From (2.1473) we obtain

$$d\mathbf{f}_\tau^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau) \zeta_j^{(i)} d\tau. \tag{2.1474}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{2.1475}$$

where $p_1, \ldots, p_k \in \mathbf{N}$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$
d\mathbf{w}_{\tau}^{(i)p}=\begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for} \quad i=1,\ldots,m \\ \\ d\tau & \text{for} \quad i=0 \end{cases}, \qquad p\in\mathbf{N}, \tag{2.1476}
$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (2.1474).

Let us substitute (2.1476) into (2.1475)

$$
\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{2.1477}
$$

where $p_1,\ldots,p_k\in\mathbf{N}$,

$$
\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$
C_{j_k\ldots j_1}=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k
$$

is the Fourier coefficient.

To best of our knowledge [73]-[75] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [75] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (2.1473) were not considered in [73], [74] (also see [75], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [75] for approximations of the Wiener process based on its series expansion (2.1472) (also see (6.16)) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (2.1477) to the iterated Stratonovich stochastic integral (2.6) does not follow from the results of the papers [73], [74] (also see [75], Theorems 7.1, 7.2) even for the case $p_1=\ldots=p_k=p$.

From the other hand, Theorems 1.1, 1.16, 2.1–2.9, 2.33–2.36, 2.41, 2.45–2.48, 2.50, 2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.65 from this monograph can be considered as the proof of the Wong–Zakai approximation based on the iterated Riemann–Stieltjes integrals (2.1475) as well as the Wiener process approximation (2.1473) on the base of its series expansion. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 1.1, 1.16, 2.1–2.9, 2.33–2.36, 2.41, 2.45–2.48, 2.50, 2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.65) to the appropriate Stratonovich stochastic integrals (2.6). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (2.1472), (2.1473), and Theorems 1.1, 2.1, 2.2, 2.4–2.9, 2.33–2.36, 2.41, 2.64, 2.65) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. In Theorems 1.16, 2.3, 2.47, 2.48, 2.50, 2.51, 2.53, 2.55, 2.57, 2.59, 2.61–2.63 the system $\{\phi_j(x)\}_{j=0}^{\infty}$ can be arbitrary.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m.$

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [73]-[75]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m,$ i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where $\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\ \ t\in[k\Delta,(k+1)\Delta),\ \ k=0,1,\ldots,N-1.$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{2.1478}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (2.1478) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=
$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \qquad (2.1479)
$$

Using (2.1479), it is not difficult to show (see Lemma 1.1, Remark 1.2, and (2.8)) that

$$
\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \qquad (2.1480)
$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (2.1480) agrees with Theorem 7.1 (see [75], p. 486).

The next example relates to the approximation (2.1473) of the Wiener process based on its series expansion (2.1472), where $t=0$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \qquad (2.1481)
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (2.1474).

Let us substitute (2.1474) into (2.1481)

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \qquad (2.1482)
$$

where

$$
C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (2.1477).

As we noted above, approximations of the Wiener process that are similar to (2.1473) were not considered in [73], [74] (also see Theorems 7.1, 7.2 in [75]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [75] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of this book. More precisely, using Theorem 2.3, we obtain from (2.1482) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{2.1483}$$

From the other hand, by Theorem 1.16 (see (1.321)) for the case $k=2$ we obtain from (2.1482) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right) + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{2.1484}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from (2.8) and (2.1484) we obtain (2.1483).

2.42 Wong–Zakai Type Theorems for Iterated Stratonovich Stochastic Integrals. The Case of Approximation of the Multidimensional Wiener Process Based on its Series Expansion Using Legendre Polynomials and Trigonometric Functions

As we mentioned above, there exists a lot of publications on the subject of Wong–Zakai approximation of stochastic integrals and SDEs [73]-[75] (also see [131]-[138]). However, these works did not consider the approximation of iterated stochastic integrals and SDEs for the case of approximation of the multidimensional Wiener process based on its series expansions. Usually, as an approximation of the Wiener process in the theorems of the Wong–Zakai type, the authors [73]-[75] (also see [131]-[138]) choose a piecewise linear approximation or an approximation based on the regularization by convolution.

The Wong–Zakai approximation is widely used to approximate stochastic integrals and SDEs. In particular, the Wong–Zakai approximation can be used to approximate the iterated Stratonovich stochastic integrals in the context of numerical integration of Itô SDEs in the framework of the approach based on the Taylor–Stratonovich expansion [84], [85] (see Chapter 4). It should be noted that the authors of the works [83] (pp. 438-439), [84] (Sect. 5.8, pp. 202–204), [85] (pp. 82-84), [93] (pp. 263-264) mention the Wong–Zakai approximation [73]-[75] within the frames of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process (see Sect. 6.2). However, in these works there is no rigorous proof of convergence for approximations of the mentioned stochastic integrals of miltiplicity 3 and higher (see discussion in Sect. 6.2).

From the other hand, the theory constructed in Chapters 1 and 2 of this monograph (also see [14]-[17]) can be considered as the proof of the Wong–Zakai approximation for iterated Stratonovich stochastic integrals. At that, this approximation is based on the Wiener process series expansion using an arbitrary complete orthonormal system of functions in $L_2([t,T])$.

The subject of this section is to reformulate the main results of Chapter 2 of this book in the form of theorems on convergence of iterated Riemann–Stiltjes integrals to iterated Stratonovich stochastic integrals.

Let us reformulate Theorems 2.3–2.6, 2.8, 2.9, 2.30, 2.32–2.36, 2.41, 2.49–2.51, 2.60–2.65 of this monograph as statements on the convergence of the iterated Riemann–Stiltjes integrals (2.1475) to the iterated Stratonovich stochastic integrals (2.6).

Theorem 2.66 [17] (reformulation of Theorem 2.3). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau)$ *are continuous functions at the interval* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of second multiplicity*

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following formula

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2}$$

is valid.

Theorem 2.67 [39] (reformulation of Theorems 2.4 and 2.6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following formula

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2}d\mathbf{f}_{t_3}^{(i_3)p_3}$$

is valid.

Theorem 2.68 [39] (reformulation of Theorem 2.5). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\int\limits_t^{*T}(t-t_3)^{l_3}\int\limits_t^{*t_3}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}$$

the following formula

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \int\limits_t^T (t-t_3)^{l_3} \int\limits_t^{t_3} (t-t_2)^{l_2} \int\limits_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)p_1} d\mathbf{f}_{t_2}^{(i_2)p_2} d\mathbf{f}_{t_3}^{(i_3)p_3},$$

where $i_1, i_2, i_3 = 1, \ldots, m,$ *is valid for each of the following cases*

1. $i_1 \neq i_2,\ i_2 \neq i_3,\ i_1 \neq i_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
2. $i_1 = i_2 \neq i_3$ *and* $l_1 = l_2 \neq l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
3. $i_1 \neq i_2 = i_3$ *and* $l_1 \neq l_2 = l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
4. $i_1, i_2, i_3 = 1, \ldots, m;$ $l_1 = l_2 = l_3 = l$ *and* $l = 0, 1, 2, \ldots$

Theorem 2.69 [39] (reformulation of Theorem 2.8). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space* $L_2([t,T])$. *Furthermore, let the function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}$$

the following formula

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p}$$

is valid, where $i_1, i_2, i_3 = 1, \ldots, m.$

Theorem 2.70 [39] (reformulation of Theorem 2.9). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1 i_2 i_3 i_4)} = \int\limits_t^{*T} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}$$

the following formula

$$I_{T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p}$$

is valid, where $i_1, i_2, i_3, i_4 = 0, 1, \ldots, m.$

Theorem 2.71 (reformulation of the modified Theorem 2.30 (see Sect. 2.22)). *Assume that the continuously differentiable functions* $\psi_l(\tau)$ $(l = 1, \ldots, k)$ *at the interval* $[t, T]$ *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions in the space* $L_2([t, T])$ *are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j=0}^{\infty} \int\limits_t^s \Phi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} \Phi_1(t_1)\phi_j(t_1)dt_1 dt_2$$

holds for all $s \in (t, T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t, T]$ *and the series on the right-hand side of the above equality converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \quad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty} \int\limits_t^s \Phi_2(\tau)\phi_j(\tau) \int\limits_t^{\tau} \Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s \in (t, T)$ *and for some* $\alpha, \beta > 0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$, $j, p \in \mathbf{N}$, *and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau < \infty, \quad \int\limits_t^T \left|\Psi_2(\tau)\right| d\tau < \infty.$$

3. *The condition*

$$\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)} \Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\}\right)^2 = 0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (*see* (2.652)) *and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} \stackrel{\mathsf{def}}{=} \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following formula

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \underset{p \to \infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p} \ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

is valid, where $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Theorem 2.72 (reformulation of the modified Theorem 2.32 (see Sect. 2.22)). *Assume that the continuous functions* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *at the interval* $[t, T]$ *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of functions in the space* $L_2([t, T])$ *are such that the following condition*

$$\lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_q=0}^{p_q} \ldots \sum_{j_k=0}^{p_k} \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \times$$

$$\times \left(\sum_{j_{g_1}=0}^{\min\{p_{g_1}, p_{g_2}\}} \sum_{j_{g_3}=0}^{\min\{p_{g_3}, p_{g_4}\}} \ldots \sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}}, p_{g_{2r}}\}} C_{j_k \ldots j_1} \Bigg|_{j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} - \right.$$

$$\left. - \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l} = g_{2l-1} + 1\}} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot),\, j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} \right)^2 = 0$$

is satisfied for all $r = 1, 2, \ldots, [k/2]$. *Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity* k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following formula

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}$$

is valid, where $i_1,\ldots,i_k=0,1,\ldots,m.$

Theorem 2.73 (reformulation of Theorem 2.33). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(s),\psi_2(s),\psi_3(s)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following formula

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}$$

is valid, where $i_1,i_3,i_3=0,1,\ldots,m.$

Theorem 2.74 (reformulation of Theorem 2.34). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(s),\ldots,\psi_4(s)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following formula

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\times$$

$$\times d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p}$$

is valid, where $i_1, \ldots, i_4 = 0, 1, \ldots, m.$

Theorem 2.75 (reformulation of Theorem 2.35). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(s), \ldots, \psi_5(s)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5) \int\limits_t^{*t_5} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \times$$

$$\times d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)}$$

the following formula

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_5(t_5) \int\limits_t^{t_5} \psi_4(t_4) \int\limits_t^{t_4} \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p} \times$$

$$\times d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p} d\mathbf{w}_{t_5}^{(i_5)p}$$

is valid, where $i_1, \ldots, i_5 = 0, 1, \ldots, m.$

Theorem 2.76 (reformulation of Theorems 2.36, 2.64, 2.65). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then, for the iterated Stratonovich stochastic integral*

$$J_{T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T} \int\limits_t^{*t_k} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} d\mathbf{w}_{t_k}^{(i_k)} \quad (k = 6, 7, 8)$$

the following formula

$$J_{T,t}^{*(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \int\limits_t^{t_k} \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})p} d\mathbf{w}_{t_k}^{(i_k)p} \quad (k = 6, 7, 8)$$

is valid, where $i_1, \ldots, i_8 = 0, 1, \ldots, m.$

Theorem 2.77 (reformulation of Theorem 2.41). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(s),\psi_2(s),\psi_3(s)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following formula

$$J^*[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}d\mathbf{w}_{t_2}^{(i_2)p_2}d\mathbf{w}_{t_3}^{(i_3)p_3}$$

is valid, where $i_1,i_2,i_3=0,1,\ldots,m.$

Theorem 2.78 (reformulation of Theorem 2.49). *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *are such that*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\sum_{j_{g_3}=0}^{\min\{p_{g_3},p_{g_4}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=0$$

for all $r=1,2,\ldots,[k/2]$. *Then, for the sum of iterated Itô stochastic integrals of the form*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\stackrel{\mathsf{def}}{=}\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\qquad(2.1485)$$

the following formula

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}$$

is valid, where $i_1,\ldots,i_k = 0,1,\ldots,m.$

Theorem 2.79 (reformulation of Theorem 2.51). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)} = \int\limits_t^{*T} (t_3-t)^{l_3}\int\limits_t^{*t_3} (t_2-t)^{l_2}\int\limits_t^{*t_2} (t_1-t)^{l_1}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following formula

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T (t_3-t)^{l_3}\int\limits_t^{t_3} (t_2-t)^{l_2}\int\limits_t^{t_2} (t_1-t)^{l_1}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}$$

is valid, where $i_1,i_2,i_3 = 0,1,\ldots,m;$ $l_1,l_2,l_3 = 0,1,2,\ldots$

Theorem 2.80 (reformulation of Theorem 2.63). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{l_1l_2l_3l_4T,t}^{*(i_1i_2i_3i_4)} = \int\limits_t^{*T} (t_4-t)^{l_4}\int\limits_t^{*t_4} (t_3-t)^{l_3}\int\limits_t^{*t_3} (t_2-t)^{l_2}\int\limits_t^{*t_2} (t_1-t)^{l_1}\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following formula

$$I_{l_1l_2l_3l_4T,t}^{*(i_1i_2i_3i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T (t_4-t)^{l_4}\int\limits_t^{t_4} (t_3-t)^{l_3}\int\limits_t^{t_3} (t_2-t)^{l_2}\int\limits_t^{t_2} (t_1-t)^{l_1}\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}$$

is valid, where $i_1,\ldots,i_4 = 0,1,\ldots,m;$ $l_1,\ldots,l_4 = 0,1,2,\ldots$

Theorem 2.81 (reformulation of Theorem 2.50). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_5)}=\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}$$

the following formula

$$J_{T,t}^{*(i_1\ldots i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}d\mathbf{w}_{t_5}^{(i_5)p}$$

is valid, where $i_1,\ldots,i_5=0,1,\ldots,m$.

Theorem 2.82 (reformulation of Theorem 2.60). *Suppose that the condition* (2.1341) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals defined by* (2.1485), *we have*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_k}^{(i_k)p},$$

where $i_1,\ldots,i_k=0,1,\ldots,m$.

Theorem 2.83 (reformulation of Theorem 2.61). *Suppose that the condition* (2.1341) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity* k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following formula

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

is valid, where $i_1,\ldots,i_k=0,1,\ldots,m.$

Theorem 2.84 (reformulation of Theorem 2.62). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\int\limits_t^{*t_6}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}d\mathbf{w}_{t_6}^{(i_6)}$$

the following formula

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\int\limits_t^{t_6}\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}d\mathbf{w}_{t_5}^{(i_5)p}d\mathbf{w}_{t_6}^{(i_6)p}$$

is valid, where $i_1,\ldots,i_6=0,1,\ldots,m.$

2.43 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k. The Case $i_1=\ldots=i_k\neq 0$ and Different Continuously Differentiable Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)$

This section was written several years earlier than Sect. 2.30–2.38. The results of Sect. 2.30–2.38 somewhat depreciate the results of Sect. 2.43, but we still included it in this version of the book.

In this section, we generalize the approach considered in Sect. 2.1.2 to the case $i_1=\ldots=i_k\neq 0$ and different weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ $(k>2)$. Let us formulate the following theorem.

Theorem 2.85 [34]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover,* $\psi_1(\tau),\ldots,\psi_k(\tau)$ $(k\ge 2)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_1)}\quad (i_1=1,\ldots,m)$$

the following equality

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_1)}\right)^{2n}\right\} = 0$$

is valid, where $n \in \mathbf{N}$,

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient and

$$\zeta_j^{(i_1)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i_1)} \quad (i_1 = 1,\ldots,m)$$

are independent standard Gaussian random variables for various j.

Proof. The case $k = 2$ is proved in Theorem 2.16. Consider the case $k > 2$. First, consider the case $k = 3$ in detail. Define the auxiliary function

$$K'(t_1,t_2,t_3) = \frac{1}{6}\begin{cases} \psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1 \le t_2 \le t_3\\ \\ \psi_1(t_1)\psi_2(t_3)\psi_3(t_2), & t_1 \le t_3 \le t_2\\ \\ \psi_1(t_2)\psi_2(t_1)\psi_3(t_3), & t_2 \le t_1 \le t_3\\ \\ \psi_1(t_2)\psi_2(t_3)\psi_3(t_1), & t_2 \le t_3 \le t_1\\ \\ \psi_1(t_3)\psi_2(t_2)\psi_3(t_1), & t_3 \le t_2 \le t_1\\ \\ \psi_1(t_3)\psi_2(t_1)\psi_3(t_2), & t_3 \le t_1 \le t_2 \end{cases}, \qquad t_1,t_2,t_3 \in [t,T].$$

Using Lemma 1.1, Remark 1.1 (see Sect. 1.1.3), and (2.399), we obtain w. p. 1

$$J[K']_{T,t}^{(3)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)} =$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+\right.
$$

$$
+\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}\sum_{l_2=0}^{l_1-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
+\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}\sum_{l_3=0}^{l_1-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
+\sum_{l_2=0}^{N-1}\sum_{l_3=0}^{l_2-1}\sum_{l_1=0}^{l_3-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\sum_{l_3=0}^{l_2-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
+\sum_{l_1=0}^{N-1}\sum_{l_3=0}^{l_1-1}\sum_{l_2=0}^{l_3-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
+\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_1})\left(\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+
$$

$$
+\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1} K'(\tau_{l_1},\tau_{l_3},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}+
$$

$$
+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1} K'(\tau_{l_1},\tau_{l_2},\tau_{l_2})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}+
$$

$$
+\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1} K'(\tau_{l_3},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+
$$

$$
+\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1} K'(\tau_{l_2},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+
$$

$$
\left.+\sum_{l_2=0}^{N-1}\sum_{l_3=0}^{l_2-1} K'(\tau_{l_2},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right) =
$$

$$
= \frac{1}{6}\left(\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+\right.
$$

$$+\int\limits_t^T \psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_1)\int\limits_t^{t_1}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\int\limits_t^{t_1}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_2}^{(i_1)}+\int\limits_t^T \psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3+$$

$$\left.+\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}dt_3+\int\limits_t^T \psi_3(t_2)\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}dt_2\right)=$$

$$=\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\frac{1}{2}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_3}^{(i_1)}+\frac{1}{2}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3=$$

$$=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}J^*[\psi^{(3)}]_{T,t}, \qquad (2.1486)$$

where the multiple stochastic integral $J[K']^{(3)}_{T,t}$ is defined by (1.16) and $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (1.9).

Using Proposition 2.2 for $n=3$ (see Sect. 2.1.2) and generalizing the Fourier–Legendre expansion (2.57) for the function $K'(t_1,t_2,t_3)$, we obtain

$$K'(t_1,t_2,t_3)=\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3), \qquad (2.1487)$$

where the multiple Fourier series (2.1487) converges to the function $K'(t_1,t_2,t_3)$ in $(t,T)^3$ and the partial sums of the series (2.1487) have an integrable majorant on $[t,T]^3$ that does not depend on p. For the trigonomertic case, the above statement follows from Proposition 2.2 (the proof that the function $K'(t_1,t_2,t_3)$ belongs to the Hölder class with parameter 1 in $[t,T]^3$ is omitted and can be carried out in the same way as for the function $K'(t_1,t_2)$ in the two-dimensional case (see Sect. 2.1.2)). The proof of generalization of the Fourier–Legendre expansion (2.57) to the three-dimensional case (see (2.1487)) is omitted. The proof that the partial sums of the series (2.1487) have an integrable majorant on $[t,T]^3$ is also omitted.

Denote

$$R'_{ppp}(t_1,t_2,t_3)=K'(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3).$$

Using Lemma 1.3 and (2.1486), we get w. p. 1

$$J^*[\psi^{(3)}]_{T,t}=J[K']^{(3)}_{T,t}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}+J[R'_{ppp}]^{(3)}_{T,t}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}+J[R'_{ppp}]^{(3)}_{T,t}.$$

Then

$$\mathsf{M}\left\{\left(J[R'_{ppp}]^{(3)}_{T,t}\right)^{2n}\right\}=\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_1)}_{j_2}\zeta^{(i_1)}_{j_3}\right)^{2n}\right\},$$

where $n\in\mathbf{N}$.

Applying (we mean here the passage to the limit $\lim\limits_{p\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (2.451) for $k=3$ and $R'_{ppp}(t_1,t_2,t_3)$ instead of $R_{p_1p_2p_3}(t_1,t_2,t_3)$, we obtain

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J[R'_{ppp}]^{(3)}_{T,t}\right)^{2n}\right\}=0.$$

Theorem 2.85 is proved for the case $k=3$.

To prove Theorem 2.85 for the case $k>3$, consider the auxiliary function

$$K'(t_1,\ldots,t_k)=\frac{1}{k!}\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1\le\ldots\le t_k\\ \ldots & \\ \psi_1(t_{g_1})\ldots\psi_k(t_{g_k}), & t_{g_1}\le\ldots\le t_{g_k}\ , \qquad t_1,\ldots,t_k\in[t,T],\\ \ldots & \\ \psi_1(t_k)\ldots\psi_k(t_1), & t_k\le\ldots\le t_1\end{cases}\tag{2.1488}$$

where $\{g_1,\ldots,g_k\}=\{1,\ldots,k\}$ and we take into account all possible permutations $(g_1,\ldots,g_k)$ on the right-hand side of the formula (2.1488).

Further, we have w. p. 1

$$J[K']^{(k)}_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]^{s_r,\ldots,s_1}_{T,t},\tag{2.1489}$$

where the function $K'(t_1,\ldots,t_k)$ is defined by (2.1488); another notations are the same as in (2.387) and Theorem 2.12 ($i_1=\ldots=i_k\ne 0$ in (2.387)).

From (2.1489) and Theorem 2.12 we obtain w. p. 1

$$J^*[\psi^{(k)}]_{T,t}=J[K']^{(k)}_{T,t}.\tag{2.1490}$$

Generalizing the above reasoning to the case $k > 3$ and taking into account (2.1490), we get w. p. 1

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} \frac{1}{k!} \left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1} \right) \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_1)} + J[R'_{p\ldots p}]_{T,t}^{(k)} =$$

$$= \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_1)} + J[R'_{p\ldots p}]_{T,t}^{(k)},$$

where

$$R'_{p\ldots p}(t_1,\ldots,t_k) \stackrel{\sf def}{=} K'(t_1,\ldots,t_k) -$$

$$- \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} \frac{1}{k!} \left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1} \right) \phi_{j_1}(t_1) \ldots \phi_{j_k}(t_k),$$

the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

Further,

$$\mathsf{M}\left\{ \left(J[R'_{p\ldots p}]_{T,t}^{(k)} \right)^{2n} \right\} = \mathsf{M}\left\{ \left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_1)} \right)^{2n} \right\},$$

where $n \in \mathbf{N}$.

Applying (we mean here the passage to the limit $\lim\limits_{p\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (2.451) for $R'_{p\ldots p}(t_1,\ldots,t_k)$ instead of $R_{p_1\ldots,p_k}(t_1,\ldots,t_k)$, we obtain

$$\lim_{p\to\infty} \mathsf{M}\left\{ \left(J[R'_{p\ldots p}]_{T,t}^{(k)} \right)^{2n} \right\} = 0.$$

Theorem 2.85 is proved.

2.44 Comparison of Theorems 2.2 and 2.7 with the Representations of Iterated Stratonovich Stochastic Integrals With Respect to the Scalar Standard Wiener Process

Note that the correctness of the formulas (2.35) and (2.261) can be verified in the following way. If $i_1 = i_2 = i_3 = i = 1, \ldots, m$ and $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv \psi(\tau)$, then we can derive from (2.35) and (2.261) the well known equalities (see Sect. 6.7)

$$\int\limits_t^{*T} \psi(t_2) \int\limits_t^{*t_2} \psi(t_1) d\mathbf{f}_{t_1}^{(i)} d\mathbf{f}_{t_2}^{(i)} = \frac{1}{2!}\left(\int\limits_t^T \psi(\tau) d\mathbf{f}_\tau^{(i)}\right)^2,$$

$$\int\limits_t^{*T} \psi(t_3) \int\limits_t^{*t_3} \psi(t_2) \int\limits_t^{*t_2} \psi(t_1) d\mathbf{f}_{t_1}^{(i)} d\mathbf{f}_{t_2}^{(i)} d\mathbf{f}_{t_3}^{(i)} = \frac{1}{3!}\left(\int\limits_t^T \psi(\tau) d\mathbf{f}_\tau^{(i)}\right)^3$$

w. p. 1, where $\psi(\tau)$ is a continuous nonrandom function at the interval $[t, T]$.

From (2.35) (under the above assumptions and $p_1 = p_2 = p$) we have (see (2.442) and (1.60))

$$J^*[\psi^{(2)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i)} \zeta_{j_2}^{(i)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p} \sum_{j_2=0}^{j_1-1} \Bigl(C_{j_2 j_1} + C_{j_1 j_2}\Bigr) \zeta_{j_1}^{(i)} \zeta_{j_2}^{(i)} + \sum_{j_1=0}^{p} C_{j_1 j_1} \left(\zeta_{j_1}^{(i)}\right)^2\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p} \sum_{j_2=0}^{j_1-1} C_{j_1} C_{j_2} \zeta_{j_1}^{(i)} \zeta_{j_2}^{(i)} + \frac{1}{2} \sum_{j_1=0}^{p} C_{j_1}^2 \left(\zeta_{j_1}^{(i)}\right)^2\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2} \sum_{\substack{j_1,j_2=0\\ j_1\ne j_2}}^{p} C_{j_1} C_{j_2} \zeta_{j_1}^{(i)} \zeta_{j_2}^{(i)} + \frac{1}{2} \sum_{j_1=0}^{p} C_{j_1}^2 \left(\zeta_{j_1}^{(i)}\right)^2\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \frac{1}{2} \left(\sum_{j_1=0}^{p} C_{j_1} \zeta_{j_1}^{(i)}\right)^2 = \frac{1}{2!}\left(\int\limits_t^T \psi(\tau) d\mathbf{f}_\tau^{(i)}\right)^2 \qquad (2.1491)$$

w. p. 1. Note that the last step in (2.1491) is performed by analogy with (1.56).

From (2.261) (under the above assumptions) we obtain (see (2.443) and (1.61)–(1.63))

$$
J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\Bigl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Bigr)\times\right.
$$

$$
\times\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+
$$

$$
+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Bigl(C_{j_3j_1j_3}+C_{j_1j_3j_3}+C_{j_3j_3j_1}\Bigr)\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+
$$

$$
\left.+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Bigl(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\Bigr)\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\sum_{j_1=0}^{p}C_{j_1j_1j_1}\left(\zeta_{j_1}^{(i)}\right)^3\right) =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.
$$

$$
+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+
$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right) =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{6}\sum_{\substack{j_1,j_2,j_3=0\\ j_1\ne j_2,j_2\ne j_3,j_1\ne j_3}}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.
$$

$$
+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+
$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right) =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{6}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\right.
$$

$$-\frac{1}{6}\left(3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\right.$$

$$\left.+\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right)+$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \frac{1}{6}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3=\frac{1}{3!}\left(\int\limits_t^T\psi(\tau)d\mathbf{f}_\tau^{(i)}\right)^3 \qquad (2.1492)$$

w. p. 1. Note that the last step in (2.1492) is performed by analogy with (1.59).

2.45 One Result on the Expansion of Multiple Stratonovich Stochastic Integrals of Multiplicity k. The Case $i_1=\ldots=i_k=1,\ldots,m$

Let us consider the multiple stochastic integral (1.16)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J[\Phi]_{T,t}^{(i_1\ldots i_k)}, \qquad (2.1493)$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbf{R}$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\{\tau_j\}_{j=0}^N$ is a partition of $[t,T]$ which satisfies the condition (1.9) and $i_1,\ldots,i_k=0,1,\ldots,m$.

The stochastic integral with respect to the scalar standard Wiener process ($i_1=\ldots=i_k\neq 0$) and similar to (2.1493) (the function $\Phi(t_1,\ldots,t_k)$ is assumed to be symmetric on the hypercube $[t,T]^k$) has been considered in literature (see, for example, Remark 1.5.7 [139]). The integral (2.1493) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral (see Lemma 1.3)

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)}=J[\varphi_1]_{T,t}^{(i_1)}\ldots J[\varphi_k]_{T,t}^{(i_k)}\quad \text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k)=\varphi_1(t_1)\ldots\varphi_k(t_k)$ and

$$J[\varphi_l]_{T,t}^{(i_l)}=\int\limits_t^T\varphi_l(s)d\mathbf{w}_s^{(i_l)}\quad (l=1,\ldots,k).$$

It is not difficult to see that for the case $i_1=\ldots=i_k\ne 0$ we have w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_1)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\frac{1}{k!}\left(\sum_{(j_1,\ldots,j_k)}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\right)\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_1)},$$

i.e.

$$J[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=J[\tilde\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}\quad\text{w. p. 1},\qquad (2.1494)$$

where

$$\tilde\Phi(t_1,\ldots,t_k)=\frac{1}{k!}\left(\sum_{(t_1,\ldots,t_k)}\Phi(t_1,\ldots,t_k)\right)$$

is the symmetrization of the function $\Phi(t_1,\ldots,t_k)$; the expression

$$\sum_{(a_1,\ldots,a_k)}$$

means the sum with respect to all possible permutations $(a_1,\ldots,a_k)$.

Due to (2.1494) the condition of symmetry of the function $\Phi(t_1,\ldots,t_k)$ need not be required in the case $i_1=\ldots=i_k\ne 0$.

Definition 2.1 [140]. *Let $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is a symmetric function and $q=1,2,\ldots,[k/2]$ (q is fixed). Suppose that for every complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ the following sum*

$$\sum_{j_1,\ldots,j_q=0}^{p}\ \sum_{j_{2q+1},\ldots,j_k=0}^{p}\ \int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_1}(t_2)\ldots\phi_{j_q}(t_{2q-1})\phi_{j_q}(t_{2q})\times$$

$$\times\phi_{j_{2q+1}}(t_{2q+1})\ldots\phi_{j_k}(t_k)dt_1\ldots dt_k\cdot\phi_{j_{2q+1}}(t_{2q+1})\ldots\phi_{j_k}(t_k)\qquad (2.1495)$$

converges in $L_2([t,T]^{k-2q})$ *if* $p\to\infty$ *to a limit, which is independent of the choice of the complete orthonormal system of functions* $\{\phi_j(x)\}_{j=0}^{\infty}$ *in the space* $L_2([t,T])$. *Then we say that the qth limiting trace for* $\Phi(t_1,\ldots,t_k)$ *exists, which by definition is the limit of the sum* (2.1495) *and is denoted as* $\overrightarrow{Tr}^{q}\Phi$. *Moreover,* $\overrightarrow{Tr}^{0}\Phi\stackrel{\sf def}{=}\Phi$.

Consider the following Theorem using our notations.

Theorem 2.86 [140]. *Let* $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ *is a symmetric nonrandom function. Furthermore, let all limiting traces for* $\Phi(t_1,\ldots,t_k)$ (*see Definition* 2.1) *exist. Then the following expansion*

$$J^{\circ}[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_1)}$$

that converges in the mean-square sense is valid, where

$$J^{\circ}[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}$$

is the multiple Stratonovich stochastic integral defined as in [141] (1993) (*also see* [140], *pp.* 910–911),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1=1,\ldots,m$,

$$\zeta_j^{(i_1)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i_1)}$$

are independent standard Gaussian random variables for various j.

In addition to the conditions of Theorem 2.86, we assume that the function $\Phi(t_1,\ldots,t_k)$ is continuous on $[t,T]^k$. Then [139]

$$J^{\circ}[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=J[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}\quad\text{w. p. 1},\qquad(2.1496)$$

where the multiple Stratonovich stochastic integral

$$J[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}$$

is defined by (2.1493). As a result, we get the following expansion

$$J[\Phi]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_1)}. \qquad (2.1497)$$

It should be noted that the expansion (2.1497) is valid provided that for the function $\Phi(t_1,\ldots,t_k)$ there exist all limiting traces that do not depend on the choice of the complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$. The last condition is essential for the proof of the equality (2.1496) (this proof follows from Theorem 1.5.3, Remark 1.5.7, and Propositions 2.2.3, 2.2.5, 4.1.2 [139]). More precisely, in [139], to prove Proposition 4.1.2 (p. 65) a special basis $\{\phi_j(x)\}_{j=0}^{\infty}$ was used. This means that the existence of a limit of the sum (2.1495) for the function $\Phi(t_1,\ldots,t_k)$ in the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ requires a separate proof.

It is not difficult to show that (see (2.1490))

$$J^*[\psi^{(k)}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})} = J[K']_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})} \quad \text{w. p. 1}, \qquad (2.1498)$$

where

$$J^*[\psi^{(k)}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}$$

is the iterated Stratonovich stochastic integral (2.373) ($i_1=\ldots=i_k\neq 0$),

$$J[K']_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}$$

is the multiple Stratonovich stochastic integral (2.1493) ($i_1=\ldots=i_k\neq 0$) for the continuous function $K'(t_1,\ldots,t_k)$ defined by (2.1488).

If we assume that the limiting traces from Theorem 2.86 exist, then we can write (see (2.1498))

$$J^*[\psi^{(k)}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})} = J[K']_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} \left(\frac{1}{k!}\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right)\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_1)} =$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_1)}, \tag{2.1499}
$$

where

$$
C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{2.1500}
$$

is the Fourier coefficient,

$$
\sum_{(j_1,\ldots,j_k)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$; another notations are the same as in Theorem 2.86.

The equality (2.1499) agrees with Theorems 2.49, 2.59, 2.61 for the particular case $i_1 = \ldots = i_k \neq 0$.

From the other hand, the following expansion (see Theorem 2.49)

$$
J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{2.1501}
$$

is valid, where $C_{j_k\ldots j_1}$ has the form (2.1500), $i_1,\ldots,i_k = 0,1,\ldots,m$; another notations are the same as in Theorem 2.49.

2.46 About Hypotheses 2.4 and 2.5

In the previous section, we saw that in a number of papers (see, for example, [139]-[141]) the conditions of theorems related to multiple stochastic integrals (see Theorem 2.86 in Sect. 2.45) are formulated in terms of limiting traces (see Definition 2.1). In addition to limiting traces, the concept of Hilbert space valued traces (integral traces) is introduced in [139]. The concepts of traces considered in [139]-[141] are close to some expressions that we used in this chapter. For example, the following integral (see (2.10))

$$
\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \int\limits_t^T K^*(t_1,t_1)dt_1
$$

is an example of a trace introduced in [139] (Definition 2.2.1). However, the function $K^*(t_1, t_2)$ defined by (2.95) is not symmetric compared with [139] (Definition 2.2.1). In addition, the expression (see (2.10))

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \sum_{j_1=0}^{\infty} \int\limits_{[t,T]^2} K(t_1, t_2)\phi_{j_1}(t_1)\phi_{j_1}(t_2)dt_1 dt_2$$

is an example of a limiting trace (see Definition 2.1) for the function $K(t_1, t_2)$, which is not symmetric (see (2.96)).

In this section, we will talk about Hypotheses 2.4 and 2.5 again. It should be noted that a significant part of Chapter 2 is devoted to the proof of Hypothesis 2.5 for various special cases (Theorems 2.1–2.9, 2.33–2.36, 2.41, 2.45–2.48, 2.50, 2.51, 2.59, 2.61–2.65). In order to prove these theorems, we developed a number of approaches for expansion of iterated Stratonovich stochastic integrals.

More precisely, in Theorems 2.1, 2.2, 2.4–2.9, 2.33–2.36, 2.41, 2.64, 2.65 we assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. The above systems of functions are most suitable for the expansion of iterated stochastic integrals from the Taylor–Itô and Taylor–Stratonovich expansions (see Chapter 5). In Theorems 2.3, 2.47, 2.48, 2.50, 2.51, 2.59, 2.61–2.63 the system $\{\phi_j(x)\}_{j=0}^{\infty}$ can be arbitrary.

Note that Theorems 2.1–2.3 are special cases of Hypothesis 2.5 for $k = 2$ and $p_1, p_2 \to \infty$. At that $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t, T]$ and $\psi_1(\tau)$ is twice continuously differentiable nonrandom function on $[t, T]$ (Theorem 2.1). In Theorem 2.2, the functions $\psi_1(\tau)$ and $\psi_2(\tau)$ are assumed to be continuously differentiable only one time on $[t, T]$. Theorem 2.3 is a generalization of Theorems 2.1, 2.2. More precisely, in Theorem 2.3 we assume that $\psi_1(\tau)$ and $\psi_2(\tau)$ are continuous functions on $[t, T]$.

Theorems 2.4–2.8, 2.33, 2.41, 2.47, 2.51 are special cases of Hypothesis 2.5 for $k = 3$. In Theorems 2.4 and 2.6, the case $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$ and $p_1, p_2, p_3 \to \infty$ is considered. Theorem 2.8 is a special case of Hypothesis 2.5 for the case when $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t, T]$ and $\psi_1(\tau), \psi_3(\tau)$ are twice continuously differentiable nonrandom functions on $[t, T]$ ($p_1 = p_2 = p_3 = p \to \infty$). Theorem 2.33 is an analogue of Theorem 2.8 for continuously differentiable functions $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ and $p_1 = p_2 = p_3 = p \to \infty$. Theorem 2.41 is a generalization of Theorem 2.33 for the case $p_1, p_2, p_3 \to \infty$. In Theorems 2.5, 2.7 and 2.51, we consider narrower particular cases of the functions $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$. For example, the functions $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ have a binomial form (Theorems 2.5, 2.51).

Theorems 2.9, 2.34, 2.35, 2.48, 2.50, 2.63 are special cases of Hypothesis 2.5 for $k = 4$ and $k = 5$. The functions $\psi_1(\tau), \ldots, \psi_5(\tau)$ are continuously differentiable on $[t, T]$ in Theorems 2.34, 2.35, $\psi_1(\tau), \ldots, \psi_5(\tau) \equiv 1$ in Theorems 2.9, 2.48, 2.50, and $\psi_1(\tau), \ldots, \psi_4(\tau)$ have a binomial form in Theorem 2.63.

In Theorems 2.36, 2.62, 2.64, 2.65 the cases $k = 6, 7, 8$ of Hypothesis 2.5 is considered. At that $\psi_1(\tau), \ldots, \psi_8(\tau) \equiv 1$ in these Theorems.

Theorems 2.59, 2.61 prove Hypothesis 2.5 for $k \in \mathbf{N}$ but under one additional condition (see (2.1310) or (2.1341)).

Let us conclude this section with a few remarks.

Remark 2.5. *The equalities* (1.10) *and* (1.54) *imply that the equality* (2.1292) *is equivalent to the relation*

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = -\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{2.1502}$$

w. p. 1, *where notations are the same as in Theorems* 1.2, 1.16 *and* 2.12.

Remark 2.6. *Applying Theorems* 1.14, 1.16, *we can reformulate the equality* (2.1502) *as follows*

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \times$$

$$\times \left(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} - \prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}} + \mathbf{1}_{\{m_l>0\}} \begin{cases} \prod\limits_{s=1}^{d_l} H_{n_{s,l}}\left(\zeta_{j_{h_{s,l}}}^{(i_l)}\right), & \text{if} \quad i_l\neq 0\\ \\ \prod\limits_{s=1}^{d_l} \left(\zeta_{j_{h_{s,l}}}^{(0)}\right)^{n_{s,l}}, & \text{if} \quad i_l = 0\end{cases}\right)\right)$$

w. p. 1, where notations are the same as in Theorems 1.14, 1.16 and 2.12 ($H_n(x)$ is the Hermite polynomial (1.267)).

Remark 2.7. *Recently, in [142], an approach to the proof of expansion similar to (2.1313) was proposed. In particular, this approach uses the representation of the multiple Stratonovich stochastic integral (2.978) as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k. Note that a similar representation in a different form is defined by the formula (2.965).*

It should be noted that an expansion similar to (2.1313) was considered in [142] for an arbitrary k. The system of basis functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ can also be arbitrary. However, in [142], the condition on convergence of trace series is used as a sufficient condition for the validity of expansion similar to (2.1313) (see [142] for details). Note that the verification of the above condition for the kernel (1.6) is a separate problem.

In Theorems 2.24–2.29, 2.36–2.40 the rate of mean-square convergence of expansions of iterated Stratonovich stochastic integrals is found. Determining the rate of mean-square convergence in the approach [142] is an open problem.

2.47 The Connection of Condition (2.1294) with the Concept of Limiting Traces from the Work of G.W. Johnson and G. Kallianpur [141]

Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

By analogy with Definition 2.1 (see Sect. 2.45), we define the rth limiting trace of the function $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ of type (1.310) by the following expression

$$T^{k-2r}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}}) \stackrel{\sf def}{=} \lim_{p\to\infty} T^{k-2r,p}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}}) \qquad (2.1503)$$

in $L_2([t,T]^{k-2r})$, where

$$T^{k-2r,p}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}}) =$$

$$= \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \cdot \phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}),$$

where $r=1,2,\ldots,[k/2]$ and $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$ (see (2.652)), $C_{j_k\ldots j_1}$ is the Fourier coefficient (2.1345). In addition we write $T^{k-2r}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}})=K(t_1,\ldots,t_k)$ for $r=0$.

Note that in [139]-[141] the Wiener process is scalar, while in this book the Wiener process is a multidimensional process with independent components. One of the main results of work [141] (Theorem 5.1) is obtained under the condition of existence of limiting traces (see Definition 2.1 in Sect. 2.45).

Further, we will show that the condition (2.1294) implies the existence of limiting traces (2.1503) (for all $r=1,2,\ldots,[k/2]$ and for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652))) for the case of a multidimensional Wiener process.

Here it is also appropriate to recall the formula (2.967)

$$
\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}
\tag{2.1504}
$$

w. p. 1, where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ is the multiple Wiener stochastic integral defined by (1.304), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (2.963), $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ (see Theorem 2.49).

The equality (2.1504) is an analogue of the formula (5.1) (see [141], Theorem 5.1) for the case of a multidimensional Wiener process.

We have

$$
T^{k-2r,p}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}})=
$$

$$
=\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)\times
$$

$$
\times\phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})+
$$

$$
+\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times\phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\stackrel{\mathsf{def}}{=}
$$

$$
\stackrel{\mathsf{def}}{=}F^{(p)}_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}})+G^{(p)}_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}}). \qquad (2.1505)
$$

Denote

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\stackrel{\mathsf{def}}{=}C^{g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}.
$$

Applying transformations (2.1315), (2.1316) (see Sect. 2.30) iteratively to $C^{g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}$ for integrations not involving the basis functions $\phi_{j_{q_1}},\ldots,\phi_{j_{q_{k-2r}}}$, we obtain

$$
C^{g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}=\sum_{d=1}^{2^r}(-1)^{d-1}\left(\hat{C}^{(d)g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}-\bar{C}^{(d)g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}\right), \qquad (2.1506)
$$

where some terms in the sum

$$
\sum_{d=1}^{2^r}
$$

can be identically equal to zero due to the remark to (2.1315), (2.1316).

Using (2.1506), we get

$$
G^{(p)}_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\times
$$

$$
\times\sum_{d=1}^{2^r}(-1)^{d-1}\Biggl(\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\hat{C}^{(d)g_1\ldots g_{2r}}_{j_{q_{k-2r}}\ldots j_{q_1}}\cdot\phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})-
$$

$$
-\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \bar{C}_{j_{q_{k-2r}}\ldots j_{q_1}}^{(d)g_1\ldots g_{2r}} \cdot \phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\Biggr) \to
$$

$$
\to \frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\times
$$

$$
\times\sum_{d=1}^{2^r}(-1)^{d-1}\Biggl(\hat{F}_{g_1,\ldots,g_{2r}}^{(d)}(t_{q_1},\ldots,t_{q_{k-2r}})-\bar{F}_{g_1,\ldots,g_{2r}}^{(d)}(t_{q_1},\ldots,t_{q_{k-2r}})\Biggr)\stackrel{\mathsf{def}}{=}
$$

$$
\stackrel{\mathsf{def}}{=} G_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}}) \ \ \text{if} \ \ p\to\infty \ \ (\text{in } L_2([t,T]^{k-2r})), \qquad (2.1507)
$$

where $G_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$, $\hat{F}_{g_1,\ldots,g_{2r}}^{(d)}(t_{q_1},\ldots,t_{q_{k-2r}})$, $\bar{F}_{g_1,\ldots,g_{2r}}^{(d)}(t_{q_1},\ldots,t_{q_{k-2r}}) \in L_2([t,T]^{k-2r})$, $d=1,\ldots,2^r$.

Futhermore,

$$
\left\|F_{g_1,\ldots,g_{2r}}^{(p)}\right\|_{L_2([t,T]^{k-2r})}^2=
$$

$$
=\int\limits_{[t,T]^{k-2r}}\Biggl(\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)\times
$$

$$
\times\phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\Biggr)^2 dt_{q_1}\ldots dt_{q_{k-2r}}=
$$

$$
=\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-
$$

$$
-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)^2.
$$

This means that the condition (2.1294) is equivalent to

$$\lim_{p\to\infty}\left\|F^{(p)}_{g_1,\ldots,g_{2r}}\right\|_{L_2([t,T]^{k-2r})}=0. \qquad (2.1508)$$

Suppose that the condition (2.1294) (or (2.1508)) is fulfilled. Applying (2.1505), (2.1507) and (2.1508), we obtain

$$\left\|T^{k-2r,p}_{g_1,\ldots,g_{2r}}K-G_{g_1,\ldots,g_{2r}}\right\|_{L_2([t,T]^{k-2r})}=$$

$$=\left\|F^{(p)}_{g_1,\ldots,g_{2r}}+G^{(p)}_{g_1,\ldots,g_{2r}}-G_{g_1,\ldots,g_{2r}}\right\|_{L_2([t,T]^{k-2r})}\le$$

$$\le\left\|F^{(p)}_{g_1,\ldots,g_{2r}}\right\|_{L_2([t,T]^{k-2r})}+\left\|G^{(p)}_{g_1,\ldots,g_{2r}}-G_{g_1,\ldots,g_{2r}}\right\|_{L_2([t,T]^{k-2r})}\ \to 0$$

if $p\to\infty$.

Thus, the limiting trace $T^{k-2r}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}})$ exists under the condition (2.1294), i.e.

$$T^{k-2r}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}})=\lim_{p\to\infty}T^{k-2r,p}_{g_1,\ldots,g_{2r}}K(t_{q_1},\ldots,t_{q_{k-2r}})=$$

$$=G_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$$

in $L_2([t,T]^{k-2r})$, where $G_{g_1,\ldots,g_{2r}}(t_{q_1},\ldots,t_{q_{k-2r}})\in L_2([t,T]^{k-2r})$, $r=1,2,\ldots,[k/2]$ and all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (2.652).

2.48 New Representations of the Hu–Meyer Formulas for the Case of a Multidimensional Wiener Process

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$.

Let us generalize the definition of the limiting trace from the previous section.

We define the rth limiting trace of the function $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the following expression

$$T^{k-2r}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})\stackrel{\sf def}{=}\lim_{p\to\infty}T^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})$$

in $L_2([t,T]^{k-2r})$, where

$$T^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})=$$

$$=\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\cdot\phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}),$$

where $r=1,2,\ldots,[k/2]$ and $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$ (see (2.652)),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \qquad (2.1509)$$

is the Fourier coefficient. Also we write $T^{k-2r}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})=\Phi(t_1,\ldots,t_k)$ and $T^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})=\Phi_p(t_1,\ldots,t_k)$ for $r=0$, where

$$\Phi_p(t_1,\ldots,t_k)=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k).$$

Let us consider a variant of the formula (2.965) for the case $p_1=\ldots=p_k=p$ and replace $K(t_1,\ldots,t_k)$ with $\Phi(t_1,\ldots,t_k)$ in it. Thus, we have

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1}, \qquad (2.1510)$$

where $k\ge 2$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$, $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals (see (1.304)), $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$ and $C_{j_k\ldots j_1}$ is defined by (2.1509).

Note that sometimes the multiple Stratonovich stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is defined as the limit in probability as $p\to\infty$ of the following expression (the case of a scalar standard Wiener process)

$$
\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i)},
$$

where $i_1=\ldots=i_k=i$ (see, for example, Definition 5.9 in [146]).

By analogy with [146], we define the multiple Stratonovich stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ (the case of a multidimensional Wiener process) as the following mean-square limit

$$
\hat J^S[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\ \hat J_p^S[\Phi]_{T,t}^{(i_1\ldots i_k)},
$$

where

$$
\hat J_p^S[\Phi]_{T,t}^{(i_1\ldots i_k)}=\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},
$$

$C_{j_k\ldots j_1}$ is the Fourier coefficient defined by (2.1509).

Let us rewrite (2.1510) in the form

$$
\hat J_p^S[\Phi]_{T,t}^{(i_1\ldots i_k)}=J'[\Phi_p]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times J'\Bigl[T_{g_1,g_2\ldots,g_{2r-1},g_{2r}}^{k-2r,p}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1},\tag{2.1511}
$$

where $J'\Bigl[T_{g_1,g_2\ldots,g_{2r-1},g_{2r}}^{k-2r,p}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\text{def}}{=}T_{g_1,g_2\ldots,g_{2r-1},g_{2r}}^{k-2r,p}\Phi$ for $k=2r$.

Assume that all limiting traces $T_{g_1,\ldots,g_{2r}}^{k-2r}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})$ (for all $r=1,2,\ldots,$ $[k/2]$ and for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652))) exist.

Recall the well-known property of the multiple Wiener stochastic integral (1.304)

$$
\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le C_k\int\limits_{[t,T]^k}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k,\tag{2.1512}
$$

where $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ and C_k is a constant.

Using (2.1511), (2.1512) and the existence of limiting traces, we have

$$
\mathsf{M}\Biggl\{\Biggl(\hat J_p^S[\Phi]_{T,t}^{(i_1\ldots i_k)}-J'[\Phi]_{T,t}^{(i_1\ldots i_k)}-
$$

$$
-\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J'\Bigl[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\Biggr)^2\Biggr\}=
$$

$$
=\mathsf{M}\Biggl\{\Biggl(J'[\Phi_p-\Phi]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J'\Bigl[T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi-T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\Biggr)^2\Biggr\}\le
$$

$$
\le C_k'\Biggl(\Bigl\|\Phi_p-\Phi\Bigr\|_{L_2([t,T]^k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\Bigl\|T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi-T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\Bigr\|_{L_2([t,T]^{k-2r})}\Biggr)\ \to\ 0 \qquad (2.1513)
$$

if $p\to\infty$, where $J'\Bigl[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi$ for $k=2r$, and C_k' is a constant.

Applying (2.1513), we obtain the following new representation of the Hu–Meyer formula for the case of a multidimensional Wiener process

$$
\hat{J}^S[\Phi]_{T,t}^{(i_1\ldots i_k)} = J'[\Phi]_{T,t}^{(i_1\ldots i_k)} +
$$

$$
+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J'\Bigl[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\Bigr]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \hbox{w. p. 1.} \qquad (2.1514)
$$

The equality (2.1514) is consistent with Theorem 5.1 [141] (the case of a scalar Wiener process).

Further, let us obtain the inverse version of the Hu–Meyer formula (2.1514), i.e. a formula expressing the multiple Wiener stochastic integral through the sum of multiple Stratonovich stochastic integrals.

Let us give the following definition of the limiting trace. We define the rth limiting trace of the function $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the following expression

$$
\tilde{T}^{k-2r}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}}) \stackrel{\sf def}{=} \lim_{p\to\infty} \tilde{T}^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}})
$$

in $L_2([t,T]^{k-2r})$, where

$$
\tilde{T}^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}}) =
$$

$$
= \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \cdot \phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}) =
$$

$$
= \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \tilde{C}_{j_{q_{k-2r}}\ldots j_{q_1}} \cdot \phi_{j_{q_1}}(t_{q_1})\ldots\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}),
$$

where

$$
\tilde{C}_{j_{q_{k-2r}}\ldots j_{q_1}} = \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =
$$

$$= \lim_{p\to\infty} \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

$r = 1,2,\ldots,[k/2]$, $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\} = \{1,2,\ldots,k\}$ (see (2.652)), $C_{j_k\ldots j_1}$ has the form (2.1509). Also we write $\tilde{T}^{k-2r}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}}) = \Phi(t_1,\ldots,t_k)$ and $\tilde{T}^{k-2r,p}_{g_1,\ldots,g_{2r}}\Phi(t_{q_1},\ldots,t_{q_{k-2r}}) = \Phi_p(t_1,\ldots,t_k)$ for $r=0$, where

$$\Phi_p(t_1,\ldots,t_k) = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k).$$

Note that

$$\left\|T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde{T}^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right\|^2_{L_2([t,T]^{k-2r})} =$$

$$= \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \tilde{C}_{j_{q_{k-2r}}\ldots j_{q_1}}\right)^2, \qquad (2.1515)$$

and suppose that

$$\lim_{p\to\infty}\left\|T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde{T}^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right\|_{L_2([t,T]^{k-2r})} = 0 \qquad (2.1516)$$

for all $r=1,2,\ldots,[k/2]$ and for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (2.652)).

Next, we assume that all limiting traces $T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi$ and $\tilde{T}^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi$ exist for all $r=1,2,\ldots,[k/2]$ and for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$.

Applying (2.1512), (2.1516), we obtain for $k>2r$

$$\mathsf{M}\left\{\left(J'\left[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t} - J'\left[\tilde{T}^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\right)^2\right\} \le$$

$$\le 3\mathsf{M}\left\{\left(J'\left[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\right)^2\right\} +$$

$$+3\mathsf{M}\left\{\left(J'\left[T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde{T}^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\right)^2\right\} +$$

$$+3\mathsf{M}\left\{\left(J'\left[\tilde{T}^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde{T}^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\right)^2\right\} \le$$

$$
\le C_{k-2r}\Biggl(\left\|T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right\|^2_{L_2([t,T]^{k-2r})}+
$$

$$
+\left\|T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right\|^2_{L_2([t,T]^{k-2r})}+
$$

$$
+\left\|\tilde T^{k-2r,p}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi - \tilde T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right\|^2_{L_2([t,T]^{k-2r})}\Biggr) \ \to\ 0
$$

if $p\to\infty$, where C_{k-2r} is a constant. Thus, under the above conditions, we have w. p. 1

$$
J'\left[T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t} = J'\left[\tilde T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}. \quad (2.1517)
$$

Combining (2.1514) and (2.1517), we get

$$
J'[\Phi]^{(i_1\ldots i_k)}_{T,t} = \hat J^{S}[\Phi]^{(i_1\ldots i_k)}_{T,t} -
$$

$$
-\sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times J'\left[\tilde T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\quad \text{w. p. 1.} \qquad (2.1518)
$$

By iteratively applying the formula (2.1518), we obtain the following inverse version of the Hu–Meyer formula (2.1514)

$$
J'[\Phi]^{(i_1\ldots i_k)}_{T,t} = \hat J^{S}[\Phi]^{(i_1\ldots i_k)}_{T,t} +
$$

$$
+\sum_{r=1}^{[k/2]}(-1)^r\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times
$$

$$
\times \hat J^{S}\left[\tilde T^{k-2r}_{g_1,g_2\ldots,g_{2r-1},g_{2r}}\Phi\right]^{(i_{q_1}\ldots i_{q_{k-2r}})}_{T,t}\quad \text{w. p. 1.} \qquad (2.1519)
$$

It is interesting to compare the formulas (2.1514) and (2.1519) with similar formulas (2.389) and (4.53) that connect the iterated Stratonovich and Itô stochastic integrals.

2.49 Invariance of Expansions of Iterated Itô and Stratonovich Stochastic Integrals from Theorems 1.16 and 2.59

In this section, we consider the invariance of expansions of iterated Itô and Stratonovich stochastic integrals from Theorems 1.16 and 2.59 (or 2.61).

Consider the multiple Wiener stochastic integral $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ defined by (1.304) ($\Phi(t_1,\ldots,t_k)=\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)$), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$.

Taking into account (1.309) and (1.319), we obtain

$$
J'[K]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1},\qquad(2.1520)
$$

where $J'[K]_{T,t}^{(i_1\ldots i_k)}$ and $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ are multiple Wiener stochastic integrals defined by (1.304), the function $K(t_1,\ldots,t_k)$ has the form (1.310).

On the other hand, the expansion (2.1313) can be written as follows

$$
\bar J^S[K]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\bar J^S[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1},\qquad(2.1521)
$$

where $\bar J^S[K]_{T,t}^{(i_1\ldots i_k)}$ and $\bar J^S[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ are multiple Stratonovich stochastic integrals defined by (2.978), the function $K(t_1,\ldots,t_k)$ has the form (2.979).

Therefore, the expansions (2.1520) and (2.1521) have the same form. At that the expansion (2.1520) is formulated using multiple Wiener stochastic integrals and the expansion (2.1521) is formulated using multiple Stratonovich stochastic integrals.

The expansions (2.1520) and (2.1521) can be written in a slightly different way. Using (1.327), we obtain

$$
J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\times
$$

$$
\times\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad\text{w. p. 1},\qquad(2.1522)
$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (1.309),

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in Theorem 1.16.

The iterated Stratonovich stochastic integrals

$$\int\limits_t^{*T}\phi_{j_k}(t_k)\ldots\int\limits_t^{*t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

satisfy the following equality

$$\bar{J}^S[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^{*T}\phi_{j_k}(t_k)\ldots\int\limits_t^{*t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},\qquad (2.1523)$$

where $\phi_j(x)$ $(j=0,1,2,\ldots)$ are continuous functions on $[t,T]$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}\quad (i=0,1,\ldots,m,\quad j=0,1,\ldots)$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), the expression

$$\sum_{(j_1,\ldots,j_k)}$$

has the same meaning as in (2.1522).

For the case $i_1=\ldots=i_k=0$ we obtain from (2.1523) the following well known formula from the classical integral calculus (see (1.38))

$$\int\limits_{[t,T]^k}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)dt_1\ldots dt_k=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k=$$

$$
=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)dt_1\ldots dt_k, \qquad (2.1524)
$$

where

$$
\sum_{(j_1,\ldots,j_k)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$ and permutations $(t_1,\ldots,t_k)$ when summing

$$
\sum_{(t_1,\ldots,t_k)}
$$

(see (2.1524)) are performed only in the values $dt_1\ldots dt_k$ (at the same time the indices near upper limits of integration in the iterated integrals are changed correspondently).

Let us check the formula (2.1523) for the cases $k=2$ and $k=3$. Using (1.46), (2.398), and (2.1522) ($k=2$), we have

$$
\sum_{(j_1,j_2)}\int\limits_t^{*T}\phi_{j_2}(t_2)\int\limits_t^{*t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}=
$$

$$
=\int\limits_t^{*T}\phi_{j_2}(t_2)\int\limits_t^{*t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^{*T}\phi_{j_1}(t_2)\int\limits_t^{*t_2}\phi_{j_2}(t_1)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}=
$$

$$
=\int\limits_t^{T}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^{T}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{T}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=
$$

$$
=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}=
$$

$$
=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\quad \text{w. p. 1.}
$$

Applying (1.47), (2.399), (2.1522) ($k=3$), and Itô's formula, we obtain w. p. 1

$$\sum_{(j_1,j_2,j_3)} \int\limits_t^{*T} \phi_{j_3}(t_3) \int\limits_t^{*t_3} \phi_{j_2}(t_2) \int\limits_t^{*t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \sum_{(j_1,j_2,j_3)} \int\limits_t^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} +$$

$$+ \mathbf{1}_{\{i_1=i_2\neq 0\}} \left(\int\limits_t^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_1)\phi_{j_1}(t_1) dt_1 d\mathbf{w}_{t_3}^{(i_3)} + \right.$$

$$\left. + \int\limits_t^{T} \phi_{j_2}(t_1)\phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_3}(t_3) d\mathbf{w}_{t_3}^{(i_3)} dt_1 \right) +$$

$$+ \mathbf{1}_{\{i_1=i_3\neq 0\}} \left(\int\limits_t^{T} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\phi_{j_3}(t_1) dt_1 d\mathbf{w}_{t_2}^{(i_2)} + \right.$$

$$\left. + \int\limits_t^{T} \phi_{j_3}(t_1)\phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_2}(t_2) d\mathbf{w}_{t_2}^{(i_2)} dt_1 \right) +$$

$$+ \mathbf{1}_{\{i_2=i_3\neq 0\}} \left(\int\limits_t^{T} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_2}(t_3)\phi_{j_3}(t_3) dt_3 d\mathbf{w}_{t_1}^{(i_1)} + \right.$$

$$\left. + \int\limits_t^{T} \phi_{j_3}(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} dt_3 \right) =$$

$$= \sum_{(j_1,j_2,j_3)} \int\limits_t^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} +$$

$$+ \mathbf{1}_{\{i_1=i_2\neq 0\}} \left(\int\limits_t^{T} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_1)\phi_{j_1}(t_1) dt_1 d\mathbf{w}_{t_3}^{(i_3)} + \right.$$

$$+\int\limits_t^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 d\mathbf{w}_{t_3}^{(i_3)}\Biggr)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\Biggl(\int\limits_t^T \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)\phi_{j_3}(t_1)dt_1 d\mathbf{w}_{t_2}^{(i_2)}+$$

$$+\int\limits_t^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_1}(t_1)\phi_{j_3}(t_1)dt_1 d\mathbf{w}_{t_2}^{(i_2)}\Biggr)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(\int\limits_t^T \phi_{j_1}(t_1)\int\limits_t^{t_1} \phi_{j_2}(t_3)\phi_{j_3}(t_3)dt_3 d\mathbf{w}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_3)\phi_{j_3}(t_3)dt_3 d\mathbf{w}_{t_1}^{(i_1)}\Biggr)=$$

$$=\sum_{(j_1,j_2,j_3)}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^T \phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T \phi_{j_2}(t_2)\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_3}(t_1)dt_1 d\mathbf{w}_{t_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_t^T \phi_{j_2}(t_3)\phi_{j_3}(t_3)dt_3 d\mathbf{w}_{t_1}^{(i_1)}=$$

$$=\sum_{(j_1,j_2,j_3)}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}=$$

$$=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}=$$

$$=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}.$$

Using (2.1523), we can write the expansion (2.1521) as follows

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\times$$

$$\times\sum_{(j_1,\ldots,j_k)}\int\limits_t^{*T}\phi_{j_k}(t_k)\ldots\int\limits_t^{*t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},\qquad (2.1525)$$

where $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Stratonovich stochastic integral; another notations are the same as in (2.1522).

Obviously, the expansions (2.1522) and (2.1525) have the same form. At that the expansion (2.1522) is formulated using iterated Itô stochastic integrals and the expansion (2.1525) is formulated using iterated Stratonovich stochastic integrals.

2.50 Expansion of Multiple Stratonovich Stochastic Integrals of Arbitrary Multiplicity k. The case of a multidimensional Wiener process and a smooth function $\Phi(t_1,\ldots,t_k)$

As we have seen in this chapter, one of the main difficulties in obtaining expansions of iterated Stratonovich stochastic integrals is related to the properties of the kernel (1.6). The kernel (1.6) is discontinuous, which causes difficulties in applying the theory of multiple Fourier series converging pointwise. Moreover, the Volterra integral operator $\mathbb{V}: L_2([0,1])\to L_2([0,1])$ of the form

$$\left(\mathbb{V}f\right)(x)=\int\limits_0^x f(\tau)d\tau\quad (f(\tau)\in L_2([0,1]))$$

with the kernel

$$K(\tau,x)=\begin{cases}1, & \tau<x\\ \\ 0, & \text{otherwise}\end{cases}\qquad (\tau,x\in[0,1])$$

is not a trace class operator [150] (see Sect. 2.27).

Thus, one cannot count on the fact that operators of the more general form (2.1193) (from the same family of operators as the Volterra integral operator) with the kernel (1.6) are operators of the trace class. It is well known [150] that for trace class operators the equality of matrix and integral traces holds (this equality is very useful for expansion of iterated Stratonovich stochastic integrals (see Sect. 2.27)). As a result, the proof of the equalities of matrix and integral traces for Volterra–type integral operators (2.1193) provides a way to calculate the matrix traces of these operators however is obviously a problem.

Let us assume that the function $\Phi(t_1,\ldots,t_k):[t,T]^k\to\mathbf{R}$ satisfies sufficient conditions for its expansion into a multiple Fourier series

$$\lim_{p\to\infty}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\tag{2.1526}$$

converging pointwise in $(t,T)^k$ to the function $\Phi(t_1,\ldots,t_k)$. Also we suppose that the partial sums

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

of the multiple Fourier series (2.1526) have an integrable majorant on $[t,T]^k$ that does not depend on p. Here $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k\tag{2.1527}$$

is the Fourier coefficient.

The mentioned conditions for $k=1$ and $k\ge 2$ (trigonometric case) are given in Sect. 2.1.1, 2.1.2.

Consider the multiple Stratonovich stochastic integral (1.16) (also see (2.1493))

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\rm def}{=} J[\Phi]_{T,t}^{(i_1\ldots i_k)},$$

where we suppose that the function $\Phi(t_1,\ldots,t_k)$ is the same as above, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (1.9) and $i_1,\ldots,i_k = 0,1,\ldots,m.$

Denote

$$R_p(t_1,\ldots,t_k) = \Phi(t_1,\ldots,t_k) - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l), \qquad (2.1528)$$

where $C_{j_k\ldots j_1}$ has the form (2.1527).

Applying (we mean here the passage to the limit $\lim\limits_{p\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (2.451) for $R_p(t_1,\ldots,t_k)$ (see (2.1528)) instead of $R_{p_1\ldots,p_k}(t_1,\ldots,t_k)$ (see (2.453)), we obtain

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J[\Phi]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}\right)^{2n}\right\} =$$

$$= \lim_{p\to\infty} \mathsf{M}\left\{\left(J[R_p]_{T,t}^{(i_1\ldots i_k)}\right)^{2n}\right\} = 0, \qquad (2.1529)$$

where $n \in \mathbf{N}$ and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$).

Note that the equality (2.1529) will also be satisfied if the multiple Fourier series (2.1526) converges to the function $\Phi(t_1,\ldots,t_k)$ almost everywhere (with respect to Lebesgue's measure) on the hypercubes $[t,T]^{k-r}$ $(r = 0,1,\ldots,[k/2])$ that are domains of integration for the integrals on the right-hand side of the inequality (2.451).

Chapter 3

Integration Order Replacement Technique for Iterated Itô Stochastic Integrals and Iterated Stochastic Integrals with Respect to Martingales

This chapter is devoted to the integration order replacement technique for iterated Itô stochastic integrals and iterated stochastic integrals with respect to martingales. We consider the class of iterated Itô stochastic integrals, for which with probability 1 the formulas on integration order replacement corresponding to the rules of classical integral calculus are correct. The theorems on integration order replacement for the class of iterated Itô stochastic integrals are proved. Many examples of these theorems usage have been considered. The mentioned results are generalized for the class of iterated stochastic integrals with respect to martingales.

3.1 Introduction

In this chapter we performed rather laborious work connected with the theorems on integration order replacement for iterated Itô stochastic integrals. However, there may appear a question about a practical usefulness of this theory, since the significant part of its conclusions directly follows from the Itô formula.

It is not difficult to see that to obtain various relations for iterated Itô stochastic integrals (see, for example, Sect. 3.6) using the Itô formula, first of all these relations should be guessed. Then it is necessary to introduce corresponding Itô processes and afterwards to use the Itô formula. It is clear that this process requires intellectual expenses and it is not always trivial.

On the other hand, the technique on integration order replacement introduced in this chapter is formally comply with the similar technique for Riemann integrals, although it is related to Itô integrals, and it provides a possibility to perform transformations naturally (as with Riemann integrals) with iterated Itô stochastic integrals and to obtain various relations for them.

So, in order to implementation of transformations of the specific class of Itô processes, which is represented by iterated Itô stochastic integrals, it is more naturally and easier to use the theorems on integration order replacement, than the Itô formula.

Many examples of these theorems usage are presented in Sect. 3.6.

Note that in Chapters 1, 2, and 4 the integration order replacement technique for iterated Itô stochastic integrals has been successfully applied for the proof and development of the method of approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series (see Chapters 1 and 2) as well as for the construction of the so-called unified Taylor–Itô and Taylor–Stratonovich expansions (see Chapter 4).

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega): [0,T]\times\Omega \rightarrow \mathbf{R}^1$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$. Further, we will use the following notation: $f(t,\omega) \stackrel{\rm def}{=} f_t$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let us recall that the class $\mathrm{M}_2([0,T])$ (see Sect. 1.1.2) consists of functions $\xi: [0,T]\times\Omega\rightarrow \mathbf{R}^1$, which satisfy the conditions:

1. The function $\xi(t,\omega)$ is measurable with respect to the pair of variables (t,ω).

2. The function $\xi(t,\omega)$ is F_t-measurable for all $t\in[0,T]$ and $\xi(\tau,\omega)$ is independent with increments $f_{t+\Delta}-f_t$ for $t\ge\tau$, $\Delta>0$.

3. The following relation is fulfilled

$$\int\limits_0^T \mathsf{M}\left\{(\xi(t,\omega))^2\right\}dt<\infty.$$

4. $\mathsf{M}\left\{(\xi(t,\omega))^2\right\}<\infty$ for all $t\in[0,T]$.

Let us recall (see Sect. 1.1.2) that the stochastic integrals

$$\int\limits_0^T \xi_\tau df_\tau \quad \text{and} \quad \int\limits_0^T \xi_\tau d\tau, \tag{3.1}$$

where $\xi_t\in \mathrm{M}_2([0,T])$ and the first integral in (3.1) is the Itô stochastic integral, can be defined in the mean-square sense by the relations (1.2) and (1.4).

We will introduce the class $\mathrm{S}_2([0,T])$ of functions $\xi:[0,T]\times\Omega\to\mathbf{R}^1$, which satisfy the conditions:

1. $\xi(\tau,\omega)\in \mathrm{M}_2([0,T])$.

2. $\xi(\tau,\omega)$ is the mean-square continuous random process at the interval $[0,T]$.

As we noted above, the Itô stochastic integral exists in the mean-square sense (see (1.2)), if the random process $\xi(\tau,\omega)\in \mathrm{M}_2([0,T])$, i.e., perhaps this process does not satisfy the property of the mean-square continuity on the interval $[0,T]$. In this chapter we will formulate and prove the theorems on integration order replacement for the special class of iterated Itô stochastic integrals. At the same time, the condition of the mean-square continuity of integrand in the innermost stochastic integral will be significant.

Let us introduce the following class of iterated stochastic integrals

$$J[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T \psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\phi_\tau dw_\tau^{(k+1)}dw_{t_k}^{(k)}\ldots dw_{t_1}^{(1)},$$

where $\phi(\tau,\omega)\stackrel{\rm def}{=}\phi_\tau$, $\phi_\tau\in \mathrm{S}_2([t,T])$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continous nonrandom function at the interval $[t,T]$, here and further $w_\tau^{(l)}=f_\tau$ or $w_\tau^{(l)}=\tau$ for $\tau\in[t,T]$ $(l=1,\ldots,k+1)$, $(\psi_1,\ldots,\psi_k)\stackrel{\rm def}{=}\psi^{(k)}$, $\psi^{(1)}\stackrel{\rm def}{=}\psi_1$.

We will call the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$ as the iterated Itô stochastic integral.

It is well known that for the iterated Riemann integral in the case of specific conditions the formula on integration order replacement is correct. In particular, if the nonrandom functions $f(x)$ and $g(x)$ are continuous at the interval $[a,b]$, then

$$\int\limits_a^b f(x)\int\limits_a^x g(y)dydx=\int\limits_a^b g(y)\int\limits_y^b f(x)dxdy. \tag{3.2}$$

If we suppose that for the Itô stochastic integral

$$J[\phi,\psi_1]_{T,t}=\int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)}$$

the formula on integration order replacement, which is similar to (3.2), is valid, then we will have

$$\int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)}=\int\limits_t^T \phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}dw_\tau^{(2)}. \tag{3.3}$$

If, in addition $w_s^{(1)},\ w_s^{(2)}=f_s$ $(s\in[t,T])$ in (3.3), then the stochastic process

$$\eta_\tau=\phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}$$

does not belong to the class $\mathrm{M}_2([t,T])$, and, consequently, for the Itô stochastic integral

$$\int\limits_t^T \eta_\tau dw_\tau^{(2)}$$

on the right-hand side of (3.3) the conditions of its existence are not fulfilled.

At the same time

$$\int\limits_t^T df_s\int\limits_t^T ds=\int\limits_t^T (s-t)df_s+\int\limits_t^T (f_s-f_t)ds\quad \text{w. p. 1}, \tag{3.4}$$

and we can obtain this equality, for example, using the Itô formula, but (3.4) can be considered as a result of integration order replacement (see below).

Actually, we can demonstrate that

$$\int_t^T (f_s - f_t) ds = \int_t^T \int_t^s df_\tau ds = \int_t^T \int_\tau^T ds df_\tau \quad \text{w. p. 1.}$$

Then

$$\int_t^T (s-t) df_s + \int_t^T (f_s - f_t) ds = \int_t^T \int_t^\tau ds df_\tau + \int_t^T \int_\tau^T ds df_\tau = \int_t^T df_s \int_t^T ds \quad \text{w. p. 1.}$$

The aim of this chapter is to establish the strict mathematical sense of the formula (3.3) for the case $w_s^{(1)}$, $w_s^{(2)} = f_s$ $(s \in [t,T])$ as well as its analogue corresponding to the iterated Itô stochastic integral $J[\phi, \psi^{(k)}]_{T,t}$, $k \ge 2$. At that, we will use the definition of the Itô stochastic integral which is more general than (1.2).

Let us consider the partition $\tau_j^{(N)}$, $j = 0, 1, \ldots, N$ of the interval $[t, T]$ such that

$$t = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \qquad \max_{0 \le j \le N-1} \left| \tau_{j+1}^{(N)} - \tau_j^{(N)} \right| \to 0 \ \text{ if } \ N \to \infty. \tag{3.5}$$

In [114] Stratonovich R.L. introduced the definition of the so-called combined stochastic integral for the specific class of integrated processes. Taking this definition as a foundation, let us consider the following construction of stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j} \left(f_{\tau_{j+1}} - f_{\tau_j} \right) \theta_{\tau_{j+1}} \stackrel{\text{def}}{=} \int_t^T \phi_\tau df_\tau \theta_\tau, \tag{3.6}$$

where $\phi_\tau,\ \theta_\tau \in \mathrm{S}_2([t,T])$, $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition (3.5) (here and sometimes further for simplicity we write τ_j instead of $\tau_j^{(N)}$).

Further, we will prove existence of the integral (3.6) for $\phi_\tau \in \mathrm{S}_2([t,T])$ and θ_τ from a little bit narrower class of processes than $\mathrm{S}_2([t,T])$. In addition, the integral defined by (3.6) will be used for the formulation and proof of the theorem on integration order replacement for the iterated Itô stochastic integrals $J[\phi, \psi^{(k)}]_{T,t}$, $k \ge 1$.

Note that under the appropriate conditions the following properties of stochastic integrals defined by the formula (3.6) can be proved

$$\int\limits_t^T \phi_\tau df_\tau g(\tau) = \int\limits_t^T \phi_\tau g(\tau) df_\tau \quad \text{w. p. 1},$$

where $g(\tau)$ is a continuous nonrandom function at the interval $[t, T]$,

$$\int\limits_t^T (\alpha\phi_\tau + \beta\psi_\tau)\, df_\tau \theta_\tau = \alpha \int\limits_t^T \phi_\tau df_\tau \theta_\tau + \beta \int\limits_t^T \psi_\tau df_\tau \theta_\tau \quad \text{w. p. 1},$$

$$\int\limits_t^T \phi_\tau df_\tau\, (\alpha\theta_\tau + \beta\psi_\tau) = \alpha \int\limits_t^T \phi_\tau df_\tau \theta_\tau + \beta \int\limits_t^T \phi_\tau df_\tau \psi_\tau \quad \text{w. p. 1},$$

where $\alpha,\ \beta \in \mathbf{R}^1$.

At that, we suppose that the stochastic processes ϕ_τ, θ_τ, and ψ_τ are such that the integrals included in the mentioned properties exist.

3.2 Formulation of the Theorem on Integration Order Replacement for Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$)

Let us define the stochastic integrals $\hat{I}[\psi^{(k)}]_{T,t}$, $k \ge 1$ of the form

$$\hat{I}[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) dw_{t_k}^{(k)} \int\limits_{t_k}^T \psi_{k-1}(t_{k-1}) dw_{t_{k-1}}^{(k-1)} \ldots \int\limits_{t_2}^T \psi_1(t_1) dw_{t_1}^{(1)}$$

in accordance with the definition (3.6) by the following recurrence relation

$$\hat{I}[\psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_k(\tau_l) \Delta w_{\tau_l}^{(k)} \hat{I}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \tag{3.7}$$

where $k \ge 1$, $\hat{I}[\psi^{(0)}]_{T,s} \stackrel{\text{def}}{=} 1$, $[s, T] \subseteq [t, T]$, here and further $\Delta w_{\tau_l}^{(i)} = w_{\tau_{l+1}}^{(i)} - w_{\tau_l}^{(i)}$, $i = 1, \ldots, k+1$, $l = 0, 1, \ldots, N-1$.

Then, we will define the iterated stochastic integral $\hat{J}[\phi, \psi^{(k)}]_{T,t}$, $k \ge 1$

$$\hat{J}[\phi, \psi^{(k)}]_{T,t} = \int\limits_t^T \phi_s dw_s^{(k+1)} \hat{I}[\psi^{(k)}]_{T,s}$$

similarly in accordance with the definition (3.6)

$$\hat{J}[\phi,\psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}.$$

Let us formulate the theorem on integration order replacement for iterated Itô stochastic integrals.

Theorem 3.1 [123] (1997), (also see [1]-[17], [77], [124]). *Suppose that $\phi_\tau \in \text{S}_2([t,T])$ and every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the stochastic integral $\hat{J}[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$ exists and*

$$J[\phi,\psi^{(k)}]_{T,t} = \hat{J}[\phi,\psi^{(k)}]_{T,t} \quad \textit{w. p. } 1.$$

3.3 Proof of Theorem 3.1 for the Case of Iterated Itô Stochastic Integrals of Multiplicity 2

At first, let us prove Theorem 3.1 for the case $k=1$. We have

$$J[\phi,\psi_1]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l)\Delta w_{\tau_l}^{(1)} \int\limits_t^{\tau_l} \phi_\tau dw_\tau^{(2)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l)\Delta w_{\tau_l}^{(1)} \sum_{j=0}^{l-1} \int\limits_{\tau_j}^{\tau_{j+1}} \phi_\tau dw_\tau^{(2)}, \tag{3.8}$$

$$\hat{J}[\phi,\psi_1]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)} \int\limits_{\tau_{j+1}}^{T} \psi_1(s)dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)} \sum_{l=j+1}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s)dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s)dw_s^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)}. \tag{3.9}$$

It is clear that if the difference ε_N of prelimit expressions on the right-hand sides of (3.8) and (3.9) tends to zero when $N\to\infty$ in the mean-square sense, then the stochastic integral $\hat{J}[\phi,\psi_1]_{T,t}$ exists and

$$J[\phi,\psi_1]_{T,t} = \hat{J}[\phi,\psi_1]_{T,t} \quad \text{w. p. } 1.$$

The difference ε_N can be represented in the form $\varepsilon_N = \tilde{\varepsilon}_N + \hat{\varepsilon}_N$, where

$$\tilde{\varepsilon}_N = \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w_{\tau_l}^{(1)} \sum_{j=0}^{l-1} \int\limits_{\tau_j}^{\tau_{j+1}} \left(\phi_\tau - \phi_{\tau_j}\right) dw_\tau^{(2)};$$

$$\hat{\varepsilon}_N = \sum_{l=0}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \left(\psi_1(\tau_l) - \psi_1(s)\right) dw_s^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j} \Delta w_{\tau_j}^{(2)}.$$

We will demonstrate that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0.$$

In order to do it we will analyze four cases:

1. $w_\tau^{(2)} = f_\tau,\ \Delta w_{\tau_l}^{(1)} = \Delta f_{\tau_l}$.
2. $w_\tau^{(2)} = \tau,\ \Delta w_{\tau_l}^{(1)} = \Delta f_{\tau_l}$.
3. $w_\tau^{(2)} = f_\tau,\ \Delta w_{\tau_l}^{(1)} = \Delta \tau_l$.
4. $w_\tau^{(2)} = \tau,\ \Delta w_{\tau_l}^{(1)} = \Delta \tau_l$.

Let us recall the well known standard moment properties of stochastic integrals [100]

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^2\right\} = \int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^2\right\} d\tau,$$

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau d\tau\right|^2\right\} \le (T-t) \int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^2\right\} d\tau, \qquad (3.10)$$

where $\xi_\tau \in \text{M}_2([t,T])$.

For Case 1 using standard moment properties for the Itô stochastic integral as well as mean-square continuity (which means uniform mean-square continuity) of the process ϕ_τ on the interval $[t,T]$, we obtain

$$\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\} = \sum_{k=0}^{N-1} \psi_1^2(\tau_k) \Delta\tau_k \sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} \mathsf{M}\left\{\left|\phi_\tau - \phi_{\tau_j}\right|^2\right\} d\tau <$$

$$< C^2 \varepsilon \sum_{k=0}^{N-1} \Delta\tau_k \sum_{j=0}^{k-1} \Delta\tau_j < C^2 \varepsilon \frac{(T-t)^2}{2},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Let us consider Case 2. Using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (3.10) for the stochastic integral, we have

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} = \sum_{k=0}^{N-1} \psi_1^2(\tau_k)\Delta\tau_k \mathsf{M}\left\{\left(\sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_{\tau_j}) d\tau\right)^2\right\} \le$$

$$\le \sum_{k=0}^{N-1} \psi_1^2(\tau_k)\Delta\tau_k \left(\sum_{j=0}^{k-1}\left(\mathsf{M}\left\{\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_{\tau_j}) d\tau\right)^2\right\}\right)^{1/2}\right)^2 <$$

$$< C^2\varepsilon \sum_{k=0}^{N-1} \Delta\tau_k \left(\sum_{j=0}^{k-1}\Delta\tau_j\right)^2 < C^2\varepsilon\frac{(T-t)^3}{3},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

For Case 3 using the Minkowski inequality, standard moment properties for the Itô stochastic integral as well as uniform mean-square continuity of the process ϕ_τ, we find

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \le \left(\sum_{k=0}^{N-1} |\psi_1(\tau_k)|\,\Delta\tau_k \left(\mathsf{M}\left\{\left(\sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_{\tau_j}) df_\tau\right)^2\right\}\right)^{1/2}\right)^2 =$$

$$= \left(\sum_{k=0}^{N-1} |\psi_1(\tau_k)|\Delta\tau_k \left(\sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} \mathsf{M}\left\{|\phi_\tau - \phi_{\tau_j}|^2\right\} d\tau\right)^{1/2}\right)^2 <$$

$$< C^2\varepsilon \left(\sum_{k=0}^{N-1} \Delta\tau_k \left(\sum_{j=0}^{k-1}\Delta\tau_j\right)^{1/2}\right)^2 < C^2\varepsilon\frac{4(T-t)^3}{9},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Finally, for Case 4 using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (3.10) for the stochastic integral, we obtain

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \le \left(\sum_{k=0}^{N-1}\sum_{j=0}^{k-1}|\psi_1(\tau_k)|\Delta\tau_k\left(\mathsf{M}\left\{\left(\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau - \phi_{\tau_j})d\tau\right)^2\right\}\right)^{1/2}\right)^2 <$$

$$< C^2\varepsilon\left(\sum_{k=0}^{N-1}\Delta\tau_k\sum_{j=0}^{k-1}\Delta\tau_j\right)^2 < C^2\varepsilon\frac{(T-t)^4}{4},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Thus, we have proved that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \tilde{\varepsilon}_N = 0.$$

Analogously, taking into account the uniform continuity of the function $\psi_1(\tau)$ on the interval $[t, T]$, we can demonstrate that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \hat{\varepsilon}_N = 0.$$

Consequently,

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0 \quad \text{w. p. 1}.$$

Theorem 3.1 is proved for the case $k = 1$.

Remark 3.1. *Proving Theorem* 3.1, *we used the fact that if the stochastic process ϕ_t is mean-square continuous at the interval $[t, T]$, then it is uniformly mean-square continuous at this interval, i.e. $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ such that for all $t_1, t_2 \in [t, T]$ satisfying the condition $|t_1 - t_2| < \delta(\varepsilon)$ the inequality*

$$\mathsf{M}\left\{|\phi_{t_1} - \phi_{t_2}|^2\right\} < \varepsilon$$

is fulfilled (here $\delta(\varepsilon)$ does not depend on t_1 and t_2).

Proof. *Suppose that the stochastic process ϕ_t is mean-square continuous at the interval $[t,T]$, but not uniformly mean-square continuous at this interval. Then for some $\varepsilon > 0$ and $\forall\ \delta(\varepsilon) > 0\ \exists\ t_1, t_2 \in [t,T]$ such that $|t_1 - t_2| < \delta(\varepsilon)$, but*

$$\mathsf{M}\left\{|\phi_{t_1} - \phi_{t_2}|^2\right\} \ge \varepsilon.$$

Consequently, for $\delta = \delta_n = 1/n$ $(n \in \mathbf{N})$ $\exists\ t_1^{(n)},\ t_2^{(n)} \in [t,T]$ such that

$$\left|t_1^{(n)} - t_2^{(n)}\right| < \frac{1}{n},$$

but

$$\mathsf{M}\left\{\left|\phi_{t_1^{(n)}} - \phi_{t_2^{(n)}}\right|^2\right\} \ge \varepsilon.$$

The sequence $t_1^{(n)}$ $(n \in \mathbf{N})$ is bounded, consequently, according to the Bolzano–Weierstrass Theorem, we can choose from it the subsequence $t_1^{(k_n)}$ $(n \in \mathbf{N})$ that converges to a certain number $\tilde{t}$ (it is simple to demonstrate that $\tilde{t} \in [t,T]$). Similarly to it and in virtue of the inequality

$$\left|t_1^{(n)} - t_2^{(n)}\right| < \frac{1}{n}$$

we have $t_2^{(k_n)} \to \tilde{t}$ when $n \to \infty$.

According to the mean-square continuity of the process ϕ_t at the moment $\tilde{t}$ and the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, we obtain

$$0 \le \mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}} - \phi_{t_2^{(k_n)}}\right|^2\right\} \le$$

$$\le 2\left(\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}} - \phi_{\tilde{t}}\right|^2\right\} + \mathsf{M}\left\{\left|\phi_{t_2^{(k_n)}} - \phi_{\tilde{t}}\right|^2\right\}\right) \to 0$$

when $n \to \infty$. Then

$$\lim_{n\to\infty} \mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}} - \phi_{t_2^{(k_n)}}\right|^2\right\} = 0.$$

It is impossible by virtue of the fact that

$$\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}} - \phi_{t_2^{(k_n)}}\right|^2\right\} \ge \varepsilon > 0.$$

The obtained contradiction proves the required statement.

3.4 Proof of Theorem 3.1 for the Case of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$)

Let us prove Theorem 3.1 for the case $k > 1$. In order to do it we will introduce the following notations

$$I[\psi_q^{(r+1)}]_{\theta,s} \stackrel{\rm def}{=} \int\limits_s^{\theta} \psi_q(t_1) \ldots \int\limits_s^{t_r} \psi_{q+r}(t_{r+1}) dw_{t_{r+1}}^{(q+r)} \ldots dw_{t_1}^{(q)},$$

$$J[\phi, \psi_q^{(r+1)}]_{\theta,s} \stackrel{\rm def}{=} \int\limits_s^{\theta} \psi_q(t_1) \ldots \int\limits_s^{t_r} \psi_{q+r}(t_{r+1}) \int\limits_s^{t_{r+1}} \phi_\tau dw_\tau^{(q+r+1)} dw_{t_{r+1}}^{(q+r)} \ldots dw_{t_1}^{(q)},$$

$$G[\psi_q^{(r+1)}]_{n,m} = \sum_{j_q=m}^{n-1} \sum_{j_{q+1}=m}^{j_q-1} \ldots \sum_{j_{q+r}=m}^{j_{q+r-1}-1} \prod_{l=q}^{r+q} I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$(\psi_q, \ldots, \psi_{q+r}) \stackrel{\rm def}{=} \psi_q^{(r+1)}, \quad \psi_q^{(1)} \stackrel{\rm def}{=} \psi_q,$$

$$(\psi_1, \ldots, \psi_{r+1}) \stackrel{\rm def}{=} \psi_1^{(r+1)}, \quad \psi_1^{(r+1)} \stackrel{\rm def}{=} \psi^{(r+1)}.$$

Note that according to notations introduced above, we have

$$I[\psi_l]_{s,\theta} = \int\limits_\theta^s \psi_l(\tau) dw_\tau^{(l)}.$$

To prove Theorem 3.1 for $k > 1$ it is enough to show that

$$J[\phi, \psi^{(k)}]_{T,t} = \underset{N\to\infty}{\rm l.i.m.}\ S[\phi, \psi^{(k)}]_N = \hat{J}[\phi, \psi^{(k)}]_{T,t} \quad \text{w. p. 1}, \qquad (3.11)$$

where

$$S[\phi, \psi^{(k)}]_N = G[\psi^{(k)}]_{N,0} \sum_{l=0}^{j_k-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)},$$

where $\Delta w_{\tau_l}^{(k+1)} = w_{\tau_{l+1}}^{(k+1)} - w_{\tau_l}^{(k+1)}$.

At first, let us prove the right equality in (3.11). We have

$$\hat{J}[\phi, \psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.}\ \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}. \qquad (3.12)$$

On the basis of the inductive hypothesis we obtain that

$$I[\psi^{(k)}]_{T,\tau_{l+1}} = \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}} \quad \text{w. p. 1}, \tag{3.13}$$

where $\hat{I}[\psi^{(k)}]_{T,s}$ is defined in accordance with (3.7) and

$$I[\psi^{(k)}]_{T,s} = \int\limits_s^T \psi_1(t_1)\ldots \int\limits_s^{t_{k-2}} \psi_{k-1}(t_{k-1}) \int\limits_s^{t_{k-1}} \psi_k(t_k) dw_{t_k}^{(k)} dw_{t_{k-1}}^{(k-1)} \ldots dw_{t_1}^{(1)}.$$

Let us note that when $k \geq 4$ (for $k = 2, 3$ the arguments are similar) due to additivity of the Itô stochastic integral the following equalities are correct

$$I[\psi^{(k)}]_{T,\tau_{l+1}} = \sum_{j_1=l+1}^{N-1} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(t_1) \int\limits_{\tau_{l+1}}^{t_1} \psi_2(t_2) I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}} dw_{t_2}^{(2)} dw_{t_1}^{(1)} =$$

$$= \sum_{j_1=l+1}^{N-1} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(t_1) \left(\sum_{j_2=l+1}^{j_1-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} + \int\limits_{\tau_{j_1}}^{t_1} \right) \psi_2(t_2) I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}} dw_{t_2}^{(2)} dw_{t_1}^{(1)} =$$

$$= \ldots = G[\psi^{(k)}]_{N,l+1} + H[\psi^{(k)}]_{N,l+1} \quad \text{w. p. 1}, \tag{3.14}$$

where

$$H[\psi^{(k)}]_{N,l+1} = \sum_{j_1=l+1}^{N-1} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(s) \int\limits_{\tau_{j_1}}^{s} \psi_2(\tau) I[\psi_3^{(k-2)}]_{\tau,\tau_{l+1}} dw_{\tau}^{(2)} dw_s^{(1)} +$$

$$+ \sum_{r=2}^{k-2} G[\psi^{(r-1)}]_{N,l+1} \sum_{j_r=l+1}^{j_{r-1}-1} \int\limits_{\tau_{j_r}}^{\tau_{j_r+1}} \psi_r(s) \int\limits_{\tau_{j_r}}^{s} \psi_{r+1}(\tau) I[\psi_{r+2}^{(k-r-1)}]_{\tau,\tau_{l+1}} dw_{\tau}^{(r+1)} dw_s^{(r)} +$$

$$+ G[\psi^{(k-2)}]_{N,l+1} \sum_{j_{k-1}=l+1}^{j_{k-2}-1} I[\psi_{k-1}^{(2)}]_{\tau_{j_{k-1}+1},\tau_{j_{k-1}}}. \tag{3.15}$$

Next, substitute (3.14) into (3.13) and (3.13) into (3.12). Then w. p. 1

$$\hat{J}[\phi, \psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} \left(G[\psi^{(k)}]_{N,l+1} + H[\psi^{(k)}]_{N,l+1} \right). \tag{3.16}$$

Since

$$\sum_{j_1=0}^{N-1}\sum_{j_2=0}^{j_1-1}\ldots\sum_{j_k=0}^{j_{k-1}-1} a_{j_1\ldots j_k}=\sum_{j_k=0}^{N-1}\ \sum_{j_{k-1}=j_k+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1} a_{j_1\ldots j_k}, \tag{3.17}$$

where $a_{j_1\ldots j_k}$ are scalars, then

$$G[\psi^{(k)}]_{N,l+1}=\sum_{j_k=l+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1}\prod_{l=1}^{k} I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}. \tag{3.18}$$

Let us substitute (3.18) into

$$\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}G[\psi^{(k)}]_{N,l+1}$$

and use again the formula (3.17). Then

$$\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}G[\psi^{(k)}]_{N,l+1}=S[\phi,\psi^{(k)}]_N. \tag{3.19}$$

Suppose that the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ S[\phi,\psi^{(k)}]_N \tag{3.20}$$

exists (its existence will be proved further).

Then from (3.19) and (3.16) it follows that for proof of the right equality in (3.11) we have to demonstrate that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}H[\psi^{(k)}]_{N,l+1}=0. \tag{3.21}$$

Analyzing the second moment of the prelimit expression on the left-hand side of (3.21) and taking into account (3.15), the independence of ϕ_{τ_l}, $\Delta w_{\tau_l}^{(k+1)}$, and $H[\psi^{(k)}]_{N,l+1}$ as well as the standard estimates for second moments of stochastic integrals and the Minkowski inequality, we find that (3.21) is correct. Thus, by the assumption of existence of the limit (3.20) we obtain that the right equality in (3.11) is fulfilled.

Let us demonstrate that the left equality in (3.11) is also fulfilled.

We have

$$J[\phi,\psi^{(k)}]_{T,t}\stackrel{\text{def}}{=}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l=0}^{N-1}\psi_1(\tau_l)\Delta w_{\tau_l}^{(1)}J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}. \tag{3.22}$$

Let us use for the integral $J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}$ in (3.22) the same arguments, which resulted to the relation (3.14) for the integral $I[\psi^{(k)}]_{T,\tau_{l+1}}$. After that let us substitute the expression obtained for the integral $J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}$ into (3.22). Further, using the Minkowski inequality and standard estimates for second moments of stochastic integrals it is easy to obtain that

$$J[\phi,\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}}\ R[\phi,\psi^{(k)}]_N \quad \text{w. p. 1}, \tag{3.23}$$

where

$$R[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1}\psi_1(\tau_{j_1})\Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0}\sum_{l=0}^{j_k-1}\int\limits_{\tau_l}^{\tau_{l+1}}\phi_\tau dw_\tau^{(k+1)}.$$

We will demonstrate that

$$\underset{N\to\infty}{\text{l.i.m.}}\ R[\phi,\psi^{(k)}]_N = \underset{N\to\infty}{\text{l.i.m.}}\ S[\phi,\psi^{(k)}]_N \quad \text{w. p. 1}. \tag{3.24}$$

It is easy to see that

$$R[\phi,\psi^{(k)}]_N = U[\phi,\psi^{(k)}]_N + V[\phi,\psi^{(k)}]_N + S[\phi,\psi^{(k)}]_N \quad \text{w. p. 1}, \tag{3.25}$$

where

$$U[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1}\psi_1(\tau_{j_1})\Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0}\sum_{l=0}^{j_k-1} I[\Delta\phi]_{\tau_{l+1},\tau_l},$$

$$V[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1} I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}} G[\psi_2^{(k-1)}]_{j_1,0}\sum_{l=0}^{j_k-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)},$$

$$I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}} = \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}(\psi_1(\tau_{j_1})-\psi_1(\tau))dw_\tau^{(1)},$$

$$I[\Delta\phi]_{\tau_{l+1},\tau_l} = \int\limits_{\tau_l}^{\tau_{l+1}}(\phi_\tau-\phi_{\tau_l})dw_\tau^{(k+1)}.$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals, the condition that the process ϕ_τ belongs to the class $\text{S}_2([t,T])$ as well as continuity (which means uniform continuity) of the function $\psi_1(\tau)$, we obtain that

$$\underset{N\to\infty}{\text{l.i.m.}}\ V[\phi,\psi^{(k)}]_N = \underset{N\to\infty}{\text{l.i.m.}}\ U[\phi,\psi^{(k)}]_N = 0 \quad \text{w. p. 1}.$$

Then, considering (3.25), we obtain (3.24). From (3.24) and (3.23) it follows that the left equality in (3.11) is fulfilled.

Note that the limit (3.20) exists because it is equal to the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$, which exists under the conditions of Theorem 3.1. So, the chain of equalities (3.11) is proved. Theorem 3.1 is proved.

3.5 Corollaries and Generalizations of Theorem 3.1

Assume that $D_k=\{(t_1,\ldots,t_k):\ t\le t_1<\ldots<t_k\le T\}$ and the following conditions are fulfilled:

AI. $\xi_\tau\in \mathrm{S}_2([t,T])$.

AII. $\Phi(t_1,\ldots,t_{k-1})$ is a continuous nonrandom function in the closed domain D_{k-1} (recall that we use the same symbol D_{k-1} to denote the open and closed domains corresponding to the domain D_{k-1}).

Let us define the following stochastic integrals

$$\hat J[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\xi_{t_k}d\mathbf{w}_{t_k}^{(i_k)}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\ \underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta\mathbf{w}_{\tau_l}^{(i_k)}\int\limits_{\tau_{l+1}}^T d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)}$$

for $k\ge 3$ and

$$\hat J[\xi,\Phi]_{T,t}^{(2)}=\int\limits_t^T\xi_{t_2}d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1)d\mathbf{w}_{t_1}^{(i_1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\ \underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta\mathbf{w}_{\tau_l}^{(i_2)}\int\limits_{\tau_{l+1}}^T\Phi(t_1)d\mathbf{w}_{t_1}^{(i_1)}$$

for $k=2$. Here $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are F_τ-measurable for all $\tau\in[0,T]$ independent standard Wiener processes, $0\le t<T$, $i_1,\ldots,i_k=0,1,\ldots,m$.

Let us denote

$$J[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-1}}\Phi(t_1,\ldots,t_{k-1})\xi_{t_k}d\mathbf{w}_{t_k}^{(i_k)}\ldots d\mathbf{w}_{t_1}^{(i_1)},\quad k\ge 2,\quad (3.26)$$

where the right-hand side of (3.26) is the iterated Itô stochastic integral.

Let us introduce the following iterated stochastic integrals

$$\tilde J[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\underset{N\to\infty}{\hbox{\rm l.i.m.}}\sum_{l=0}^{N-1}\Delta\mathbf{w}_{\tau_l}^{(i_{k-1})}\int\limits_{\tau_{l+1}}^T d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)},$$

$$J'[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-2}}\Phi(t_1,\ldots,t_{k-1})d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\ldots d\mathbf{w}_{t_1}^{(i_1)},\quad k\ge 2.$$

Similarly to the proof of Theorem 3.1 it is easy to demonstrate that under the condition AII the stochastic integral $\tilde J[\Phi]_{T,t}^{(k-1)}$ exists and

$$J'[\Phi]_{T,t}^{(k-1)}=\tilde J[\Phi]_{T,t}^{(k-1)}\quad \hbox{w. p. 1.}\qquad (3.27)$$

Moreover, using (3.27) the following generalization of Theorem 3.1 can be proved similarly to the proof of Theorem 3.1.

Theorem 3.2 [123] (1997) (also see [1]-[17], [77], [124]). *Suppose that the conditions* AI, AII *of this section are fulfilled. Then, the stochastic integral* $\hat J[\xi,\Phi]_{T,t}^{(k)}$ *exists and for* $k\ge 2$

$$J[\xi,\Phi]_{T,t}^{(k)}=\hat J[\xi,\Phi]_{T,t}^{(k)}\quad \textit{w. p. } 1.$$

Let us consider the following stochastic integrals

$$I=\int\limits_t^T d\mathbf{f}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi_1(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)},\quad J=\int\limits_t^T\int\limits_t^{t_2}\Phi_2(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}.$$

If we consider

$$\int\limits_{t_2}^{T} \Phi_1(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of I and

$$\int\limits_{t}^{t_2} \Phi_2(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of J, then, due to independence of these integrands we may mistakenly think that $\mathsf{M}\{IJ\}=0$. But it is not the fact. Actually, using the integration order replacement technique in the stochastic integral I, we have w. p. 1

$$I=\int\limits_{t}^{T}\int\limits_{t}^{t_1} \Phi_1(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)}=\int\limits_{t}^{T}\int\limits_{t}^{t_2} \Phi_1(t_2,t_1)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}.$$

So, using the standard properties of the Itô stochastic integral [100], we get

$$\mathsf{M}\{IJ\}=\mathbf{1}_{\{i_1=i_2\}}\int\limits_{t}^{T}\int\limits_{t}^{t_2} \Phi_1(t_2,t_1)\Phi_2(t_1,t_2)dt_1dt_2,$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the following statement.

Theorem 3.3 [123] (1997) (also see [1]-[17], [77], [124]). *Let the conditions of Theorem* 3.1 *are fulfilled and $h(\tau)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$\int\limits_{t}^{T}\phi_\tau dw_\tau^{(k+1)}h(\tau)\hat{I}[\psi^{(k)}]_{T,\tau}=\int\limits_{t}^{T}\phi_\tau h(\tau)dw_\tau^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau}\quad \text{w. p. 1},\qquad (3.28)$$

where stochastic integrals on the left-hand side of (3.28) *as well as on the right-hand side of* (3.28) *exist.*

Proof. According to Theorem 3.1, the iterated stochastic integral on the right-hand side of (3.28) exists. In addition

$$\int\limits_{t}^{T}\phi_\tau h(\tau)dw_\tau^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau}=\int\limits_{t}^{T}\phi_\tau dw_\tau^{(k+1)}h(\tau)\hat{I}[\psi^{(k)}]_{T,\tau}-$$

$$
-\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l=0}^{N-1}\phi_{\tau_l}\Delta h(\tau_l)\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}\quad \text{w. p. } 1,
$$

where $\Delta h(\tau_l)=h(\tau_{l+1})-h(\tau_l)$.

Using the arguments which resulted to the right equality in (3.11), we obtain

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l=0}^{N-1}\phi_{\tau_l}\Delta h(\tau_l)\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}=
$$

$$
=\underset{N\to\infty}{\text{l.i.m.}}\ G[\psi^{(k)}]_{N,0}\sum_{l=0}^{j_k-1}\phi_{\tau_l}\Delta h(\tau_l)\Delta w_{\tau_l}^{(k+1)}\quad \text{w. p. } 1. \tag{3.29}
$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals as well as continuity of the function $h(\tau)$, we obtain that the second moment of the prelimit expression on the right-hand side of (3.29) tends to zero when $N\to\infty$. Theorem 3.3 is proved.

Let us consider one corollary of Theorem 3.1.

Theorem 3.4 [123] (1997) (also see [1]-[17], [77], [124]). *Under the conditions of Theorem* 3.3 *the following equality*

$$
\int\limits_t^T h(t_1)\int\limits_t^{t_1}\phi_\tau dw_\tau^{(k+2)}dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}=
$$

$$
=\int\limits_t^T \phi_\tau dw_\tau^{(k+2)}\int\limits_\tau^T h(t_1)dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}\quad w.\ p.\ 1 \tag{3.30}
$$

is fulfilled. Moreover, the stochastic integrals in (3.30) *exist.*

Proof. Using Theorem 3.1 two times, we obtain

$$
\int\limits_t^T \phi_\tau dw_\tau^{(k+2)}\int\limits_\tau^T h(t_1)dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}=
$$

$$
=\int\limits_t^T \psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\rho_\tau dw_\tau^{(k+1)}dw_{t_k}^{(k)}\ldots dw_{t_1}^{(1)}=
$$

$$
=\int\limits_t^T \rho_\tau dw_\tau^{(k+1)}\int\limits_\tau^T \psi_k(t_k)dw_{t_k}^{(k)}\ldots\int\limits_{t_2}^T \psi_1(t_1)dw_{t_1}^{(1)}\quad \text{w. p. 1},
$$

where

$$
\rho_\tau \stackrel{\rm def}{=} h(\tau)\int\limits_t^\tau \phi_s dw_s^{(k+2)}.
$$

Theorem 3.4 is proved.

3.6 Examples of Integration Order Replacement Technique for the Concrete Iterated Itô Stochastic Integrals

As we mentioned above, the formulas from this section could be obtained using the Itô formula. However, the method based on Theorem 3.1 is more simple and familiar, since it deals with usual rules of the integration order replacement for Riemann integrals.

Using the integration order replacement technique for iterated Itô stochastic integrals (Theorem 3.1), we obtain the following equalities which are fulfilled w. p. 1

$$
\int\limits_t^T\int\limits_t^{t_2} df_{t_1}dt_2=\int\limits_t^T (T-t_1)df_{t_1}, \tag{3.31}
$$

$$
\int\limits_t^T \cos(t_2-T)\int\limits_t^{t_2} df_{t_1}dt_2=\int\limits_t^T \sin(T-t_1)df_{t_1},
$$

$$
\int\limits_t^T \sin(t_2-T)\int\limits_t^{t_2} df_{t_1}dt_2=\int\limits_t^T \left(\cos(T-t_1)-1\right)df_{t_1},
$$

$$
\int\limits_t^T e^{\alpha(t_2-T)}\int\limits_t^{t_2} df_{t_1}dt_2=\frac{1}{\alpha}\int\limits_t^T \left(1-e^{\alpha(t_1-T)}\right)df_{t_1},\quad \alpha\ne 0,
$$

$$
\int\limits_t^T (t_2-T)^{\alpha}\int\limits_t^{t_2} df_{t_1}dt_2=-\frac{1}{\alpha+1}\int\limits_t^T (t_1-T)^{\alpha+1}df_{t_1},\quad \alpha\ne -1,
$$

$$J_{(100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)^2df_{t_1},\quad J_{(010)T,t}=\int\limits_t^T(t_1-t)(T-t_1)df_{t_1},$$

$$J_{(110)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}, \tag{3.32}$$

$$J_{(101)T,t}=\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2},\quad J_{(1011)T,t}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1101)T,t}=\int\limits_t^T\int\limits_t^{t_3}(t_3-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1110)T,t}=\int\limits_t^T(T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},\quad J_{(1100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_2)^2\int\limits_t^{t_2}df_{t_1}df_{t_2},$$

$$J_{(1010)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}, \tag{3.33}$$

$$J_{(1001)T,t}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)^2df_{t_1}df_{t_2},\quad J_{(0110)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0101)T,t}=\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0010)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)(t_1-t)^2df_{t_1},\quad J_{(0100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)^2(t_1-t)df_{t_1},$$

$$J_{(1000)T,t}=\frac{1}{3!}\int\limits_t^T(T-t_1)^3df_{t_1},$$

$$J_{(1\underbrace{0\ldots0}_{k-1})T,t}=\frac{1}{(k-1)!}\int\limits_t^T(T-t_1)^{k-1}df_{t_1},$$

$$J_{(11\underbrace{0\ldots0}_{k-2})T,t}=\frac{1}{(k-2)!}\int\limits_t^T(T-t_2)^{k-2}\int\limits_t^{t_2}df_{t_1}df_{t_2},$$

$$J_{(\underbrace{1\ldots1}_{k-1}0)T,t}=\int\limits_t^T(T-t_1)J_{(\underbrace{1\ldots1}_{k-2})t_1,t}df_{t_1},$$

$$J_{(1\underbrace{0\ldots0}_{k-2}1)T,t}=\frac{1}{(k-2)!}\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)^{k-2}df_{t_1}df_{t_2},$$

$$J_{(10\underbrace{1\ldots1}_{k-2})T,t}=\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}\ldots df_{t_{k-1}},$$

$$J_{(\underbrace{1\ldots1}_{k-2}01)T,t}=\int\limits_t^T\int\limits_t^{t_{k-1}}(t_{k-1}-t_{k-2})\int\limits_t^{t_{k-2}}\ldots\int\limits_t^{t_2}df_{t_1}\ldots df_{t_{k-3}}df_{t_{k-2}}df_{t_{k-1}},$$

$$J_{(10)T,t}+J_{(01)T,t}=(T-t)J_{(1)T,t},$$

$$J_{(110)T,t}+J_{(101)T,t}+J_{(011)T,t}=(T-t)J_{(11)T,t},$$

$$J_{(001)T,t}+J_{(010)T,t}+J_{(100)T,t}=\frac{(T-t)^2}{2}J_{(1)T,t},$$

$$J_{(1100)T,t}+J_{(1010)T,t}+J_{(1001)T,t}+J_{(0110)T,t}+$$

$$+J_{(0101)T,t}+J_{(0011)T,t}=\frac{(T-t)^2}{2}J_{(11)T,t},$$

$$J_{(1000)T,t}+J_{(0100)T,t}+J_{(0010)T,t}+J_{(0001)T,t}=\frac{(T-t)^3}{3!}J_{(1)T,t},$$

$$J_{(1110)T,t}+J_{(1101)T,t}+J_{(1011)T,t}+J_{(0111)T,t}=(T-t)J_{(111)T,t},$$

$$\sum_{l=1}^{k}J_{(\underbrace{0\ldots0}_{l-1}1\underbrace{0\ldots0}_{k-l})T,t}=\frac{1}{(k-1)!}(T-t)^{k-1}J_{(1)T,t},$$

$$\sum_{l=1}^{k} J_{(\underbrace{1\ldots 1}_{l-1}\,0\,\underbrace{1\ldots 1}_{k-l})T,t} = (T-t)J_{(\underbrace{1\ldots 1}_{k-1})T,t},$$

$$\sum_{\substack{l_1+\ldots+l_k=m\\ l_i\in\{0,\ 1\},\ i=1,\ldots,k}} J_{(l_1\ldots l_k)T,t} = \frac{(T-t)^{k-m}}{(k-m)!}J_{(\underbrace{1\ldots 1}_{m})T,t},$$

where

$$J_{(l_1\ldots l_k)T,t} = \int\limits_t^T \ldots \int\limits_t^{t_2} dw_{t_1}^{(1)}\ldots dw_{t_k}^{(k)},$$

$l_i = 1$ when $w_{t_i}^{(i)} = f_{t_i}$ and $l_i = 0$ when $w_{t_i}^{(i)} = t_i$ $(i = 1,\ldots,k)$, f_τ is a standard Wiener process.

Let us consider two examples and show explicitly the technique on integration order replacement for iterated Itô stochastic integrals.

Example 3.1. *Let us prove the equality* (3.32). *Using Theorems* 3.1 *and* 3.3, *we obtain*:

$$J_{(110)T,t} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} df_{t_1}df_{t_2}dt_3 = \int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}\int\limits_{t_2}^T dt_3 =$$

$$= \int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}(T-t_2) = \int\limits_t^T df_{t_1}\int\limits_{t_1}^T (T-t_2)df_{t_2} =$$

$$= \int\limits_t^T (T-t_2)\int\limits_t^{t_2} df_{t_1}df_{t_2}\quad \text{w. p. 1.} \tag{3.34}$$

Example 3.2. *Let us prove the equality* (3.33). *Using Theorems* 3.1 *and* 3.3, *we obtain*

$$J_{(1010)T,t} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} df_{t_1}dt_2df_{t_3}dt_4 = \int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}\int\limits_{t_3}^T dt_4 =$$

$$= \int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}(T-t_3) = \int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T (T-t_3)df_{t_3} =$$

$$
=\int\limits_t^T (T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2df_{t_3}=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2\right)df_{t_3}=
$$

$$
=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}\int\limits_{t_1}^{t_3} dt_2\right)df_{t_3}=
$$

$$
=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}(t_3-t_1)\right)df_{t_3}=
$$

$$
=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} (t_3-t_1)df_{t_1}\right)df_{t_3}=
$$

$$
=\int\limits_t^T (T-t_2)\int\limits_t^{t_2} (t_2-t_1)df_{t_1}df_{t_2}\quad \hbox{w. p. 1.}
$$

3.7 Integration Order Replacement Technique for Iterated Stochastic Integrals with Respect to Martingale

In this section, we will generalize the theorems on integration order replacement for iterated Itô stochastic integrals to the class of iterated stochastic integrals with respect to martingale.

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing family of σ-algebras defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$. Suppose that M_t, $t\in[0,T]$ is an F_t-measurable martingale for all $t\in[0,T]$, which satisfies the condition $\mathsf{M}\{|M_t|\}<\infty$. Moreover, for all $t\in[0,T]$ there exists an F_t- measurable and nonnegative w. p. 1 stochastic process ρ_t, $t\in[0,T]$ such that

$$
\mathsf{M}\left\{(M_s-M_t)^2\ |\ \mathrm{F}_t\right\}=\mathsf{M}\left\{\int\limits_t^s \rho_\tau d\tau\Bigg|\mathrm{F}_t\right\}\quad \hbox{w. p. 1,}
$$

where $0\le t<s\le T$.

Let us consider the class $H_2(\rho,[0,T])$ of stochastic processes $\varphi_t,\ t\in[0,T]$, which are F_t-measurable for all $t\in[0,T]$ and satisfy the condition

$$
\mathsf{M}\left\{\int\limits_0^T \varphi_t^2\rho_t dt\right\}<\infty.
$$

For any partition $\tau_j^{(N)},\ j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$
0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty \tag{3.35}
$$

we will define the sequence of step functions

$$
\varphi^{(N)}(t,\omega)=\varphi_j\left(\omega\right)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),
$$

where $\varphi^{(N)}(t,\omega)\in H_2(\rho,[0,T]),\ j=0,1,\ldots,N-1,\ N=1,2,\ldots$

Let us define the stochastic integral with respect to martingale for $\varphi(t,\omega)\in H_2(\rho,[0,T])$ as the following mean-square limit [100]

$$
\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\varphi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(M\left(\tau_{j+1}^{(N)},\omega\right)-M\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\varphi_\tau dM_\tau,
$$

where $\varphi^{(N)}(t,\omega)$ is any step function from the class $H_2(\rho,[0,T])$, which converges to the function $\varphi(t,\omega)$ in the following sense

$$
\lim_{N\to\infty}\int\limits_0^T\mathsf{M}\left\{\left|\varphi^{(N)}(t,\omega)-\varphi(t,\omega)\right|^2\right\}\rho_t dt=0.
$$

It is well known [100] that the stochastic integral

$$
\int\limits_0^T\varphi_\tau dM_\tau
$$

exists and it does not depend on the selection of sequence $\varphi^{(N)}(t,\omega)$.

Let $\tilde{H}_2(\rho,[0,T])$ be the class of stochastic processes $\varphi_\tau,\ \tau\in[0,T]$, which are mean-square continuous for all $\tau\in[0,T]$ and belong to the class $H_2(\rho,[0,T])$.

Let us consider the following iterated stochastic integrals

$$S[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\phi_\tau dM_\tau^{(k+1)}dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}, \quad (3.36)$$

$$S[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}. \quad (3.37)$$

Here $\phi_\tau\in\tilde{H}_2(\rho,[t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous nonrandom functions at the interval $[t,T]$, $M_\tau^{(l)}=M_\tau$ or $M_\tau^{(l)}=\tau$ if $\tau\in[t,T]$, $l=1,\ldots,k+1$, M_τ is the martingale defined above.

Let us define the iterated stochastic integral $\hat{S}[\psi^{(k)}]_{T,s}$, $0\le t\le s\le T$, $k\ge 1$ with respect to martingale

$$\hat{S}[\psi^{(k)}]_{T,s}=\int\limits_s^T\psi_k(t_k)dM_{t_k}^{(k)}\ldots\int\limits_{t_2}^T\psi_1(t_1)dM_{t_1}^{(1)}$$

by the following recurrence relation

$$\hat{S}[\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\psi_k(\tau_l)\Delta M_{\tau_l}^{(k)}\hat{S}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \quad (3.38)$$

where $k\ge 1$, $\hat{S}[\psi^{(0)}]_{T,s}\stackrel{\rm def}{=}1$, $[s,T]\subseteq[t,T]$, here and further $\Delta M_{\tau_l}^{(i)}=M_{\tau_{l+1}}^{(i)}-M_{\tau_l}^{(i)}$, $i=1,\ldots,k+1$, $l=0,1,\ldots,N-1$, $\{\tau_l\}_{l=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition similar to (3.35), another notations are the same as in (3.36), (3.37).

Further, let us define the iterated stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$ of the form

$$\hat{S}[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\phi_s dM_s^{(k+1)}\hat{S}[\psi^{(k)}]_{T,s}$$

by the equality

$$\hat{S}[\phi,\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta M_{\tau_l}^{(k+1)}\hat{S}[\psi^{(k)}]_{T,\tau_{l+1}},$$

where the sense of notations included in (3.36)–(3.38) is saved.

Let us formulate the theorem on integration order replacement for the iterated stochastic integrals with respect to martingale, which is the generalization of Theorem 3.1.

Theorem 3.5 [151] (1999) (also see [1]-[17], [124]). *Let $\phi_\tau \in \tilde{H}_2(\rho, [t, T])$, every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[t, T]$, and $|\rho_\tau| \le K < \infty$ w. p. 1 for all $\tau \in [t, T]$. Then, the stochastic integral $\hat{S}[\phi, \psi^{(k)}]_{T,t}$ exists and*

$$S[\phi, \psi^{(k)}]_{T,t} = \hat{S}[\phi, \psi^{(k)}]_{T,t} \quad \text{w. p. } 1.$$

The proof of Theorem 3.5 is similar to the proof of Theorem 3.1.

Remark 3.2. *Let us note that we can propose another variant of the conditions in Theorem 3.5. For example, if we not require the boundedness of the process ρ_τ, then it is necessary to require the fulfillment of the following additional conditions*:

1. $\mathsf{M}\{|\rho_\tau|\} < \infty$ *for all* $\tau \in [t, T]$.
2. *The process ρ_τ is independent with the processes ϕ_τ and M_τ.*

Remark 3.3. *Note that it is well known the construction of stochastic integral with respect to the Wiener process with integrable process, which is not an F_τ-measurable stochastic process — the so-called Stratonovich stochastic integral* [114].

The stochastic integral $\hat{S}[\phi, \psi^{(k)}]_{T,t}$ is also the stochastic integral with integrable process, which is not an F_τ-measurable stochastic process. However, under the conditions of Theorem 3.5

$$S[\phi, \psi^{(k)}]_{T,t} = \hat{S}[\phi, \psi^{(k)}]_{T,t} \quad \text{w. p. } 1,$$

where $S[\phi, \psi^{(k)}]_{T,t}$ is a usual iterated stochastic integral with respect to martingale. If, for example, $M_\tau, \tau \in [t, T]$ is the Wiener process, then the question on connection between stochastic integral $\hat{S}[\phi, \psi^{(k)}]_{T,t}$ and Stratonovich stochastic integral is solving as a standard question on connection between Stratonovich and Itô stochastic integrals [114].

Let us consider several statements, which are the generalizations of theorems formulated in the previous sections.

Assume that $D_k = \{(t_1, \ldots, t_k) : \ t \le t_1 < \ldots < t_k \le T\}$ and the following conditions are fulfilled:

BI. $\xi_\tau \in \tilde{H}_2(\rho, [t, T])$.

BII. $\Phi(t_1,\ldots,t_{k-1})$ is a continuous nonrandom function in the closed domain D_{k-1} (recall that we use the same symbol D_{k-1} to denote the open and closed domains corresponding to the domain D_{k-1}).

Let us define the following stochastic integrals with respect to martingale

$$\hat{S}[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T \xi_{t_k}dM_{t_k}^{(k)}\ldots\int\limits_{t_3}^T dM_{t_2}^{(2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})dM_{t_1}^{(1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta M_{\tau_l}^{(k)}\int\limits_{\tau_{l+1}}^T dM_{t_{k-1}}^{(k-1)}\ldots\int\limits_{t_3}^T dM_{t_2}^{(2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})dM_{t_1}^{(1)}$$

for $k\ge 3$ and

$$\hat{S}[\xi,\Phi]_{T,t}^{(2)}=\int\limits_t^T \xi_{t_2}dM_{t_2}^{(2)}\int\limits_{t_2}^T \Phi(t_1)dM_{t_1}^{(1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta M_{\tau_l}^{(2)}\int\limits_{\tau_{l+1}}^T \Phi(t_1)dM_{t_1}^{(1)}$$

for $k=2$, where the sense of notations included in (3.36)–(3.38) is saved. Moreover, the stochastic process ξ_τ, $\tau\in[t,T]$ belongs to the class $\tilde{H}_2(\rho,[t,T])$.

In addition, let

$$S[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-1}}\Phi(t_1,\ldots,t_{k-1})\xi_{t_k}dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)},\quad k\ge 2,\quad (3.39)$$

where the right-hand side of (3.39) is the iterated stochastic integral with respect to martingale.

Let us introduce the following iterated stochastic integrals with respect to martingale

$$\tilde{S}[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T dM_{t_{k-1}}^{(k-1)}\ldots\int\limits_{t_3}^T dM_{t_2}^{(2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})dM_{t_1}^{(1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l=0}^{N-1}\Delta M_{\tau_l}^{(k-1)}\int\limits_{\tau_{l+1}}^T dM_{t_{k-2}}^{(k-2)}\ldots\int\limits_{t_3}^T dM_{t_2}^{(2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})dM_{t_1}^{(1)},$$

$$
S'[\Phi]_{T,t}^{(k-1)} = \int\limits_t^T \ldots \int\limits_t^{t_{k-2}} \Phi(t_1,\ldots,t_{k-1}) dM_{t_{k-1}}^{(k-1)} \ldots dM_{t_1}^{(1)}, \quad k \ge 2.
$$

It is easy to demonstrate similarly to the proof of Theorem 3.5 that under the condition BII the stochastic integral $\tilde{S}[\Phi]_{T,t}^{(k-1)}$ exists and

$$
S'[\Phi]_{T,t}^{(k-1)} = \tilde{S}[\Phi]_{T,t}^{(k-1)} \quad \text{w. p. 1.}
$$

In its turn, using this fact we can prove the following theorem similarly to the proof of Theorem 3.5.

Theorem 3.6 [151] (1999) (also see [1]-[17], [124]). *Let the conditions* BI, BII *of this section are fulfilled and* $|\rho_\tau| \le K < \infty$ *w. p.* 1 *for all* $\tau \in [t,T]$. *Then, the stochastic integral* $\hat{S}[\xi,\Phi]_{T,t}^{(k)}$ *exists and for* $k \ge 2$

$$
S[\xi,\Phi]_{T,t}^{(k)} = \hat{S}[\xi,\Phi]_{T,t}^{(k)} \quad \textit{w. p. } 1.
$$

Theorem 3.6 is the generalization of Theorem 3.2 for the case of iterated stochastic integrals with respect to martingale.

Let us consider two statements.

Theorem 3.7 [151] (1999) (also see [1]-[17], [124]). *Let the conditions of Theorem* 3.5 *are fulfilled and* $h(\tau)$ *is a continuous nonrandom function at the interval* $[t,T]$. *Then*

$$
\int\limits_t^T \phi_\tau dM_\tau^{(k+1)} h(\tau) \hat{S}[\psi^{(k)}]_{T,\tau} = \int\limits_t^T \phi_\tau h(\tau) dM_\tau^{(k+1)} \hat{S}[\psi^{(k)}]_{T,\tau} \quad \textit{w. p. } 1, \tag{3.40}
$$

where the stochastic integrals in (3.40) *exist.*

Theorem 3.8 [151] (1999) (also see [1]-[17], [124]). *Under the conditions of Theorem* 3.5

$$
\int\limits_t^T h(t_1) \int\limits_t^{t_1} \phi_\tau dM_\tau^{(k+2)} dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} =
$$

$$
= \int\limits_t^T \phi_\tau dM_\tau^{(k+2)} \int\limits_\tau^T h(t_1) dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} \ \textit{w. p. } 1, \tag{3.41}
$$

where the stochastic integrals in (3.41) *exist.*

The proofs of Theorems 3.7 and 3.8 are similar to the proofs of Theorems 3.3 and 3.4 correspondingly.

Remark 3.4. *The integration order replacement technique for iterated Itô stochastic integrals* (*Theorems* 3.1–3.4) *has been successfully applied for construction of the so-called unified Taylor–Itô and Taylor–Stratonovich expansions* (*see Chapter* 4) *as well as for proof and development of the mean-square approximation method for iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series* (*see Chapters* 1 *and* 2).

Chapter 4

Four New Forms of the Taylor–Itô and Taylor–Stratonovich Expansions and its Application to the High-Order Strong Numerical Methods for Itô Stochastic Differential Equations

The problem of the Taylor–Itô and Taylor–Stratonovich expansions of the Itô stochastic processes in a neighborhood of a fixed time moment is considered in this chapter. The classical forms of the Taylor–Itô and Taylor–Stratonovich expansions are transformed to four new representations, which include the minimal sets of different types of iterated Itô and Stratonovich stochastic integrals. Therefore, these representations (the so-called unified Taylor–Itô and Taylor–Stratonovich expansions) are more convenient for constructing of the high-order strong numerical methods for Itô SDEs. Explicit one-step strong numerical schemes with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0 based on the unified Taylor–Itô and Taylor–Stratonovich expansions are derived.

4.1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a non-decreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent.

Consider an Itô SDE in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{4.1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the Itô SDE (4.1). The nonrandom functions $\mathbf{a} : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$, $B : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^{n \times m}$ guarantee the existence and uniqueness (up to stochastic equivalence) of a strong solution to the equation (4.1) [100]. The second integral on the right-hand side of (4.1) is interpreted as an Itô stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$. Also we assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [84], [85], [93], [152], [153] (also see [13]) that Itô SDEs are adequate mathematical models of dynamic systems of different physical nature that are affected by random perturbations. For example, Itô SDEs are used as mathematical models in stochastic mathematical finance, hydrology, seismology, geophysics, chemical kinetics, population dynamics, electrodynamics, medicine and other fields [84], [85], [93], [152], [153] (also see [13]).

Numerical integration of Itô SDEs based on the strong convergence criterion of approximations [84] is widely used for the numerical simulation of sample trajectories of solutions to Itô SDEs (which is required for constructing new mathematical models on the basis of such equations and for the numerical solution of different mathematical problems connected with Itô SDEs). Among these problems, we note the following: filtering of signals under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis [82], [84], [85], [92], [93], [129], [154]-[158].

Exact solutions of Itô SDEs are known in rather rare cases. For this reason it is necessary to construct numerical procedures for solving these equations.

In this chapter, a promising approach [82], [84], [85], [92], [93] to the numerical integration of Itô SDEs based on the stochastic analogues of the Taylor formula (Taylor–Itô and Taylor–Stratonovich expansions) [159], [160] (also see [52], [77], [161]-[165]) is used. This approach uses a finite discretization of the time variable and the numerical simulation of the solution to the Itô SDE at discrete time moments using the stochastic analogues of the Taylor formula mentioned above. A number of works (e.g., [82]-[85], [92], [93]) describe numerical schemes with the strong convergence orders $1.5, 2.0, 2.5$, and 3.0 for Itô SDEs; however, they do not contain efficient procedures of the mean-square approximation of the iterated stochastic integrals for the case of multidimensional nonadditive noise.

In this chapter, we consider the unified Taylor–Itô and Taylor–Stratonovich expansions [161], [163] (also see [52], [77]) which makes it possible (in contrast with its classical analogues [84], [159]) to use the minimal sets of iterated Itô and Stratonovich stochastic integrals; this is a simplifying factor for the numerical methods implementation. We prove the unified Taylor–Itô expansion [161] with using of the slightly different approach (which is taken from [163]) in comparison with the approach from [161]. Moreover, we obtain another (second) version of the unified Taylor–Itô expansion [81], [164]. In addition we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions [163]).

It should be noted that in Chapter 5 on the base of the results of Chapters 1, 2 we study methods of numerical simulation of specific iterated Itô and Stratonovich stochastic integrals of multiplicities $1, 2, 3, 4, 5$, and 6 from the Taylor–Itô and Taylor–Stratonovich expansions. These stochastic integrals are used in the strong numerical methods for Itô SDEs [82], [84], [85], [92] (also see [13]). To approximate the iterated Itô and Stratonovich stochastic integrals appearing in the numerical schemes with the strong convergence orders $1.0, 1.5, 2.0, 2.5$, and 3.0, the method of generalized multiple Fourier series (see Chapter 1) and especially method of multiple Fourier–Legendre series will be applied in Chapter 5. It is important that the method of generalized multiple Fourier series (Theorems 1.1, 1.16) does not lead to the partitioning of the integration interval of the iterated Itô and Stratonovich stochastic integrals under consideration; this interval length is the integration step of the numerical methods used to solve Itô SDEs; therefore, it is already fairly small and does not need to be partitioned. Computational experiments [1] show that the numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors. Also note that the Legendre polynomials have essential advantage (in a number of aspects) over the trigonomentric functions (see Chapter 5) constructing the mean-square approximations of iterated Itô and Stratonovich stochastic integrals in the framework of the method of generalized multiple Fourier series (Theorems 1.1, 1.16).

Let us consider the following iterated Itô and Stratonovich stochastic integrals:

$$J[\psi^{(k)}]_{s,t} = \int\limits_t^s \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{4.2}$$

$$J^*[\psi^{(k)}]_{s,t} = \int\limits_t^{*s} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{4.3}$$

where $0 \le t < s \le T$, every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[0, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

It should be noted that one of the main problems when constructing the high-order strong numerical methods for Itô SDEs on the base of the Taylor–Itô and Taylor–Stratonovich expansions is the mean-square approximation of the iterated Itô and Stratonovich stochastic integrals (4.2) and (4.3). Obviously, in the absence of procedures for the numerical simulation of stochastic integrals, the mentioned numerical methods are unrealizable in practice. For this reason, in Chapter 5 we give the extensive practical material on expansions and mean-square approximations of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich expansions. In Chapter 5, the main focus is on approximations based on multiple Fourier–Legendre series. Such approximations is more effective in comparison with the trigonometric approximations (see Sect. 5.2) at least for the numerical methods with the strong convergence order 1.5 and higher [21], [40].

The rest of this Chapter is organized as follows. In Sect. 4.1 (below) we consider a brief review of publications on the problem of construction of the Taylor–Itô and Taylor–Stratonovich expansions for the solutions of Itô SDEs. Sect. 4.2 is devoted to some auxiliary lemmas. In Sect. 4.3 we consider the classical Taylor–Itô expansion while Sect. 4.4 and Sect. 4.5 are devoted to first and second forms of the so-called unified Taylor–Itô expansion correspondingly. The classical Taylor–Stratonovich expansion is considered in Sect. 4.6. First and second forms of the unified Taylor–Stratonovich expansion are derived in Sect. 4.7 and Sect. 4.8. In Sect. 4.9 we give a comparative analysis of the unified Taylor–Itô and Taylor–Stratonovich expansions with the classical Taylor–Itô and Taylor–Stratonovich expansions. Application of the first form of the unified Taylor–Itô expansion to the high-order strong numerical methods for Itô SDEs is considered in Sect. 4.10. In Sect. 4.11 we construct the high-order strong numerical methods for Itô SDEs on the base of the first form of the unified Taylor–Stratonovich expansion.

Let us give a brief review of publications on the problem of construction of the Taylor–Itô and Taylor–Stratonovich expansions for the solutions of Itô SDEs. A few variants of a stochastic analog of the Taylor formula have been obtained in [159], [160] (also see [82], [84]) for the stochastic processes in the form $R(\mathbf{x}_s, s)$, where $\mathbf{x}_s$ is a solution of the Itô SDE (4.1) and $R : \mathbf{R}^n \times [0, T] \to \mathbf{R}$ is a sufficiently smooth nonrandom function.

The first result in this direction called the Itô–Taylor expansion has been obtained in [160] (also see [159]). This result gives an expansion of the process $R(\mathbf{x}_s, s)$ into a series such that every term (if $k > 0$) contains the iterated Itô stochastic integral

$$\int\limits_t^s \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4.4}$$

as a multiplier factor, where $0 \le t < s \le T$, $i_1, \ldots, i_k = 0, 1, \ldots, m$. Obviously, the iterated Itô stochastic integral (4.4) is a particular case of (4.2) for $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$.

In [159] another expansion of the stochastic process $R(\mathbf{x}_s, s)$ into a series has been derived. The iterated Stratonovich stochastic integrals

$$\int\limits_t^{*s} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4.5}$$

were used instead of the iterated Itô stochastic integrals; the corresponding expansion was called the Stratonovich–Taylor expansion. In the formula (4.5) the indices $i_1, \ldots, i_k$ take values $0, 1, \ldots, m$.

In [161] the Itô–Taylor expansion [159] is reduced to the interesting and unexpected form (the so-called unified Taylor–Itô expansion) by special transformations (see Chapter 3). Every term of this expansion (if $k > 0$) contains the iterated Itô stochastic integral

$$\int\limits_t^s (s - t_k)^{l_k} \ldots \int\limits_t^{t_2} (s - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{4.6}$$

where $l_1, \ldots, l_k = 0, 1, 2, \ldots$ and $i_1, \ldots, i_k = 1, \ldots, m$.

It is worth to mention another form of the unified Taylor–Itô expansion [81], [164] (also see [1]-[17]). Terms of the latter expansion contain iterated Itô stochastic integrals of the form

$$\int\limits_t^s (t-t_k)^{l_k}\ldots \int\limits_t^{t_2}(t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{4.7}$$

where $l_1,\ldots,l_k=0,1,2,\ldots$ and $i_1,\ldots,i_k=1,\ldots,m.$

Obviously that some of the iterated Itô stochastic integrals (4.4) or (4.5) are connected by linear relations, while this is not the case for integrals defined by (4.6), (4.7). In this sense, the total quantity of stochastic integrals defined by (4.6) or (4.7) is minimal. Futhermore, in this chapter we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions) [165] (also see [163]) such that every term (if $k>0$) contains as a multiplier the iterated Stratonovich stochastic integral of one of two types

$$\int\limits_t^{*s} (t-t_k)^{l_k}\ldots \int\limits_t^{*t_2}(t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{4.8}$$

$$\int\limits_t^{*s} (s-t_k)^{l_k}\ldots \int\limits_t^{*t_2}(s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{4.9}$$

where $l_1,\ldots,l_k=0,1,2,\ldots,$ $i_1,\ldots,i_k=1,\ldots,m,$ and $k=1,2,\ldots$

It is not difficult to see that for the sets of iterated Stratonovich stochastic integrals (4.8) and (4.9) the property of minimality (see above) also holds as for the sets of iterated Itô stochastic integrals (4.6), (4.7).

As we noted above, the main problem in implementation of high-order strong numerical methods for Itô SDEs is the mean-square approximation of the iterated stochastic integrals (4.4)–(4.9). Obviously, these stochastic integrals are particular cases of the stochastic integrals (4.2), (4.3).

Taking into account the results of Chapters 1, 2, 3, 5 and the minimality of the sets of stochastic integrals (4.6)–(4.9), we conclude that the unified Taylor–Itô and Taylor–Stratonovich expansions based on the iterated stochastic integrals (4.6)–(4.9) can be useful for constructing of high-order strong numerical methods with the convergence orders 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, ... for Itô SDEs.

4.2 Auxiliary Lemmas

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) \stackrel{\mathsf{def}}{=} f_t : [0,T]\times\Omega \to \mathbf{R}$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$.

Consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let us consider the class $\mathrm{M}_2([t,T])$ $(t\ge 0)$ of random functions $\xi(\tau,\omega) \stackrel{\mathsf{def}}{=} \xi_\tau : [t,T]\times\Omega\to\mathbf{R}$ (see Sect. 1.1.2).

Recall (see Sect. 2.1.1) that the class $\mathrm{Q}_m([t,T])$ $(t\ge 0)$ consists of Itô processes η_τ, $\tau\in[t,T]$ of the form

$$\eta_\tau = \eta_t + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s df_s, \tag{4.10}$$

where $(a_\tau)^m, (b_\tau)^m \in \mathrm{M}_2([t,T])$ and $\lim\limits_{s\to\tau}\mathsf{M}\left\{|b_s-b_\tau|^4\right\}=0$ for all $\tau\in[t,T]$. The second integral on the right-hand side of (4.10) is the Itô stochastic integral.

Also note that the definition of the Stratonovich stochastic integral in the mean-square sense is given by (2.3) (Sect. 2.1.1) and the relation between Stratonovich and Itô stochastic integrals (see Sect. 2.1.1) has the following form [114] (also see [84])

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau + \frac{1}{2}\int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau \quad \text{w. p. 1.} \tag{4.11}$$

If the Wiener processes in (4.10) and (4.11) are independent, then

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau \quad \text{w. p. 1.} \tag{4.12}$$

Recall that a possible variant of conditions providing the correctness of the formulas (4.11) and (4.12) consists of the conditions: $\eta_\tau\in\mathrm{Q}_4([t,T])$, $F(\eta_\tau,\tau)\in$

$\mathrm{M}_2([t,T])$, $F(x,\tau)\in C^{2,1}(\mathbf{R}\times[t,T])$, where $C^{2,1}(\mathbf{R}\times[t,T])$ $(t\ge 0)$ is the space of functions $F(x,\tau):\mathbf{R}\times[t,T]\to\mathbf{R}$ such that

$$\left|\frac{\partial F}{\partial x}(x,\tau)\right|\le K,\quad \left|\frac{\partial^2 F}{\partial x^2}(x,\tau)\right|\le K,\quad \left|\frac{\partial F}{\partial \tau}(x,\tau)\right|\le K,\quad \left|\frac{\partial^2 F}{\partial \tau\partial x}(x,\tau)\right|\le K$$

for all $x\in\mathbf{R}$ and $\tau\in[t,T]$, where constant K does not depend on x,τ.

Remark 4.1. *Note that if $F(x,\tau)=F_1(x)F_2(\tau)$, then it suffices to require that $F(x,\tau)$ be twice differentiable with respect to x (with bounded derivatives) and continuous with respect to τ (instead of the condition $F(x,\tau)\in C^{2,1}(\mathbf{R}\times[t,T])$).*

Also remind that $\mathrm{S}_2([t,T])$ $(t\ge 0)$ is a subset of $\mathrm{M}_2([t,T])$ and $\mathrm{S}_2([t,T])$ consists of the mean-square continuous random functions (see Sect. 3.1).

Let us apply Theorem 3.1 (see Sect. 3.2) to derive one property for Itô stochastic integrals.

Lemma 4.1 [14]-[17], [52]. *Let $h(\tau)$, $g(\tau)$, $G(\tau):[t,s]\to\mathbf{R}$ be continuous nonrandom functions at the interval $[t,s]$ and let $G(\tau)$ be a antiderivative of the function $g(\tau)$. Furthermore, let $\xi_\tau\in\mathrm{S}_2([t,s])$. Then*

$$\int\limits_t^s g(\tau)\int\limits_t^\tau h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau=\int\limits_t^s (G(s)-G(\theta))h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}$$

w. p. 1, where $i,j=1,2$ and $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ are independent standard Wiener processes that are F_τ–measurable for all $\tau\in[t,s]$.

Proof. Applying Theorem 3.1 two times and Theorem 3.3, we get the following relations

$$\int\limits_t^s g(\tau)\int\limits_t^\tau h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau=\int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau=$$

$$=G(s)\int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}-\int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s G(\theta)h(\theta)d\mathbf{f}_\theta^{(j)}=$$

$$=G(s)\int\limits_t^s h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}-\int\limits_t^s G(\theta)h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}=$$

$$
= \int\limits_t^s (G(s) - G(\theta))h(\theta) \int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} \quad \text{w. p. 1.}
$$

The proof of Lemma 4.1 is completed.

Let us consider an analogue of Lemma 4.1 for Stratonovich stochastic integrals.

Lemma 4.2 [163] (also see [1]-[5], [12]-[17], [52]). *Let $h(\tau)$, $g(\tau)$, $G(\tau)$: $[t, s] \to \mathbf{R}$ be continuous nonrandom functions at the interval $[t, s]$ and let $G(\tau)$ be a antiderivative of the function $g(\tau)$. Furthermore, let $\xi_\tau^{(l)} \in \mathrm{Q}_4([t, s])$ and*

$$
\xi_\tau^{(l)} = \int\limits_t^\tau a_u du + \int\limits_t^\tau b_u d\mathbf{f}_u^{(l)}, \quad l = 1, 2.
$$

Then

$$
\int\limits_t^s g(\tau) \int\limits_t^{*\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} d\tau = \int\limits_t^{*s} (G(s) - G(\theta))h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} \tag{4.13}
$$

w. p. 1, where $i, j, l = 1, 2$ and $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ are independent standard Wiener processes that are F_τ–measurable for all $\tau \in [t, s]$.

Proof. Under the conditions of Lemma 4.2, we can apply the equalities (4.11) and (4.12) with $F(x, \theta) \equiv xh(\theta)$ and

$$
\eta_\theta = \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)},
$$

since the function $xh(\theta)$ is sufficiently smooth (see Remark 4.1) and the following obvious inclusions hold: $\eta_\theta \in \mathrm{Q}_4([t, s])$, $\eta_\theta h(\theta) \in \mathrm{M}_2([t, s])$.

Thus, we have the equalities

$$
\int\limits_t^{*\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} = \int\limits_t^{\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} + \frac{1}{2} \mathbf{1}_{\{i=j\}} \int\limits_t^{\tau} h(\theta) \xi_\theta^{(l)} d\theta, \tag{4.14}
$$

$$
\int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} = \int\limits_t^{\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} + \frac{1}{2} \mathbf{1}_{\{l=i\}} \int\limits_t^{\theta} b_u du \tag{4.15}
$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of the set A.

Substituting the formulas (4.14) and (4.15) into the left-hand side of the equality (4.13) and applying Theorem 3.1 twice and Theorem 3.3, we get the following relations

$$
\int\limits_t^s g(\tau)\int\limits_t^{*\tau} h(\theta)\int\limits_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau = \int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta\int\limits_\theta^s g(\tau)d\tau=
$$

$$
=G(s)\left(\int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta+\right.
$$

$$
\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\right)-
$$

$$
-\left(\int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s G(\theta)h(\theta)d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)}d\theta+\right.
$$

$$
\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)G(\theta)d\mathbf{f}_\theta^{(j)}\right)=
$$

$$
=G(s)\left(\int\limits_t^s h(\theta)\int\limits_t^\theta \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta+\right.
$$

$$
\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s h(\theta)\int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right)-
$$

$$
-\left(\int\limits_t^s G(\theta)h(\theta)\int\limits_t^\theta \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)}d\theta+\right.
$$

$$
\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s h(\theta)G(\theta)\int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right) \tag{4.16}
$$

w. p. 1. Applying successively the formulas (4.14), (4.15) together with the formula (4.14) in which $h(\theta)$ replaced by $G(\theta)h(\theta)$ as well as the relation (4.16), we obtain the equality (4.13). The proof of Lemma 4.2 is completed.

4.3 The Taylor–Itô Expansion

In this section, we use the Taylor–Itô expansion [159] and introduce some necessary notations. At that we will use the original notations introduced by the author of this book.

Let $C^{2,1}(\mathbf{R}^n\times[0,T]) \stackrel{\sf def}{=} \mathrm{L}$ be the space of functions $R(\mathbf{x},t):\mathbf{R}^n\times[0,T]\to\mathbf{R}$ with the following property: these functions are continuous and twice continuously differentiable in $\mathbf{x}$ and have one continuous derivative in t. We consider the following operators on the space L

$$
L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}a^{(i)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(i)}}+
$$

$$
+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}}, \tag{4.17}
$$

$$
G_0^{(i)}=\sum_{j=1}^{n}B^{(ji)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(j)}},\quad i=1,\ldots,m, \tag{4.18}
$$

where $\mathbf{x}^{(j)}$ is the jth component of $\mathbf{x}$, $a^{(j)}(\mathbf{x},t)$ is the jth component of $a(\mathbf{x},t)$, and $B^{(ij)}(\mathbf{x},t)$ is the ijth element of $B(\mathbf{x},t)$.

By the Itô formula, we have the equality

$$
R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\int\limits_t^s LR(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^{m}\int\limits_t^s G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)} \tag{4.19}
$$

w. p. 1, where $\mathbf{x}_t$ is a strong solution of the Itô SDE (4.1), $0\le t<s\le T$. In the formula (4.19) it is assumed that the functions $\mathbf{a}(\mathbf{x},t)$, $B(\mathbf{x},t)$, and $R(\mathbf{x},t)$ satisfy the following condition: $LR(\mathbf{x}_\tau,\tau)$, $G_0^{(i)}R(\mathbf{x}_\tau,\tau)\in \mathrm{M}_2([0,T])$ for $i=1,\ldots,m$.

Introduce the following notations

$$
{}^{(k)}A=\Biggl\Vert A^{(i_1\ldots i_k)}\Biggr\Vert_{i_1=1,\ldots,i_k=1}^{m_1\ \ldots\ \ m_k},\quad m_1,\ldots,m_k\ge 1, \tag{4.20}
$$

$$
{}^{(k+l)}A \overset{l}{\cdot} {}^{(l)}B^{(k)} = \begin{cases} \left\Vert \sum\limits_{i_1=1}^{m_1} \ldots \sum\limits_{i_l=1}^{m_l} A^{(i_1\ldots i_{k+l})} B^{(i_1\ldots i_l)} \right\Vert_{i_{l+1}=1,\ldots,i_{l+k}=1}^{m_{l+1}\ \ldots\ m_{l+k}} & \text{for} \quad k \ge 1 \\ \sum\limits_{i_1=1}^{m_1} \ldots \sum\limits_{i_l=1}^{m_l} A^{(i_1\ldots i_l)} B^{(i_1\ldots i_l)} & \text{for} \quad k = 0 \end{cases},
$$

$$
\left\Vert A_{k+1} D_k^{(i_k)} A_k \ldots A_2 D_1^{(i_1)} A_1 R(\mathbf{x}, t) \right\Vert_{i_1=1,\ldots,i_k=1}^{m_1\ \ldots\ m_k} = {}^{(k)}A_{k+1} D_k A_k \ldots A_2 D_1 A_1 R(\mathbf{x}, t), \tag{4.21}
$$

where A_p and $D_q^{(i_q)}$ are operators defined on the space L for $p = 1, \ldots, k+1$, $q = 1, \ldots, k$, and $i_q = 1, \ldots, m_q$. It is assumed that the left-hand side of (4.21) exists. The symbol $\overset{0}{\cdot}$ is treated as the usual multiplication. If $m_l = 0$ in (4.20) for some $l \in \{1, \ldots, k\}$, then the right-hand side of (4.20) is treated as

$$
\left\Vert A^{(i_1\ldots i_{l-1} i_{l+1}\ldots i_k)} \right\Vert_{i_1=1,\ldots,i_{l-1}=1, i_{l+1}=1,\ldots,i_k=1}^{m_1\ \ldots\ m_{l-1}\ \ m_{l+1}\ \ldots\ m_k},
$$

(shortly, ${}^{(k-1)}A$).

We also introduce the following notations

$$
\left\Vert Q_{\lambda_l}^{(i_l)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) \right\Vert_{i_1=\lambda_1,\ldots,i_l=\lambda_l}^{m\lambda_1\ \ldots\ m\lambda_l} \stackrel{\text{def}}{=} {}^{(p_l)}Q_{\lambda_l} \ldots Q_{\lambda_1} R(\mathbf{x}, t),
$$

$$
{}^{(p_k)}J_{(\lambda_k\ldots\lambda_1)s,t} = \left\Vert J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} \right\Vert_{i_1=\lambda_1,\ldots,i_k=\lambda_k}^{m\lambda_1\ \ldots\ m\lambda_k},
$$

$$
M_k = \left\{ (\lambda_k, \ldots, \lambda_1) : \lambda_l = 1 \text{ or } \lambda_l = 0;\ l = 1, \ldots, k \right\}, \quad k \ge 1,
$$

$$
J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} = \int\limits_t^s \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_k)} \ldots d\mathbf{w}_{t_k}^{(i_1)}, \quad k \ge 1,
$$

where $\lambda_l = 1$ or $\lambda_l = 0$, $Q_{\lambda_l}^{(i_l)} = L$ and $i_l = 0$ for $\lambda_l = 0$, $Q_{\lambda_l}^{(i_l)} = G_0^{(i_l)}$ and $i_l = 1, \ldots, m$ for $\lambda_l = 1$,

$$
p_l = \sum_{j=1}^{l} \lambda_j \quad \text{for} \quad l = 1, \ldots, r+1, \quad r \in \mathbf{N},
$$

$\mathbf{w}_\tau^{(i)}$ $(i=1,\ldots,m)$ are F_τ-measurable for all $\tau\in[0,T]$ independent standard Wiener processes and $\mathbf{w}_\tau^{(0)}=\tau.$

Applying (4.19) to the process $R(\mathbf{x}_s,s)$ repeatedly, we obtain the following Taylor–Itô expansion [159]

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}{}^{(p_k)}Q_{\lambda_k}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\overset{p_k}{\cdot}{}^{(p_k)}J_{(\lambda_k\ldots\lambda_1)s,t}+$$

$$+\left(D_{r+1}\right)_{s,t} \tag{4.22}$$

w. p. 1, where $s,t\in[0,T],\ s>t,$

$$\left(D_{r+1}\right)_{s,t}=$$

$$=\sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}}\int\limits_t^s\ldots\left(\int\limits_t^{t_2}{}^{(p_{r+1})}Q_{\lambda_{r+1}}\ldots Q_{\lambda_1}R(\mathbf{x}_{t_1},t_1)\overset{\lambda_{r+1}}{\cdot}d\mathbf{w}_{t_1}\right)\ldots\overset{\lambda_1}{\cdot}d\mathbf{w}_{t_{r+1}}. \tag{4.23}$$

It is assumed that the right-hand sides of (4.22), (4.23) exist.

A possible variant of the conditions, under which the right-hand sides of (4.22), (4.23) exist is as follows

(i) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\in\mathrm{L}$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r}M_g;$

(ii) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_\tau,\tau)\in\mathrm{M}_2([0,T])$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r+1}M_g.$

Let us rewrite the expansion (4.22) in the another form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}+$$

$$+\left(D_{r+1}\right)_{s,t}\quad \text{w. p. 1.}$$

Denote

$$G_{rk}=\left\{(\lambda_k,\ldots,\lambda_1):\ r+1\le 2k-\lambda_1-\ldots-\lambda_k\le 2r\right\},$$

$$E_{qk}=\left\{(\lambda_k,\ldots,\lambda_1):\ 2k-\lambda_1-\ldots-\lambda_k=q\right\},$$

where $\lambda_l=1$ or $\lambda_l=0$ $(l=1,\ldots,k).$

The Taylor–Itô expansion ordered according to the order of smallness (in the mean-square sense when $s \downarrow t$) of its terms has the form

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} + (H_{r+1})_{s,t} \quad (4.24)$$

w. p. 1, where

$$(H_{r+1})_{s,t} = \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} + (D_{r+1})_{s,t}. \quad (4.25)$$

4.4 The First Form of the Unified Taylor–Itô Expansion

In this section, we transform the right-hand side of (4.22) by Theorem 3.1 and Lemma 4.1 to a representation including the iterated Itô stochastic integrals (4.7).

Denote

$$I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} = \int_t^s (t-t_k)^{l_k} \ldots \int_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1 \quad (4.26)$$

and

$$I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} = 1 \quad \text{for} \quad k = 0,$$

where $i_1, \ldots, i_k = 1, \ldots, m$. Moreover, let

$${}^{(k)}I_{l_1\ldots l_k s,t} = \left\| I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} \right\|_{i_1,\ldots,i_k=1}^{m},$$

$$G_p^{(i)} \stackrel{\mathsf{def}}{=} \frac{1}{p}\left(G_{p-1}^{(i)} L - L G_{p-1}^{(i)}\right), \quad p = 1, 2, \ldots, \quad i = 1, \ldots, m, \quad (4.27)$$

where L and $G_0^{(i)}$, $i = 1, \ldots, m$ are determined by the equalities (4.17), (4.18).

Denote

$$A_q \stackrel{\mathsf{def}}{=} \left\{ (k, j, l_1, \ldots, l_k) : \; k + j + \sum_{p=1}^{k} l_p = q; \; k, j, l_1, \ldots, l_k = 0, 1, \ldots \right\},$$

$$\left\|G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}G_{l_1}\ldots G_{l_k}L^jR(\mathbf{x},t),$$

$$L^jR(\mathbf{x},t)\stackrel{\sf def}{=}\begin{cases}\underbrace{L\ldots L}_{j}R(\mathbf{x},t) & \text{for } j\ge 1\\ & \\ R(\mathbf{x},t) & \text{for } j=0\end{cases}.$$

Theorem 4.1. *Let conditions* (i), (ii) *be satisfied. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *and for any positive integer* r, *the following expansion takes place w. p. 1*

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}+\left(D_{r+1}\right)_{s,t}, \tag{4.28}$$

where $\left(D_{r+1}\right)_{s,t}$ *is defined by* (4.23).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}{}^{(p_q)}Q_{\lambda_q}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J_{(\lambda_q\ldots\lambda_1)s,t}=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} \tag{4.29}$$

w. p. 1. The equality (4.29) is valid for $q=1$. Assume that (4.29) is valid for some $q>1$. In this case, using the induction hypothesis, we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}}{}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_{q+1}}{\cdot}{}^{(p_{q+1})}J_{(\lambda_1\ldots\lambda_{q+1})s,t}=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^s\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}\left({}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J_{(\lambda_1\ldots\lambda_q)\theta,t}\right)\stackrel{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^s\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\frac{(\theta-t)^j}{j!}\times$$

$$\times\left({}^{(k+\lambda_{q+1})}G_{l_1}\ldots G_{l_k}L^jQ_{\lambda_{q+1}}R(\mathbf{x}_t,t)\overset{k}{\cdot}{}^{(k)}I_{l_1\ldots l_{k\,s,t}}\right)\overset{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\left({}^{(k)}G_{l_1}\ldots G_{l_k}L^{j+1}R(\mathbf{x}_t,t)\overset{k}{\cdot}\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta+\right.$$

$$\left.+\left({}^{(k+1)}G_{l_1}\ldots G_{l_k}L^jG_0R(\mathbf{x}_t,t)\overset{k}{\cdot}\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}\right)\overset{1}{\cdot}d\mathbf{f}_\theta\right)\qquad(4.30)$$

w. p. 1.

Using Lemma 4.1, we obtain

$$\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta=$$

$$=\frac{1}{(j+1)!}\begin{cases}(s-t)^{j+1} & \text{for } k=0\\ \\ (s-t)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k\,s,t}}-(-1)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0\end{cases}\qquad(4.31)$$

w. p. 1. In addition (see (4.26)) we get

$$\int\limits_t^s\frac{(\theta-t)^j}{j!}I^{(i_1\ldots i_k)}_{l_1\ldots l_{k\,\theta,t}}d\mathbf{f}^{(i_{k+1})}_\theta=\frac{(-1)^j}{j!}I^{(i_1\ldots i_k i_{k+1})}_{l_1\ldots l_k j_{s,t}}\qquad(4.32)$$

in the notations just introduced. Substitute (4.31) and (4.32) into the formula (4.30). Grouping summands in the obtained expression with equal lower indices at iterated Itô stochastic integrals and using (4.27) as well as the equality

$$G^{(i)}_pR(\mathbf{x},t)=\frac{1}{p!}\sum_{q=0}^p(-1)^qC^q_pL^qG^{(i)}_0L^{p-q}R(\mathbf{x},t),\quad C^q_p=\frac{p!}{q!(p-q)!}\qquad(4.33)$$

(this equality follows from (4.27)), we note that the obtained expression equals to

$$\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}}\frac{(s-t)^j}{j!}{}^{(k)}G_{l_1}\ldots G_{l_k}L^j\{\eta_t\}\overset{k}{\cdot}{}^{(k)}I_{l_1\ldots l_{k\,s,t}}$$

w. p. 1. Summing the equalities (4.29) for $q=1,2,\ldots,r$ and applying the formula (4.22), we obtain the expression (4.28). The proof is completed.

Let us order terms of the expansion (4.28) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t)+$$

$$+\sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} + (H_{r+1})_{s,t} \tag{4.34}$$

w. p. 1, where

$$(H_{r+1})_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}+$$

$$+ (D_{r+1})_{s,t} \,,$$

$$\mathrm{D}_q = \left\{(k,j,l_1,\ldots,l_k) : k + 2\left(j + \sum_{p=1}^{k} l_p\right) = q;\; k,j,l_1,\ldots,l_k = 0,1,\ldots\right\}, \tag{4.35}$$

$$\mathrm{U}_r = \left\{(k,j,l_1,\ldots,l_k) : k + j + \sum_{p=1}^{k} l_p \le r,\right.$$

$$\left. k + 2\left(j + \sum_{p=1}^{k} l_p\right) \ge r+1;\; k,j,l_1,\ldots,l_k = 0,1,\ldots\right\}, \tag{4.36}$$

and $(D_{r+1})_{s,t}$ is defined by (4.23). Note that the remainder term $(H_{r+1})_{s,t}$ in (4.34) has a higher order of smallness in the mean-square sense as $s \downarrow t$ than the terms of the main part of the expansion (4.34).

4.5 The Second Form of the Unified Taylor–Itô Expansion

Consider iterated Itô stochastic integrals of the form

$$J_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} = \int\limits_t^s (s-t_k)^{l_k} \ldots \int\limits_t^{t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1$$

and

$$J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=1\quad \text{for}\quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m.$

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=\sum_{j_1=0}^{l_1}\ldots\sum_{j_k=0}^{l_k}\prod_{g=1}^{k}C_{l_g}^{j_g}(t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\ J^{(i_1\ldots i_k)}_{j_1\ldots j_k s,t}\quad \text{w. p. 1},\tag{4.37}$$

where

$$C_l^k=\frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Itô expansion of the process $\eta_s=R(\mathbf{x}_s,s),\ s\in[0,T]$ can be constructed either using the iterated stochastic integrals $I^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}$ similarly to the previous section or using the iterated stochastic integrals $J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}$. This is the main subject of this section.

Denote

$$\left\|J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}J_{l_1\ldots l_k s,t},$$

$$\left\|L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}L^jG_{l_1}\ldots G_{l_k}R(\mathbf{x},t).$$

Theorem 4.2. *Let conditions* (i), (ii) *be satisfied. Then for any $s,t\in[0,T]$ such that $s>t$ and for any positive integer r, the following expansion is valid w. p. 1*

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x}_t,t)J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}+\left(D_{r+1}\right)_{s,t},\tag{4.38}$$

where $\left(D_{r+1}\right)_{s,t}$ is defined by (4.23).

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x}_t,t)J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} \quad \text{w. p. 1} \qquad (4.39)$$

for $q=1,2,\ldots,r$. To check (4.39), substitute the expression (4.37) into the right-hand side of (4.39) and then use the formulas (4.27), (4.33).

Let us order terms of the expansion (4.38) according to their smallness orders as $s\downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} + (H_{r+1})_{s,t}$$

w. p. 1, where

$$(H_{r+1})_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)}+$$

$$+\left(D_{r+1}\right)_{s,t}.$$

The remainder term $\left(D_{r+1}\right)_{s,t}$ is defined by (4.23); the sets D_q and U_r are defined by (4.35) and (4.36), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Itô expansion (4.22) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s,s)$ as $r\to\infty$ for all $s,t\in[0,T]$ such that $s>t$ and $T<\infty$ has been proved in [84] (Proposition 5.9.2). Since the expansions (4.28) and (4.38) are obtained from the Taylor–Itô expansion (4.22) without any additional conditions, the truncated expansions (4.28) and (4.38) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of Proposition 5.9.2 [84] converge to the process $R(\mathbf{x}_s,s)$ w. p. 1 as $r\to\infty$ for all $s,t\in[0,T]$ such that $s>t$ and $T<\infty$.

4.6 The Taylor–Stratonovich Expansion

In this section, we use the Taylor–Stratonovich expansion [159] and introduce some necessary notations. At that we will use the original notations introduced by the author of this book.

Let us consider two classic results.

Proposition 4.1 [100]. *Suppose that the following conditions are satisfied.*

AI. *The functions* $\mathbf{a}(\mathbf{x},t),\ B_j(\mathbf{x},t):\ \mathbf{R}^n\times[0,T]\to\mathbf{R}^n$ $(j=1,\ldots,m)$ *are measurable for all* $(\mathbf{x},t)\in\mathbf{R}^n\times[0,T]$, *where* $B_j(\mathbf{x},t)$ *is the* j*th column of the matrix* $B(\mathbf{x},t)$ (*see* (4.1)).

AII. *For all* $\mathbf{x},\ \mathbf{y}\in\mathbf{R}^n$ *there exists a constant* $K<\infty$ *such that*

$$\left|\mathbf{a}(\mathbf{x},t)-\mathbf{a}(\mathbf{y},t)\right|+\sum_{j=1}^m\left|B_j(\mathbf{x},t)-B_j(\mathbf{y},t)\right|\le K\left|\mathbf{x}-\mathbf{y}\right|,$$

$$\left|\mathbf{a}(\mathbf{x},t)\right|^2+\sum_{j=1}^m\left|B_j(\mathbf{x},t)\right|^2\le K^2\left(1+\left|\mathbf{x}\right|^2\right),$$

where $|\cdot|$ *is the Euclidean norm of the vector.*

AIII. *A random variable* $\mathbf{x}_0$ *is* F_0*-measurable and* $\mathsf{M}\left\{\left|\mathbf{x}_0\right|^2\right\}<\infty$.

Then there exists a unique (up to stochastic equivalence) and continuous w. p. 1 *strong solution of the Itô SDE* (4.1).

Proposition 4.2 [84]. *Suppose that the conditions* AI–AIII (*see Proposition* 4.1) *are satisfied and* $\mathsf{M}\left\{\left|\mathbf{x}_{t_0}\right|^{2n}\right\}<\infty$ $(n\ge 1)$. *Then*

$$\mathsf{M}\left\{\left|\mathbf{x}_t\right|^{2n}\right\}\le\left(1+\mathsf{M}\left\{\left|\mathbf{x}_{t_0}\right|^{2n}\right\}\right)e^{C(t-t_0)},$$

$$\mathsf{M}\left\{\left|\mathbf{x}_t-\mathbf{x}_{t_0}\right|^{2n}\right\}\le C_1\left(1+\mathsf{M}\left\{\left|\mathbf{x}_{t_0}\right|^{2n}\right\}\right)(t-t_0)^n e^{C(t-t_0)},$$

where $\mathbf{x}_t$ *is the solution of the Itô SDE* (4.1), $t\in[t_0,T]$, $T<\infty$, *constant* C_1 $(C_1\in(0,\infty))$ *depends only on* $n,K,T-t_0$, $C=2n(2n+1)K^2$, $K<\infty$ *is a constant.*

Assume that $R(\mathbf{x},t)\in\mathrm{L}$, $LR(\mathbf{x}_\tau,\tau)$, $G_0^{(i)}R(\mathbf{x}_\tau,\tau)\in\mathrm{M}_2([0,T])$ for $i=1,\ldots,m$ and consider the Itô formula (4.19).

In addition, suppose that the function $G_0^{(i)}R(\mathbf{x},t)$ $(i=1,\ldots,m)$ is such that the formulas (4.11) and (4.12) can be applied. For example, assume that

1. $G_0^{(i)}R(\mathbf{x},t)\in\mathrm{L}$, $i=1,\ldots,m$.

2. For all $\mathbf{x},\mathbf{y}\in\mathbf{R}^n$, $t,s\in[0,T]$, $i_1,i_2=1,\ldots,m$ and for some $\nu>0$

$$\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x},t)-G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{y},t)\right|\le K_1\left|\mathbf{x}-\mathbf{y}\right|,$$

$$\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x},t)\right|+\left|LG_0^{(i_1)}R(\mathbf{x},t)\right|\le K_1\left(1+\left|\mathbf{x}\right|\right),$$

$$\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x},s)-G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x},t)\right|\le K_1\left|s-t\right|^{\nu}\left(1+\left|\mathbf{x}\right|\right),$$

where $K_1 < \infty$ is a constant.

3. Conditions AI, AII are fulfilled (see Proposition 4.1).

4. $\mathsf{M}\{|\mathbf{x}_0|^8\} < \infty$.

Indeed, using the above conditions, Proposition 4.2 and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$
\mathsf{M}\left\{\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_s,s) - G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_t,t)\right|^4\right\} \le
$$

$$
\le 8\mathsf{M}\left\{\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_s,s) - G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_t,s)\right|^4\right\} +
$$

$$
+8\mathsf{M}\left\{\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_t,s) - G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_t,t)\right|^4\right\} \le
$$

$$
\le 8K_1^4\mathsf{M}\left\{|\mathbf{x}_s - \mathbf{x}_t|^4\right\} + 8K_1^4\left|s-t\right|^{4\nu}\mathsf{M}\left\{\left(1+|\mathbf{x}_t|\right)^4\right\} \le
$$

$$
\le C_2\left|s-t\right|^2 + C_3\left|s-t\right|^{4\nu} \to 0 \quad \text{if} \quad s-t\to 0, \tag{4.40}
$$

$$
\mathsf{M}\left\{\left|G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_s,s)\right|^8\right\} \le K_1^8\mathsf{M}\left\{\left(1+|\mathbf{x}_s|\right)^8\right\} \le
$$

$$
\le C_4\left(1+\mathsf{M}\left\{|\mathbf{x}_s|^8\right\}\right) \le C_5\left(1+\left(1+\mathsf{M}\left\{|\mathbf{x}_0|^8\right\}\right)e^{Cs}\right) < \infty, \tag{4.41}
$$

where $C_2,\ldots,C_5 < \infty$ are constants, $t,s\in[0,T]$.

Analogously, we get

$$
\mathsf{M}\left\{\left|LG_0^{(i_1)}R(\mathbf{x}_s,s)\right|^8\right\} < \infty, \quad s\in[0,T]. \tag{4.42}
$$

Applying the Itô formula, we obtain w. p. 1

$$
R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t) + \int\limits_t^s LR(\mathbf{x}_\tau,\tau)d\tau + \sum_{i_1=1}^m \int\limits_t^s G_0^{(i_1)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i_1)}, \tag{4.43}
$$

$$
G_0^{(i_1)}R(\mathbf{x}_s,s) = G_0^{(i_1)}R(\mathbf{x}_t,t) + \int\limits_t^s LG_0^{(i_1)}R(\mathbf{x}_\tau,\tau)d\tau +
$$

$$
+\sum_{i_2=1}^m \int\limits_t^s G_0^{(i_2)}G_0^{(i_1)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i_2)}, \tag{4.44}
$$

where $i_1, i_2 = 1, \ldots, m$.

Thus, using (4.40)–(4.42), (4.11) and (4.12), we have

$$\int\limits_t^s G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} = \int\limits_t^{*s} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} - \frac{1}{2} \int\limits_t^s G_0^{(i)} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\tau \quad (4.45)$$

w. p. 1, where $s, t \in [0, T]$, $s > t$, $i = 1, \ldots, m$.

Using the relation (4.45), let us write (4.43) in the following form

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \int\limits_t^s \bar{L} R(\mathbf{x}_\tau, \tau) d\tau + \sum_{i=1}^m \int\limits_t^{*s} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} \quad \text{w. p. 1}, \quad (4.46)$$

where

$$\bar{L} R(\mathbf{x}, t) = L R(\mathbf{x}, t) - \frac{1}{2} \sum_{i=1}^m G_0^{(i)} G_0^{(i)} R(\mathbf{x}, t). \quad (4.47)$$

Introduce the following notations

$$\left\| D_{\lambda_l}^{(i_l)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) \right\|_{i_1=\lambda_1, \ldots, i_l=\lambda_l}^{m\lambda_1 \ \ldots \ m\lambda_l} \stackrel{\sf def}{=} {}^{(p_l)} D_{\lambda_l} \ldots D_{\lambda_1} R(\mathbf{x}, t),$$

$${}^{(p_k)} J^*_{(\lambda_k \ldots \lambda_1) s,t} = \left\| J^{*(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1) s,t} \right\|_{i_1=\lambda_1, \ldots, i_k=\lambda_k}^{m\lambda_1 \ \ldots \ m\lambda_k},$$

$$M_k = \left\{ (\lambda_k, \ldots, \lambda_1) : \lambda_l = 1 \text{ or } \lambda_l = 0; \ l = 1, \ldots, k \right\}, \quad k \ge 1,$$

$$J^{*(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1) s,t} = \int\limits_t^{*s} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_k)} \ldots d\mathbf{w}_{t_k}^{(i_1)}, \quad k \ge 1,$$

where $\lambda_l = 1$ or $\lambda_l = 0$, $D_{\lambda_l}^{(i_l)} = \bar{L}$ and $i_l = 0$ for $\lambda_l = 0$, $D_{\lambda_l}^{(i_l)} = G_0^{(i_l)}$ and $i_l = 1, \ldots, m$ for $\lambda_l = 1$,

$$p_l = \sum_{j=1}^l \lambda_j \quad \text{for} \quad l = 1, \ldots, r+1, \quad r \in \mathbf{N},$$

$\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are F_τ-measurable for all $\tau \in [0, T]$ independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$.

Applying the formula (4.46) to the process $R(\mathbf{x}_s, s)$ repeatedly, we obtain the following Taylor–Stratonovich expansion [159]

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} {}^{(p_k)}D_{\lambda_k}\ldots D_{\lambda_1}R(\mathbf{x}_t, t) \stackrel{p_k}{\cdot} {}^{(p_k)}J^*_{(\lambda_k\ldots\lambda_1)s,t} + \\ + \left(D_{r+1}\right)_{s,t} \qquad (4.48)$$

w. p. 1, where $s, t \in [0, T]$, $s > t$,

$$\left(D_{r+1}\right)_{s,t} =$$

$$= \sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}} \int\limits_t^{*s} \ldots \left(\int\limits_t^{*t_2} {}^{(p_{r+1})}D_{\lambda_{r+1}}\ldots D_{\lambda_1}R(\mathbf{x}_{t_1}, t_1) \stackrel{\lambda_{r+1}}{\cdot} d\mathbf{w}_{t_1}\right)\ldots \stackrel{\lambda_1}{\cdot} d\mathbf{w}_{t_{r+1}}. \qquad (4.49)$$

Let us rewrite the expansion (4.48) in another form w. p. 1

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D^{(i_k)}_{\lambda_k}\ldots D^{(i_1)}_{\lambda_1}R(\mathbf{x}_t, t) J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t} + \left(D_{r+1}\right)_{s,t}. \qquad (4.50)$$

Denote

$$G_{rk} = \Bigl\{(\lambda_k, \ldots, \lambda_1) :\ r + 1 \le 2k - \lambda_1 - \ldots - \lambda_k \le 2r\Bigr\},$$

$$E_{qk} = \Bigl\{(\lambda_k, \ldots, \lambda_1) :\ 2k - \lambda_1 - \ldots - \lambda_k = q\Bigr\},$$

where $\lambda_l = 1$ or $\lambda_l = 0$ $(l = 1, \ldots, k)$.

Let us order terms of the Taylor–Stratonovich expansion (4.48) or (4.50) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D^{(i_k)}_{\lambda_k}\ldots D^{(i_1)}_{\lambda_1}R(\mathbf{x}_t, t) J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t} + \left(H_{r+1}\right)_{s,t} \qquad (4.51)$$

w. p. 1, where

$$\left(H_{r+1}\right)_{s,t} = \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D^{(i_k)}_{\lambda_k}\ldots D^{(i_1)}_{\lambda_1}R(\mathbf{x}_t, t) J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t} +$$

$$+\left(D_{r+1}\right)_{s,t}. \tag{4.52}$$

The following two questions seem interesting.

1. Under what conditions do the right-hand sides of the formulas (4.51) and (4.52) exist for $r \geq 2$?

2. Is it possible to obtain another representation of the remainder term (4.52) for $r \geq 2$?

Below we will provide compelling arquments in favor of the following two facts.

(A). First, one can construct the Taylor–Stratonovich expansion (4.51) ($r \geq 2$) in such a way that its remainder term will coincide w. p. 1 with the remainder term (4.25) ($r \geq 2$) of the Taylor–Itô expansion (4.24) ($r \geq 2$).

(B). Second, the truncated Taylor–Stratonovich expansion (4.51) ($r \geq 2$) (without the remainder term (4.52) ($r \geq 2$)) will coincide w. p. 1 with the truncated Taylor–Itô expansion (4.24) ($r \geq 2$) (without the remainder term (4.25) ($r \geq 2$)).

This means that the right-hand side of (4.51) ($r \geq 2$) (in which the remainder term will have the form (4.25) ($r \geq 2$)) will exist under the conditions (i), (ii) (see Sect. 4.3).

Let us begin our reasoning with Theorem 2.12 (see Sect. 2.4.1). This theorem allows us to represent the iterated Stratonovich stochastic integral of multiplicity k ($k \in \mathbf{N}$) as a sum of iterated Itô stochastic integrals and its mathematical expectation. It is obvious that it is possible to obtain an inverse formula that will express the iterated Itô stochastic integral (2.374) as a sum of iterated Stratonovich stochastic integrals (2.373). Below we present the corresponding proposition.

Proposition 4.3. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous function at the interval $[t, T]$. Then, the following relation between iterated Itô and Stratonovich stochastic integrals*

$$J[\psi^{(k)}]_{T,t} = J^*[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{(-1)^r}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J^*[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1} \tag{4.53}$$

is correct, where $\sum\limits_{\emptyset}$ is supposed to be equal to zero, $J[\psi^{(k)}]_{T,t}$ and $J^*[\psi^{(k)}]_{T,t}$ are defined by (2.374) *and* (2.373)*, respectively,*

$$J^*[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{l} {\bf 1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times {\int\limits_t^{*}}^{T} \psi_k(t_k)\ldots {\int\limits_t^{*}}^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times {\int\limits_t^{*}}^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots {\int\limits_t^{*}}^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times {\int\limits_t^{*}}^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots {\int\limits_t^{*}}^{t_2} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d{\bf w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d{\bf w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d{\bf w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d{\bf w}_{t_k}^{(i_k)},$$

where

$${\rm A}_{k,l}=\bigl\{(s_l,\ldots,s_1):\ \ s_l>s_{l-1}+1,\ldots,s_2>s_1+1,\ \ s_l,\ldots,s_1=1,\ldots,k-1\bigr\},$$

$$(s_l,\ldots,s_1)\in{\rm A}_{k,l},\quad l=1,\ldots,[k/2]\,,\quad i_s=0,1,\ldots,m,\quad s=1,\ldots,k,$$

$[x]$ is an integer part of a real number x, ${\bf 1}_A$ is the indicator of the set A.

For example, from Proposition 4.3 for $k=1,2,3,4$ we obtain the following equalities w. p. 1

$$\int\limits_t^T \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)} = {\int\limits_t^{*}}^{T} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)},$$

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}d{\bf w}_{t_2}^{(i_2)} = {\int\limits_t^{*}}^{T} \psi_2(t_2){\int\limits_t^{*}}^{t_2} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}d{\bf w}_{t_2}^{(i_2)}-$$

$$-\frac{1}{2}{\bf 1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_2(t_2)\psi_1(t_2)dt_2,$$

$$\int\limits_t^T \psi_3(t_3)\ldots\int\limits_t^{t_2} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_3}^{(i_3)} = {\int\limits_t^{*}}^{T} \psi_3(t_3)\ldots{\int\limits_t^{*}}^{t_2} \psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_3}^{(i_3)}-$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{T}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{*t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3,
$$

$$
\int\limits_t^{T}\psi_4(t_4)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}=\int\limits_t^{*T}\psi_4(t_4)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{*t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3d\mathbf{w}_{t_4}^{(i_4)}-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{T}\psi_4(t_4)\psi_3(t_4)\int\limits_t^{*t_4}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_4+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{T}\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4
$$

Further, using Proposition 4.3, we obtain for $r\ge 2$

$$
R(\mathbf{x}_t,t)+\sum_{q,k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in\mathrm{E}_{qk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}=
$$

$$
=R(\mathbf{x}_t,t)+\sum_{q,k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in\mathrm{E}_{qk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_k}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)}\qquad(4.54)
$$

w. p. 1, where notations are the same as in (4.24), (4.51). Thus, (A) and (B) take place.

4.7 The First Form of the Unified Taylor–Stratonovich Expansion

In this section, we transform the right-hand side of (4.48) by Theorem 3.1 and Lemma 4.2 to a representation including the iterated Stratonovich stochastic integrals (4.8). Moreover, we will use the remainder term $(D_{r+1})_{s,t}$ of the form (4.49).

Denote

$$I^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} = \int\limits_t^{*s} (t-t_k)^{l_k}\ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k} \quad \text{for} \quad k\ge 1 \tag{4.55}$$

and

$$I^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} = 1 \quad \text{for} \quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m.$

Futhermore, let

$$^{(k)}I^{*}_{l_1\ldots l_k s,t} = \left\| I^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t}\right\|_{i_1,\ldots,i_k=1}^{m},$$

$$\bar{G}^{(i)}_p \stackrel{\text{def}}{=} \frac{1}{p}\left(\bar{G}^{(i)}_{p-1}\bar{L}-\bar{L}\bar{G}^{(i)}_{p-1}\right), \quad p=1,2,\ldots, \quad i=1,\ldots,m, \tag{4.56}$$

where $\bar{G}^{(i)}_0 \stackrel{\text{def}}{=} G^{(i)}_0$, $i=1,\ldots,m$. The operators $\bar{L}$ and $G^{(i)}_0$, $i=1,\ldots,m$ are determined by the equalities (4.17), (4.18), and (4.47).

Denote

$$A_q \stackrel{\text{def}}{=} \left\{(k,j,l_1,\ldots,l_k):\ k+j+\sum_{p=1}^{k} l_p=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\},$$

$$\left\|\bar{G}^{(i_1)}_{l_1}\ldots\bar{G}^{(i_k)}_{l_k}\bar{L}^j R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\text{def}}{=} {}^{(k)}\bar{G}_{l_1}\ldots\bar{G}_{l_k}\bar{L}^j R(\mathbf{x},t),$$

$$\bar{L}^j R(\mathbf{x},t) \stackrel{\text{def}}{=} \begin{cases} \underbrace{\bar{L}\ldots\bar{L}}_{j} R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases}.$$

Theorem 4.3 [163] (also see [1]-[17], [52], [165]). *Suppose that sufficient conditions are satisfied under which the right-hand sides of* (4.48), (4.49) *exist. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *and for any positive integer* r, *the following expansion takes place w. p. 1*

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar G_{l_1}^{(i_1)}\ldots\bar G_{l_k}^{(i_k)}\bar L^j R(\mathbf{x}_t,t)I_{l_1\ldots l_{k}s,t}^{*(i_1\ldots i_k)}+\left(D_{r+1}\right)_{s,t}, \tag{4.57}$$

where $(D_{r+1})_{s,t}$ *is defined by* (4.49).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}{}^{(p_q)}D_{\lambda_q}\ldots D_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J^*_{(\lambda_q\ldots\lambda_1)s,t}=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar G_{l_1}^{(i_1)}\ldots\bar G_{l_k}^{(i_k)}\bar L^j R(\mathbf{x}_t,t)I_{l_1\ldots l_{k}s,t}^{*(i_1\ldots i_k)} \tag{4.58}$$

w. p. 1. The equality (4.58) is valid for $q=1$. Assume that (4.58) is valid for some $q>1$. In this case using the induction hypothesis we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}}{}^{(p_{q+1})}D_{\lambda_1}\ldots D_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_{q+1}}{\cdot}{}^{(p_{q+1})}J^*_{(\lambda_1\ldots\lambda_{q+1})s,t}=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^{*s}\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}\left({}^{(p_{q+1})}D_{\lambda_1}\ldots D_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J^*_{(\lambda_1\ldots\lambda_q)\theta,t}\right)\stackrel{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^{*s}\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\frac{(\theta-t)^j}{j!}\times$$

$$\times\left({}^{(k+\lambda_{q+1})}\bar G_{l_1}\ldots\bar G_{l_k}\bar L^j D_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{k}{\cdot}{}^{(k)}I^*_{l_1\ldots l_{k s,t}}\right)\stackrel{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\Biggl({}^{(k)}\bar G_{l_1}\ldots\bar G_{l_k}\bar L^{j+1}R(\mathbf{x}_t,t)\stackrel{k}{\cdot}\int\limits_t^{s}\frac{(\theta-t)^j}{j!}{}^{(k)}I^*_{l_1\ldots l_{k\theta,t}}d\theta+$$

$$+\left({}^{(k+1)}\bar G_{l_1}\ldots\bar G_{l_k}\bar L^j\bar G_0R(\mathbf{x}_t,t)\stackrel{k}{\cdot}\int\limits_t^{*s}\frac{(\theta-t)^j}{j!}{}^{(k)}I^*_{l_1\ldots l_{k\theta,t}}\right)\stackrel{1}{\cdot}d\mathbf{f}_\theta\Biggr) \tag{4.59}$$

w. p. 1.

Using Lemma 4.2, we obtain

$$\int\limits_t^s \frac{(\theta-t)^j}{j!}{}^{(k)}I^*_{l_1\ldots l_{k_{\theta,t}}}d\theta=$$

$$=\frac{1}{(j+1)!}\begin{cases}(s-t)^{j+1} & \text{for } k=0\\ (s-t)^{j+1}\cdot{}^{(k)}I^*_{l_1\ldots l_{k_{s,t}}}-(-1)^{j+1}\cdot{}^{(k)}I^*_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0\end{cases} \tag{4.60}$$

w. p. 1. In addition (see (4.55)) we get

$$\int\limits_t^{*s} \frac{(\theta-t)^j}{j!}I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{\theta,t}}}d\mathbf{f}_\theta^{(i_{k+1})}=\frac{(-1)^j}{j!}I^{*(i_1\ldots i_k i_{k+1})}_{l_1\ldots l_k j_{s,t}} \tag{4.61}$$

in the notations just introduced. Substitute (4.60) and (4.61) into the formula (4.59). Grouping summands in the obtained expression with equal lower indices at iterated Stratonovich stochastic integrals and using (4.56) as well as the equality

$$\bar{G}_p^{(i)}R(\mathbf{x},t)=\frac{1}{p!}\sum_{q=0}^p(-1)^qC_p^q\bar{L}^q\bar{G}_0^{(i)}\bar{L}^{p-q}R(\mathbf{x},t),\quad C_p^q=\frac{p!}{q!(p-q)!} \tag{4.62}$$

(this equality follows from (4.56)), we note that the obtained expression equals to

$$\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}}\frac{(s-t)^j}{j!}{}^{(k)}\bar{G}_{l_1}\ldots\bar{G}_{l_k}\bar{L}^j\{\eta_t\}\stackrel{k}{\cdot}{}^{(k)}I^*_{l_1\ldots l_{k_{s,t}}}$$

w. p. 1. Summing the equalities (4.58) for $q=1,2,\ldots,r$ and applying the formula (4.48), we obtain the expression (4.57). The proof is completed.

Let us order terms of the expansion (4.57) according to their smallness orders as $s\downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^r\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^m\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^jR(\mathbf{x}_t,t)I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{s,t}}}+(H_{r+1})_{s,t} \tag{4.63}$$

w. p. 1, where

$$(H_{r+1})_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} + \\ + (D_{r+1})_{s,t}\ ,$$

$$\mathrm{D}_q = \left\{ (k,j,l_1,\ldots,l_k) : k + 2\left(j+\sum_{p=1}^{k} l_p\right) = q;\ \ k,j,l_1,\ldots,l_k = 0,1,\ldots \right\}, \tag{4.64}$$

$$\mathrm{U}_r = \left\{ (k,j,l_1,\ldots,l_k) : k+j+\sum_{p=1}^{k} l_p \le r, \right.$$

$$\left. k + 2\left(j+\sum_{p=1}^{k} l_p\right) \ge r+1;\ \ k,j,l_1,\ldots,l_k = 0,1,\ldots \right\}, \tag{4.65}$$

and $(D_{r+1})_{s,t}$ is defined by (4.49). Note that the remainder term $(H_{r+1})_{s,t}$ in (4.63) has a higher order of smallness in the mean-square sense as $s \downarrow t$ than the terms of the main part of the expansion (4.63).

4.8 The Second Form of the Unified Taylor–Stratonovich Expansion

Consider iterated Stratonovich stochastic integrals of the form

$$J_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = {\int\limits_t^{*}}^{s} (s-t_k)^{l_k} \ldots {\int\limits_t^{*}}^{t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1$$

and

$$J_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = 1 \quad \text{for} \quad k = 0,$$

where $i_1,\ldots,i_k = 1,\ldots,m$.

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = \sum_{j_1=0}^{l_1} \ldots \sum_{j_k=0}^{l_k} \prod_{g=1}^{k} C_{l_g}^{j_g} (t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\ J_{j_1\ldots j_{k_{s,t}}}^{*(i_1\ldots i_k)} \quad \text{w. p. 1}, \tag{4.66}$$

where

$$C_l^k = \frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Stratonovich expansion of the process $\eta_s = R(\mathbf{x}_s, s)$, $s \in [0, T]$ can be constructed either using the iterated stochastic integrals $I^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t}$ similarly to the previous section or using the iterated stochastic integrals $J^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t}$. This is the main subject of this section.

Denote

$$\left\| J^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t} \right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\sf def}{=} {}^{(k)} J^{*}_{l_1 \ldots l_k s,t},$$

$$\left\| \bar{L}^j \bar{G}^{(i_1)}_{l_1} \ldots \bar{G}^{(i_k)}_{l_k} R(\mathbf{x}, t) \right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\sf def}{=} {}^{(k)} \bar{L}^j \bar{G}_{l_1} \ldots \bar{G}_{l_k} R(\mathbf{x}, t).$$

Theorem 4.4 [163] (also see [1]-[17], [52], [165]). *Suppose that sufficient conditions are satisfied under which the right-hand sides of* (4.48), (4.49) *exist. Then for any $s, t \in [0, T]$ such that $s > t$ and for any positive integer r, the following expansion is valid w. p. 1*

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}^{(i_1)}_{l_1} \ldots \bar{G}^{(i_k)}_{l_k} R(\mathbf{x}_t, t) J^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t} + \left(D_{r+1}\right)_{s,t}, \tag{4.67}$$

where $\left(D_{r+1}\right)_{s,t}$ is defined by (4.49).

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k) \in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}^{(i_1)}_{l_1} \ldots \bar{G}^{(i_k)}_{l_k} R(\mathbf{x}_t, t) J^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t} =$$

$$= \sum_{(k,j,l_1,\ldots,l_k) \in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}^{(i_1)}_{l_1} \ldots \bar{G}^{(i_k)}_{l_k} \bar{L}^j R(\mathbf{x}_t, t) I^{*(i_1 \ldots i_k)}_{l_1 \ldots l_k s,t} \quad \text{w. p. 1} \tag{4.68}$$

for $q = 1, 2, \ldots, r$. To check (4.68), substitute the expression (4.66) into the right-hand side of (4.68) and then use the formulas (4.56), (4.62).

Let us order terms of the expansion (4.67) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+\sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_{k s,t}} + \left(H_{r+1}\right)_{s,t}$$

w. p. 1, where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_{k s,t}} +$$

$$+ \left(D_{r+1}\right)_{s,t}.$$

The remainder term $\left(D_{r+1}\right)_{s,t}$ is defined by (4.49); the sets D_q and U_r are defined by (4.64) and (4.65), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Stratonovich expansion (4.48) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s, s)$ as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$ has been proved in [84] (Proposition 5.10.2). Since the expansions (4.57) and (4.67) are obtained from the Taylor–Stratonovich expansion (4.48) without any additional conditions, the truncated expansions (4.57) and (4.67) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of Proposition 5.10.2 [84] converge to the process $R(\mathbf{x}_s, s)$ w. p. 1 as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$.

4.9 A Remark on Theorems 4.3 and 4.4

Note that when proving Theorems 4.3 and 4.4 we established the following equalities w. p. 1

$$R(\mathbf{x}_t, t) + \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t} =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k s,t}}, \tag{4.69}$$

$$R(\mathbf{x}_t, t) + \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t} =$$

$$
= R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)}. \tag{4.70}
$$

It is easy to see that by analogy with (4.69) and (4.70) the following equalities can be obtained w. p. 1

$$
R(\mathbf{x}_t,t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t,t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} =
$$

$$
= R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)}, \tag{4.71}
$$

$$
R(\mathbf{x}_t,t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t,t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} =
$$

$$
= R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)}. \tag{4.72}
$$

Recall the equality (4.54)

$$
R(\mathbf{x}_t,t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t,t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} =
$$

$$
= R(\mathbf{x}_t,t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t,t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} \tag{4.73}
$$

w. p. 1, where $r \ge 2$.

Combining (4.71)–(4.73), we obtain

$$
R(\mathbf{x}_t,t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t,t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} =
$$

$$
= R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)} =
$$

$$
= R(\mathbf{x}_t,t) + \sum_{q=1}^{r}\ \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)}\ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} \tag{4.74}
$$

w. p. 1, where $r\ge 2$.

The equality (4.74) means that we have the following theorem (see (4.24), (4.25)).

Theorem 4.5. *Let conditions* (i), (ii) (*see Sect.* 4.3) *be satisfied. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *the following unified Taylor–Stratonovich expansions take place w. p.* 1

$$
R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t)+
$$

$$
+\sum_{q=1}^{r}\ \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)}\ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t) I^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} + \left(H_{r+1}\right)_{s,t},
$$

$$
R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t)+
$$

$$
+\sum_{q=1}^{r}\ \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)}\ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} + \left(H_{r+1}\right)_{s,t},
$$

where $r\ge 2$, *the reainder term* $\left(H_{r+1}\right)_{s,t}$ *is definded by the relations* (4.25) *and* (4.23); *another notations are the same as in Sect.* 4.3, 4.6–4.8.

4.10 Comparison of the Unified Taylor–Itô and Taylor–Stratonovich Expansions with the Classical Taylor–Itô and Taylor–Stratonovich Expansions

Note that the truncated unified Taylor–Itô and Taylor–Stratonovich expansions contain the less number of various iterated Itô and Stratonovich stochastic integrals (moreover, their major part will have less multiplicity) in comparison with the classical Taylor–Itô and Taylor–Stratonovich expansions [159].

It is easy to notice that the stochastic integrals from the sets (4.4), (4.5) are connected by linear relations. However, the stochastic integrals from the sets (4.6), (4.7) cannot be connected by linear relations. This also holds for the stochastic integrals from the sets (4.8), (4.9). Therefore, we will call the sets (4.6)–(4.9) as the *stochastic bases.*

Let us call the numbers $\text{rank}_{\text{A}}(r)$ and $\text{rank}_{\text{D}}(r)$ of various iterated Itô and Stratonovich stochastic integrals, which are included in the sets (4.6)–(4.9) as the *ranks of stochastic bases* when summation in the stochastic expansions is performed using the sets A_q $(q = 1, \ldots, r)$ and D_q $(q = 1, \ldots, r)$ correspondingly. Here r is a fixed natural number.

At the beginning, let us analyze several examples related to the Taylor–Itô expansions (obviously, the same conclusions will hold for the Taylor–Stratonovich expansions).

Assume that the summation in the unified Taylor–Itô expansions is performed using the sets D_q $(q = 1, \ldots, r)$. It is easy to see that the truncated unified Taylor–Itô expansion (4.34), where the summation is performed using the sets D_q when $r = 3$ includes 4 $(\text{rank}_{\text{D}}(3) = 4)$ various iterated Itô stochastic integrals

$$I_{0_{s,t}}^{(i_1)}, \quad I_{00_{s,t}}^{(i_1 i_2)}, \quad I_{1_{s,t}}^{(i_1)}, \quad I_{000_{s,t}}^{(i_1 i_2 i_3)}.$$

The same truncated classical Taylor–Itô expansion (4.24) [84] contains 5 various iterated Itô stochastic integrals

$$J_{(1)s,t}^{(i_1)}, \quad J_{(11)s,t}^{(i_1 i_2)}, \quad J_{(10)s,t}^{(i_1 0)}, \quad J_{(01)s,t}^{(0 i_1)}, \quad J_{(111)s,t}^{(i_1 i_2 i_3)}.$$

For $r = 4$ we have 7 $(\text{rank}_{\text{D}}(4) = 7)$ stochastic integrals

$$I_{0_{s,t}}^{(i_1)}, \quad I_{00_{s,t}}^{(i_1 i_2)}, \quad I_{1_{s,t}}^{(i_1)}, \quad I_{000_{s,t}}^{(i_1 i_2 i_3)}, \quad I_{01_{s,t}}^{(i_1 i_2)}, \quad I_{10_{s,t}}^{(i_1 i_2)}, \quad I_{0000_{s,t}}^{(i_1 i_2 i_3 i_4)}$$

against 9 stochastic integrals

$$J_{(1)s,t}^{(i_1)}, \quad J_{(11)s,t}^{(i_1 i_2)}, \quad J_{(10)s,t}^{(i_1 0)}, \quad J_{(01)s,t}^{(0 i_1)}, \quad J_{(111)s,t}^{(i_1 i_2 i_3)}, \quad J_{(101)s,t}^{(i_1 0 i_3)}, \quad J_{(110)s,t}^{(i_1 i_2 0)}, \quad J_{(011)s,t}^{(0 i_1 i_2)}, \quad J_{(1111)s,t}^{(i_1 i_2 i_3 i_4)}.$$

For $r = 5$ $(\text{rank}_{\text{D}}(5) = 12)$ we get 12 integrals against 17 integrals and for $r = 6$ and $r = 7$ we have 20 against 29 and 33 against 50 correspondingly.

We will obtain the same results when compare the unified Taylor–Stratonovich expansions [163] (also see [1]-[17], [52], [165]) with their classical analogues [84], [159] (see previous sections).

Note that the summation with respect to the sets D_q is usually used while constructing strong numerical methods (built according to the mean-square criterion of convergence) for Itô SDEs [82], [84] (also see [13]). The summation with respect to the sets A_q is usually used when building weak numerical methods (built in accordance with the weak criterion of convergence) for Itô SDEs [82], [84]. For example, $\text{rank}_{\text{A}}(4) = 15$ while the total number of various iterated Itô stochastic integrals (included in the classical Taylor–Itô expansion [84] when $r = 4$) equals to 26.

Let us show that [3]-[17], [52]

$$\mathrm{rank}_{\mathrm{A}}(r) = 2^r - 1.$$

Let $(l_1, \ldots, l_k)$ be an ordered set such that $l_1, \ldots,\ l_k = 0, 1, \ldots$ and $k = 1, 2, \ldots$ Consider $S(k) \stackrel{\mathsf{def}}{=} l_1 + \ldots + l_k = p$ (p is a fixed natural number or zero). Let $N(k, p)$ be a number of all ordered combinations $(l_1, \ldots, l_k)$ such that $l_1, \ldots, l_k = 0, 1, \ldots,\ k = 1, 2, \ldots,$ and $S(k) = p$. First, let us show that

$$N(k, p) = C_{p+k-1}^{k-1},$$

where

$$C_n^m = \frac{n!}{m!(n-m)!}$$

is a binomial coefficient.

It is not difficult to see that

$$N(1, p) = 1 = C_{p+1-1}^{1-1},$$

$$N(2, p) = p + 1 = C_{p+2-1}^{2-1},$$

$$N(3, p) = \frac{(p+1)(p+2)}{2} = C_{p+3-1}^{3-1}.$$

Moreover,

$$N(k+1, p) = \sum_{l=0}^{p} N(k, l) = \sum_{l=0}^{p} C_{l+k-1}^{k-1} = C_{p+k}^{k},$$

where we used the induction assumption and the well known property of binomial coefficients.

Then

$$\mathrm{rank}_A(r) =$$

$$= N(1, 0) + (N(1, 1) + N(2, 0)) + (N(1, 2) + N(2, 1) + N(3, 0)) + \ldots$$

$$\ldots + (N(1, r-1) + N(2, r-2) + \ldots + N(r, 0)) =$$

$$= C_0^0 + (C_1^0 + C_1^1) + (C_2^0 + C_2^1 + C_2^2) + \ldots$$

$$\ldots + (C_{r-1}^0 + C_{r-1}^1 + C_{r-1}^2 + \ldots + C_{r-1}^{r-1}) =$$

$$= 2^0 + 2^1 + 2^2 + \ldots + 2^{r-1} = 2^r - 1.$$

Let $n_{\rm M}(r)$ be the total number of various iterated stochastic integrals included in the classical Taylor–Itô expansion (4.22) [84], where summation is performed with respect to the set

$$\bigcup_{k=1}^{r} {\rm M}_k.$$

If we exclude from the consideration the integrals, which are equal to

$$\frac{(s-t)^j}{j!},$$

then

$$n_M(r)=$$
$$=(2^1-1)+(2^2-1)+(2^3-1)+\ldots+(2^r-1)=$$
$$=2(1+2+2^2+\ldots+2^{r-1})-r=2(2^r-1)-r.$$

It means that

$$\lim_{r\to\infty}\frac{n_M(r)}{{\rm rank}_A(r)}=2.$$

The numbers

$${\rm rank}_{\rm A}(r),\quad n_{\rm M}(r),\quad f(r)=n_{\rm M}(r)/{\rm rank}_{\rm A}(r)$$

for various values r are shown in Table 4.1.

Let us show that [3]-[17], [52]

$${\rm rank}_{\rm D}(r)=\begin{cases}\displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{(r-1)/2+[s/2]} C_l^s & \text{for}\quad r=1,\ 3,\ 5,\ldots\\ \\ \displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{r/2-1+[(s+1)/2]} C_l^s & \text{for}\quad r=2,\ 4,\ 6,\ldots\end{cases},\qquad (4.75)$$

where $[x]$ is an integer part of a real number x and C_n^m is a binomial coefficient.

For the proof of (4.75) we rewrite the condition

$$k+2(j+S(k))\le r,$$

where $S(k)\stackrel{\sf def}{=}l_1+\ldots+l_k$ $(k,j,l_1,\ldots,l_k=0,1,\ldots)$ in the form

$$j+S(k)\le (r-k)/2$$

Table 4.1: Numbers $\text{rank}_{\text{A}}(r)$, $n_{\text{M}}(r)$, $f(r) = n_{\text{M}}(r)/\text{rank}_{\text{A}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{A}}(r)$	1	3	7	15	31	63	127	255	511	1023
$n_{\text{M}}(r)$	1	4	11	26	57	120	247	502	1013	2036
$f(r)$	1	1.3333	1.5714	1.7333	1.8387	1.9048	1.9449	1.9686	1.9824	1.9902

and perform the consideration of all possible combinations with respect to $k = 1, \ldots, r$. Moreover, we take into account the above reasoning.

Let us calculate the number $n_E(r)$ of all different iterated Itô stochastic integrals from the classical Taylor–Itô expansion (4.24) [84] if the summation in this expansion is performed with respect to the set

$$\bigcup_{q,k=1}^{r} E_{qk}.$$

The summation condition can be rewritten in this case in the form

$$0 \le p + 2q \le r,$$

where q is a total number of integrations with respect to time while p is a total number of integrations with respect to the Wiener processes in the selected iterated stochastic integral from the Taylor–Itô expansion (4.24) [84]. At that the multiplicity of the mentioned stochastic integral equals to $p+q$ and it is not more than r. Let us rewrite the above condition $(0 \le p + 2q \le r)$ in the form: $0 \le q \le (r-p)/2 \Leftrightarrow 0 \le q \le [(r-p)/2]$, where $[x]$ means an integer part of a real number x. Then, performing the consideration of all possible combinations with respect to $p = 1, \ldots, r$ and using the combinatorial reasoning, we come to the formula

$$n_E(r) = \sum_{s=1}^{r} \sum_{l=0}^{[(r-s)/2]} C^s_{[(r-s)/2]+s-l}, \tag{4.76}$$

where $[x]$ means an integer part of a real number x.

The numbers

$$\text{rank}_{\text{D}}(r), \quad n_{\text{E}}(r), \quad g(r) = n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$$

for various values r are shown in Table 4.2.

Table 4.2: Numbers $\text{rank}_{\text{D}}(r)$, $n_{\text{E}}(r)$, $g(r) = n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{D}}(r)$	1	2	4	7	12	20	33	54	88	143
$n_{\text{E}}(r)$	1	2	5	9	17	29	50	83	138	261
$g(r)$	1	1	1.2500	1.2857	1.4167	1.4500	1.5152	1.5370	1.5682	1.8252

4.11 Application of First Form of the Unified Taylor–Itô Expansion to the High-Order Strong Numerical Methods for Itô SDEs

Let us rewrite (4.34) for all $s, t \in [0, T]$ such that $s > t$ in the following form

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \text{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1 \ldots l_k s,t}^{(i_1 \ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d \in \mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t) + \left(\bar{H}_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \qquad (4.77)$$

where

$$\left(\bar{H}_{r+1}\right)_{s,t} = \left(H_{r+1}\right)_{s,t} - \mathbf{1}_{\{r=2d-1, d \in \mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t).$$

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} |\tau_{j+1} - \tau_j|.$$

From (4.77) for $s = \tau_{p+1}$, $t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (4.1), which is based on first form of the unified Taylor–Itô expansion

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \text{D}_q} \frac{(\tau_{p+1}-\tau_p)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{y}_p \ \hat{I}_{l_1 \ldots l_k \tau_{p+1},\tau_p}^{(i_1 \ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d \in \mathbf{N}\}} \frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \qquad (4.78)$$

where $\hat{I}^{(i_1\ldots i_k)}_{l_1\ldots l_k{}_{\tau_{p+1},\tau_p}}$ is an approximation of iterated Itô stochastic integral $I^{(i_1\ldots i_k)}_{l_1\ldots l_k{}_{\tau_{p+1},\tau_p}}$ of the form

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_k{}_{s,t}} = \int\limits_t^s (t-t_k)^{l_k}\ldots \int\limits_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}.$$

Note that we understand the equality (4.78) componentwise with respect to the components $\mathbf{y}^{(i)}_p$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $T=\tau_N$, $p=0,1,\ldots,N$.

It is known [84] that under the appropriate conditions the numerical scheme (4.78) has strong order of convergence $r/2$ ($r\in\mathbf{N}$).

Let $B_j(\mathbf{x},t)$ is the jth column of the matrix function $B(\mathbf{x},t)$.

Below we consider particular cases of the numerical scheme (4.78) for $r = 2,3,4,5$, and 6, i.e. explicit one-step strong numerical schemes for the Itô SDE (4.1) with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. At that for simplicity we will write $\mathbf{a}$, $L\mathbf{a}$, B_i, $G^{(i)}_0 B_j$, $\ldots$ instead of $\mathbf{a}(\mathbf{y}_p,\tau_p)$, $L\mathbf{a}(\mathbf{y}_p,\tau_p)$, $B_i(\mathbf{y}_p,\tau_p)$, $G^{(i)}_0 B_j(\mathbf{y}_p,\tau_p)$, $\ldots$ correspondingly. Moreover, the operators L and $G^{(i)}_0$, $i=1,\ldots,m$ are determined by the equalities (4.17), (4.18).

Scheme with strong order 1.0 (Milstein Scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1}\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^m G^{(i_2)}_0 B_{i_1}\hat{I}^{(i_2 i_1)}_{00_{\tau_{p+1},\tau_p}}. \qquad (4.79)$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1}\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^m G^{(i_2)}_0 B_{i_1}\hat{I}^{(i_2 i_1)}_{00_{\tau_{p+1},\tau_p}} +$$

$$+\sum_{i_1=1}^m \left[G^{(i_1)}_0\mathbf{a}\left(\Delta\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \hat{I}^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right) - LB_{i_1}\hat{I}^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right] +$$

$$+\sum_{i_1,i_2,i_3=1}^m G^{(i_3)}_0 G^{(i_2)}_0 B_{i_1}\hat{I}^{(i_3 i_2 i_1)}_{000_{\tau_{p+1},\tau_p}} +$$

$$
+\frac{\Delta^2}{2}L\mathbf{a}. \tag{4.80}
$$

Scheme with strong order 2.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\right.
$$

$$
\left.+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}. \tag{4.81}
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\right.
$$

$$
\left.+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4 i_3 i_2 i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+
$$

$$
+\frac{1}{2}LLB_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\Bigr)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Bigl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}\right)-
$$

$$
-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{100_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}\Bigr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{00000_{\tau_{p+1},\tau_p}}^{(i_5 i_4 i_3 i_2 i_1)}+
$$

$$
+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{4.82}
$$

Scheme with strong order 3.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+\right.
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \qquad (4.83)
$$

where

$$
\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+
$$

$$
+\frac{1}{2}LLB_{i_1}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\left(\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-
$$

$$
-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{00000_{\tau_{p+1},\tau_p}}^{(i_5i_4i_3i_2i_1)}+
$$

$$
+\frac{\Delta^3}{6}LL\mathbf{a},
$$

and

$$
\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{\Delta^2}{2}\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+
$$

$$
+\frac{1}{2}LLG_0^{(i_2)}B_{i_1}\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+G_0^{(i_2)}LG_0^{(i_1)}\mathbf{a}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right)+
$$

$$
+LG_0^{(i_2)}LB_{i_1}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+
$$

$$
+G_0^{(i_2)}LLB_{i_1}\left(\frac{1}{2}\hat{I}_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{1}{2}\hat{I}_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-
$$

$$
\left.-LG_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}\left[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\hat{I}_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+\right.
$$

$$
+G_0^{(i_4)}G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat{I}_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)-
$$

$$
-LG_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+
$$

$$
+G_0^{(i_4)}LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+
$$

$$
\left.+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000000_{\tau_{p+1},\tau_p}}^{(i_6i_5i_4i_3i_2i_1)}.
$$

It is well known [84] that under the standard conditions the numerical schemes (4.79)–(4.83) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Itô stochastic integrals from the numerical schemes (4.79)–(4.83) [84] (also see [13])

$$
\mathsf{M}\left\{\left(I_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{(i_1\ldots i_k)}-\hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^{r+1}, \tag{4.84}
$$

where constant C is independent of Δ and $r/2$ are strong orders of convergence for the numerical schemes (4.79)–(4.83), i.e. $r/2=1.0, 1.5, 2.0, 2.5,$ and 3.0.

As we mentioned above, the numerical schemes (4.79)–(4.83) are unrealizable in practice without procedures for the numerical simulation of iterated Itô stochastic integrals from (4.77).

In Chapter 5 we give an extensive material on the mean-square approximation of specific iterated Itô stochastic integrals from the numerical schemes (4.79)–(4.83). The mentioned material based on the results of Chapter 1.

4.12 Application of First Form of the Unified Taylor–Stratonovich Expansion to the High-Order Strong Numerical Methods for Itô SDEs

Let us rewrite (4.63) for all $s,t\in[0,T]$ such that $s>t$ in the following from

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^jR(\mathbf{x}_t,t)I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)}+$$

$$+\mathbf{1}_{\{r=2d-1,d\in\mathbf{N}\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t)+\left(\bar{H}_{r+1}\right)_{s,t}\quad \text{w. p. 1},\qquad (4.85)$$

where

$$\left(\bar{H}_{r+1}\right)_{s,t}=\left(H_{r+1}\right)_{s,t}-\mathbf{1}_{\{r=2d-1,d\in\mathbf{N}\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t).$$

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0,T]$ such that

$$0=\tau_0<\tau_1<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\left|\tau_{j+1}-\tau_j\right|.$$

From (4.85) for $s=\tau_{p+1},\ t=\tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (4.1), which is based on first form of the unified Taylor–Stratonovich expansion

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(\tau_{p+1}-\tau_p)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^j\mathbf{y}_p\ \hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)}+$$

$$+\mathbf{1}_{\{r=2d-1,d\in\mathbf{N}\}}\frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}\mathbf{y}_p,\qquad (4.86)$$

where $\hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)}$ is an approximation of iterated Stratonovich stochastic integral $I_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)}$ of the form

$$I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)}=\int\limits_t^{*s}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}.$$

Note that we understand the equality (4.86) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $T = \tau_N$, $p = 0, 1, \ldots, N$.

It is known [84] that under the appropriate conditions the numerical scheme (4.86) has strong order of convergence $r/2$ $(r \in \mathbf{N})$.

Denote

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t) - \frac{1}{2}\sum_{j=1}^{m} G_0^{(j)} B_j(\mathbf{x}, t),$$

where $B_j(\mathbf{x}, t)$ is the jth column of the matrix function $B(\mathbf{x}, t)$.

It is not difficult to show that (see (4.47))

$$\bar{L}R(\mathbf{x}, t) = \frac{\partial R}{\partial t}(\mathbf{x}, t) + \sum_{j=1}^{n} \bar{\mathbf{a}}^{(j)}(\mathbf{x}, t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x}, t), \tag{4.87}$$

where $\bar{\mathbf{a}}^{(j)}(\mathbf{x}, t)$ is the jth component of the vector function $\bar{\mathbf{a}}(\mathbf{x}, t)$.

Below we consider particular cases of the numerical scheme (4.86) for $r = 2, 3, 4, 5$, and 6, i.e. explicit one-step strong numerical schemes for the Itô SDE (4.1) with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. At that, for simplicity we will write $\bar{\mathbf{a}}$, $\bar{L}\bar{\mathbf{a}}$, $L\mathbf{a}$, B_i, $G_0^{(i)}B_j$, $\ldots$ instead of $\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $\bar{L}\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $L\mathbf{a}(\mathbf{y}_p, \tau_p)$, $B_i(\mathbf{y}_p, \tau_p)$, $G_0^{(i)}B_j(\mathbf{y}_p, \tau_p)$, $\ldots$ correspondingly. Moreover, the operators $\bar{L}$ and $G_0^{(i)}$, $i = 1, \ldots, m$ are determined by the equalities (4.17), (4.18), and (4.87).

Scheme with strong order 1.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)}. \tag{4.88}$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} +$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right) - \bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right] +$$

$$
+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+
$$

$$
+\frac{\Delta^2}{2}L\mathbf{a}. \qquad (4.89)
$$

Scheme with strong order 2.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\right.
$$

$$
\left.+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}. \qquad (4.90)
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)+
$$

$$
+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)-
$$

$$
-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{00000_{\tau_{p+1},\tau_p}}^{*(i_5i_4i_3i_2i_1)}+
$$

$$
+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{4.91}
$$

Scheme with strong order 3.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \qquad (4.92)
$$

where

$$
\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)+
$$

$$
+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)-
$$

$$
-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{00000_{\tau_{p+1},\tau_p}}^{*(i_5i_4i_3i_2i_1)}+
$$

$$
+\frac{\Delta^3}{6}\bar{L}\bar{L}\bar{\mathbf{a}},
$$

and

$$
\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}_{02_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\frac{\Delta^2}{2}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)+
$$

$$
+\frac{1}{2}\bar{L}\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}+
$$

$$
+G_0^{(i_2)}\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)\right)+
$$

$$
+\bar{L}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}\right)+
$$

$$
+G_0^{(i_2)}\bar{L}\bar{L}B_{i_1}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\frac{1}{2}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)-
$$

$$
\left.-\bar{L}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}\left[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)+\right.
$$

$$
+G_0^{(i_4)}G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}\right)-
$$

$$
-\bar{L}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}+
$$

$$
+G_0^{(i_4)}\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}\right)+
$$

$$
\left.+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_6i_5i_4i_3i_2i_1)}_{000000_{\tau_{p+1},\tau_p}}.
$$

It is well known [84] that under the standard conditions the numerical schemes (4.88)–(4.92) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from the numerical schemes (4.88)–(4.92) [84] (also see [13])

$$
\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}-\hat{I}^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}\right)^2\right\}\le C\Delta^{r+1},
$$

where constant C is independent of Δ and $r/2$ are strong orders of convergence for the numerical schemes (4.88)–(4.92), i.e. $r/2=1.0, 1.5, 2.0, 2.5,$ and 3.0.

As we mentioned above, the numerical schemes (4.88)–(4.92) are unrealizable in practice without procedures for the numerical simulation of iterated Stratonovich stochastic integrals from (4.85).

In Chapter 5 we give an extensive material on the mean-square approximation of specific iterated Itô and Stratonovich stochastic integrals from the numerical schemes (4.79)–(4.83), (4.88)–(4.92). The mentioned material based on the results of Chapters 1 and 2.

Chapter 5

Mean-Square Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of Multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich Expansions Based on Theorems From Chapters 1 and 2

5.1 Mean-Square Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of multiplicities 1 to 6 Based on Legendre Polynomials

This section is devoted to the extensive practical material on expansions and mean-square approximations of specific iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 on the base of Theorems 1.1, 2.1–2.9, 2.33–2.36, 2.50, 2.51, 2.62, 2.63 and multiple Fourier–Legendre series. The considered iterated Itô and Stratonovich stochastic integrals are part of the Taylor–Itô and Taylor–Stratonovich expansions. Therefore, the results of this section can be useful for the numerical solution of Itô SDEs with non-commutative noise.

Consider the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.1}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.2}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes; $i_1,\ldots,i_k=0,1,\ldots,m.$

As we saw in Chapter 4, $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ in (5.1), (5.2) if we consider the iterated stochastic integrals from the classical Taylor–Itô and Taylor–Stratonovich expansions [84]. At the same time $\psi_l(\tau)\equiv(t-\tau)^{q_l}$ $(l=1,\ldots,k,\ q_1,\ldots,q_k=0,1,2,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ for the iterated stochastic integrals from the unified Taylor–Itô and Taylor–Stratonovich expansions [1]-[17], [52], [161], [163].

Thus, in this section, we will consider the following collections of iterated Itô and Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},\tag{5.3}$$

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},\tag{5.4}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ l_1,\ldots,l_k=0,1,\ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots,\tag{5.5}$$

where

$$P_j(x)=\frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j\tag{5.6}$$

is the Legendre polynomial.

Let us recall some properties of Legendre polynomials [121] (see Sect. 2.1.2)

$$P_j(1)=1,\quad P_{j+1}(-1)=-P_j(-1),\quad j=0,1,2,\ldots,$$

$$\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),$$

$$xP_j(x)=\frac{(j+1)P_{j+1}(x)+jP_{j-1}(x)}{2j+1},\quad j=1,2,\ldots,$$

$$\int\limits_{-1}^{1} x^k P_j(x)dx = 0, \quad k = 0, 1, \ldots, j-1,$$

$$\int\limits_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if} \ \ k \neq j \\ \\ 2/(2j+1) & \text{if} \ \ k = j \end{cases},$$

$$P_n(x)P_m(x) = \sum_{k=0}^{m} K_{m,n,k} P_{n+m-2k}(x),$$

where

$$K_{m,n,k} = \frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}} \cdot \frac{2n+2m-4k+1}{2n+2m-2k+1}, \quad a_k = \frac{(2k-1)!!}{k!}, \quad m \leq n.$$

Applying the above properties of the Legendre polynomial system (5.5) and Theorems 1.1, 2.1–2.9, 2.33–2.36, 2.50, 2.51, 2.62, 2.63, we obtain the following expansions of iterated Itô and Stratonovich stochastic integrals from the sets (5.3), (5.4)

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.7}$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.8}$$

$$I_{(2)T,t}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{5.9}$$

$$I_{(00)T,t}^{*(i_1 i_2)} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{5.10}$$

$$I_{(00)T,t}^{(i_1 i_2)} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.11}$$

$$I_{(01)T,t}^{*(i_1 i_2)} = -\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)} - \frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)} + \right.$$

$$\left. + \sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{5.12}$$

$$
I_{(10)T,t}^{*(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{*(i_1 i_2)} - \frac{(T-t)^2}{4} \left(\frac{1}{\sqrt{3}} \zeta_0^{(i_2)} \zeta_1^{(i_1)} + \right.
$$

$$
\left. + \sum_{i=0}^{\infty} \left(\frac{(i+1)\zeta_{i+2}^{(i_2)} \zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)} \zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right) \tag{5.13}
$$

or

$$
I_{(01)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{01} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},
$$

$$
I_{(10)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{10} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},
$$

where

$$
C_{j_2 j_1}^{01} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{01},
$$

$$
C_{j_2 j_1}^{10} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{10}, \tag{5.14}
$$

$$
\bar{C}_{j_2 j_1}^{01} = -\int\limits_{-1}^{1} (1+y) P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy,
$$

$$
\bar{C}_{j_2 j_1}^{10} = -\int\limits_{-1}^{1} P_{j_2}(y) \int\limits_{-1}^{y} (1+x) P_{j_1}(x) dx dy;
$$

$$
I_{(10)T,t}^{(i_1 i_2)} = I_{(10)T,t}^{*(i_1 i_2)} + \frac{1}{4} \mathbf{1}_{\{i_1=i_2\}} (T-t)^2 \quad \text{w. p. 1},
$$

$$
I_{(01)T,t}^{(i_1 i_2)} = I_{(01)T,t}^{*(i_1 i_2)} + \frac{1}{4} \mathbf{1}_{\{i_1=i_2\}} (T-t)^2 \quad \text{w. p. 1},
$$

$$
I_{(01)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)} - \frac{(T-t)^2}{4} \left(\frac{1}{\sqrt{3}} \zeta_0^{(i_1)} \zeta_1^{(i_2)} + \right.
$$

$$
\left. + \sum_{i=0}^{\infty} \left(\frac{(i+2)\zeta_i^{(i_1)} \zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)} \zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right), \tag{5.15}
$$

$$
I_{(10)T,t}^{(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1 i_2)}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+\right.
$$

$$
\left.+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right) \qquad (5.16)
$$

or

$$
I_{(01)T,t}^{(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{01}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(10)T,t}^{(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{10}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(000)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \qquad (5.17)
$$

$$
I_{(000)T,t}^{(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\right.
$$

$$
\left.-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \qquad (5.18)
$$

$$
I_{(000)T,t}^{(i_1 i_1 i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right)\quad \text{w. p. 1},
$$

$$
I_{(000)T,t}^{*(i_1 i_1 i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\zeta_0^{(i_1)}\right)^3\quad \text{w. p. 1}, \qquad (5.19)
$$

where

$$
C_{j_3 j_2 j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1}, \qquad (5.20)
$$

$$
\bar{C}_{j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz; \qquad (5.21)
$$

here and further in this section

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}\quad (i=1,\ldots,m,\ j=0,1,\ldots)$$

are independent standard Gaussian random variables for various i or j;

$$I_{(000)T,t}^{(i_1i_2i_3)}=I_{(000)T,t}^{*(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}I_{(1)T,t}^{(i_3)}-$$
$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left((T-t)I_{(0)T,t}^{(i_1)}+I_{(1)T,t}^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(02)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(01)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+\right.$$
$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$
$$\left.\left.+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right], \tag{5.22}$$

$$I_{(20)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(10)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+\right.$$
$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$
$$\left.\left.+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right], \tag{5.23}$$

$$I_{(11)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+$$
$$+\frac{(T-t)^3}{8}\left[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.\right.$$

$$
+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr] \qquad (5.24)
$$

or

$$
I_{(02)T,t}^{*(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{02}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
I_{(20)T,t}^{*(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{20}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
I_{(11)T,t}^{*(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{11}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

where

$$
C_{j_2j_1}^{02}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{02},
$$

$$
C_{j_2j_1}^{20}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{20},
$$

$$
C_{j_2j_1}^{11}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{11},
$$

$$
\bar{C}_{j_2j_1}^{02}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)^2\int\limits_{-1}^{y}P_{j_1}(x)dxdy,
$$

$$
\bar{C}_{j_2j_1}^{20}=\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)^2dxdy,
$$

$$
\bar{C}_{j_2j_1}^{11}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdy,
$$

$$
I_{(11)T,t}^{*(i_1i_1)}=\frac{1}{2}\left(I_{(1)T,t}^{(i_1)}\right)^2\quad \text{w. p. 1},
$$

$$
I_{(02)T,t}^{(i_1i_2)}=I_{(02)T,t}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3\quad \text{w. p. 1}, \qquad (5.25)
$$

$$I_{(20)T,t}^{(i_1i_2)}=I_{(20)T,t}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3\quad\text{w. p. 1},\tag{5.26}$$

$$I_{(11)T,t}^{(i_1i_2)}=I_{(11)T,t}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3\quad\text{w. p. 1},$$

$$I_{(02)T,t}^{(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1i_2)}-(T-t)I_{01_{T,t}}^{(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,\tag{5.27}$$

$$I_{(20)T,t}^{(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1i_2)}-(T-t)I_{(10)T,t}^{(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,\tag{5.28}$$

$$I_{(11)T,t}^{(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1i_2)}-\frac{T-t}{2}\left(I_{(10)T,t}^{(i_1i_2)}+I_{(01)T,t}^{(i_1i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$
\left.\left.+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right]-
$$

$$
-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \tag{5.29}
$$

or

$$
I_{(02)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{02}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(20)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{20}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(11)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{11}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(3)T,t}^{(i_1)}=-\frac{(T-t)^{7/2}}{4}\left(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\right), \tag{5.30}
$$

$$
I_{(0000)T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},
$$

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\right.
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right), \tag{5.31}
$$

$$I_{(0000)T,t}^{(i_1 i_1 i_1 i_1)}=\frac{1}{24}(T-t)^2\left(\left(\zeta_0^{(i_1)}\right)^4-6\left(\zeta_0^{(i_1)}\right)^2+3\right)\quad \text{w. p. 1},$$

$$I_{(0000)T,t}^{*(i_1 i_1 i_1 i_1)}=\frac{1}{24}(T-t)^2\left(\zeta_0^{(i_1)}\right)^4\quad \text{w. p. 1}, \tag{5.32}$$

where

$$C_{j_4 j_3 j_2 j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2\bar{C}_{j_4 j_3 j_2 j_1}, \tag{5.33}$$

$$\bar{C}_{j_4 j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu; \tag{5.34}$$

$$I_{(001)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(010)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(100)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(001)T,t}^{(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.35}$$

$$I_{(010)T,t}^{(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{010}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.36}$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}^{100}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.37}$$

where

$$C_{j_3 j_2 j_1}^{001}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3 j_2 j_1}^{001},$$

$$C_{j_3 j_2 j_1}^{010}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3 j_2 j_1}^{010},$$

$$C_{j_3 j_2 j_1}^{100}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3 j_2 j_1}^{100},$$

$$\bar C_{j_3 j_2 j_1}^{100}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydz,$$

$$\bar C_{j_3 j_2 j_1}^{010}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar C_{j_3 j_2 j_1}^{001}=-\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz;$$

$$I_{(lll)T,t}^{(i_1 i_1 i_1)}=\frac{1}{6}\left(\left(I_{(l)T,t}^{(i_1)}\right)^3-3I_{(l)T,t}^{(i_1)}\Delta_{l(T,t)}\right)\quad \text{w. p. 1},$$

$$I_{(lll)T,t}^{*(i_1 i_1 i_1)}=\frac{1}{6}\left(I_{(l)T,t}^{(i_1)}\right)^3\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{(i_1 i_1 i_1 i_1)}=\frac{1}{24}\left(\left(I_{(l)T,t}^{(i_1)}\right)^4-6\left(I_{(l)T,t}^{(i_1)}\right)^2\Delta_{(l)T,t}+3\left(\Delta_{(l)T,t}\right)^2\right)\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{*(i_1 i_1 i_1 i_1)}=\frac{1}{24}\left(I_{(l)T,t}^{(i_1)}\right)^4\quad \text{w. p. 1},$$

where

$$I_{(l)T,t}^{(i_1)}=\sum_{j=0}^{l} C_j^l \zeta_j^{(i_1)} \quad \text{w. p. 1}, \tag{5.38}$$

$$\Delta_{l(T,t)}=\int_t^T (t-s)^{2l}ds, \quad C_j^l=\int_t^T (t-s)^l \phi_j(s)ds;$$

$$I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)}=\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}=\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{5.39}$$

$$I_{(00000)T,t}^{(i_1 i_1 i_1 i_1 i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right) \quad \text{w. p. 1},$$

$$I_{(00000)T,t}^{*(i_1 i_1 i_1 i_1 i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\zeta_0^{(i_1)}\right)^5 \quad \text{w. p. 1},$$

where

$$C_{j_5 j_4 j_3 j_2 j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar C_{j_5 j_4 j_3 j_2 j_1},$$

$$\bar C_{j_5 j_4 j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv;$$

$$I_{(0001)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0010)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0100)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(1000)T,t}^{*(i_1 i_2 i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$
I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0010}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

$$
I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{0100}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

$$
I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4 j_3 j_2 j_1}^{1000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

where

$$C_{j_4j_3j_2j_1}^{0001}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0001},$$

$$C_{j_3j_2j_1}^{0010}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0010},$$

$$C_{j_4j_3j_2j_1}^{0100}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_3j_2j_1}^{0100},$$

$$C_{j_4j_3j_2j_1}^{1000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{1000},$$

$$\bar{C}_{j_4j_3j_2j_1}^{1000}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0100}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0010}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0001}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu;$$

$$I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)},$$

$$I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$

$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$

$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),$$

$$I_{(000000)T,t}^{(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\left(\zeta_0^{(i_1)}\right)^6-15\left(\zeta_0^{(i_1)}\right)^4+45\left(\zeta_0^{(i_1)}\right)^2-15\right)\quad \text{w. p. 1},$$

$$I_{(000000)T,t}^{*(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\zeta_0^{(i_1)}\right)^6\quad \text{w. p. 1},$$

where

$$
C_{j_6 j_5 j_4 j_3 j_2 j_1}=
$$

$$
=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3\bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1},
$$

$$
\bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1}=
$$

$$
=\int\limits_{-1}^{1}P_{j_6}(w)\int\limits_{-1}^{w}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudvdw.
$$

It should be noted that instead of the expansion (5.17) we can consider the following expansion, which is derived by direct calculation

$$
I_{(000)T,t}^{*(i_1 i_2 i_3)}=-\frac{1}{T-t}\left(I_{(0)T,t}^{(i_3)}I_{(10)T,t}^{*(i_2 i_1)}+I_{(0)T,t}^{(i_1)}I_{(10)T,t}^{*(i_2 i_3)}\right)+\frac{1}{2}I_{(0)T,t}^{(i_3)}\left(I_{(00)T,t}^{*(i_1 i_2)}-I_{(00)T,t}^{*(i_2 i_1)}\right)-
$$

$$
-(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)}+\sqrt{3}\zeta_1^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right)+\frac{1}{4}D_{T,t}^{(i_1 i_2 i_3)}\right), \qquad (5.40)
$$

where

$$
D_{T,t}^{(i_1 i_2 i_3)}=\sum_{\substack{i=1,\ j=0,\ k=i\\ 2i\ge k+i-j\ge -2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i+1,k+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+
$$

$$
+\sum_{\substack{i=1,\ j=0,\ 1\le k\le i-1\\ 2k\ge k+i-j\ge -2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k+1,i+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-
$$

$$
-\sum_{\substack{i=1,\ j=0,\ k=i+2\\ 2i+2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i+1,k-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-
$$

$$
-\sum_{\substack{i=1,\ j=0,\ 1\le k\le i+1\\ 2k-2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k-1,i+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-
$$

$$
-\sum_{\substack{i=1,\ j=0,\ k=i-2,k\ge 1\\ 2i-2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i-1,k+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-
$$

$$
-\sum_{\substack{i=1,\ j=0,\ 1\le k\le i-3\\ 2k+2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k+1,i-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+
$$

$$
+\sum_{\substack{i=1,\ j=0,\ k=i\\ 2i\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty} N_{ijk}K_{i-1,k-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+
$$

$$
+\sum_{\substack{i=1,\ j=0\ 1\le k\le i-1\\ 2k\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty} N_{ijk}K_{k-1,i-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)},
$$

where

$$
N_{ijk}=\sqrt{\frac{1}{(2k+1)(2j+1)(2i+1)}},
$$

$$
K_{m,n,k}=\frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1},\quad a_k=\frac{(2k-1)!!}{k!},\quad m\le n.
$$

However, as we will see further the expansion (5.18) is more convenient for the practical implementation then (5.40).

Also note the following relation between iterated Itô and Stratonovich stochastic integrals

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)}=I_{(0000)T,t}^{*(i_1i_2i_3i_4)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}I_{(10)T,t}^{*(i_3i_4)}-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I_{(10)T,t}^{*(i_1i_4)}-I_{(01)T,t}^{*(i_1i_4)}\right)-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left((T-t)I_{(00)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+\frac{1}{8}(T-t)^2\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}}\quad \text{w. p. 1.}
$$

Let us denote as

$$
I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)q},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}
$$

the approximations of iterated Itô and Stratonovich stochastic integrals

$$
I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}
$$

defined by (5.3), (5.4), i.e. we replace ∞ on q in the expansions of these stochastic integrals. For example, $I_{(00)T,t}^{*(i_1i_2)q}$ is the approximation of the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1i_2)}$ obtained from (5.10) by replacing ∞ on q, etc.

It is easy to prove that

$$
\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2).\quad (5.41)
$$

Moreover, using Theorem 1.3, we obtain for $i_1 \ne i_2$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)}-I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)}-I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\right.
$$

$$
\left.-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \tag{5.42}
$$

For the case $i_1 = i_2$ using Theorem 1.3, we have

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_1)}-I_{(10)T,t}^{(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_1)}-I_{(01)T,t}^{(i_1 i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{5.43}
$$

In Tables 5.1–5.3 we have calculations according to the formulas (5.41)–(5.43) for various values of q. In the given tables ε means the right-hand sides of these formulas. Obviously, these results are consistent with the estimate (1.225).

Let us consider (5.12), (5.13) for $i_1 = i_2$

$$
I_{(01)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.
$$

$$
\left.+\sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}-\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right), \tag{5.44}
$$

$$
I_{(10)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.
$$

$$
\left.+\sum_{i=0}^{\infty}\left(-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}+\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right). \tag{5.45}
$$

Table 5.1: Confirmation of the formula (5.41)

$2\varepsilon/(T-t)^2$	0.1667	0.0238	0.0025	$2.4988 \cdot 10^{-4}$	$2.4999 \cdot 10^{-5}$
q	1	10	100	1000	10000

Table 5.2: Confirmation of the formula (5.42)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450 \cdot 10^{-4}$	$6.2495 \cdot 10^{-5}$
q	1	10	100	1000	10000

From (5.44), (5.45), considering (5.7) and (5.8), we obtain

$$I_{(10)T,t}^{*(i_1 i_1)}+I_{(01)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{2}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}\right)=I_{(0)T,t}^{(i_1)}I_{(1)T,t}^{(i_1)}\quad \text{w. p. 1.} \tag{5.46}$$

Obtaining (5.46) we supposed that the formulas (5.12), (5.13) are valid w. p. 1. The complete proof of this fact is given in Sect. 1.7.2 (Theorem 1.10).

Note that it is easy to obtain the equality (5.46) using the Itô formula and standard relations between iterated Itô and Stratonovich stochastic integrals.

Using the Itô formula, we obtain

$$I_{(11)T,t}^{*(i_1 i_1)}=\frac{\left(I_{(1)T,t}^{(i_1)}\right)^2}{2}\quad \text{w. p. 1.} \tag{5.47}$$

In addition, using the Itô formula, we have

$$I_{(20)T,t}^{(i_1 i_1)}+I_{(02)T,t}^{(i_1 i_1)}=I_{(0)T,t}^{(i_1)}I_{(2)T,t}^{(i_1)}-\frac{(T-t)^3}{3}\quad \text{w. p. 1.} \tag{5.48}$$

From (5.48), considering the formulas (5.25), (5.26), we obtain

$$I_{(20)T,t}^{*(i_1 i_1)}+I_{(02)T,t}^{*(i_1 i_1)}=I_{(0)T,t}^{(i_1)}I_{(2)T,t}^{(i_1)}\quad \text{w. p. 1.} \tag{5.49}$$

Let us check whether the formulas (5.47), (5.49) follow from (5.22)–(5.24),

Table 5.3: Confirmation of the formula (5.43)

$16\varepsilon/(T-t)^4$	0.0070	$4.3551\cdot 10^{-5}$	$6.0076\cdot 10^{-8}$	$6.2251\cdot 10^{-11}$	$6.3178\cdot 10^{-14}$
q	1	10	100	1000	10000

if we suppose $i_1 = i_2$ in the last ones. From (5.22)–(5.24) for $i_1 = i_2$ we get

$$I_{(20)T,t}^{*(i_1 i_1)} + I_{(02)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{2} I_{(00)T,t}^{*(i_1 i_1)} - (T-t)\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) +$$

$$+\frac{(T-t)^3}{4}\left(\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2 + \frac{2}{3\sqrt{5}}\zeta_2^{(i_1)}\zeta_0^{(i_1)}\right), \tag{5.50}$$

$$I_{(11)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{*(i_1 i_1)} - \frac{T-t}{2}\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) + \frac{(T-t)^3}{24}\left(\zeta_1^{(i_1)}\right)^2. \tag{5.51}$$

It is easy to see that considering (5.46) and (5.7)–(5.10), we actually obtain the equalities (5.47) and (5.49) from (5.50) and (5.51). This fact indirectly confirms the correctness of the formulas (5.22)–(5.24).

Obtaining (5.47), (5.49) we supposed that the formulas (5.22)–(5.24) are valid w. p. 1. The complete proof of this fact is given in Sect. 1.7.2 (Theorem 1.10).

On the basis of the presented expansions of iterated stochastic integrals we can see that increasing of multiplicities of these integrals or degree indices of their weight functions leads to noticeable complication of formulas for the mentioned expansions.

However, increasing of the mentioned parameters leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities in expansions of iterated stochastic integrals, which are required for achieving the acceptable accuracy of approximation. In this context, let us consider the approach to the approximation of iterated stochastic integrals, which provides a possibility to obtain the mean-square approximations of the required accuracy without using the complex expansions like (5.40).

Let us analyze the following approximation of iterated Itô stochastic integral of multiplicity 3 using (5.18)

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.52}$$

where $C_{j_3 j_2 j_1}$ is defined by (5.20), (5.21).

In particular, from (5.52) for $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ we obtain

$$
I_{(000)T,t}^{(i_1 i_2 i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{5.53}
$$

Furthermore, using Theorem 1.3 for $k=3$, we get

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2\quad (i_1\ne i_2, i_1\ne i_3, i_2\ne i_3), \tag{5.54}
$$

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_2 j_3 j_1}C_{j_3 j_2 j_1}\quad (i_1\ne i_2=i_3), \tag{5.55}
$$

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}C_{j_1 j_2 j_3}\quad (i_1=i_3\ne i_2), \tag{5.56}
$$

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_1 j_2}C_{j_3 j_2 j_1}\quad (i_1=i_2\ne i_3). \tag{5.57}
$$

From the other hand, from Theorem 1.4 for $k=3$ we obtain

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2\right), \tag{5.58}
$$

where $i_1,i_2,i_3=1,\ldots,m$.

We can act similarly with more complicated iterated stochastic integrals. For example, for the approximation of stochastic integral $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}$ we can write (see (5.31))

$$
I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2} = \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \tag{5.59}
$$

where $C_{j_4 j_3 j_2 j_1}$ is defined by (5.33), (5.34).

Moreover, according to Theorem 1.4 for $k = 4$, we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} \le 24 \left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2 \right),
$$

where $i_1, i_2, i_3, i_4 = 1, \ldots, m$.

For pairwise different $i_1, i_2, i_3, i_4 = 1, \ldots, m$ from Theorem 1.3 we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2. \tag{5.60}
$$

Using Theorem 1.3, we can calculate exactly the left-hand side of (5.60) for any possible combinations of i_1, i_2, i_3, i_4. These relations were obtained in Sect. 1.2. For example

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} =
$$

$$
= \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_4 j_3 j_2 j_1} \right) \right),
$$

where $i_1 = i_2 \neq i_3 = i_4$ and

$$
\sum_{(j_1,j_2)}
$$

means the sum with respect to permutations (j_1, j_2).

Table 5.4: Coefficients $\bar{C}_{0j_2j_1}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$	$j_1 = 3$	$j_1 = 4$	$j_1 = 5$	$j_1 = 6$
$j_2 = 0$	$\frac{4}{3}$	$\frac{-2}{3}$	$\frac{2}{15}$	0	0	0	0
$j_2 = 1$	0	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{4}{105}$	0	0	0
$j_2 = 2$	$\frac{-4}{15}$	$\frac{2}{15}$	$\frac{2}{105}$	$\frac{-2}{35}$	$\frac{2}{105}$	0	0
$j_2 = 3$	0	$\frac{-2}{35}$	$\frac{2}{35}$	$\frac{2}{315}$	$\frac{-2}{63}$	$\frac{8}{693}$	0
$j_2 = 4$	0	0	$\frac{-8}{315}$	$\frac{2}{63}$	$\frac{2}{693}$	$\frac{-2}{99}$	$\frac{10}{1287}$
$j_2 = 5$	0	0	0	$\frac{-10}{693}$	$\frac{2}{99}$	$\frac{2}{1287}$	$\frac{-2}{143}$
$j_2 = 6$	0	0	0	0	$\frac{-4}{429}$	$\frac{2}{143}$	$\frac{2}{2145}$

Table 5.5: Coefficients $\bar{C}_{1j_2j_1}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$	$j_1 = 3$	$j_1 = 4$	$j_1 = 5$	$j_1 = 6$
$j_2 = 0$	$\frac{2}{3}$	$\frac{-4}{15}$	0	$\frac{2}{105}$	0	0	0
$j_2 = 1$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2 = 2$	$\frac{-2}{15}$	$\frac{8}{105}$	0	$\frac{-2}{105}$	0	$\frac{4}{1155}$	0
$j_2 = 3$	$\frac{-2}{35}$	0	$\frac{8}{315}$	0	$\frac{-38}{3465}$	0	$\frac{20}{9009}$
$j_2 = 4$	0	$\frac{-4}{315}$	0	$\frac{46}{3465}$	0	$\frac{-64}{9009}$	0
$j_2 = 5$	0	0	$\frac{-4}{693}$	0	$\frac{74}{9009}$	0	$\frac{-32}{6435}$
$j_2 = 6$	0	0	0	$\frac{-10}{3003}$	0	$\frac{4}{715}$	0

Table 5.6: Coefficients $\bar{C}_{2j_2j_1}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$	$j_1 = 3$	$j_1 = 4$	$j_1 = 5$	$j_1 = 6$
$j_2 = 0$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2 = 1$	$\frac{2}{15}$	$\frac{-4}{105}$	0	$\frac{-2}{315}$	0	$\frac{8}{3465}$	0
$j_2 = 2$	$\frac{2}{105}$	0	0	0	$\frac{-2}{495}$	0	$\frac{4}{3003}$
$j_2 = 3$	$\frac{-2}{35}$	$\frac{8}{315}$	0	$\frac{-2}{3465}$	0	$\frac{-116}{45045}$	0
$j_2 = 4$	$\frac{-8}{315}$	0	$\frac{4}{495}$	0	$\frac{-2}{6435}$	0	$\frac{-16}{9009}$
$j_2 = 5$	0	$\frac{-4}{693}$	0	$\frac{38}{9009}$	0	$\frac{-8}{45045}$	0
$j_2 = 6$	0	0	$\frac{-8}{3003}$	0	$\frac{118}{45045}$	0	$\frac{-4}{36465}$

Table 5.7: Coefficients $\bar{C}_{3j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	$\frac{2}{105}$	0	$\frac{-4}{315}$	0	$\frac{2}{693}$	0
$j_2=1$	$\frac{4}{105}$	0	$\frac{-2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{10}{9009}$
$j_2=2$	$\frac{2}{35}$	$\frac{-2}{105}$	0	$\frac{4}{3465}$	0	$\frac{-74}{45045}$	0
$j_2=3$	$\frac{2}{315}$	0	$\frac{-2}{3465}$	0	$\frac{16}{45045}$	0	$\frac{-10}{9009}$
$j_2=4$	$\frac{-2}{63}$	$\frac{46}{3465}$	0	$\frac{-32}{45045}$	0	$\frac{2}{9009}$	0
$j_2=5$	$\frac{-10}{693}$	0	$\frac{38}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{122}{765765}$
$j_2=6$	0	$\frac{-10}{3003}$	0	$\frac{20}{9009}$	0	$\frac{-226}{765765}$	0

Table 5.8: Coefficients $\bar{C}_{4j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	$\frac{2}{315}$	0	$\frac{-4}{693}$	0	$\frac{2}{1287}$
$j_2=1$	0	$\frac{2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{-10}{9009}$	0
$j_2=2$	$\frac{2}{105}$	0	$\frac{-2}{495}$	0	$\frac{4}{6435}$	0	$\frac{-38}{45045}$
$j_2=3$	$\frac{2}{63}$	$\frac{-38}{3465}$	0	$\frac{16}{45045}$	0	$\frac{2}{9009}$	0
$j_2=4$	$\frac{2}{693}$	0	$\frac{-2}{6435}$	0	0	0	$\frac{2}{13923}$
$j_2=5$	$\frac{-2}{99}$	$\frac{74}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{-2}{153153}$	0
$j_2=6$	$\frac{-4}{429}$	0	$\frac{118}{45045}$	0	$\frac{-4}{13923}$	0	$\frac{-2}{188955}$

Table 5.9: Coefficients $\bar{C}_{5j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	$\frac{2}{693}$	0	$\frac{-4}{1287}$	0
$j_2=1$	0	0	$\frac{8}{3465}$	0	$\frac{-10}{9009}$	0	$\frac{-4}{6435}$
$j_2=2$	0	$\frac{4}{1155}$	0	$\frac{-74}{45045}$	0	$\frac{16}{45045}$	0
$j_2=3$	$\frac{8}{693}$	0	$\frac{-116}{45045}$	0	$\frac{2}{9009}$	0	$\frac{8}{58905}$
$j_2=4$	$\frac{2}{99}$	$\frac{-64}{9009}$	0	$\frac{2}{9009}$	0	$\frac{4}{153153}$	0
$j_2=5$	$\frac{2}{1287}$	0	$\frac{-8}{45045}$	0	$\frac{-2}{153153}$	0	$\frac{4}{415701}$
$j_2=6$	$\frac{-2}{143}$	$\frac{4}{715}$	0	$\frac{-226}{765765}$	0	$\frac{-8}{415701}$	0

Table 5.10: Coefficients $\bar{C}_{6j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	0	$\frac{2}{1287}$	0	$\frac{-4}{2145}$
$j_2=1$	0	0	0	$\frac{10}{9009}$	0	$\frac{-4}{6435}$	0
$j_2=2$	0	0	$\frac{4}{3003}$	0	$\frac{-38}{45045}$	0	$\frac{8}{36465}$
$j_2=3$	0	$\frac{20}{9009}$	0	$\frac{-10}{9009}$	0	$\frac{8}{58905}$	0
$j_2=4$	$\frac{10}{1287}$	0	$\frac{-16}{9009}$	0	$\frac{2}{13923}$	0	$\frac{4}{188955}$
$j_2=5$	$\frac{2}{143}$	$\frac{-32}{6435}$	0	$\frac{122}{765765}$	0	$\frac{4}{415701}$	0
$j_2=6$	$\frac{2}{2145}$	0	$\frac{-4}{36465}$	0	$\frac{-2}{188955}$	0	0

Table 5.11: Coefficients $\bar{C}_{00j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{3}$	$\frac{-2}{5}$	$\frac{2}{15}$
$j_2=1$	$\frac{-2}{15}$	$\frac{2}{15}$	$\frac{-2}{21}$
$j_2=2$	$\frac{-2}{15}$	$\frac{2}{35}$	$\frac{2}{105}$

Table 5.12: Coefficients $\bar{C}_{10j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{5}$	$\frac{-2}{9}$	$\frac{2}{35}$
$j_2=1$	$\frac{-2}{45}$	$\frac{2}{35}$	$\frac{-2}{45}$
$j_2=2$	$\frac{-2}{21}$	$\frac{2}{45}$	$\frac{2}{315}$

Table 5.13: Coefficients $\bar{C}_{02j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{15}$	$\frac{2}{21}$	$\frac{-4}{105}$
$j_2=1$	$\frac{2}{35}$	$\frac{-4}{105}$	$\frac{2}{105}$
$j_2=2$	$\frac{4}{105}$	$\frac{-2}{105}$	0

Table 5.14: Coefficients $\bar{C}_{01j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{45}$	$\frac{-2}{105}$	$\frac{2}{315}$
$j_2=2$	$\frac{-2}{35}$	$\frac{2}{63}$	$\frac{-2}{315}$

Table 5.15: Coefficients $\bar{C}_{11j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Assume that $q_1 = 6$. In Tables 5.4–5.10 we have the exact values of coefficients $\bar{C}_{j_3j_2j_1}$ $(j_1, j_2, j_3 = 0, 1, \ldots, 6)$. Here and further in this section the Fourier–Legendre coefficients have been calculated exactly using computer algebra system Derive. Note that in [53], [54] the database with 270,000 exactly calculated Fourier–Legendre coefficients was described. This database was used in the software package, which is written in the Python programming language for the implementation of the numerical schemes (4.79)-(4.83), (4.88)-(4.92).

Calculating the value on the right-hand side of (5.54) for $q_1 = 6$ $(i_1 \ne i_2$, $i_1 \ne i_3$, $i_3 \ne i_2)$, we obtain the following approximate equality

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} \approx 0.01956(T-t)^3.$$

Let us choose, for example, $q_2 = 2$. In Tables 5.11–5.19 we have the exact values of coefficients $\bar{C}_{j_4j_3j_2j_1}$ $(j_1, j_2, j_3, j_4 = 0, 1, 2)$. In the case of pairwise

Table 5.16: Coefficients $\bar{C}_{20j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 5.17: Coefficients $\bar{C}_{21j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{21}$	$\frac{-2}{45}$	$\frac{2}{315}$
$j_2=1$	$\frac{2}{315}$	$\frac{2}{315}$	$\frac{-2}{225}$
$j_2=2$	$\frac{-2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 5.18: Coefficients $\bar{C}_{12j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{35}$	$\frac{2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{63}$	$\frac{-2}{105}$	$\frac{2}{225}$
$j_2=2$	$\frac{2}{105}$	$\frac{-2}{225}$	$\frac{-2}{3465}$

different i_1,i_2,i_3,i_4 we obtain from (5.60) the following approximate equality

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}\approx 0.0236084(T-t)^4. \tag{5.61}$$

Let us analyze the following four approximations of the iterated Itô stochastic integrals (see (5.35)–(5.39))

$$I_{(001)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.62}$$

$$I_{(010)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{010}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.63}$$

$$I_{(100)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{100}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.64}$$

Table 5.19: Coefficients $\bar{C}_{22j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{105}$	$\frac{-2}{315}$	0
$j_2=1$	$\frac{2}{315}$	0	$\frac{-2}{1155}$
$j_2=2$	0	$\frac{2}{3465}$	0

$$I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr). \tag{5.65}$$

Assume that $q_3=2$, $q_4=1$. In Tables 5.20–5.36 we have the exact values of Fourier–Legendre coefficients $\bar{C}_{j_3j_2j_1}^{001}$, $\bar{C}_{j_3j_2j_1}^{010}$, $\bar{C}_{j_3j_2j_1}^{100}$ $(j_1,j_2,j_3=0,1,2)$, $\bar{C}_{j_5j_4j_3j_2j_1}$ $(j_1,\ldots,j_5=0,1)$.

In the case of pairwise different $i_1,\ldots,i_5$ from Tables 5.20–5.36 we obtain

Table 5.20: Coefficients $\bar{C}_{0j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	-2	$\frac{14}{15}$	$\frac{-2}{15}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{15}$	$\frac{6}{35}$
$j_2 = 2$	$\frac{2}{5}$	$\frac{-22}{105}$	$\frac{-2}{105}$

Table 5.21: Coefficients $\bar{C}_{1j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-6}{5}$	$\frac{22}{45}$	$\frac{-2}{105}$
$j_2 = 1$	$\frac{-2}{9}$	$\frac{-2}{105}$	$\frac{26}{315}$
$j_2 = 2$	$\frac{22}{105}$	$\frac{-38}{315}$	$\frac{-2}{315}$

Table 5.22: Coefficients $\bar{C}_{2j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{5}$	$\frac{2}{21}$	$\frac{4}{105}$
$j_2 = 1$	$\frac{-22}{105}$	$\frac{4}{105}$	$\frac{2}{105}$
$j_2 = 2$	0	$\frac{-2}{105}$	0

Table 5.23: Coefficients $\bar{C}_{0j_2j_1}^{100}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{3}$	$\frac{2}{15}$	$\frac{2}{15}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{45}$	$\frac{2}{35}$
$j_2 = 2$	$\frac{2}{15}$	$\frac{-2}{35}$	$\frac{-4}{105}$

Table 5.24: Coefficients $\bar{C}_{1j_2j_1}^{100}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{5}$	$\frac{2}{45}$	$\frac{2}{21}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2 = 2$	$\frac{2}{35}$	$\frac{-2}{63}$	$\frac{-2}{105}$

Table 5.25: Coefficients $\bar{C}_{2j_2j_1}^{100}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2=1$	$\frac{-2}{21}$	$\frac{-2}{315}$	$\frac{2}{105}$
$j_2=2$	$\frac{-2}{105}$	$\frac{-2}{315}$	0

Table 5.26: Coefficients $\bar{C}_{0j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{3}$	$\frac{8}{15}$	0
$j_2=1$	$\frac{-4}{15}$	0	$\frac{8}{105}$
$j_2=2$	$\frac{4}{15}$	$\frac{-16}{105}$	0

Table 5.27: Coefficients $\bar{C}_{1j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{5}$	$\frac{4}{15}$	$\frac{4}{105}$
$j_2=1$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2=2$	$\frac{4}{35}$	$\frac{-8}{105}$	0

Table 5.28: Coefficients $\bar{C}_{2j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2=1$	$\frac{-4}{21}$	$\frac{4}{105}$	$\frac{4}{315}$
$j_2=2$	$\frac{-4}{105}$	0	0

Table 5.29: Coefficients $\bar{C}_{000j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{15}$	$\frac{-8}{45}$
$j_2=1$	$\frac{-4}{45}$	$\frac{8}{105}$

Table 5.30: Coefficients $\bar{C}_{010j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{45}$	$\frac{-16}{315}$
$j_2=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.31: Coefficients $\bar{C}_{110j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{105}$	$\frac{-2}{45}$
$j_2=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.32: Coefficients $\bar{C}_{011j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{315}$	$\frac{-4}{315}$
$j_2=1$	0	$\frac{2}{945}$

Table 5.33: Coefficients $\bar{C}_{001j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	0	$\frac{4}{315}$
$j_2=1$	$\frac{8}{315}$	$\frac{-2}{105}$

Table 5.34: Coefficients $\bar{C}_{100j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{45}$	$\frac{-4}{35}$
$j_2=1$	$\frac{-16}{315}$	$\frac{2}{45}$

Table 5.35: Coefficients $\bar{C}_{101j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{315}$	0
$j_2=1$	$\frac{4}{315}$	$\frac{-8}{945}$

Table 5.36: Coefficients $\bar{C}_{111j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{2}{105}$	$\frac{-8}{945}$
$j_2=1$	$\frac{2}{945}$	0

$$
\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=
$$

$$
=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{100}\right)^2\approx 0.00815429(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=
$$

$$
=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{010}\right)^2\approx 0.01739030(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=
$$

$$
=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{001}\right)^2\approx 0.02528010(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}=
$$

$$
=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.
$$

Note that from Theorem 1.4 for $k=5$ we have

$$
\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^2\right),
$$

where $i_1,\ldots,i_5=1,\ldots,m$.

Moreover, from Theorem 1.4 we obtain the following useful estimates

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(20))T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(1000)T,t}^{(i_1i_2i_3i_4)}-I_{(1000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0100)T,t}^{(i_1i_2i_3i_4)}-I_{(0100)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0010)T,t}^{(i_1i_2i_3i_4)}-I_{(0010)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0001)T,t}^{(i_1i_2i_3i_4)}-I_{(0001)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right),$$

$$
\mathsf{M}\left\{\left(I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)}-I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)q}\right)^2\right\}\le
$$

$$
\le 720\left(\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q} C_{j_6 j_5 j_4 j_3 j_2 j_1}^2\right).
$$

In addition, from Theorem 1.3 for $k=2$ we get

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_2)}-I_{(10)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{10}\right)^2-\sum_{j_1,j_2=0}^{q} C_{j_2 j_1}^{10} C_{j_1 j_2}^{10}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_2)}-I_{(10)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{10}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_2)}-I_{(01)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{01}\right)^2-\sum_{j_1,j_2=0}^{q} C_{j_2 j_1}^{01} C_{j_1 j_2}^{01}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_2)}-I_{(01)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{01}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1 i_2)}-I_{(20)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2-\sum_{j_1,j_2=0}^{q} C_{j_2 j_1}^{20} C_{j_1 j_2}^{20}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1 i_2)}-I_{(20)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)}-I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
= \frac{(T-t)^6}{18} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2j_1}^{11}\right)^2 - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}^{11} C_{j_1j_2}^{11} \quad (i_1 = i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)} - I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\} = \frac{(T-t)^6}{18} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2j_1}^{11}\right)^2 \quad (i_1 \ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(02)}^{(i_1i_2)} - I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\} =
$$

$$
= \frac{(T-t)^6}{6} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2j_1}^{02}\right)^2 - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}^{02} C_{j_1j_2}^{02} \quad (i_1 = i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)} - I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\} = \frac{(T-t)^6}{6} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2j_1}^{02}\right)^2 \quad (i_1 \ne i_2).
$$

Clearly, expansions for iterated Stratonovich stochastic integrals (see Theorems 1.1, 2.1–2.9, 2.33–2.36, 2.50, 2.51, 2.62–2.65) are simpler than expansions for iterated Itô stochastic integrals (see Theorems 1.1, 1.2, 1.16 and (1.45)–(1.51)). However, the calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Itô stochastic integrals. Below we consider how we can estimate or calculate exactly (for some particular cases) the mean-square approximation error for iterated Stratonovich stochastic integrals.

Consider the iterated Stratonovich stochastic integral of multiplicity 2

$$
J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_1)} \quad (i_1 = 1, \ldots, m),
$$

where $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions on $[t, T]$.

By Theorem 2.2 we have

$$
J^*[\psi^{(2)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_1)}.
$$

Consider the following approximation of the stochastic integral $J^*[\psi^{(2)}]_{T,t}$

$$
J^*[\psi^{(2)}]_{T,t}^q = \sum_{j_1,j_2=0}^{q} C_{j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_1)}.
$$

According to the standard relation between Stratonovich and Itô stochastic integrals (see (2.478)) and (1.96), we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-J^*[\psi^{(2)}]_{T,t}^q\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}+\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q+\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q\right)^2\right\}+\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2=
$$

$$
=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}C_{j_1j_2}+
$$

$$
+\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2,
$$

where

$$
J[\psi^{(2)}]_{T,t}^q=\sum_{j_1,j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\sum_{j_1=0}^{q}C_{j_1j_1}
$$

is the approximation (see (1.46)) of the iterated Itô stochastic integral

$$
J[\psi^{(2)}]_{T,t}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}\quad(i_1=1,\ldots,m).
$$

It is not difficult to see that the value

$$
\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-J^*[\psi^{(2)}]_{T,t}^q\right)^2\right\}
$$

is greater than the value

$$
\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q\right)^2\right\}
$$

by

$$E_q^{(i_1)}=\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2.$$

For some particular cases $E_q^{(i_1)}=0$. For example, for the case $\psi_1(\tau),\ \psi_2(\tau)\equiv 1$ ($\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$) we have

$$\sum_{j_1=0}^{q}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{q}\left(C_{j_1}\right)^2=\frac{1}{2}\left(C_0\right)^2=\frac{1}{2}(T-t)=\frac{1}{2}\int\limits_t^T ds.$$

However, $E_q^{(i_1)}\ne 0$ in a general case.

Consider the following iterated Stratonovich stochastic integral of multiplicity 3

$$I_{(000)T,t}^{*(i_1i_2i_3)}=\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m).$$

Taking into account the standard relations between Itô and Stratonovich stochastic integrals (see (2.399)) and Theorem 1.1 (the case $k=3$) together with Theorem 2.8, we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+I_{(000)T,t}^{(i_1i_2i_3)q}+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}+\right.\right.$$

$$\left.\left.+\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\},\qquad (5.66)$$

where the approximations $I_{(000)T,t}^{*(i_1i_2i_3)q},\ I_{(000)T,t}^{(i_1i_2i_3)q}$ are defined by the relations (see (5.17), (5.18))

$$
I_{(000)T,t}^{(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.67}
$$

$$
I_{(000)T,t}^{*(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{5.68}
$$

Substituting (5.67) and (5.68) into (5.66) yields

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)} - I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)q} + \mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right) + \right.\right.
$$

$$
\left.\left. +\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau - \sum_{j_1,j_3=0}^{q} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right) - \mathbf{1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{q} C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \tag{5.69}
$$

$$
\le 4\left(\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\} + \mathbf{1}_{\{i_1=i_2\}}F_q^{(i_3)} + \mathbf{1}_{\{i_2=i_3\}}G_q^{(i_1)} + \mathbf{1}_{\{i_1=i_3\}}H_q^{(i_2)}\right), \tag{5.70}
$$

where

$$
F_q^{(i_3)} = \mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}, \tag{5.71}
$$

$$
G_q^{(i_1)} = \mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau - \sum_{j_1,j_3=0}^{q} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}, \tag{5.72}
$$

$$
H_q^{(i_2)} = \mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q} C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \tag{5.73}
$$

In the cases of Legendre polynomials or trigonometric functions, we have (see Theorem 2.8) the equalities

$$
\lim_{q\to\infty} F_q^{(i_3)} = 0, \quad \lim_{q\to\infty} G_q^{(i_1)} = 0, \quad \lim_{q\to\infty} H_q^{(i_2)} = 0.
$$

However, in accordance with (5.70) the value

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}
$$

with a finite q can be estimated by terms of a rather complex structure (see (5.71)-(5.73)). As is easily observed, this peculiarity will also apply to the iterated Stratonovich stochastic integrals of multiplicities $k\ge 4$ with the only difference that the number of additional terms like (5.71)-(5.73) will be considerably higher and their structure will be more complicated (the exact calculation of the mean-square error of approximation for iterated Stratonovich stochastic integrals of multiplicities 1 to 4 is presented in Sect. 5.5, 5.6).

Therefore, the payment for a relatively simple approximation of the iterated Stratonovich stochastic integrals (Theorems 2.1–2.9, 2.30, 2.33–2.36, 2.50, 2.51, 2.62–2.65) in comparison with the iterated Itô stochastic integrals (Theorems 1.1, 1.2, 1.16) is a much more difficult calculation or estimation procedure of their mean-square approximation errors.

As we mentioned above, on the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated Stratonovich stochastic integrals ($T-t\ll 1$ because the length $T-t$ of integration interval $[t,T]$ of the iterated Stratonovich stochastic integrals plays the role of integration step for the numerical methods for Itô SDEs, i.e. $T-t$ is already fairly small). This leads to a sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals which are required for achieving the acceptable accuracy of approximation.

From (5.41) ($i_1\ne i_2$) we obtain

$$
\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le
$$

$$
\le\frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx=-\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{(T-t)^2}{q},\quad (5.74)
$$

where constant C_1 does not depend on q.

It is easy to notice that for a sufficiently small $T-t$ (recall that $T-t\ll 1$ since it is a step of integration for the numerical schemes for Itô SDEs) there exists a constant C_2 such that

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C_2\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\},\qquad(5.75)$$

where $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}$ is an approximation of the iterated Stratonovich stochastic integral $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$.

From (5.74) and (5.75) we finally obtain

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C\frac{(T-t)^2}{q},\qquad(5.76)$$

where constant C is independent of $T-t$.

The same idea can be found in [84] in the framework of the method of approximation of iterated Stratonovich stochastic integrals based on the trigonometric expansion of the Brownian bridge process. Note that, in contrast to the estimate (5.76), the constant C in Theorems 2.38–2.40 does not depend on p.

We can get more information about the numbers q (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since for pairwise different $i_1,\ldots,i_k=1,\ldots,m$

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}\quad \text{w. p. 1},$$

where $J[\psi^{(k)}]_{T,t}$, $J^*[\psi^{(k)}]_{T,t}$ are defined by (5.1) and (5.2) correspondingly, then for pairwise different $i_1,\ldots,i_6=1,\ldots,m$ from Theorem 1.3 we obtain

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2,\qquad(5.77)$$

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2,\qquad(5.78)$$

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{*(i_1i_2i_3)}-I_{(100)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(010))T,t}-I^{*(i_1i_2i_3)q}_{(010)T,t}\right)^2\right\}=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{010}_{j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(001)T,t}-I^{*(i_1i_2i_3)q}_{(001)T,t}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{001}_{j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4i_5)}_{(00000)T,t}-I^{*(i_1i_2i_3i_4i_5)q}_{(00000)T,t}\right)^2\right\}=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C^2_{j_5i_4i_3i_2j_1},$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(20)T,t}-I^{*(i_1i_2)q}_{(20)T,t}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C^{20}_{j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(11)T,t}-I^{*(i_1i_2)q}_{(11)T,t}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C^{11}_{j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(02)T,t}-I^{*(i_1i_2)q}_{(02)T,t}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C^{02}_{j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(1000)T,t}-I^{*(i_1i_2i_3i_4)q}_{(1000)T,t}\right)^2\right\}=\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C^{1000}_{j_4j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0100)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0100)T,t}\right)^2\right\}=\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C^{0100}_{j_4j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0010)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0010)T,t}\right)^2\right\}=\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C^{0010}_{j_4j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0001)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0001)T,t}\right)^2\right\}=\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C^{0001}_{j_4j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4i_5i_6)}_{(000000)T,t}-I^{*(i_1i_2i_3i_4i_5i_6)q}_{(000000)T,t}\right)^2\right\}=\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C^2_{j_6j_5j_4j_3j_2j_1}.$$

Recall that the systems of iterated stochastic integrals (5.1)–(5.4) are part of the Taylor–Itô and Taylor–Stratonovich expansions (see Chapter 4).

The function $K(t_1,\ldots,t_k)$ from Theorem 1.1 for the set (5.3) is defined by

$$K(t_1,\ldots,t_k)=(t-t_k)^{l_k}\ldots(t-t_1)^{l_1}\ \mathbf{1}_{\{t_1<\ldots<t_k\}},\quad t_1,\ldots,t_k\in[t,T],\quad (5.79)$$

where $\mathbf{1}_A$ is the indicator of the set A.

In particular, for the stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$, $I_{(00)T,t}^{(i_1i_2)}$, $I_{(000)T,t}^{(i_1i_2i_3)}$, $I_{(01)T,t}^{(i_1i_2)}$, $I_{(10)T,t}^{(i_1i_2)}$, $I_{(0000)T,t}^{(i_1\ldots i_4)}$, $I_{(20)T,t}^{(i_1i_2)}$, $I_{(11)T,t}^{(i_1i_2)}$, $I_{(02)T,t}^{(i_1i_2)}$ $(i_1,\ldots,i_4=1,\ldots,m)$ the functions $K(t_1,\ldots,t_k)$ defined by (5.79) look as follows

$$K_1(t_1)=t-t_1,\quad K_2(t_1)=(t-t_1)^2,\quad K_{00}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2\}}, \tag{5.80}$$

$$K_{000}(t_1,t_2,t_3)=\mathbf{1}_{\{t_1<t_2<t_3\}},\quad K_{01}(t_1,t_2)=(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{5.81}$$

$$K_{10}(t_1,t_2)=(t-t_1)\mathbf{1}_{\{t_1<t_2\}},\quad K_{0000}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}, \tag{5.82}$$

$$K_{20}(t_1,t_2)=(t-t_1)^2\mathbf{1}_{\{t_1<t_2\}},\quad K_{11}(t_1,t_2)=(t-t_1)(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{5.83}$$

$$K_{02}(t_1,t_2)=(t-t_2)^2\mathbf{1}_{\{t_1<t_2\}}, \tag{5.84}$$

where $t_1,\ldots,t_4\in[t,T]$.

It is obviously that the most simple expansion for the polynomial of a finite degree into the Fourier series using the complete orthonormal system of functions in the space $L_2([t,T])$ will be its Fourier–Legendre expansion (finite sum). The polynomial functions are included in the functions (5.80)–(5.84) as their components if $l_1^2+\ldots+l_k^2>0$. So, it is logical to expect that the most simple expansions for the functions (5.80)–(5.84) into generalized multiple Fourier series will be Fourier–Legendre expansions of these functions when $l_1^2+\ldots+l_k^2>0$. Note that the given assumption is confirmed completely (compare the formulas (5.8), (5.9) with the formulas (5.85), (5.90) (see below) correspondently). So, usage of Legendre polynomials for the approximation of iterated Itô and Stratonovich stochastic integrals is a step forward.

5.2 Mean-Square Approximation of Specific Iterated Stratonovich Stochastic Integrals of multiplicities 1 to 3 Based on Trigonometric System of Functions

In [1]-[17], [50] on the base of Theorems 1.1, 2.2, 2.6, and 2.8 the author obtained (also see early publications [76] (1997), [77] (1998), [80] (1994), [81] (1996)) the following expansions of the iterated Stratonovich stochastic integrals (5.4) (independently from the papers [82]-[85], [92] excepting the method, in which the additional random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ are introduced)

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{*(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \qquad (5.85)$$

$$I_{(00)T,t}^{*(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr), \qquad (5.86)$$

$$I_{(000)T,t}^{*(i_1i_2i_3)q}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\right.$$

$$\left.+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+D_{T,t}^{(i_1i_2i_3)q}\Biggr), \qquad (5.87)$$

where

$$D_{T,t}^{(i_1i_2i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\neq l}}^{q}\Biggl(\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+$$

$$
+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{4\sqrt{2}\pi^2}\Biggl(\sum_{r,m=1}^{q}\Biggl(\frac{2}{rm}\left(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\right.
$$

$$
\left.+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{m(r+m)}\left(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)\Biggr)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl(\frac{1}{m(l-m)}\left(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{l(l-m)}\left(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)\Biggr)\Biggr),
$$

$$
I_{(10)T,t}^{*(i_1i_2)q}=-(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\Biggr)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\Biggr)-
$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\neq l}}^{q}\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\Biggr)+
$$

$$
+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr),
\tag{5.88}
$$

$$I_{(01)T,t}^{*(i_1i_2)q}=(T-t)^2\Biggl(-\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-2\xi_q^{(i_2)}\zeta_0^{(i_1)}\right)+$$
$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)-$$
$$-\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}\right)-\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$
$$+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr)-$$
$$-\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)-\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),\qquad(5.89)$$

$$I_{(2)T,t}^{*(i_1)q}=(T-t)^{5/2}\Biggl(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\Biggr)-$$
$$-\frac{1}{\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr),\qquad(5.90)$$

where

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},\quad \mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},$$

$$\beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},\quad \zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)},$$

where $\phi_j(s)$ is defined by (1.69) and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,$ $i=1,\ldots,m)$ are independent standard Gaussian random variables $(i_1,i_2,i_3=1,\ldots,m)$.

Note that (5.88), (5.89) imply the following

$$\sum_{j=0}^{\infty}C_{jj}^{10}=\sum_{j=0}^{\infty}C_{jj}^{01}=-\frac{(T-t)^2}{4},\qquad(5.91)$$

where

$$C_{jj}^{10} = \int\limits_t^T \phi_j(x) \int\limits_t^x \phi_j(y)(t-y)dydx,$$

$$C_{jj}^{01} = \int\limits_t^T \phi_j(x)(t-x) \int\limits_t^x \phi_j(y)dydx.$$

Note that the formulas (5.91) are particular cases of the more general relation (2.10), which has been applied for the proof of Theorems 2.1–2.3.

Let us consider the mean-square errors of approximations (5.86)–(5.89). From the relations (5.86)–(5.89) when $i_1 \ne i_2$, $i_2 \ne i_3$, $i_1 \ne i_3$ by direct calculation we obtain

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)} - I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^q \frac{1}{r^2}\right), \tag{5.92}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = (T-t)^3\left(\frac{1}{4\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^q \frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{55}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^q \frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{r,l=1\\ r\ne l}}^{\infty} - \sum_{\substack{r,l=1\\ r\ne l}}^{q}\right)\frac{5l^4 + 4r^4 - 3l^2r^2}{r^2 l^2 (r^2 - l^2)^2}\right), \tag{5.93}$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)} - I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\} = (T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^q \frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{5}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^q \frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty} - \sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2 + k^2}{k^2 (l^2 - k^2)^2}\right), \tag{5.94}$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)} - I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\} = (T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^q \frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{5}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^q \frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty} - \sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2 + k^2}{l^2 (l^2 - k^2)^2}\right). \tag{5.95}$$

It is easy to demonstrate that the relations (5.93), (5.94), and (5.95) can be represented using Theorem 1.3 in the following form

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=(T-t)^3\Biggl(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

$$
-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\Biggr), \tag{5.96}
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)}-I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\Biggl(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

$$
-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\Biggr), \tag{5.97}
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)}-I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\Biggl(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

$$
-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}\Biggr). \tag{5.98}
$$

Comparing (5.96)–(5.98) and (5.93)–(5.95), we note that

$$
\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}=\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}\frac{l^2+k^2}{l^2\left(l^2-k^2\right)^2}=\frac{\pi^4}{48}, \tag{5.99}
$$

$$
\sum_{\substack{r,l=1\\ r\ne l}}^{\infty}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}=\frac{9\pi^4}{80}. \tag{5.100}
$$

Let us consider approximations of stochastic integrals $I_{(10)T,t}^{*(i_1 i_1)}$, $I_{(01)T,t}^{*(i_1 i_1)}$ and conditions for selecting number q using the trigonometric system of functions

$$
I_{(10)T,t}^{*(i_1 i_1)q}=-(T-t)^2\Biggl(\frac{1}{6}\left(\zeta_0^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-
$$

$$
-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{\pi^2 r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)-
$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr),
$$

$$
I_{(01)T,t}^{*(i_1 i_1)q}=(T-t)^2\Biggl(-\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-
$$

$$
-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{\pi^2 r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr).
$$

Furthermore, we have

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_1)}-I_{(01)T,t}^{*(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_1)}-I_{(10)T,t}^{*(i_1 i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{2}{\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{\pi^4}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^2+
$$

$$
+\frac{1}{\pi^4}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{\infty}-\sum_{\substack{k,l=1\\ k\neq l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr). \tag{5.101}
$$

Considering (5.99), we can rewrite the relation (5.101) in the following form

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_1)}-I_{(01)T,t}^{*(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_1)}-I_{(10)T,t}^{*(i_1 i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{17}{240}-\frac{1}{3\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\frac{2}{\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}+
$$

Table 5.37: Confirmation of the formula (5.96)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.38: Confirmation of the formulas (5.97), (5.98)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

$$
+\frac{1}{\pi^4}\left(\sum_{r=1}^{q}\frac{1}{r^2}\right)^2-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr). \tag{5.102}
$$

In Tables 5.37–5.39 we confirm numerically the formulas (5.96)–(5.98), (5.102) for various values of q. In Tables 5.37–5.39 the number ε means right-hand sides of the mentioned formulas. Obviously, these results are consistent with the estimate (1.225).

The formulas (5.99), (5.100) appear to be interesting. Let us confirm numerically their correctness in Tables 5.40 and 5.41 (the number ε_q is an absolute deviation of multiple partial sums with the upper limit of summation q for the series (5.99), (5.100) from the right-hand sides of the formulas (5.99), (5.100); convergence of multiple series is regarded here when $p_1 = p_2 = q \to \infty$, which is acceptable according to Theorems 1.1, 2.2, 2.6, and 2.8).

Using the trigonometric system of functions, let us consider approximations of iterated stochastic integrals of the following form

$$
J^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)T,t}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where $\lambda_l = 1$ if $i_l = 1,\ldots,m$ and $\lambda_l = 0$ if $i_l = 0$, $l = 1,\ldots,k$ ($\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$).

Table 5.39: Confirmation of the formula (5.102)

$4\varepsilon/(T-t)^4$	0.0268	0.0034	$3.3955\cdot 10^{-4}$	$3.3804\cdot 10^{-5}$	$3.3778\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.40: Confirmation of the formula (5.99)

ε_q	2.0294	0.3241	0.0330	0.0033	$3.2902\cdot 10^{-4}$
q	1	10	100	1000	10000

Table 5.41: Confirmation of the formula (5.100)

ε_q	10.9585	1.8836	0.1968	0.0197	0.0020
q	1	10	100	1000	10000

It is easy to see that the approximations

$$J^{*(i_1 i_2)q}_{(\lambda_1\lambda_2)T,t},\qquad J^{*(i_1 i_2 i_3)q}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

of the stochastic integrals

$$J^{*(i_1 i_2)}_{(\lambda_1\lambda_2)T,t},\qquad J^{*(i_1 i_2 i_3)}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

are defined by the right-hand sides of the formulas (5.86), (5.87), where it is necessary to take

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{5.103}$$

and $i_1,i_2,i_3=0,1,\ldots,m.$

Since

$$\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(0)}=\begin{cases}\sqrt{T-t} & \text{if } j=0\\ \\ 0 & \text{if } j\ne 0\end{cases},$$

then it is easy to get from (5.86) and (5.87), considering that in these equalities $\zeta_j^{(i)}$ is defined by (5.103) and $i_1,i_2,i_3=0,1,\ldots,m$, the following family of formulas

$$J^{*(i_1 0)q}_{(10)T,t}=\frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.104}$$

$$J^{*(0 i_2)q}_{(01)T,t}=\frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_2)}+\sqrt{\alpha_q}\xi_q^{(i_2)}\right)\right), \tag{5.105}$$

$$J_{(001)T,t}^{*(00i_3)q}=(T-t)^{5/2}\Biggl(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_3)}+\sqrt{\beta_q}\mu_q^{(i_3)}\Biggr)-$$

$$-\frac{1}{2\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_3)}+\sqrt{\alpha_q}\xi_q^{(i_3)}\Biggr)\Biggr),$$

$$J_{(010)T,t}^{*(0i_20)q}=(T-t)^{5/2}\Biggl(\frac{1}{6}\zeta_0^{(i_2)}-\frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_2)}+\sqrt{\beta_q}\mu_q^{(i_2)}\Biggr)\Biggr),$$

$$J_{(100)T,t}^{*(i_100)q}=(T-t)^{5/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\Biggr)+$$

$$+\frac{1}{2\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr),$$

$$J_{(011)T,t}^{*(0i_2i_3)q}=(T-t)^{2}\Biggl(\frac{1}{6}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_3)}\zeta_0^{(i_2)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_3)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_3)}\Biggr)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\right)\Biggr)-$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_2)}\zeta_{2l}^{(i_3)}+\frac{l}{r}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_2)}\zeta_{2r-1}^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\right)+$$

$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr), \qquad (5.106)$$

$$J_{(110)T,t}^{*(i_1i_20)q}=(T-t)^{2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$

$$+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_1)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+$$

$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),$$

$$J_{(101)T,t}^{*(i_10i_3)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}\right)\Biggr)-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_3)}-$$

$$-\sum_{r=1}^{q}\frac{1}{4\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\right)\Biggr).$$

5.3 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Numerical Solution of Itô SDEs

The section is devoted to comparative analysis of efficiency of application the Legendre polynomials and trigonometric functions for the numerical integration of Itô SDEs in the framework of the method of approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series (Theorems 1.1, 2.1–2.9, 2.30, 2.33–2.36, 2.50, 2.51, 2.62–2.65). This section is written on the base of the papers [21], [40], and [32] (Sect. 4).

Using the iterated Itô stochastic integrals of multiplicities 1 to 3 appearing in the Taylor–Itô expansion as an example, it is shown that their expansions obtained using multiple Fourier–Legendre series are significantly simpler and less computationally costly than their analogues obtained on the basis of multiple trigonometric Fourier series.

Let us consider the following set of iterated Itô and Stratonovich stochastic integrals from the classical Taylor–Itô and Taylor–Stratonovich expansions [84]

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)}, \tag{5.107}$$

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)}, \tag{5.108}$$

where ${\bf w}_{\tau}^{(i)}={\bf f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and ${\bf w}_{\tau}^{(0)}=\tau,$ $i_1,\ldots,i_k=0,1,\ldots,m,$ $\lambda_l=0$ for $i_l=0$ and $\lambda_l=1$ for $i_l=1,\ldots,m$ $(l=1,\ldots,k)$.

In [82] Milstein G.N. obtained the following expansion of $J_{(11)T,t}^{(i_1i_2)}$ on the base of the Karhunen–Loève expansion of the Brownian bridge process (we will discuss the method [82] in detail in Sect. 6.2)

$$J_{(11)T,t}^{(i_1i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{5.109}$$

where the series converges in the mean-square sense, $i_1\ne i_2,$ $i_1,i_2=1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d{\bf f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,\ldots,m,\ j=0,1,\ldots)$,

$$\phi_j(s)=\frac{1}{\sqrt{T-t}}\begin{cases}1 & \text{for } j=0\\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)) & \text{for } j=2r-1,\\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)) & \text{for } j=2r\end{cases} \tag{5.110}$$

where $r=1,2,\ldots$

Moreover,

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

where $i_1=1,\ldots,m.$

In principle, for implementing the strong numerical method with the convergence order 1.0 (Milstein method [82], see Sect. 4.10) for Itô SDEs we can take the following approximations

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.111}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{5.112}$$

where $i_1\ne i_2,\ i_1,i_2=1,\ldots,m.$

It is not difficult to show that

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{5.113}$$

However, this approach has an obvious drawback. Indeed, we have too complex formulas for the stochastic integrals with Gaussian distribution

$$J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{5.114}$$

$$J_{(001)T,t}^{(00i_1)}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

where the meaning of notations used in (5.112) is retained.

In [82] Milstein G.N. proposed the following mean-square approximations on the base of (5.109), (5.114)

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),\tag{5.115}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\right),\tag{5.116}$$

where $i_1\neq i_2$ in (5.116), and

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},\tag{5.117}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $r=1,\ldots,q$, $i=1,\ldots,m$ are independent standard Gaussian random variables.

Obviously, for the approximations (5.115) and (5.116) we obtain [82]

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,$$

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right).\tag{5.118}$$

This idea has been developed in [83]-[85]. For example, the approximation $J_{(001)T,t}^{(00i_1)q}$, which corresponds to (5.115), (5.116) is defined by [83]-[85]

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.$$

$$\left.-\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),\tag{5.119}$$

where $\xi_q^{(i)}$, α_q have the form (5.117),

$$\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},$$

$\phi_j(s)$ is defined by (5.110), and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,\ i=1,\ldots,m)$ are independent standard Gaussian random variables.

Moreover,

$$\mathsf{M}\left\{\left(J_{(001)T,t}^{(00i_1)}-J_{(001)T,t}^{(00i_1)q}\right)^2\right\}=0.$$

Nevetheless, the expansions (5.115), (5.119) are too complex for the approximation of two Gaussian random variables $J_{(01)T,t}^{(0i_1)}$, $J_{(001)T,t}^{(00i_1)}$.

Further, we will see that the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of stochastic integral $J_{(111)T,t}^{(i_1i_2i_3)}$ $(i_1,i_2,i_3=1,\ldots,m)$. This is due to the fact that the number q is fixed for stochastic integrals included into the considered collection. However, it is clear that due to the smallness of $T-t$, the number q for $J_{(111)T,t}^{(i_1i_2i_3)}$ could be taken significantly less than in the formula (5.116). This feature is also valid for the formulas (5.115), (5.119).

On the other hand, the following very simple formulas are well known (see (5.7)–(5.9))

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},\tag{5.120}$$

$$J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),\tag{5.121}$$

$$J_{(001)T,t}^{(00i_1)}=\frac{(T-t)^{5/2}}{6}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),\tag{5.122}$$

where $\zeta_0^{(i)}$, $\zeta_1^{(i)}$, $\zeta_2^{(i)}$ $(i=1,\ldots,m)$ are indepentent standard Gaussian random variables. Obviously, that the formulas (5.120)-(5.122) are part of the method based on Theorem 1.1 (also see Sect. 5.1).

To obtain the Milstein expansion for the stochastic integral (5.2) the truncated expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals in (5.2), and the integrals must be calculated starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (5.2) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple integrals (5.2) were obtained [82]-[85], [92], [93] by the Milstein approach [82] based on the Karhunen–Loève expansion of the Brownian bridge process.

At that, in [82], [92] the case $\psi_1(s)$, $\psi_2(s) \equiv 1$ and $i_1, i_2 = 0, 1, \ldots, m$ $(i_1 \neq i_2)$ is considered. In [83]-[85], [93] the attempt to consider the case $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s) \equiv 1$ and $i_1, i_2, i_3 = 0, 1, \ldots, m$ is realized. Note that, generally speaking, the mean-square convergence of $J^{*(i_1 i_2 i_3)q}_{(111)T,t}$ to $J^{*(i_1 i_2 i_3)}_{(111)T,t}$ if $q \to \infty$ was not proved rigorously in [83]-[85], [93] within the frames of the Milstein approach [82] together with the Wong–Zakai approximation [73]-[75] (see discussions in Sect. 2.41, 2.42, 6.2).

5.3.1 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integral $J^{(i_1 i_2)}_{(11)T,t}$

Using Theorem 1.1 and complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$, we have (see (5.11))

$$
J^{(i_1 i_2)}_{(11)T,t} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.123}
$$

where series converges in the mean-square sense, $i_1, i_2 = 1, \ldots, m$,

$$
\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{f}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j,

$$
\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \tag{5.124}
$$

where $P_j(x)$ is the Legendre polynomial.

The formula (5.123) has been derived for the first time in [76] (1997) with using Theorem 2.10.

Remind the formula (5.41) [76] (1997)

$$
\mathsf{M}\left\{\left(J^{(i_1 i_2)}_{(11)T,t} - J^{(i_1 i_2)q}_{(11)T,t}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{5.125}
$$

Table 5.42: Numbers q_{trig}, q^*_{trig}, q_{pol}

$T-t$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}	2^{-11}	2^{-12}
q_{trig}	3	4	7	14	27	53	105	209
q^*_{trig}	6	11	20	40	79	157	312	624
q_{pol}	5	9	17	33	65	129	257	513

where

$$J_{(11)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right). \tag{5.126}$$

Let us compare (5.126) with (5.116) and (5.125) with (5.118). Consider minimal natural numbers q_{trig} and q_{pol}, which satisfy to (see Table 5.42)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q_{\text{pol}}}\frac{1}{4i^2-1}\right) \le (T-t)^3, \tag{5.127}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q_{\text{trig}}}\frac{1}{r^2}\right) \le (T-t)^3.$$

Thus, we have

$$\frac{q_{\text{pol}}}{q_{\text{trig}}} \approx \quad 1.67, \quad 2.22, \quad 2.43, \quad 2.36, \quad 2.41, \quad 2.43, \quad 2.45, \quad 2.45.$$

From the other hand, the formula (5.116) includes $(4q+4)m$ independent standard Gaussian random variables. At the same time the folmula (5.126) includes only $(2q+2)m$ independent standard Gaussian random variables. Moreover, the formula (5.126) is simpler than the formula (5.116). Thus, in this case we can talk about approximately equal computational costs for the formulas (5.116) and (5.126).

There is one important feature. As we mentioned above, further we will see that the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of stochastic integral $J_{(111)T,t}^{(i_1 i_2 i_3)}$ $(i_1, i_2, i_3 = 1, \ldots, m)$. This is due to the fact that the number q is fixed for all stochastic integrals, which included into the considered collection. However, it is clear that due to the smallness of $T-t$, the number q for $J_{(111)T,t}^{(i_1 i_2 i_3)}$ could be chosen significantly less than in the formula (5.116). This feature is also valid for the formulas (5.115), (5.119). However, for the case of Legendre polynomials we can choose different numbers q for different stochastic integrals.

From the other hand, if we will not introduce the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1 i_2)}$ will be three times larger (see (5.113)). Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0 i_1)}$, $J_{(001)T,t}^{(00 i_1)}$ (with Gaussian distribution) will be approximated worse.

Consider minimal natural numbers $q_{\rm trig}^{*}$, which satisfy to (see Table 5.42)

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}^{*}}\frac{1}{r^2}\right)\le (T-t)^3.$$

In this situation we can talk about the advantage of Legendre polynomials ($q_{\rm trig}^{*} > q_{pol}$ and (5.116) is more complex than (5.126)).

5.3.2 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integrals $J_{(1)T,t}^{(i_1)}$, $J_{(11)T,t}^{(i_1 i_2)}$, $J_{(01)T,t}^{(0 i_1)}$, $J_{(10)T,t}^{(i_1 0)}$, $J_{(111)T,t}^{(i_1 i_2 i_3)}$

It is well known [82]-[85], [92] (also see [14]-[17]) that for the numerical realization of strong Taylor–Itô numerical methods with the convergence order 1.5 for Itô SDEs we need to approximate the following collection of iterated Itô stochastic integrals (see Sect. 4.10)

$$J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1 i_2)},\quad J_{(01)T,t}^{(0 i_1)},\quad J_{(10)T,t}^{(i_1 0)},\quad J_{(111)T,t}^{(i_1 i_2 i_3)}.$$

Using Theorem 1.1 for the system of trigonometric functions, we have (see Sect. 5.2)

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{5.128}$$

$$J_{(11)T,t}^{(i_1 i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$
$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+$$
$$+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\Biggr), \tag{5.129}$$

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.130}$$

$$J_{(10)T,t}^{(i_10)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.131}$$

$$J_{(111)T,t}^{(i_1i_2i_3)q}=(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$

$$\left.+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\right)+$$

$$+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$

$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\right.$$

$$\left.\left.\left.+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\right)+D_{T,t}^{(i_1i_2i_3)q}\right), \tag{5.132}$$

where in (5.132) we suppose that $i_1\ne i_2$, $i_1\ne i_3$, $i_2\ne i_3$,

$$D_{T,t}^{(i_1i_2i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\left(\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+\right.\right.$$

$$\left.\left.+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)+$$

$$+\frac{1}{4\sqrt{2}\pi^2}\left(\sum_{r,m=1}^{q}\left(\frac{2}{rm}\left(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\right.\right.\right.$$

$$
+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{m(r+m)}\Biggl(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)\Biggr)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl(\frac{1}{m(l-m)}\Biggl(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{l(l-m)}\Biggl(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)\Biggr)\Biggr),
$$

where

$$
\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},
$$

$$
\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},
$$

and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,\ i=1,\ldots,m)$ are independent standard Gaussian random variables.

The mean-square errors of approximations (5.129)–(5.132) are represented by the formulas

$$
\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,
$$

$$
\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=0,
$$

$$
\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{5.133}
$$

Table 5.43: Confirmation of the formula (5.134)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

$$
{\sf M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)}-J_{(111)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\}=(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right), \tag{5.134}
$$

where $i_1\ne i_2$, $i_1\ne i_3$, $i_2\ne i_3$.

In Table 5.43 we can see the numerical confirmation of the formula (5.134) (ε means the right-hand side of (5.134)).

Note that the formulas (5.128), (5.129) have been obtained for the first time in [82]. Using (5.128), (5.129), we can realize numerically an explicit one-step strong numerical method with the convergence order 1.0 for Itô SDEs (Milstein method [82]; also see Sect. 4.10).

An analogue of the formula (5.132) has been obtained for the first time in [83], [84].

As we mentioned above, the Milstein expansion (i.e. expansion based on the Karhunen–Loève expansion of the Brownian bridge process) for iterated stochastic integrals leads to iterated application of the operation of limit transition. An analogue of (5.132) for iterated Stratonovich stochastic integrals has been derived in [83], [84] on the base of the Milstein expansion together with the Wong–Zakai approximation [73]-[75] (without rigorous proof). It means that the authors in [83], [84] formally could not use the double sum with the upper limit q in the analogue of (5.132). From the other hand, the correctness of (5.132) follows directly from Theorem 1.1. Note that (5.132) has been obtained reasonably for the first time in [1]. The version of (5.132) but without the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ can be found in [76] (1997).

Note that the formula (5.133) appears for the first time in [82]. The mean-square error (5.134) has been obtained for the first time in [81] (1996) on the base of the simplified variant of Theorem 1.1 (the case of pairwise different $i_1,\ldots,i_k$).

The number q as we noted above must be the same in (5.129)–(5.132). This is the main drawback of this approach, because really the number q in (5.132) can be chosen essentially smaller than in (5.129).

Note that in (5.132) we can replace $J_{(111)T,t}^{(i_1 i_2 i_3)q}$ with $J_{(111)T,t}^{*(i_1 i_2 i_3)q}$ and (5.132) then will be valid for any $i_1, i_2, i_3 = 0, 1, \ldots, m$ (see Theorems 2.6–2.8).

Consider now approximations of iterated stochastic integrals

$$
J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1 i_2)},\quad J_{(01)T,t}^{(0 i_1)},\quad J_{(10)T,t}^{(i_1 0)},\quad J_{(111)T,t}^{(i_1 i_2 i_3)}\quad (i_1, i_2, i_3 = 1, \ldots, m)
$$

on the base of Theorem 1.1 (the case of Legendre polynomials) [1]-[17], [32]

$$
J_{(1)T,t}^{(i_1)} = \sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{5.135}
$$

$$
J_{(11)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.136}
$$

$$
J_{(01)T,t}^{(0 i_1)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.137}
$$

$$
J_{(10)T,t}^{(i_1 0)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.138}
$$

$$
\begin{aligned}
J_{(111)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \\
&-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad q_1 \ll q,
\end{aligned} \tag{5.139}
$$

$$
J_{(111)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right),
$$

where

$$
\begin{aligned}
C_{j_3 j_2 j_1} &= \int\limits_t^T \phi_{j_3}(z)\int\limits_t^z \phi_{j_2}(y)\int\limits_t^y \phi_{j_1}(x)dxdydz = \\
&= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},
\end{aligned}
$$

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz, \tag{5.140}$$

$\phi_j(x)$ is defined by (5.124) and $P_i(x)$ is the Legendre polynomial ($i = 0, 1, 2, \ldots$).

The mean-square errors of approximations (5.136), (5.139) are represented by the formulas (see Theorems 1.3 and 1.4; also see Sect. 5.1)

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)} - J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \quad (i_1 \ne i_2), \tag{5.141}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)} - J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}^2 \quad (i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3), \tag{5.142}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)} - J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_2j_3j_1}C_{j_3j_2j_1} \ (i_1 \ne i_2 = i_3), \tag{5.143}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)} - J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \ne i_2), \tag{5.144}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)} - J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \ne i_3), \tag{5.145}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)} - J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3j_2j_1}^2\right), \tag{5.146}$$

where $i_1, i_2, i_3 = 1, \ldots, m$ in (5.146).

Let us compare the efficiency of application of Legendre polynomials and trigonometric functions for the approximation of iterated stochastic integrals $J_{(11)T,t}^{(i_1 i_2)}$, $J_{(111)T,t}^{(i_1 i_2 i_3)}$.

Consider the following conditions $(i_1 \ne i_2,\ i_1 \ne i_3,\ i_2 \ne i_3)$

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le (T-t)^4, \tag{5.147}$$

$$(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}\frac{\left(C_{j_3 j_2 j_1}\right)^2}{(T-t)^3}\right) \le (T-t)^4, \tag{5.148}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p}\frac{1}{r^2}\right) \le (T-t)^4, \tag{5.149}$$

$$(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{p_1}\frac{1}{r^2}-\frac{55}{32\pi^4}\sum_{r=1}^{p_1}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{p_1}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right) \le (T-t)^4, \tag{5.150}$$

where

$$C_{j_3 j_2 j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_i(x)$ is the Legendre polynomial.

In Tables 5.44 and 5.45 we can see the minimal numbers q, q_1, p, p_1, which satisfy the conditions (5.147)–(5.150). As we mentioned above, the numbers q, q_1 are different. At that $q_1 \ll q$ (the case of Legendre polynomials). As we saw in the previous sections, we cannot take different numbers p, p_1 for the case of trigonometric functions. Thus, we should choose $q = p$ in (5.129)–(5.132). This leads to huge computational costs (see the fairly complicated formula (5.132)).

From the other hand, we can take different numbers q in (5.129)–(5.132). At that we should exclude random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (5.129)–(5.132). At this situation for the case $i_1 \ne i_2$, $i_2 \ne i_3$, $i_1 \ne i_3$ we have

Table 5.44: Numbers q, q_1

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 5.45: Numbers p, p_1, p^*, p_1^*

$T-t$	0.08222	0.05020	0.02310	0.01956
p	8	21	96	133
p_1	1	1	3	4
p^*	23	61	286	398
p_1^*	1	2	4	5

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p^*}\frac{1}{r^2}\right)\le (T-t)^4, \tag{5.151}$$

$$(T-t)^3\left(\frac{5}{36}-\frac{1}{2\pi^2}\sum_{r=1}^{p_1^*}\frac{1}{r^2}-\frac{79}{32\pi^4}\sum_{r=1}^{p_1^*}\frac{1}{r^4}-\right.$$
$$\left.-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{p_1^*}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \tag{5.152}$$

where the left-hand sides of (5.151), (5.152) correspond to (5.129), (5.132) but without $\xi_q^{(i)}$, $\mu_q^{(i)}$. In Table 5.45 we can see minimal numbers p^*, p_1^*, which satisfy the conditions (5.151), (5.152).

Moreover,

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=$$

Table 5.46: Confirmation of the formula (5.152)

$\varepsilon/(T-t)^3$	0.0629	0.0097	0.0010	$1.0129\cdot 10^{-4}$	$1.0132\cdot 10^{-5}$
q	1	10	100	1000	10000

$$=\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)\neq 0, \tag{5.153}$$

where $J_{(01)T,t}^{(0i_1)q}$, $J_{(10)T,t}^{(i_10)q}$ are defined by (5.130), (5.131) but without $\xi_q^{(i)}$.

It is not difficult to see that the numbers $q_{\rm trig}$ in Table 5.42 correspond to minimal numbers $q_{\rm trig}$, which satisfy the condition (compare with (5.153))

$$\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^4.$$

From the other hand, the right-hand sides of (5.137), (5.138) include only 2 random variables. In this situation we again can talk about the advantage of Ledendre polynomials.

In Table 5.46 we can see the numerical confirmation of the formula (5.152) (ε means the left-hand side of (5.152)).

5.3.3 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integral $J_{(011)T,t}^{*(0i_1i_2)}$

In this section, we compare computational costs for approximation of the iterated Stratonovich stochastic integral $J_{(011)T,t}^{*(0i_1i_2)}$ $(i_1,i_2=1,\ldots,m)$ within the framework of the method of generalized multiple Fourier series for the Legendre polynomial system and the system of trigomomenric functions.

Using Theorem 2.1 for the case of trigonometric system of functions, we obtain [6]-[17], [40]

$$J_{(011)T,t}^{*(0i_1i_2)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right)-$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right)+$$

Table 5.47: Confirmation of the formula (5.155)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.48: Confirmation of the formula (5.157)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450\cdot 10^{-4}$	$6.2495\cdot 10^{-5}$
q	1	10	100	1000	10000

$$
+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+
$$

$$
+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr). \tag{5.154}
$$

For the case $i_1\ne i_2$ from Theorem 1.3 we get [6]-[18], [31], [40]

$$
\mathsf{M}\left\{\left(J_{(011)T,t}^{*(0i_1i_2)}-J_{(011)T,t}^{*(0i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\Biggl(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

$$
-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\Biggr). \tag{5.155}
$$

Analogues of the formulas (5.154), (5.155) for the case of Legendre polynomials will look as follows [6]-[18], [31], [40]

$$
J_{(011)T,t}^{*(0i_1i_2)q}=\frac{T-t}{2}J_{(11)T,t}^{*(i_1i_2)q}+\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+
$$

$$
+\sum_{i=0}^{q}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr), \tag{5.156}
$$

where

$$
J_{(11)T,t}^{*(i_1i_2)q}=\frac{T-t}{2}\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\Biggr),
$$

$$
\mathsf{M}\left\{\left(J_{(011)T,t}^{*(0i_1i_2)}-J_{(011)T,t}^{*(0i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.
$$

$$
\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right), \qquad (5.157)
$$

where $i_1\ne i_2$.

In Tables 5.47 and 5.48 we can see the numerical confirmation of the formulas (5.155) and (5.157) (ε means the right-hand side of (5.155) or (5.157)).

Let us compare the complexity of the formulas (5.154) and (5.156). The formula (5.154) includes the double sum

$$
\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right).
$$

Thus, the formula (5.154) is more complex, than the formula (5.156) even if we take identical numbers q in these formulas. As we noted above, the number q in (5.154) must be equal to the number q from the formula (5.129), so it is much larger than the number q from the formula (5.156). As a result, we have obvious advantage of the formula (5.156) in computational costs.

As we mentioned above, if we will not introduce the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the number q in (5.154) can be chosen smaller, but the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1i_2)}$ will be three times larger (see (5.113)). Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0i_1)}$, $J_{(10)T,t}^{(i_10)}$, $J_{(001)T,t}^{(00i_1)}$ (with Gaussian distribution) will be approximated worse. In this situation, we can again talk about the advantage of Ledendre polynomials.

5.3.4 Conclusions

Summing up the results of previous sections we can come to the following conclusions.

1. We can talk about approximately equal computational costs for the formulas (5.129) and (5.136). This means that computational costs for realizing the Milstein scheme (explicit one-step strong numerical method with the convergence order $\gamma=1.0$ for Itô SDEs; see Sect. 4.10) for the case of Legendre

polynomials and for the case of trigonometric functions are approximately the same.

2. If we will not introduce the random variables $\xi_q^{(i)}$ (see (5.129)), then the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1 i_2)}$ will be three times larger (see (5.113)). In this situation, we can talk about the advantage of Ledendre polynomials in the Milstein method. Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0i_1)}$, $J_{(10)T,t}^{(i_1 0)}$, $J_{(001)T,t}^{(00i_1)}$ (with Gaussian distribution) will be approximated worse.

3. If we talk about the explicit one-step strong numerical scheme with the convergence order $\gamma = 1.5$ for Itô SDEs (see Sect. 4.10), then the numbers q, q_1 (see (5.136), (5.139)) are different. At that $q_1 \ll q$ (the case of Legendre polynomials). The number q must be the same in (5.129)–(5.132) (the case of trigonometric functions). This leads to huge computational costs (see the fairly complicated formula (5.132)). From the other hand, we can take different numbers q in (5.129)–(5.132). At that we should exclude the random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (5.129)–(5.132). This leads to another problems which we discussed above (see Conclusion 2).

4. In addition, the author supposes that effect described in Conclusion 3 will be more impressive when analyzing more complex sets of iterated Itô and Stratonovich stochastic integrals (when $\gamma = 2.0, 2.5, 3.0, \ldots$). This supposition is based on the fact that the polynomial system of functions has the significant advantage (in comparison with the trigonometric system) when approximating the iterated stochastic integrals for which not all weight functions are equal to 1 (see Sect 5.4.3 and conclusion at the end of Sect. 5.1).

5.4 Optimization of the Mean-Square Approximation Procedures for Iterated Itô Stochastic Integrals Based on Theorem 1.1 and Multiple Fourier–Legendre Series

This section is devoted to optimization of the mean-square approximation procedures for iterated Itô stochastic integrals (5.3) of multiplicities 1 to 4 based on Theorem 1.1 and multiple Fourier–Legendre series. The mentioned stochastic integrals are part of strong numerical methods with convergence orders 1.0, 1.5, and 2.0 for Itô SDEs with multidimensional non-commutative noise (see (4.79)–(4.81)). We show that the lengths of sequences of independent standard

Gaussian random variables required for the mean-square approximation of iterated Itô stochastic integrals (5.3) can be significantly reduced without the loss of the mean-square accuracy of approximation for these stochastic integrals. This section is written on the base of paper [56]. An extension of the mentioned results to iterated Itô stochastic integrals of multiplicity 5 can be found in [55].

Using Theorem 1.1 and the system of Legendre polynomials, we obtain the following approximations of iterated Itô stochastic integrals (5.3)

$$
I_{(0)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},
$$

$$
I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),
$$

$$
I_{(00)T,t}^{(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.158}
$$

$$
I_{(000)T,t}^{(i_1i_2i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.159}
$$

$$
I_{(10)T,t}^{(i_1i_2)q_2}=\sum_{j_1,j_2=0}^{q_2}C_{j_2j_1}^{10}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{5.160}
$$

$$
I_{(01)T,t}^{(i_1i_2)\bar q_2}=\sum_{j_1,j_2=0}^{\bar q_2}C_{j_2j_1}^{01}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{5.161}
$$

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)q_3}=\sum_{j_1,j_2,j_3,j_4=0}^{q_3}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$\left.+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right), \qquad (5.162)$$

where $\mathbf{1}_A$ is the indicator of the set A,

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)} \quad (i=1,\ldots,m,\ j=0,1,\ldots)$$

are independent standard Gaussian random variables for various i or j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ (see (5.5)),

$$C_{j_3j_2j_1}=\frac{1}{8}L_{j_1j_2j_3}(T-t)^{3/2}\bar{C}_{j_3j_2j_1}, \quad C_{j_2j_1}^{01}=\frac{1}{8}L_{j_1j_2}(T-t)^{2}\bar{C}_{j_2j_1}^{01}, \quad (5.163)$$

$$C_{j_2j_1}^{10}=\frac{1}{8}L_{j_1j_2}(T-t)^{2}\bar{C}_{j_2j_1}^{10}, \quad C_{j_4j_3j_2j_1}=\frac{1}{16}L_{j_1j_2j_3j_4}(T-t)^{2}\bar{C}_{j_4j_3j_2j_1}, \quad (5.164)$$

$$L_{j_1j_2}=\sqrt{(2j_1+1)(2j_2+1)}, \quad L_{j_1j_2j_3}=\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)},$$

$$L_{j_1j_2j_3j_4}=\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_2j_1}^{01}=-\int\limits_{-1}^{1}(1+y)P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2j_1}^{10}=-\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}(1+x)P_{j_1}(x)dxdy,$$

$P_j(x)$ is the Legendre polynomial (see (5.6)).

Combining the estimates (4.84) and (1.129) for $p_1 = \ldots = p_k = p$, we obtain

$$k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1,\ldots,j_k=0}^{p} C^2_{j_k\ldots j_1}\right) \le C(T-t)^{r+1}, \quad (5.165)$$

where $K(t_1,\ldots,t_k)$ is defined by (5.79) (see (5.80)–(5.82)), $r/2$ is the strong convergence orders for the numerical schemes (4.79)–(4.81), i.e. $r/2 = 1.0, 1.5,$ and 2.0; constant C is independent of $T-t$.

It is not difficult to see that the multiplier factor $k!$ on the left-hand side of the inequality (5.165) leads to a significant increase of computational costs for approximation of iterated Itô stochastic integrals. The mentioned problem can be overcome if we calculate the mean-square approximation error for iterated Itô stochastic integrals exactly (see Theorem 1.3 and Sect. 1.2.3). In this section, we discuss how to essentially minimize the numbers q, q_1, q_2, $\bar q_2$, q_3 from (5.158)–(5.162). At that we will use the results from Sect. 1.2.3.

Denote

$$E_p^{(l_1\ldots l_k)} \stackrel{\mathrm{def}}{=} \mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)} - I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}\right)^2\right\}, \quad (5.166)$$

where $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (5.3) and $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$ is the mean-square approximation of this stochastic integral. More precisely, the approximations $I_{(00)T,t}^{(i_1 i_2)q}$, $I_{(000)T,t}^{(i_1 i_2 i_3)q_1}$, $I_{(10)T,t}^{(i_1 i_2)q_2}$, $I_{(01)T,t}^{(i_1 i_2)\bar q_2}$, $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_3}$ are defined by (5.158)-(5.162).

The results of Sect. 1.2.3 give the following formulas for the case of Legendre polynomials

$$E_q^{(00)} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \quad i_1 \ne i_2, \quad (5.167)$$

$$E_{q_{1,1}}^{(000)} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_{1,1}} L^2_{j_1 j_2 j_3}\left(\bar C_{j_3 j_2 j_1}\right)^2\right), \quad (5.168)$$

where $i_1 \ne i_2,\ i_1 \ne i_3,\ i_2 \ne i_3$,

$$E_{q_{1,2}}^{(000)} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_{1,2}} L^2_{j_1 j_2 j_3}\left(\left(\bar C_{j_3 j_2 j_1}\right)^2 + \bar C_{j_3 j_1 j_2}\bar C_{j_3 j_2 j_1}\right)\right), \quad (5.169)$$

where $i_1 = i_2 \ne i_3$,

$$E_{q_{1,3}}^{(000)} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_{1,3}} L^2_{j_1 j_2 j_3}\left(\left(\bar C_{j_3 j_2 j_1}\right)^2 + \bar C_{j_2 j_3 j_1}\bar C_{j_3 j_2 j_1}\right)\right), \quad (5.170)$$

where $i_1 \neq i_2 = i_3$,

$$E_{q_{1,4}}^{(000)} = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{q_{1,4}} L_{j_1 j_2 j_3}^2 \left(\left(\bar{C}_{j_3 j_2 j_1} \right)^2 + \bar{C}_{j_3 j_2 j_1} \bar{C}_{j_1 j_2 j_3} \right) \right), \quad (5.171)$$

where $i_1 = i_3 \neq i_2$,

$$E_{q_{2,1}}^{(10)} = (T-t)^4 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1,j_2=0}^{q_{2,1}} L_{j_1 j_2}^2 \left(\bar{C}_{j_2 j_1}^{10} \right)^2 \right), \quad i_1 \neq i_2, \quad (5.172)$$

$$E_{q_{2,2}}^{(10)} = (T-t)^4 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1,j_2=0}^{q_{2,2}} L_{j_1 j_2}^2 \bar{C}_{j_2 j_1}^{10} \left(\sum_{(j_1,j_2)} \bar{C}_{j_2 j_1}^{10} \right) \right), \quad i_1 = i_2, \quad (5.173)$$

$$E_{\bar{q}_{2,1}}^{(01)} = (T-t)^4 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1,j_2=0}^{\bar{q}_{2,1}} L_{j_1 j_2}^2 \left(\bar{C}_{j_2 j_1}^{01} \right)^2 \right), \quad i_1 \neq i_2, \quad (5.174)$$

$$E_{\bar{q}_{2,2}}^{(01)} = (T-t)^4 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1,j_2=0}^{\bar{q}_{2,2}} L_{j_1 j_2}^2 \bar{C}_{j_2 j_1}^{01} \left(\sum_{(j_1,j_2)} \bar{C}_{j_2 j_1}^{01} \right) \right), \quad i_1 = i_2, \quad (5.175)$$

$$E_{q_{3,1}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,1}} L_{j_1 \ldots j_4}^2 \left(\bar{C}_{j_4 \ldots j_1} \right)^2 \right), \quad (5.176)$$

where $i_1, \ldots, i_4$ are pairwise different,

$$E_{q_{3,2}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,2}} L_{j_1 \ldots j_4}^2 \bar{C}_{j_4 \ldots j_1} \left(\sum_{(j_1,j_2)} \bar{C}_{j_4 \ldots j_1} \right) \right), \quad (5.177)$$

where $i_1 = i_2 \neq i_3, i_4;\ i_3 \neq i_4$,

$$E_{q_{3,3}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,3}} L_{j_1 \ldots j_4}^2 \bar{C}_{j_4 \ldots j_1} \left(\sum_{(j_1,j_3)} \bar{C}_{j_4 \ldots j_1} \right) \right), \quad (5.178)$$

where $i_1 = i_3 \neq i_2, i_4;\ i_2 \neq i_4$,

$$E_{q_{3,4}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,4}} L_{j_1 \ldots j_4}^2 \bar{C}_{j_4 \ldots j_1} \left(\sum_{(j_1,j_4)} \bar{C}_{j_4 \ldots j_1} \right) \right), \quad (5.179)$$

where $i_1 = i_4 \ne i_2, i_3;\ \ i_2 \ne i_3,$

$$E_{q_{3,5}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,5}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_2,j_3)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.180)$$

where $i_2 = i_3 \ne i_1, i_4;\ \ i_1 \ne i_4,$

$$E_{q_{3,6}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,6}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_2,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.181)$$

where $i_2 = i_4 \ne i_1, i_3;\ \ i_1 \ne i_3,$

$$E_{q_{3,7}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,7}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_3,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.182)$$

where $i_3 = i_4 \ne i_1, i_2;\ \ i_1 \ne i_2,$

$$E_{q_{3,8}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,8}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.183)$$

where $i_1 = i_2 = i_3 \ne i_4,$

$$E_{q_{3,9}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,9}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_2,j_3,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.184)$$

where $i_2 = i_3 = i_4 \ne i_1,$

$$E_{q_{3,10}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,10}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.185)$$

where $i_1 = i_2 = i_4 \ne i_3,$

$$E_{q_{3,11}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,11}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_3,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right), \qquad (5.186)$$

where $i_1 = i_3 = i_4 \ne i_2,$

$$E_{q_{3,12}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,12}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} \bar{C}_{j_4\ldots j_1}\right)\right)\right), \qquad (5.187)$$

where $i_1 = i_2 \neq i_3 = i_4$,

$$E_{q_{3,13}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,13}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} \bar{C}_{j_4\ldots j_1} \right)\right)\right), \tag{5.188}$$

where $i_1 = i_3 \neq i_2 = i_4$,

$$E_{q_{3,14}}^{(0000)} = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{q_{3,14}} L_{j_1\ldots j_4}^2 \bar{C}_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} \bar{C}_{j_4\ldots j_1} \right)\right)\right), \tag{5.189}$$

where $i_1 = i_4 \neq i_2 = i_3$.

Obviously, the conditions (5.167)–(5.189) do not contain the multiplier factors 2!, 3!, and 4! in contrast to the estimate (1.129) (see Theorem 1.4). However, the number of the mentioned conditions is quite large, which is inconvenient for practice. In this section, we propose the hypothesis [53]-[56] that all the formulas (5.167)–(5.189) can be replaced by the formulas (5.167), (5.168), (5.172), (5.174), (5.176) in which we can suppose that $i_1, i_2, i_3, i_4 = 1, \ldots, m$. At that we will not have a noticeable loss of the mean-square approximation accuracy of iterated Itô stochastic integrals.

It should be noted that unlike the method based on Theorem 1.1, existing approaches to the mean-square approximation of iterated stochastic integrals based on Fourier series (see, for example, [82]-[85], [92], [96]) do not allow to choose different numbers p (see (5.166)) for approximations of different iterated stochastic integrals with multiplicities $k = 2, 3, 4, \ldots$ Moreover, the noted approaches exclude the possibility for obtaining of approximate and exact expressions similar to (1.76), (1.129) (see Theorems 1.3, 1.4). The detailed comparison of Theorem 1.1 with methods from [82]-[85], [92]-[94], [96], [97], [99] is given in Chapter 6 of this monograph.

Consider the following conditions

$$E_q^{(00)} \leq (T-t)^4, \quad E_{q_{1,i}}^{(000)} \leq (T-t)^4, \quad i = 1, \ldots, 4, \tag{5.190}$$

and

$$E_q^{(00)} \leq (T-t)^5, \quad E_{q_{1,i}}^{(000)} \leq (T-t)^5, \quad E_{q_{2,j}}^{(10)} \leq (T-t)^5, \tag{5.191}$$

$$E_{\bar{q}_{2,j}}^{(01)} \leq (T-t)^5, \quad E_{q_{3,k}}^{(0000)} \leq (T-t)^5, \tag{5.192}$$

where $i = 1, \ldots, 4;\ \ j = 1, 2;\ \ k = 1, \ldots, 14.$

Let us show by numerical experiments that in most situations the following inequalities are fulfilled (under conditions (5.190) and (5.191), (5.192))

$$q_{1,1} \ge q_{1,i}, \quad i = 2, 3, 4, \tag{5.193}$$

$$q_{2,1} \ge q_{2,2}, \quad \bar{q}_{2,1} \ge \bar{q}_{2,2}, \tag{5.194}$$

$$q_{3,1} \ge q_{3,k}, \quad k = 2, \ldots, 14, \tag{5.195}$$

where $q_{1,i},\ q_{2,j},\ \bar{q}_{2,j},\ q_{3,k}$ $(i = 1, \ldots, 4;\ \ j = 1, 2;\ \ k = 1, \ldots, 14)$ are minimal natural numbers satisfying the conditions (5.190) and (5.191), (5.192).

In Tables 5.49–5.56 we can see the results of numerical experiments. These results confirm the hypothesis proposed earlier in this section. Note that in Tables 5.54–5.56 we calculate the mean-square approximation errors of iterated Itô stochastic integrals in the case when

$$q_{1,i} = q_{1,1}, \quad i = 2, 3, 4,$$

$$q_{2,2} = q_{2,1}, \quad \bar{q}_{2,2} = \bar{q}_{2,1},$$

$$q_{3,k} = q_{3,1}, \quad k = 2, \ldots, 14,$$

where $q_{1,1},\ q_{2,1},\ \bar{q}_{2,1},\ q_{3,1}$ are minimal natural numbers satisfying the conditions (5.190) and (5.191), (5.192). In this case, there is no noticeable loss of the mean-square approximation accuracy of iterated Itô stochastic integrals (see Tables 5.54–5.56). This means that all the formulas (5.167)–(5.189) can be replaced by the formulas (5.167), (5.168), (5.172), (5.174), (5.176) in which we can suppose that $i_1, i_2, i_3, i_4 = 1, \ldots, m$.

Let $q_{1,1}$ and $q_{3,1}$ be minimal natural numbers satisfying the conditions

$$E_{q_{1,1}}^{(000)} \le (T - t)^4, \tag{5.196}$$

$$E_{q_{3,1}}^{(0000)} \le (T - t)^5, \tag{5.197}$$

where the left-hand sides of these inequalities are defined by the formulas (5.168) and (5.176), respectively.

Let $p_{1,1}$ and $p_{3,1}$ be minimal natural numbers satisfying the conditions

$$3! \cdot E_{p_{1,1}}^{(000)} \le (T - t)^4, \tag{5.198}$$

$$4! \cdot E_{p_{3,1}}^{(0000)} \le (T - t)^5, \tag{5.199}$$

Table 5.49: Conditions $E_{q_{1,i}}^{(000)} \le (T-t)^4$, $i=1,\ldots,4$.

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q_{1,1}$	12	16	28	36	47	50
$q_{1,2}$	6	8	14	18	23	25
$q_{1,3}$	6	8	14	18	23	25
$q_{1,4}$	12	16	28	36	47	51

where the values $E_{p_{1,1}}^{(000)}$ and $E_{p_{3,1}}^{(0000)}$ on the left-hand sides of these inequalities are defined by the formulas (5.168) and (5.176), respectively.

In Tables 5.57, 5.58 we can see the numerical comparison of the numbers $q_{1,1}$ and $q_{3,1}$ with the numbers $p_{1,1}$ and $p_{3,1}$, respectively. Obviously, excluding of the multiplier factors 3! and 4! essentially (in many times) reduces the calculation costs for the mean-square approximations of iterated Itô stochastic integrals. Note that in this section we use the exactly calculated Fourier–Legendre coefficients using the Python programming language [53], [54].

As we mentioned above, existing approaches to the mean-square approximation of iterated stochastic integrals based on Fourier series (see, for example, [82]-[85], [92], [96]) do not allow to choose different numbers p (see Theorem 1.3) for approximations of different iterated stochastic integrals with multiplicities $k=2,3,4,\ldots$ and exclude the possibility for obtaining of approximate and exact expressions similar to the formulas (1.76), (1.129) (see Theorems 1.3, 1.4). This leads to unnecessary terms usage in the expansions of iterated Itô stochastic integrals and, as a consequence, to essential increase of computational costs for the implementation of numerical methods for Itô SDEs.

In this section (also see [55], [56]) we have optimized the method based on Theorems 1.1 and 1.3, which makes it possible to correctly choose the lengths of sequences of standard Gaussian random variables required for the approximation of iterated Itô stochastic integrals. Thus, the computational costs for the implementation of numerical methods for Itô SDEs are significantly reduced.

On the base of the obtained results we recommend to use in practice the following conditions (for any $i_1,\ldots,i_4=1,\ldots,m$) for correct choosing the minimal natural numbers q, q_1, q_2, $\bar{q}_2$, q_3

$$E_q^{(00)} \le C(T-t)^3$$

Table 5.50: Conditions $E_{q_{3,k}}^{(0000)} \le (T-t)^5$, $i=1,\ldots,14$.

$T-t$	0.011	0.008	0.0045	0.0042	0.0040
$q_{3,1}$	6	8	14	15	16
$q_{3,2}$	4	5	10	11	11
$q_{3,3}$	6	8	14	15	16
$q_{3,4}$	6	8	14	15	16
$q_{3,5}$	3	5	9	9	10
$q_{3,6}$	6	8	14	15	16
$q_{3,7}$	4	5	10	11	11
$q_{3,8}$	2	3	4	5	5
$q_{3,9}$	2	3	4	5	5
$q_{3,10}$	4	6	10	11	11
$q_{3,11}$	4	6	10	11	11
$q_{3,12}$	2	3	5	6	6
$q_{3,13}$	6	8	14	15	16
$q_{3,14}$	3	5	9	9	10

Table 5.51: The conditions (5.191), (5.192).

$T-t$	0.010	0.005	0.0025
$q_{2,1}$	4	8	16
$q_{2,2}$	1	1	1
$\bar{q}_{2,1}$	4	8	16
$\bar{q}_{2,2}$	1	1	1

Table 5.52: The condition (5.190).

$T-t$	2^{-1}	2^{-3}	2^{-5}	2^{-8}
q	1	8	128	8192
$q_{1,1}$	0	1	4	32
$q_{1,2}$	0	0	2	16
$q_{1,3}$	0	0	2	16
$q_{1,4}$	0	0	4	33

Table 5.53: The conditions (5.191), (5.192).

$T-t$	2^{-1}	2^{-3}	2^{-5}	2^{-8}
q	1	8	64	512
$q_{1,1}$	0	2	4	32
$q_{1,2}$	0	1	4	16
$q_{1,3}$	0	1	4	16
$q_{1,4}$	0	2	8	33
$\bar{q}_{2,1}$	0	0	1	1
$\bar{q}_{2,2}, q_{2,1}, q_{2,2}$	0	0	0	0
$q_{3,1}, \ldots, q_{3,14}$	0	0	0	0

Table 5.54: Values $E_{q_{1,i}}^{(000)} \cdot (T-t)^{-3} \stackrel{\sf def}{=} E_{q_{1,i}}$, $i=1,\ldots,4$.

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q_{1,1}$	12	16	28	36	47	50
$E_{q_{1,1}}$	0.010154	0.007681	0.004433	0.003456	0.002652	0.002494
$q_{1,2}$	12	16	28	36	47	50
$E_{q_{1,2}}$	0.005077	0.003841	0.002216	0.001728	0.001326	0.001247
$q_{1,3}$	12	16	28	36	47	50
$E_{q_{1,3}}$	0.005077	0.003841	0.002216	0.001728	0.001326	0.001247
$q_{1,4}$	12	16	28	36	47	50
$E_{q_{1,4}}$	0.010308	0.007787	0.004480	0.003488	0.002673	0.002513

Table 5.55: Values $E^{(01)}_{\bar q_{2,j}}\cdot (T-t)^{-4} \stackrel{\sf def}{=} E_{\bar q_{2,j}}$, $E^{(10)}_{q_{2,j}}\cdot (T-t)^{-4} \stackrel{\sf def}{=} E_{q_{2,j}}$, $j=1,2.$

$T-t$	0.010	0.005	0.0025
$\bar q_{2,1}$	4	8	16
$E_{\bar q_{2,1}}$	0.008950	0.004660	0.002383
$\bar q_{2,2}$	4	8	16
$E_{\bar q_{2,2}}$	0.000042	0.000006	0.000001
$q_{2,1}$	4	8	16
$E_{q_{2,1}}$	0.008950	0.004660	0.002383
$q_{2,2}$	4	8	16
$E_{q_{2,2}}$	0.000042	0.000006	0.000001

(for the Milstein scheme (4.79)),

$$E^{(00)}_{q} \le (T-t)^4,\quad E^{(000)}_{q_{1,1}} \le C(T-t)^4$$

(for the strong scheme (4.80) with order 1.5), and

$$E^{(00)}_{q} \le C(T-t)^5,\quad E^{(000)}_{q_{1,1}} \le C(T-t)^5,\quad E^{(10)}_{q_{2,1}} \le C(T-t)^5,$$
$$E^{(01)}_{\bar q_{2,1}} \le C(T-t)^5,\quad E^{(0000)}_{q_{3,1}} \le C(T-t)^5$$

(for the strong scheme (4.81) with order 2.0). Here the left-hand sides of the above inequalities are defined by the relations (5.167), (5.168), (5.172), (5.174), (5.176) and C is a constant from the condition (4.84).

Taking into account the results of this section (also see [55]), we recommend to use in practice the following condition (for any $i_1,\ldots,i_k=1,\ldots,m$) on the mean-square approximation accuracy for iterated Itô stochastic integrals

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p} C^2_{j_k\ldots j_1}\le C(T-t)^{r+1},$$

Table 5.56: Values $E_{q_{3,k}}^{(0000)}\cdot(T-t)^{-4}\stackrel{\sf def}{=}E_{q_{3,k}},\ k=1,\ldots,14.$

$T-t$	0.011	0.008	0.0045	0.0042
$q_{3,1}$	6	8	14	15
$E_{q_{3,1}}$	0.009636	0.007425	0.004378	0.004096
$q_{3,2}$	6	8	14	15
$E_{q_{3,2}}$	0.006771	0.005191	0.003041	0.002843
$q_{3,3}$	6	8	14	15
$E_{q_{3,3}}$	0.009722	0.007502	0.004424	0.004139
$q_{3,4}$	6	8	14	15
$E_{q_{3,4}}$	0.009641	0.007427	0.004379	0.004097
$q_{3,5}$	6	8	14	15
$E_{q_{3,5}}$	0.005997	0.004614	0.002720	0.002545
$q_{3,6}$	6	8	14	15
$E_{q_{3,6}}$	0.009722	0.007502	0.004424	0.004139
$q_{3,7}$	6	8	14	15
$E_{q_{3,7}}$	0.006771	0.005191	0.003041	0.002843
$q_{3,8}$	6	8	14	15
$E_{q_{3,8}}$	0.003095	0.002364	0.001379	0.001290
$q_{3,9}$	6	8	14	15
$E_{q_{3,9}}$	0.003095	0.002364	0.001379	0.001290
$q_{3,10}$	6	8	14	15
$E_{q_{3,10}}$	0.006885	0.005282	0.003090	0.002889
$q_{3,11}$	6	8	14	15
$E_{q_{3,11}}$	0.006885	0.005282	0.003090	0.002889
$q_{3,12}$	6	8	14	15
$E_{q_{3,12}}$	0.003690	0.002834	0.001663	0.001555
$q_{3,13}$	6	8	14	15
$E_{q_{3,13}}$	0.009756	0.007545	0.004457	0.004170
$q_{3,14}$	6	8	14	15
$E_{q_{3,14}}$	0.006010	0.004621	0.002722	0.002547

Table 5.57: Comparison of numbers $q_{1,1}$ and $p_{1,1}$.

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$q_{1,1}$	0	0	1	2	4	8
$(q_{1,1}+1)^3$	1	1	8	27	125	729
$p_{1,1}$	1	3	6	12	24	48
$(p_{1,1}+1)^3$	8	64	343	2197	15625	117649

Table 5.58: Comparison of numbers $q_{3,1}$ and $p_{3,1}$.

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$q_{3,1}$	0	0	0	0	0	0
$(q_{3,1}+1)^4$	1	1	1	1	1	1
$p_{3,1}$	3	4	6	9	12	17
$(p_{3,1}+1)^4$	256	625	2401	10000	28561	104976

where $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$ is the iterated Itô stochastic integral (5.3), $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$ is the mean-square approximation of this stochastic integral based on Theorem 1.1 and multiple Fourier–Legendre series, p and $k\in\mathbf{N}$,

$$K(t_1,\ldots,t_k)=(t-t_k)^{l_k}\ldots(t-t_1)^{l_1}\ \mathbf{1}_{\{t_1<\ldots<t_k\}},\quad t_1,\ldots,t_k\in[t,T],$$

$\mathbf{1}_A$ is the indicator of the set A, $l_1,\ldots,l_k=0,1,\ldots,$ C and r have the same meaning as in the formula (4.84).

5.5 Exact Calculation of the Mean-Square Approximation Errors for Iterated Stratonovich Stochastic Integrals $I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{*(i_1)}$, $I_{(00)T,t}^{*(i_1i_2)}$, $I_{(000)T,t}^{*(i_1i_2i_3)}$

Consider the question on the exact calculation of the mean-square approximation errors for the following iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)},\quad I_{(1)T,t}^{*(i_1)},\quad I_{(00)T,t}^{*(i_1i_2)},\quad I_{(000)T,t}^{*(i_1i_2i_3)},\quad i_1,i_2,i_3=1,\ldots,m. \tag{5.200}$$

We assume that the stochastic integrals (5.200) are approximated using Theorems 1.1, 2.1, 2.8 and the Legendre polynomial system. Since $I_{(0)T,t}^{(i_1)}=$

$I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{(i_1)}=I_{(1)T,t}^{*(i_1)}$ w. p. 1, we can use (5.7), (5.8) to approximate the stochastic integrals $I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{*(i_1)}$. In this case, we will have zero mean-square approximation errors.

To approximate the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1i_2)}$ we can use the formula (see (5.10))

$$I_{(00)T,t}^{*(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \quad (5.201)$$

The mean-square approximation error for (5.201) will be determined by the formula (5.41) ($i_1\ne i_2$). For the case $i_1=i_2$ we can use the formula (see (6.75))

$$I_{(00)T,t}^{*(i_1i_1)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\right)^2\quad \text{w. p. 1.}$$

Consider now the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1i_2i_3)}$ of multiplicity 3 ($i_1,i_2,i_3=1,\ldots,m$). For the case of pairwise different i_1,i_2,i_3 we can use the formula (5.77). In the case $i_1=i_2=i_3$, to approximate the stochastic integral $I_{(000)T,t}^{*(i_1i_1i_1)}$, we use the formula (5.19).

Thus, it remains to consider the following three cases

$$i_1=i_2\ne i_3, \quad (5.202)$$

$$i_1\ne i_2=i_3, \quad (5.203)$$

$$i_1=i_3\ne i_2. \quad (5.204)$$

Consider the case (5.202). From (5.69) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_\tau^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \quad (5.205)$$

According to the formulas (1.77), (1.86), the quantity

$$I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}$$

includes only iterated Itô stochastic integrals of multiplicity 3. At the same time, the quantity

$$
\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}
$$

contains only iterated Itô stochastic integrals of multiplicity 1. This means that from (5.205) we get

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(\tau-t)d\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \qquad (5.206)
$$

The relation (1.103) implies that

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-
$$

$$
-\sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_1j_2}C_{j_3j_2j_1}, \qquad (5.207)
$$

where $i_1=i_2\ne i_3$.

We have

$$
\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(\tau-t)d\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}=\frac{1}{4}\int\limits_t^T(\tau-t)^2d\tau-
$$

$$
-\sum_{j_1,j_3=0}^{q} C_{j_3j_1j_1}\int\limits_t^T(\tau-t)\phi_{j_3}(\tau)d\tau+\sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_3j_1j_1}\right)^2, \qquad (5.208)
$$

where $\phi_{j_3}(\tau)$ is the Legendre polynomial defined by (5.5).

According to (2.155), we obtain

$$
\int\limits_t^T(\tau-t)\phi_{j_3}(\tau)d\tau=\frac{(T-t)^{3/2}}{2}\begin{cases}1, & j_3=0\\ 1/\sqrt{3}, & j_3=1.\\ 0, & j_3\ge 2\end{cases} \qquad (5.209)
$$

Combining (5.206)–(5.209), we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{4}-$$

$$-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_1j_2}C_{j_3j_2j_1}-$$

$$-\frac{(T-t)^{3/2}}{2}\sum_{j_1=0}^{q}\left(C_{0j_1j_1}+\frac{1}{\sqrt{3}}C_{1j_1j_1}\right)+$$

$$+\sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_3j_1j_1}\right)^2, \tag{5.210}$$

where $i_1=i_2\ne i_3$.

Consider the case (5.203). From (5.69) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+\frac{1}{2}\int\limits_t^T(T-s)d\mathbf{f}_s^{(i_1)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(T-s)d\mathbf{f}_s^{(i_1)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+$$

$$+\frac{1}{4}\int\limits_t^T(T-s)^2ds-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\int\limits_t^T(T-s)\phi_{j_1}(s)ds+$$

$$
+\sum_{j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_1}\right)^2, \tag{5.211}
$$

where $\phi_{j_1}(\tau)$ is the Legendre polynomial defined by (5.5).

The relation (1.104) implies that

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-
$$

$$
-\sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2-\sum_{j_1,j_2,j_3=0}^{q} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1}, \tag{5.212}
$$

where $i_1 \neq i_2=i_3$.

Moreover,

$$
\int_t^T (T-s)\phi_{j_1}(s)ds=\frac{(T-t)^{3/2}}{2}\begin{cases}1, & j_1=0\\ -1/\sqrt{3}, & j_1=1.\\ 0, & j_1 \geq 2\end{cases} \tag{5.213}
$$

Combining (5.211)–(5.213), we get

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=\frac{(T-t)^3}{4}-
$$

$$
-\sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2-\sum_{j_1,j_2,j_3=0}^{q} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1}-
$$

$$
-\frac{(T-t)^{3/2}}{2}\sum_{j_3=0}^{q}\left(C_{j_3 j_3 0}-\frac{1}{\sqrt{3}} C_{j_3 j_3 1}\right)+
$$

$$
+\sum_{j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_1}\right)^2, \tag{5.214}
$$

where $i_1 \neq i_2=i_3$.

Consider the case (5.204). From (5.69) we obtain

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=
$$

$$
= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} - \sum_{j_1,j_2=0}^{q} C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} + \mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q} C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} + \sum_{j_2=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_1 j_2 j_1}\right)^2. \qquad (5.215)
$$

The relation (1.105) implies that

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} = \frac{(T-t)^3}{6} -
$$

$$
- \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3}, \qquad (5.216)
$$

where $i_1 = i_3 \neq i_2$.

Combining (5.215) and (5.216), we obtain

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{(T-t)^3}{6} -
$$

$$
- \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} +
$$

$$
+ \sum_{j_2=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_1 j_2 j_1}\right)^2, \qquad (5.217)
$$

where $i_1 = i_3 \neq i_2$.

Thus, the exact calculaton of the mean-square approximation error for the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1 i_2 i_3)}$ $(i_1, i_2, i_3 = 1, \ldots, m)$ is given by the formulas (5.77), (5.210), (5.214), and (5.217).

5.6 Exact Calculation of the Mean-Square Approximation Error for Iterated Stratonovich Stochastic Integral $I_{(0000)T,t}^{*(i_1i_2i_3i_4)}$

Consider now the iterated Stratonovich stochastic integral $I_{(0000)T,t}^{*(i_1i_2i_3i_4)}$ of multiplicity 4 $(i_1, i_2, i_3, i_4 = 1, \ldots, m)$. For the case of pairwise different i_1, i_2, i_3, i_4 we can use the formula (5.78). In the case $i_1 = i_2 = i_3 = i_4$, to approximate the iterated stochastic integral $I_{(0000)T,t}^{*(i_1i_1i_1i_1)}$, we use the formula (5.32).

Thus, it remains to consider the following 13 cases

$$i_1 = i_2 \ne i_3, i_4;\ \ i_3 \ne i_4, \tag{5.218}$$

$$i_1 = i_3 \ne i_2, i_4;\ \ i_2 \ne i_4, \tag{5.219}$$

$$i_1 = i_4 \ne i_2, i_3;\ \ i_2 \ne i_3, \tag{5.220}$$

$$i_2 = i_3 \ne i_1, i_4;\ \ i_1 \ne i_4, \tag{5.221}$$

$$i_2 = i_4 \ne i_1, i_3;\ \ i_1 \ne i_3, \tag{5.222}$$

$$i_3 = i_4 \ne i_1, i_2;\ \ i_1 \ne i_2, \tag{5.223}$$

$$i_1 = i_2 = i_3 \ne i_4, \tag{5.224}$$

$$i_2 = i_3 = i_4 \ne i_1, \tag{5.225}$$

$$i_1 = i_2 = i_4 \ne i_3, \tag{5.226}$$

$$i_1 = i_3 = i_4 \ne i_2, \tag{5.227}$$

$$i_1 = i_2 \ne i_3 = i_4, \tag{5.228}$$

$$i_1 = i_3 \ne i_2 = i_4, \tag{5.229}$$

$$i_1 = i_4 \ne i_2 = i_3. \tag{5.230}$$

By analogy with (5.69) and using (2.400), (1.48), we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)} - I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} + \right.\right.$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_4}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2 - I_{(0000)T,t}^{(i_1i_2i_3i_4)q}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_2,j_1=0}^{q}C_{j_2j_1j_2j_1}+
$$

$$
\left.\left.+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2\right\}, \tag{5.231}
$$

where $I_{(0000)T,t}^{(i_1i_2i_3i_4)q}$ is defined by (5.162).

Consider the case (5.218). From (5.231) we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}-\right.\right.
$$

$$
\left.\left.-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}. \tag{5.232}
$$

Note that

$$
\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}+\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_4)d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_3}^{(i_3)}
$$

$$
\tag{5.233}
$$

w. p. 1, where $i_3 \ne i_4$.

According to the formulas (1.77), (1.86), the quantity

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}$$

includes only iterated Itô stochastic integrals of multiplicity 4. At the same time (see (5.233)), the quantity

$$\frac{1}{2}\int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} - \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{p} C_{j_4 j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

contains only iterated Itô stochastic integrals of multiplicity 2. This means that from (5.232) we have

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T \int\limits_t^{t_4} (t_3 - t) d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} - \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\} + \frac{1}{4}\int\limits_t^T \int\limits_t^{t_4} (t_3 - t)^2 dt_3 dt_4 +$$

$$+ \sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\right)^2 -$$

$$- \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3 - t) dt_3 dt_4 =$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\} + \frac{(T-t)^4}{48} + \sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\right)^2 +$$

$$+ \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1} C_{j_4 j_3}^{10}, \tag{5.234}$$

where (see (5.14))

$$C_{j_4 j_3}^{10} = \int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t - t_3) dt_3 dt_4. \tag{5.235}$$

Using (1.110) and (5.234), we finally get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2)}C_{j_4j_3j_2j_1}\right)+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+
$$

$$
+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}C_{j_4j_3}^{10}, \tag{5.236}
$$

where $i_1=i_2\ne i_3,i_4;\ \ i_3\ne i_4$.

Consider the cases (5.219), (5.220) by analogy with the case (5.218) using (1.111), (1.112). We have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3)}C_{j_4j_3j_2j_1}\right)+\sum_{j_4,j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_1j_2j_1}\right)^2,
$$

where $i_1=i_3\ne i_2,i_4$ and $i_2\ne i_4$;

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_4)}C_{j_4j_3j_2j_1}\right)+\sum_{j_3,j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\right)^2,
$$

where $i_1=i_4\ne i_2,i_3$ and $i_2\ne i_3$.

Consider the case (5.221) by analogy with the case (5.218). We have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2-
$$

$$
-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)(t_4-t_1)dt_3dt_4.
$$

Then using (1.113), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_3)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(C_{j_4j_1}^{10}-C_{j_4j_1}^{01}\right),
$$

where $i_2=i_3\neq i_1,i_4$ and $i_1\neq i_4$; $C_{j_4j_1}^{10}$ is defined by (5.235) and

$$
C_{j_4j_1}^{01}=\int\limits_t^T\phi_{j_4}(t_4)(t-t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4. \tag{5.237}
$$

For the case (5.222) by analogy with the case (5.218) and using (1.114), we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_4)}C_{j_4j_3j_2j_1}\right)+\sum_{j_3,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_2j_3j_2j_1}\right)^2,
$$

where $i_2=i_4\neq i_1,i_3$ and $i_1\neq i_3$.

Consider the case (5.223) by analogy with the case (5.218). We have (see Example 3.1 in Sect. 3.6)

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(T-t_2)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_2,j_1=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2-
$$

$$
-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\int\limits_t^T(T-t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2.
$$

Then using (1.115), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_3,j_4)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_2,j_1=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left((T-t)C_{j_2j_1}+C_{j_2j_1}^{01}\right),
$$

where $i_3=i_4\neq i_1,i_2$ and $i_1\neq i_2$; $C_{j_2j_1}^{01}$ is defined by (5.237) and

$$
C_{j_2j_1}=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2.
$$

Consider the case (5.224). From (5.231) we have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_1i_1i_4)}-I_{(0000)T,t}^{*(i_1i_1i_1i_4)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_1i_4)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}+\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)} - I_{(0000)T,t}^{(i_1i_1i_1i_4)q} - \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
\left.\left. - \sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_1)}\zeta_{j_4}^{(i_4)} - \sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}. \tag{5.238}
$$

Furthermore,

$$
\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1d\mathbf{w}_{t_3}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)} + \int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)} =
$$

$$
= \int\limits_t^T\int\limits_t^{t_4}(t_1-t)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)} + \int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)} =
$$

$$
= \int\limits_t^T(t_4-t)\int\limits_t^{t_4} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)} \quad \text{w. p. 1.} \tag{5.239}
$$

From (5.238) and (5.239) we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_1i_1i_4)} - I_{(0000)T,t}^{*(i_1i_1i_1i_4)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_1i_4)} - I_{(0000)T,t}^{(i_1i_1i_1i_4)q}\right)^2\right\} +
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(t_4-t)\int\limits_t^{t_4} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)} -\right.\right.
$$

$$
\left.\left. - \sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2} + C_{j_4j_2j_1j_2} + C_{j_4j_2j_2j_1}\right)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_1i_4)} - I_{(0000)T,t}^{(i_1i_1i_1i_4)q}\right)^2\right\} + \frac{(T-t)^4}{16} +
$$

$$
+ \sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2} + C_{j_4j_2j_1j_2} + C_{j_4j_2j_2j_1}\right)\right)^2 -
$$

$$
- \sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2} + C_{j_4j_2j_1j_2} + C_{j_4j_2j_2j_1}\right)\int\limits_t^T(t_4-t)\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4. \tag{5.240}
$$

Using (1.116) and (5.240), we finally obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{5(T-t)^4}{48} -
$$

$$
- \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4 j_3 j_2 j_1}\right) +
$$

$$
+ \sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4 j_1 j_2 j_2} + C_{j_4 j_2 j_1 j_2} + C_{j_4 j_2 j_2 j_1}\right)\right)^2 +
$$

$$
+ \sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4 j_1 j_2 j_2} + C_{j_4 j_2 j_1 j_2} + C_{j_4 j_2 j_2 j_1}\right) C_{j_4 j_2}^{01},
$$

where $i_1 = i_2 = i_3 \neq i_4$.

Consider the case (5.225). From (5.231) we have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_2 i_2)} - I_{(0000)T,t}^{*(i_1 i_2 i_2 i_2)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_2 i_2)} + \frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 d\mathbf{w}_{t_4}^{(i_2)} + \right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 - I_{(0000)T,t}^{(i_1 i_2 i_2 i_2)q} - \sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_4 j_2 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_2)} -
$$

$$
\left.\left. - \sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_2)} - \sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \tag{5.241}
$$

Moreover,

$$
\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 d\mathbf{w}_{t_4}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 =
$$

$$
= \int\limits_t^T\int\limits_t^{t_4}(t_4 - t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_4}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_4}(T - t_4) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_4}^{(i_2)} =
$$

$$
= \int\limits_t^T\int\limits_t^{t_4}(T - t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_4}^{(i_2)} \quad \text{w. p. 1.} \tag{5.242}
$$

From (5.241) and (5.242) we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_2i_2)}-I_{(0000)T,t}^{*(i_1i_2i_2i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_2)}-I_{(0000)T,t}^{(i_1i_2i_2i_2)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(T-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}-\right.\right.
$$

$$
\left.\left.-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_2)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_2)}-I_{(0000)T,t}^{(i_1i_2i_2i_2)q}\right)^2\right\}+\frac{(T-t)^4}{16}+
$$

$$
+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\right)^2-
$$

$$
-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}(T-t_1)\phi_{j_1}(t_1)dt_1dt_4.
\tag{5.243}
$$

Using (1.117) and (5.243), we finally obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{5(T-t)^4}{48}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_3,j_4)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\right)^2-
$$

$$
-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\left((T-t)C_{j_4j_1}+C_{j_4j_1}^{10}\right),
$$

where $i_2=i_3=i_4\ne i_1$.

For the cases (5.226), (5.227) by analogy with the case (5.225) and using (1.118), (1.119), we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2,j_4)} C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}\left(C_{j_4j_3j_1j_1}+C_{j_1j_3j_4j_1}+C_{j_1j_3j_1j_4}\right)\right)^2+
$$

$$
+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}\left(C_{j_4j_3j_1j_1}+C_{j_1j_3j_4j_1}+C_{j_1j_3j_1j_4}\right)C_{j_4j_3}^{10},
$$

where $i_1=i_2=i_4\ne i_3$;

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_2=0}^{q}\left(\sum_{j_1=0}^{q}\left(C_{j_4j_1j_2j_1}+C_{j_1j_4j_2j_1}+C_{j_1j_1j_2j_4}\right)\right)^2-
$$

$$
-\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q}\left(C_{j_4j_1j_2j_1}+C_{j_1j_4j_2j_1}+C_{j_1j_1j_2j_4}\right)\left((T-t)C_{j_2j_3}+C_{j_2j_3}^{01}\right),
$$

where $i_1=i_3=i_4\ne i_2$.

Let us consider the case (5.228). Using (5.231), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_1i_3i_3)}-I_{(0000)T,t}^{*(i_1i_1i_3i_3)q}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_3i_3)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}+\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3+\frac{(T-t)^2}{8}-I_{(0000)T,t}^{(i_1i_1i_3i_3)q}-
$$

$$
\left.\left.-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}+\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2\right\}=
$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)} - I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)q} + \frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)} - \right.\right.$$

$$- \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)} - \mathbf{1}_{\{j_3=j_4\}}\right) +$$

$$+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3 - \sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}}\right) +$$

$$\left.\left. + \frac{(T-t)^2}{8} - \sum_{j_3,j_1=0}^{q} C_{j_3 j_3 j_1 j_1}\right)^2\right\}. \tag{5.244}$$

Note that

$$\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)} - \mathbf{1}_{\{j_3=j_4\}} =$$

$$= \int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)} + \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_4)d\mathbf{w}_{t_4}^{(i_3)}d\mathbf{w}_{t_3}^{(i_3)}, \tag{5.245}$$

$$\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}} =$$

$$= \int\limits_t^T \phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)} + \int\limits_t^T \phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_2}(t_2)d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_1}^{(i_1)} \tag{5.246}$$

w. p. 1.

The relations (5.244)–(5.246) and Example 3.1 in Sect. 3.6 imply the following

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)} - I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)} - I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)q}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)} - \right.\right.$$

$$\left.\left. - \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)} - \mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} +$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3-\right.\right.
$$

$$
\left.\left.-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+
$$

$$
+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_3i_3)}-I_{(0000)T,t}^{(i_1i_1i_3i_3)q}\right)^2\right\}+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}-\right.\right.
$$

$$
\left.\left.-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(T-t_2)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}-\right.\right.
$$

$$
\left.\left.-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+
$$

$$
+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_3i_3)}-I_{(0000)T,t}^{(i_1i_1i_3i_3)q}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(C_{j_3j_4}^{10}+C_{j_4j_3}^{10}\right)+
$$

$$
+\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+
$$

$$
+\frac{(T-t)^4}{48}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left((T-t)\left(C_{j_1j_2}+C_{j_2j_1}\right)+C_{j_1j_2}^{01}+C_{j_2j_1}^{01}\right)+
$$

$$
+\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+
$$

$$+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q} C_{j_3j_3j_1j_1}\right)^2. \qquad (5.247)$$

Furthermore,

$$\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}-2\left(\sum_{j_3,j_1=0}^{q} C_{j_3j_3j_1j_1}\right)^2+$$

$$+\left(\sum_{j_3,j_1=0}^{q} C_{j_3j_3j_1j_1}\right)^2=$$

$$=\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}-\left(\sum_{j_3,j_1=0}^{q} C_{j_3j_3j_1j_1}\right)^2, \qquad (5.248)$$

$$\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^2\right\}-\left(\sum_{j_1,j_3=0}^{q} C_{j_3j_3j_1j_1}\right)^2. \qquad (5.249)$$

Using (2.324), we get

$$\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}=$$

$$=\left(\sum_{j_3,j_1=0}^{q} C_{j_3j_3j_1j_1}\right)^2+\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q} C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q} C_{j_4j_3j_1j_1}\right)^2+$$

$$+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_4j_4j_1j_1}\right)^2. \qquad (5.250)$$

From (5.248) and (5.250) we have

$$
\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}=
$$

$$
=\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q}C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_4j_1j_1}\right)^2. \quad (5.251)
$$

By analogy with (5.251) we obtain

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}=
$$

$$
=\sum_{j_2=0}^{q}\sum_{j_1=0}^{j_2-1}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_1j_2}+\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2+2\sum_{j_2=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_2}\right)^2. \quad (5.252)
$$

Combining (1.120), (5.247), (5.251), and (5.252), we finally have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{12}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)}C_{j_4j_3j_2j_1}\right)\right)+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(C_{j_3j_4}^{10}+C_{j_4j_3}^{10}\right)+
$$

$$
+\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q}C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_4j_1j_1}\right)^2-
$$

$$
-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left((T-t)C_{j_1}C_{j_2}+C_{j_1j_2}^{01}+C_{j_2j_1}^{01}\right)+
$$

$$
+\sum_{j_2=0}^{q}\sum_{j_1=0}^{j_2-1}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_1j_2}+\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2+2\sum_{j_2=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_2}\right)^2+
$$

$$
+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2,
$$

where $i_1 = i_2 \neq i_3 = i_4$ and

$$C_j = \int\limits_t^T \phi_j(\tau)d\tau = \begin{cases} \sqrt{T-t}, & j=0 \\ \\ 0, & j \neq 0 \end{cases}.$$

Consider the case (5.229) by analogy with the case (5.228). Using (5.231), we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_1i_2)} - I_{(0000)T,t}^{*(i_1i_2i_1i_2)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_1i_2)} - I_{(0000)T,t}^{(i_1i_2i_1i_2)q} -\right.\right.$$

$$\left.\left.-\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} + \sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_1i_2)} - I_{(0000)T,t}^{(i_1i_2i_1i_2)q} - \sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \mathbf{1}_{\{j_2=j_4\}}\right) -\right.\right.$$

$$\left.\left.-\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right) - \sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_1i_2)} - I_{(0000)T,t}^{(i_1i_2i_1i_2)q}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \mathbf{1}_{\{j_2=j_4\}}\right)\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right)\right)^2\right\} +$$

$$+\left(\sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2. \qquad (5.253)$$

Using (1.121) and (5.253), we finally get

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)} - I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\} = \frac{(T-t)^4}{24} -$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4j_3j_2j_1}\right)\right)+
$$

$$
+\sum_{j_4=0}^{q}\sum_{j_2=0}^{j_4-1}\left(\sum_{j_1=0}^{q} C_{j_2j_1j_4j_1}+\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_4j_1j_4j_1}\right)^2+
$$

$$
+\sum_{j_3=0}^{q}\sum_{j_1=0}^{j_3-1}\left(\sum_{j_2=0}^{q} C_{j_2j_1j_2j_3}+\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\right)^2+2\sum_{j_3=0}^{q}\left(\sum_{j_2=0}^{q} C_{j_2j_3j_2j_3}\right)^2+
$$

$$
+\left(\sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2,
$$

where $i_1=i_3\ne i_2=i_4$.

Consider the case (5.230) by analogy with the cases (5.228) and (5.229). Using (5.231), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_2i_1)}-I_{(0000)T,t}^{*(i_1i_2i_2i_1)q}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}-\right.\right.
$$

$$
\left.\left.-\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_1j_3j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}+\sum_{j_2,j_1=0}^{q} C_{j_1j_2j_2j_1}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}+\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_1)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)-
$$

$$
\left.\left.-\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)-\sum_{j_2,j_1=0}^{q} C_{j_1j_2j_2j_1}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}\right)^2\right\}+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_1)}-\right.\right.
$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)\Biggr)^2\Biggr\}+$$

$$+\mathsf{M}\left\{\left(\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)\right)^2\right\}+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}\right)^2\right\}+\frac{(T-t)^4}{48}-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}(t_4-t_1)\phi_{j_1}(t_1)dt_1dt_4+\right.$$

$$\left.+\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}(t_4-t_1)\phi_{j_4}(t_1)dt_1dt_4\right)+$$

$$+\mathsf{M}\left\{\left(\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)\right)^2\right\}+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2. \tag{5.254}$$

Using (1.122) and (5.254), we finally get

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-$$

$$-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)}C_{j_4j_3j_2j_1}\right)\right)-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(C_{j_4j_1}^{10}+C_{j_1j_4}^{10}-C_{j_4j_1}^{01}-C_{j_1j_4}^{01}\right)+$$

$$+\sum_{j_4=0}^{q}\sum_{j_1=0}^{j_4-1}\left(\sum_{j_2=0}^{q}C_{j_1j_2j_2j_4}+\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_4}\right)^2+$$

$$+\sum_{j_3=0}^{q}\sum_{j_2=0}^{j_3-1}\left(\sum_{j_1=0}^{q}C_{j_1j_2j_3j_1}+\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\right)^2+2\sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_3j_3j_1}\right)^2+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2,$$

where $i_1=i_4\ne i_2=i_3$.

5.7 Optimization of the Mean-Square Approximation Procedures for Iterated Stratonovich Stochastic Integrals Based on Theorems 2.2, 2.8 and Multiple Fourier–Legendre Series

This section is devoted to optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals (5.4) of multiplicities 1 to 3 based on Theorems 2.2, 2.8 and multiple Fourier–Legendre series [68][1].

The mentioned stochastic integrals are part of strong numerical methods with convergence orders 1.0 and 1.5 for Itô SDEs with multidimensional noncommutative noise (see (4.88), (4.89)).

We show that the lengths of sequences of independent standard Gaussian random variables required for the mean-square approximation of iterated Stratonovich stochastic integrals (5.4) can be significantly reduced without the loss of the mean-square accuracy of approximation for these stochastic integrals.

Using Theorems 2.2, 2.8 and the system of Legendre polynomials, we obtain the following approximations of iterated Stratonovich stochastic integrals (5.4)

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{*(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

[1]The results of this section were obtained jointly with Kuznetsov M.D., who is also a co-author of the publications [53]-[56], [58], [60], [68].

$$I_{(00)T,t}^{*(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \quad (5.255)$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \quad (5.256)$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)} \quad (i = 1, \ldots, m, \ j = 0, 1, \ldots)$$

are independent standard Gaussian random variables for various i or j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ (see (5.5)),

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dx dy dz,$$

$P_j(x)$ is the Legendre polynomial (see (5.6)).

Denote

$$E_p^{*(l_1\ldots l_k)} \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)} - I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}\right)^2\right\},$$

where $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$ is the iterated Stratonovich stochastic integral (5.4) and $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}$ is the mean-square approximation of this stochastic integral. More precisely, the approximations $I_{(00)T,t}^{*(i_1 i_2)q}$, $I_{(000)T,t}^{*(i_1 i_2 i_3)q_1}$ are defined by (5.255), (5.256).

Using (5.41), (5.77), (5.210), (5.214), (5.217), we get

$$E_q^{*(00)} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \quad (i_1 \ne i_2), \quad (5.257)$$

$$E_{q_{1,1}}^{*(000)} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_{1,1}} C_{j_3 j_2 j_1}^2 \quad (i_1 \ne i_2, \ i_1 \ne i_3, \ i_2 \ne i_3), \quad (5.258)$$

$$E_{q_{1,2}}^{*(000)} = \frac{(T-t)^3}{4} - \sum_{j_1,j_2,j_3=0}^{q_{1,2}} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_{1,2}} C_{j_3 j_1 j_2}C_{j_3 j_2 j_1} -$$

$$
-\frac{(T-t)^{3/2}}{2}\sum_{j_1=0}^{q_{1,2}}\left(C_{0j_1j_1}+\frac{1}{\sqrt{3}}C_{1j_1j_1}\right)+\sum_{j_3=0}^{q_{1,2}}\left(\sum_{j_1=0}^{q_{1,2}}C_{j_3j_1j_1}\right)^2 \quad (i_1=i_2\neq i_3), \tag{5.259}
$$

$$
E_{q_{1,3}}^{*(000)}=\frac{(T-t)^3}{4}-\sum_{j_1,j_2,j_3=0}^{q_{1,3}}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_{1,3}}C_{j_2j_3j_1}C_{j_3j_2j_1}-
$$

$$
-\frac{(T-t)^{3/2}}{2}\sum_{j_3=0}^{q_{1,3}}\left(C_{j_3j_30}-\frac{1}{\sqrt{3}}C_{j_3j_31}\right)+\sum_{j_1=0}^{q_{1,3}}\left(\sum_{j_3=0}^{q_{1,3}}C_{j_3j_3j_1}\right)^2 \quad (i_1\neq i_2=i_3), \tag{5.260}
$$

$$
E_{q_{1,4}}^{*(000)}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_{1,4}}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_{1,4}}C_{j_3j_2j_1}C_{j_1j_2j_3}+
$$

$$
+\sum_{j_2=0}^{q_{1,4}}\left(\sum_{j_1=0}^{q_{1,4}}C_{j_1j_2j_1}\right)^2 \quad (i_1=i_3\neq i_2). \tag{5.261}
$$

Note that the number of conditions (5.258)–(5.261) is quite large, which is inconvenient for practice. In this section, we propose the hypothesis that all the formulas (5.258)–(5.261) can be replaced by the formula (5.258) in which we can suppose that $i_1, i_2, i_3 = 1, \ldots, m$. At that we will not have a noticeable loss of the mean-square approximation accuracy of iterated Stratonovich stochastic integrals.

Consider the following condition

$$
E_q^{*(00)}\le (T-t)^4, \quad E_{q_{1,i}}^{*(000)}\le (T-t)^4, \quad i=1,\ldots,4. \tag{5.262}
$$

Let us show by numerical experiments that in most situations the following inequality is fulfilled

$$
q_{1,1}\ge q_{1,i}, \quad i=2,3,4, \tag{5.263}
$$

where $q_{1,i}$ $(i=1,\ldots,4)$ are minimal natural numbers satisfying the condition (5.262).

In Tables 5.59–5.61 we can see the results of numerical experiments. These results confirm the hypothesis proposed earlier in this section. Note that in Table 5.61 we calculate the mean-square approximation errors of iterated Stratonovich stochastic integrals in the case when

$$
q_{1,i}=q_{1,1}, \quad i=2,3,4,
$$

Table 5.59: Conditions $E_{q_{1,i}}^{*(000)} \le (T-t)^4$, $i=1,\ldots,4$.

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q_{1,1}$	12	16	28	36	47	50
$q_{1,2}$	6	8	14	18	23	25
$q_{1,3}$	6	8	14	18	23	25
$q_{1,4}$	12	16	28	36	47	51

Table 5.60: The condition (5.262).

$T-t$	2^{-1}	2^{-3}	2^{-5}	2^{-8}
q	1	8	128	8192
$q_{1,1}$	0	1	4	32
$q_{1,2}$	0	0	2	16
$q_{1,3}$	0	0	2	16
$q_{1,4}$	0	0	4	33

Table 5.61: Values $E_{q_{1,i}}^{*(000)} \cdot (T-t)^{-3} \stackrel{\mathsf{def}}{=} E_{q_{1,i}}^{*}$, $i=1,\ldots,4$.

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q_{1,1}$	12	16	28	36	47	50
$E_{q_{1,1}}^{*}$	0.010154	0.007681	0.004433	0.003456	0.002652	0.002494
$q_{1,2}$	12	16	28	36	47	50
$E_{q_{1,2}}^{*}$	0.005102	0.003855	0.002221	0.001731	0.001328	0.001248
$q_{1,3}$	12	16	28	36	47	50
$E_{q_{1,3}}^{*}$	0.005102	0.003855	0.002221	0.001731	0.001328	0.001248
$q_{1,4}$	12	16	28	36	47	50
$E_{q_{1,4}}^{*}$	0.010407	0.007845	0.004500	0.003501	0.002680	0.002519

Table 5.62: Comparison of numbers $q_{1,1}$ and $p_{1,1}$.

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$q_{1,1}$	0	0	1	2	4	8
$(q_{1,1}+1)^3$	1	1	8	27	125	729
$p_{1,1}$	1	3	6	12	24	48
$(p_{1,1}+1)^3$	8	64	343	2197	15625	117649

where $q_{1,1}$ is the minimal natural number satisfying the condition (5.262). In this case, there is no noticeable loss of the mean-square approximation accuracy of iterated Stratonovich stochastic integrals (see Table 5.61). This means that all the formulas (5.258)–(5.261) can be replaced by the formula (5.258) in which we can suppose that $i_1, i_2, i_3 = 1, \ldots, m$.

Let $q_{1,1}$ be the minimal natural number satisfying the condition

$$E_{q_{1,1}}^{(000)} \le (T-t)^4, \tag{5.264}$$

where the left-hand side of (5.264) is defined by the formula (5.258).

Let $p_{1,1}$ be the minimal natural number satisfying the condition

$$3! \cdot E_{p_{1,1}}^{(000)} \le (T-t)^4, \tag{5.265}$$

where the value $E_{p_{1,1}}^{(000)}$ on the left-hand side of (5.265) is defined by the formula (5.258) (recall that 3! is included in the inequality (5.165) for the case $k=3$).

In Table 5.62 we can see the numerical comparison of numbers $q_{1,1}$ and $p_{1,1}$. Obviously, excluding of the multiplier factor 3! essentially (in many times) reduces the calculation costs for the mean-square approximations of iterated Stratonovich stochastic integrals. Note that in this section we use the exactly calculated Fourier–Legendre coefficients using the Python programming language [53], [54].

Chapter 6

Other Methods of Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of Multiplicities 1 to 4

6.1 New Simple Method for Obtainment an Expansion of Iterated Itô Stochastic integrals of Multiplicity 2 Based on the Wiener Process Expansion Using Legendre Polynomials and Trigonometric Functions

This section is devoted to the expansion of iterated Itô stochastic integrals of multiplicity 2 based on the Wiener process expansion using complete orthonormal systems of functions in $L_2([t,T])$. The expansions of these stochastic integrals using Legendre polynomials and trigonometric functions are considered. In contrast to the method of expansion of iterated Itô stochastic integrals based on the Karhunen–Loève expansion of the Brownian bridge process [82]-[84], this method allows the use of different systems of basis functions, not only the trigonometric system of functions. The proposed method makes it possible to obtain expansions of iterated Itô stochastic integrals of multiplicity 2 much easier than the method based on generalized multiple Fourier series (see Chapters 1 and 2). The latter involve the calculation of coefficients of multiple Fourier series, which is a time-consuming task. However, the proposed method can be applied only to iterated Itô stochastic integrals of multiplicity 2.

It is well known that the idea of representing of the Wiener process as a functional series with random coefficients (that are independent standard Gaussian random variables) with using the complete orthonormal system of trigonometric functions in $L_2([0,T])$ goes back to the works of Wiener [166] (1924) and Lévy [167] (1951). The specified series was used in [166] and [167] for construction of the Brownian motion process (Wiener process). A little later, Itô and McKean in [168] (1965) used for this purpose the complete orthonormal system of Haar functions in $L_2([0,T])$.

Let $\mathbf{f}_\tau$, $\tau \in [0,T]$ be an m-dimestional standard Wiener process with independent components $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m$.

We have

$$\mathbf{f}_s^{(i)} - \mathbf{f}_t^{(i)} = \int\limits_t^s d\mathbf{f}_\tau^{(i)} = \int\limits_t^T \mathbf{1}_{\{\tau<s\}} d\mathbf{f}_\tau^{(i)},$$

where

$$\int\limits_t^T \mathbf{1}_{\{\tau<s\}} d\mathbf{f}_\tau^{(i)}$$

is the Itô stochastic integral, $t \ge 0$, and

$$\mathbf{1}_{\{\tau<s\}} = \begin{cases} 1, & \tau < s \\ & \\ 0, & \text{otherwise} \end{cases}, \quad \tau, s \in [t,T].$$

Consider the Fourier expansion of $\mathbf{1}_{\{\tau<s\}} \in L_2([t,T])$ at the interval $[t,T]$ (see, for example, [130])

$$\sum_{j=0}^{\infty} \int\limits_t^T \mathbf{1}_{\{\tau<s\}} \phi_j(\tau) d\tau \phi_j(\tau) = \sum_{j=0}^{\infty} \int\limits_t^s \phi_j(\tau) d\tau \phi_j(\tau), \tag{6.1}$$

where $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$ and the series (6.1) converges in the mean-square sense, i.e.

$$\int\limits_t^T \left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^{q} \int\limits_t^s \phi_j(\tau) d\tau \phi_j(\tau) \right)^2 d\tau \to 0 \quad \text{if} \quad q \to \infty.$$

Let $\mathbf{f}_{s,t}^{(i)q}$ be the mean-square approximation of the process $\mathbf{f}_s^{(i)} - \mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_{s,t}^{(i)q} = \int\limits_t^T \left(\sum_{j=0}^{q} \int\limits_t^s \phi_j(\tau) d\tau \phi_j(\tau) \right) d\mathbf{f}_\tau^{(i)} = \sum_{j=0}^{q} \int\limits_t^s \phi_j(\tau) d\tau \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}. \tag{6.2}$$

Moreover,

$$\mathsf{M}\left\{\left(\mathbf{f}_s^{(i)}-\mathbf{f}_t^{(i)}-\mathbf{f}_{s,t}^{(i)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^q\int\limits_t^s\phi_j(\tau)d\tau\phi_j(\tau)\right)d\mathbf{f}_\tau^{(i)}\right)^2\right\}=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^q\int\limits_t^s\phi_j(\tau)d\tau\phi_j(\tau)\right)^2 d\tau\to 0\quad\hbox{if } q\to\infty. \tag{6.3}$$

In [87] it was proposed to use an expansion similar to (6.2) for the expansion of iterated Itô stochastic integrals

$$I_{(00)T,t}^{(i_1i_2)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad(i_1,i_2=1,\ldots,m). \tag{6.4}$$

At that, to obtain the mentioned expansion of (6.4), the truncated expansions (6.2) of components of the Wiener process $\mathbf{f}_s$ have been iteratively substituted in the single integrals [87]. This procedure leads to the calculation of coefficients of the double Fourier series, which is a time-consuming task for not too complex problem of expansion of the iterated Itô stochastic integral (6.4). In [87] the expansions on the base of Haar functions and trigonometric functions have been considered.

In contrast to [87] we subsitute the expansion (6.2) only one time and only into the innermost integral in (6.4). This procedure leads to the simple calculation of the coefficients

$$\int\limits_t^s\phi_j(\tau)d\tau\quad(j=0,1,2,\ldots)$$

of the usual (not double) Fourier series.

Moreover, we use the Legendre polynomials [51], [71] for the construction of the expansion of (6.4). For the first time the Legendre polynomials have been applied in the framework of the mentioned problem in the author's papers [76] (1997), [77] (1998), [78] (2000), [79] (2001) (also see [1]-[71]) while in the papers of other author's these polynomials have not been considered as the basis functions for the construction of expansions of iterated Itô or Stratonovich stochastic integrals.

Theorem 6.1 [14]-[17], [51], [71]. *Let $\phi_j(\tau)$ $(j=0,1,\ldots)$ be an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Let*

$$\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \sum_{j=0}^{q} \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i_1)} \int\limits_t^T \int\limits_t^s \phi_j(\tau) d\tau d\mathbf{f}_s^{(i_2)} \qquad (6.5)$$

be an approximation of the iterated Itô stochastic integral (6.4) for $i_1 \ne i_2$. Then

$$I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} \quad (i_1 \ne i_2),$$

where $i_1, i_2 = 1, \ldots, m$.

Proof. Using the standard properties of the Itô stochastic integral as well as (6.3) and the property of orthonormality of functions $\phi_j(\tau)$ $(j=0,1,\ldots)$ at the interval $[t,T]$, we obtain

$$\mathsf{M}\left\{\left(\int\limits_t^T \int\limits_t^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} - \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)}\right)^2\right\} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left(\mathbf{f}_s^{(i_1)} - \mathbf{f}_t^{(i_1)} - \mathbf{f}_{s,t}^{(i_1)q}\right)^2\right\} ds =$$

$$= \int\limits_t^T \int\limits_t^T \left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^{q} \int\limits_t^s \phi_j(\tau) d\tau \phi_j(\tau)\right)^2 d\tau ds =$$

$$= \int\limits_t^T \left((s-t) - \sum_{j=0}^{q} \left(\int\limits_t^s \phi_j(\tau) d\tau\right)^2\right) ds. \qquad (6.6)$$

Using the continuity of the functions $u_q(s)$ (see below), the nondecreasing property of the functional sequence

$$u_q(s) = \sum_{j=0}^{q} \left(\int\limits_t^s \phi_j(\tau) d\tau\right)^2,$$

and the continuity of the limit function $u(s) = s-t$ according to Dini's Theorem, we have the uniform convergence $u_q(s)$ to $u(s)$ at the interval $[t, T]$.

Then from this fact as well as from (6.6) we obtain

$$I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)}. \qquad (6.7)$$

Note that we could also use Lebesgue's Dominated Convergence Theorem in (6.6) to obtain (6.7). Theorem 6.1 is proved.

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$, which has the form (5.5). Then

$$\int\limits_t^s \phi_j(\tau)d\tau = \frac{T-t}{2}\left(\frac{\phi_{j+1}(s)}{\sqrt{(2j+1)(2j+3)}} - \frac{\phi_{j-1}(s)}{\sqrt{4j^2-1}}\right) \quad \text{for } \ j \ge 1. \ (6.8)$$

Let us denote

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)} \quad (i = 1, \ldots, m).$$

From (6.5) and (6.8) we obtain

$$\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)} \int\limits_t^T (s-t)\mathbf{f}_s^{(i_2)} +$$

$$+\frac{T-t}{2}\sum_{j=1}^{q} \zeta_j^{(i_1)} \left(\frac{1}{\sqrt{(2j+1)(2j+3)}}\zeta_{j+1}^{(i_2)} - \frac{1}{\sqrt{4j^2-1}}\zeta_{j-1}^{(i_2)}\right) =$$

$$= \frac{T-t}{2}\zeta_0^{(i_1)}\left(\zeta_0^{(i_2)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right) +$$

$$+\frac{T-t}{2}\sum_{j=1}^{q} \zeta_j^{(i_1)} \left(\frac{1}{\sqrt{(2j+1)(2j+3)}}\zeta_{j+1}^{(i_2)} - \frac{1}{\sqrt{4j^2-1}}\zeta_{j-1}^{(i_2)}\right) =$$

$$= \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{j=1}^{q}\frac{1}{\sqrt{4j^2-1}}\left(\zeta_{j-1}^{(i_1)}\zeta_j^{(i_2)} - \zeta_j^{(i_1)}\zeta_{j-1}^{(i_2)}\right)\right) +$$

$$+\frac{T-t}{2}\zeta_q^{(i_1)}\zeta_{q+1}^{(i_2)}\frac{1}{\sqrt{(2q+1)(2q+3)}}. \tag{6.9}$$

Then from (6.7) and (6.9) we get

$$I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} =$$

$$= \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{j=1}^{\infty}\frac{1}{\sqrt{4j^2-1}}\left(\zeta_{j-1}^{(i_1)}\zeta_j^{(i_2)} - \zeta_j^{(i_1)}\zeta_{j-1}^{(i_2)}\right)\right). \tag{6.10}$$

It is not difficult to see that the relation (6.10) has been obtained in Sect. 5.1 (see (5.11)).

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be a complete orthonormal system of trigonomertic functions in the space $L_2([t,T])$, which has the form (5.110).

We have

$$\int\limits_t^s \phi_j(\tau)d\tau = \frac{T-t}{2\pi r}\begin{cases} \phi_{2r-1}(s), & j=2r \\ \\ \sqrt{2}\phi_0(s)-\phi_{2r}(s), & j=2r-1 \end{cases}, \tag{6.11}$$

where $j\ge 1$ and $r=1,2,\ldots$

From (6.5) and (6.11) we obtain

$$\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\int\limits_t^T (s-t)\mathbf{f}_s^{(i_2)} +$$

$$+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right) =$$

$$= \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right) +$$

$$+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right) =$$

$$= \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} +$$

$$
+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr)\Biggr)-\frac{T-t}{\pi\sqrt{2}}\zeta_0^{(i_1)}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}. \qquad (6.12)
$$

From (6.12) and (6.7) we get

$$
I_{(00)T,t}^{(i_1 i_2)}=\underset{q\to\infty}{\text{l.i.m.}}\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q}d\mathbf{f}_s^{(i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+
$$

$$
+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr)\Biggr), \qquad (6.13)
$$

where $i_1\neq i_2$.

It is obvious that (6.13) is consistent with (5.86) for $i_1\neq i_2$ (we consider here (5.86) without the random variables $\xi_q^{(i)}$).

6.2 Milstein method of Expansion of Iterated Itô and Stratonovich Stochastic Integrals

The method that is considered in this section was proposed by Milstein G.N. [82] (1988) and probably until the mid-2000s remained one of the most famous methods for strong approximation of iterated stochastic integrals (also see [83]-[85], [91]-[93], [96], [97]). However, in light of the results of Chapters 1 and 2 as well as Sect. 5.1 and 5.3, it can be argued that the method based on Theorem 1.1 is more general and effective.

The mentioned Milstein method [82] is based on the expansion of the Brownian bridge process into the trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loève expansion).

Let us consider the Brownian bridge process

$$
\mathbf{f}_t-\frac{t}{\Delta}\mathbf{f}_\Delta,\quad t\in[0,\Delta],\quad \Delta>0, \qquad (6.14)
$$

where $\mathbf{f}_t$ is a standard Wiener process with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$.

The componentwise Karhunen–Loève expansion of the process (6.14) has the following form

$$
\mathbf{f}_t^{(i)}-\frac{t}{\Delta}\mathbf{f}_\Delta^{(i)}=\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \qquad (6.15)
$$

where the series converges in the mean-square sense and

$$a_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,$$

$$b_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds,$$

$r=0,1,\ldots,\ i=1,\ldots,m.$

It is easy to demonstrate [82] that the random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\}=\mathsf{M}\left\{a_{i,r}b_{i,k}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}a_{i,k}\right\}=\mathsf{M}\left\{b_{i,r}b_{i,k}\right\}=0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\}=\mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}^2\right\}=\mathsf{M}\left\{b_{i,r}^2\right\}=\frac{\Delta}{2\pi^2r^2},$$

where $i,i_1,i_2=1,\ldots,m,\ r\ne k,\ i_1\ne i_2.$

According to (6.15), we have

$$\mathbf{f}_t^{(i)}=\mathbf{f}_{\Delta}^{(i)}\frac{t}{\Delta}+\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right),\tag{6.16}$$

where the series converges in the mean-square sense.

Note that the trigonometric functions are the eigenfunctions of the covariance operator of the Brownian bridge process. That is why the basis functions are the trigonometric functions in the considered approach.

Using the relation (6.16), it is easy to get the following expansions [82]-[84]

$$\int\limits_0^t d\mathbf{f}_{\tau}^{(i)}=\frac{t}{\Delta}\mathbf{f}_{\Delta}^{(i)}+\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right),\tag{6.17}$$

$$\int\limits_0^t\int\limits_0^{\tau}d\mathbf{f}_{\tau_1}^{(i)}d\tau=\frac{t^2}{2\Delta}\mathbf{f}_{\Delta}^{(i)}+\frac{t}{2}a_{i,0}+$$

$$+\frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(a_{i,r}\sin\frac{2\pi rt}{\Delta}-b_{i,r}\left(\cos\frac{2\pi rt}{\Delta}-1\right)\right),\tag{6.18}$$

$$\int\limits_0^t\int\limits_0^{\tau}d\tau_1 d\mathbf{f}_{\tau}^{(i)}=t\int\limits_0^t d\mathbf{f}_{t}^{(i)}-\int\limits_0^t\int\limits_0^{\tau}d\mathbf{f}_{\tau_1}^{(i)}d\tau=\frac{t^2}{2\Delta}\mathbf{f}_{\Delta}^{(i)}+$$

$$+t\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right)-$$

$$-\frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(a_{i,r}\sin\frac{2\pi rt}{\Delta}-b_{i,r}\left(\cos\frac{2\pi rt}{\Delta}-1\right)\right), \qquad (6.19)$$

$$\int\limits_0^t\int\limits_0^{\tau}d\mathbf{f}_{\tau_1}^{(i_1)}d\mathbf{f}_{\tau}^{(i_2)}=\frac{1}{\Delta}\mathbf{f}_{\Delta}^{(i_1)}\int\limits_0^t\int\limits_0^{\tau}d\tau_1 d\mathbf{f}_{\tau}^{(i_2)}+\frac{1}{2}a_{i_1,0}\int\limits_0^t d\mathbf{f}_{\tau}^{(i_2)}+$$

$$+\frac{t\pi}{\Delta}\sum_{r=1}^{\infty}r\left(a_{i_1,r}b_{i_2,r}-b_{i_1,r}a_{i_2,r}\right)+$$

$$+\frac{1}{4}\sum_{r=1}^{\infty}\Biggl(\left(a_{i_1,r}a_{i_2,r}-b_{i_1,r}b_{i_2,r}\right)\left(1-\cos\frac{4\pi rt}{\Delta}\right)+$$

$$+\left(a_{i_1,r}b_{i_2,r}+b_{i_1,r}a_{i_2,r}\right)\sin\frac{4\pi rt}{\Delta}+$$

$$+\frac{2}{\pi r}\mathbf{f}_{\Delta}^{(i_2)}\left(a_{i_1,r}\sin\frac{2\pi rt}{\Delta}+b_{i_1,r}\left(\cos\frac{2\pi rt}{\Delta}-1\right)\right)\Biggr)+$$

$$+\sum_{k=1}^{\infty}\sum_{r=1 (r\ne k)}^{\infty}k\Biggl(a_{i_1,r}a_{i_2,k}\left(\frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)}+\frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}-\frac{k}{k^2-r^2}\right)+$$

$$+a_{i_1,r}b_{i_2,k}\left(\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)}+\frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right)+$$

$$+b_{i_1,r}b_{i_2,k}\left(\frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}-\frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)}-\frac{r}{k^2-r^2}\right)+$$

$$+\frac{\Delta}{2\pi}b_{i_1,r}a_{i_2,k}\left(\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)}-\frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right)\Biggr) \qquad (6.20)$$

converging in the mean-square sense, where we suppose that $i_1 \neq i_2$ in (6.20).

It is necessary to pay a special attention to the fact that the double series in (6.20) should be understood as the iterated one, and not as a multiple series (as in Theorem 1.1), i.e. as the iterated passage to the limit for the sequence of double partial sums. So, the Milstein method of approximation of iterated stochastic integrals [82] leads to iterated application of the limit transition (in contrast with the method of generalized multiple Fourier series (Theorem 1.1), for which the limit transition is implemented only once) starting at least from the second or third multiplicity of iterated stochastic integrals (we mean at least double or triple integration with respect to components of the Wiener process). Multiple series are more preferential for approximation than the iterated ones, since the partial sums of multiple series converge for any possible case of joint converging to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, as we will see further in this section in [83] (pp. 438-439), [84] (Sect. 5.8, pp. 202–204), [85] (pp. 82-84), [93] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein method [82] together with the Wong–Zakai approximation [73]-[75] (also see discussions in Sect. 2.41, 2.42). Furthermore, in order to obtain the Milstein expansion for iterated stochastic integral, the truncated expansions (6.16) of components of the Wiener process $\mathbf{f}_t$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to the expansion of iterated stochastic integral of multiplicity k ($k \in \mathbf{N}$).

Assume that $t = \Delta$ in the relations (6.17)–(6.20) (at that double partial sums of iterated series in (6.20) will become zero). As a result, we get

$$\int\limits_0^{\Delta} d\mathbf{f}_{\tau}^{(i)} = \mathbf{f}_{\Delta}^{(i)}, \tag{6.21}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \frac{1}{2}\Delta\left(\mathbf{f}_{\Delta}^{(i)} + a_{i,0}\right), \tag{6.22}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i)} = \frac{1}{2}\Delta\left(\mathbf{f}_{\Delta}^{(i)} - a_{i,0}\right), \tag{6.23}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{1}{2}\mathbf{f}_{\Delta}^{(i_1)}\mathbf{f}_{\Delta}^{(i_2)} - \frac{1}{2}\left(a_{i_2,0}\mathbf{f}_{\Delta}^{(i_1)} - a_{i_1,0}\mathbf{f}_{\Delta}^{(i_2)}\right) +$$

$$+\pi\sum_{r=1}^{\infty} r\left(a_{i_1,r}b_{i_2,r} - b_{i_1,r}a_{i_2,r}\right) \tag{6.24}$$

converging in the mean-square sense, where we suppose that $i_1 \ne i_2$ in (6.24).

Deriving (6.21)–(6.24), we used the relation

$$a_{i,0} = -2\sum_{r=1}^{\infty} a_{i,r}, \tag{6.25}$$

which results from (6.15) when $t = \Delta$.

Let us compare expansions of some iterated stochastic integrals of first and second multiplicity obtained by Milstein method [82] and method based on generalized multiple Fourier series (Theorem 1.1).

Let us denote

$$\zeta_{2r-1}^{(i)} = \sqrt{\frac{2}{\Delta}}\int\limits_0^{\Delta} \sin\frac{2\pi rs}{\Delta} d\mathbf{f}_s^{(i)}, \quad \zeta_{2r}^{(i)} = \sqrt{\frac{2}{\Delta}}\int\limits_0^{\Delta} \cos\frac{2\pi rs}{\Delta} d\mathbf{f}_s^{(i)}, \tag{6.26}$$

$$\zeta_0^{(i)} = \frac{1}{\sqrt{\Delta}}\int\limits_0^{\Delta} d\mathbf{f}_s^{(i)}, \tag{6.27}$$

where $r = 1, 2, \ldots,\ i = 1, \ldots, m$.

Using the Itô formula, it is not difficult to show that

$$a_{i,r} = -\frac{1}{\pi r}\sqrt{\frac{\Delta}{2}}\zeta_{2r-1}^{(i)}, \quad b_{i,r} = \frac{1}{\pi r}\sqrt{\frac{\Delta}{2}}\zeta_{2r}^{(i)} \quad \text{w. p. 1.} \tag{6.28}$$

From (6.25) we get

$$a_{i,0} = \frac{\sqrt{2\Delta}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)}. \tag{6.29}$$

After substituting (6.28), (6.29) into (6.21)–(6.24) and taking into account (6.26), (6.27), we have

$$\int\limits_0^{\Delta} d\mathbf{f}_{\tau}^{(i_1)} = \sqrt{\Delta}\zeta_0^{(i_1)}, \tag{6.30}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i_1)} = \frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{6.31}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\tau = \frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{6.32}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.$$

$$\left.\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\right). \tag{6.33}$$

Obviously, the formulas (6.30)–(6.33) are consistent with the formulas (5.7), (5.86), (5.104), (5.105). It testifies that at least for the considered iterated stochastic integrals and trigonometric system of functions, the Milstein method and the method based on generalized multiple Fourier series (Theorem 1.1) give the same result (it is an interesting fact, although it is rather expectable).

Further, we will discuss the usage of Milstein method for the iterated stochastic integrals of third multiplicity.

First, we note that the authors of the monograph [84] based on the results of Wong E. and Zakai M. [73], [74] (also see [75]) concluded (without rigorous proof) that the expansions of iterated stochastic integrals on the basis of (6.16) (the case $i_1, i_2, i_3 = 1, \ldots, m$) converge to the iterated Stratonovich stochastic integrals (see discussions in Sect. 2.41, 2.42). It is obvious that this conclusion is consistent with the results given above in this section for the case $i_1 \neq i_2$.

As we mentioned before, the technical peculiarities of the Milstein method [82] may result to the iterated series of products of standard Gaussian random variables (in contradiction to multiple series as in Theorem 1.1). In the case of simplest stochastic integral of second multiplicity this problem was avoided as we saw above. However, the situation is not the same for the simplest stochastic integrals of third multiplicity.

Let us denote

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\lambda_l = 1$ if $i_l = 1, \ldots, m$ and $\lambda_l = 0$ if $i_l = 0$, $l = 1, \ldots, k$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Let us consider the expansion of iterated Stratonovich stochastic integral of third multiplicity obtained in [83]-[85], [93] by the Milstein method [82]

$$
J^{*(i_1 i_2 i_3)}_{(111)\Delta,0} = \frac{1}{\Delta} J^{*(i_1)}_{(1)\Delta,0} J^{*(0 i_2 i_3)}_{(011)\Delta,0} +
$$

$$
+\frac{1}{2} a_{i_1,0} J^{*(i_2 i_3)}_{(11)\Delta,0} + \frac{1}{2\pi} b_{i_1} J^{(i_2)}_{(1)\Delta,0} J^{*(i_3)}_{(1)\Delta,0} - \Delta J^{*(i_2)}_{(1)\Delta,0} B_{i_1 i_3} +
$$

$$
+\Delta J^{*(i_3)}_{(1)\Delta,0} \left(\frac{1}{2} A_{i_1 i_2} - C_{i_2 i_1} \right) + \Delta^{3/2} D_{i_1 i_2 i_3}, \tag{6.34}
$$

where

$$
J^{*(0 i_2 i_3)}_{(011)\Delta,0} = \frac{1}{6} J^{*(i_2)}_{(1)\Delta,0} J^{*(i_3)}_{(1)\Delta,0} - \frac{1}{\pi} \Delta J^{*(i_3)}_{(1)\Delta,0} b_{i_2} +
$$

$$
+\Delta^2 B_{i_2 i_3} - \frac{1}{4} \Delta a_{i_3,0} J^{*(i_2)}_{(1)\Delta,0} + \frac{1}{2\pi} \Delta b_{i_3} J^{*(i_2)}_{(1)\Delta,0} + \Delta^2 C_{i_2 i_3} + \frac{1}{2} \Delta^2 A_{i_2 i_3},
$$

$$
A_{i_2 i_3} = \frac{\pi}{\Delta} \sum_{r=1}^{\infty} r \left(a_{i_2,r} b_{i_3,r} - b_{i_2,r} a_{i_3,r} \right),
$$

$$
C_{i_2 i_3} = -\frac{1}{\Delta} \sum_{l=1}^{\infty} \sum_{r=1 (r \neq l)}^{\infty} \frac{r}{r^2 - l^2} \left(r a_{i_2,r} a_{i_3,l} + l b_{i_2,r} b_{i_3,l} \right),
$$

$$
B_{i_2 i_3} = \frac{1}{2\Delta} \sum_{r=1}^{\infty} \left(a_{i_2,r} a_{i_3,r} + b_{i_2,r} b_{i_3,r} \right), \ \ b_i = \sum_{r=1}^{\infty} \frac{1}{r} b_{i,r},
$$

$$
D_{i_1 i_2 i_3} = -\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} l \Biggl(a_{i_2,l} \left(a_{i_3,l+r} b_{i_1,r} - a_{i_1,r} b_{i_3,l+r} \right) +
$$

$$
+ b_{i_2,l} \left(a_{i_1,r} a_{i_3,r+l} + b_{i_1,r} b_{i_3,l+r} \right) \Biggr) +
$$

$$
+\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{l-1} l \Biggl(a_{i_2,l} \left(a_{i_1,r} b_{i_3,l-r} + a_{i_3,l-r} b_{i_1,r} \right) -
$$

$$
-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\Biggr)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{\infty}\sum_{r=l+1}^{\infty}l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+
$$

$$
+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\Biggr).
$$

From the expansion (6.34) and expansion of the stochastic integral $J^{*(0i_2i_3)}_{(011)\Delta,0}$ we can conclude that they include iterated (double) series. Moreover, for approximation of the stochastic integral $J^{*(i_1i_2i_3)}_{(111)\Delta,0}$ in the works [83] (pp. 438-439), [84] (Sect. 5.8, pp. 202–204), [85] (pp. 82-84), [93] (pp. 263-264) it is proposed to put upper limits of summation by equal q (on the base of the Wong–Zakai approximation [73]-[75] but without rigorous proof; also see discussions in Sect. 2.41, 2.42).

For example, the value $D_{i_1i_2i_3}$ is approximated in [83]-[85], [93] by the double sums of the form

$$
D^q_{i_1i_2i_3}=-\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{q}l\Biggl(a_{i_2,l}\left(a_{i_3,l+r}b_{i_1,r}-a_{i_1,r}b_{i_3,l+r}\right)+
$$

$$
+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r+l}+b_{i_1,r}b_{i_3,l+r}\right)\Biggr)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{l-1}l\Biggl(a_{i_2,l}\left(a_{i_1,r}b_{i_3,l-r}+a_{i_3,l-r}b_{i_1,r}\right)-
$$

$$
-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\Biggr)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=l+1}^{2q}l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+
$$

$$
+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\Biggr).
$$

We can avoid the mentioned problem (iterated application of the operation of limit transition) using the method based on Theorems 1.1, 2.1–2.9, 2.33–2.36, 2.50, 2.51, 2.62, 2.63.

From the other hand, if we prove that the members of the expansion (6.34) coincide with the members of its analogue obtained using Theorem 1.1, then we can replace the iterated series in (6.34) by the multiple series (see Theorems 1.1, 2.1–2.9, 2.33–2.36, 2.50, 2.51, 2.62, 2.63) as was made formally in [83]-[85], [93]. However, it requires the separate argumentation.

6.3 Usage of Integral Sums for Approximation of Iterated Itô Stochastic Integrals

It should be noted that there is an approach to the mean-square approximation of iterated stochastic integrals based on multiple integral sums (see, for example, [82], [92], [94], [169]). This method implies the partitioning of the integration interval $[t, T]$ of the iterated stochastic integral under consideration; this interval is the integration step of the numerical methods used to solve Itô SDEs (see Chapter 4); therefore, it is already fairly small and does not need to be partitioned. Computational experiments [1] (also see below in this section) show that the application of the method [82], [92], [94], [169] to stochastic integrals with multiplicities $k \geq 2$ leads to unacceptably high computational cost and accumulation of computation errors.

As we noted in the introduction to this book, considering the modern state of question on the approximation of iterated stochastic integrals, the method analyzed in this section is hardly important for practice. However, we will consider this method in order to get the overall view. In this section, we will analyze one of the simplest modifications of the mentioned method.

Let the functions $\psi_l(\tau)$, $l = 1, \ldots, k$ satisfy the Lipschitz condition at the interval $[t, T]$ with constants C_l

$$|\psi_l(\tau_1) - \psi_l(\tau_2)| \leq C_l|\tau_1 - \tau_2| \quad \text{for all} \quad \tau_1, \tau_2 \in [t, T]. \tag{6.35}$$

Then, according to Lemma 1.1 (see Sect. 1.1.3), the following equality is correct

$$J[\psi^{(k)}]_{T,t} = \underset{N \to \infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1},$$

where notations are the same as in (1.12).

Let us consider the following approximation

$$J[\psi^{(k)}]_{T,t}^N=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^k\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \tag{6.36}$$

of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}$. The relation (6.36) can be rewritten as

$$J[\psi^{(k)}]_{T,t}^N=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^k\sqrt{\Delta\tau_{j_l}}\psi_l(\tau_{j_l})\mathbf{u}_{j_l}^{(i_l)}, \tag{6.37}$$

where $\mathbf{u}_j^{(i)}\stackrel{\rm def}{=}\left(\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}\right)/\sqrt{\Delta\tau_j},\ \ i=1,\ldots,m$ are independent standard Gaussian random variables for various i or j, $\mathbf{u}_j^{(0)}=\sqrt{\Delta\tau_j}$.

Assume that

$$\tau_j=t+j\Delta,\quad j=0,1,\ldots,N,\quad \tau_N=T,\quad \Delta>0. \tag{6.38}$$

Then

$$J[\psi^{(k)}]_{T,t}^N=\Delta^{k/2}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^k\psi_l(t+j_l\Delta)\mathbf{u}_{j_l}^{(i_l)}, \tag{6.39}$$

where $\mathbf{u}_j^{(i)}\stackrel{\rm def}{=}\left(\mathbf{w}_{t+(j+1)\Delta}^{(i)}-\mathbf{w}_{t+j\Delta}^{(i)}\right)/\sqrt{\Delta},\ i=1,\ldots,m,\ \mathbf{u}_j^{(0)}=\sqrt{\Delta}.$

Lemma 6.1. *Suppose that the functions $\psi_l(\tau),\ l=1,\ldots k$ satisfy the Lipschitz condition (6.35) and $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$, which satisfies the condition (6.38). Then for a sufficiently small value $T-t$ there exists a constant $H_k<\infty$ such that*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^N\right)^2\right\}\le\frac{H_k(T-t)^2}{N}.$$

Proof. It is easy to see that in the case of a sufficiently small value $T-t$ there exists a constant L_k such that

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^N\right)^2\right\}\le L_k\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^N\right)^2\right\},$$

where

$$J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^N=\sum_{j=1}^3 S_j^N,$$

$$
S_1^N=\sum_{j_1=0}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_2(t_2)\int\limits_{\tau_{j_1}}^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)},
$$

$$
S_2^N=\sum_{j_1=0}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\psi_2(t_2)-\psi_2(\tau_{j_1})\right)d\mathbf{w}_{t_2}^{(i_2)}\sum_{j_2=0}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)},
$$

$$
S_3^N=\sum_{j_1=0}^{N-1}\psi_2(\tau_{j_1})\Delta\mathbf{w}_{\tau_{j_1}}^{(i_2)}\sum_{j_2=0}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\left(\psi_1(t_1)-\psi_1(\tau_{j_2})\right)d\mathbf{w}_{t_1}^{(i_1)}.
$$

Therefore, according to the Minkowski inequality, we have

$$
\left(\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^N\right)^2\right\}\right)^{1/2}\le\sum_{j=1}^{3}\left(\mathsf{M}\left\{\left(S_j^N\right)^2\right\}\right)^{1/2}.
$$

Using standard moment properties of stochastic integrals (see (1.26), (1.27)), let us estimate the values $\mathsf{M}\left\{\left(S_j^N\right)^2\right\}$, $j=1,2,3.$

Let us consider four cases.

Case 1. $i_1,i_2\ne 0:$

$$
\mathsf{M}\left\{\left(S_1^N\right)^2\right\}\le\frac{\Delta}{2}(T-t)\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),
$$

$$
\mathsf{M}\left\{\left(S_2^N\right)^2\right\}\le\frac{\Delta^2}{6}(T-t)^2\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),
$$

$$
\mathsf{M}\left\{\left(S_3^N\right)^2\right\}\le\frac{\Delta^2}{6}(T-t)^2\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).
$$

Case 2. $i_1\ne 0,\ i_2=0:$

$$
\mathsf{M}\left\{\left(S_1^N\right)^2\right\}\le\frac{\Delta}{2}(T-t)^2\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),
$$

$$
\mathsf{M}\left\{\left(S_2^N\right)^2\right\}\le\frac{\Delta^2}{3}(T-t)^3\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),
$$

$$
\mathsf{M}\left\{\left(S_3^N\right)^2\right\}\le\frac{\Delta^2}{3}(T-t)^3\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).
$$

Case 3. $i_2 \neq 0,\ i_1 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)^3\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{8}(T-t)^3\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

Case 4. $i_1 = i_2 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^2\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^4\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{16}(T-t)^4\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

According to the obtained estimates, we have

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^N\right)^2\right\} \le H_k(T-t)\Delta = \frac{H_k(T-t)^2}{N},$$

where $H_k < \infty$. Lemma 6.1 is proved.

It is easy to check that the following relation is correct

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)N}\right)^2\right\} = \frac{(T-t)^2}{2N}, \qquad (6.40)$$

where $i_1, i_2 = 1, \ldots, m$ and $I_{(00)T,t}^{(i_1 i_2)N}$ is the approximation of the iterated stochastic integral $I_{(00)T,t}^{(i_1 i_2)}$ (see (6.4)) obtained according to the formula (6.39).

Finally, we will demonstrate that the method based on generalized multiple Fourier series (Theorem 1.1) is signficiantly better, than the method based on multiple integral sums in the sense of computational costs on modeling of iterated stochastic integrals.

Let us consider the approximations of iterated Itô stochastic integrals obtained using the method based on multiple integral sums

$$I_{(0)T,t}^{(1)q} = \sqrt{\Delta}\sum_{j=0}^{q-1}\xi_j^{(1)}, \qquad (6.41)$$

Table 6.1: Values $T_{\rm sum}/T_{\rm pol}$.

$T-t$	2^{-5}	2^{-6}	2^{-7}
$T_{\rm sum}/T_{\rm pol}$	8.67	23.25	55.86

$$
I_{(00)T,t}^{(12)q}=\Delta\sum_{j=0}^{q-1}\left(\sum_{i=0}^{j-1}\xi_i^{(1)}\right)\xi_j^{(2)}, \tag{6.42}
$$

where

$$
\xi_j^{(i)}=\left(\mathbf{f}_{t+(j+1)\Delta}^{(i)}-\mathbf{f}_{t+j\Delta}^{(i)}\right)/\sqrt{\Delta},\quad i=1,2
$$

are independent standard Gaussian random variables, $\Delta=(T-t)/q$, $I_{(00)T,t}^{(12)q}$, $I_{(0)T,t}^{(1)q}$ are approximations of the iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$ (see (6.4)), $I_{(0)T,t}^{(1)}=\mathbf{f}_T^{(1)}-\mathbf{f}_t^{(1)}$.

Let us choose the number q (see (6.41), (6.42)) from the condition

$$
{\sf M}\left\{\left(I_{(00)T,t}^{(12)}-I_{(00)T,t}^{(12)q}\right)^2\right\}=\frac{(T-t)^2}{2q}\le (T-t)^3.
$$

Let us implement 200 independent numerical modelings of the collection of iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$, $I_{(0)T,t}^{(1)}$ using the formulas (6.41), (6.42) for $T-t=2^{-j}$, $j=5,6,7$. We denote by $T_{\rm sum}$ the computer time which is necessary for performing this task.

Let us repeat the above experiment for the case when the approximations of iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$, $I_{(0)T,t}^{(1)}$ are defined by (5.135), (5.136) and the number q is chosen from the condition (5.127) (method based on Theorem 1.1, the case of Legendre polynomials). Let $T_{\rm pol}$ be the computer time which is necessary for performing this task.

Considering the results from Table 6.1, we come to conclusion that the method based on multiple integral sums even when $T-t=2^{-7}$ is more than 50 times worse in terms of computer time for modeling the collection of iterated Itô stochastic integrals $I_{00_{T,t}}^{(12)}$, $I_{0_{T,t}}^{(1)}$, than the method based on generalized multiple Fourier series.

It is not difficult to see that this effect will be more essential if we consider iterated stochastic integrals of multiplicities $3,4,\ldots$ or choose value $T-t$ smaller than 2^{-7}.

6.4 Iterated Itô Stochastic Integrals as Solutions of Systems of Linear Itô SDEs

Milstein G.N. [82] (also see [99])) proposed an approach to numerical modeling of iterated Itô stochastic integrals based on their representation in the form of systems of linear Itô SDEs. Let us consider this approach using the following set of iterated Itô stochastic integrals

$$I_{(0)s,t}^{(i_1)}=\int\limits_t^s d\mathbf{f}_{t_1}^{(i_1)},\quad I_{(00)s,t}^{(i_1 i_2)}=\int\limits_t^s\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)},\tag{6.43}$$

where $i_1, i_2 = 1,\ldots,m$, $0\le t<s\le T$, $\mathbf{f}_s^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes.

Obviously, we have the following representation

$$d\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}d\mathbf{f}_s^{(i_2)}+\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}d\begin{pmatrix}\mathbf{f}_s^{(i_1)}\\ \mathbf{f}_s^{(i_2)}\end{pmatrix}.\tag{6.44}$$

It is well known [82], [84] that the solution of system (6.44) has the following integral form

$$\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}=\int\limits_t^s e^{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\left(\mathbf{f}_s^{(i_2)}-\mathbf{f}_\theta^{(i_2)}\right)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}d\begin{pmatrix}\mathbf{f}_\theta^{(i_1)}\\ \mathbf{f}_\theta^{(i_2)}\end{pmatrix},\tag{6.45}$$

where e^A is a matrix exponent

$$e^A\stackrel{\rm def}{=}\sum_{k=0}^{\infty}\frac{A^k}{k!},$$

A is a square matrix, and $A^0\stackrel{\rm def}{=}I$ is a unity matrix.

Numerical modeling of the right-hand side of (6.45) is unlikely simpler task than the jointly numerical modeling of the collection of stochastic integrals (6.43). We have to perform numerical modeling of (6.43) within the frames of the considered approach by numerical integration of the system of linear Itô SDEs (6.44). This procedure can be realized using the Euler (Euler–Maruyama) method [82]. Note that the expressions of more accurate numerical methods for the system (6.44) (see Chapter 4) contain the iterated Itô stochastic integrals (6.43) and therefore they useless in our situation.

Let $\{\tau_j\}_{j=0}^{N}$ be the partition of $[t,s]$ such that

$$\tau_j = t + j\Delta, \quad j = 0, 1, \ldots, N, \quad \tau_N = s.$$

Let us consider the Euler method for the system of linear Itô SDEs (6.44)

$$\begin{pmatrix} {\bf y}_{p+1}^{(i_1)} \\ \\ {\bf y}_{p+1}^{(i_1 i_2)} \end{pmatrix} = \begin{pmatrix} {\bf y}_{p}^{(i_1)} \\ \\ {\bf y}_{p}^{(i_1 i_2)} \end{pmatrix} + \begin{pmatrix} \Delta{\bf f}_{\tau_p}^{(i_1)} \\ \\ {\bf y}_{p}^{(i_1)}\Delta{\bf f}_{\tau_p}^{(i_2)} \end{pmatrix}, \quad {\bf y}_0^{(i_1)} = 0, \quad {\bf y}_0^{(i_1 i_2)} = 0, \qquad (6.46)$$

where

$${\bf y}_{\tau_p}^{(i_1)} \stackrel{\rm def}{=} {\bf y}_{p}^{(i_1)}, \quad {\bf y}_{\tau_p}^{(i_1 i_2)} \stackrel{\rm def}{=} {\bf y}_{p}^{(i_1 i_2)}$$

are approximations of the iterated Itô stochastic integrals $I_{(0)\tau_p,t}^{(i_1)}$, $I_{(00)\tau_p,t}^{(i_1 i_2)}$ obtained using the numerical scheme (6.46), $\Delta{\bf f}_{\tau_p}^{(i)} = {\bf f}_{\tau_{p+1}}^{(i)} - {\bf f}_{\tau_p}^{(i)}$, $i = 1, \ldots, m$.

Iterating the expression (6.46), we have

$${\bf y}_N^{(i_1)} = \sum_{l=0}^{N-1} \Delta{\bf f}_{\tau_l}^{(i_1)}, \quad {\bf y}_N^{(i_1 i_2)} = \sum_{q=0}^{N-1}\sum_{l=0}^{q-1} \Delta{\bf f}_{\tau_l}^{(i_1)}\Delta{\bf f}_{\tau_q}^{(i_2)}, \qquad (6.47)$$

where $\sum\limits_{\emptyset} \stackrel{\rm def}{=} 0$.

Obviously, the formulas (6.47) are formulas for approximations of the iterated Itô stochastic integrals (6.43) obtained using the method based on multiple integral sums (see (6.41), (6.42)).

Consequently, the efficiency of methods for the approximation of iterated Itô stochastic integrals based on multiple integral sums and numerical integration of systems of linear Itô SDEs on the base of the Euler method turns out to be equivalent.

6.5 Combined Method of the Mean-Square Approximation of Iterated Itô Stochastic Integrals

This section is written of the base of the work [170] (also see [17]) and devoted to the combined method of approximation of iterated Itô stochastic integrals based on Theorem 1.1 and the method of multiple integral sums (see Sect. 6.3).

The combined method of approximation of iterated Itô stochastic integrals provides a possibility to minimize significantly the total number of the Fourier–Legendre coefficients which are necessary for the approximation of iterated Itô stochastic integrals. However, in this connection the computational costs for approximation of the mentioned stochastic integrals are become bigger.

Using the additive property of the Itô stochastic integral, we have

$$I_{(0)T,t}^{(i_1)} = \sqrt{\Delta} \sum_{k=0}^{N-1} \zeta_{0,k}^{(i_1)} \quad \text{w. p. 1}, \tag{6.48}$$

$$I_{(1)T,t}^{(i_1)} = \sum_{k=0}^{N-1} \left(I_{(1)\tau_{k+1},\tau_k}^{(i_1)} - \Delta^{3/2} k \zeta_{0,k}^{(i_1)} \right) \quad \text{w. p. 1}, \tag{6.49}$$

$$I_{(00)T,t}^{(i_1 i_2)} = \Delta \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \zeta_{0,l}^{(i_1)} \zeta_{0,k}^{(i_2)} + \sum_{k=0}^{N-1} I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)} \quad \text{w. p. 1}, \tag{6.50}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)} = \Delta^{3/2} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \sum_{q=0}^{l-1} \zeta_{0,q}^{(i_1)} \zeta_{0,l}^{(i_2)} \zeta_{0,k}^{(i_3)} +$$

$$+\sqrt{\Delta} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)} \zeta_{0,k}^{(i_3)} + \zeta_{0,l}^{(i_1)} I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)} \right) + \sum_{k=0}^{N-1} I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)} \quad \text{w. p. 1}, \tag{6.51}$$

where stochastic integrals

$$I_{(0)T,t}^{(i_1)}, \quad I_{(1)T,t}^{(i_1)}, \quad I_{(00)T,t}^{(i_1 i_2)}, \quad I_{(000)T,t}^{(i_1 i_2 i_3)}$$

have the form (5.3), $i_1, \ldots, i_k = 1, \ldots, m$, $T - t = N\Delta$, $\tau_k = t + k\Delta$,

$$\zeta_{0,k}^{(i)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{\Delta}} \int\limits_{\tau_k}^{\tau_{k+1}} d\mathbf{f}_s^{(i)},$$

$k = 0, 1, \ldots, N-1$, the sum with respect to the empty set is equal to zero.

Substituting the relation

$$I_{(1)\tau_{k+1},\tau_k}^{(i_1)} = -\frac{\Delta^{3/2}}{2} \left(\zeta_{0,k}^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_{1,k}^{(i_1)} \right) \quad \text{w. p. 1}$$

into (6.49), where $\zeta_{0,k}^{(i_1)}$, $\zeta_{1,k}^{(i_1)}$ are independent standard Gaussian random variables, we get

$$I_{(1)T,t}^{(i_1)} = -\Delta^{3/2} \sum_{k=0}^{N-1} \left(\left(\frac{1}{2} + k \right) \zeta_{0,k}^{(i_1)} + \frac{1}{2\sqrt{3}} \zeta_{1,k}^{(i_1)} \right) \quad \text{w. p. 1}. \tag{6.52}$$

Consider approximations of the following iterated Itô stochastic integrals using the method based on multiple Fourier–Legendre series (Theorem 1.1)

$$I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)},\quad I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)},\quad I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)}.$$

As a result, we get

$$I_{(00)T,t}^{(i_1 i_2)N,q}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\zeta_{0,l}^{(i_1)}\zeta_{0,k}^{(i_2)}+\sum_{k=0}^{N-1}I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q},\tag{6.53}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)N,q_1,q_2}=\Delta^{3/2}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\sum_{q=0}^{l-1}\zeta_{0,q}^{(i_1)}\zeta_{0,l}^{(i_2)}\zeta_{0,k}^{(i_3)}+$$

$$+\sqrt{\Delta}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1}\zeta_{0,k}^{(i_3)}+\zeta_{0,l}^{(i_1)}I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1}\right)+\sum_{k=0}^{N-1}I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2},\tag{6.54}$$

where we suppose that the approximations

$$I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q},\quad I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q_1},\quad I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}$$

are obtained using Theorem 1.1 (the case of Legendre polynomials).

In particular, when $N=2$, the formulas (6.48), (6.52)-(6.54) will look as follows

$$I_{(0)T,t}^{(i_1)}=\sqrt{\Delta}\left(\zeta_{0,0}^{(i_1)}+\zeta_{0,1}^{(i_1)}\right)\quad \text{w. p. 1},\tag{6.55}$$

$$I_{(1)T,t}^{(i_1)}=-\Delta^{3/2}\left(\frac{1}{2}\zeta_{0,0}^{(i_1)}+\frac{3}{2}\zeta_{0,1}^{(i_1)}+\frac{1}{2\sqrt{3}}\left(\zeta_{1,0}^{(i_1)}+\zeta_{1,1}^{(i_1)}\right)\right)\quad \text{w. p. 1},\tag{6.56}$$

$$I_{(00)T,t}^{(i_1 i_2)2,q}=\Delta\left(\zeta_{0,0}^{(i_1)}\zeta_{0,1}^{(i_2)}+I_{(00)\tau_1,\tau_0}^{(i_1 i_2)q}+I_{(00)\tau_2,\tau_1}^{(i_1 i_2)q}\right),\tag{6.57}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)2,q_1,q_2}=\sqrt{\Delta}\left(I_{(00)\tau_1,\tau_0}^{(i_1 i_2)q_1}\zeta_{0,1}^{(i_3)}+\zeta_{0,0}^{(i_1)}I_{(00)\tau_2,\tau_1}^{(i_2 i_3)q_1}\right)+$$

$$+I_{(000)\tau_1,\tau_0}^{(i_1 i_2 i_3)q_2}+I_{(000)\tau_2,\tau_1}^{(i_1 i_2 i_3)q_2},\tag{6.58}$$

where $\Delta=(T-t)/2$, $\tau_k=t+k\Delta$, $k=0,1,2$.

Note that if $N=1$, then (6.48), (6.52)-(6.54) are the formulas for numerical modeling of the mentioned stochastic integrals using the method based on Theorem 1.1.

Further, we will demonstrate that modeling of the iterated Itô stochastic integrals

$$I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1 i_2)},\quad I_{(000)T,t}^{(i_1 i_2 i_3)}$$

using the formulas (6.55)–(6.58) results in abrupt decrease of the total number of Fourier–Legendre coefficients, which are necessary for approximation of these stochastic integrals using the method based on Theorem 1.1.

From the other hand, the formulas (6.57), (6.58) include two approximations of iterated Itô stochastic integrals of second and third multiplicity, and each one of them should be obtained using the method based on Theorem 1.1. Obviously, this leads to an increase in computational costs for the approximation.

Let us calculate the mean-square approximation errors for the formulas (6.53), (6.54). We have

$$E_N^q \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)}-I_{(00)T,t}^{(i_1 i_2)N,q}\right)^2\right\}=\sum_{k=0}^{N-1}\mathsf{M}\left\{\left(I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)}-I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q}\right)^2\right\}=$$

$$=N\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{l=1}^{q}\frac{1}{4l^2-1}\right)=\frac{(T-t)^2}{2N}\left(\frac{1}{2}-\sum_{l=1}^{q}\frac{1}{4l^2-1}\right), \tag{6.59}$$

$$E_N^{q_1,q_2} \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)N,q_1,q_2}\right)^2\right\}=$$

$$=\mathsf{M}\Biggl\{\Biggl(\sum_{k=0}^{N-1}\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)}-I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1}\right)+$$

$$+\zeta_{0,l}^{(i_1)}\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)}-I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1}\right)\Biggr)\Biggr)+I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)}-I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}\Biggr)\Biggr)^2\Biggr\}=$$

$$=\sum_{k=0}^{N-1}\mathsf{M}\Biggl\{\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)}-I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1}\right)+$$

$$+\zeta_{0,l}^{(i_1)}\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)}-I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1}\right)\Biggr)+I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)}-I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}\Biggr)^2\Biggr\}=$$

$$
=\sum_{k=0}^{N-1}\left(\Delta\mathsf{M}\left\{\left(\zeta_{0,k}^{(i_3)}\sum_{l=0}^{k-1}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1i_2)}-I_{(00)\tau_{l+1},\tau_l}^{(i_1i_2)q_1}\right)\right)^2\right\}+\right.
$$

$$
\left.+\Delta\mathsf{M}\left\{\left(\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2i_3)}-I_{(00)\tau_{k+1},\tau_k}^{(i_2i_3)q_1}\right)\sum_{l=0}^{k-1}\zeta_{0,l}^{(i_1)}\right)^2\right\}+H_{k,q_2}^{(i_1i_2i_3)}\right)=
$$

$$
=\sum_{k=0}^{N-1}\left(\Delta\sum_{l=0}^{k-1}\mathsf{M}\left\{\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1i_2)}-I_{(00)\tau_{l+1},\tau_l}^{(i_1i_2)q_1}\right)^2\right\}+\right.
$$

$$
\left.+k\Delta\mathsf{M}\left\{\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2i_3)}-I_{(00)\tau_{k+1},\tau_k}^{(i_2i_3)q_1}\right)^2\right\}+H_{k,q_2}^{(i_1i_2i_3)}\right)=
$$

$$
=\sum_{k=0}^{N-1}\left(2k\Delta\mathsf{M}\left\{\left(I_{(00)\tau_{k+1},\tau_k}^{(i_1i_2)}-I_{(00)\tau_{k+1},\tau_k}^{(i_1i_2)q_1}\right)^2\right\}+H_{k,q_2}^{(i_1i_2i_3)}\right)=
$$

$$
=\sum_{k=0}^{N-1}\left(2k\Delta\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+H_{k,q_2}^{(i_1i_2i_3)}\right)=
$$

$$
=\Delta^3\frac{N(N-1)}{2}\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+\sum_{k=0}^{N-1}H_{k,q_2}^{(i_1i_2i_3)}=
$$

$$
=\frac{1}{2}(T-t)^3\left(\frac{1}{N}-\frac{1}{N^2}\right)\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+
$$

$$
+\sum_{k=0}^{N-1}H_{k,q_2}^{(i_1i_2i_3)}, \tag{6.60}
$$

where

$$
H_{k,q_2}^{(i_1i_2i_3)}=\mathsf{M}\left\{\left(I_{(000)\tau_{k+1},\tau_k}^{(i_1i_2i_3)}-I_{(000)\tau_{k+1},\tau_k}^{(i_1i_2i_3)q_2}\right)^2\right\}.
$$

Moreover, we suppose that $i_1\neq i_2$ in (6.59) and not all indices i_1,i_2,i_3 in (6.60) are equal. Otherwise there are simple relationships for modeling the integrals $I_{(00)T,t}^{(i_1i_2)}$, $I_{(000)T,t}^{(i_1i_2i_3)}$

$$
I_{(00)T,t}^{(i_1i_1)}=\frac{1}{2}(T-t)\left(\left(\zeta_0^{(i_1)}\right)^2-1\right)\quad \text{w. p. 1},
$$

$$
I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right) \quad \text{w. p. 1},
$$

where

$$
\zeta_0^{(i_1)} = \frac{1}{\sqrt{T-t}}\int\limits_t^T d\mathbf{f}_s^{(i_1)}
$$

is a standard Gaussian random variable.

For definiteness, assume that i_1, i_2, i_3 are pairwise different in (6.60) (other cases are represented by (5.55)–(5.58)). Then from Theorem 1.3 we have

$$
H_{k,q_2}^{(i_1 i_2 i_3)} = \Delta^3\left(\frac{1}{6} - \sum_{j_1,j_2,j_3=0}^{q_2}\frac{C_{j_3 j_2 j_1}^2}{\Delta^3}\right), \tag{6.61}
$$

where

$$
C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar{C}_{j_3 j_2 j_1},
$$

$$
\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,
$$

and $P_i(x)$ $(i = 0, 1, 2, \ldots)$ is the Legendre polynomial.

Substituting (6.61) into (6.60), we obtain

$$
E_N^{q_1,q_2} = \frac{1}{2}(T-t)^3\left(\frac{1}{N} - \frac{1}{N^2}\right)\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) +
$$

$$
+\frac{(T-t)^3}{N^2}\left(\frac{1}{6} - \sum_{j_1,j_2,j_3=0}^{q_2}\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{64}\bar{C}_{j_3 j_2 j_1}^2\right). \tag{6.62}
$$

Note that for $N = 1$ the formulas (6.59), (6.62) pass into the corresponding formulas for the mean-square approximation errors of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ (see Theorem 1.3).

Let us consider modeling the integrals $I_{(0)T,t}^{(i_1)}$, $I_{(00)T,t}^{(i_1 i_2)}$. To do it we can use the relations (6.48), (6.53). At that, the mean-square approximation error for the integral $I_{(00)T,t}^{(i_1 i_2)}$ is defined by the formula (6.59) for the case of Legendre polynomials. Let us calculate the value E_N^q for various N and q

$$
E_3^2 \approx 0.0167(T-t)^2, \quad E_2^3 \approx 0.0179(T-t)^2, \tag{6.63}
$$

Table 6.2: $T-t=0.1$.

N	q	q_1	q_2	M
1	13	–	1	21
2	6	0	0	7
3	4	0	0	5

Table 6.3: $T-t=0.05$.

N	q	q_1	q_2	M
1	50	–	2	77
2	25	2	0	26
3	17	1	0	18

$$E_1^6 \approx 0.0192(T-t)^2. \tag{6.64}$$

Note that the combined method (see (6.63)) requires calculation of a significantly smaller number of the Fourier–Legendre coefficients than the method based on Theorem 1.1 (see (6.64)).

Assume that the mean-square approximation error of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ equals to $(T-t)^4$.

In Tables 6.2–6.4 we can see the values N, q, q_1, q_2, which satisfy the system of inequalities

$$\begin{cases} E_N^q \le (T-t)^4 \\ \\ E_N^{q_1,q_2} \le (T-t)^4 \end{cases} \tag{6.65}$$

as well as the total number M of the Fourier–Legendre coefficients, which are nesessary for approximation of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ when $T-t=0.1, 0.05, 0.02$ (the numbers q, q_1, q_2 were taken in such a manner that the number M was the smallest one).

From Tables 6.2–6.4 it is clear that the combined method with the small N

Table 6.4: $T-t=0.02$.

N	q	q_1	q_2	M
1	312	–	6	655
2	156	4	2	183
3	104	6	0	105

$(N = 2)$ provides a possibility to decrease significantly the total number of the Fourier–Legendre coefficients, which are necessary for the approximation of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ in comparison with the method based on Theorem 1.1 $(N = 1)$. However, as we noted before, as a result the computational costs for the approximation are increased. The approximation accuracy of iterated Itô stochastic integrals for the combined method and the method based on Theorem 1.1 was taken $(T-t)^4$.

6.6 Representation of Iterated Itô Stochastic Integrals of Multiplicity k with Respect to the Scalar Standard Wiener Process Based on Hermite Polynomials

In Chapters 1, 2, and 5 we analyzed the general theory of the approximation of iterated Itô and Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. However, in some narrow special cases we can get exact expressions for iterated Itô and Stratonovich stochastic integrals in the form of polynomials of finite degrees from one standard Gaussian random variable. This and next sections will be devoted to this question. The results described in them can be found, for example, in [108] (also see [82], [84]).

Let us consider the set of polynomials $H_n(x, y)$, $n = 0, 1, \ldots$ defined by

$$H_n(x,y)=\left.\left(\frac{d^n}{d\alpha^n}e^{\alpha x-\alpha^2 y/2}\right)\right|_{\alpha=0}.$$

It is well known that polynomials $H_n(x, y)$ are connected with the Hermite polynomials

$$h_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

by the formula

$$H_n(x,y)=\left(\frac{y}{2}\right)^{n/2}h_n\left(\frac{x}{\sqrt{2y}}\right)=y^{n/2}H_n\left(\frac{x}{\sqrt{y}}\right),$$

where $H_n(x)$ is the Hermite polynomial (1.267).

Using the recurrent formulas

$$\frac{dh_n}{dz}(z)=2nh_{n-1}(z),\quad n=1,\ 2,\ldots,$$

$$h_n(z) = 2zh_{n-1}(z) - 2(n-1)h_{n-2}(z), \quad n = 2,\ 3, \ldots,$$

it is easy to get the following recurrent relations for polynomials $H_n(x, y)$

$$\frac{\partial H_n}{\partial x}(x, y) = nH_{n-1}(x, y), \quad n = 1, 2, \ldots, \tag{6.66}$$

$$\frac{\partial H_n}{\partial y}(x, y) = \frac{n}{2y}H_n(x, y) - \frac{nx}{2y}H_{n-1}(x, y), \quad n = 1, 2, \ldots, \tag{6.67}$$

$$\frac{\partial H_n}{\partial y}(x, y) = -\frac{n(n-1)}{2}H_{n-2}(x, y), \quad n = 2, 3, \ldots \tag{6.68}$$

From (6.66) – (6.68) it follows that

$$\frac{\partial H_n}{\partial y}(x, y) + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(x, y) = 0, \quad n = 2, 3, \ldots \tag{6.69}$$

Using the Itô formula, we have

$$H_n(f_t, t) - H_n(0, 0) = \int\limits_0^t \frac{\partial H_n}{\partial x}(f_s, s)df_s + \int\limits_0^t \left(\frac{\partial H_n}{\partial y}(f_s, s) + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(f_s, s)\right) ds \tag{6.70}$$

w. p. 1, where $t \in [0, T]$ and f_t is a scalar standard Wiener process.

Note that $H_n(0, 0) = 0$, $n = 2, 3, \ldots$ Then from (6.69) and (6.70) we get

$$H_n(f_t, t) = \int\limits_0^t nH_{n-1}(f_s, s)df_s, \quad \text{w. p. 1} \quad (n = 2, 3, \ldots). \tag{6.71}$$

Furthermore, by induction it is easy to get the following relation (see (6.71))

$$\int\limits_0^t \ldots \int\limits_0^{t_2} df_{t_1} \ldots df_{t_n} = \frac{H_n(f_t, t)}{n!} \quad \text{w. p. 1} \quad (n = 1, 2, \ldots). \tag{6.72}$$

Let us consider one generalization of the formula (6.72) [108]

$$J_t^{(n)} \stackrel{\sf def}{=} \int\limits_0^t \psi(t_n) \ldots \int\limits_0^{t_2} \psi(t_1)df_{t_1} \ldots df_{t_n} = \frac{H_n(x_t, y(t))}{n!} \quad \text{w. p. 1}, \tag{6.73}$$

where $t \in [0, T]$, $n = 1, 2, \ldots$, and

$$x_t \stackrel{\text{def}}{=} \int\limits_0^t \psi(s) df_s, \quad y(t) \stackrel{\text{def}}{=} \int\limits_0^t \psi^2(s) ds,$$

where $\psi(s) \in L_2([0, T])$.

To prove the equality (6.73), we apply the Itô formula. Using the Itô formula and (6.66), (6.69), we obtain w. p. 1 ($H_n(0,0) = 0$, $n = 2, 3, \ldots$)

$$H_n(x_t, y(t)) - H_n(0,0) = \int\limits_0^t \psi(s) \frac{\partial H_n}{\partial x}(x_s, y(s)) df_s +$$

$$+ \int\limits_0^t \left(\frac{\partial H_n}{\partial s}(x_s, y(s)) + \frac{1}{2} \psi^2(s) \frac{\partial^2 H_n}{\partial x^2}(x_s, y(s)) \right) ds =$$

$$= \int\limits_0^t \psi(s) \frac{\partial H_n}{\partial x}(x_s, y(s)) df_s +$$

$$+ \int\limits_0^t \left(\frac{\partial H_n}{\partial y(s)}(x_s, y(s)) y'(s) + \frac{1}{2} \psi^2(s) \frac{\partial^2 H_n}{\partial x^2}(x_s, y(s)) \right) ds =$$

$$= \int\limits_0^t \psi(s) \frac{\partial H_n}{\partial x}(x_s, y(s)) df_s +$$

$$+ \int\limits_0^t \psi^2(s) \left(\frac{\partial H_n}{\partial y(s)}(x_s, y(s)) + \frac{1}{2} \frac{\partial^2 H_n}{\partial x^2}(x_s, y(s)) \right) ds =$$

$$= \int\limits_0^t \psi(s) \frac{\partial H_n}{\partial x}(x_s, y(s)) df_s = \int\limits_0^t \psi(s) n H_{n-1}(x_s, y(s)) df_s =$$

$$= \int\limits_0^t \psi(s) \int\limits_0^s \psi(\tau) n(n-1) H_{n-2}(x_\tau, y(\tau)) df_\tau df_s = \ldots$$

$$\ldots = n! \int\limits_0^t \psi(t_n) \ldots \int\limits_0^{t_2} \psi(t_1) df_{t_1} \ldots df_{t_n}. \tag{6.74}$$

From (6.74) we get (6.73).

It is easy to check that first eight formulas from the set (6.73) have the following form

$$J_t^{(1)} = \frac{1}{1!}x_t,$$

$$J_t^{(2)} = \frac{1}{2!}\left((x_t)^2 - y(t)\right),$$

$$J_t^{(3)} = \frac{1}{3!}\left((x_t)^3 - 3x_t y(t)\right),$$

$$J_t^{(4)} = \frac{1}{4!}\left((x_t)^4 - 6(x_t)^2 y(t) + 3y^2(t)\right),$$

$$J_t^{(5)} = \frac{1}{5!}\left((x_t)^5 - 10(x_t)^3 y(t) + 15x_t y^2(t)\right),$$

$$J_t^{(6)} = \frac{1}{6!}\left((x_t)^6 - 15(x_t)^4 y(t) + 45(x_t)^2 y^2(t) - 15y^3(t)\right),$$

$$J_t^{(7)} = \frac{1}{7!}\left((x_t)^7 - 21(x_t)^5 y(t) + 105(x_t)^3 y^2(t) - 105x_t y^3(t)\right),$$

$$J_t^{(8)} = \frac{1}{8!}\left((x_t)^8 - 28(x_t)^6 y(t) + 210(x_t)^4 y^2(t) - 420(x_t)^2 y^3(t) + 105y^4(t)\right)$$

w. p. 1. As follows from the results of Sect. 1.1.6, for the case $\psi_1(\tau),\ldots,\psi_k(\tau) \equiv \psi(\tau)$ and $i_1 = \ldots = i_k = 1,\ldots,m$ the formula (1.54) transforms into (6.73).

6.7 Representation of Iterated Stratonovich Stochastic Integrals of Multiplicity k with Respect to the Scalar Standard Wiener Process

Let us prove the following relation for iterated Stratonovich stochastic integrals (see, for example, [84])

$$\int\limits_0^{*t} \ldots \int\limits_0^{*t_2} df_{t_1}\ldots df_{t_n} = \frac{\left(f_t\right)^n}{n!} \quad \text{w. p. 1}, \tag{6.75}$$

where $t\in[0,T]$.

At first, we will consider the case $n=2$. Using Theorem 2.12, we obtain

$$\int\limits_0^{*t}\int\limits_0^{*t_2} df_{t_1}df_{t_2} = \int\limits_0^{t}\int\limits_0^{t_2} df_{t_1}df_{t_2} + \frac{1}{2}\int\limits_0^{t} dt_1 \quad \text{w. p. 1}. \tag{6.76}$$

From the relation (6.72) for $n = 2$ it follows that

$$\int\limits_0^t \int\limits_0^{t_2} df_{t_1} df_{t_2} = \frac{\left(f_t\right)^2}{2!} - \frac{1}{2}\int\limits_0^t dt_1 \quad \text{w. p. 1.} \tag{6.77}$$

Substituting (6.77) into (6.76), we have

$$\int\limits_0^{*t} \int\limits_0^{*t_2} df_{t_1} df_{t_2} = \frac{\left(f_t\right)^2}{2!} \quad \text{w. p. 1.}$$

So, the formula (6.75) is correct for $n = 2$. Using the induction assumption and (2.4), we obtain

$$\int\limits_0^{*t} \ldots \int\limits_0^{*t_2} df_{t_1} \ldots df_{t_{n+1}} = \int\limits_0^{*t} \frac{\left(f_\tau\right)^n}{n!} df_\tau = \int\limits_0^{t} \frac{\left(f_\tau\right)^n}{n!} df_\tau + \frac{1}{2}\int\limits_0^{t} \frac{\left(f_\tau\right)^{n-1}}{(n-1)!} d\tau \tag{6.78}$$

w. p. 1. From the other hand, using the Itô formula, we get

$$\frac{\left(f_t\right)^{n+1}}{(n+1)!} = \int\limits_0^{t} \frac{\left(f_\tau\right)^{n-1}}{2(n-1)!} d\tau + \int\limits_0^{t} \frac{\left(f_\tau\right)^n}{n!} df_\tau \quad \text{w. p. 1.} \tag{6.79}$$

From (6.78) and (6.79) we obtain (6.75). It is easy to see that the formula (6.75) admits the following generalization

$$\int\limits_0^{*t} \psi(t_n) \ldots \int\limits_0^{*t_2} \psi(t_1) df_{t_1} \ldots df_{t_n} = \frac{1}{n!}\left(\int\limits_0^{t} \psi(\tau) df_\tau\right)^n \quad \text{w. p. 1,} \tag{6.80}$$

where $t \in [0, T]$ and $\psi(\tau)$ is a continuous nonrandom function at the interval $[0, T]$.

To prove the equality (6.80), first consider the case $n = 2$. Using Theorem 2.12, we get

$$\int\limits_0^{*t} \psi(t_2) \int\limits_0^{*t_2} \psi(t_1) df_{t_1} df_{t_2} = \int\limits_0^{t} \psi(t_2) \int\limits_0^{t_2} \psi(t_1) df_{t_1} df_{t_2} + \frac{1}{2}\int\limits_0^{t} \psi^2(s) ds \quad \text{w. p. 1.} \tag{6.81}$$

From the relation (6.73) for $n=2$ it follows that

$$\int\limits_0^t \psi(t_2)\int\limits_0^{t_2}\psi(t_1)df_{t_1}df_{t_2}=\frac{1}{2!}\left(\int\limits_0^t\psi(s)df_s\right)^2-\frac{1}{2}\int\limits_0^t\psi^2(s)ds\quad \text{w. p. 1.} \tag{6.82}$$

Substituting (6.82) into (6.81), we obtain

$$\int\limits_0^{*t} \psi(t_2)\int\limits_0^{*t_2}\psi(t_1)df_{t_1}df_{t_2}=\frac{1}{2!}\left(\int\limits_0^t\psi(s)df_s\right)^2\quad \text{w. p. 1.}$$

Thus the formula (6.80) is proved for $n=2$. Applying the induction assumption and (2.4), we have

$$\int\limits_0^{*t}\psi(t_{n+1})\ldots\int\limits_0^{*t_2}\psi(t_1)df_{t_1}\ldots df_{t_{n+1}}=\int\limits_0^{*t}\psi(\tau)\frac{1}{n!}\left(\int\limits_0^{\tau}\psi(s)df_s\right)^n df_{\tau}=$$

$$=\int\limits_0^{t}\psi(\tau)\frac{1}{n!}\left(\int\limits_0^{\tau}\psi(s)df_s\right)^n df_{\tau}+\frac{1}{2}\int\limits_0^t\psi^2(\tau)\frac{1}{(n-1)!}\left(\int\limits_0^{\tau}\psi(s)df_s\right)^{n-1}d\tau \tag{6.83}$$

w. p. 1. Applying the Itô formula, we obtain

$$\frac{1}{(n+1)!}\left(\int\limits_0^t\psi(s)df_s\right)^{n+1}=\int\limits_0^t\psi^2(\tau)\frac{1}{2(n-1)!}\left(\int\limits_0^{\tau}\psi(s)df_s\right)^{n-1}d\tau+$$

$$+\int\limits_0^{t}\psi(\tau)\frac{1}{n!}\left(\int\limits_0^{\tau}\psi(s)df_s\right)^n df_{\tau}\quad \text{w. p. 1.} \tag{6.84}$$

From (6.83) and (6.84) we get (6.80).

6.8 Weak Approximation of Iterated Itô Stochastic Integrals of Multiplicity 1 to 4

In the previous chapters of the book and previous sections of this chapter we analyzed in detail the methods of so-called strong or mean-square approximation of iterated stochastic integrals. For numerical integration of Itô SDEs the so-called weak approximations of iterated Itô stochastic integrals from the Taylor–Itô expansions (see Chapter 4) are also interesting.

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener process, which is F_t-measurable for all $t\in[0,T]$. We suppose that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider an Itô SDE in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau,\quad \mathbf{x}_0=\mathbf{x}(0,\omega), \tag{6.85}$$

where $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the Itô SDE (6.85), the nonrandom functions $\mathbf{a}:\mathbf{R}^n\times[0,T]\to\mathbf{R}^n$, $B:\mathbf{R}^n\times[0,T]\to\mathbf{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (6.85) [100], $\mathbf{x}_0$ is an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\bigl\{|\mathbf{x}_0|^2\bigr\}<\infty$, $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ are independent for $t>0$.

Let us consider the iterated Itô stochastic integrals from the classical Taylor–Itô expansion (see Chapter 4)

$$J_{(\lambda_1\ldots\lambda_k)s,t}^{(i_1\ldots i_k)}=\int\limits_t^s\ldots\int\limits_t^{\tau_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (k\ge 1),$$

where $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_l=0$ if $\lambda_l=0$ and $i_l=1,\ldots,m$ if $\lambda_l=1$ $(l=1,\ldots,k)$. Moreover, let

$$\mathrm{M}_k=\Biggl\{(\lambda_1,\ldots,\lambda_k):\ \lambda_l=0 \text{ or } 1,\ l=1,\ldots,k\Biggr\}.$$

Weak approximations of iterated Itô stochastic stochastic integrals are formed or selected from the specific moment conditions [82], [84], [85], [92], [93] (see below) and they are significantly simpler than their mean-square analogues. However, weak approximations are focused on the numerical solution of other problems [82], [84], [85], [92], [93] connected with Itô SDEs in comparison with mean-square approximations.

We will say that the set of weak approximations

$$\hat{J}_{(\lambda_1\ldots\lambda_k)s,t}^{(i_1\ldots i_k)}$$

of the iterated Itô stochastic integrals

$$J^{(i_1\ldots,i_k)}_{(\lambda_1\ldots\lambda_k)s,t}$$

from the Taylor–Itô expansion (4.22) has the order r, if [82], [84] for $t\in[t_0,T]$ and $r\in\mathbf{N}$ there exists a constant $K\in(0,\infty)$ such that the condition

$$\left|\mathsf{M}\left\{\prod_{g=1}^{l} J^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t,t_0}-\prod_{g=1}^{l}\hat J^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t,t_0}\Bigg|\mathrm{F}_{t_0}\right\}\right|\le K(t-t_0)^{r+1}\quad \text{w. p. 1}\quad (6.86)$$

is satisfied for all $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)\in\mathrm{M}_{k_g},\ i_1^{(g)},\ldots,i_{k_g}^{(g)}=0,1,\ldots,m,\ k_g\le r,\ g=1,\ldots,l,\ l=1,2,\ldots,2r+1.$

If we talk about the unified Taylor–Itô expansion (4.28), then we will say that the set of weak approximations

$$\hat I^{(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}}$$

of the iterated Itô stochastic integrals

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}}=\int\limits_t^s(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}\quad (i_1,\ldots,i_k=1,\ldots,m)$$

has the order r, if for $t\in[t_0,T]$ and $r\in\mathbf{N}$ there exists a constant $K\in(0,\infty)$ such that the condition

$$\left|\mathsf{M}\left\{\prod_{g=1}^{l}\frac{(t-t_0)^{j_g}}{j_g!}\ I^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{l_1^{(g)}\ldots\ l_{k_g\ t,t_0}^{(g)}}-\prod_{g=1}^{l}\frac{(t-t_0)^{j_g}}{j_g!}\ \hat I^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{l_1^{(g)}\ldots\ l_{k_g\ t,t_0}^{(g)}}\Bigg|\mathrm{F}_{t_0}\right\}\right|\le$$

$$\le K(t-t_0)^{r+1}\quad \text{w. p. 1}\qquad (6.87)$$

is satisfied for all $\left(k_g,j_g,l_1^{(g)},\ldots,\ l_{k_g}^{(g)}\right)\in\mathrm{A}_{q_g},\ i_1^{(g)},\ldots,i_{k_g}^{(g)}=1,\ldots,m,\ q_g\le r,$ $g=1,\ldots,l,\ l=1,2,\ldots,2r+1,$ where

$$\mathrm{A}_q=\left\{(k,j,l_1,\ldots,l_k):k+j+\sum_{p=1}^{k}l_p=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\}.$$

The theory of weak approximations of iterated Itô stochastic integrals is not so rich as the theory of mean-square approximations. On the one hand, it is connected with the sufficiency for practical needs of already found approximations [82], [84], [92], and on the other hand, it is connected with the complexity of their formation owing to the necessity to satisfy a lot of moment conditions.

Let us consider the basic results in this area.

In [84] (also see [82]) the authors found the weak approximations with the orders $r = 1, 2$ when $m, n \ge 1$ as well as with the order $r = 3$ when $m = 1$, $n \ge 1$ for iterated Itô stochastic integrals

$$J_{(\lambda_1\ldots\lambda_k)t,t_0}^{(i_1\ldots i_k)}.$$

Recall that n is a dimension of the Itô process $\mathbf{x}_t$, which is a solution of the Itô SDE (6.85) and m is a dimension of the Wiener process in (6.85).

Further, we will consider the mentioned weak approximations as well as weak approximations with the order $r = 4$ when $m = 1$, $n \ge 1$ [171] (2000) for iterated Itô stochastic integrals

$$I_{l_1\ldots l_{k t,t_0}}^{(i_1\ldots i_k)}.$$

In order to shorten the record let us write

$$\mathsf{M}\left\{\prod_{g=1}^{l} J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t_0+\Delta,t_0}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\Bigg|\mathrm{F}_{t_0}\right\} \stackrel{\rm def}{=} \mathsf{M}'\left\{\prod_{g=1}^{l} J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}, \qquad (6.88)$$

where $\Delta \in [0, T-t_0]$, $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right) \in \mathrm{M}_{k_g}$, $k_g \le r$, $g = 1,\ldots,l$.

Further in this section, equalities and inequalities for conditional expectations are understood w. p. 1. As before, $\mathbf{1}_A$ means the indicator of the set A.

Let us consider the exact values of conditional expectations (6.88) calculated in [82], [84] and necessary to form weak approximations

$$\hat{J}_{(\lambda_1\ldots\lambda_k)t_0+\Delta,t_0}^{(i_1\ldots i_k)}$$

of the orders $r = 1, 2$ when $m, n \ge 1$

$$\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}\right\} = \Delta\mathbf{1}_{\{i_1=i_2\}}, \qquad (6.89)$$

$$\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(01)}^{(0i_2)}\right\} = \mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(10)}^{(i_20)}\right\} = \frac{1}{2}\Delta^2\mathbf{1}_{\{i_1=i_2\}}, \qquad (6.90)$$

$$\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}\right\} = \frac{1}{2}\Delta^2\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \qquad (6.91)$$

$$
{\sf M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\begin{cases}\Delta^2 & \text{when } i_1=\ldots=i_4\\ & \\ \Delta^2/2 & \begin{matrix}\text{when } i_3\ne i_4,\ i_1=i_3,\ i_2=i_4\\ \text{or } i_3\ne i_4,\ i_1=i_4,\ i_2=i_3\end{matrix}\\ & \\ 0 & \text{otherwise}\end{cases}, \tag{6.92}
$$

$$
{\sf M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}=\begin{cases}3\Delta^2 & \text{when } i_1=\ldots=i_4\\ & \\ \Delta^2 & \begin{matrix}\text{if among } i_1,\ldots,i_4 \text{ there are}\\ \text{two pairs of identical numbers}\end{matrix}\\ & \\ 0 & \text{otherwise}\end{cases}, \tag{6.93}
$$

$$
{\sf M}'\left\{J_{(10)}^{(i_1 0)}J_{(01)}^{(0 i_2)}\right\}=\frac{1}{6}\Delta^3\mathbf{1}_{\{i_1=i_2\}}, \tag{6.94}
$$

$$
{\sf M}'\left\{J_{(10)}^{(i_1 0)}J_{(10)}^{(i_2 0)}\right\}={\sf M}'\left\{J_{(01)}^{(0 i_1)}J_{(01)}^{(0 i_2)}\right\}=\frac{1}{3}\Delta^3\mathbf{1}_{\{i_1=i_2\}}, \tag{6.95}
$$

$$
{\sf M}'\left\{J_{(01)}^{(0 i_1)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}={\sf M}'\left\{J_{(10)}^{(i_1 0)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}=
$$

$$
=\begin{cases}3\Delta^3/2 & \text{when } i_1=\ldots=i_4\\ & \\ \Delta^3/2 & \begin{matrix}\text{if among } i_1,\ldots,i_4 \text{ there are}\\ \text{two pairs of identical numbers}\end{matrix}\\ & \\ 0 & \text{otherwise}\end{cases}, \tag{6.96}
$$

$$
{\sf M}'\left\{J_{(01)}^{(0 i_1)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\frac{1}{6}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \tag{6.97}
$$

$$
{\sf M}'\left\{J_{(10)}^{(i_1 0)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\frac{1}{3}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \tag{6.98}
$$

$$
\mathsf{M}'\left\{J_{(1)}^{(i_1)}\ldots J_{(1)}^{(i_6)}\right\}=\begin{cases}15\Delta^3 & \text{when } i_1=\ldots=i_6\\[2ex] 3\Delta^3 & \begin{matrix}\text{if among } i_1,\ldots,i_6 \text{ there is a pair}\\ \text{and a quad of identical numbers}\end{matrix}\\[2ex] \Delta^3 & \begin{matrix}\text{if among } i_1,\ldots,i_6 \text{ there are three}\\ \text{pairs of identical numbers}\end{matrix}\\[2ex] 0 & \text{otherwise}\end{cases}, \qquad (6.99)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_5i_6)}\right\}=
$$

$$
=\frac{1}{6}\Delta^3\Biggl(\mathbf{1}_{\{i_2=i_4\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_3=i_6\}}+\mathbf{1}_{\{i_1=i_6\}}\mathbf{1}_{\{i_3=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_6\}}\left(\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_4=i_5\}}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i_3=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_4=i_6\}}\left(\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_5\}}+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{i_1=i_5\}}\right)\Biggr), \qquad (6.100)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}\right\}=
$$

$$
=\frac{1}{2}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{i_5=i_6\}}+
$$

$$
+\frac{1}{6}\Delta^3\Biggl(2\cdot\mathbf{1}_{\{i_1=i_3\}}\left(\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{i_4=i_6\}}+\mathbf{1}_{\{i_2=i_6\}}\mathbf{1}_{\{i_4=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_4=i_6\}}+\mathbf{1}_{\{i_1=i_6\}}\mathbf{1}_{\{i_4=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\left(\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{i_2=i_6\}}+\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{i_2=i_5\}}\right)+
$$

$$
+2\cdot\mathbf{1}_{\{i_2=i_4\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_3=i_6\}}+\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{i_1=i_6\}}\right)\Biggr), \qquad (6.101)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(1)}^{(i_3)}\ldots J_{(1)}^{(i_6)}\right\}=
$$

$$
=\frac{1}{2}\left(\mathsf{M}'\left\{J_{(1)}^{(i_1)}\ldots J_{(1)}^{(i_6)}\right\}-\Delta\mathbf{1}_{\{i_1=i_2\}}\mathsf{M}'\left\{J_{(1)}^{(i_3)}\ldots J_{(1)}^{(i_6)}\right\}\right). \qquad (6.102)
$$

Let us explain the formula (6.101). From the following equality

$$J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}=J_{(11)}^{(i_5i_6)}+J_{(11)}^{(i_6i_5)}+\Delta\mathbf{1}_{\{i_5=i_6\}}\quad\text{w. p. 1}$$

we obtain

$$\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}\right\}=\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_5i_6)}\right\}+$$

$$+\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_6i_5)}\right\}+\Delta\mathbf{1}_{\{i_5=i_6\}}\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}\right\}. \qquad (6.103)$$

Applying (6.91), (6.100) to the right-hand side of (6.103) gives (6.101). It is necessary to note [82], [84] that

$$\mathsf{M}'\left\{\prod_{g=1}^{l}J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}=0$$

if the number of units included in all multi-indices $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)$ is odd ($k_g\le r$, $g=1,\ldots,l$). In addition [82], [84]

$$\left|\mathsf{M}'\left\{\prod_{g=1}^{l}J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}\right|\le K\Delta^{\gamma_l},$$

where $\gamma_l=\delta_l/2+\rho_l$, δ_l is a number of units and ρ_l is a number of zeros included in all multi-indices $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)$, $k_g\le r$, $g=1,\ldots,l$, $K\in(0,\infty)$ is a constant.

In the case $n,m\ge 1$ and $r=1$ we can put [82], [84]

$$\hat{J}_{(1)}^{(i)}=\Delta\tilde{\mathbf{f}}^{(i)}\quad(i=1,\ldots,m),$$

where $\Delta\tilde{\mathbf{f}}^{(i)}$, $i=1,\ldots,m$ are independent discrete random variables for which

$$\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)}=\pm\sqrt{\Delta}\right\}=\frac{1}{2}.$$

It is not difficult to see that the approximation

$$\hat{J}_{(1)}^{(i)}=\sqrt{\Delta}\zeta_0^{(i)}\quad(i=1,\ldots,m)$$

also satisfies the condition (6.86) when $r=1$. Here $\zeta_0^{(i)}$ are independent standard Gaussian random variables.

In the case $n, m \ge 1$ and $r = 2$ as the approximations $\hat{J}_{(1)}^{(i_1)}$, $\hat{J}_{(11)}^{(i_1 i_2)}$, $\hat{J}_{(10)}^{(i_1 0)}$, $\hat{J}_{(01)}^{(0 i_1)}$ are taken the following ones [84]

$$\hat{J}_{(1)}^{(i_1)} = \Delta\tilde{\mathbf{f}}^{(i_1)}, \quad \hat{J}_{(10)}^{(i_1 0)} = \hat{J}_{(01)}^{(0 i_1)} = \frac{1}{2}\Delta \cdot \Delta\tilde{\mathbf{f}}^{(i_1)}, \tag{6.104}$$

$$\hat{J}_{(11)}^{(i_1 i_2)} = \frac{1}{2}\left(\Delta\tilde{\mathbf{f}}^{(i_1)}\Delta\tilde{\mathbf{f}}^{(i_2)} + V^{(i_1 i_2)}\right), \tag{6.105}$$

where $\Delta\tilde{\mathbf{f}}^{(i)}$ are independent Gaussian random variables with zero expectation and variance Δ or independent discrete random variables for which the following conditions are fulfilled

$$\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)} = \pm\sqrt{3\Delta}\right\} = \frac{1}{6},$$

$$\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)} = 0\right\} = \frac{2}{3},$$

$V^{(i_1 i_2)}$ are independent discrete random variables satisfying the conditions

$$\mathsf{P}\left\{V^{(i_1 i_2)} = \pm\Delta\right\} = \frac{1}{2} \quad \text{when} \quad i_2 < i_1,$$

$$V^{(i_1 i_1)} = -\Delta, \quad V^{(i_1 i_2)} = -V^{(i_2 i_1)} \quad \text{when} \quad i_1 < i_2,$$

where $i_1, i_2 = 1, \ldots, m$.

Let us consider the case $r = 3$ and $m = 1, n \ge 1$. In this situation in addition to the formulas (6.89)–(6.103) we need a number of formulas for the conditional expectations (6.88) when $m = 1$.

We have [82], [84]

$$\mathsf{M}'\{J_{(1)}J_{(111)}\} = \mathsf{M}'\{J_{(01)}J_{(111)}\} = \mathsf{M}'\{J_{(10)}J_{(111)}\} = 0,$$

$$\mathsf{M}'\{J_{(011)}J_{(11)}\} = \mathsf{M}'\{J_{(101)}J_{(11)}\} = \mathsf{M}'\{J_{(110)}J_{(11)}\} = \frac{1}{6}\Delta^3,$$

$$\mathsf{M}'\{J_{(001)}J_{(1)}\} = \mathsf{M}'\{J_{(010)}J_{(1)}\} = \mathsf{M}'\{J_{(100)}J_{(1)}\} = \frac{1}{6}\Delta^3,$$

$$\mathsf{M}'\{J_{(100)}J_{(10)}\} = \mathsf{M}'\{J_{(001)}J_{(01)}\} = \frac{1}{8}\Delta^4, \quad \mathsf{M}'\{J_{(111)}J_{(11)}\} = 0,$$

$$\mathsf{M}'\{J_{(010)}J_{(10)}\} = \mathsf{M}'\{J_{(010)}J_{(01)}\} = \frac{1}{6}\Delta^4, \quad \mathsf{M}'\left\{\left(J_{(111)}\right)^2\right\} = \frac{1}{6}\Delta^3,$$

$$\mathsf{M}'\{J_{(100)}J_{(01)}\} = \mathsf{M}'\{J_{(001)}J_{(10)}\} = \frac{1}{24}\Delta^4,$$

$$\mathsf{M}'\{J_{(110)}J_{(10)}\} = \mathsf{M}'\{J_{(110)}J_{(01)}\} = \mathsf{M}'\{J_{(101)}J_{(10)}\} = 0,$$

$$\mathsf{M}'\{J_{(101)}J_{(01)}\} = \mathsf{M}'\{J_{(011)}J_{(10)}\} = \mathsf{M}'\{J_{(011)}J_{(01)}\} = 0,$$

$$\mathsf{M}'\left\{J_{(011)}\left(J_{(1)}\right)^2\right\} = \mathsf{M}'\left\{J_{(101)}\left(J_{(1)}\right)^2\right\} = \mathsf{M}'\left\{J_{(110)}\left(J_{(1)}\right)^2\right\} = \frac{1}{6}\Delta^3,$$

$$\mathsf{M}'\left\{J_{(111)}\left(J_{(1)}\right)^3\right\} = \Delta^3, \quad \mathsf{M}'\left\{J_{(111)}J_{(11)}J_{(1)}\right\} = \frac{1}{2}\Delta^3,$$

where

$$J_{(\lambda_1\ldots\lambda_k)} \stackrel{\rm def}{=} \int\limits_{t_0}^{t_0+\Delta}\ldots\int\limits_{t_0}^{t_2} df_{t_1}^{(\lambda_1)}\ldots df_{t_k}^{(\lambda_k)},$$

$f_t^{(0)} \stackrel{\rm def}{=} t$, $f_t^{(1)} \stackrel{\rm def}{=} f_t$ is standard scalar Wiener process, $\lambda_l = 0$ or $\lambda_l = 1$, $l = 1,\ldots,k$.

In [82], [84] using the given moment relations the authors proposed the following weak approximations of iterated Itô stochastic integrals for $r = 3$ when $m = 1$, $n \ge 1$

$$\hat J_{(1)} = \Delta\tilde f, \tag{6.106}$$

$$\hat J_{(10)} = \Delta\hat f, \quad \hat J_{(01)} = \Delta\cdot\Delta\tilde f - \Delta\hat f, \tag{6.107}$$

$$\hat J_{(11)} = \frac{1}{2}\left(\left(\Delta\tilde f\right)^2 - \Delta\right), \quad \hat J_{(001)} = \hat J_{(010)} = \hat J_{(100)} = \frac{1}{6}\Delta^2\cdot\Delta\tilde f,$$

$$\hat J_{(110)} = \hat J_{(101)} = \hat J_{(011)} = \frac{1}{6}\Delta\left(\left(\Delta\tilde f\right)^2 - \Delta\right),$$

$$\hat J_{(111)} = \frac{1}{6}\Delta\tilde f\left(\left(\Delta\tilde f\right)^2 - 3\Delta\right),$$

where

$$\Delta\tilde f \sim {\rm N}(0,\Delta), \quad \Delta\hat f \sim {\rm N}\left(0,\frac{1}{3}\Delta^3\right), \quad \mathsf{M}\left\{\Delta\tilde f\Delta\hat f\right\} = \frac{1}{2}\Delta^2.$$

Here ${\rm N}(0,\sigma^2)$ is a Gaussian distribution with zero expectation and variance σ^2.

Finally, we will form the weak approximations of iterated Itô stochastic integrals for $r = 4$ when $m = 1$, $n \ge 1$ [1]-[17].

The truncated Taylor–Itô expansion (4.22) when $r = 4$ and $m = 1$ includes 26 various iterated Itô stochastic integrals. The formation of weak approximations for these stochastic integrals satisfying the condition (6.87) when $r = 4$ is extremely difficult due to the necessity to consider a lot of moment conditions. However, this problem can be simplified if we consider the truncated unified Taylor–Itô expansion (4.28) when $r=4$ and $m=1$, since this expansion includes only 15 various iterated Itô stochastic integrals

$$
I_0,\ I_1,\ I_{00},\ I_{000},\ I_2,\ I_{10},\ I_{01},\ I_3,\ I_{11},\ I_{20},\ I_{02},\ I_{100},\ I_{010},\ I_{001},\ I_{0000},
$$

where

$$
I_{l_1\ldots l_k} \stackrel{\rm def}{=} \int\limits_{t_0}^{t_0+\Delta}(t_0-t_k)^{l_k}\ldots\int\limits_{t_0}^{t_2}(t_0-t_1)^{l_1}df_{t_1}\ldots df_{t_k}\quad (k\ge 1)
$$

and f_t is standard scalar Wiener process.

It is not difficult to notice that the condition (6.87) will be satisfied for $r=4$ and $i_1=\ldots=i_4$ if the following more strong condition is fulfilled

$$
\left|{\sf M}\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}-\prod_{g=1}^{l}\hat I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\Biggl|{\rm F}_{t_0}\right\}\right|\le K(t-t_0)^5\quad \hbox{w. p. 1}\qquad (6.108)
$$

for all $l_1^{(g)}\ldots\ l_{k_g}^{(g)}\in A,\ k_g\le 4,\ g=1,\ldots,l,\ l=1,2,\ldots,9,$ where $K\in(0,\infty)$ and

$$
A=\Biggl\{0,1,00,000,2,10,01,3,11,20,02,100,010,001,0000\Biggr\}
$$

is the set of multi-indices.

Let (see Sect. 5.1 and 6.6) [14]-[17], [171]

$$
\hat I_0=\sqrt{\Delta}\zeta_0,\quad \hat I_{00}=\frac{1}{2}\Delta\biggl(\bigl(\zeta_0\bigr)^2-1\biggr),\qquad (6.109)
$$

$$
\hat I_1=-\frac{\Delta^{3/2}}{2}\left(\zeta_0+\frac{1}{\sqrt{3}}\zeta_1\right),\quad \hat I_{000}=\frac{\Delta^{3/2}}{6}\biggl(\bigl(\zeta_0\bigr)^3-3\zeta_0\biggr),\qquad (6.110)
$$

$$
\hat I_{0000}=\frac{\Delta^2}{24}\biggl(\bigl(\zeta_0\bigr)^4-6\bigl(\zeta_0\bigr)^2+3\biggr).\qquad (6.111)
$$

Here and further

$$
\zeta_0 \stackrel{\rm def}{=} \frac{1}{\sqrt{\Delta}}\int\limits_{t_0}^{t_0+\Delta}df_s,\quad \zeta_1 \stackrel{\rm def}{=} \frac{2\sqrt{3}}{\Delta^{3/2}}\int\limits_{t_0}^{t_0+\Delta}\left(s-t_0-\frac{\Delta}{2}\right)df_s,
$$

where f_s is scalar standard Wiener process.

It is not difficult to see that ζ_0, ζ_1 are independent standard Gaussian random variables. In addition, the approximations (6.109)–(6.111) equal w. p. 1

to the iterated Itô stochastic integrals corresponding to these approximations. This implies that all products

$$\prod_{g=1}^{l} \hat{I}_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}},$$

which contain only the approximations (6.109)–(6.111) will convert the left-hand side of (6.108) to zero w. p. 1, i.e. the condition (6.108) will be fulfilled automatically.

For forming the approximations

$$\hat{I}_{100},\ \hat{I}_{010},\ \hat{I}_{001},\ \hat{I}_{10},\ \hat{I}_{01},\ \hat{I}_{11},\ \hat{I}_{20},\ \hat{I}_{02},\ \hat{I}_{2},\ \hat{I}_{3}$$

it is necessary to calculate several conditional expectations

$$\mathsf{M}\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\Bigg|\mathrm{F}_{t_0}\right\}, \tag{6.112}$$

where $l_1^{(g)}\ldots\ l_{k_g}^{(g)}\in A$.

We will denote (6.112) (as before) as follows

$$\mathsf{M}'\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\right\}.$$

We have

$$\mathsf{M}'\{I_3\}=\mathsf{M}'\{I_3(I_0)^2\}=\mathsf{M}'\{I_3I_{00}\}=0,\quad \mathsf{M}'\{I_3I_0\}=-\frac{\Delta^4}{4},$$

$$\mathsf{M}'\{I_2(I_0)^2\}=\mathsf{M}'\{I_2I_{00}\}=\mathsf{M}'\{I_2I_{000}\}=\mathsf{M}'\{I_2I_{0000}\}=0,$$

$$\mathsf{M}'\{I_2(I_{00})^2\}=\mathsf{M}'\{I_2(I_0)^4\}=\mathsf{M}'\{I_2I_{000}I_0\}=0,$$

$$\mathsf{M}'\{I_2I_{00}(I_0)^2\}=\mathsf{M}'\{I_2I_{10}\}=\mathsf{M}'\{I_2I_{01}\}=\mathsf{M}'\{I_2I_1I_0\}=\mathsf{M}'\{I_2\}=0,$$

$$\mathsf{M}'\{I_2I_0\}=\frac{\Delta^3}{3},\quad \mathsf{M}'\{I_2(I_0)^3\}=\Delta^4,\quad \mathsf{M}'\{I_2I_{00}I_0\}=\frac{\Delta^4}{3},$$

$$\mathsf{M}'\{I_2I_1\}=-\frac{\Delta^4}{4},\quad \mathsf{M}'\{I_\mu\}=\mathsf{M}'\{I_\mu I_0\}=\mathsf{M}'\{I_\mu I_{000}\}=\mathsf{M}'\{I_\mu(I_0)^3\}=0,$$

$$\mathsf{M}'\{I_\mu I_{00}I_0\}=\mathsf{M}'\{I_\mu I_1\}=0,\quad \mathsf{M}'\{I_{20}(I_0)^2\}=\frac{\Delta^4}{6},\quad \mathsf{M}'\{I_{20}I_{00}\}=\frac{\Delta^4}{12},$$

$$
\mathsf{M}'\{I_{11}(I_0)^2\}=\frac{\Delta^4}{4},\quad \mathsf{M}'\{I_{11}I_{00}\}=\frac{\Delta^4}{8},\quad \mathsf{M}'\{I_{02}(I_0)^2\}=\frac{\Delta^4}{2},
$$

$$
\mathsf{M}'\{I_{02}I_{00}\}=\frac{\Delta^4}{4},\quad \mathsf{M}'\{I_{\lambda}\}=\mathsf{M}'\{I_{\lambda}I_0\}=\mathsf{M}'\{I_{\lambda}(I_0)^2\}=\mathsf{M}'\{I_{\lambda}I_{00}\}=0,
$$

$$
\mathsf{M}'\{I_{\lambda}I_1\}=\mathsf{M}'\{I_{\lambda}I_{0000}\}=\mathsf{M}'\{I_{\lambda}(I_{00})^2\}=\mathsf{M}'\{I_{\lambda}(I_0)^4\}=0,
$$

$$
\mathsf{M}'\{I_{\lambda}I_{000}I_0\}=\mathsf{M}'\{I_{\lambda}I_{00}(I_0)^2\}=\mathsf{M}'\{I_{\lambda}I_{10}\}=0,
$$

$$
\mathsf{M}'\{I_{\lambda}I_{01}\}=\mathsf{M}'\{I_{\lambda}I_1I_0\}=0,
$$

$$
\mathsf{M}'\{I_{100}I_{000}\}=-\frac{\Delta^4}{24},\quad \mathsf{M}'\{I_{100}(I_0)^3\}=-\frac{\Delta^4}{4},\quad \mathsf{M}'\{I_{100}I_{00}I_0\}=-\frac{\Delta^4}{8},
$$

$$
\mathsf{M}'\{I_{010}I_{000}\}=-\frac{\Delta^4}{12},\quad \mathsf{M}'\{I_{010}(I_0)^3\}=-\frac{\Delta^4}{2},\quad \mathsf{M}'\{I_{010}I_{00}I_0\}=-\frac{\Delta^4}{4},
$$

$$
\mathsf{M}'\{I_{001}I_{000}\}=-\frac{\Delta^4}{8},\quad \mathsf{M}'\{I_{001}(I_0)^3\}=-\frac{3\Delta^4}{4},\quad \mathsf{M}'\{I_{001}I_{00}I_0\}=-\frac{3\Delta^4}{8},
$$

$$
\mathsf{M}'\{I_{\rho}I_0\}=\mathsf{M}'\{I_{\rho}I_{000}\}=\mathsf{M}'\{I_{\rho}(I_0)^3\}=\mathsf{M}'\{I_{\rho}I_{00}I_0\}=0,
$$

$$
\mathsf{M}'\{I_{\rho}I_1\}=\mathsf{M}'\{I_{\rho}I_{0000}\}=\mathsf{M}'\{I_{\rho}(I_0)^5\}=\mathsf{M}'\{I_{\rho}(I_{00})^2I_0\}=0,
$$

$$
\mathsf{M}'\{I_{\rho}I_{00}(I_0)^3\}=\mathsf{M}'\{I_{\rho}I_{000}(I_0)^2\}=\mathsf{M}'\{I_{\rho}I_{0000}I_0\}=0,
$$

$$
\mathsf{M}'\{I_{\rho}I_{000}I_{00}\}=\mathsf{M}'\{I_{\rho}I_{100}\}=\mathsf{M}'\{I_{\rho}I_{010}\}=0,
$$

$$
\mathsf{M}'\{I_{\rho}I_{001}\}=\mathsf{M}'\{I_{\rho}I_2\}=\mathsf{M}'\{(I_{\rho})^2I_0\}=\mathsf{M}'\{I_{\rho}I_{00}I_1\}=0,
$$

$$
\mathsf{M}'\{I_{10}I_{01}I_0\}=\mathsf{M}'\{I_{\rho}\}=\mathsf{M}'\{I_{\rho}I_1(I_0)^2\}=0,
$$

$$
\mathsf{M}'\{I_{10}(I_0)^2\}=-\frac{\Delta^3}{3},\quad \mathsf{M}'\{I_{10}I_{00}\}=-\frac{\Delta^3}{6},\quad \mathsf{M}'\{I_{10}(I_{00})^2\}=-\frac{\Delta^4}{3},
$$

$$
\mathsf{M}'\{I_{10}(I_0)^4\}=-2\Delta^4,\quad \mathsf{M}'\{I_{10}I_{000}I_0\}=-\frac{\Delta^4}{6},
$$

$$
\mathsf{M}'\{I_{10}I_{00}(I_0)^2\}=-\frac{5\Delta^4}{6},
$$

$$
\mathsf{M}'\{(I_{10})^2\}=\frac{\Delta^4}{12},\quad \mathsf{M}'\{I_{10}I_{01}\}=\frac{\Delta^4}{8},\quad \mathsf{M}'\{I_{10}I_1I_0\}=\frac{5\Delta^4}{24},
$$

$$
\mathsf{M}'\{I_{01}(I_0)^2\}=-\frac{2\Delta^3}{3},\quad \mathsf{M}'\{I_{01}I_{00}\}=-\frac{\Delta^3}{3},\quad \mathsf{M}'\{I_{01}(I_{00})^2\}=-\frac{2\Delta^4}{3},
$$

$$
\mathsf{M}'\{I_{01}(I_0)^4\}=-4\Delta^4,\quad \mathsf{M}'\{I_{01}I_{000}I_0\}=-\frac{\Delta^4}{3},
$$

$$\mathsf{M}'\{I_{01}I_{00}(I_0)^2\} = -\frac{5\Delta^4}{3}, \quad \mathsf{M}'\{(I_{01})^2\} = \frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_{01}I_1I_0\} = \frac{3\Delta^4}{8},$$

where

$$\mu = 02,\ 11,\ 20, \quad \lambda = 100,\ 010,\ 001, \quad \rho = 10,\ 01$$

(these recordings should be understood as sequences of digits).

The above relations are obtained using the standard properties of the Itô stochastic integral and the following equalities resulting from the Itô formula

$$(I_0)^4 = 24I_{0000} + 12\Delta I_{00} + 3\Delta^2, \quad (I_{00})^2 = 6I_{0000} + 2\Delta I_{00} + \frac{\Delta^2}{2},$$

$$I_{00}(I_0)^2 = 12I_{0000} + 5\Delta I_{00} + \Delta^2, \quad I_1I_0 = I_{10} + I_{01} - \frac{\Delta^2}{2},$$

$$I_{00}(I_0)^3 = 60I_{00000} + 27\Delta I_{000} + 6\Delta^2 I_0,$$

$$(I_0)^5 = 120I_{00000} + 60\Delta I_{000} + 15\Delta^2 I_0,$$

$$(I_{00})^2 I_0 = 30I_{00000} + 12\Delta I_{000} + \frac{10\Delta^2}{4} I_0,$$

$$I_{000}(I_0)^2 = 20I_{00000} + 7\Delta I_{000} + \Delta^2 I_0, \quad I_{0000}I_0 = 5I_{00000} + \Delta I_{000},$$

$$I_{000}I_{00} = 10I_{00000} + 3\Delta I_{000} + \frac{\Delta^2}{2} I_0, \quad I_{00}I_1 = I_{001} + I_{010} + I_{100} - \frac{\Delta^2}{2} I_0,$$

$$(I_0)^3 = 6I_{000} + 3\Delta I_0, \quad I_{00}I_0 = 3I_{000} + \Delta I_0,$$

$$I_{10}I_0 = I_{010} + I_{100} + \Delta I_1 + I_2, \quad I_{000}I_0 = 4I_{0000} + \Delta I_{00}, \quad (I_0)^2 = 2I_{00} + \Delta,$$

$$I_{01}I_0 = 2I_{001} + I_{010} - \frac{1}{2}\left(I_2 + \Delta^2 I_0\right)$$

w. p. 1.

Using the given before moment relations, we can form the weak approximations $\hat I_{100}$, $\hat I_{010}$, $\hat I_{001}$, $\hat I_{10}$, $\hat I_{01}$, $\hat I_{11}$, $\hat I_{20}$, $\hat I_{02}$, $\hat I_2$, $\hat I_3$ [14]-[17], [171]

$$\hat I_{100} = -\frac{\Delta^{5/2}}{24}\left(\left(\zeta_0\right)^3 - 3\zeta_0\right), \quad \hat I_{010} = -\frac{\Delta^{5/2}}{12}\left(\left(\zeta_0\right)^3 - 3\zeta_0\right), \qquad (6.113)$$

$$\hat I_{001} = -\frac{\Delta^{5/2}}{8}\left(\left(\zeta_0\right)^3 - 3\zeta_0\right), \quad \hat I_{11} = \frac{\Delta^3}{8}\left(\left(\zeta_0\right)^2 - 1\right), \qquad (6.114)$$

$$\hat I_{20} = \frac{\Delta^3}{12}\left(\left(\zeta_0\right)^2 - 1\right), \quad \hat I_{02} = \frac{\Delta^3}{4}\left(\left(\zeta_0\right)^2 - 1\right), \qquad (6.115)$$

$$\hat{I}_3 = -\frac{\Delta^{7/2}}{4}\zeta_0, \quad \hat{I}_2 = \frac{\Delta^{5/2}}{3}\left(\zeta_0 + \frac{\sqrt{3}}{2}\zeta_1\right), \tag{6.116}$$

$$\hat{I}_{10} = \Delta^2\left(-\frac{1}{6}\left(\left(\zeta_0\right)^2 - 1\right) - \frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \pm \frac{1}{12\sqrt{2}}\left(\left(\zeta_1\right)^2 - 1\right)\right), \tag{6.117}$$

$$\hat{I}_{01} = \Delta^2\left(-\frac{1}{3}\left(\left(\zeta_0\right)^2 - 1\right) - \frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \mp \frac{1}{12\sqrt{2}}\left(\left(\zeta_1\right)^2 - 1\right)\right), \tag{6.118}$$

where ζ_0, ζ_1 are the same random variables as in (6.109)–(6.111).

It is easy to check that the approximations (6.109)–(6.111), (6.113)–(6.118) satisfy the condition (6.108) for $r = 4$ and $m = 1$, $n \geq 1$, i.e. they are weak approximations of the order $r = 4$ for the case $m = 1$, $n \geq 1$.

Chapter 7

Approximation of Iterated Stochastic Integrals with Respect to the Q-Wiener Process. Application to the High-Order Strong Numerical Methods for Non-Commutative Semilinear SPDEs with Nonliear Multiplicative Trace Class Noise

7.1 Introduction

There exists a lot of publications on the subject of numerical integration of stochastic partial differential equations (SPDEs) (see, for example [172]-[196]).

One of the perspective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for semilinear SPDEs is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [178], [180]-[183]. A significant step in this direction was made in [182] (2015), [183] (2016), where the exponential Milstein and Wagner–Platen methods for semilinear SPDEs with nonlinear multiplicative trace class noise were constructed. Under the appropriate conditions [182], [183] these methods have strong orders of convergence $1.0-\varepsilon$ and $1.5-\varepsilon$ correspondingly with respect to the temporal variable (where ε is an arbitrary small posilive real number). It should be noted that in [187] (2007) the convergence with strong order 1.0 of the exponential Milstein scheme for semilinear SPDEs was proved under additional smoothness assumptions.

An important feature of the mentioned numerical methods is the presence in them the so-called iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [189]. Approximation of these stochastic integrals is a complex problem. The problem of numerical modeling of these stochastic integrals with multiplicities 1 to 3 was solved in [182], [183] for the case when special commutativity conditions for semilinear SPDE with nonlinear multiplicative trace class noise are fulfilled.

If the mentioned commutativity conditions are not fulfilled, which often corresponds to SPDEs in numerous applications, the numerical modeling of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process becomes much more difficult. Note that the exponential Milstein scheme [182] contains the iterated stochastic integrals of multiplicities 1 and 2 with respect to the infinite-dimensional Q-Wiener process and the exponential Wagner–Platen scheme [183] contains the mentioned stochastic integrals of multiplicities 1 to 3 (see Sect. 7.2).

In [195] (2017), [196] (2018) two methods of the mean-square approximation of simplest iterated (double) stochastic integrals from the exponential Milstein scheme for semilinear SPDEs with nonlinear multiplicative trace class noise and without the commutativity conditions are considered and theorems on the convergence of these methods are given. At that, the basic idea (first of the mentioned methods [195], [196]) about the Karhunen–Loève expansion of the Brownian bridge process was taken from the monograph [82] (Milstein approach, see Sect. 6.2). The second of the mentioned methods [195], [196] is based on the results of Wiktorsson M. [88], [89] (2001).

Note that the mean-square error of approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process consists of two components [195], [196]. The first component is related with the finite-dimentional approximation of the infinite-dimentional Q-Wiener process while the second one is connected with the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions.

It is important to note that the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be reduced to the approximation of iterated Itô stochastic integrals with respect to the finite-dimensional Wiener process. In a lot of author's publications [1]-[63] (see Chapters 1, 2, and 5) an effective method of the mean-square approximation of iterated Itô (and Stratonovich) stochastic integrals with respect to the finite-dimensional Wiener process was proposed and developed. This method is based on the generalized multiple Fourier series, in particular, on the multiple Fourier–Legendre series (see Sect. 5.1).

The purpose of this chapter is an adaptation of the method [1]-[63] for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process. In the author's publications [24], [48] (see Sect. 7.3) the problem of the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process in the sense of the second component of approximation error (see above) has been solved for arbitraty multiplicity k ($k \in \mathbf{N}$) of stochastic integrals and without the assumptions of commutativity for SPDE. More precisely, in [24], [48] the method of generalized multiple Fourier series (Theorems 1.1, 1.2, 1.16) for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process (in the sense of the second component of approximation error).

In Sect. 7.4 (also see [25], [49]), we extend the method [195], [196] and estimate the first component of approximation error for iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. In addition, we combine the obtained results with the results from [24], [48] (see Sect. 7.3). Thus, the results of this chapter can be applied to the implementation of exponential Milstein and Wagner–Platen schemes for semilinear SPDEs with nonlinear multiplicative trace class noise and without the commutativity conditions.

Let U, H be separable $\mathbf{R}$-Hilbert spaces and $L_{HS}(U,H)$ be a space of Hilbert–Schmidt operators mapping from U to H. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$ [189], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here and further $L(U)$ denotes all bounded linear operators on U. Let U_0 be an $\mathbf{R}$-Hilbert space defined as $U_0 = Q^{1/2}(U)$. At that, a scalar product in U_0 is given by the relation [183]

$$\langle u, w\rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w\right\rangle_U$$

for all $u, w \in U_0$.

Consider the semilinear parabolic SPDE with nonlinear multiplicative trace class noise

$$dX_t = (AX_t + F(X_t))\, dt + B(X_t) d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}], \tag{7.1}$$

where nonlinear operators F, B ($F: H \to H$, $B: H \to L_{HS}(U_0, H)$), the linear operator $A: D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the SPDE mild solution (see [183], Assumptions A1–A4).

It is well known [192] that Assumptions A1–A4 [183] guarantee the existence and uniqueness (up to modifications) of the mild solution $X_t : [0, \bar{T}] \times \Omega \to H$ of the SPDE (7.1)

$$X_t = \exp(At)\xi + \int\limits_0^t \exp(A(t-\tau))F(X_\tau)d\tau + \int\limits_0^t \exp(A(t-\tau))B(X_\tau)d\mathbf{W}_\tau \quad (7.2)$$

w. p. 1 for all $t \in [0, \bar{T}]$, where $\exp(At)$, $t \geq 0$ is the semigroup generated by the operator A.

As we mentioned earlier, numerical methods of high orders of accuracy (with respect to the temporal discretization) for approximating the mild solution of the SPDE (7.1), which are based on the Taylor formula for operators and an exponential formula for the mild solution of SPDEs, contain iterated stochastic integrals with respect to the Q-Wiener process [178], [180]-[183], [187].

Note that the exponential Milstein type numerical scheme [182] and the exponential Wagner-Platen type numerical scheme [183] contain, for example, the following iterated stochastic integrals (see Sect. 7.2)

$$\int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \quad \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \quad (7.3)$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}, \quad \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \quad (7.4)$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \quad (7.5)$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \quad (7.6)$$

where $0 \leq t < T \leq \bar{T}$, $Z: \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and F', B', B'' denote Frêchet derivatives. At that, the exponential Milstein type scheme [182] contains integrals (7.3) while the exponential Wagner–Platen type scheme [183] contains integrals (7.3)–(7.6) (see Sect. 7.2).

It is easy to notice that the numerical schemes for SPDEs with higher orders of convergence (with respect to the temporal discretization) in contrast with the numerical schemes from [182], [183] will include iterated stochastic integrals (with respect to the Q-Wiener process) with multiplicities $k > 3$ [181] (2011). So, this chapter is partially devoted to the approximation of iterated stochastic integrals of the form

$$I[\Phi^{(k)}(Z)]_{T,t} = \int\limits_t^T \Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\left(\int\limits_t^{t_2}\Phi_1(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)\ldots\right)d\mathbf{W}_{t_k}, \tag{7.7}$$

where $0 \le t < T \le \bar{T}$, $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v))\ldots\ ))$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0 \times \ldots \times U_0}_{k \text{ times}}$ to H for all $v \in H$.

In Sect. 7.3.1 we consider the approximation of more general iterated stochastic integrals than (7.7). In Sect. 7.3.2 and 7.3.3 some other types of iterated stochastic integrals of multiplicities 2–4 with respect to the Q-Wiener process will be considered.

Note that the stochastic integral (7.6) is not a special case of the stochastic integral (7.7) for $k = 3$. Nevertheless, the extended representation for approximation of the stochastic integral (7.6) is similar to (7.12) (see below) for $k = 3$. Moreover, the mentioned representation for approximation of the stochastic integral (7.6) contains the same iterated Itô stochastic integrals of third multiplicity as in (7.12) for $k = 3$ (see Sect. 7.3.2). These conclusions mean that one of the main results of this chapter (Theorem 7.1, Sect. 7.3.1) for $k = 3$ can be reformulated naturally for the stochastic integral (7.6) (see Sect. 7.3.2).

It should be noted that by developing the approach from the work [183], which uses the Taylor formula for operators and a formula for the mild solution of the SPDE (7.1), we obviously obtain a number of other iterated stochastic integrals. For example, the following stochastic integrals

$$\int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2},$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3},$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3},$$

$$\int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)dt_3,$$

$$\int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2,$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}$$

will be considered in Sect. 7.3.3. Here $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and B', B'', B''', F', F'' are Frêchet derivatives.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i=(i_1,\ldots,i_d)\in J$, $x=(x_1,\ldots,x_d)$, and $J=\bigl\{i:\ i\in\mathbf{N}^d \text{ and } \lambda_i>0\bigr\}$.

The series representation of the Q-Wiener process has the form [189]

$$\mathbf{W}(t,x)=\sum_{i\in J} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar{T}]$$

or in the shorter notations

$$\mathbf{W}_t=\sum_{i\in J} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar{T}],$$

where $\mathbf{w}_t^{(i)}$, $i\in J$ are independent standard Wiener processes.

Note that eigenfunctions e_i, $i\in J$ form an orthonormal basis of U [189].

Consider the finite-dimensional approximation of $\mathbf{W}_t$ [189]

$$\mathbf{W}_t^M=\sum_{i\in J_M} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar{T}], \tag{7.8}$$

where

$$J_M=\bigl\{i:\ 1\le i_1,\ldots,i_d\le M \text{ and } \lambda_i>0\bigr\}. \tag{7.9}$$

Using (7.8) and the relation [189]

$$\mathbf{w}_t^{(i)} = \frac{1}{\sqrt{\lambda_i}} \langle e_i, \mathbf{W}_t \rangle_U, \quad i \in J, \tag{7.10}$$

we obtain

$$\mathbf{W}_t^M = \sum_{i \in J_M} e_i \ \langle e_i, \mathbf{W}_t \rangle_U, \quad t \in [0, \bar{T}], \tag{7.11}$$

where $\langle \cdot, \cdot \rangle_U$ is a scalar product in U.

Taking into account (7.10) and (7.11), we note that the approximation $I[\Phi^{(k)}(Z)]_{T,t}^M$ of the iterated stochastic integral $I[\Phi^{(k)}(Z)]_{T,t}$ (see (7.7)) can be written w. p. 1 in the following form

$$I[\Phi^{(k)}(Z)]_{T,t}^M =$$

$$= \int\limits_t^T \Phi_k(Z) \left(\ldots \left(\int\limits_t^{t_3} \Phi_2(Z) \left(\int\limits_t^{t_2} \Phi_1(Z) d\mathbf{W}_{t_1}^M \right) d\mathbf{W}_{t_2}^M \right) \ldots \right) d\mathbf{W}_{t_k}^M =$$

$$= \sum_{r_1, \ldots, r_k \in J_M} \Phi_k(Z) \left(\ldots \left(\Phi_2(Z) \left(\Phi_1(Z) e_{r_1} \right) e_{r_2} \right) \ldots \right) e_{r_k} \times$$

$$\times \int\limits_t^T \ldots \int\limits_t^{t_3} \int\limits_t^{t_2} d\langle e_{r_1}, \mathbf{W}_{t_1} \rangle_U \ d\langle e_{r_2}, \mathbf{W}_{t_2} \rangle_U \ \ldots \ d\langle e_{r_k}, \mathbf{W}_{t_k} \rangle_U =$$

$$= \sum_{r_1, \ldots, r_k \in J_M} \Phi_k(Z) \left(\ldots \left(\Phi_2(Z) \left(\Phi_1(Z) e_{r_1} \right) e_{r_2} \right) \ldots \right) e_{r_k} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times$$

$$\times \int\limits_t^T \ldots \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)} d\mathbf{w}_{t_2}^{(r_2)} \ldots d\mathbf{w}_{t_k}^{(r_k)}, \tag{7.12}$$

where $0 \le t < T \le \bar{T}$.

Remark 7.1. *Obviously, without loss of generality, we can write* $J_M = \{1, 2, \ldots, M\}$.

As we mentioned before, when special conditions of commutativity for the SPDE (7.1) be fulfilled, it is proposed to simulate numerically the stochastic integrals (7.3)–(7.6) using the simple formulas [182], [183]. In this case, the numerical simulation of the mentioned stochastic integrals requires the use of increments of the Q-Wiener process only. However, if these commutativity conditions are not fulfilled (which often corresponds to SPDEs in numerous applications), the numerical simulation of the stochastic integrals (7.3)–(7.6) becomes much more difficult. Recall that in [195], [196] two methods for the mean-square approximation of simplest iterated (double) stochastic integrals defined by (7.3) are proposed. In this chapter, we consider a substantially more general and effective method (based on the results of Chapters 1 and 5) for the mean-square approximation of iterated stochastic integrals of multiplicity k ($k \in \mathbf{N}$) with respect to the Q-Wiener process. The convergence analysis in the transition from J_M to J, i.e., from the finite-dimensional Wiener process to the infinite-dimensional one will be carried out in Sect. 7.4 for integrals of multiplicities 1 to 3 similar to the proof of Theorem 1 [196].

7.2 Exponential Milstein and Wagner–Platen Numerical Schemes for Non-Commutative Semilinear SPDEs

Let assumptions of Sect. 7.1 are fulfilled. Let $\Delta > 0$, $\tau_p = p\Delta$ ($p = 0, 1, \ldots, N$), and $N\Delta = \bar{T}$. Consider the exponential Milstein numerical scheme [182]

$$Y_{p+1} = \exp\left(A\Delta\right)\left(Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$\left. + \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s\right) \qquad (7.13)$$

and the exponential Wagner–Platen numerical scheme [183]

$$Y_{p+1} = \exp\left(\frac{A\Delta}{2}\right)\left(\exp\left(\frac{A\Delta}{2}\right)Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$+ \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s +$$

$$
+\frac{\Delta^2}{2}F'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)+\int\limits_{\tau_p}^{\tau_{p+1}}F'(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}\right)ds+
$$

$$
+\frac{\Delta^2}{4}\sum_{i\in J}\lambda_i F''(Y_p)\Bigl(B(Y_p)e_i,B(Y_p)e_i\Bigr)+
$$

$$
+A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}ds-\frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}}B(Y_p)d\mathbf{W}_s\right)+
$$

$$
+\Delta\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_s-\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_{\tau}ds+
$$

$$
+\frac{1}{2}\int\limits_{\tau_p}^{\tau_{p+1}}B''(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau},\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}\right)d\mathbf{W}_s+
$$

$$
+\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\left(\int\limits_{\tau_p}^{s}B'(Y_p)\left(\int\limits_{\tau_p}^{\tau}B(Y_p)d\mathbf{W}_{\theta}\right)d\mathbf{W}_{\tau}\right)d\mathbf{W}_s\Biggr) \qquad (7.14)
$$

for the SPDE (7.1), where Y_p is an approximation of X_{τ_p} (mild solution (7.2) at the time moment τ_p), $p=0,1,\ldots,N$, and B', B'', F', F'' are Frêchet derivatives.

Note that in addition to the temporal discretization, the implementation of numerical schemes (7.13) and (7.14) also requires a discretization of the infinite-dimensional Hilbert space H (approximation with respect to the space domain) and a finite-dimensional approximation of the Q-Wiener process. Let us focus on the approximation connected with the Q-Wiener process.

Consider the following iterated Itô stochastic integrals

$$
J_{(1)T,t}^{(r_1)}=\int\limits_t^T d\mathbf{w}_{t_1}^{(r_1)},\quad J_{(10)T,t}^{(r_10)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}dt_2,\quad J_{(01)T,t}^{(0r_2)}=\int\limits_t^T\int\limits_t^{t_2}dt_1d\mathbf{w}_{t_2}^{(r_2)}, \qquad (7.15)
$$

$$
J_{(11)T,t}^{(r_1r_2)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)},\quad J_{(111)T,t}^{(r_1r_2r_3)}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}d\mathbf{w}_{t_3}^{(r_3)}, \quad (7.16)
$$

where $r_1, r_2, r_3 \in J_M$, $0 \le t < T \le \bar{T}$, and J_M is defined by (7.9).

Let us replace the infinite-dimensional Q-Wiener process in the iterated stochastic integrals from (7.13), (7.14) by its finite-dimensional approximation (7.8). Then we have w. p. 1

$$\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s^M = \sum_{r_1\in J_M} B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}J_{(1)\tau_{p+1},\tau_p}^{(r_1)}, \tag{7.17}$$

$$A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M ds - \frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s^M\right) =$$

$$= A\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)\left(\frac{\tau_{p+1}}{2} - s + \frac{\tau_p}{2}\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1\in J_M} AB(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\frac{\Delta}{2}J_{(1)\tau_{p+1},\tau_p}^{(r_1)} - J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{7.18}$$

$$\Delta\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)d\mathbf{W}_s^M - \int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)d\mathbf{W}_\tau^M ds =$$

$$= \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\int\limits_{\tau_p}^{s}\Bigl(AY_p + F(Y_p)\Bigr)d\tau d\mathbf{W}_s^M =$$

$$= \sum_{r_1\in J_M} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}, \tag{7.19}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} F'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)ds =$$

$$= \sum_{r_1\in J_M} F'(Y_p)B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\Delta J_{(1)\tau_{p+1},\tau_p}^{(r_1)} - J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{7.20}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2\in J_M} B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)\tau_{p+1},\tau_p}^{(r_1r_2)}, \qquad (7.21)$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B'(Y_p)\left(\int\limits_{\tau_p}^{\tau} B(Y_p)d\mathbf{W}_\theta^M\right)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B'(Y_p)\left(B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)}, \qquad (7.22)$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1}, B(Y_p)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}. \qquad (7.23)$$

Note that in (7.18)–(7.20) we used the Itô formula. Moreover, using the Itô formula we obtain

$$\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)} = J_{(11)s,\tau_p}^{(r_1r_2)} + J_{(11)s,\tau_p}^{(r_2r_1)} + \mathbf{1}_{\{r_1=r_2\}}(s-\tau_p) \quad \text{w. p. 1}, \qquad (7.24)$$

where $\mathbf{1}_A$ is the indicator of the set A. From (7.24) we have w. p. 1

$$\int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)} = J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + J_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)\tau_{p+1},\tau_p}^{(0r_3)}.$$

$$(7.25)$$

After substituting (7.25) into (7.23), we obtain w. p. 1

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =
$$

$$
= \sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1}, B(Y_p)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times
$$

$$
\times\left(J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + J_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)\tau_{p+1},\tau_p}^{(0r_3)}\right). \tag{7.26}
$$

Thus, for the implementation of numerical schemes (7.13) and (7.14) we need to approximate the following collection of iterated Itô stochastic integrals

$$
J_{(1)T,t}^{(r_1)},\quad J_{(01)T,t}^{(0r_1)},\quad J_{(11)T,t}^{(r_1r_2)},\quad J_{(111)T,t}^{(r_1r_2r_3)}, \tag{7.27}
$$

where $r_1,r_2,r_3\in J_M,\ \ 0\le t<T\le \bar{T}$.

The problem of the mean-square approximation of iterated Itô stochastic integrals (7.27) is considered completely in Chapters 1 and 5.

7.3 Approximation of Iterated Stochastic Integrals of Multiplicity k ($k\in \mathbf{N}$) with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

In this section, we consider a method for the approximation of iterated stochastic integrals of multiplicity k ($k\in\mathbf{N}$) with respect to the finite-dimensional approximation $\mathbf{W}_t^M$ of the Q-Wiener process $\mathbf{W}_t$ using the mean-square approximation method of iterated Itô stochastic integrals based on Theorems 1.1, 1.2, 1.16.

7.3.1 Theorem on the Mean-Square Approximation of Iterated Stochastic Integrals of Multiplicity k ($k\in\mathbf{N}$) with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

Consider the iterated stochastic integral with respect to the Q-Wiener process in the following form

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}=\int\limits_t^T \Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\right)\psi_2(t_2)d\mathbf{W}_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}, \tag{7.28}$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $\mathbf{W}_\tau$ is the Q-Wiener process, $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v\in H$, and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M$ be the approximation of the iterated stochastic integral (7.28)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M=\int\limits_t^T \Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}^M\right)\psi_2(t_2)d\mathbf{W}_{t_2}^M\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}^M=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^k\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}, \tag{7.29}$$

where $0\le t<T\le\bar{T}$ and

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated Itô stochastic integral (1.5).

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}$ be the approximation of the iterated stochastic integral (7.29)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}, \qquad (7.30)$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as the expression before passing to the limit on the right-hand side of (1.321)

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}+\sum_{m=1}^{[k/2]}(-1)^m\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{m}\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}\Biggr). \qquad (7.31)$$

Let U, H be separable $\mathbf{R}$-Hilbert spaces, $U_0=Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let

$$L(U,H)_0=\bigl\{T|_{U_0}:\ T\in L(U,H)\bigr\},$$

where $T|_{U_0}$ is the restriction of operator T to the space U_0. It is known [189] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 7.1 [14]-[17], [24], [48]. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Furthermore, let the following conditions be satisfied:*

1. $Q\in L(U)$ is a nonnegative and symmetric trace class operator (λ_i and e_i $(i\in J)$ are its eigenvalues and eigenfunctions (which form an orthonormal basis of U) correspondingly) and $\mathbf{W}_\tau$, $\tau\in[0,\bar T]$ is an U-valued Q-Wiener process.

2. $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping.

3. $\Phi_1\in L(U,H)_0$, $\Phi_2\in L(H,L(U,H)_0)$, and $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v\in H$ such that

$$
\left\| \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H^2 \le L_k < \infty
$$

w. p. 1 for all $r_1, r_2, \ldots, r_k \in J_M,\ M \in \mathbf{N}$.

Then

$$
\mathsf{M}\left\{\left\| I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}\right\|_H^2\right\} \le
$$

$$
\le L_k (k!)^2 \left(\mathrm{tr}\ Q\right)^k \left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right), \tag{7.32}
$$

where

$$
\mathrm{tr}\ Q = \sum_{i\in J}\lambda_i < \infty,
$$

$$
I_k = \|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k,
$$

$$
C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k
$$

is the Fourier coefficient,

$$
K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l < t_{l+1}\}},
$$

where $t_1,\ldots,t_k \in [t,T]$ $(k \ge 2)$ *and* $K(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t,T]$ ($\mathbf{1}_A$ *denotes the indicator of the set* A).

Remark 7.2. *It should be noted that the right-hand side of the inequality* (7.32) *is independent of* M *and tends to zero if* $p_1,\ldots,p_k \to \infty$ *due to the Parseval equality.*

Remark 7.3. *Recall the estimate (1.328), which we will use in the proof of Theorem 7.1*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k) p_1,\ldots,p_k}\right)^2\right\} \le
$$

$$
\le k!\left(I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2\right),
$$

where $J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)}$ *is defined by (1.5) and* $J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k) p_1,\ldots,p_k}$ *is defined by (7.31).*

Proof. Using (1.129), we obtain

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M} - I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}\right\|_H^2\right\} =
$$

$$
= \mathsf{M}\left\{\left\|\sum_{r_1,r_2,\ldots,r_k \in J_M} \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2} \times\right.\right.
$$

$$
\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k) p_1,\ldots,p_k}\right)\right\|_H^2\right\} = \qquad (7.33)
$$

$$
= \left|\mathsf{M}\left\{\sum_{r_1,r_2,\ldots,r_k \in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2} \times\right.\right.
$$

$$
\times\Biggl\langle \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\ ,
$$

$$
\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{\prime}}\right)e_{r_2^{\prime}}\right)\ldots\right)e_{r_k^{\prime}}\Biggr\rangle_H \times
$$

$$
\times\ \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi^{(k)}]_{T,t}^{(r_1 r_2 \ldots r_k) p_1,\ldots,p_k}\right)\times\right.
$$

$$
\left.\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime} r_2^{\prime} \ldots r_k^{\prime})} - J[\psi^{(k)}]_{T,t}^{(r_1^{\prime} r_2^{\prime} \ldots r_k^{\prime}) p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right\}\right| \le \qquad (7.34)
$$

$$
\le \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times
$$

$$
\times\mathsf{M}\Biggl\{\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H\times
$$

$$
\times\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{\prime}}\right)e_{r_2^{\prime}}\right)\ldots\right)e_{r_k^{\prime}}\right\|_H\times
$$

$$
\times\left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.\right.
$$

$$
\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\Biggr|\Biggr\}\le
$$

$$
\le L_k\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times
$$

$$
\times\mathsf{M}\left\{\left|\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.\right.
$$

$$
\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)\right|\right\}\le
$$

$$
\le L_k\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times
$$

$$
\times\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} k!\ \lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)=
$$

$$
=L_k\,(k!)^2\sum_{r_1,r_2,\ldots,r_k\in J_M}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\le
$$

$$
\le L_k\,(k!)^2\,(\mathrm{tr}\ Q)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),
$$

where $\langle\cdot,\cdot\rangle_H$ is a scalar product in H, and

$$
\sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}}
$$

means the sum with respect to all possible permutations $(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime})$ such that $\{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}$.

The transition from (7.33) to (7.34) is based on the following theorem.

Theorem 7.2 [14]-[17], [24], [48]. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then, the following equality is true*

$$
\mathsf{M}\Biggl\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}\right)\times
$$

$$
\times\left(J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}-J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\right)\Biggl|\mathbf{F}_t\Biggr\}=0 \qquad (7.35)
$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Proof. Using the standard moment properties of the Itô stochastic integral, we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\Bigg|\mathbf{F}_t\right\}=0 \tag{7.36}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $(r_1,\ldots,r_k)\ne(m_1,\ldots,m_k)$.

Using (1.311), (1.318), (1.335), and (1.327), we obtain

$$J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(m_1\ldots m_k)}, \tag{7.37}$$

where

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(m_1\ldots m_k)}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\quad\text{w. p. 1}, \tag{7.38}$$

and

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then m_r swapped with m_q in the permutation $(m_1,\ldots,m_k)$. Another notations are the same as in Theorems 1.1, 1.2, 1.16 ($J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(m_1\ldots m_k)}$ is defined by (1.304)).

Then w. p. 1

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\Bigg|\mathbf{F}_t\right\}.$$

From the standard moment properties of the Itô stochastic integral it follows that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\Bigg|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Then

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=0 \tag{7.39}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Using (7.37), (7.38), we have

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{q_1=0}^{p_1}\ldots\sum_{q_k=0}^{p_k}C_{q_k\ldots q_1}\times$$

$$\times\mathsf{M}\left\{\left(\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\right)\times\right.$$

$$\left.\times\left(\sum_{(q_1,\ldots,q_k)}\int\limits_t^T\phi_{q_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{q_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\right)\Bigg|\mathbf{F}_t\right\}=0 \tag{7.40}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

From (7.36), (7.39), and (7.40) we obtain (7.35). Theorem 7.2 is proved.

Corollary 7.1 [14]-[17], [24], [48]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then, the following equality is true*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}\right)\times\right.$$

$$\left.\times\left(J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)}-J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)q_1,\ldots,q_l}\right)\Bigg|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $l=1,2,\ldots,k-1$ and $r_1,\ldots,r_k$, $m_1,\ldots,m_l\in J_M$, $p_1,\ldots,p_k$, $q_1,\ldots,q_l=0,1,2,\ldots$

7.3.2 Approximation of Some Iterated Stochastic Integrals of Miltiplicities 2 and 3 with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 7.1)

$$I_0[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}^M, \qquad (7.41)$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2, \qquad (7.42)$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M. \qquad (7.43)$$

Let Conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B''(v)(B(v),B(v))$ be a 3-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0$ to H for all $v\in H$. Then we have w. p. 1 (see (7.29))

$$I_0[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M} B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}J_{(01)T,t}^{(0r_1)}, \qquad (7.44)$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M} F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}J_{(10)T,t}^{(r_10)}, \qquad (7.45)$$

$$I_2[B(Z)]_{T,t}^M=\sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}. \qquad (7.46)$$

Using the Itô formula, we obtain

$$\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}=J_{(11)s,t}^{(r_1r_2)}+J_{(11)s,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}(s-t)\quad \text{w. p. 1}. \qquad (7.47)$$

From (7.47) we have

$$\int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)} \int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right) d\mathbf{w}_s^{(r_3)} = J_{(111)T,t}^{(r_1 r_2 r_3)} + J_{(111)T,t}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1=r_2\}} J_{(01)T,t}^{(0 r_3)} \quad \text{w. p. 1.} \tag{7.48}$$

Note that in (7.44), (7.45), (7.47), and (7.48) we use the notations from Sect. 7.2 (see (7.15), (7.16)). After substituting (7.48) into (7.46), we have

$$I_2[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(J_{(111)T,t}^{(r_1 r_2 r_3)} + J_{(111)T,t}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1=r_2\}} J_{(01)T,t}^{(0 r_3)}\right) \quad \text{w. p. 1.} \tag{7.49}$$

Taking into account (5.137), (5.138), we put for $q=1$

$$J_{(01)T,t}^{(0 r_3)q} = J_{(01)T,t}^{(0 r_3)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(r_3)}\right) \quad \text{w. p. 1,} \tag{7.50}$$

$$J_{(10)T,t}^{(r_1 0)q} = J_{(10)T,t}^{(r_1 0)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right) \quad \text{w. p. 1,} \tag{7.51}$$

where $J_{(01)T,t}^{(0 r_3)q}$, $J_{(10)T,t}^{(r_1 0)q}$ denote the approximations of corresponding iterated Itô stochastic integrals.

Denote by $I_0[B(Z),F(Z)]_{T,t}^{M,q}$, $I_1[B(Z),F(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$ the approximations of iterated stochastic integrals (7.44), (7.45), (7.49)

$$I_0[B(Z),F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}} J_{(01)T,t}^{(0 r_1)q}, \tag{7.52}$$

$$I_1[B(Z),F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}} J_{(10)T,t}^{(r_1 0)q}, \tag{7.53}$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(J_{(111)T,t}^{(r_1 r_2 r_3)q} + J_{(111)T,t}^{(r_2 r_1 r_3)q} + \mathbf{1}_{\{r_1=r_2\}} J_{(01)T,t}^{(0 r_3)q}\right), \tag{7.54}$$

where $q = 1$ in (7.52), (7.53) and the approximations $J_{(111)T,t}^{(r_1 r_2 r_3)q}$, $J_{(111)T,t}^{(r_2 r_1 r_3)q}$ are defined by (1.108) for some $q \ge 1$.

From (7.44), (7.45), (7.49), (7.52)–(7.54) we have

$$I_0[B(Z),F(Z)]_{T,t}^{M} - I_0[B(Z),F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. 1},$$

$$I_1[B(Z),F(Z)]_{T,t}^{M} - I_1[B(Z),F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. 1},$$

$$I_2[B(Z)]_{T,t}^{M} - I_2[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(\left(J_{(111)T,t}^{(r_1 r_2 r_3)} - J_{(111)T,t}^{(r_1 r_2 r_3)q} \right) + \left(J_{(111)T,t}^{(r_2 r_1 r_3)} - J_{(111)T,t}^{(r_2 r_1 r_3)q} \right) \right) \quad \text{w. p. 1}.$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k = 3$, we obtain

$$\mathsf{M}\left\{ \left\| I_2[B(Z)]_{T,t}^{M} - I_2[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \le$$

$$\le 4C(3!)^2 \left(\text{tr } Q\right)^3 \left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2 \right),$$

where here and further constant C has the same meaning as constant L_k in Theorem 7.1 (k is the multiplicity of the iterated stochastic integral), and

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

where $P_j(x)$ is the Legendre polynomial.

7.3.3 Approximation of Some Iterated Stochastic Integrals of Miltiplicities 3 and 4 with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

In this section, we consider the approximation of iterated stochastic integrals of the following form (see Sect. 7.1)

$$I_3[B(Z)]_{T,t}^M = \int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M,$$

$$I_4[B(Z)]_{T,t}^M =$$

$$= \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M,$$

$$I_5[B(Z)]_{T,t}^M =$$

$$= \int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M,$$

$$I_6[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)dt_3,$$

$$I_7[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2,$$

$$I_8[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M.$$

Consider the stochastic integral $I_3[B(Z)]_{T,t}^M$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B'''(v)(B(v),B(v),B(v))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$
\times \int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)} \int\limits_t^s d\mathbf{w}_\tau^{(r_2)} \int\limits_t^s d\mathbf{w}_\tau^{(r_3)} \right) d\mathbf{w}_s^{(r_4)} \quad \text{w. p. 1.} \tag{7.55}
$$

By analogy with (2.416) or using the Itô formula, we obtain

$$
J_{(1)s,t}^{(r_1)} J_{(1)s,t}^{(r_2)} J_{(1)s,t}^{(r_3)} = J_{(111)s,t}^{(r_1r_2r_3)} + J_{(111)s,t}^{(r_1r_3r_2)} + J_{(111)s,t}^{(r_2r_1r_3)} + J_{(111)s,t}^{(r_2r_3r_1)} + J_{(111)s,t}^{(r_3r_1r_2)} + J_{(111)s,t}^{(r_3r_2r_1)} +
$$

$$
+\mathbf{1}_{\{r_1=r_2\}} \left(J_{(10)s,t}^{(r_30)} + J_{(01)s,t}^{(0r_3)} \right) + \mathbf{1}_{\{r_1=r_3\}} \left(J_{(10)s,t}^{(r_20)} + J_{(01)s,t}^{(0r_2)} \right) +
$$

$$
+\mathbf{1}_{\{r_2=r_3\}} \left(J_{(10)s,t}^{(r_10)} + J_{(01)s,t}^{(0r_1)} \right) =
$$

$$
= \sum_{(r_1,r_2,r_3)} J_{(111)s,t}^{(r_1r_2r_3)} + (s-t) \left(\mathbf{1}_{\{r_2=r_3\}} J_{(1)s,t}^{(r_1)} + \mathbf{1}_{\{r_1=r_3\}} J_{(1)s,t}^{(r_2)} + \mathbf{1}_{\{r_1=r_2\}} J_{(1)s,t}^{(r_3)} \right) \tag{7.56}
$$

w. p. 1, where

$$
\sum_{(r_1,r_2,r_3)}
$$

means the sum with respect to all possible permutations (r_1, r_2, r_3). We also use the notations from Sect. 7.2 (see (7.15), (7.16)).

After substituting (7.56) into (7.55), we obtain

$$
I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4 \in J_M} B'''(Z) \left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3} \right) e_{r_4} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \times
$$

$$
\times \left(\sum_{(r_1,r_2,r_3)} J_{(1111)T,t}^{(r_1r_2r_3r_4)} - \mathbf{1}_{\{r_1=r_2\}} I_{(01)T,t}^{(r_3r_4)} - \mathbf{1}_{\{r_1=r_3\}} I_{(01)T,t}^{(r_2r_4)} - \mathbf{1}_{\{r_2=r_3\}} I_{(01)T,t}^{(r_1r_4)} \right) \tag{7.57}
$$

w. p. 1, where $J_{(1111)T,t}^{(r_1r_2r_3r_4)}$ is defined by (5.107) and

$$
I_{(01)T,t}^{(r_1r_2)} = \int\limits_t^T (t-s) \int\limits_t^s d\mathbf{w}_\tau^{(r_1)} d\mathbf{w}_s^{(r_2)}. \tag{7.58}
$$

Denote by $I_3[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.57), which has the following form

$$I_3[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2},B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)} J_{(1111)T,t}^{(r_1r_2r_3r_4)q}-\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(r_3r_4)q}-\mathbf{1}_{\{r_1=r_3\}}I_{(01)T,t}^{(r_2r_4)q}-\mathbf{1}_{\{r_2=r_3\}}I_{(01)T,t}^{(r_1r_4)q}\right), \tag{7.59}$$

where the approximations $J_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $I_{(01)T,t}^{(r_1r_2)q}$ are based on Theorem 1.1 and Legendre polynomials (see (5.15) and (5.59)).

For example, from (5.15) we have (here we use the notation $I_{(01)T,t}^{(r_1r_2)}$ from the formula (5.15))

$$I_{(01)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}J_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_1)}\zeta_1^{(r_2)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(r_1)}\zeta_{i+2}^{(r_2)}-(i+1)\zeta_{i+2}^{(r_1)}\zeta_i^{(r_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \tag{7.60}$$

$$J_{(11)T,t}^{(r_1r_2)q}=\frac{T-t}{2}\left(\zeta_0^{(r_1)}\zeta_0^{(r_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(r_1)}\zeta_i^{(r_2)}-\zeta_i^{(r_1)}\zeta_{i-1}^{(r_2)}\right)-\mathbf{1}_{\{r_1=r_2\}}\right), \tag{7.61}$$

where notations are the same as in Theorem 1.1. For $r_1\ne r_2$ we get (see (5.42))

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \tag{7.62}$$

From (1.129) and (7.62) we obtain

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}\le\frac{(T-t)^4}{8}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right),$$

where $r_1,r_2=1,\ldots,M$.

From (7.57) and (7.59) it follows that

$$I_3[B(Z)]_{T,t}^{M} - I_3[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\sum_{(r_1,r_2,r_3)}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)} - J_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right) - \mathbf{1}_{\{r_1=r_2\}}\left(I_{(01)T,t}^{(r_3r_4)} - I_{(01)T,t}^{(r_3r_4)q}\right) -$$

$$-\mathbf{1}_{\{r_1=r_3\}}\left(I_{(01)T,t}^{(r_2r_4)} - I_{(01)T,t}^{(r_2r_4)q}\right) - \mathbf{1}_{\{r_2=r_3\}}\left(I_{(01)T,t}^{(r_1r_4)} - I_{(01)T,t}^{(r_1r_4)q}\right)\Biggr)\quad \text{w. p. 1.} \quad (7.63)$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$, we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M} - I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\text{tr}\;Q)^4\left(6^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + 3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.62) and

$$C_{j_4j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2\bar{C}_{j_4j_3j_2j_1}, \quad (7.64)$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

where $P_j(x)$ is the Legendre polynomial.

Consider the stochastic integral $I_4[B(Z)]_{T,t}^{M}$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B'(v)(B''(v)(B(v),B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_4[B(Z)]_{T,t}^{M} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\int\limits_t^T\int\limits_t^s\left(\int\limits_t^{\tau}d\mathbf{w}_u^{(r_1)}\int\limits_t^{\tau}d\mathbf{w}_u^{(r_2)}\right)d\mathbf{w}_{\tau}^{(r_3)}d\mathbf{w}_s^{(r_4)}\quad \text{w. p. 1.}\quad (7.65)$$

From (7.48) and (7.65) we obtain

$$I_4[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M}B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)}+J_{(1111)T,t}^{(r_2r_1r_3r_4)}-\mathbf{1}_{\{r_1=r_2\}}I_{(10)T,t}^{(r_3r_4)}\right)\quad \text{w. p. 1,}\quad (7.66)$$

where

$$I_{(10)T,t}^{(r_3r_4)}=\int\limits_t^T\int\limits_t^s(t-\tau)d\mathbf{w}_{\tau}^{(r_3)}d\mathbf{w}_s^{(r_4)}. \quad (7.67)$$

Denote by $I_4[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.66), which has the following form

$$I_4[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M}B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)q}+J_{(1111)T,t}^{(r_2r_1r_3r_4)q}-\mathbf{1}_{\{r_1=r_2\}}I_{(10)T,t}^{(r_3r_4)q}\right)\quad \text{w. p. 1,}\quad (7.68)$$

where the approximations $J_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $I_{(10)T,t}^{(r_1r_2)q}$ are based on Theorem 1.1 and Legendre polynomials.

For example, from (5.16) we have (here we use the notation $I_{(10)T,t}^{(r_1r_2)}$ from the formula (5.16))

$$I_{(10)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}J_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(r_2)}\zeta_1^{(r_1)}+$$

$$+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(r_2)}\zeta_i^{(r_1)}-(i+2)\zeta_i^{(r_2)}\zeta_{i+2}^{(r_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\Biggr),\quad (7.69)$$

where the approximation $J_{(11)T,t}^{(r_1r_2)q}$ is defined by (7.61).

Moreover,

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(r_1r_2)}-I_{(10)T,t}^{(r_1r_2)q}\right)^2\right\}=E_q\quad (r_1\ne r_2), \tag{7.70}$$

where E_q is the right-hand side of (7.62) (see (5.42)).

From (7.66), (7.68) we have

$$I_4[B(Z)]_{T,t}^{M}-I_4[B(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)}-J_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right)+\left(J_{(1111)T,t}^{(r_2r_1r_3r_4)}-J_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right)-$$

$$-\mathbf{1}_{\{r_1=r_2\}}\left(I_{(10)T,t}^{(r_3r_4)}-I_{(10)T,t}^{(r_3r_4)q}\right)\Biggr)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$, we obtain

$$\mathsf{M}\left\{\left\|I_4[B(Z)]_{T,t}^{M}-I_4[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le C\,(\text{tr }Q)^4\left(2^2(4!)^2\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right)+(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.62) and $C_{j_4j_3j_2j_1}$ is defined by (7.64).

Consider the stochastic integral $I_5[B(Z)]_{T,t}^{M}$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B''(v)(B(v),B'(v)(B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_5[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)}\quad \text{w. p. 1.} \tag{7.71}$$

Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and Example 3.1) or the Itô formula, we obtain

$$\int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)} \int\limits_t^s \int\limits_t^\tau d\mathbf{w}_u^{(r_2)} d\mathbf{w}_\tau^{(r_1)}\right) d\mathbf{w}_s^{(r_4)} =$$

$$= J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)} +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)} - I_{(01)T,t}^{(r_2 r_4)}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)} \quad \text{w. p. 1}, \tag{7.72}$$

where we use the notations from Sect. 7.2 (see (7.16)) and $I_{(01)T,t}^{(r_1 r_2)}$, $I_{(10)T,t}^{(r_1 r_2)}$ are defined by (7.58), (7.67).

After substituting (7.72) into (7.71), we obtain

$$I_5[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\Biggl(J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)} +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)} - I_{(01)T,t}^{(r_2 r_4)}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)}\Biggr) \quad \text{w. p. 1}. \tag{7.73}$$

Denote by $I_5[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.73), which has the following form

$$I_5[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\Biggl(J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)q} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)q} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)q} +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)q} - I_{(01)T,t}^{(r_2 r_4)q}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)q}\Biggr) \quad \text{w. p. 1}, \tag{7.74}$$

where the approximations $J_{(1111)T,t}^{(r_1 r_2 r_3 r_4)q}$, $I_{(01)T,t}^{(r_1 r_2)q}$, and $I_{(10)T,t}^{(r_1 r_2)q}$ are based on Theorem 1.1 and Legendre polynomials.

From (7.73), (7.74) it follows that

$$I_5[B(Z)]_{T,t}^{M}-I_5[B(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\left(J_{(1111)T,t}^{(r_2r_1r_3r_4)}-J_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right)+\left(J_{(1111)T,t}^{(r_2r_3r_1r_4)}-J_{(1111)T,t}^{(r_2r_3r_1r_4)q}\right)+\left(J_{(1111)T,t}^{(r_3r_2r_1r_4)}-J_{(1111)T,t}^{(r_3r_2r_1r_4)q}\right)+$$

$$+\mathbf{1}_{\{r_1=r_3\}}\left(\left(I_{(10)T,t}^{(r_2r_4)}-I_{(10)T,t}^{(r_2r_4)q}\right)-\left(I_{(01)T,t}^{(r_2r_4)}-I_{(01)T,t}^{(r_2r_4)q}\right)\right)-$$

$$-\mathbf{1}_{\{r_2=r_3\}}\left(I_{(10)T,t}^{(r_1r_4)}-I_{(10)T,t}^{(r_1r_4)q}\right)\Biggr)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$ and taking into account (7.70), we obtain

$$\mathsf{M}\left\{\left\|I_5[B(Z)]_{T,t}^{M}-I_5[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le C\,(\text{tr}\;Q)^4\left(3^2(4!)^2\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right)+3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.62) and $C_{j_4j_3j_2j_1}$ is defined by (7.64).

Consider the stochastic integral $I_6[B(Z),F(Z)]_{T,t}^{M}$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled. We have (see (7.29))

$$I_6[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\int\limits_t^T\int\limits_t^s\int\limits_t^{\tau}d\mathbf{w}_u^{(r_1)}d\mathbf{w}_{\tau}^{(r_2)}ds\quad \text{w. p. 1.}\qquad (7.75)$$

Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and Example 3.1) or the Itô formula, we obtain

$$\int\limits_t^T\int\limits_t^s\int\limits_t^{\tau}d\mathbf{w}_u^{(r_1)}d\mathbf{w}_{\tau}^{(r_2)}ds=(T-t)J_{(11)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_1r_2)}\quad \text{w. p. 1.}\qquad (7.76)$$

After substituting (7.76) into (7.75), we have

$$I_6[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(11)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_1r_2)}\right)\quad \text{w. p. 1.} \tag{7.77}$$

Denote by $I_6[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.77), which has the following form

$$I_6[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(11)T,t}^{(r_1r_2)q}+I_{(01)T,t}^{(r_1r_2)q}\right), \tag{7.78}$$

where the approximations $I_{(01)T,t}^{(r_1r_2)q}$, $J_{(11)T,t}^{(r_1r_2)q}$ are defined by (7.60), (7.61).

From (7.77), (7.78) we get

$$I_6[B(Z),F(Z)]_{T,t}^{M}-I_6[B(Z),F(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)\left(J_{(11)T,t}^{(r_1r_2)}-J_{(11)T,t}^{(r_1r_2)q}\right)+\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)\right)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_6[B(Z),F(Z)]_{T,t}^{M}-I_6[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2C(2!)^2\left(\text{tr }Q\right)^2\left((T-t)^2G_q+E_q\right),$$

where G_q and E_q are the right-hand sides of (5.41) and (7.62) correspondingly.

Consider the stochastic integral $I_7[B(Z),F(Z)]_{T,t}^{M}$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled.

Then we have (see (7.29))

$$I_7[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds\quad \text{w. p. 1.} \tag{7.79}$$

From (7.47) and (7.76) we get w. p. 1

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds=$$

$$=\int\limits_t^T J_{(11)s,t}^{(r_1r_2)}ds+\int\limits_t^T J_{(11)s,t}^{(r_2r_1)}ds+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)\left(J_{(11)T,t}^{(r_1r_2)}+J_{(11)T,t}^{(r_2r_1)}\right)+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)\left(J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}-\mathbf{1}_{\{r_1=r_2\}}(T-t)\right)+$$

$$+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}. \tag{7.80}$$

After substituting (7.80) into (7.79), we obtain

$$I_7[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right)\quad \text{w. p. 1.} \tag{7.81}$$

Denote by $I_7[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.81), which has the following form

$$I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)q}+I_{(01)T,t}^{(r_2r_1)q}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right),\qquad(7.82)$$

where the approximation $I_{(01)T,t}^{(r_1r_2)q}$ is defined by (7.60).

From (7.81), (7.82) it follows that

$$I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}}\left(\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)+\left(I_{(01)T,t}^{(r_2r_1)}-I_{(01)T,t}^{(r_2r_1)q}\right)\right)\quad\text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 4C(2!)^2\left(\text{tr }Q\right)^2E_q,$$

where E_q is the right-hand side of (7.62).

Consider the stochastic integral $I_8[B(Z),F(Z)]_{T,t}^{M}$. Let Conditions 1, 2 of Theorem 7.1 be fulfilled.

Then we have w. p. 1 (see (7.29))

$$I_8[B(Z),F(Z)]_{T,t}^{M}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(01)T,t}^{(r_1r_2)}.\quad(7.83)$$

Denote by $I_8[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.83), which has the following form

$$I_8[B(Z),F(Z)]_{T,t}^{M,q}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(01)T,t}^{(r_1r_2)q},\qquad(7.84)$$

where the approximation $I_{(01)T,t}^{(r_1r_2)q}$ is defined by (7.60).

From (7.83), (7.84) we get

$$I_8[B(Z),F(Z)]_{T,t}^{M}-I_8[B(Z),F(Z)]_{T,t}^{M,q}=$$

$$=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_8[B(Z),F(Z)]_{T,t}^{M}-I_8[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le C(2!)^2\left(\text{tr }Q\right)^2E_q,$$

where E_q is the right-hand side of (7.62).

7.4 Approximation of Iterated Stochastic Integrals of Miltiplicities 1 to 3 with Respect to the Infinite-Dimensional Q-Wiener Process

This section is devoted to the application of Theorem 1.1 and multiple Fourier–Legendre series for the approximation of iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. These iterated stochastic integrals are part of the exponential Milstein and Wagner–Platen numerical methods for semilinear SPDEs with nonlinear multiplicative trace class noise (see Sect. 7.2). Theorem 7.3 (see below) on the mean-square convergence of approximations of iterated stochastic integrals of multiplicities 2 and 3 with respect to the infinite-dimensional Q-Wiener process is formulated and proved. The results of this section can be applied to the implementation of high-order strong numerical methods for non-commutative semilinear SPDEs with nonlinear multiplicative trace class noise.

7.4.1 Formulas for the Numerical Modeling of Iterated Stochastic Integrals of Miltiplicities 1 to 3 with Respect to the Infinite-Dimensional Q-Wiener Process Based on Theorem 1.1 and Legendre Polynomials

This section is devoted to the formulas for numerical modeling of iterated stochastic integrals from the Milstein type scheme (7.13) and the Wagner–Platen type scheme (7.14) for non-commutative semilinear SPDEs. These integrals have the following form (below we introduce new notations for the stochastic integrals (7.88)-(7.91) and their approximations)

$$J_1[B(Z)]_{T,t} = \int\limits_t^T B(Z) d\mathbf{W}_{t_1}, \tag{7.85}$$

$$J_2[B(Z)]_{T,t} = A \left(\int\limits_t^T \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} dt_2 - \frac{(T-t)}{2} \int\limits_t^T B(Z) d\mathbf{W}_{t_1} \right), \tag{7.86}$$

$$J_3[B(Z), F(Z)]_{T,t} = (T-t) \int\limits_t^T B'(Z) \Bigl(AZ + F(Z) \Bigr) d\mathbf{W}_{t_1} -$$

$$- \int\limits_t^T \int\limits_t^{t_2} B'(Z) \Bigl(AZ + F(Z) \Bigr) d\mathbf{W}_{t_1} dt_2, \tag{7.87}$$

$$J_4[B(Z), F(Z)]_{T,t} = \int\limits_t^T F'(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) dt_2, \tag{7.88}$$

$$I_1[B(Z)]_{T,t} = \int\limits_t^T B'(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2}, \tag{7.89}$$

$$I_2[B(Z)]_{T,t} = \int\limits_t^T B'(Z) \left(\int\limits_t^{t_3} B'(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) d\mathbf{W}_{t_3}, \tag{7.90}$$

$$I_3[B(Z)]_{T,t} = \int\limits_t^T B''(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2}, \tag{7.91}$$

where $Z : \Omega \rightarrow H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $0 \le t < T \le \bar{T}$.

Note that according to (7.17)–(7.20), (5.7), (5.137), and (5.138), we can write the following relatively simple formulas for numerical modeling [25], [49]

$$J_1[B(Z)]_{T,t}^M = \int\limits_t^T B(Z) d\mathbf{W}_s^M =$$

$$= (T-t)^{1/2}\sum_{r_1\in J_M} B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_0^{(r_1)},$$

$$J_2[B(Z)]_{T,t}^M = A\left(\int\limits_t^T\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M dt_2 - \frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_{t_1}^M\right) =$$

$$= -\frac{(T-t)^{3/2}}{2\sqrt{3}}\sum_{r_1\in J_M} AB(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_1^{(r_1)}, \qquad (7.92)$$

$$J_3[B(Z),F(Z)]_{T,t}^M = (T-t)\int\limits_t^T B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}^M -$$

$$-\int\limits_t^T\int\limits_t^{t_2} B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}^M dt_2 =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} B'(Z)\Bigl(AZ+F(Z)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \qquad (7.93)$$

$$J_4[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2 =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} F'(Z)B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \qquad (7.94)$$

where $\zeta_0^{(r_1)}$, $\zeta_1^{(r_1)}$ $(r_1\in J_M)$ are independent standard Gaussian random variables.

Further, consider the stochastic integrals (7.89)–(7.91), which are more complicate, in detail.

Let $I_1[B(Z)]_{T,t}^M$, $I_2[B(Z)]_{T,t}^M$, $I_3[B(Z)]_{T,t}^M$ be approximations of the stochastic integrals (7.89)–(7.91), which have the following form (see (7.21), (7.22), and (7.26))

$$I_1[B(Z)]_{T,t}^M = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M =$$

$$=\sum_{r_1,r_2\in J_M} B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)T,t}^{(r_1r_2)}, \qquad (7.95)$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M=$$

$$=\sum_{r_1,r_2,r_3\in J_M} B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)T,t}^{(r_1r_2r_3)}, \qquad (7.96)$$

$$I_3[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M=$$

$$=\sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)}+J_{(111)T,t}^{(r_2r_1r_3)}+\mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)}\right). \qquad (7.97)$$

Let $I_1[B(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M.q}$, $I_3[B(Z)]_{T,t}^{M,q}$ be approximations of the stochastic integrals (7.95)–(7.97), which look as follows

$$I_1[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M} B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)T,t}^{(r_1r_2)q},$$

$$I_2[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M} B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)T,t}^{(r_1r_2r_3)q}, \qquad (7.98)$$

$$I_3[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)q}+J_{(111)T,t}^{(r_2r_1r_3)q}+\mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)}\right),$$

where the approximations $J_{(11)T,t}^{(r_1r_2)q}$, $J_{(111)T,t}^{(r_1r_2r_3)q}$, $J_{(111)T,t}^{(r_2r_1r_3)q}$ of the stochastic integrals (7.16) are defined by (5.136), (5.139) and $J_{(01)T,t}^{(0r_3)}$ has the form (5.137), $q\ge 1$.

7.4.2 Theorem on the Mean-Square Approximation of Iterated Stochastic Integrals of Multiplicities 2 and 3 with Respect to the Ininite-Dimensional Q-Wiener Process

Recall that $L_{HS}(U_0,H)$ is a space of Hilbert–Schmidt operators mapping from U_0 to H. Moreover, let $L_{HS}^{(2)}(U_0,H)$ and $L_{HS}^{(3)}(U_0,H)$ be spaces of bilinear and 3-linear Hilbert–Schmidt operators mapping from $U_0\times U_0$ to H and from $U_0\times U_0\times U_0$ to H correspondingly. Furthermore, let

$$\|\cdot\|_{L_{HS}(U_0,H)},\quad \|\cdot\|_{L_{HS}^{(2)}(U_0,H)},\quad \|\cdot\|_{L_{HS}^{(3)}(U_0,H)}$$

be operator norms in these spaces.

Theorem 7.3 [14]-[17], [25], [49], [57], [58]. *Let Conditions* 1, 2 *of Theorem* 7.1 *be fulfilled. Furthermore, let*

$$B(v)\in L_{HS}(U_0,H),\quad B'(v)(B(v))\in L_{HS}^{(2)}(U_0,H),$$

$$B'(v)(B'(v)(B(v))),\quad B''(v)(B(v),B(v))\in L_{HS}^{(3)}(U_0,H)$$

for all $v\in H$ (we suppose that Frêchet derivatives B', B'' exist; see Sect. 7.1*). Moreover, let there exists a constant C such that w. p.* 1

$$\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}<C,\quad \left\|B'(Z)(B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)}<C,$$

$$\left\|B'(Z)(B'(Z)(B(Z)))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C,$$

$$\left\|B''(Z)(B(Z),B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C$$

for some $\alpha>0$. Then

$$\mathsf{M}\left\{\left\|I_1[B(Z)]_{T,t}-I_1[B(Z)]_{T,t}^{M,p}\right\|_H^2\right\}\le$$

$$\le (T-t)^2\left(C_0\,(\text{tr }Q)^2\left(\frac{1}{2}-\sum_{j=1}^{p}\frac{1}{4j^2-1}\right)+K_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right),\quad (7.99)$$

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le (T-t)^3\left(C_1\,(\mathrm{tr}\ Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+L_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \quad (7.100)$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le (T-t)^3\left(C_2\,(\mathrm{tr}\ Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+M_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \quad (7.101)$$

where $p,q\in\mathbf{N},\ C_0,C_1,C_2,K_Q,L_Q,M_Q<\infty,$ *and*

$$\hat{C}_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_j(x)$ $(j=0,1,2,\ldots)$ *is the Legendre polynomial.*

Remark 7.4. *Note that the estimate similar to* (7.99) *has been derived in* [195], [196] (*also see* [182]) *with the difference connected with the first term on the right-hand side of* (7.99)*. In* [196] *the authors used the Karhunen–Loève expansion of the Brownian bridge process for the approximation of iterated Itô stochastic integrals with respect to the finite-dimensional Wiener process* (*Milstein approach, see Sect.* 6.2)*. In this section, we apply Theorem* 1.1 *and the system of Legendre polynomials to obtain the first term on the right-hand side of* (7.99).

Remark 7.5. *If we assume that* $\lambda_i\le C'i^{-\gamma}$ $(\gamma>1,C'<\infty)$ *for* $i\in J$*, then the parameter* $\alpha>0$ *obviously increases with decreasing of* γ [195].

Proof. The estimate (7.99) follows directly from (7.32) for $k=2$ (the first term on the right-hand side of (7.99)) and Theorem 1 from [196] (the second term on the right-hand side of (7.99)). Further C_3, C_4, ... denote various constants.

Let us prove the estimates (7.100), (7.101). Using Theorem 7.1, we obtain

$$
\begin{gathered}
\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+\\
+2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le\\
\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+\\
+C_3(T-t)^3\left(\operatorname{tr} Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \qquad (7.102)
\end{gathered}
$$

$$
\begin{gathered}
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+\\
+2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}. \qquad (7.103)
\end{gathered}
$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=3$, we have (also see Sect. 7.3.2)

$$
\begin{gathered}
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le\\
\le 4\tilde{C}(3!)^2\left(\operatorname{tr} Q\right)^3(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \qquad (7.104)
\end{gathered}
$$

where constant $\tilde{C}$ has the same meaning as constant L_k in Theorem 7.1 (k is the multiplicity of the iterated stochastic integral).

Combining (7.103) and (7.104), we obtain

$$
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+
$$

$$
+C_4(T-t)^3\left(\text{tr}\ Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right). \qquad (7.105)
$$

Let us estimate the right-hand sides of (7.102) and (7.105). Using the elementary inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$, we obtain

$$
\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(E_{T,t}^{1,M}+E_{T,t}^{2,M}+E_{T,t}^{3,M}\right), \qquad (7.106)
$$

$$
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(G_{T,t}^{1,M}+G_{T,t}^{2,M}+G_{T,t}^{3,M}\right), \qquad (7.107)
$$

where

$$
E_{T,t}^{1,M}=
$$

$$
=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},
$$

$$
E_{T,t}^{2,M}=
$$

$$
=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},
$$

$$
E_{T,t}^{3,M}=
$$

$$
=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\left(\mathbf{W}_{t_3}-\mathbf{W}_{t_3}^M\right)\right\|_H^2\right\},
$$

$$G_{T,t}^{1,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{2,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right), \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{3,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2} - \mathbf{W}_{t_2}^M\right)\right\|_H^2\right\}.$$

We have

$$E_{T,t}^{1,M} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3 \le$$

$$\le C_5\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\}dt_3 =$$

$$= C_5\int\limits_t^T\int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2dt_3 \le$$

$$\le C_6\int\limits_t^T\int\limits_t^{t_3} \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2dt_3 \le \qquad (7.108)$$

$$\le C_6\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1dt_2dt_3 \le \quad (7.109)$$

$$
\le C_7 \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha} (T-t)^3. \tag{7.110}
$$

Note that the transition from (7.108) to (7.109) was made by analogy with the proof of Theorem 1 in [196] (also see [182]). More precisely, taking into account the relation $Q^\alpha e_i = \lambda_i^\alpha e_i$, we have (see [196], Sect. 3.1)

$$
\mathsf{M}\left\{ \left\| \int\limits_t^{t_2} B(Z) d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right) \right\|_H^2 \right\} =
$$

$$
= \mathsf{M}\left\{ \left\| \sum_{i \in J \backslash J_M} \sqrt{\lambda_i} \int\limits_t^{t_2} B(Z) e_i d\mathbf{w}_{t_1}^{(i)} \right\|_H^2 \right\} =
$$

$$
= \sum_{i \in J \backslash J_M} \lambda_i \int\limits_t^{t_2} \mathsf{M}\left\{ \left\| B(Z) Q^{-\alpha} Q^\alpha e_i \right\|_H^2 \right\} dt_1 =
$$

$$
= \sum_{i \in J \backslash J_M} \lambda_i^{1+2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{ \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 =
$$

$$
= \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{ \sum_{i \in J \backslash J_M} \lambda_i \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 \le
$$

$$
\le \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{ \sum_{i \in J} \lambda_i \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 =
$$

$$
= \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{ \left\| B(Z) Q^{-\alpha} \right\|_{L_{HS}(U_0,H)}^2 \right\} dt_1. \tag{7.111}
$$

Further, we also will use the estimate similar to (7.111).

We have

$$
E_{T,t}^{2,M} =
$$

$$
= \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3 \le
$$

$$
\le C_8 \int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right\|_H^2\right\} dt_3 \le
$$

$$
\le C_8 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_2 dt_3 \le
$$

$$
\le C_8 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(B(Z)\right)Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)}^2\right\} (t_2-t)dt_2 dt_3 \le
$$

$$
\le C_9 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (7.112)
$$

Moreover,

$$
E_{T,t}^{3,M} \le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times
$$

$$
\times \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M\right) Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_3 \le
$$

$$
\le C_{10} \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times
$$

$$
\times \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(B'(Z)\left(B(Z)\right)\right)Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}^2\right\} \frac{(t_3-t)^2}{2} dt_3 \le
$$

$$
\leq C_{11}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.113)
$$

Combining (7.102), (7.106), (7.110), (7.112), and (7.113), we obtain (7.100).

We have

$$
G_{T,t}^{1,M}=
$$

$$
=\int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1},\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3\leq
$$

$$
\leq C_{12}\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^2\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_3\leq
$$

$$
\leq C_{12}\int\limits_t^T\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^4\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\leq
$$

$$
\leq C_{13}\int\limits_t^T\int\limits_t^{t_2}\left(\mathsf{M}\left\{\left\|B(Z)\right\|_{L_{HS}(U_0,H)}^4\right\}\right)^{1/2}dt_1\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\leq
$$

$$
\leq C_{14}\int\limits_t^T(t_2-t)\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3. \qquad (7.114)
$$

Let us estimate the right-hand side of (7.114). If $s>t$, then for fixed $M\in\mathbf{N}$ and for some $N>M$ $(N\in\mathbf{N})$ we have

$$\mathsf{M}\left\{\left\|\int\limits_t^s B(Z)d\left(\mathbf{W}_{t_1}^N-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}=$$

$$=\mathsf{M}\left\{\left\langle\sum_{j\in J_N\backslash J_M}\sqrt{\lambda_j}B(Z)e_j\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right),\right.\right.$$

$$\left.\left.\sum_{j'\in J_N\backslash J_M}\sqrt{\lambda_{j'}}B(Z)e_{j'}\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\right\rangle_H^2\right\}=$$

$$=\sum_{j,j',l,l'\in J_N\backslash J_M}\sqrt{\lambda_j\lambda_{j'}\lambda_l\lambda_{l'}}\ \mathsf{M}\Biggl\{\Bigl\langle B(Z)e_j,B(Z)e_{j'}\Bigr\rangle_H\Bigl\langle B(Z)e_l,B(Z)e_{l'}\Bigr\rangle_H\times$$

$$\times\mathsf{M}\left\{\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right)\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\left(\mathbf{w}_s^{(l)}-\mathbf{w}_t^{(l)}\right)\left(\mathbf{w}_s^{(l')}-\mathbf{w}_t^{(l')}\right)\Bigl|\mathbf{F}_t\right\}\Biggr\}=$$

$$=3(s-t)^2\sum_{j\in J_N\backslash J_M}\lambda_j^2\mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^4\right\}+$$

$$+(s-t)^2\sum_{j,j'\in J_N\backslash J_M(j\ne j')}\lambda_j\lambda_{j'}\Biggl(\mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^2\Bigl\|B(Z)e_{j'}\Bigr\|_H^2\right\}+$$

$$+2\Bigl\langle B(Z)e_j,B(Z)e_{j'}\Bigr\rangle_H^2\Biggr)\le$$

$$\le 3(s-t)^2\Biggl(\sum_{j\in J_N\backslash J_M}\lambda_j^2\mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^4\right\}+$$

$$+\sum_{j,j'\in J_N\backslash J_M(j\ne j')}\lambda_j\lambda_{j'}\mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^2\Bigl\|B(Z)e_{j'}\Bigr\|_H^2\right\}\Biggr)=$$

$$=3(s-t)^2\mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M}\lambda_j\Bigl\|B(Z)e_j\Bigr\|_H^2\right)^2\right\}\le$$

$$\le 3(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha}\mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M}\lambda_j\Bigl\|B(Z)Q^{-\alpha}e_j\Bigr\|_H^2\right)^2\right\}\le$$

$$\leq C_{15}(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|^4_{L_{HS}(U_0,H)}\right\}. \qquad (7.115)$$

Using (7.115), we obtain

$$\mathsf{M}\left\{\left\|\int\limits_t^s B(Z)d\left(\mathbf{W}^N_{t_1}-\mathbf{W}^M_{t_1}\right)\right\|^4_H\right\}\to 0$$

if $N,M\to\infty$ $(N>M)$. This means that

$$\int\limits_t^s B(Z)d\mathbf{W}^N_{t_1}$$

is a Cauchy sequence in the L_4–space of H–valued stochastic processes.

It is well known [226] that L_p–space $(1\leq p<\infty)$ of Banach space valued stochastic processes is a Banach space, i.e. a complete space. Then, carrying out the passage to the limit $\lim\limits_{N\to\infty}$ in (7.115), we get

$$\mathsf{M}\left\{\left\|\int\limits_t^s B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}^M_{t_1}\right)\right\|^4_H\right\}=$$

$$=\lim_{N\to\infty}\mathsf{M}\left\{\left\|\int\limits_t^s B(Z)d\left(\mathbf{W}^N_{t_1}-\mathbf{W}^M_{t_1}\right)\right\|^4_H\right\}\leq$$

$$\leq C_{15}(s-t)^2\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{4\alpha}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|^4_{L_{HS}(U_0,H)}\right\}. \qquad (7.116)$$

Combining (7.114) and (7.116), we obtain

$$G^{1,M}_{T,t}\leq C_{16}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.117)$$

Absolutely analogously we get

$$G^{2,M}_{T,t}\leq C_{17}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.118)$$

Let us estimate $G_{T,t}^{3,M}$. We have

$$G_{T,t}^{3,M} \le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times$$

$$\times \int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2 \le$$

$$\le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \sum_{i\in J}\sum_{j,l\in J_M} \lambda_i\lambda_j\lambda_l \times$$

$$\times \int\limits_t^T (t_2-t)^2 \left(\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H^2\right\}+\right.$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_j)Q^{-\alpha}e_i\right\|_H \left\|B''(Z)(B(Z)e_l, B(Z)e_l)Q^{-\alpha}e_i\right\|_H\right\}+$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H \times\right.$$

$$\left.\left.\times\left\|B''(Z)(B(Z)e_l, B(Z)e_j)Q^{-\alpha}e_i\right\|_H\right\}\right)dt_2 \le$$

$$\le C_{18}\left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.119)$$

Combining (7.105), (7.107), and (7.117)–(7.119), we obtain (7.101). Theorem 7.3 is proved.

Let us consider the convergence analysis for the stochastic integrals (7.86)–(7.88) (convergence for the stochastic integral (7.85) follows from (7.111) (see Theorem 1 in [196] or [182])).

Using the Itô formula, we obtain w. p. 1 [183]

$$J_2[B(Z)]_{T,t} = \int\limits_t^T \left(\frac{T}{2} - s + \frac{t}{2}\right) AB(Z)d\mathbf{W}_s,$$

$$J_3[B(Z),F(Z)]_{T,t}=\int\limits_t^T(s-t)B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_s.$$

Suppose that

$$\mathsf{M}\left\{\left\|B'(Z)\Bigl(AZ+F(Z)\Bigr)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}<\infty,$$

$$\mathsf{M}\left\{\left\|AB(Z)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}<\infty$$

for some $\alpha>0$.

Then by analogy with (7.111) we get

$$\mathsf{M}\left\{\left\|J_2[B(Z)]_{T,t}-J_2[B(Z)]^M_{T,t}\right\|^2_H\right\}\le$$

$$\le C_{19}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},$$

$$\mathsf{M}\left\{\left\|J_3[B(Z),F(Z)]_{T,t}-J_3[B(Z),F(Z)]^M_{T,t}\right\|^2_H\right\}\le$$

$$\le C_{20}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},$$

where $J_2[B(Z)]^M_{T,t}$, $J_3[B(Z),F(Z)]^M_{T,t}$ are defined by (7.92), (7.93).

Moreover, under the conditions of Theorem 7.3 we obtain for some $\alpha>0$

$$\mathsf{M}\left\{\left\|J_4[B(Z),F(Z)]_{T,t}-J_4[B(Z),F(Z)]^M_{T,t}\right\|^2_H\right\}=$$

$$=\mathsf{M}\left\{\left\|\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}^M_{t_1}\right)\right)dt_2\right\|^2_H\right\}\le$$

$$
\leq (T-t)\int\limits_t^T \mathsf{M}\left\{\left\|F'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_H^2\right\}dt_2 \leq
$$

$$
\leq C_{21}(T-t)\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2 \leq
$$

$$
\leq C_{21}(T-t)\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1dt_2 \leq
$$

$$
\leq C_{22}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},
$$

where $J_4[B(Z),F(Z)]_{T,t}^M$ is defined by (7.94).

Epilogue

The results presented in this book were developed [53], [54] in the form of a software package in the Python programming language that implements the numerical methods (4.79)–(4.83), (4.88)–(4.92) (see Chapter 4) with the orders 1.0, 1.5, 2.0, 2.5, and 3.0 of strong convergence based on the unified Taylor–Itô and Taylor–Stratonovich expansions. At that for the numerical simulation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 we used [53], [54] the formulas from Sect. 5.1, i.e. method based on Theorem 1.1 and multiple Fourier–Legendre series. Note that in [53], [54] we used the database with 270,000 exactly calculated Fourier–Legendre coefficients.

Using computational experiments it was shown in [55], [56] (also see Sect. 5.4) that in most cases all the exact formulas from Sect. 1.2.3 for the mean-square approximation errors of iterated Itô stochastic integarls can be replaced by the formula (1.81) for $k = 1, \ldots, 5$. This allows us to neglect the multiplier factor $k!$ (see the formula (1.129)). As a result, the computational costs for the approximation of iterated Itô stochastic integrals are significantly reduced. For the same reason, we can replace the multiplier factor $(k!)^2$ by $k!$ in the formula (7.32) in practical calculations.

Iterated stochastic integrals are a fundamental tool for describing and studying the dynamics of various types of stochastic equations. In recent years and decades, numerical methods of high orders of accuracy have been constructed using iterated stochastic integrals not only for Itô SDEs, but also for SDEs with jumps [93], SPDEs with multiplicative trace class noise [182], [183], [187], McKean SDEs [197], SDEs with switchings [198], mean-field SDEs [199], Itô–Volterra SDEs [187], etc.

Bibliography

[1] Kuznetsov, D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-227
Available at: http://www.sde-kuznetsov.spb.ru/06.pdf

[2] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-228
Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf

[3] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, xxxii+770 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf

[4] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, xxxiv+768 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf

[5] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, xxx+786 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf

[6] Kuznetsov, D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Differential Equations and Control Processes, 3 (2010), A.1-A.257.
Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[7] Kuznetsov, D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-232
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf

[8] Kuznetsov, D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-233
Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf

[9] Kuznetsov, D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing, Saarbrucken, 2012, 409 pp. Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[10] Kuznetsov, D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf

[11] Kuznetsov, D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier–Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Differential Equations and Control Processes, 1 (2017), A.1-A.385. Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[12] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Differential Equations and Control Processes, 2 (2017), A.1-A.1000. Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[13] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Differential Equations and Control Processes, 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[14] Kuznetsov, D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Differential Equations and Control Processes, 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[15] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Differential Equations and Control Processes, 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[16] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Differential Equations and Control Processes, 1 (2023), A.1-A.947. Available at:
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[17] Kuznetsov, D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution

of Itô SDEs and Semilinear SPDEs. arXiv.2003.14184 [math.PR], 2025, 1218 pp. [In English].

[18] Kuznetsov, D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070.
DOI: http://doi.org/10.1134/S0965542518070096

[19] Kuznetsov, D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254.
DOI: http://doi.org/10.1134/S0005117918070056

[20] Kuznetsov, D.F. On Numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[21] Kuznetsov, D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[22] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49

[23] Kuznetsov, D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[24] Kuznetsov, D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Differential Equations and Control Processes. 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[25] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Differential Equations and Control Processes, 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[26] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. [In English]. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878

[27] Kuznetsov, D.F. The proof of convergence with probability 1 in the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series. [In English]. Differential Equations and Control Processes, 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[28] Kuznetsov, D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier–Legendre series summarized by Prinsheim method. [In Russian]. Differential Equations and Control Processes, 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[29] Kuznetsov, D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv.2003.14184 [math.PR]. 2023, 145 pp. [in English].

[30] Kuznetsov, D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv.1712.08991 [math.PR]. 2023, 58 pp. [in English].

[31] Kuznetsov, D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv.1801.01079 [math.PR]. 2023, 71 pp. [in English].

[32] Kuznetsov, D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. arXiv.1801.00231 [math.PR]. 2017, 106 pp. [in English].

[33] Kuznetsov, D.F. Expansions of Iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 8 and beyond. arXiv.1712.09516 [math.PR]. 2025, 389 pp. [in English].

[34] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv.1801.00784 [math.PR]. 2023, 80 pp. [in English].

[35] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv.1801.01564 [math.PR]. 2018, 66 pp. [in English].

[36] Kuznetsov, D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. arXiv.1801.05654 [math.PR]. 2018, 46 pp. [In English].

[37] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv.1801.07248 [math.PR]. 2018, 20 pp. [In English].

[38] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of fifth, sixth, seventh and eighth multiplicities based on generalized multiple Fourier series. arXiv.1802.00643 [math.PR]. 2025, 304 pp. [in English].

[39] Kuznetsov, D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv.1801.03195 [math.PR]. 2025, 315 pp. [in English].

[40] Kuznetsov, D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv.1901.02345 [math.GM], 2019, 40 pp. [In English].

[41] Kuznetsov, D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv.1801.06501 [math.PR]. 2018, 40 pp. [In English].

[42] Kuznetsov, D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv.1802.00888 [math.PR]. 2018, 29 pp. [In English].

[43] Kuznetsov, D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor–Stratonovich expansion. arXiv.1806.10705 [math.PR]. 2018, 29 pp. [In English].

[44] Kuznetsov, D.F. Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor–Ito expansion. arXiv.1805.12527 [math.PR]. 2018, 29 pp. [In English].

[45] Kuznetsov, D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor–Ito and Taylor–Stratonovich expansions. arXiv.1802.04844 [math.PR]. 2018, 37 pp. [in English].

[46] Kuznetsov, D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. arXiv.1807.02190 [math.PR], 2018, 44 pp. [in English].

[47] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier–Legendre series summarized by Pringsheim method. arXiv.1801.01962 [math.PR]. 2023, 49 pp. [in English].

[48] Kuznetsov, D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv.1905.03724 [math.GM], 2019, 41 pp. [In English].

[49] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv.1912.02612 [math.PR], 2019, 32 pp. [In English].

[50] Kuznetsov, D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor–Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. arXiv.1801.08862 [math.PR], 2018, 36 pp. [In English].

[51] Kuznetsov, D.F. New simple method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on expansion of the Brownian motion using Legendre polynomials and trigonometric functions. arXiv.1807.00409 [math.PR], 2019, 23 pp. [In English].

[52] Kuznetsov, D.F. Four new forms of the Taylor–Ito and Taylor–Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv.2001.10192 [math.PR], 2023, 95 pp. [In English].

[53] Kuznetsov, M.D., Kuznetsov, D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[54] Kuznetsov, M.D., Kuznetsov, D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. arXiv.2009.14011 [math.PR], 2020, 347 pp. [In English].

[55] Kuznetsov, M.D., Kuznetsov, D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series arXiv.2010.13564 [math.PR], 2020, 63 pp. [In English].

[56] Kuznetsov, D.F., Kuznetsov, M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[57] Kuznetsov, D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[58] Kuznetsov, D.F., Kuznetsov, M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[59] Kuznetsov, D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv.2006.16040 [math.PR], 2020, 33 pp. [In English].

[60] Kuznetsov, D.F., Kuznetsov, M.D. A software package for implementation of strong numerical methods of convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with

non-commutative multi-dimensional noise. [In English]. Abstracts of the 19th International Conference "Aviation and Cosmonautics" AviaSpace-2020 (Moscow, MAI, 23-27 November, 2020), Publishing House "Pero", 2020, pp. 569-570.
Available at: http://www.sde-kuznetsov.spb.ru/20h.pdf

[61] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs with nonlinear multiplicative trace class noise. [In English]. Proceedings of the 5th International Conference on Stochastic Methods ICSM-5 (Russia, Moscow, November 23-27, 2020), RUDN Press, 2020, pp. 88-92. Available at:
http://www.sde-kuznetsov.spb.ru/20i.pdf

[62] Kuznetsov, D.F. Application of the Fourier method for the numerical solution of stochastic differential equations. [In English]. Abstracts of the 2nd International Conference on Mathematical Modeling in Applied Sciences ICMMAS 19 (Belgorod, Russia, August 20-24, 2019), Belgorod State University & Alpha-Publishing, 2019, pp. 236-237.
Available at: http://www.sde-kuznetsov.spb.ru/19c.pdf

[63] Kuznetsov, D.F. Strong approximation of multiple Ito and Stratonovich stochastic integrals. [In English]. Abstracts of the International Conference on Mathematical Modeling in Applied Sciences ICMMAS 17 (Saint-Petersburg, Russia, July 24-28, 2017), Polytechnic University Publishing House, 2017, pp. 141-142. Available at:
http://www.sde-kuznetsov.spb.ru/17.pdf

[64] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Differential Equations and Control Processes, 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[65] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Differential Equations and Control Processes, 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[66] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. The case of arbitrary complete orthonormal systems in Hilbert space. [In English]. Differential Equations and Control Processes, 2 (2024), 73–170. Available at:
http://diffjournal.spbu.ru/EN/numbers/2024.2/article.1.6.html
DOI: http://doi.org/10.21638/11701/spbu35.2024.206

[67] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. The case of arbitrary complete orthonormal systems in Hilbert space. II [In English]. Differential Equations and Control Processes, 4 (2024), 104-190. Available at:
http://diffjournal.spbu.ru/EN/numbers/2024.4/article.1.6.html
DOI: http://doi.org/10.21638/11701/spbu35.2024.406

[68] Kuznetsov, D.F., Kuznetsov, M.D. Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to

components of the multi-dimensional Wiener process based on Multiple Fourier-Legendre series. [In English]. MATEC Web of Conferences, Vol. 362 (2022), article id: 01014, 10 pp. DOI: http://doi.org/10.1051/matecconf/202236201014

[69] Kuznetsov, D.F. A new proof of the expansion of iterated Itô stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. Differential Equations and Control Processes, 4 (2023), 67-124. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.4/article.1.5.html

[70] Kuznetsov, D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. arXiv.2307.11006 [math.PR], 2023, 58 pp. [In English].

[71] Kuznetsov, D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. [In Russian]. Differential Equations and Control Processes, 4 (2019), 32-52. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[72] Clark, J.M.C., Cameron, R.J. The maximum rate of convergence of discrete approximations for stochastic differential equations. Stochastic Differential Systems Filtering and Control. Lecture Notes in Control and Information Sciences, vol 25. Ed. Grigelionis B. Springer, Berlin, Heidelberg, 1980, 162-171.

[73] Wong, E., Zakai, M. On the convergence of ordinary integrals to stochastic integrals. The Annals of Mathematical Statistics, 36, 5 (1965), 1560-1564.

[74] Wong, E., Zakai, M. On the relation between ordinary and stochastic differential equations. International Journal of Engineering Science, 3 (1965), 213-229.

[75] Ikeda, N., Watanabe, S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[76] Kuznetsov, D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Differential Equations and Control Processes, 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[77] Kuznetsov, D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Differential Equations and Control Processes, 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html

[78] Kuznetsov, D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86.
DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
Available at: http://www.sde-kuznetsov.spb.ru/00a.pdf

[79] Kuznetsov, D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[80] Kulchitskiy, O.Yu., Kuznetsov, D.F. Approximation of multiple Ito stochastic integrals. [In Russian]. VINITI, 1679-V94 (1994), 42 pp.

[81] Kuznetsov, D.F. Methods of numerical simulation of stochastic differential Ito equations solutions in problems of mechanics. [In Russian]. Ph. D. Thesis, Saint-Petersburg State Technical University, 1996, 260 pp.

[82] Milstein, G.N. Numerical Integration of Stochastic Differential Equations. [In Russian]. Ural University Press, Sverdlovsk, 1988, 225 pp.

[83] Kloeden, P.E., Platen, E., Wright, I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[84] Kloeden, P.E., Platen, E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992, 632 pp.

[85] Kloeden, P.E., Platen, E., Schurz, H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[86] Averina, T.A., Prigarin, S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[87] Prigarin, S.M., Belov, S.M. One application of series expansions of Wiener process. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[88] Wiktorsson, M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487.

[89] Ryden, T., Wiktorsson, M. On the simulation of iterated Ito integrals. Stochastic Processes and Their Applications, 91, 1 (2001), 151-168.

[90] Gaines, J.G., Lyons, T.J. Random generation of stochastic area integrals. SIAM Journal of Applied Mathematics, 54 (1994), 1132-1146.

[91] Gilsing, H., Shardlow, T. SDELab: A package for solving stochastic differential equations in MATLAB. Journal of Computational and Applied Mathematics, 2, 205 (2007), 1002-1018.

[92] Milstein, G.N., Tretyakov, M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[93] Platen, E., Bruti-Liberati, N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010, 868 pp.

[94] Allen, E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[95] Rybakov, K. Spectral representations of iterated stochastic integrals and their application for modeling nonlinear stochastic dynamics. Mathematics, 11, 19 (2023), 4047. DOI: http://doi.org/10.3390/math11194047

[96] Tang, X., Xiao, A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics, 45 (2019), 813-846.

[97] Zahri, M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science, 8, 2 (2014), 186-198.

[98] Chernykh, N.V., Pakshin, P.V. Numerical solution algorithms for stochastic differential systems with switching diffusion. Automation and Remote Control, 74, 12 (2013), 2037-2063. DOI: http://doi.org/10.1134/S0005117913120072

[99] Li, C.W., Liu, X.Q. Approximation of multiple stochastic integrals and its application to stochastic differential equations. Nonlinear Analysis: Theory, Methods & Applications, 30, 2 (1997), 697-708.

[100] Gihman, I.I., Skorohod, A.V. Stochastic Differential Equations. Springer-Verlag, Berlin, Heidelberg, 1972, 356 pp.

[101] Gihman, I.I., Skorohod, A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[102] Skorohod, A.V. Stochastc Processes with Independent Increments. Nauka, Moscow, 1964, 280 pp.

[103] Koroluk, V.S., Portenko, N.I., Skhorohod, A.V., Turbin, A.F. HandBook on Probability Theory and Mathematical Statistics. Nauka, Moscow, 1985, 640 pp.

[104] Shiryaev, A.N. Probability. Springer-Verlag, New York, 1996, 624 pp.

[105] Kloeden, P.E., Neuenkirch, A. The pathwise convergence of approximation schemes for stochastic differential equations. LMS Journal of Computation and Mathematics, 10 (2007), 235-253.

[106] Itô, K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.

[107] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differential Equations and Control Processes, 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[108] Chung, K.L., Williams, R.J. Introduction to Stochastic Integration. 2nd Edition. Probability and its Applications. Ed. Liggett T., Newman C., Pitt L. Birkhauser, Boston, Basel, Berlin, 1990, 276 pp.

[109] Kuo, H.-H. Introduction to Stochastic Integration. Universitext (UTX), Springer. N. Y., 2006, 289 pp.

[110] Fox, R., Taqqu, M.S. Multiple stochastic integrals with dependent integrators. Journal of Multivariate Analysis, 21 (1987), 105-127.

[111] Major, P. The theory of Wiener–Itô integrals in vector valued Gaussian stationary random fields. Part I. Moscow Mathematical Journal, 20, 4 (2020), 749-812.

[112] Major, P. Multiple Wiener–Itô Integrals With Applications to Limit Theorems. Second Edition. Springer. Cham, Heidelberg, New York, Dordrecht, London. 2014, 126 pp.

[113] Major, P. Wiener–Itô integral representation in vector valued Gaussian stationary random fields. arXiv.1901.04084 [math.PR], 2019, 90 pp.

[114] Stratonovich, R.L. Conditional Markov Processes and Their Application to the Theory of Optimal Control. Elsevier, N. Y., 1968, 350 pp.

[115] Hobson, E.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge University Press, Cambridge, 1931, 502 pp.

[116] Reed, M, Simon, B. Functional Analysis. Vol 1. Academic Press, San Diego, 1980, 400 pp.

[117] Rybakov, K.A. On the orthogonal expansion of the iterated Stratonovich stochastic integrals. Vestnik DSU. Series 1. Natural sciences, 37, 2 (2022), 27-32.
DOI: http://doi.org/10.21779/2542-0321-2022-37-2-27-32

[118] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

[119] Rybakov, K.A., private communication, March, 2024.

[120] Starchenko, T.K. About conditions of convergence of double Fourier–Legendre series. Proceedings of the Mathematical Institute of NAS of Belarus. Analitical methods of analysis and differential equations, Minsk, 5 (2005), 124-126.

[121] Suetin, P.K. Classical orthogonal polynomials. 3rd Edition. Fizmatlit, Moscow, 2005, 480 pp.

[122] Ilin, V.A., Poznyak, E.G. Foundations of mathematical analysis. Part II. Nauka, Moscow, 1973, 448 pp.

[123] Kuznetsov, D.F. Theorems about integration order replacement in multiple Ito stochastic integrals. [In Russian]. VINITI, 3607-V97 (1997), 31 pp.

[124] Kuznetsov, D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. arXiv.1801.04634 [math.PR], 2018, 28 pp. [In English].

[125] Bari, N.K. Trigonometric Series. Fiz.– Mat. Lit., Moscow, 1961, 936 pp.

[126] Sjölin, P. Convergence almost everywhere of certain singular integrals and multiple Fourier series. Arkiv för Matematik, 9, 1-2 (1971), 65-90.

[127] Gohberg, I.C., Krein, M.G. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space. Fizmatlit, Moscow, 1965, 448 pp.

[128] Gohberg, I., Goldberg, S., Krupnik, N. Traces and Determinants of Linear Operators. Birkhauser Verlag, Basel, Boston, Berlin, 2000, 258 pp.

[129] Liptser, R.Sh., Shirjaev, A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. Nauka, Moscow, 1974, 696 pp.

[130] Luo, W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. Ph. D. Thesis, California Institute of Technology, 2006, 225 pp.

[131] Wong, E., Zakai, M. Riemann-Stieltjes approximations of stochastic integrals. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 12 (1969), 87-97.

[132] Ikeda, N., Nakao, S., Yamato, Y. A class of approximations of Brownian motion. Publications of the Research Institute for Mathematical Sciences, Kyoto University, 13 (1977), 285-300.

[133] Koneczny, F. On the Wong–Zakai approximation of stochastic differential equations. Journal of Multivariate Analysis, 13 (1983), 605-611.

[134] Karatzas, I., Shreve, S.E. Brownian Motion and Stochastic Calculus. 2nd Edition. Springer-Verlag, N. Y., Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, 1991, 470 pp.

[135] Mackevicius, V., Zibaitis, B. Gaussian approximations of Brownian motion in a stochastic integral. Lithuanian Mathematical Journal, 33 (1993), 393-406.
DOI: http://doi.org/10.1007/BF00995993

[136] Twardowska, K. Wong–Zakai approximations for stochastic differential equations. Acta Applicandae Mathematica, 43 (1996), 317-359. DOI: http://doi.org/10.1007/BF00047670

[137] Gyöngy, I., Michaletzky, G. On Wong-Zakai approximations with δ-martingales. Proceedings of the Royal Society of London. Series A. 460, 2041 (2004), 309-324.

[138] Gyöngy, I., Shmatkov, A. Rate of convergence of Wong–Zakai approximations for stochastic partial differential equations. Applied Mathematics and Optimization, 54 (2006), 315-341.

[139] Budhiraja, A. Multiple stochastic integrals and Hilbert space valued traces with applications to asymptotic statistics and non-linear filtering. Ph. D. Thesis, The University of North Caroline at Chapel Hill, 1994, VII+132 pp.

[140] Budhiraja, A., Kallianpur, G. Two results on multiple Stratonovich integrals. Statistica Sinica, 7 (1997), 907-922.

[141] Johnson, G.W., Kallianpur, G. Homogeneous chaos, p-forms, scaling and the Feynman integral. Transactions of the American Mathematical Society, 340 (1993), 503-548.

[142] Rybakov, K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Differential Equations and Control Processes, 4 (2021), 81-115. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.5.html

[143] Solé, J. LL., Utzet, F. Stratonovich integral and trace. Stochastics: An International Journal of Probability and Stochastic Processes, 29, 2 (1990), 203-220.

[144] Bardina, X., Jolis, M. Weak convergence to the multiple Stratonovich integral. Stochastic Processes and their Applications, Elsevier, 90, 2 (2000), 277-300.

[145] Bardina, X., Rovira, C. On the strong convergence of multiple ordinary integrals to multiple Stratonovich integrals. Publicacions Matemàtiques, 65 (2021), 859-876. DOI: http://doi.org/10.5565/PUBLMAT6522114

[146] Hu, Y. Analysis on Gaussian Spaces. World Scientific, New Jersey, London, Singapore, 2017, 470 pp.

[147] Pugachev, V.S. Lectures on Fuctional Analysis. MAI, Moscow, 1996, 744 pp.

[148] Rudin, W. Real and Complex Analysis. McGraw–Hill, 1987, 416 pp.

[149] Hairer, M. On Malliavin's proof of Hörmander's theorem. Bulletin Des Sciences Mathématiques, 135, 6-7 (2011), 650-666.

[150] Brislawn, C. Kernels of trace class operators. Proceedings of the American Mathematical Society, 104, 4 (1988), 1181-1190.

[151] Kuznetsov, D.F. Replacement the order of integration in iterated stochastic integrals with respect to martingale. [In Russian]. Preprint. SPbGTU Publishing House, Saint-Petersburg, 1999, 11 pp. DOI: http://doi.org/10.13140/RG.2.2.19936.58889 Available at: http://www.sde-kuznetsov.spb.ru/99c.pdf

[152] Arato, M. Linear Stochastic Systems with Constant Coefficients. A Statistical Approach. Springer-Verlag, Berlin, Heidelberg, N. Y., 1982, 289 pp.

[153] Shiryaev, A.N. Essentials of Stochastic Finance: Facts, Models, Theory. World Scientific Publishing Co United States, 1999, 852 pp.

[154] Chugai, K.N., Kosachev, I.M., Rybakov, K.A. Approximate filtering methods in continuous-time stochastic systems. Smart Innovation, Systems and Technologies, vol. 173, Eds. Jain L.C., Favorskaya M.N., Nikitin I.S., Reviznikov D.L. Springer, 2020, pp. 351-371. DOI: http://doi.org/10.1007/978-981-15-2600-8_24

[155] Averina, T.A. Statistical Modeling of Solutions of Stochastic Differential Equations and Systems with Random Structure. Siberian Branch of the Russian Academy of Sciences, Novosibirsk, 2019, 350 pp.

[156] Nasyrov, F.S. Local Times, Symmetric Integrals and Stochastic Analysis. Fizmatlit, Moscow, 2011, 212 pp.

[157] Kagirova, G.R., Nasyrov, F.S. On an optimal filtration problem for one-dimensional diffusion processes. Siberian Advances in Mathematics, 28, 3 (2018), 155-165.

[158] Asadullin, E.M., Nasyrov, F.S. About filtering problem of diffusion processes. Ufa Mathematical Journal, 3, 2 (2011), 3-9.

[159] Kloeden, P.E., Platen, E. The Stratonovich and Ito–Taylor expansions. Mathematische Nachrichten, 151 (1991), 33-50.

[160] Platen, E., Wagner, W. On a Taylor formula for a class of Ito processes. Probability and Mathematical Statistics, 3 (1982), 37-51.

[161] Kulchitskiy, O.Yu., Kuznetsov, D.F. The unified Taylor–Ito expansion. [In English]. Journal of Mathematical Sciences (N. Y.), 99, 2 (2000), 1130-1140.
DOI: http://doi.org/10.1007/BF02673635

[162] Kulchitskiy, O.Yu., Kuznetsov, D.F. Unified Taylor–Ito expansion. [In Russian]. Differential Equations and Control Processes, 1 (1997), 1-17. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.1.html

[163] Kuznetsov, D.F. New representations of the Taylor–Stratonovich expansion. [In English]. Journal of Mathematical Sciences (N. Y.), 118, 6 (2003), 5586-5596.
DOI: http://doi.org/10.1023/A:1026138522239

[164] Kulchitski, O.Yu., Kuznetsov, D.F. Expansion of Ito processes into a Taylor–Ito series in a neighborhood of a fixed time moment. [In Russian]. VINITI, 2637-93 (1993), 26 pp.

[165] Kuznetsov, D.F. Two new representations of the Taylor–Stratonovich expansion. [In Russian]. Preprint. SPbGTU Publishing House, Saint-Petersburg, 1999, 13 pp.
DOI: http://doi.org/10.13140/RG.2.2.18258.86729
Available at: http://www.sde-kuznetsov.spb.ru/99b.pdf

[166] Wiener, N. Un problème de probabilités dénombrables. Bulletin de la Société Mathématique de France, 52 (1924), 569-578.

[167] Lévy, P. Wiener's random function and other Laplacian random functions. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1951, pp. 171-187.

[168] Ito, K., McKean, H. Diffusion Processes and Their Sample Paths. Springer-Verlag, Berlin, Heidelberg, N. Y., 1965, 395 pp.

[169] Neckel, T., Parra Hinojosa, A., Rupp, F. Path-wise algorithms for random and stochastic ODEs with applications to ground-motion-induced excitations of multi-storey buildings. Report number TUM-I1758, TU Munchen, 2017, 33 pp.

[170] Kuznetsov, D.F, Combined method of strong approximation of multiple stochastic integrals. [In English]. Journal of Automation and Information Sciences (Begell House), 2002, 34 (Issue 8), 6 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v34.i8.40
Available at: http://www.sde-kuznetsov.spb.ru/02b.pdf

[171] Kuznetsov, D.F. The order 4.0 weak numerical method for stochastic differential Ito equations with scalar nonadditive noise. [In Russian]. Vestnik Molodykh Uchenykh. Applied Mathematics and Mechanics, 4 (2000), 47-52. Available at:
http://www.sde-kuznetsov.spb.ru/00b.pdf

[172] Gyöngy, I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. Potential Analysis, 9, 1 (1998), 1-25.

[173] Gyöngy, I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. Potential Analysis, 11, 1 (1999), 1-37.

[174] Gyöngy, I. and Krylov N. An accelerated splitting-up method for parabolic equations. SIAM Journal of Mathematical Analysis, 37, 4 (2005), 1070-1097.

[175] Hausenblas, E. Numerical analysis of semilinear stochastic evolution equations in Banach spaces. Journal of Computational and Applied Mathematics, 147, 2 (2002), 485-516.

[176] Hausenblas, E. Approximation for semilinear stochastic evolution equations. Potential Analysis, 18, 2 (2003), 141-186.

[177] Jentzen, A. Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Analysis, 31, 4 (2009), 375-404.

[178] Jentzen, A. Taylor expansions of solutions of stochastic partial differential equations. arXiv.0904.2232 [math.NA], 2009, 32 pp.

[179] Jentzen, A. and Kloeden P.E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. Proceedings of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences, 465, 2102 (2009), 649-667.

[180] Jentzen, A. and Kloeden P.E. Taylor expansions of solutions of stochastic partial differential equations with additive noise. The Annals of Probability, 38, 2 (2010), 532-569.

[181] Jentzen, A., Kloeden, P.E. Taylor Approximations for Stochastic Partial Differential Equations. SIAM, Philadelphia, 2011, 224 pp.

[182] Jentzen, A., Röckner, M. A Milstein scheme for SPDEs. Foundations of Computational Mathematics, 15, 2 (2015), 313-362.

[183] Becker, S., Jentzen, A., Kloeden, P.E. An exponential Wagner–Platen type scheme for SPDEs. SIAM Journal of Numerical Analysis, 54, 4 (2016), 2389-2426.

[184] Zhang, Z., Karniadakis, G. Numerical Methods for Stochastic Partial Differential Equations with White Noise. Springer, 2017, 398 pp.

[185] Jentzen, A., Röckner, M. Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. Journal of Differential Equations, 252, 1 (2012), 114-136.

[186] Lord, G.J., Tambue, A.A modified semi-implict Euler-Maruyama scheme for finite element discretization of SPDEs. arXiv.1004.1998 [math.NA], 2010, 23 pp.

[187] Mishura, Y.S., Shevchenko, G.M. Approximation schemes for stochastic differential equations in a Hilbert space. Theory of Probability and its Applications, 51, 3 (2007), 442-458.

[188] Müller-Gronbach, T., Ritter, K., Wagner, T. Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. Monte Carlo and Quasi-Monte Carlo Methods 2006. Springer, Berlin, 2007, pp. 577-589.

[189] Prévôt C., Röckner, M. A Concise Course on Stochastic Partial Differential Equations, Vol. 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007, 148 pp.

[190] Shardlow, T. Numerical methods for stochastic parabolic PDEs. Numerical Functional Analysis and Optimization, 20, 1-2 (1999), 121-145.

[191] Da Prato, G., Jentzen, A., Röckner, M. A mild Itô formula for SPDEs. arXiv.1009.3526 [math.PR], 2012, 39 pp.

[192] Da Prato, G., Zabczyk, J. Stochastic Equations in Infinite Dimensions. 2nd Edition. Cambridge University Press, Cambridge, 2014, 493 pp.

[193] Kruse, R. Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise. IMA Journal of Numerical Analysis, 34, 1 (2014), 217-251.

[194] Kruse, R. Consistency and stability of a Milstein-Galerkin finite element scheme for semilinear SPDE. Stochastic Partial Differential Equations: Analysis and Computations, 2, 4 (2014), 471-516.

[195] Leonhard, C. Derivative-free numerical schemes for stochastic partial differential equations. Ph. D. Thesis, Institute of Mathematics of the University of Lübeck, 2017, 131 pp.

[196] Leonhard, C., Rößler, A. Iterated stochastic integrals in infinite dimensions: approximation and error estimates. Stochastic Partial Differential Equations: Analysis and Computations, 7, 2 (2018), 209-239.

[197] Bao, J., Reisinger, C., Renz, P., Stockinger, W. First order convergence of Milstein schemes for McKean equations and interacting particle systems. arXiv.2004.03325 [math.PR], 2020, 27 pp.

[198] Son, L.N., Tuan, A.H., Dung, T.N., Yin G. Milstein-type procedures for numerical solutions of stochastic differential equations with Markovian switching. SIAM Journal of Numerical Analysis, 55, 2 (2017), 953-979.

[199] Sun, Y., Yang, J., Zhao W. Ito-Taylor schemes for solving mean-field stochastic differential equations. Numerical Mathematics: Theory, Methods and Applications, 10, 4 (2017), 798-828.

[200] Rybakov, K.A. Spectral method of analysis and optimal estimation in linear stochastic systems. International Journal of Modeling, Simulation, and Scientific Computing, 11, 3 (2020), 2050022. DOI: http://doi.org/10.1142/S1793962320500221

[201] Rybakov, K.A. Modeling linear nonstationary stochastic systems by spectral method. Differential Equations and Control Processes, 3 (2020), 98-128. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.5.html

[202] Averina, T.A., Rybakov, K.A. Using maximum cross section method for filtering jump-diffusion random processes. Russian Journal of Numerical Analysis and Mathematical Modelling, 35, 2 (2020), 55-67. DOI: http://doi.org/10.1515/rnam-2020-0005

[203] Rybakov, K.A. Modeling and analysis of output processes of linear continuous stochastic systems based on orthogonal expansions of random functions. Journal of Computer and Systems Sciences International, 59, 3 (2020), 322-337.
DOI: http://doi.org/10.1134/S1064230720030156

[204] Kuznetsov, D.F. The three-step strong numerical methods of the orders of accuracy 1.0 and 1.5 for Ito stochastic differential equations. [In English]. Journal of Automation and Information Sciences (Begell House), 2002, 34 (Issue 12), 14 pp.
DOI: http://doi.org/10.1615/JAutomatInfScien.v34.i12.30
Available at: http://www.sde-kuznetsov.spb.ru/02a.pdf

[205] Kuznetsov, D.F. Finite-difference strong numerical methods of order 1.5 and 2.0 for stochastic differential Ito equations with nonadditive multidimensional noise. [In English]. Journal of Automation and Information Sciences (Begell House), 2001, 33 (Issue 5-8), 13 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v33.i5-8.180
Available at: http://www.sde-kuznetsov.spb.ru/01c.pdf

[206] Kuznetsov, D.F. Expansion of the Stratonovich multiple stochastic integrals based on the Fourier multiple series. [In English]. Journal of Mathematical Sciences (N. Y.), 109, 6 (2002), 2148-2165. DOI: http://doi.org/10.1023/A:1014581416903

[207] Kuznetsov, D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. SPbGTU Publishing House, Saint-Petersburg, 1998, 204 pp. ISBN 5-7422-0045-5. Hard cover edition of the publication:
http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html

[208] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In Russian]. Abstracts of talks given at the 7th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2022), Theory of Probability and its Applications, 67, 4 (2022), 834-835. DOI: http://doi.org/10.4213/tvp5585

[209] Kuznetsov, D.F. Recent results on a new approach to series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of a multidimensional Wiener process. [In Russian]. Abstracts of talks given at the 8th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 1-8, 2023), Theory of Probability and its Applications, 68, 4 (2023), 850-850.
DOI: http://doi.org/10.4213/tvp5677

[210] Kuznetsov, D.F. Expansions of iterated Itô and Stratonovich stochastic integrals. The case of arbitrary CONS in $L_2[t,T]$. [In Russian]. Abstracts of talks given at the 9th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-8, 2024), Theory of Probability and its Applications, 69, 4 (2024), 813-814.
DOI: http://doi.org/10.4213/tvp5736

[211] Rybakov, K.A. Using spectral form of mathematical description to represent iterated Stratonovich stochastic integrals. Applied Mathematics and Computational Mechanics for Smart Applications. Smart Innovation, Systems and Technologies, vol. 217. Eds. Jain L.C., Favorskaya M.N., Nikitin I.S., Reviznikov D.L. Springer, Singapore, 2021, pp. 287-304. DOI: http://doi.org/10.1007/978-981-33-4826-4_20

[212] Yang, G., Burrage, K., Komori, Y., Burrage, P., Ding, X. A class of new Magnus-type methods for semi-linear non-commutative Ito stochastic differential equations. Numerical Algorithms (2021). DOI: http://doi.org/10.1007/s11075-021-01089-7

[213] Cao, W., Zhang, Z., Karniadakis G. Numerical methods for stochastic delay differential equations via the Wong-Zakai approximation. SIAM Journal on Scientific Computing, 37, 1 (2015), A295-A318. DOI: http://doi.org/10.1137/130942024

[214] Schurz, H. Basic concepts of numerical analysis of stochastic differential equations explained by balanced implicit theta methods. Stochastic Differential Equations and Processes. Eds. Zili M., Filatova D. Springer Proceedings in Mathematics, vol. 7. Springer, Berlin, Heidelberg, 2012, pp. 1-139. DOI: http://doi.org/10.1007/978-3-642-22368-6_1

[215] Mishura, Y.S., Shvai, A.V. An estimate of the rate of convergence of an approximating scheme applied to a stochastic differential equation with an additional parameter. Theory of Probability and Mathematical Statistics, 82 (2011), 75-85.
DOI: http://doi.org/10.1090/S0094-9000-2011-00828-9

[216] Gaines, J.G. A basis for iterated stochastic integrals. Mathematics and Computers in Simulation, 38 (1995), 7-11.

[217] Foster, J., Habermann, K. Brownian bridge expansions for Lévy area approximations and particular values of the Riemann zeta function. Combinatorics, Probability and Computing (2022), 1-28. DOI: http://doi.org/10.1017/S096354832200030X

[218] Kastner, F., Rößler, A. An analysis of approximation algorithms for iterated stochastic integrals and a Julia and MATLAB simulation toolbox. arXiv.2201.08424 [math.NA], 2022, 43 pp.

[219] Malham, S.J.A., Wiese A. Efficient almost-exact Lévy area sampling. Statistics & Probability Letters, 88 (2014), 50-55.

[220] Stump, D.M., Hill J.M. On an infinite integral arising in the numerical integration of stochastic differential equations. Proceedings of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences 461, 2054 (2005), 397-413.

[221] Rybakov, K.A. Features of the expansion of multiple stochastic Stratonovich integrals using Walsh and Haar functions. Differential Equations and Control Processes, 1 (2023), 137-150. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.9.html

[222] Kamrani, M., Jamshidi, N. Implicit Milstein method for stochastic differential equations via the Wong–Zakai approximation. Numerical Algorithms, 79 (2018), 357-374.

[223] Leonhard, C., Mißfeldt, R, Rößler, A. An Exponential Stochastic Runge–Kutta Type Method of Order up to 1.5 for SPDEs of Nemytskii-type. arXiv.2412.08299 [math.NA], 2024, 36 pp.

[224] Mrongowius, J, Rößler, A. On the approximation and simulation of iterated stochastic integrals and the corresponding Lévy areas in terms of a multidimensional Brownian motion. arXiv.2101.09542 [math.PR], 2021, 25 pp.

[225] Fihtengolc, G.M. Differential and Integral Calculus Course. Vol II. Fizmatlit, Moscow, 1970, 800 pp.

[226] Pisier, G. Martingales in Banach Spaces. Cambridge University Press, Cambridge, 2016, 580 pp.