Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs

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Dedicated to My Family

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Preface

The book is devoted to the problem of strong (mean-square) approximation of iterated Itô and Stratonovich stochastic integrals in the context of numerical integration of Itô stochastic differential equations (SDEs) and non-commutative semilinear stochastic partial differential equations (SPDEs) with nonlinear multiplicative trace class noise. The presented monograph open a new direction in researching of iterated stochastic integrals and summarizes the author's research on the mentioned problem carried out in the period 1994–2020.

The basis of this book composes on the monographs [1]-[15] and recent author's results [16]-[54].

This monograph (also see books [6]-[11], [14], [15]) is the first monograph where the problem of strong (mean-square) approximation of iterated Itô and Stratonovich stochastic integrals is sistematically analyzed in application to the numerical solution of SDEs.

For the first time we successfully use the generalized multiple Fourier series (Fourier–Legendre series as well as trigonometric Fourier series) converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ for the expansion and strong approximation of iterated Itô stochastic integrals of arbitrary multiplicity k, $k \in \mathbf{N}$ (Chapter 1).

This result has been adapted for iterated Stratonovich stochastic integrals of multiplicities 1 to 5 for the Legendre polynomial system and the system of trigonometric functions (Chapter 2) as well as for some other types of iterated stochastic integrals (Chapter 1).

Two theorems on expansions of iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbf{N}$) based on generalized iterated Fourier series with the pointwise convergence are formulated and proved (Chapter 2).

The integration order replacement technique for the class of iterated Itô stochastic integrals has been introduced (Chapter 3). This result is generalized for the class of iterated stochastic integrals with respect to martingales.

We derived the exact and approximate expressions for the mean-square approximation error of iterated Itô stochastic integrals of multiplicity k, $k \in \mathbf{N}$

(Chapter 1). Furthermore, we provided a significant practical material (Chapter 5) devoted to the expansions and approximations of specific iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich expansions (Chapter 4) using the system of Legendre polynomials and the system of trigonometric functions.

The methods formulated in this book have been compared with some existing methods of strong approximation of iterated Itô and Stratonovich stochastic integrals (Chapter 6).

The results of Chapter 1 were applied (Chapter 7) to the approximation of iterated stochastic integrals with respect to the finite-dimensional approximation $\mathbf{W}_t^M$ of the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of arbitrary multiplicity k, $k \in \mathbf{N}$) and to the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of multiplicities 1 to 3).

This book will be interesting for specialists dealing with the theory of stochastic processes, applied and computational mathematics as well as senior students and postgraduates of technical institutes and universities.

The importance of the problem of numerical integration of Itô SDEs and semilinear SPDEs is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems of various physical nature under random disturbances and to the application of these equations for solving various mathematical problems, among which we mention signal filtering in the background of random noise, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems, etc.

It is well known that one of the effective and perspective approaches to the numerical integration of Itô SDEs and semilinear SPDEs is an approach based on the stochastic analogues of the Taylor formula for solutions of these equations. This approach uses the finite discretization of temporal variable and performs numerical modeling of solutions of Itô SDEs and semilinear SPDEs in discrete moments of time using stochastic analogues of the Taylor formula.

Speaking about Itô SDEs, note that the most important feature of the mentioned stochastic analogues of the Taylor formula for solutions of Itô SDEs is a presence in them of the so-called iterated Itô and Stratonovich stochastic integrals which are the functionals of a complex structure with respect to components of the multidimensional Wiener process. These iterated stochastic integrals are subject for study in this book. The mentioned iterated Itô and Stratonovich stochastic integrals are defined by the following formulas

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{(Itô integrals)},$$

$$\int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{(Stratonovich integrals)},$$

where $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are continuous nonrandom functions at the interval $[t, T]$ (as a rule, in the applications they are identically equal to 1 or have a binomial form (see Chapter 4)), $\mathbf{w}_\tau$ is a random vector with an $m+1$ components: $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

The above iterated stochastic integrals are the specific objects in the theory of stochastic processes. From the one side, nonrandomness of weight functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is the factor simplifying their structure. From the other side, nonscalarity of the Wiener process $\mathbf{f}_\tau$ with independent components $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ and the fact that the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are different for various l $(l = 1, \ldots, k)$ are essential complicating factors of the structure of iterated stochastic integrals. Taking into account features mentioned above, the systems of iterated Itô and Stratonovich stochastic integrals play the extraordinary and perhaps the key role for solving the problem of numerical integration of Itô SDEs.

A natural question arises: is it possible to construct a numerical scheme for Itô SDE that includes only increments of the Wiener processes $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$, but has a higher order of convergence than the Euler method? It is known that this is impossible for $m > 1$ in the general case. This fact is called the "Clark–Cameron paradox" [55] and explains the need to use iterated stochastic integrals for constructing high-order numerical methods for Itô SDEs.

We want to mention in short that there are two main criteria of numerical methods convergence for Itô SDEs: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Itô SDE, simply stated, but the distribution of Itô SDE solution. Both mentioned criteria are independent, i.e. in general it is impossible to state that from the execution of strong criterion follows the execution of weak criterion and vice versa. Each of two convergence criteria is oriented on the solution of specific classes of mathematical problems connected with Itô SDEs.

Numerical integration of Itô SDEs based on the strong convergence criterion of approximation is widely used for the numerical simulation of sample trajectories of solutions to Itô SDEs (which is required for constructing new mathematical models based on such equations and for the numerical solution of different mathematical problems connected with Itô SDEs). Among these problems, we note the following: signal filtering under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis.

Exact solutions of Itô SDEs and semilinear SPDEs are known in rather rare cases. Therefore, the need arises to construct numerical procedures for solving these equations.

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Itô or Stratonovich stochastic integrals is very important and difficult from theoretical and computing point of view.

Seems that iterated stochastic integrals may be approximated by multiple integral sums. However, this approach implies the partitioning of the interval of integration $[t,T]$ for iterated stochastic integrals. The length $T-t$ of this interval is already fairly small (because it is a step of integration of numerical methods for Itô SDEs) and does not need to be partitioned. Computational experiments show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors.

The problem of effective decreasing of the mentioned cost (in several times or even in several orders) is very difficult and requires new complex investigations. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s),\ldots,\psi_k(s) \equiv \psi(s)$. This case allows the investigation with using of the Itô formula. In the more general case, when not all numbers $i_1,\ldots,i_k$ are equal, the mentioned problem turns out to be more complex (it cannot be solved using the Itô formula and requires more deep and complex investigation). Note that even for the case $i_1 = \ldots = i_k \neq 0$, but for different functions $\psi_1(s),\ldots,\psi_k(s)$ the mentioned difficulties persist and simple sets of iterated Itô and Stratonovich stochastic integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (with respect to the mean-square approximation) using the system of standard Gaussian random variables. The Itô formula is also useless in this case and as a result we need to use more complex but effective expansions.

Why the problem of the mean-square approximation of iterated stochastic integrals is so complex?

Firstly, the mentioned stochastic integrals (in the case of fixed limits of integration) are the random variables, whose density functions are unknown in the general case. The exception is connected with the narrow particular case which is the simplest iterated Itô stochastic integral with multiplicity 2 and $\psi_1(s), \psi_2(s) \equiv 1$; $i_1, i_2 = 1, \ldots, m$. Nevertheless, the knowledge of this density function not gives a simple way for approximation of iterated Itô stochastic integral of multiplicity 2.

Secondly, we need to approximate not only one stochastic integral, but several iterated stochastic integrals that are complexly dependent in a probabilistic sense.

Often, the problem of combined mean-square approximation of iterated Itô and Stratonovich stochastic integrals occurs even in cases when the exact solution of Itô SDE is known. It means that even if you know the solution of Itô SDE exactly, you cannot model it numerically without the combined numerical modeling of iterated stochastic integrals.

Note that for a number of special types of Itô SDEs the problem of approximation of iterated stochastic integrals may be simplified but cannot be solved. Equations with additive vector noise, with non-additive scalar noise, with additive scalar noise, with a small parameter are related to such types of equations. In these cases, simplifications are connected to the fact that some members from stochastic Taylor expansions are equal to zero or we may neglect some members from these expansions due to the presence of a small parameter.

Furthermore, the problem of combined numerical modeling (with respect to the mean-square convergence criterion) of iterated Itô and Stratonovich stochastic integrals is rather new.

One of the main and unexpected achievements of this book is the successful usage of functional analysis methods (more concretely, we mean generalized multiple and iterated Fourier series (convergence in $L_2([t,T]^k)$ and pointwise correspondently) through various systems of basis functions) in this scientific field.

The problem of combined numerical modeling (with respect to the mean-square convergence criterion) of systems of iterated Itô and Stratonovich stochastic integrals was systematically analyzed in the context of the problem of numerical integration of Itô SDEs in the following monographs:

[I] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Kluwer Academic Publishers. Dordrecht. 1995 (Russian Ed. 1988).

[II] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer-Verlag. Berlin. 1992 (2nd Ed. 1995, 3rd Ed. 1999).

[III] Milstein G.N., Tretyakov M. V. Stochastic Numerics for Mathematical Physics. Springer-Verlag. Berlin. 2004.

[IV] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. Polytechnical University Publ. St.-Petersburg. 2007. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 (2nd Ed. 2007 [3], 3rd Ed. 2009 [4], 4th Ed. 2010 [5], 5th Ed. 2017 [12], 6th Ed. 2018 [13]).

Note that the initial version of the book [IV] has been published in 2006 [1]. Also we mention the books [6] (2010), [7] (2011), [8] (2011), [9] (2012), [10] (2013), [11] (2017) and [14], [15] (2020).

The books [I] and [III] analyze the problem of the mean-square approximation of iterated stochastic integrals only for two simplest iterated Itô stochastic integrals of 1st and 2nd multiplicities ($k = 1$ and 2, $\psi_1(s)$ and $\psi_2(s) \equiv 1$) for the multidimensional case: $i_1, i_2 = 0, 1, \ldots, m$. In addition, the main idea is based on the expansion of the so-called Brownian bridge process into the trigonometric Fourier series (version of the so-called Karhunen–Loève expansion). This method is called in [I] and [III] as the Fourier method.

In [II] using the Fourier method [I], the attempt was made to obtain the mean-square approximation of elementary iterated stochastic integrals of multiplicities 1 to 3 ($k = 1, \ldots, 3$, $\psi_1(s), \ldots, \psi_3(s) \equiv 1$) for the multidimensional case: $i_1, \ldots, i_3 = 0, 1, \ldots, m$. However, as we can see in the presented book, the results of the monograph [II], related to the mean-square approximation of iterated stochastic integrals of 3rd multiplicity, cause a number of critical remarks (see discussions in Sect. 2.6.2, 6.2).

The main purpose of this book is to construct and develop newer and more effective methods (than presented in the books [I]–[III]) of combined mean-square approximation of iterated Itô and Stratonovich stochastic integrals.

Talking about the history of solving the problem of combined mean-square approximation of iterated stochastic integrals, the idea to find a basis of random variables using which we may represent iterated stochastic integrals turned out to be useful. This idea was transformed several times during last decades.

Attempts to approximate the iterated stochastic integrals using various integral sums were made until 1980s and later, i.e. the interval of integration $[t, T]$ of the stochastic integral was divided into n parts and the iterated stochastic integral was represented approximately by the multiple integral sum, which included the system of independent standard Gaussian random variables, whose numerical modeling is not a problem.

However, as we noted above, it is obvious that the length $T - t$ of integration interval $[t, T]$ of the iterated stochastic integrals is a step of integration of numerical methods for Itô SDEs, which is already a rather small value even without the additional splitting. Numerical experiments demonstrate that such approach results in drastic increasing of computational costs accompanied by the growth of multiplicity of the stochastic integrals (beginning from 2nd and 3rd multiplicity) that is necessary for construction of high-order strong numerical methods for Itô SDEs or in the case of decrease of integration step of numerical methods, and thereby it is almost useless for practice.

The new step for solution of the problem of combined mean-square approximation of iterated stochastic integrals was made by Milstein G.N. in his monograph [I] (1988). He proposed to use converging in the mean-square sense trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karhunen–Loève expansion), which we may use to expand the iterated stochastic integrals. Using this method, the expansions of two simplest iterated Itô stochastic integrals of 1st and 2nd multiplicities into the series of products of standard Gaussian random variables were obtained and their mean-square convergence was proved in [I].

As we noted above, the attempt to develop this idea together with the Wong–Zakai approximation [56]-[58] was made in the monograph [II] (1992), where the expansions of simplest iterated Stratonovich stochastic integrals of multiplicities 1 to 3 were obtained. However, due to a number of limitations and technical difficulties which are typical for the method [I], in [II] and following publications this problem was not solved more completely. In addition, the author has reasonable doubts about application of the Wong–Zakai approximation [56]-[58] for the iterated stochastic integrals of 3rd multiplicity in the monograph [II] (see discussions in Sect. 2.6.2, 6.2).

It is necessary to note that the computational cost for the method [I] is significantly less than for the method of multiple integral sums.

Regardless of the method [I] positive features, the number of its limitations are also outlined. Among them let us mention the following.

1. The absence of explicit formula for calculation of expansion coefficients for iterated stochastic integrals.

2. The practical impossibility of exact calculation of the mean-square approximation error of iterated stochastic integrals with the exception of simplest integrals of 1st and 2nd multiplicity (as a result, it is necessary to consider redundant terms of expansions and it results to the growth of computational cost and complication of the numerical methods for Itô SDEs).

3. There is a hard limitation on the system of basis functions — it may be only the trigonometric functions.

4. There are some technical problems if we use this method for iterated stochastic integrals whose multiplicity is greater than 2nd.

Nevertheless it should be noted that the analyzed method is a concrete step forward in this scientific field.

The author thinks that the method presented by him in [IV] (for the first time this method is appeared in the final form in [1] (2006)) and in this book (hereafter this method is reffered to as the method of generalized multiple Fourier series) is a breakthrough in solution of the problem of combined mean-square approximation of iterated Itô stochastic integrals.

The idea of this method is as follows: the iterated Itô stochastic integral of multiplicity k ($k \in \mathbf{N}$) is represented as the multiple stochastic integral from the certain nonrandom discontinuous function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Itô stochastic integral. Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Itô stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Itô stochastic integral.

As a result, we obtain the following new possibilities and advantages in comparison with the Fourier method [I].

1. There is an explicit formula for calculation of expansion coefficients of iterated Itô stochastic integral with any fixed multiplicity k. In other words, we can calculate (without any preliminary and additional work) the expansion coefficient with any fixed number in the expansion of iterated Itô stochastic integral of the preset fixed multiplicity. At that, we do not need any knowledge about coefficients with other numbers or about other iterated Itô stochastic integrals included in the considered set.

2. We have new possibilies for obtainment the exact and approximate expressions for the mean-square approximation errors of iterated Itô stochastic integrals. These possibilities are realized by the exact and estimate formulas for the mentioned mean-square approximation errors. As a result, we would not need to consider redundant terms of expansions that may complicate approximations of iterated Itô stochastic integrals.

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t, T]^k)$, we have new possibilities for approximation — we can use not only the trigonometric functions as in [I] but the Legendre polynomials as well as the systems of Haar and Rademacher–Walsh functions.

4. As it turned out, it is more convenient to work with Legendre polynomials for approximation of iterated Itô stochastic integrals. The approximations themselves are simpler than their analogues based on the system of trigonometric functions. For the systems of Haar and Rademacher–Walsh functions the expansions of iterated stochastic integrals become too complex and ineffective for practice [IV].

5. The question about what kind of functions (polynomial or trigonometric) is more convenient in the context of computational costs for approximation turns out to be nontrivial, since it is necessary to compare approximations not for one stochastic integral but for several stochastic integrals at the same time. At that there is a possibility that computational costs for some integrals will be smaller for the system of Legendre polynomials and for others — for the system of trigonometric functions. The author proved [19] (also see Sect. 5.3 in this book) that the computational costs are significantly less for the system of Legendre polynomials at least in the case of approximation of the special set of iterated Itô stochastic integrals, which are necessary for the implementation of strong numerical methods for Itô SDEs with the order of convergence $\gamma = 1.5$. In addition, the author supposes that this effect will be more impressive when

analyzing more complex sets of iterated Itô stochastic integrals ($\gamma = 2.0,\ 2.5,\ 3.0,\ \ldots$). This supposition is based on the fact that the polynomial system of functions has the significant advantage (in comparison with the trigonometric system of functions) in approximation of iterated Itô stochastic integrals for which not all weight functions are equal to 1.

6. The Milstein approach [I] for approximation of iterated Itô stochastic integrals leads to iterated applicaton of the operation of limit transition (in contrast with the method of generalized multiple Fourier series, for which the operation of limit transition is implemented only once) starting at least from the second or third multiplicity of iterated Itô stochastic integrals (we mean at least double or triple integration with respect to components of the multidimensional Wiener process). Multiple series are more preferential for approximation than the iterated ones, since the partial sums of multiple series converge for any possible case of joint converging to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series it is obviously not the case. However, in [II] the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein approach [I] together with the Wong–Zakai approximation [56]-[58] (see discussions in Sect. 2.6.2, 6.2).

7. The convergence in the mean of degree $2n$ ($n \in \mathbf{N}$) as well as the convergence with probability 1 of approximations from the method of generalized multiple Fourier series are proved.

8. The method of generalized multiple Fourier series has been applied for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson random measures and iterated stochastic integrals with respect to martingales) as well as for approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process.

9. Another modification of the method of generalized multiple Fourier series is connected with the application of complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t, T]^k)$.

10. As it turned out, the method of generalized multiple Fourier series can be adapted for iterated Stratonovich stochastic integrals of multiplicities 1 to 5 (see Chapter 2).

11. The results of Chapter 1 (Theorems 1.1, 1.2) and Chapter 2 (Theorems 2.1–2.9) can be considered from the point of view of the Wong–Zakai approximation [56]-[58] for the case of a multidimensional Wiener process and the Wiener

process approximation based on its series expansion using Legendre polynomials and trigonometric functions (see discussions in Sect. 2.6.2, 6.2). These results overcome a number of difficulties that were noted above and relate to the Fourier method [I].

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Basic Notations

$\mathbf{N}$	set of natural numbers
$\mathbf{R}$	set of real numbers
$\mathbf{R}^n$	n-dimensional Euclidean space
$n!$	$1 \cdot 2 \cdot \ldots \cdot n$ for $n \in \mathbf{N}$ $(0! = 1)$
$(2n-1)!!$	$1 \cdot 3 \cdot \ldots \cdot (2n-1)$ for $n \in \mathbf{N}$
$\stackrel{\mathsf{def}}{=}$	equal by definition
$\equiv$	identically equal to
C_n^m	binomial coefficient $n!/(m!(n-m)!)$
$\emptyset$	empty set
$\mathbf{1}_{\mathcal{A}}$	indicator of the set A
$x \in X$	x is an element of the set X
$X \bigcup Y$	union of sets X and Y
$X \times Y$	Cartesian product of sets X and Y
$\overline{\lim\limits_{n\to\infty}}$	$\limsup\limits_{n\to\infty}$
$\underline{\lim\limits_{n\to\infty}}$	$\liminf\limits_{n\to\infty}$
$x \ll y$	x much less than y
$[x]$	largest integer number not exceeding x
$\|x\|$	absolute value of the real number x
$F: \ X \to Y$	function F from X into Y

$A^{(ij)}$	ijth element of the matrix A
A_i	ith colomn of the matrix A
$\mathbf{x}^{(i)}$	ith component of the vector $\mathbf{x} \in \mathbf{R}^n$
$O(x)$	value with the property $\lim\limits_{x\to 0} O(x)/x = \text{const}$
$\sum\limits_{(i_1,\ldots,i_k)}$	sum with respect to all possible permutations $(i_1,\ldots,i_k)$
$\mathsf{M}\{\xi\}$	expectation of ξ
$\mathsf{M}\{\xi\|\mathrm{F}\}$	conditional expectation of ξ with respect to F
$\xi \sim \mathrm{N}(m,\sigma^2)$	Gaussian random variable ξ with expectation m and variance σ^2
$\underset{n\to\infty}{\text{l.i.m.}}$	limit in the mean square sense
$\mathcal{B}(X)$	σ-algebra of Borel subsets of X
f_t	scalar standard Wiener process
$\mathbf{f}_t$	vector standard Wiener process with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$
w. p. 1	with probability 1
$\mathbf{w}_t$	vector with components $\mathbf{w}_t^{(i)}$, $i=0,1,\ldots,m$ and property $\mathbf{w}_t^{(i)} = \mathbf{f}_t^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_t^{(0)} = t$
$\dfrac{\partial F}{\partial \mathbf{x}^{(i)}}$	partial derivative of $F : \mathbf{R}^n \to \mathbf{R}$
$\dfrac{\partial^2 F}{\partial \mathbf{x}^{(i)} \partial \mathbf{x}^{(j)}}$	2nd order partial derivative of $F : \mathbf{R}^n \to \mathbf{R}$
$\int\limits_t^T \ldots d\mathbf{w}_\tau^{(i)}$	Itô stochastic integral
$\int\limits_t^{*T} \ldots d\mathbf{w}_\tau^{(i)}$	Stratonovich stochastic integral
$\mathbf{W}_t$	Q-Wiener process

$J[\psi^{(k)}]_{T,t},\quad I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$	iterated Itô stochastic integrals
$J^*[\psi^{(k)}]_{T,t},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$	iterated Stratonovich stochastic integrals
$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k},\quad I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$	approximations of iterated Itô stochastic integrals
$J^*[\psi^{(k)}]_{T,t}^{p},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}$	approximations of iterated Stratonovich stochastic integrals
$L_2(D)$	Hilbert space of square integrable functions on D
$\Vert\cdot\Vert_{L_2(D)}$	norm in the Hilbert space $L_2(D)$
$\mathrm{tr}\ A$	trace of the operator A
$\Vert\cdot\Vert_H$	norm in the Hilbert space H
$\langle u,v\rangle_H$	scalar product in the Hilbert space H
$L_{HS}(U,H)$	space of Hilbert–Schmidt operators from U to H
$\Vert\cdot\Vert_{L_{HS}(U,H)}$	operator norm in the space of Hilbert–Schmidt operators from U to H
$\int\limits_t^T \ldots d\mathbf{W}_\tau$	stochastic integral with respect to the Q-Wiener process

Contents

Chapter 1

Method of Expansion and Mean-Square Approximation of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series

This chapter is devoted to the expansions of iterated Itô stochastic integrals with respect to components of the multidimensional Wiener process based on generalized multiple Fourier series converging in the sense of norm in the space $L_2([t, T]^k)$, $k \in \mathbf{N}$. The method of generalized multiple Fourier series for expansion and mean-square approximation of iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) is proposed and developed. The obtained expansions contain only one operation of the limit transition in contrast to existing analogues. In this chapter it is also obtained the generalization of the proposed method for the case of discontinuous complete orthonormal systems of functions in the space $L_2([t, T]^k)$, $k \in \mathbf{N}$ as well as for the case of complete orthonormal with weight $r(t_1) \ldots r(t_k) \geq 0$ systems of functions in the space $L_2([t, T]^k)$, $k \in \mathbf{N}$. It is shown that in the case of scalar Wiener process the proposed method leads to the well known expansion of iterated Itô stochastic integrals based on the Itô formula and Hermite polynomials. The convergence in the mean of degree $2n$ ($n \in \mathbf{N}$) as well as the convergence with probability 1 of the proposed method are proved. The exact and approximate expressions for the mean-square approximation error of iterated Itô stochastic integrals of multiplicity k ($k \in \mathbf{N}$) have been derived. The considered method has been applied for other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson random measures and iterated stochastic integrals with respect to martingales).

1.1 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series Converging in the Mean

1.1.1 Introduction

The idea of representing the iterated Itô and Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using multiple and iterated Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of author's publications [1]-[54] (also see early publications [59] (1997), [60] (1998), [61] (2000), [62] (2001), [63] (1994), [64] (1996)). Note that another approaches to the mean-square approximation of iterated Itô and Stratonovich stochastic integrals can be found in [65]-[82].

Specifically, the approach [1]-[54] appeared for the first time in [63], [64]. In these works the mentioned idea is formulated more likely at the level of guess (without any satisfactory grounding), and as a result the papers [63], [64] contain rather fuzzy formulations and a number of incorrect conclusions. Note that in [63], [64] we used the trigonometric multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$, $k=1,2,3$. It should be noted that the results of [63], [64] are correct for a sufficiently narrow particular case when the numbers $i_1,...,i_k$ are pairwise different, $i_1,\ldots,i_k=1,\ldots,m$ (see Theorem 1.1 below).

Usage of Fourier series with respect to the system of Legendre polynomials for approximation of iterated stochastic integrals took place for the first time in the publications of the author [59]-[62] (also see [1]-[54]).

The question about what integrals (Itô or Stratonovich) are more suitable for expansions within the frames of distinguished direction of researches has turned out to be rather interesting and difficult.

On the one side, the results of Chapter 1 (see Theorem 1.1) conclusively demonstrate that the structure of iterated Itô stochastic integrals is rather convenient for expansions into multiple series with respect to the system of standard Gaussian random variables regardless of multiplicity k of iterated Itô stochastic integrals.

On the other side, the results of Chapter 2 [6]-[21], [24], [26], [28], [30]-[37],

[40], [41], [43]-[45], [50], [59]-[62] convincingly testify that there is a doubtless relation between multiplier factor 1/2, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Itô stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its Fourier series converges to the value $(f(x-0)+f(x+0))/2$. In addition, as it is demonstrated in Chapter 2 [6]-[21], [24], [26], [28], [30]-[37], [40], [41], [43]-[45], [50], the final formulas for expansions of iterated Stratonovich stochastic integrals (of second multiplicity in the general case and of third, fourth, and fifth multiplicity in some particular cases) are more compact than their analogues for iterated Itô stochastic integrals.

1.1.2 Itô Stochastic Integral

Let $(\Omega, \text{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega):[0,T]\times\Omega\rightarrow\mathbf{R}$ be the standard Wiener process defined on the probability space $(\Omega, \text{F}, \mathsf{P})$. Further, we will use the following notation: $f(t,\omega)\stackrel{\rm def}{=}f_t$.

Let us consider the right-continous family of σ-algebras $\{\text{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \text{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\text{F}_s\subset\text{F}_t\subset\text{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let us introduce the class $\text{M}_2([0,T])$ of functions $\xi:[0,T]\times\Omega\rightarrow\mathbf{R}$, which satisfy the conditions:

1. The function $\xi(t,\omega)$ is measurable with respect to the pair of variables (t,ω).

2. The function $\xi(t,\omega)$ is F_t-measurable for all $t\in[0,T]$ and $\xi(\tau,\omega)$ is independent with increments $f_{t+\Delta}-f_t$ for $t\ge\tau$, $\Delta>0$.

3. The following relation is fulfilled

$$\int\limits_0^T \mathsf{M}\left\{(\xi(t,\omega))^2\right\}dt<\infty.$$

4. $\mathsf{M}\left\{(\xi(t,\omega))^2\right\}<\infty$ for all $t\in[0,T]$.

For any partition $\tau_j^{(N)}$, $j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty \tag{1.1}$$

we will define the sequence of step functions

$$\xi^{(N)}(t,\omega)=\xi\left(\tau_j^{(N)},\omega\right)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),$$

where $j=0,1,\ldots,N-1$, $N=1,2,\ldots$ Here and further, w. p. 1 means with probability 1.

Let us define the Itô stochastic integral for $\xi(t,\omega)\in \text{M}_2([0,T])$ as the following mean-square limit [83], [84] (also see [67])

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(f\left(\tau_{j+1}^{(N)},\omega\right)-f\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\xi_\tau df_\tau, \tag{1.2}$$

where $\xi^{(N)}(t,\omega)$ is any step function, which converges to the function $\xi(t,\omega)$ in the following sense

$$\lim_{N\to\infty}\int\limits_0^T \mathsf{M}\left\{\left|\xi^{(N)}(t,\omega)-\xi(t,\omega)\right|^2\right\}dt=0. \tag{1.3}$$

Further, we will denote $\xi(\tau,\omega)$ as ξ_τ.

It is well known [83] that the Itô stochastic integral exists as the limit (1.2) and it does not depend on the selection of sequence $\xi^{(N)}(t,\omega)$. Furthermore, the Itô stochastic integral satisfies w. p. 1 to the following properties [83]

$$\mathsf{M}\left\{\int\limits_0^T\xi_t df_t\biggl|\text{F}_0\right\}=0,$$

$$\mathsf{M}\left\{\left|\int\limits_0^T\xi_t df_t\right|^2\biggl|\text{F}_0\right\}=\mathsf{M}\left\{\int\limits_0^T\xi_t^2 dt\biggl|\text{F}_0\right\},$$

$$\int\limits_0^T(\alpha\xi_t+\beta\psi_t)df_t=\alpha\int\limits_0^T\xi_t df_t+\beta\int\limits_0^T\psi_t df_t,$$

where $\xi_t,\ \phi_t \in \mathrm{M}_2([0,T]),\ \alpha,\ \ \beta \in \mathbf{R}^1$.

Let us define the stochastic integral for $\xi_\tau \in \mathrm{M}_2([0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(\tau_{j+1}^{(N)}-\tau_j^{(N)}\right) \stackrel{\text{def}}{=} \int\limits_0^T \xi_\tau d\tau, \tag{1.4}$$

where $\xi^{(N)}(t,\omega)$ is any step function from the class $\mathrm{M}_2([0,T])$, which converges in the sense (1.3) to the function $\xi(t,\omega)$.

1.1.3 Theorem on Expansion of Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a non-decreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider the following iterated Itô stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.5}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau,\ i_1,\ldots,i_k = 0,1,\ldots,m$.

Let us consider the approach to expansion of the iterated Itô stochastic integrals (1.5) [1]-[54] (the so-called method of generalized multiple Fourier series). The idea of this method is as follows: the iterated Itô stochastic integral (1.5) of multiplicity k $(k \in \mathbf{N})$ is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$. Here $[t,T]$ is the interval of integration of the iterated Itô stochastic integral (1.5). Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Itô stochastic integral (1.5) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Itô stochastic integral (1.5).

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}=\prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{1.6}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{1.7}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{1.8}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\ \int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{1.9}$$

Theorem 1.1 [1] (2006) (also see [2]-[54]). *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (1.10)$$

where

$$\mathrm{G}_k = \mathrm{H}_k\backslash \mathrm{L}_k, \quad \mathrm{H}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g\ne r);\ g,r = 1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \qquad (1.11)$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (1.8), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of $[t,T]$, which satisfies the condition* (1.9).

Proof. At first, let us prove preparatory lemmas.

Lemma 1.1. *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1}, \qquad (1.12)$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval $[t,T]$ satisfying the condition* (1.9).

Proof. It is easy to notice that using the property of stochastic integrals additivity, we can write

$$J[\psi^{(k)}]_{T,t} = \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} + \varepsilon_N \quad \text{w. p. 1}, \qquad (1.13)$$

where

$$J[\psi_l]_{s,\theta}=\int\limits_{\theta}^{s}\psi_l(\tau)d\mathbf{w}_{\tau}^{(i_l)}$$

and

$$\varepsilon_N=\sum_{j_k=0}^{N-1}\int\limits_{\tau_{j_k}}^{\tau_{j_k+1}}\psi_k(s)\int\limits_{\tau_{j_k}}^{s}\psi_{k-1}(\tau)J[\psi^{(k-2)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k-1})}d\mathbf{w}_{s}^{(i_k)}+$$

$$+\sum_{r=1}^{k-3}G[\psi_{k-r+1}^{(k)}]_N\times$$

$$\times\sum_{j_{k-r}=0}^{j_{k-r+1}-1}\int\limits_{\tau_{j_{k-r}}}^{\tau_{j_{k-r}+1}}\psi_{k-r}(s)\int\limits_{\tau_{j_{k-r}}}^{s}\psi_{k-r-1}(\tau)J[\psi^{(k-r-2)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k-r-1})}d\mathbf{w}_{s}^{(i_{k-r})}+$$

$$+G[\psi_3^{(k)}]_N\sum_{j_2=0}^{j_3-1}J[\psi^{(2)}]_{\tau_{j_2+1},\tau_{j_2}},$$

where

$$G[\psi_m^{(k)}]_N=\sum_{j_k=0}^{N-1}\sum_{j_{k-1}=0}^{j_k-1}\ldots\sum_{j_m=0}^{j_{m+1}-1}\prod_{l=m}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$(\psi_m,\psi_{m+1},\ldots,\psi_k)\stackrel{\rm def}{=}\psi_m^{(k)},\quad (\psi_1,\ldots,\psi_k)=\psi_1^{(k)}\stackrel{\rm def}{=}\psi^{(k)}.$$

Using the standard estimates (1.25), (1.26) (see below) for the moments of stochastic integrals, we obtain w. p. 1

$$\underset{N\to\infty}{\rm l.i.m.}\ \varepsilon_N=0. \tag{1.14}$$

Comparing (1.13) and (1.14), we get

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\rm l.i.m.}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}\quad \text{w. p. 1}. \tag{1.15}$$

Let us rewrite $J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}$ in the form

$$J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}=\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}+\int\limits_{\tau_{j_l}}^{\tau_{j_l+1}}(\psi_l(\tau)-\psi_l(\tau_{j_l}))d\mathbf{w}_{\tau}^{(i_l)}$$

and substitute it into (1.15). Then, due to the moment properties of stochastic integrals and continuity (which means uniform continuity) of the functions $\psi_l(s)$ $(l = 1, \ldots, k)$ it is easy to see that the prelimit expression on the right-hand side of (1.15) is a sum of the prelimit expression on the right-hand side of (1.12) and the value which tends to zero in the mean-square sense if $N \to \infty$. Lemma 1.1 is proved.

Remark 1.1. *It is easy to see that if* $\Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.12) *for some* $l \in \{1, \ldots, k\}$ *is replaced with* $\left(\Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p = 2,\ i_l \ne 0)$, *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\psi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p = 3, 4, \ldots,$ *then the right-hand side of the formula* (1.12) *will become zero w. p.* 1. *If we replace* $\Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.12) *for some* $l \in \{1, \ldots, k\}$ *with* $(\Delta \tau_{j_l})^p$ $(p = 2, 3, \ldots)$, *then the right-hand side of the formula* (1.12) *also will be equal to zero w. p.* 1.

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J[\Phi]_{T,t}^{(k)}. \tag{1.16}$$

Assume that $D_k = \{(t_1, \ldots, t_k) :\ t \le t_1 < \ldots < t_k \le T\}$. We will say that the domain D_k is closed if $D_k = \{(t_1, \ldots, t_k) :\ t \le t_1 \le \ldots \le t_k \le T\}$. We write the same symbol D_k when we consider the closed or not closed domain D_k. However, we always specify what domain we consider (closed or not closed).

Also we will write $\Phi(t_1, \ldots, t_k) \in C(D_k)$ if $\Phi(t_1, \ldots, t_k)$ is a continuous in the closed domain D_k nonrandom function of k variables.

Let us consider the iterated Itô stochastic integral

$$I[\Phi]_{T,t}^{(k)} \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.17}$$

where $\Phi(t_1, \ldots, t_k) \in C(D_k)$.

It is easy to check that the stochastic integral (1.17) exists in the mean-square sense if the following condition is fulfilled

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1, \ldots, t_k) dt_1 \ldots dt_k < \infty.$$

Using the arguments which similar to the arguments used in the proof of Lemma 1.1 it is easy to demonstrate that if $\Phi(t_1, \ldots, t_k) \in C(D_k)$, then the following equality is fulfilled

$$I[\Phi]_{T,t}^{(k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1.} \tag{1.18}$$

In order to explain this, let us check the correctness of the equality (1.18) when $k=3$. For definiteness we will suppose that $i_1, i_2, i_3 = 1,\ldots,m$. We have

$$I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \int\limits_t^{\tau_{j_3}} \int\limits_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \left(\int\limits_t^{\tau_{j_2}} + \int\limits_{\tau_{j_2}}^{t_2} \right) \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \tag{1.19}$$

Let us demonstrate that the second limit on the right-hand side of (1.19) equals to zero.

Actually, for the second moment of its prelimit expression we get

$$\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi^2(t_1,t_2,\tau_{j_3}) dt_1 dt_2 \Delta\tau_{j_3} \le M^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \frac{1}{2} \left(\Delta\tau_{j_2}\right)^2 \Delta\tau_{j_3} \to 0$$

when $N\to\infty$. Here M is a constant, which restricts the module of the function $\Phi(t_1,t_2,t_3)$ due to its continuity, $\Delta\tau_j = \tau_{j+1} - \tau_j$.

Considering the obtained conclusions, we have

$$I[\Phi]_{T,t}^{(3)} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,\tau_{j_2},\tau_{j_3}) - \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3}) \Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (1.20)$$

In order to get the sought result, we just have to demonstrate that the first two limits on the right-hand side of (1.20) equal to zero. Let us prove that the first one of them equals to zero (proof for the second limit is similar).

The second moment of prelimit expression of the first limit on the right-hand side of (1.20) equals to the following expression

$$\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3}. \qquad (1.21)$$

Since the function $\Phi(t_1,t_2,t_3)$ is continuous in the closed bounded domain D_3, then it is uniformly continuous in this domain. Therefore, if the distance between two points of the domain D_3 is less than $\delta(\varepsilon)$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on mentioned points), then the corresponding oscillation of the function $\Phi(t_1,t_2,t_3)$ for these two points of the domain D_3 is less than ε.

If we assume that $\Delta\tau_j < \delta(\varepsilon)$ ($j = 0,1,\ldots,N-1$), then the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$ is obviously less than $\delta(\varepsilon)$. In this case

$$|\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})| < \varepsilon.$$

Consequently, when $\Delta\tau_j < \delta(\varepsilon)$ $(j = 0,\ 1, \ldots, N-1)$ the expression (1.21) is estimated by the following value

$$\varepsilon^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1} \Delta\tau_{j_2} \Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.$$

Therefore, the first limit on the right-hand side of (1.20) equals to zero. Similarly, we can prove that the second limit on the right-hand side of (1.20) equals to zero.

Consequently, the equality (1.18) is proved for $k = 3$. The cases $k = 2$ and $k > 3$ are analyzed absolutely similarly.

It is necessary to note that the proof of correctness of (1.18) is similar when the nonrandom function $\Phi(t_1, \ldots, t_k)$ is continuous in the open domain D_k and bounded at its boundary.

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\rm def}{=} J'[\Phi]_{T,t}^{(k)}. \qquad (1.22)$$

According to (1.18), we get the following equality

$$J'[\Phi]_{T,t}^{(k)} = \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \biggl(\Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}\biggr) \quad \text{w. p. 1}, \qquad (1.23)$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1, \ldots, t_k)$. At the same time the summation with respect to permutations $(t_1, \ldots, t_k)$ is performed in (1.23) only in the expression, which is enclosed in parentheses, and the nonrandom function $\Phi(t_1, \ldots, t_k)$ is assumed to be continuous in the corresponding domains of integration.

It is not difficult to see that (1.23) can be rewritten in the form

$$J'[\Phi]_{T,t}^{(k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \qquad (1.24)$$

where permutations $(t_1, \ldots, t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1, \ldots, t_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$.

Lemma 1.2. *Suppose that the following condition is satisfied*

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1, \ldots, t_k) dt_1 \ldots dt_k < \infty,$$

where $\Phi(t_1, \ldots, t_k)$ is a nonrandom function. Then

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^2\right\} \le C_k \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1, \ldots, t_k) dt_1 \ldots dt_k, \quad C_k < \infty,$$

where $I[\Phi]_{T,t}^{(k)}$ is defined by the formula (1.17).

Proof. Using standard properties and estimates of stochastic integrals for $\xi_\tau \in \mathrm{M}_2([t,T])$, we have [84]

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^2\right\} = \int\limits_t^T \mathsf{M}\{|\xi_\tau|^2\} d\tau, \tag{1.25}$$

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau d\tau\right|^2\right\} \le (T-t) \int\limits_t^T \mathsf{M}\{|\xi_\tau|^2\} d\tau. \tag{1.26}$$

Let us denote

$$\xi[\Phi]_{t_{l+1},\ldots,t_k,t}^{(l)} = \int\limits_t^{t_{l+1}} \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_l}^{(i_l)},$$

where $l = 1, \ldots, k-1$ and

$$\xi[\Phi]_{t_1,\ldots,t_k,t}^{(0)} \stackrel{\text{def}}{=} \Phi(t_1, \ldots, t_k).$$

By induction it is easy to demonstrate that

$$\xi[\Phi]_{t_{l+1},\ldots,t_k,t}^{(l)} \in \mathrm{M}_2([t,T])$$

with respect to the variable t_{l+1}. Further, using the estimates (1.25), (1.26) repeatedly we obtain the statement of Lemma 1.2.

It is not difficult to see that in the case $i_1,\ldots,i_k=1,\ldots,m$ from the proof of Lemma 1.2 we obtain

$$
\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^2\right\}=\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{1.27}
$$

Lemma 1.3. *Suppose that every $\varphi_l(s)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Then*

$$
\prod_{l=1}^k J[\varphi_l]_{T,t}=J[\Phi]_{T,t}^{(k)}\quad \text{w. p. 1}, \tag{1.28}
$$

where

$$
J[\varphi_l]_{T,t}=\int\limits_t^T\varphi_l(s)d\mathbf{w}_s^{(i_l)},\qquad \Phi(t_1,\ldots,t_k)=\prod_{l=1}^k\varphi_l(t_l),
$$

and the integral $J[\Phi]_{T,t}^{(k)}$ is defined by the equality (1.16).

Proof. Let at first $i_l\ne 0,\ l=1,\ldots,k$. Let us denote

$$
J[\varphi_l]_N\stackrel{\text{def}}{=}\sum_{j=0}^{N-1}\varphi_l(\tau_j)\Delta\mathbf{w}_{\tau_j}^{(i_l)}.
$$

Since

$$
\prod_{l=1}^k J[\varphi_l]_N-\prod_{l=1}^k J[\varphi_l]_{T,t}=
$$

$$
=\sum_{l=1}^k\left(\prod_{g=1}^{l-1}J[\varphi_g]_{T,t}\right)\left(J[\varphi_l]_N-J[\varphi_l]_{T,t}\right)\left(\prod_{g=l+1}^{k}J[\varphi_g]_N\right), \tag{1.29}
$$

then due to the Minkowski inequality and the inequality of Cauchy–Bunyakovsky we obtain

$$
\left(\mathsf{M}\left\{\left|\prod_{l=1}^k J[\varphi_l]_N-\prod_{l=1}^k J[\varphi_l]_{T,t}\right|^2\right\}\right)^{1/2}\le
$$

$$\leq C_k \sum_{l=1}^{k} \left(\mathsf{M} \left\{ \left| J[\varphi_l]_N - J[\varphi_l]_{T,t} \right|^4 \right\} \right)^{1/4}, \tag{1.30}$$

where C_k is a constant.

Note that

$$J[\varphi_l]_N - J[\varphi_l]_{T,t} = \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j},$$

$$J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))\, d\mathbf{w}_s^{(i_l)}.$$

Since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ are independent for various j, then [85]

$$\mathsf{M} \left\{ \left| \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^4 \right\} = \sum_{j=0}^{N-1} \mathsf{M} \left\{ \left| J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^4 \right\} +$$

$$+ 6 \sum_{j=0}^{N-1} \mathsf{M} \left\{ \left| J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^2 \right\} \sum_{q=0}^{j-1} \mathsf{M} \left\{ \left| J[\Delta\varphi_l]_{\tau_{q+1},\tau_q} \right|^2 \right\}. \tag{1.31}$$

Moreover, since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ is a Gaussian random variable, we have

$$\mathsf{M} \left\{ \left| J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^2 \right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds,$$

$$\mathsf{M} \left\{ \left| J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^4 \right\} = 3 \left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds \right)^2.$$

Using these relations and continuity (which means uniform continuity) of the functions $\varphi_l(s)$, we get

$$\mathsf{M} \left\{ \left| \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} \right|^4 \right\} \leq \varepsilon^4 \left(3 \sum_{j=0}^{N-1} (\Delta\tau_j)^2 + 6 \sum_{j=0}^{N-1} \Delta\tau_j \sum_{q=0}^{j-1} \Delta\tau_q \right) <$$

$$< 3\varepsilon^4 \left(\delta(\varepsilon)(T-t) + (T-t)^2 \right),$$

where $\Delta\tau_j < \delta(\varepsilon),\ j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$). Then the right-hand side of the formula (1.31) tends to zero when $N \to \infty$. Considering this fact as well as (1.30), we obtain (1.28).

If $\mathbf{w}_{t_l}^{(i_l)} = t_l$ for some $l \in \{1, \ldots, k\}$, then the proof of Lemma 1.3 becomes obviously simpler and it is performed similarly. Lemma 1.3 is proved.

Remark 1.2. *It is easy to see that if* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.28) *for some* $l \in \{1, \ldots, k\}$ *is replaced with* $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ ($p = 2,\ i_l \ne 0$), *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\Phi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p = 3, 4, \ldots,$ *then the right-hand side of the formula* (1.28) *will become zero w. p.* 1.

Let us consider the case $p = 2$ in detail. Let $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (1.28) for some $l \in \{1, \ldots, k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^2$ ($i_l \ne 0$) and

$$
J[\varphi_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2, \quad J[\varphi_l]_{T,t} \stackrel{\text{def}}{=} \int\limits_t^T \varphi_l(s)ds.
$$

We have

$$
\left(\mathsf{M}\left\{\left|J[\varphi_l]_N - J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4} =
$$

$$
= \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \int\limits_t^T \varphi_l(s)ds\right|^4\right\}\right)^{1/4} =
$$

$$
= \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \int\limits_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(s) - \varphi_l(\tau_j) + \varphi_l(\tau_j)\right) ds\right)\right|^4\right\}\right)^{1/4} \le
$$

$$
\le \left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} \varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2 - \Delta\tau_j\right)\right|^4\right\}\right)^{1/4} +
$$

$$
+ \left|\sum_{j=0}^{N-1} \int\limits_{\tau_j}^{\tau_{j+1}} \left(\varphi_l(\tau_j) - \varphi_l(s)\right) ds\right|. \qquad (1.32)
$$

From the relation, which is similar to (1.31), we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)\right|^4\right\}=$$

$$=\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^4\right\}+$$

$$+6\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^2\right\}\times$$

$$\times\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_q}^{(i_l)}\right)^2-\Delta\tau_q\right)^2\right\}=60\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\left(\Delta\tau_j\right)^4+$$

$$+24\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\left(\Delta\tau_j\right)^2\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\left(\Delta\tau_q\right)^2\le C\left(\Delta_N\right)^2\ \to 0 \qquad (1.33)$$

if $N\to\infty$, where constant C does not depend on N.

The second term on the right-hand side of (1.32) tends to zero if $N\to\infty$ due to continuity (which means uniform continuity) of the function $\varphi_l(s)$ at the interval $[t,T]$. Then, taking into account (1.29), (1.30), (1.33), we come to the affirmation of Remark 1.2.

Let us prove Theorem 1.1. According to Lemma 1.1, we have

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,} \qquad (1.34)$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression enclosed in parentheses.

It is easy to see that (1.34) can be rewritten in the form

$$J[\psi^{(k)}]_{T,t}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1,} \qquad (1.35)$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Since integration of bounded function with respect to the set with measure zero for Riemann integrals gives zero result, then the following formula is correct for these integrals

$$\int\limits_{[t,T]^k}G(t_1,\ldots,t_k)dt_1\ldots dt_k=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}G(t_1,\ldots,t_k)dt_1\ldots dt_k, \qquad (1.36)$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $dt_1\ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals are changed correspondently and $G(t_1,\ldots,t_k)$ is the integrable function on the hypercube $[t,T]^k$.

According to Lemmas 1.1–1.3 and (1.23), (1.24), (1.34), (1.35), we get the following representation

$$J[\psi^{(k)}]_{T,t}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}= \tag{1.37}
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.
$$

$$
\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times
$$

$$
\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+
$$

$$
+R_{T,t}^{p_1,\ldots,p_k}\quad \text{w. p. 1}, \tag{1.38}
$$

where

$$
R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)\times
$$

$$
\times\, d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.39}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 1.2, we have

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times
$$

$$
\times dt_1\ldots dt_k=
$$

$$
=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k\to 0 \tag{1.40}
$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}$. Theorem 1.1 is proved.

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 1.1 we obtain

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.
$$

1.1.4 Expansions of Iterated Itô Stochastic Integrals with Multiplicities 1 to 7 Based on Theorem 1.1

In order to evaluate the significance of Theorem 1.1 for practice we will demonstrate its transformed particular cases (see Remark 1.2) for $k=1,\ldots,7$ [1]-[54]

$$
J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{1.41}
$$

$$
J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{1.42}
$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad (1.43)$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \quad (1.44)$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \qquad (1.45)
$$

$$
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\left(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\right.
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr),
\end{aligned}
\tag{1.46}
$$

$$
J[\psi^{(7)}]_{T,t} = \underset{p_1,\ldots,p_7\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_7=0}^{p_7} C_{j_7\ldots j_1}\Biggl(\prod_{l=1}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0,j_1=j_6\}}\prod_{\substack{l=1\\ l\ne 1,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6\}}\prod_{\substack{l=1\\ l\ne 2,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_3=i_6\ne 0,j_3=j_6\}}\prod_{\substack{l=1\\ l\ne 3,6}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_4=i_6\ne 0,j_4=j_6\}}\prod_{\substack{l=1\\ l\ne 4,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_5=i_6\ne 0,j_5=j_6\}}\prod_{\substack{l=1\\ l\ne 5,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2\}}\prod_{\substack{l=1\\ l\ne 1,2}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3\}}\prod_{\substack{l=1\\ l\ne 1,3}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4\}}\prod_{\substack{l=1\\ l\ne 1,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5\}}\prod_{\substack{l=1\\ l\ne 1,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3\}}\prod_{\substack{l=1\\ l\ne 2,3}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4\}}\prod_{\substack{l=1\\ l\ne 2,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5\}}\prod_{\substack{l=1\\ l\ne 2,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0,j_3=j_4\}}\prod_{\substack{l=1\\ l\ne 3,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_3=i_5\ne 0,j_3=j_5\}}\prod_{\substack{l=1\\ l\ne 3,5}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_4=i_5\ne 0,j_4=j_5\}}\prod_{\substack{l=1\\ l\ne 4,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1\}}\prod_{\substack{l=1\\ l\ne 1,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2\}}\prod_{\substack{l=1\\ l\ne 2,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3\}}\prod_{\substack{l=1\\ l\ne 3,7}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4\}}\prod_{\substack{l=1\\ l\ne 4,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5\}}\prod_{\substack{l=1\\ l\ne 7,5}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6\}}\prod_{\substack{l=1\\ l\ne 7,6}}^{7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0,j_6=j_2,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0,j_6=j_3,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0,j_6=j_4,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0,j_6=j_5,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_7=j_1,i_7=i_1\ne 0,j_5=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=4,2,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+
$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}-$$

$$
-\Biggl(\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_1}^{(i_1)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_6\ne 0,j_1=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_2}^{(i_2)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_5\ne 0,j_2=j_5,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_6\ne 0,j_2=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_3}^{(i_3)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5\}}\Biggr)\zeta_{j_4}^{(i_4)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_5}^{(i_5)}-
$$

$$
-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}+
$$

$$
+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_6}^{(i_6)}-
$$

$$-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_5=i_6\ne 0,j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_4=i_5\ne 0,j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6\}}+$$

$$+\mathbf{1}_{\{i_4=i_1\ne 0,j_4=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5\}}+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_5=i_1\ne 0,j_5=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\Biggr)\zeta_{j_7}^{(i_7)}\Biggr), \qquad (1.47)$$

where $\mathbf{1}_A$ is the indicator of the set A.

1.1.5 Expansion of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$) Based on Theorem 1.1

Consider a generalization of the formulas (1.41)–(1.47) for the case of arbitrary multiplicity k for $J[\psi^{(k)}]_{T,t}$. In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \qquad (1.48)$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (1.48) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \qquad (1.49)$$

Below there are several examples of sums in the form (1.49)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=
$$

$$
=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=
$$

$$
=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+
$$

$$
+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=
$$

$$
=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+
$$

$$
+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+
$$

$$
+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.
$$

Now we can formulate Theorem 1.1 (see (1.10)) using alternative form.

Theorem 1.2 [4] (2009) (also see [5]-[14], [22], [27], [37], [46], [47]). *In the conditions of Theorem* 1.1 *the following expansion*

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\quad(1.50)
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x.

In particular, from (1.50) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (1.45).

1.1.6 Comparison of Theorem 1.2 with the Representations of Iterated Itô Stochastic Integrals Based on Hermite Polynomials

Note that the correctness of the formulas (1.41)–(1.47) can be verified in the following way. If $i_1 = \ldots = i_7 = i = 1,\ldots,m$ and $\psi_1(s),\ldots,\psi_7(s)\equiv\psi(s)$, then we can derive from (1.41)–(1.47) [2]-[14], [27] the well known equalities

$$J[\psi^{(1)}]_{T,t} = \frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t} = \frac{1}{2!}\left(\delta_{T,t}^2 - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t} = \frac{1}{3!}\left(\delta_{T,t}^3 - 3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t} = \frac{1}{4!}\left(\delta_{T,t}^4 - 6\delta_{T,t}^2\Delta_{T,t} + 3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t} = \frac{1}{5!}\left(\delta_{T,t}^5 - 10\delta_{T,t}^3\Delta_{T,t} + 15\delta_{T,t}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t} = \frac{1}{6!}\left(\delta_{T,t}^6 - 15\delta_{T,t}^4\Delta_{T,t} + 45\delta_{T,t}^2\Delta_{T,t}^2 - 15\Delta_{T,t}^3\right),$$

$$J[\psi^{(7)}]_{T,t} = \frac{1}{7!}\left(\delta_{T,t}^7 - 21\delta_{T,t}^5\Delta_{T,t} + 105\delta_{T,t}^3\Delta_{T,t}^2 - 105\delta_{T,t}\Delta_{T,t}^3\right)$$

w. p. 1, where

$$\delta_{T,t} = \int\limits_t^T \psi(s)d\mathbf{f}_s^{(i)},\quad \Delta_{T,t} = \int\limits_t^T \psi^2(s)ds,$$

which can be independently obtained using the Itô formula and Hermite polynomials [98].

When $k=1$ everything is evident. Let us consider the cases $k=2$ and $k=3$ in detail. When $k=2$ and $p_1=p_2=p$ we have [2]-[14], [27]

$$J[\psi^{(2)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}-\sum_{j_1=0}^{p}C_{j_1j_1}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\left(C_{j_2j_1}+C_{j_1j_2}\right)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\sum_{j_1=0}^{p}C_{j_1j_1}\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\frac{1}{2}\sum_{\substack{j_1,j_2=0\\ j_1\ne j_2}}^{p}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\frac{1}{2}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^2-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\right)=$$

$$=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right). \tag{1.51}$$

Let us explain the last step in (1.51). For the Itô stochastic integral the following estimate [99] is valid

$$\mathsf{M}\left\{\left|\int_t^T\xi_\tau df_\tau\right|^q\right\}\le K_q\mathsf{M}\left\{\left(\int_t^T|\xi_\tau|^2d\tau\right)^{q/2}\right\}, \tag{1.52}$$

where $q>0$ is a fixed number, f_τ is a scalar standard Wiener process, $\xi_\tau\in\text{M}_2([t,T])$, K_q is a constant depending only on q,

$$\int_t^T|\xi_\tau|^2d\tau<\infty\quad \text{w. p. } 1,$$

$$\mathsf{M}\left\{\left(\int_t^T|\xi_\tau|^2d\tau\right)^{q/2}\right\}<\infty.$$

Since

$$\delta_{T,t} - \sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)} = \int\limits_t^T \left(\psi(s) - \sum_{j_1=0}^{p} C_{j_1}\phi_{j_1}(s)\right) d\mathbf{f}_s^{(i)},$$

then applying the estimate (1.52) to the right-hand side of this expression and considering that

$$\int\limits_t^T \left(\psi(s) - \sum_{j_1=0}^{p} C_{j_1}\phi_{j_1}(s)\right)^2 ds \ \to 0$$

if $p \to \infty$, we obtain

$$\int\limits_t^T \psi(s) d\mathbf{f}_s^{(i)} = q\ \text{-}\ \underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}, \quad q > 0. \tag{1.53}$$

Here $q\ \text{-}\ \underset{p\to\infty}{\text{l.i.m.}}$ is a limit in the mean of degree q. Hence, if $q = 4$, then it is easy to conclude that

$$\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^2 = \delta_{T,t}^2.$$

This equality as well as Parseval's equality were used in the last step of the formula (1.51).

When $k = 3$ and $p_1 = p_2 = p_3 = p$ we obtain [2]-[14], [27]

$$J[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} - \right.$$

$$\left. - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i)} - \sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1}\zeta_{j_1}^{(i)} - \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}\zeta_{j_2}^{(i)}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} - \sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_1} + C_{j_1j_1j_3} + C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\left(C_{j_3j_2j_1} + C_{j_3j_1j_2} + C_{j_2j_1j_3} + C_{j_2j_3j_1} + C_{j_1j_2j_3} + C_{j_1j_3j_2}\right)\times\right.$$

$$\times\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} +$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Biggl(C_{j_3j_1j_3}+C_{j_1j_3j_3}+C_{j_3j_3j_1}\Biggr)\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\Biggl(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\Biggr)\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\sum_{j_1=0}^{p}C_{j_1j_1j_1}\left(\zeta_{j_1}^{(i)}\right)^3-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\frac{1}{6}\sum_{\substack{j_1,j_2,j_3=0\\ j_1\ne j_2,j_2\ne j_3,j_1\ne j_3}}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\frac{1}{6}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\right.$$

$$-\frac{1}{6}\left(3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\right.$$

$$\left.+\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right)+$$

$$
+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+
$$

$$
\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\ \left(\frac{1}{6}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\sum_{j_3=0}^{p}C_{j_3}\zeta_{j_3}^{(i)}\right)=
$$

$$
=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right). \tag{1.54}
$$

The last step in (1.54) follows from Parseval's equality, Theorem 1.1 for $k=1$, and the equality

$$
\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3=\delta_{T,t}^3,
$$

which can be obtained easily when $q=8$ (see (1.53)).

In addition, we used the following relations between Fourier coefficients for the considered case

$$
C_{j_1j_2}+C_{j_2j_1}=C_{j_1}C_{j_2},\quad 2C_{j_1j_1}=C_{j_1}^2,
$$

$$
C_{j_1j_2j_3}+C_{j_1j_3j_2}+C_{j_2j_3j_1}+C_{j_2j_1j_3}+C_{j_3j_2j_1}+C_{j_3j_1j_2}=C_{j_1}C_{j_2}C_{j_3}, \tag{1.55}
$$

$$
2\left(C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}\right)=C_{j_1}^2C_{j_3},
$$

$$
6C_{j_1j_1j_1}=C_{j_1}^3.
$$

1.1.7 On Usage of Discontinuous Complete Orthonormal Systems of Functions in Theorem 1.1

Analyzing the proof of Theorem 1.1, we can ask the question: can we weaken the continuity condition for the functions $\phi_j(x)$, $j=1,2,\ldots$?

We will say that the function $f(x):[t,T]\to\mathbf{R}$ *satisfies the condition* $(\star)$, *if it is continuous at the interval* $[t,T]$ *except may be for the finite number of points of the finite discontinuity as well as it is right-continuous at the interval* $[t,T]$.

Furthermore, let us suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

It is easy to see that continuity of the functions $\phi_j(x)$ was used substantially for the proof of Theorem 1.1 in two places. More precisely, we mean Lemma 1.3 and the formula (1.18). It is clear that without the loss of generality the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ in Lemma 1.3 and (1.18) can be taken so "dense" that among the points τ_j of this partition there will be all points of jumps of the functions $\varphi_1(\tau)=\phi_{j_1}(\tau),\ \ldots,\ \varphi_k(\tau)=\phi_{j_k}(\tau)$ $(j_1,\ldots,j_k<\infty)$ and among the points $(\tau_{j_1},\ldots,\tau_{j_k})$ for which $0\le j_1<\ldots<j_k\le N-1$ there will be all points of jumps of the function $\Phi(t_1,\ldots,t_k)$.

Let us demonstrate how to modify the proofs of Lemma 1.3 and the formula (1.18) in the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

At first, appeal to Lemma 1.3. From the proof of this lemma it follows that

$$
{\sf M}\left\{\left|\sum_{j=0}^{N-1}J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}=\sum_{j=0}^{N-1}{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}+
$$

$$
+6\sum_{j=0}^{N-1}{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1}{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}, \tag{1.56}
$$

$$
{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}=\int\limits_{\tau_j}^{\tau_{j+1}}(\varphi_l(\tau_j)-\varphi_l(s))^2ds,
$$

$$
{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}=3\left(\int\limits_{\tau_j}^{\tau_{j+1}}(\varphi_l(\tau_j)-\varphi_l(s))^2ds\right)^2.
$$

Suppose that the functions $\varphi_l(s)$ $(l=1,\ldots,k)$ satisfy the condition $(\star)$ and the partition $\{\tau_j\}_{j=0}^{N}$ includes all points of jumps of the functions $\varphi_l(s)$ $(l=1,\ldots,k)$. It means that for the integral

$$
\int\limits_{\tau_j}^{\tau_{j+1}}(\varphi_l(\tau_j)-\varphi_l(s))^2ds
$$

the integrand function is continuous at the interval $[\tau_j, \tau_{j+1}]$, except possibly the point τ_{j+1} of finite discontinuity.

Let $\mu \in (0, \Delta\tau_j)$ be fixed. Due to continuity (which means uniform continuity) of the functions $\varphi_l(s)$ $(l = 1, \ldots, k)$ at the interval $[\tau_j, \tau_{j+1} - \mu]$ we have

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds =$$

$$= \int\limits_{\tau_j}^{\tau_{j+1}-\mu} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds + \int\limits_{\tau_{j+1}-\mu}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds < \varepsilon^2(\Delta\tau_j - \mu) + M^2\mu. \tag{1.57}$$

When obtaining the inequality (1.57) we supposed that $\Delta\tau_j < \delta(\varepsilon)$ for all $j = 0, 1, \ldots, N-1$ (here $\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on s),

$$|\varphi_l(\tau_j) - \varphi_l(s)| < \varepsilon$$

for $s \in [\tau_j, \tau_{j+1} - \mu]$ (due to uniform continuity of the functions $\varphi_l(s)$, $l = 1, \ldots, k$),

$$|\varphi_l(\tau_j) - \varphi_l(s)| < M$$

for $s \in [\tau_{j+1} - \mu, \tau_{j+1}]$, M is a constant (potential discontinuity point of the function $\varphi_l(s)$ is the point τ_{j+1}).

Performing the passage to the limit in the inequality (1.57) when $\mu \to +0$, we get

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds \le \varepsilon^2 \Delta\tau_j. \tag{1.58}$$

Using (1.58) to estimate the right-hand side of (1.56), we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4 \left(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\right) <$$

$$< 3\varepsilon^4\left(\delta(\varepsilon)(T-t) + (T-t)^2\right). \tag{1.59}$$

This implies that

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \to 0$$

when $N \to \infty$ and Lemma 1.3 remains correct.

Now, let us present explanations concerning the correctness of the formula (1.18), when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

Consider the case $k=3$ and the representation (1.20). Let us demonstrate that in the studied case the first limit on the right-hand side of (1.20) equals to zero (similarly, we can demonstrate that the second limit on the right-hand side of (1.20) equals to zero; proof of the second limit equality to zero on the right-hand side of the formula (1.19) is the same as for the case of continuous functions $\phi_j(x)$, $j=0,1,\ldots$).

The second moment of the prelimit expression of first limit on the right-hand side of (1.20) looks as follows

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2\Delta\tau_{j_3}.$$

Further, for the fixed $\mu\in(0,\Delta\tau_{j_2})$ and $\rho\in(0,\Delta\tau_{j_1})$ we have

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2=$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\right)\left(\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2=$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\times$$

$$\times\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2<$$

$$<\varepsilon^2\left(\Delta\tau_{j_2}-\mu\right)\left(\Delta\tau_{j_1}-\rho\right)+M^2\rho\left(\Delta\tau_{j_2}-\mu\right)+M^2\mu\left(\Delta\tau_{j_1}-\rho\right)+M^2\mu\rho, \quad (1.60)$$

where M is a constant, $\Delta\tau_j<\delta(\varepsilon)$ for $j=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$). We suppose here that the partition $\{\tau_j\}_{j=0}^N$ contains all discontinuity points of the function $\Phi(t_1,t_2,t_3)$ as points τ_j (for every variable). When obtaining the inequality (1.60) we also supposed that potential discontinuity points of this function (for every variable) are contained among the points $\tau_{j_1+1},\tau_{j_2+1},\tau_{j_3+1}$.

Let us explain in detail how we obtained the inequality (1.60). Since the function $\Phi(t_1,t_2,t_3)$ is continuous at the closed bounded set

$$Q_3=\Biggl\{(t_1,t_2,t_3):t_1\in[\tau_{j_1},\tau_{j_1+1}-\rho],t_2\in[\tau_{j_2},\tau_{j_2+1}-\mu],t_3\in[\tau_{j_3},\tau_{j_3+1}-\nu]\Biggr\},$$

where ρ,μ,ν are fixed small positive numbers such that

$$\nu\in(0,\Delta\tau_{j_3}),\quad \mu\in(0,\Delta\tau_{j_2}),\quad \rho\in(0,\Delta\tau_{j_1}),$$

then this function is also uniformly continous at this set. Moreover, the function $\Phi(t_1,t_2,t_3)$ is supposed to be bounded at the closed set D_3 (see the proof of Theorem 1.1).

Since the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})\in Q_3$ is obviously less than $\delta(\varepsilon)$ ($\Delta\tau_j<\delta(\varepsilon)$ for $j=0,1,\ldots,N-1$), then

$$|\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})|<\varepsilon.$$

This inequality was used to estimate the first double integral in (1.60). Estimating the three remaining double integrals in (1.60) we used the boundedness property for the function $\Phi(t_1,t_2,t_3)$ in the form of inequality

$$|\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})|<M.$$

Performing the passage to the limit in the inequality (1.60) when $\mu,\rho\to+0$, we obtain the estimate

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2dt_1dt_2\le\varepsilon^2\Delta\tau_{j_2}\Delta\tau_{j_1}.$$

This estimate provides

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2dt_1dt_2\Delta\tau_{j_3}\le$$

$$\le\varepsilon^2\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3}<\varepsilon^2\frac{(T-t)^3}{6}.$$

The last inequality means that in the considered case the first limit on the right-hand side of (1.20) equals to zero (similarly, we can demonstrate that the second limit on the right-hand side of (1.20) equals to zero).

Consequently, the formula (1.18) is correct when $k=3$ in the studied case. Similarly, we can perform the argumentation for the cases $k=2$ and $k>3$.

Therefore, in Theorem 1.1 we can use complete orthonormal systems of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

One of the examples of such systems of functions is a complete orthonormal system of Haar functions in the space $L_2([t,T])$

$$\phi_0(x)=\frac{1}{\sqrt{T-t}},\quad \phi_{nj}(x)=\frac{1}{\sqrt{T-t}}\ \varphi_{nj}\left(\frac{x-t}{T-t}\right),$$

where $n=0,1,\ldots,$ $j=1,2,\ldots,2^n$, and the functions $\varphi_{nj}(x)$ are defined as

$$\varphi_{nj}(x)=\begin{cases}2^{n/2}, & x\in[(j-1)/2^n,\ (j-1)/2^n+1/2^{n+1})\\ \\ -2^{n/2}, & x\in[(j-1)/2^n+1/2^{n+1},\ j/2^n)\\ \\ 0, & \text{otherwise}\end{cases},$$

$n=0,1,\ldots,$ $j=1,2,\ldots,2^n$ (we choose the values of Haar functions in the points of discontinuity in such a way that these functions will be right-continuous).

The other example of similar system of functions is a complete orthonormal system of Rademacher–Walsh functions in the space $L_2([t,T])$

$$\phi_0(x)=\frac{1}{\sqrt{T-t}},$$

$$\phi_{m_1\ldots m_k}(x)=\frac{1}{\sqrt{T-t}}\ \varphi_{m_1}\left(\frac{x-t}{T-t}\right)\ldots\varphi_{m_k}\left(\frac{x-t}{T-t}\right),$$

where $0<m_1<\ldots<m_k$, $m_1,\ldots,m_k=1,2,\ldots,$ $k=1,2,\ldots,$

$$\varphi_m(x)=(-1)^{[2^m x]},$$

$x\in[0,1]$, $m=1,2,\ldots,$ $[y]$ is an integer part of a real number y.

1.1.8 Remark on Usage of Complete Orthonormal Systems of Functions in Theorem 1.1

Note that actually the functions $\phi_j(s)$ from the complete orthonormal system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ depend not only on s, but on t and T.

For example, the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ have the following form

$$\phi_j(s,t,T)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}\right),$$

$$P_j(y)=\frac{1}{2^j j!}\frac{d^j}{dy^j}\left(y^2-1\right)^j,$$

where $P_j(y)$ $(j=0,1,2,\ldots)$ is the Legendre polynomial,

$$\phi_j(s,t,T)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(s-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(s-t)/(T-t)\right), & j=2r\end{cases}\qquad(1.61)$$

where $r=1,2,\ldots$

Note that the specified systems of functions are assumed to be used in the context of implementation of numerical methods for Itô SDEs (see Chapter 4) for the sequences of time intervals

$$[T_0,T_1],\ [T_1,T_2],\ [T_2,T_3],\ \ldots$$

and Hilbert spaces

$$L_2([T_0,T_1]),\ L_2([T_1,T_2]),\ L_2([T_2,T_3]),\ \ldots$$

We can explain that the dependence of functions $\phi_j(s,t,T)$ on t and T (hereinafter these constants will mean fixed moments of time) will not affect on the main properties of independence of random variables

$$\zeta_{(j)T,t}^{(i)}=\int\limits_t^T\phi_j(s,t,T)d\mathbf{w}_s^{(i)},$$

where $i = 1, \ldots, m$ and $j = 0, 1, 2, \ldots$

Indeed, for fixed t and T due to orthonormality of the mentioned systems of functions we have

$$\mathsf{M}\left\{\zeta_{(j)T,t}^{(i)}\zeta_{(g)T,t}^{(r)}\right\} = \mathbf{1}_{\{i=r\}}\mathbf{1}_{\{j=g\}},$$

where $i, r = 1, \ldots, m, \ \ j, g = 0, 1, 2, \ldots$

This means that $\zeta_{(j)T,t}^{(i)}$ and $\zeta_{(g)T,t}^{(r)}$ are independent for $j \neq g$ (since these random variables are Gaussian) or $i \neq r$.

From the other side, the random variables

$$\zeta_{(j)T_1,t_1}^{(i)} = \int\limits_{t_1}^{T_1} \phi_j(s, t_1, T_1) d\mathbf{w}_s^{(i)}, \qquad \zeta_{(j)T_2,t_2}^{(i)} = \int\limits_{t_2}^{T_2} \phi_j(s, t_2, T_2) d\mathbf{w}_s^{(i)}$$

are independent if $[t_1, T_1] \cap [t_2, T_2] = \emptyset$ (the case $T_1 = t_2$ is possible) according to the properties of the Itô stochastic integral.

Therefore, the important properties of random variables $\zeta_{(j)T,t}^{(i)}$, which are the basic motive of their usage, are saved.

1.1.9 Convergence in the Mean of Degree $2n$ ($n \in \mathbf{N}$) of Expansions of Iterated Itô Stochastic Integrals from Theorem 1.1

Constructing the expansions of iterated Itô stochastic integrals from Theorem 1.1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge not only in the mean-square sense but in the stronger probabilistic senses.

We will obtain the general estimate which proves convergence in the mean of degree $2n$ ($n \in \mathbf{N}$) of expansions from Theorem 1.1.

According to the notations of Theorem 1.1 (see (1.39)), we have

$$R_{T,t}^{p_1,\ldots,p_k} = J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1 \ldots p_k}(t_1, \ldots, t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{1.62}$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) \stackrel{\rm def}{=} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),$$

$J[\psi^{(k)}]_{T,t}$ is the stochastic integral (1.5), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (1.10) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$.

Note that for definiteness we consider in this section the case $i_1,\ldots,i_k = 1,\ldots,m$. Another notations from this section are the same as in the formulation and proof of Theorem 1.1.

When proving Theorem 1.1 we obtained the following estimate (see (1.40))

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k,$$

where C_k is a constant.

Assume that

$$\eta_{t_l,t}^{(l-1)} \stackrel{\rm def}{=} \int\limits_t^{t_l}\ldots\int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_{l-1}}^{(i_{l-1})}, \quad l=2,\ 3,\ldots,k+1,$$

$$\eta_{T,t}^{(k)} \stackrel{\rm def}{=} \int\limits_t^{T}\ldots\int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \quad \eta_{t_{k+1},t}^{(k)} \stackrel{\rm def}{=} \eta_{T,t}^{(k)}.$$

Using the Itô formula it is easy to demonstrate that [84]

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\} = n(2n-1)\int\limits_{t_0}^{t}\mathsf{M}\left\{\left(\int\limits_{t_0}^{s}\xi_u df_u\right)^{2n-2}\xi_s^2\right\}ds.$$

Using the Hölder inequality (under the integral sign on the right-hand side of the last equality) for $p=n/(n-1),\ q=n$ and using the increasing of the value

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}$$

with the growth of t, we get

$$
\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\le n(2n-1)\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\right)^{(n-1)/n}\times
$$

$$
\times\int\limits_{t_0}^{t}\left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds.
$$

After raising to power n the obtained inequality and dividing the result by

$$
\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\right)^{n-1},
$$

we get the following estimate

$$
\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\le (n(2n-1))^n\left(\int\limits_{t_0}^{t}\left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds\right)^{n}. \qquad (1.63)
$$

Using the estimate (1.63) repeatedly, we have

$$
\mathsf{M}\left\{\left(\eta_{T,t}^{(k)}\right)^{2n}\right\}\le (n(2n-1))^n\left(\int\limits_{t}^{T}\left(\mathsf{M}\left\{\left(\eta_{t_k,t}^{(k-1)}\right)^{2n}\right\}\right)^{1/n}dt_k\right)^{n}\le
$$

$$
\le (n(2n-1))^n\times
$$

$$
\times\left(\int\limits_{t}^{T}\left((n(2n-1))^n\left(\int\limits_{t}^{t_k}\left(\mathsf{M}\left\{\left(\eta_{t_{k-1},t}^{(k-2)}\right)^{2n}\right\}\right)^{1/n}dt_{k-1}\right)^{n}\right)^{1/n}dt_k\right)^{n}=
$$

$$
=(n(2n-1))^{2n}\left(\int\limits_{t}^{T}\int\limits_{t}^{t_k}\left(\mathsf{M}\left\{\left(\eta_{t_{k-1},t}^{(k-2)}\right)^{2n}\right\}\right)^{1/n}dt_{k-1}dt_k\right)^{n}\le\ldots
$$

$$
\ldots\le (n(2n-1))^{n(k-1)}\left(\int\limits_{t}^{T}\int\limits_{t}^{t_k}\ldots\int\limits_{t}^{t_3}\left(\mathsf{M}\left\{\left(\eta_{t_2,t}^{(1)}\right)^{2n}\right\}\right)^{1/n}dt_3\ldots dt_{k-1}dt_k\right)^{n}=
$$

$$
= (n(2n-1))^{n(k-1)}(2n-1)!! \left(\int\limits_t^T \ldots \int\limits_t^{t_2} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n \le
$$

$$
\le (n(2n-1))^{n(k-1)}(2n-1)!! \times
$$

$$
\times \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n .
$$

The penultimate step was obtained using the formula

$$
\mathsf{M}\left\{ \left(\eta^{(1)}_{t_2,t}\right)^{2n} \right\} = (2n-1)!! \left(\int\limits_t^{t_2} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1 \right)^n ,
$$

which follows from Gaussianity of

$$
\eta^{(1)}_{t_2,t} = \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}^{(i_1)}_{t_1}.
$$

Similarly, we estimate each summand on the right-hand side of (1.62). Then, from (1.62) using the Minkowski inequality, we finally get

$$
\mathsf{M}\left\{ \left(R^{p_1,\ldots,p_k}_{T,t}\right)^{2n} \right\} \le
$$

$$
\le \left(k! \left((n(2n-1))^{n(k-1)}(2n-1)!! \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n \right)^{1/2n} \right)^{2n}
$$

$$
= (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!! \times
$$

$$
\times \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n . \qquad (1.64)
$$

Using the orthonormality of the functions $\phi_j(s)$ $(j=0,1,2,\ldots)$, we obtain

$$\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k=$$

$$=\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-2\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k+$$

$$+\int\limits_{[t,T]^k}\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-2\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k+$$

$$+\sum_{j_1=0}^{p_1}\sum_{j_1'=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\sum_{j_k'=0}^{p_k}C_{j_k\ldots j_1}C_{j_k'\ldots j_1'}\prod_{l=1}^k\int\limits_t^T\phi_{j_l}(t_l)\phi_{j_l'}(t_l)dt_l=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-2\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}. \qquad (1.65)$$

Let us substitute (1.65) into (1.64)

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le
$$

$$
\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times
$$

$$
\times\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n. \qquad (1.66)
$$

Due to Parseval's equality

$$
\int\limits_{[t,T]^k}R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k=
$$

$$
=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\ \to\ 0
$$

if $p_1,\ldots,p_k\to\infty$. Therefore, the inequality (1.64) (or (1.66)) means that the expansions of iterated Itô stochastic integrals obtained using Theorem 1.1 converge in the mean of degree $2n$ ($n\in\mathbf{N}$) to the appropriate iterated Itô stochastic integrals.

1.1.10 Conclusions

Thus, we obtain the following useful possibilities and modifications of the approach based on Theorem 1.1.

1. There is an explicit formula (see (1.8)) for calculation of expansion coefficients of the iterated Itô stochastic integral (1.5) with any fixed multiplicity k ($k\in\mathbf{N}$).

2. We have possibilities for exact calculation of the mean-square approximation error of the iterated Itô stochastic integral (1.5) [14], [16], [29] (see Sect. 1.2).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only the trigonometric functions as in [65]-[68], [75], [76], [79], [80], but the Legendre polynomials.

4. As it turned out [1]-[54], it is more convenient to work with Legendre polynomials for approximation of the iterated Itô stochastic integrals (1.5) (see Chapter 5). Approximations based on Legendre polynomials essentially simpler than their analogues based on trigonometric functions [1]-[54]. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [19], [38] (see Sect. 5.3).

5. The Milstein approach [65] (see Sect. 6.2 in this book) to expansion of iterated stochastic integrals based on the Karhunen–Loève expansion of the Brownian bridge process (also see [66]-[68], [75], [76], [79], [80]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1.1) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$; $i_1, i_2, i_3 = 1, \ldots, m$) of the iterated Itô stochastic integrals (1.5). Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, in practice, we usually choose $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [66]-[68], [76] the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein approach [65] together with the Wong–Zakai approximation [56]-[58] (see discussions in Sect. 2.6.2, 6.2).

6. As we mentioned above, constructing the expansions of iterated Itô stochastic integrals from Theorem 1.1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge with probability 1. The convergence with probability 1 in Theorem 1.1 has been proved for some particular cases in [3]-[14], [30] (see Sect. 1.7.1) and for the general case of iterated Itô stochastic integrals of multiplicity k ($k \in \mathbf{N}$) in [14], [27], [29], [30], [25] (see Sect. 1.7.2).

7. The generalizations of Theorem 1.1 for complete orthonormal with weight $r(t_1) \ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t, T]^k)$ ($k \in \mathbf{N}$) [12]-[14], [39] as well as for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson measures and iterated stochastic integrals with respect to martingales) [1]-[14], [39] are presented in Sect. 1.3–1.6.

8. The adaptation of Theorem 1.1 for iterated Stratonovich stochastic in-

tegrals of multiplicities 1 to 5 was realized in [6]-[21], [26], [28], [30]-[37], [41], [43]-[45], [48], [50] (see Chapter 2).

9. Application of Theorem 1.1 for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in [14], [22], [23], [46], [47] (see Chapter 7).

1.2 Exact Calculation of the Mean-Square Error in the Method of Approximation of Iterated Itô Stochastic integrals Based on Generalized Multiple Fourier Series

This section is devoted to the obtainment of exact and approximate expressions for the mean-square approximation error in Theorem 1.1 for iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$). As a result, we do not need to use redundant terms of expansions of iterated Itô stochastic integrals.

1.2.1 Introduction

Recall that we called the method of expansion and mean-square approximation of iterated Itô stochastic integrals based on Theorem 1.1 as the method of generalized multiple Fourier series. The question about how estimate or even calculate exactly the mean-square approximation error of iterated Itô stochastic integrals for the method of generalized multiple Fourier series composes the subject of Sect. 1.2. From the one side the mentioned question is essentially difficult in the case of multidimensional Wiener process, because of we need to take into account all possible combinations of components of the multidimensional Wiener process. From the other side an effective solution of the mentioned problem allows to construct more economic expansions of iterated Itô stochastic integrals than in [65]-[70], [75]-[77], [79], [80].

Sect. 1.2.2 is devoted to the formulation and proof of Theorem 1.3, which allows to calculate exacly the mean-square approximation error of iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) for the method of generalized multiple Fourier series. The particular cases ($k = 1, \ldots, 5$) of Theorem 1.3 are considered in detail in Sect. 1.2.3. In Sect. 1.2.4 we prove the effective estimate for the mean-square approximation error of iterated Itô stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) for the method of generalized multiple

Fourier series.

1.2.2 Theorem on Exact Calculation of the Mean-Square Approximation Error for Iterated Itô Stochastic integrals

Theorem 1.3 [12]-[16], [29]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{1.67}$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right), \tag{1.68}$$

$$S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)}, \tag{1.69}$$

the Fourier coefficient $C_{j_k\ldots j_1}$ has the form (1.8),

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)} \tag{1.70}$$

are independent standard Gaussian random variables for various i or j $(i=1,\ldots,m)$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$ *(see* (1.67)*); another notations are the same as in Theorem* 1.1.

Remark 1.3. *Note that*

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=$$

$$=\mathsf{M}\left\{\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k=C_{j_k\ldots j_1}.$$

Therefore, in the case of pairwise different numbers $i_1,\ldots,i_k$ *from Theorem* 1.3 *we obtain*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^p\ldots\sum_{j_k=0}^p C_{j_k\ldots j_1}^2. \tag{1.71}$$

Moreover, if $i_1=\ldots=i_k$, *then from Theorem* 1.3 *we get*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^p\ldots\sum_{j_k=0}^p C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

For example, for the case $k=3$ *we have*

$$
\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^p\right)^2\right\}=\int\limits_t^T\psi_3^2(t_3)\int\limits_t^{t_3}\psi_2^2(t_2)\int\limits_t^{t_2}\psi_1^2(t_1)dt_1dt_2dt_3-
$$

$$
-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_3j_1}+C_{j_2j_1j_3}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr).
$$

Proof. Using Theorem 1.1 for the case $i_1,\ldots,i_k=1,\ldots,m$ and $p_1=\ldots=p_k=p$, we obtain

$$
J[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right). \tag{1.72}
$$

For $n>p$ we can write

$$
J[\psi^{(k)}]_{T,t}^n=\left(\sum_{j_1=0}^{p}+\sum_{j_1=p+1}^{n}\right)\ldots\left(\sum_{j_k=0}^{p}+\sum_{j_k=p+1}^{n}\right)C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)=
$$

$$
=J[\psi^{(k)}]_{T,t}^p+\xi[\psi^{(k)}]_{T,t}^{p+1,n}. \tag{1.73}
$$

Let us prove that due to the special structure of random variables $S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}$ (see (1.41)–(1.47), (1.69)) the following relations are correct

$$
\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right\}=0, \tag{1.74}
$$

$$
\mathsf{M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j'_l}^{(i_l)}-S_{j'_1,\ldots,j'_k}^{(i_1\ldots i_k)}\right)\right\}=0, \tag{1.75}
$$

where

$$
(j_1,\ldots,j_k)\in\mathrm{K}_p,\quad (j'_1,\ldots,j'_k)\in\mathrm{K}_n\backslash\mathrm{K}_p
$$

and

$$
\mathrm{K}_n=\left\{(j_1,\ldots,j_k):\ 0\le j_1,\ldots,j_k\le n\right\},
$$

$$\mathrm{K}_p = \left\{(j_1, \ldots, j_k):\ 0 \le j_1, \ldots, j_k \le p\right\}.$$

For the case $i_1, \ldots, i_k = 1, \ldots, m$ and $p_1 = \ldots = p_k = p$ from (1.37), (1.38) (see the proof of Theorem 1.1) we obtain

$$\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)} =$$

$$= \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{1.76}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1, \ldots, j_k)$, then i_r swapped with i_q in the permutation $(i_1, \ldots, i_k)$; another notations are the same as in Theorem 1.1.

So, we obtain (1.74) from (1.76) due to the moment property of the Itô stochastic integral.

Let us prove (1.75). From (1.76) we have

$$0 \le \left|\mathsf{M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j'_l}^{(i_l)} - S_{j'_1,\ldots,j'_k}^{(i_1\ldots i_k)}\right)\right\}\right| =$$

$$= \left|\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j'_1,\ldots,j'_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.\right.$$

$$\left.\left.\times\int\limits_t^T\phi_{j'_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j'_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}\right| \le$$

$$\le \sum_{(j'_1,\ldots,j'_k)}\int\limits_t^T\phi_{j_k}(t_k)\phi_{j'_k}(t_k)dt_k\ldots\int\limits_t^T\phi_{j_1}(t_1)\phi_{j'_1}(t_1)dt_1 =$$

$$
= \sum_{(j_1',\ldots,j_k')} \mathbf{1}_{\{j_1=j_1'\}} \ldots \mathbf{1}_{\{j_k=j_k'\}}, \tag{1.77}
$$

where $\mathbf{1}_A$ is the indicator of the set A. From (1.77) we obtain (1.75).

Let us consider in detail the case $k=3$ in (1.77). We have

$$
\left| \mathsf{M}\left\{ \sum_{(j_1,j_2,j_3)} \sum_{(j_1',j_2',j_3')} \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \times \right.\right.
$$

$$
\left.\left. \times \int\limits_t^T \phi_{j_3'}(t_3) \int\limits_t^{t_3} \phi_{j_2'}(t_2) \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \right\}\right| =
$$

$$
= \left| \int\limits_t^T \phi_{j_3}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_2}(s)\phi_{j_2'}(s)ds \int\limits_t^T \phi_{j_1}(s)\phi_{j_1'}(s)ds + \right.
$$

$$
+\mathbf{1}_{\{i_1=i_2\}} \int\limits_t^T \phi_{j_3}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_1}(s)\phi_{j_2'}(s)ds \int\limits_t^T \phi_{j_2}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_2=i_3\}} \int\limits_t^T \phi_{j_1}(s)\phi_{j_1'}(s)ds \int\limits_t^T \phi_{j_2}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_3}(s)\phi_{j_2'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_3\}} \int\limits_t^T \phi_{j_1}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_2}(s)\phi_{j_2'}(s)ds \int\limits_t^T \phi_{j_3}(s)\phi_{j_1'}(s)ds +
$$

$$
+\mathbf{1}_{\{i_1=i_2=i_3\}} \int\limits_t^T \phi_{j_2}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_1}(s)\phi_{j_2'}(s)ds \int\limits_t^T \phi_{j_3}(s)\phi_{j_1'}(s)ds +
$$

$$
\left. +\mathbf{1}_{\{i_1=i_2=i_3\}} \int\limits_t^T \phi_{j_1}(s)\phi_{j_3'}(s)ds \int\limits_t^T \phi_{j_3}(s)\phi_{j_2'}(s)ds \int\limits_t^T \phi_{j_2}(s)\phi_{j_1'}(s)ds \right| =
$$

$$
= \left| \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}} + \mathbf{1}_{\{i_1=i_2\}} \cdot \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}} + \right.
$$

$$
+\mathbf{1}_{\{i_2=i_3\}} \cdot \mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}} + \mathbf{1}_{\{i_1=i_3\}} \cdot \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}} +
$$

$$
\left. +\mathbf{1}_{\{i_1=i_2=i_3\}} \cdot \mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}} + \mathbf{1}_{\{i_1=i_2=i_3\}} \cdot \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}} \right| \le
$$

$$\le \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}} + \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+$$
$$+\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}} + \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$
$$+\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}} + \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}} =$$
$$= \sum_{(j_1',j_2',j_3')} \mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_3'\}},$$

where we used the relation

$$\int\limits_t^T \phi_g(s)\phi_q(s)ds = \mathbf{1}_{\{g=q\}}, \quad g,q=0,1,2\ldots$$

From (1.74) and (1.75) we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{p}\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right\}=0.$$

Due to (1.68), (1.72), and (1.73) we can write

$$\xi[\psi^{(k)}]_{T,t}^{p+1,n} = J[\psi^{(k)}]_{T,t}^{n} - J[\psi^{(k)}]_{T,t}^{p},$$

$$\underset{n\to\infty}{\text{l.i.m.}}\ \xi[\psi^{(k)}]_{T,t}^{p+1,n} = J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p} \stackrel{\text{def}}{=} \xi[\psi^{(k)}]_{T,t}^{p+1}.$$

We have

$$0 \le \left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1} J[\psi^{(k)}]_{T,t}^{p}\right\}\right| =$$

$$= \left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1} - \xi[\psi^{(k)}]_{T,t}^{p+1,n} + \xi[\psi^{(k)}]_{T,t}^{p+1,n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| =$$

$$\le \left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1} - \xi[\psi^{(k)}]_{T,t}^{p+1,n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| + \left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1,n} J[\psi^{(k)}]_{T,t}^{p}\right\}\right| =$$

$$= \left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| \le$$

$$\le \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}} \le$$

$$
\le \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\times
$$

$$
\times\left(\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}-J[\psi^{(k)}]_{T,t}\right)^2\right\}}+\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}}\right)\le
$$

$$
\le K\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\to 0 \quad \text{if} \quad n\to\infty, \tag{1.78}
$$

where K is a constant.

From (1.78) it follows that

$$
\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}=0
$$

or

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}=0.
$$

The last equality means that

$$
\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}. \tag{1.79}
$$

Taking into account (1.79), we obtain

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}-2\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}-
$$

$$
-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=
$$

$$
=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}. \tag{1.80}
$$

Let us consider the value

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}.$$

From (1.68) and (1.76) we have

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}. \tag{1.81}$$

After substituting (1.81) into (1.80), we get

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}.$$

Theorem 1.3 is proved.

1.2.3 Exact Calculation of the Mean-Square Approximation Errors for the Cases $k=1,\ldots,5$

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}\stackrel{\rm def}{=}E_k^p,$$

$$\|K\|_{L_2([t,T]^k)}^2=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\rm def}{=}I_k.$$

The case $k=1$

In this case from Theorem 1.3 we obtain

$$E_1^p=I_1-\sum_{j_1=0}^{p}C_{j_1}^2.$$

The case $k = 2$

In this case from Theorem 1.3 we have

(I). $i_1 \ne i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^2, \tag{1.82}$$

(II). $i_1 = i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} C_{j_1 j_2}. \tag{1.83}$$

Example 1.1. Let us consider the following iterated Itô stochastic integral

$$I_{(00)T,t}^{(i_1 i_2)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)}, \tag{1.84}$$

where $i_1, i_2 = 1, \ldots, m$.

Approximation of the iterated Itô stochastic integral (1.84) based on the expansion (1.10) (Theorem 1.1, the case of Legendre polynomials) has the following form

$$I_{(00)T,t}^{(i_1 i_2)p} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{p} \frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right). \tag{1.85}$$

Note that (1.85) has been derived for the first time in [59] (1997) (also see [60]-[62]) with using the another approach. This approach will be considered in Sect. 2.5. Later (1.85) was obtained [1] (2006), [2]-[54] on the base of Theorem 1.1.

Using (1.82), we get

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)p}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{p} \frac{1}{4i^2-1}\right), \tag{1.86}$$

where $i_1 \ne i_2$.

It should be noted that the formula (1.86) also has been obtained for the first time in [59] (1997) by direct calculation.

The case $k = 3$

In this case from Theorem 1.3 we get

(I). $i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2,$$

(II). $i_1 = i_2 = i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_3j_2j_1}\right), \tag{1.87}$$

(III).1. $i_1 = i_2 \neq i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}, \tag{1.88}$$

(III).2. $i_1 \neq i_2 = i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_1}C_{j_3j_2j_1},$$

(III).3. $i_1 = i_3 \neq i_2$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3}.$$

Note that the cases $k = 2$ and $k = 3$ (excepting the formula (1.87)) were investigated for the first time in [2] (2007) using the direct calculation.

Example 1.2. Let us consider the following iterated Itô stochastic integral

$$I_{(000)T,t}^{(i_1i_2i_3)} = \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}, \tag{1.89}$$

where $i_1, i_2, i_3 = 1, \ldots, m$.

Approximation of the iterated Itô stochastic integral (1.89) based on Theorem 1.1 (the case of Legendre polynomials and $p_1 = p_2 = p_3 = p$) has the following form [1] (2006), [2]-[54]

$$
I_{(000)T,t}^{(i_1 i_2 i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -
$$

$$
- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{1.90}
$$

where

$$
C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8} (T-t)^{3/2} \bar{C}_{j_3 j_2 j_1}, \tag{1.91}
$$

$$
\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,
$$

where $P_i(x)$ is the Legendre polynomial ($i = 0, 1, 2, \ldots$).

For example, using (1.88), we obtain

$$
\mathsf{M} \left\{ \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)p} \right)^2 \right\} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2 -
$$

$$
- \sum_{j_1,j_2,J_3=0}^{p} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3).
$$

The exact values of Fourier–Legendre coefficients $\bar{C}_{j_3 j_2 j_1}$ can be calculated using computer algebra system Derive [1]-[14], [30] (see Sect. 5.1, Tables 5.4–5.36). For more details on calculating of $\bar{C}_{j_3 j_2 j_1}$ using Python programming language see [51], [52].

For the case $i_1 = i_2 = i_3$ it is convenient to use the following well known formula

$$
I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6} (T-t)^{3/2} \left(\left(\zeta_0^{(i_1)} \right)^3 - 3 \zeta_0^{(i_1)} \right) \quad \text{w. p. 1.}
$$

The case $k = 4$

In this case from Theorem 1.3 we have

(I). $i_1, \ldots, i_4$ are pairwise different:

$$
E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4 \ldots j_1}^2,
$$

(II). $i_1 = i_2 = i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,\ldots,j_4)} C_{j_4\ldots j_1}\right),$$

(III).1. $i_1 = i_2 \ne i_3, i_4;\ \ i_3 \ne i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\right),$$

(III).2. $i_1 = i_3 \ne i_2, i_4;\ \ i_2 \ne i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\right),$$

(III).3. $i_1 = i_4 \ne i_2, i_3;\ \ i_2 \ne i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1}\right),$$

(III).4. $i_2 = i_3 \ne i_1, i_4;\ \ i_1 \ne i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right),$$

(III).5. $i_2 = i_4 \ne i_1, i_3;\ \ i_1 \ne i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right),$$

(III).6. $i_3 = i_4 \ne i_1, i_2;\ \ i_1 \ne i_2$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right),$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1} \right),$$

(IV).2. $i_2 = i_3 = i_4 \neq i_1$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1} \right),$$

(IV).3. $i_1 = i_2 = i_4 \neq i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1} \right),$$

(IV).4. $i_1 = i_3 = i_4 \neq i_2$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1} \right),$$

(V).1. $i_1 = i_2 \neq i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \right) \right),$$

(V).2. $i_1 = i_3 \neq i_2 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1} \right) \right),$$

(V).3. $i_1 = i_4 \neq i_2 = i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right) \right).$$

The case $k = 5$

In this case from Theorem 1.3 we obtain

(I). $i_1, \ldots, i_5$ are pairwise different:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}^2,$$

(II). $i_1 = i_2 = i_3 = i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,\ldots,j_5)} C_{j_5\ldots j_1}\right),$$

(III).1. $i_1 = i_2 \neq i_3, i_4, i_5$ (i_3, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_5\ldots j_1}\right),$$

(III).2. $i_1 = i_3 \neq i_2, i_4, i_5$ (i_2, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1}\right),$$

(III).3. $i_1 = i_4 \neq i_2, i_3, i_5$ (i_2, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1}\right),$$

(III).4. $i_1 = i_5 \neq i_2, i_3, i_4$ (i_2, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1}\right),$$

(III).5. $i_2 = i_3 \neq i_1, i_4, i_5$ (i_1, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1}\right),$$

(III).6. $i_2 = i_4 \neq i_1, i_3, i_5$ (i_1, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \right),$$

(III).7. $i_2 = i_5 \neq i_1, i_3, i_4$ (i_1, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} C_{j_5\ldots j_1} \right),$$

(III).8. $i_3 = i_4 \neq i_1, i_2, i_5$ (i_1, i_2, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(III).9. $i_3 = i_5 \neq i_1, i_2, i_4$ (i_1, i_2, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right),$$

(III).10. $i_4 = i_5 \neq i_1, i_2, i_3$ (i_1, i_2, i_3 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4, i_5$ ($i_4 \neq i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1} \right),$$

(IV).2. $i_1 = i_2 = i_4 \neq i_3, i_5$ ($i_3 \neq i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1} \right),$$

(IV).3. $i_1 = i_2 = i_5 \neq i_3, i_4$ $(i_3 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1} \Biggr),$$

(IV).4. $i_2 = i_3 = i_4 \neq i_1, i_5$ $(i_1 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1} \Biggr),$$

(IV).5. $i_2 = i_3 = i_5 \neq i_1, i_4$ $(i_1 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1} \Biggr),$$

(IV).6. $i_2 = i_4 = i_5 \neq i_1, i_3$ $(i_1 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1} \Biggr),$$

(IV).7. $i_3 = i_4 = i_5 \neq i_1, i_2$ $(i_1 \neq i_2)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1} \Biggr),$$

(IV).8. $i_1 = i_3 = i_5 \neq i_2, i_4$ $(i_2 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1} \Biggr),$$

(IV).9. $i_1 = i_3 = i_4 \neq i_2, i_5$ $(i_2 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1} \Biggr),$$

(IV).10. $i_1 = i_4 = i_5 \neq i_2, i_3$ $(i_2 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).1. $i_1 = i_2 = i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(V).2. $i_1 = i_2 = i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3,j_5)} C_{j_5\ldots j_1} \right),$$

(V).3. $i_1 = i_2 = i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).4. $i_1 = i_3 = i_4 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).5. $i_2 = i_3 = i_4 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(VI).1. $i_5 \neq i_1 = i_2 \neq i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VI).2. $i_5 \neq i_1 = i_3 \neq i_2 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).3. $i_5 \neq i_1 = i_4 \neq i_2 = i_3 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).4. $i_4 \neq i_1 = i_2 \neq i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).5. $i_4 \neq i_1 = i_5 \neq i_2 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).6. $i_4 \neq i_2 = i_5 \neq i_1 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VI).7. $i_3 \neq i_2 = i_5 \neq i_1 = i_4 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).8. $i_3 \neq i_1 = i_2 \neq i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).9. $i_3 \ne i_2 = i_4 \ne i_1 = i_5 \ne i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).10. $i_2 \ne i_1 = i_4 \ne i_3 = i_5 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).11. $i_2 \ne i_1 = i_3 \ne i_4 = i_5 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).12. $i_2 \ne i_1 = i_5 \ne i_3 = i_4 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).13. $i_1 \ne i_2 = i_3 \ne i_4 = i_5 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).14. $i_1 \ne i_2 = i_4 \ne i_3 = i_5 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).15. $i_1 \ne i_2 = i_5 \ne i_3 = i_4 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).1. $i_1 = i_2 = i_3 \neq i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} \left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1} \right) \right),$$

(VII).2. $i_1 = i_2 = i_4 \neq i_3 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_5)} \left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VII).3. $i_1 = i_2 = i_5 \neq i_3 = i_4$:

$$E_p = I - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_4)} \left(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VII).4. $i_2 = i_3 = i_4 \neq i_1 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VII).5. $i_2 = i_3 = i_5 \neq i_1 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VII).6. $i_2 = i_4 = i_5 \neq i_1 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VII).7. $i_3 = i_4 = i_5 \neq i_1 = i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VII).8. $i_1 = i_3 = i_5 \neq i_2 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_4)} \Biggl(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1} \Biggr)\Biggr),$$

(VII).9. $i_1 = i_3 = i_4 \neq i_2 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_5)} \Biggl(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1} \Biggr)\Biggr),$$

(VII).10. $i_1 = i_4 = i_5 \neq i_2 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_3)} \Biggl(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1} \Biggr)\Biggr).$$

1.2.4 Estimate for the Mean-Square Approximation Error of Iterated Itô Stochastic Integrals Based on Theorem 1.1

In this section, we prove the useful estimate for the mean-square approximation error in Theorem 1.1.

Theorem 1.4 [12]-[14], [29]. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then the estimate*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right) \qquad (1.92)$$

is valid for the following cases:

1. $i_1, \ldots, i_k = 1, \ldots, m$ *and* $0 < T - t < \infty$,

2. $i_1, \ldots, i_k = 0, 1, \ldots, m,$ $i_1^2 + \ldots + i_k^2 > 0,$ *and* $0 < T - t < 1$,

where $J[\psi^{(k)}]_{T,t}$ is the stochastic integral (1.5), *$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of* (1.10) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 1.1.

Proof. In the proof of Theorem 1.1 we obtained w. p. 1 the following representation (see (1.38))

$$J[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} + R_{T,t}^{p_1,\ldots,p_k},$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (1.10) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ and

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right) \times$$

$$\times\, d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (1.93)$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$, which are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots\, d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

The stochastic integrals on the right-hand side of (1.93) will be dependent in a stochastic sense ($i_1,\ldots,i_k = 1,\ldots,m,\ k\in\mathbf{N}$). Let us estimate the second moment of

$$J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}.$$

From the orthonormality of the system $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (1.65)), (1.25), (1.26), (1.93), and elementary inequality

$$(a_1+a_2+\ldots+a_p)^2 \le p\left(a_1^2+a_2^2+\ldots+a_p^2\right), \quad p\in\mathbf{N} \qquad (1.94)$$

we obtain the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k! \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1\ldots dt_k =$$

$$= k! \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= k! \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \right), \qquad (1.95)$$

where $T-t \in (0,\infty)$ and $i_1,\ldots,i_k = 1,\ldots,m.$

From the orthonormality of the system $\{\phi_j(x)\}_{j=0}^{\infty}$, (1.25), (1.26), (1.93), and (1.94) we obtain

$$\mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} \right)^2 \right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= C_k \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \right),$$

where $i_1,\ldots,i_k = 0,1,\ldots,m$, $i_1^2+\ldots+i_k^2>0$, and C_k is a constant.

It is not difficult to see that the constant C_k depends on k (k is the multiplicity of the iterated Itô stochastic integral) and $T-t$ ($T-t$ is the length of integration interval of the iterated Itô stochastic integral). Moreover, C_k has the following form

$$C_k = k! \cdot \max\left\{ (T-t)^{\alpha_1},\ (T-t)^{\alpha_2},\ \ldots,\ (T-t)^{\alpha_{k!}} \right\},$$

where $\alpha_1,\alpha_2,\ldots,\alpha_{k!} = 0,\ 1,\ldots,\ k-1.$

However, $T-t$ is an integration step of numerical procedures for Itô SDEs (see Chapter 4), which is a rather small value. For example, $0 < T-t < 1.$ Then $C_k \le k!$

It means that for the case $i_1,\ldots,i_k=0,1,\ldots,m,\ i_1^2+\ldots+i_k^2>0$, and $0<T-t<1$ we get (1.92). Theorem 1.4 is proved.

Example 1.3. The particular case of the estimate (1.92) for the iterated Itô stochastic integral $I_{(000)T,t}^{(i_1i_2i_3)}$ (see (1.89)) has the following form

$$
{\sf M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2\right),
$$

where $i_1,i_2,i_3=1,\ldots,m$ and $C_{j_3j_2j_1}$ is defined by the formula (1.91).

Let us consider the case of pairwise different $i_1,\ldots,i_k=1,\ldots,m$ and prove the following equality

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}=
$$

$$
=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2, \tag{1.96}
$$

where notations are the same as in Theorem 1.4.

The stochastic integrals on the right-hand side of (1.93) will be independent in a stochastic sense for the case of pairwise different $i_1,\ldots,i_k=1,\ldots,m$. Then

$$
{\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}=
$$

$$
={\sf M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times\right.\right.
$$

$$
\left.\left.\times d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right)^2\right\}=
$$

$$
=\sum_{(t_1,\ldots,t_k)}{\sf M}\left\{\left(\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times\right.\right.
$$

$$
\left.\left.\times d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right)^2\right\}=
$$

$$
= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2.
$$

1.3 Expansion of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Complete Orthonormal with Weight $r(t_1)\ldots r(t_k)$ Systems of Functions in the Space $L_2([t,T]^k)$

In this section, we consider a modification of Theorem 1.1 for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbf{N}$.

Let $\{\Psi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$. It is well known that the Fourier series of the function $f(x)$ $\left(f(x)\sqrt{r(x)} \in L_2([t,T])\right)$ with respect to the system $\{\Psi_j(x)\}_{j=0}^{\infty}$ converges to the function $f(x)$ in the mean-square sense with weight $r(x)$, i.e.

$$
\lim_{p\to\infty} \int\limits_t^T \left(f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \right)^2 r(x) dx = 0, \tag{1.97}
$$

where

$$
\tilde{C}_j = \int\limits_t^T f(x)\Psi_j(x) r(x) dx \tag{1.98}
$$

is the Fourier coefficient.

The relations (1.97), (1.98) can be obtained if we will expand the function $f(x)\sqrt{r(x)} \in L_2([t,T])$ into a usual Fourier series with respect to the complete orthonormal with weight 1 system of functions

$$
\left\{ \Psi_j(x)\sqrt{r(x)} \right\}_{j=0}^{\infty}
$$

in the space $L_2([t,T])$. Then

$$\lim_{p\to\infty}\int\limits_t^T\left(f(x)\sqrt{r(x)}-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\sqrt{r(x)}\right)^2dx=$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(f(x)-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\right)^2r(x)dx=0, \qquad (1.99)$$

where $\tilde C_j$ is defined by (1.98).

Let us consider an obvious generalization of this approach to the case of k variables. Let us expand the function $K(t_1,\ldots,t_k)$ such that

$$K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}\in L_2([t,T]^k)$$

using the complete orthonormal system of functions

$$\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)},\quad j_l=0,1,2,\ldots,\quad l=1,\ldots,k$$

in the space $L_2([t,T]^k)$ into the generalized multiple Fourier series.

It is well known that the mentioned generalized multiple Fourier series converges in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)^2\times$$

$$\times dt_1\ldots dt_k=$$

$$=\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\times$$

$$\times\left(\prod_{l=1}^{k}r(t_l)\right)dt_1\ldots dt_k=0, \qquad (1.100)$$

where

$$\tilde{C}_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Biggl(\Psi_{j_l}(t_l)r(t_l)\Biggr)dt_1\ldots dt_k.$$

Let us consider the following iterated Itô stochastic integrals

$$\tilde{J}[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (1.101)$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1,\ldots,i_k = 0,1,\ldots,m$.

So, we obtain the following version of Theorem 1.1.

Theorem 1.5 [13], [14], [27], [39]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then*

$$\tilde{J}[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)} -$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \text{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \quad (1.102)$$

where

$$\text{G}_k = \text{H}_k\backslash\text{L}_k, \quad \text{H}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\bigr\},$$

$$\text{L}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\tilde{\zeta}_j^{(i)} = \int\limits_t^T \Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition *of* $[t,T]$, *which satisfies the condition* (1.9),

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^k\Biggl(\Psi_{j_l}(t_l)r(t_l)\Biggr)dt_1\ldots dt_k \qquad (1.103)$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Proof. According to Lemmas 1.1–1.3 and (1.23), (1.24), (1.34), (1.35), we get the following representation

$$\tilde{J}[\psi^{(k)}]_{T,t}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)\prod_{l=1}^k\sqrt{r(t_l)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\prod_{l=1}^k\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times$$

$$\times\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \times
$$

$$
\times \left(\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} \Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} - \right.
$$

$$
\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +
$$

$$
+ \tilde{R}_{T,t}^{p_1,\ldots,p_k} =
$$

$$
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \times
$$

$$
\times \left(\prod_{l=1}^{k} \tilde{\zeta}_{j_l}^{(i_l)} - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +
$$

$$
+ \tilde{R}_{T,t}^{p_1,\ldots,p_k} \quad \text{w. p. 1},
$$

where

$$
\tilde{R}_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int_t^T \ldots \int_t^{t_2} \left(K(t_1,\ldots,t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} - \right.
$$

$$
\left. - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \left(\Psi_{j_l}(t_l)\sqrt{r(t_l)} \right) \right) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $\tilde{R}_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 1.2, we have

$$
\mathsf{M}\left\{\left(\tilde{R}_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\prod_{l=1}^k \sqrt{r(t_l)} -\right.
$$

$$
\left. - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^k \left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)^2 dt_1\ldots dt_k = \qquad (1.104)
$$

$$
= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^k \Psi_{j_l}(t_l)\right)^2 \times
$$

$$
\times \left(\prod_{l=1}^k r(t_l)\right) dt_1\ldots dt_k \to 0 \qquad (1.105)
$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral (1.101). Theorem 1.5 is proved.

Let us formulate the version of Theorem 1.4.

Theorem 1.6 [14], [27], [39]. *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect.* 1.1.7). *Then the estimate*

$$
\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t} - \tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le
$$

$$
\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^k r(t_l)\right) dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}^2\right) \qquad (1.106)
$$

is valid for the following cases:

1. $i_1,\ldots,i_k = 1,\ldots,m$ *and* $0 < T-t < \infty$,

2. $i_1,\ldots,i_k = 0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2 > 0,$ *and* $0 < T-t < 1,$

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1.101), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.102) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 1.5.

1.4 Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures Based on Generalized Multiple Fourier Series

In this section, we consider the version of Theorem 1.1 connected with the expansion of iterated stochastic integrals with respect to martingale Poisson measures.

1.4.1 Stochastic Integral with Respect to Martingale Poisson Measure

Let us consider the Poisson random measure on the set $[0,T]\times\mathbf{Y}$ ($\mathbf{R}^n \stackrel{\rm def}{=} \mathbf{Y}$). We will denote the value of this measure at the set $\Delta\times A$ ($\Delta\subseteq[0,T]$, $A\subset\mathbf{Y}$) as $\nu(\Delta,A)$. Assume that

$$\mathsf{M}\left\{\nu(\Delta,A)\right\}=|\Delta|\Pi(A),$$

where $|\Delta|$ is the Lebesgue measure of Δ, $\Pi(A)$ is a measure on σ-algebra $\mathcal{B}$ of Borel sets of $\mathbf{Y}$, and $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ consisting of sets $A\subset\mathcal{B}$ that satisfy the condition $\Pi(A)<\infty$.

Let us consider the martingale Poisson measure

$$\tilde{\nu}(\Delta,A)=\nu(\Delta,A)-|\Delta|\Pi(A).$$

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a fixed probability space, let $\{\mathrm{F}_t,\ t\in[0,T]\}$ be a non-decreasing family of σ-algebras $\mathrm{F}_t\subset\mathrm{F}$.

Assume that the following conditions are fulfilled:

1. The random variables $\nu([0,t),A)$ are F_t-measurable for all $A\subseteq\mathcal{B}_0$, $t\in[0,T]$.

2. The random variables $\nu([t,t+h),A)$, $A\subseteq\mathcal{B}_0$, $h>0$ do not depend on events of σ-algebra F_t.

Let us define the class $H_l(\Pi,[0,T])$ of random functions $\varphi:[0,T]\times\mathbf{Y}\times\Omega\to\mathbf{R}^1$ that are F_t-measurable for all $t\in[0,T]$, $\mathbf{y}\in\mathbf{Y}$ and satisfy the following condition

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\left\{|\varphi(t,\mathbf{y})|^l\right\}\Pi(d\mathbf{y})dt < \infty.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$, which satisfies the condition (1.9), and define the stochastic integral with respect to the martingale Poisson measure for $\varphi(t,\mathbf{y}) \in H_2(\Pi,[0,T])$ as the following mean-square limit [83]

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi^{(N)}(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}), \tag{1.107}$$

where $\varphi^{(N)}(t,\mathbf{y})$ is any sequence of step functions from the class $H_2(\Pi,[0,T])$ such that

$$\lim_{N\to\infty} \int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\left\{\left|\varphi(t,\mathbf{y}) - \varphi^{(N)}(t,\mathbf{y})\right|^2\right\}\Pi(d\mathbf{y})dt \to 0.$$

It is well known [83] that the stochastic integral (1.107) exists, it does not depend on selection of the sequence $\varphi^{(N)}(t,\mathbf{y})$ and it satisfies w. p. 1 to the following properties

$$\mathsf{M}\left\{\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\middle|\mathrm{F}_0\right\} = 0,$$

$$\int\limits_0^T \int\limits_{\mathbf{Y}} (\alpha\varphi_1(t,\mathbf{y}) + \beta\varphi_2(t,\mathbf{y}))\tilde{\nu}(dt,d\mathbf{y}) =$$

$$= \alpha \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi_1(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) + \beta \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi_2(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}),$$

$$\mathsf{M}\left\{\left|\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\right|^2 \middle|\mathrm{F}_0\right\} = \int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\left\{|\varphi(t,\mathbf{y})|^2 \middle|\mathrm{F}_0\right\}\Pi(d\mathbf{y})dt,$$

where $\alpha,\ \beta \in \mathbf{R}^1$ and $\varphi_1(t,\mathbf{y})$, $\varphi_2(t,\mathbf{y})$, $\varphi(t,\mathbf{y})$ from the class $H_2(\Pi,[0,T])$.

The stochastic integral

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y})$$

with respect to the Poisson measure will be defined as follows [83]

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y}) = \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) + \int\limits_0^T \int\limits_{\mathbf{Y}} \varphi(t,\mathbf{y})\Pi(d\mathbf{y})dt, \quad (1.108)$$

where we suppose that the right-hand side of the last equality exists.

According to the Itô formula for Itô processes with jumps, we get [83]

$$(z_t)^p = \int\limits_0^t \int\limits_{\mathbf{Y}} \Bigl((z_{\tau-} + \gamma(\tau,\mathbf{y}))^p - (z_{\tau-})^p\Bigr)\nu(d\tau,d\mathbf{y}) \quad \text{w. p. 1}, \quad (1.109)$$

where $p \in \mathbf{N}$, $z_{\tau-}$ means the value of the process z_τ at the point τ from the left,

$$z_t = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\nu(d\tau,d\mathbf{y}).$$

We suppose that the function $\gamma(\tau,\mathbf{y})$ satisfies the conditions of existence of the right-hand side of (1.109) [83].

Let us consider the useful estimate for moments of stochastic integrals with respect to the Poisson measure [83]

$$a_p(T) \le \max_{j\in\{p,\ 1\}} \left\{ \left(\int\limits_0^T \int\limits_{\mathbf{Y}} \left(\left((b_p(\tau,\mathbf{y}))^{1/p} + 1 \right)^p - 1 \right) \Pi(d\mathbf{y})d\tau \right)^j \right\}, \quad (1.110)$$

where

$$a_p(t) = \sup_{0\le\tau\le t} \mathsf{M}\Bigl\{|z_\tau|^p\Bigr\}, \quad b_p(\tau,\mathbf{y}) = \mathsf{M}\Bigl\{|\gamma(\tau,\mathbf{y})|^p\Bigr\}.$$

We suppose that the right-hand side of (1.110) exists. According to (see (1.108))

$$\int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\tilde{\nu}(d\tau,d\mathbf{y}) = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\nu(d\tau,d\mathbf{y}) - \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})d\tau$$

and the Minkowski inequality, we obtain

$$\left(\mathsf{M}\left\{|\tilde{z}_t|^{2p}\right\}\right)^{1/2p} \le \left(\mathsf{M}\left\{|z_t|^{2p}\right\}\right)^{1/2p} + \left(\mathsf{M}\left\{|\hat{z}_t|^{2p}\right\}\right)^{1/2p}, \qquad (1.111)$$

where

$$\tilde{z}_t = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\tilde{\nu}(d\tau,d\mathbf{y})$$

and

$$\hat{z}_t \stackrel{\rm def}{=} \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})d\tau.$$

The value $\mathsf{M}\left\{|\hat{z}_\tau|^{2p}\right\}$ can be estimated using the well known inequality [83]

$$\mathsf{M}\left\{|\hat{z}_t|^{2p}\right\} \le t^{2p-1}\int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \varphi(\tau,\mathbf{y})\Pi(d\mathbf{y})\right|^{2p}\right\} d\tau, \qquad (1.112)$$

where we suppose that

$$\int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})\right|^{2p}\right\} d\tau < \infty.$$

1.4.2 Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures

Let us consider the following iterated stochastic integrals

$$P[\chi^{(k)}]_{T,t} =$$

$$= \int\limits_t^T \int\limits_{\mathbf{X}} \chi_k(t_k,\mathbf{y}_k)\ldots \int\limits_t^{t_2} \int\limits_{\mathbf{X}} \chi_1(t_1,\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)\ldots\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y}_k), \qquad (1.113)$$

where $i_1,\ldots,i_k = 0,1,\ldots,m$, $\mathbf{R}^n \stackrel{\rm def}{=} \mathbf{X}$, $\chi_l(\tau,\mathbf{y}) = \psi_l(\tau)\varphi_l(\mathbf{y})$ $(l=1,\ldots,k)$, every function $\psi_l(\tau): [t,T] \to \mathbf{R}^1$ $(l=1,\ldots,k)$ and every function $\varphi_l(\mathbf{y}):$ $\mathbf{X} \to \mathbf{R}^1$ $(l=1,\ldots,k)$ such that

$$\chi_l(\tau,\mathbf{y})\in H_2(\Pi,[t,T])\quad (l=1,\ldots,k),$$

where definition of the class $H_2(\Pi,[t,T])$ see above, $\nu^{(i)}(dt,d\mathbf{y})$ $(i=1,\ldots,m)$ are independent Poisson measures for various i, which are defined on $[0,T]\times\mathbf{X}$,

$$\tilde{\nu}^{(i)}(dt,d\mathbf{y})=\nu^{(i)}(dt,d\mathbf{y})-\Pi(d\mathbf{y})dt\quad (i=1,\ldots,m)$$

are independent martingale Poisson measures for various i, $\tilde{\nu}^{(0)}(dt,d\mathbf{y})\stackrel{\rm def}{=}\Pi(d\mathbf{y})dt$.

Let us formulate the analogue of Theorem 1.1 for the iterated stochastic integrals (1.113).

Theorem 1.7 [1]-[14], [39]. *Suppose that the following conditions are hold:*

1. *Every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7).*

3. *For $l=1,\ldots,k$ and $q=2^{k+1}$ the following condition is satisfied*

$$\int\limits_{\mathbf{X}}|\varphi_l(\mathbf{y})|^q\,\Pi(d\mathbf{y})<\infty.$$

Then, for the iterated stochastic integral with respect to martingale Poisson measures $P[\chi^{(k)}]_{T,t}$ defined by (1.113) *the following expansion*

$$P[\chi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{g=1}^{k}\pi_{j_g}^{(g,i_g)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\int\limits_{\mathbf{X}}\varphi_g(\mathbf{y})\tilde{\nu}^{(i_g)}([\tau_{l_g},\tau_{l_g+1}),d\mathbf{y})\Biggr)\qquad (1.114)$$

that converges in the mean-square sense is valid, where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (1.9),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m$, *random variables*

$$\pi_j^{(g,i_g)} = \int\limits_t^T \phi_j(\tau) \int\limits_{\mathbf{X}} \varphi_g(\mathbf{y}) \tilde{\nu}^{(i_g)}(d\tau, d\mathbf{y})$$

are independent for various i_g $($*if* $i_g \neq 0)$ *and uncorrelated for various* j*,*

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k$$

is the Fourier coefficient,

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2,$$

and $K(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t, T]$*.*

Proof. The scheme of the proof of Theorem 1.7 is the same with the scheme of the proof of Theorem 1.1. Some differences will take place in the proof of Lemmas 1.4, 1.5 (see below) and in the final part of the proof of Theorem 1.7.

Lemma 1.4. *Suppose that every* $\psi_l(\tau)$ $(l = 1, \ldots, k)$ *is a continuous function at the interval* $[t, T]$ *and every function* $\varphi_l(\mathbf{y})$ $(l = 1, \ldots, k)$ *such that*

$$\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^2 \, \Pi(d\mathbf{y}) < \infty.$$

Then, the following equality

$$P[\bar{\chi}^{(k)}]_{T,t} = \underset{N \to \infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \chi_l(\tau_{j_l}, \mathbf{y}) \bar{\nu}^{(i_l)}([\tau_{j_l}, \tau_{j_l+1}), d\mathbf{y}) \qquad (1.115)$$

is valid w. p. 1, where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t, T]$ *satisfying the condition* (1.9)*,*

$$\bar{\nu}^{(i)}([\tau, s), d\mathbf{y}) = \begin{cases} \tilde{\nu}^{(i)}([\tau, s), d\mathbf{y}) \\ \\ \nu^{(i)}([\tau, s), d\mathbf{y}) \end{cases} \quad (i = 0, 1, \ldots, m).$$

In contrast to the integral $P[\chi^{(k)}]_{T,t}$ *defined by* (1.113)*,* $\bar{\nu}^{(i_l)}(dt_l, d\mathbf{y}_l)$ *is used in the integral* $P[\bar{\chi}^{(k)}]_{T,t}$ *instead of* $\tilde{\nu}^{(i_l)}(dt_l, d\mathbf{y}_l)$ $(l = 1, \ldots, k)$*.*

Proof. Using the moment properties of stochastic integrals with respect to the Poisson measure (see above) and the conditions of Lemma 1.4, it is easy to notice that the integral sum of the integral $P[\bar{\chi}^{(k)}]_{T,t}$ can be represented as a sum of the prelimit expression from the right-hand side of (1.115) and the value, which converges to zero in the mean-square sense if $N \to \infty$. Lemma 1.4 is proved.

Note that in the case when the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_j, $j = 0, 1, \ldots, N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l = 1, \ldots, k)$. Further, we can apply the argumentation as in Sect. 1.1.7.

Let us consider the following multiple and iterated stochastic integrals

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l(\mathbf{y}) \tilde{\nu}^{(i_l)}([\tau_{j_l}, \tau_{j_l+1}), d\mathbf{y}) \stackrel{\text{def}}{=} P[\Phi]_{T,t}^{(k)},$$

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y}) \tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}) \ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y}) \tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \hat{P}[\Phi]_{T,t}^{(k)},$$

where $\Phi(t_1, \ldots, t_k) : [t, T]^k \to \mathbf{R}^1$ is a bounded nonrandom function and the sense of notations of the formula (1.115) is remaining.

Note that if the functions $\varphi_l(\mathbf{y})$ $(l = 1, \ldots, k)$ satisfy the conditions of Lemma 1.4 and the function $\Phi(t_1, \ldots, t_k)$ is continuous in the domain of integration, then for the integral $\hat{P}[\Phi]_{T,t}^{(k)}$ the equality similar to (1.115) is valid w. p. 1.

Lemma 1.5. *Assume that the following representation takes place*:

$$g_l(\tau, \mathbf{y}) = h_l(\tau)\varphi_l(\mathbf{y}) \quad (l = 1, \ldots, k),$$

where the functions $h_l(\tau) : [t, T] \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ *satisfy the condition* $(\star)$ (*see Sect.* 1.1.7) *and the functions* $\varphi_l(\mathbf{y}) : \mathbf{X} \to \mathbf{R}^1$ $(l = 1, \ldots, k)$ *satisfy the condition*

$$\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^p \, \Pi(d\mathbf{y}) < \infty \quad \textit{for} \quad p = 2^{k+1}.$$

Then

$$\prod_{l=1}^{k}\int\limits_{t}^{T}\int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}) = P[\Phi]_{T,t}^{(k)} \quad w.\ p.\ 1, \tag{1.116}$$

where $i_l = 0, 1, \ldots, m$ $(l = 1, \ldots, k)$ *and*

$$\Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} h_l(t_l).$$

Proof. Let us introduce the following notations

$$J[\bar{g}_l]_N \stackrel{\rm def}{=} \sum_{j=0}^{N-1}\int\limits_{\mathbf{X}} g_l(\tau_j,\mathbf{y})\bar{\nu}^{(i_l)}([\tau_j,\tau_{j+1}),d\mathbf{y}),$$

$$J[\bar{g}_l]_{T,t} \stackrel{\rm def}{=} \int\limits_{t}^{T}\int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}),$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (1.9).

It is easy to see that

$$\prod_{l=1}^{k} J[\bar{g}_l]_N - \prod_{l=1}^{k} J[\bar{g}_l]_{T,t} =$$

$$= \sum_{l=1}^{k}\left(\prod_{q=1}^{l-1} J[\bar{g}_q]_{T,t}\right)\left(J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t}\right)\left(\prod_{q=l+1}^{k} J[\bar{g}_q]_N\right).$$

Using the Minkowski inequality and the inequality of Cauchy–Bunyakovsky together with the estimates of moments of stochastic integrals with respect to the Poisson measure and the conditions of Lemma 1.5, we obtain

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k} J[\bar{g}_l]_N - \prod_{l=1}^{k} J[\bar{g}_l]_{T,t}\right|^2\right\}\right)^{1/2} \le C_k\sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{1.117}$$

where $C_k < \infty$.

We have

$$J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q},$$

where

$$J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q} = \int\limits_{\tau_q}^{\tau_{q+1}} (h_l(\tau_q) - h_l(s)) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y}).$$

Let us introduce the notation

$$h_l^{(N)}(s) = h_l(\tau_q), \quad s \in [\tau_q, \tau_{q+1}), \quad q = 0, 1, \ldots, N-1.$$

Then

$$J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q} =$$

$$= \int\limits_t^T \left(h_l^{(N)}(s) - h_l(s)\right) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y}).$$

Applying the estimates (1.110) (for $p = 4$) and (1.111), (1.112) (for $p = 2$) to the value

$$\mathsf{M}\left\{\left|\int\limits_t^T \left(h_l^{(N)}(s) - h_l(s)\right) \int\limits_{\mathbf{X}} \phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds, d\mathbf{y})\right|^4\right\},$$

taking into account (1.117), the conditions of Lemma 1.5, and the estimate

$$|h_l(\tau_q) - h_l(s)| < \varepsilon, \quad s \in [\tau_q, \tau_{q+1}], \quad q = 0, 1, \ldots, N-1, \tag{1.118}$$

where ε is an arbitrary small positive real number and $|\tau_{q+1} - \tau_q| < \delta(\varepsilon)$, we obtain that the right-hand side of (1.117) converges to zero when $N \to \infty$. Therefore, we come to the affirmation of Lemma 1.5.

It should be noted that (1.118) is valid if the functions $h_l(s)$ are continuous at the interval $[t, T]$, i.e. these functions are uniformly continuous at this interval. So, $|h_l(\tau_q) - h_l(s)| < \varepsilon$ if $s \in [\tau_q, \tau_{q+1}]$, where $|\tau_{q+1} - \tau_q| < \delta(\varepsilon)$, $q = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$).

In the case when the functions $h_l(s)$ $(l = 1, \ldots, k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_q, $q = 0, 1, \ldots, N$ there are all points of jumps of the functions $h_l(s)$ $(l = 1, \ldots, k)$. Further, we can apply the argumentation as in Sect. 1.1.7.

Obviously, if $i_l = 0$ for some $l = 1, \ldots, k$, then we also come to the affirmation of Lemma 1.5. Lemma 1.5 is proved.

Proving Theorem 1.7 by the scheme of the proof of Theorem 1.1 using Lemmas 1.4, 1.5 and estimates for moments of stochastic integrals with respect to the Poisson measure, we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y})\Pi(d\mathbf{y})\times
$$

$$
\times \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)\times
$$

$$
\times dt_1\ldots dt_k =
$$

$$
= C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y})\Pi(d\mathbf{y}) \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2\times
$$

$$
\times dt_1\ldots dt_k \le
$$

$$
\le \bar{C}_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0
$$

if $p_1,\ldots,p_k \to \infty$, where constant $\bar{C}_k$ depends only on k (k is the multiplicity of the iterated stochastic integral with respect to the martingale Poisson measures). Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$
R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)\times
$$

$$
\times \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y})\ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y}), \tag{1.119}
$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.119) are performed only in the values $\varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y})\ldots\varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y})$. At the same time, the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover,

$\varphi_r(\mathbf{y})$ swapped with $\varphi_q(\mathbf{y})$ in the permutation $(\varphi_1(\mathbf{y}),\ldots,\varphi_k(\mathbf{y}))$. Theorem 1.7 is proved.

Let us consider the application of Theorem 1.7. Let $i_1\ne i_2,\ i_1,i_2=1,\ldots,m.$ Using Theorem 1.7 and the system of Legendre polynomials, we obtain

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_2(\mathbf{y}_2)\int\limits_t^{t_2}\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde\nu^{(i_1)}(dt_1,d\mathbf{y}_1)\tilde\nu^{(i_2)}(dt_2,d\mathbf{y}_2)=$$

$$=\frac{T-t}{2}\left(\pi_0^{(1,i_1)}\pi_0^{(2,i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\pi_{i-1}^{(1,i_1)}\pi_i^{(2,i_2)}-\pi_i^{(1,i_1)}\pi_{i-1}^{(2,i_2)}\right)\right),$$

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde\nu^{(i_1)}(dt_1,d\mathbf{y}_1)=\sqrt{T-t}\,\pi_0^{(1,i_1)},$$

where

$$\pi_j^{(l,i_l)}=\int\limits_t^T\phi_j(\tau)\int\limits_{\mathbf{X}}\varphi_l(\mathbf{y})\tilde\nu^{(i_l)}(d\tau,d\mathbf{y})\quad (l=1,\ 2)$$

and $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

1.5 Expansion of Iterated Stochastic Integrals with Respect to Martingales Based on Generalized Multiple Fourier Series

1.5.1 Stochastic Integral with Respect to Martingale

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a fixed probability space, let $\{\mathrm{F}_t,\ t\in[0,T]\}$ be a non-decreasing family of σ-algebras $\mathrm{F}_t\subset\mathrm{F}$, and let $\mathrm{M}_2(\rho,[0,T])$ be a class of F_t-measurable for each $t\in[0,T]$ martingales M_t satisfying the conditions

$$\mathsf{M}\left\{(M_s-M_t)^2\right\}=\int\limits_t^s\rho(\tau)d\tau,$$

$$\mathsf{M}\left\{|M_s-M_t|^p\right\}\le C_p|s-t|,\quad p=3,4,\ldots,$$

where $0 \le t < s \le T$, $\rho(\tau)$ is a non-negative and continuously differentiable nonrandom function at the interval $[0, T]$, $C_p < \infty$ is a constant.

Let us define the class $H_2(\rho, [0, T])$ of stochastic processes ξ_t, $t \in [0, T]$, which are F_t-measurable for all $t \in [0, T]$ and satisfy the condition

$$\int\limits_0^T \mathsf{M}\left\{|\xi_t|^2\right\}\rho(t)dt < \infty.$$

For any partition $\left\{\tau_j^{(N)}\right\}_{j=0}^N$ of the interval $[0, T]$ such that

$$0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0 \ \text{ if } \ N \to \infty \tag{1.120}$$

we will define the sequence of step functions $\xi^{(N)}(t, \omega)$ by the following relation

$$\xi^{(N)}(t, \omega) = \xi\left(\tau_j^{(N)}, \omega\right) \quad \text{w. p. 1} \quad \text{for} \quad t \in \left[\tau_j^{(N)}, \tau_{j+1}^{(N)}\right),$$

where $j = 0, 1, \ldots, N-1$, $N = 1, 2, \ldots$

Let us define the stochastic integral with respect to martingale from the process $\xi(t, \omega) \in H_2(\rho, [0, T])$ as the following mean-square limit [83]

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}\left(\tau_j^{(N)}, \omega\right)\left(M\left(\tau_{j+1}^{(N)}, \omega\right) - M\left(\tau_j^{(N)}, \omega\right)\right) \stackrel{\text{def}}{=} \int\limits_0^T \xi_\tau dM_\tau, \tag{1.121}$$

where $\xi^{(N)}(t, \omega)$ is any step function from the class $H_2(\rho, [0, T])$, which converges to the function $\xi(t, \omega)$ in the following sense

$$\lim_{N\to\infty} \int\limits_0^T \mathsf{M}\left\{\left|\xi^{(N)}(t, \omega) - \xi(t, \omega)\right|^2\right\}\rho(t)dt = 0.$$

It is well known [83] that the stochastic integral (1.121) exists, it does not depend on the selection of sequence $\xi^{(N)}(t, \omega)$ and it satisfies w. p. 1 to the following properties

$$\mathsf{M}\left\{\int\limits_0^T \xi_t dM_t \middle| \mathrm{F}_0\right\} = 0,$$

$$
\mathsf{M}\left\{\left|\int\limits_0^T \xi_t dM_t\right|^2 \Bigg| \mathrm{F}_0\right\} = \mathsf{M}\left\{\int\limits_0^T \xi_t^2 \rho(t)dt \Bigg| \mathrm{F}_0\right\},
$$

$$
\int\limits_0^T (\alpha\xi_t + \beta\psi_t)dM_t = \alpha\int\limits_0^T \xi_t dM_t + \beta\int\limits_0^T \psi_t dM_t,
$$

where $\xi_t,\ \psi_t \in H_2(\rho,[0,T])$, $\alpha,\beta\in\mathbf{R}^1$.

1.5.2 Expansion of Iterated Stochastic Integrals with Respect to Martingales

Let $Q_4(\rho,[0,T])$ be the class of martingales M_t, $t\in[0,T]$, which satisfy the following conditions:

1. M_t, $t\in[0,T]$ belongs to the class $\mathrm{M}_2(\rho,[0,T])$.
2. For some $\alpha>0$ the following estimate is correct

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau g(s)dM_s\right|^4\right\} \le K_4\int\limits_t^\tau |g(s)|^\alpha ds, \tag{1.122}
$$

where $0\le t<\tau\le T$, $g(s)$ is a bounded nonrandom function at the interval $[0,T]$, $K_4<\infty$ is a constant.

Let $G_n(\rho,[0,T])$ be the class of martingales M_t, $t\in[0,T]$, which satisfy the following conditions:

1. M_t, $t\in[0,T]$ belongs to the class $\mathrm{M}_2(\rho,[0,T])$.
2. The following estimate is correct

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau g(s)dM_s\right|^n\right\} < \infty,
$$

where $0\le t<\tau\le T$, $n\in\mathbf{N}$, $g(s)$ is the same function as in the definition of the class $Q_4(\rho,[0,T])$.

Let us remind that if $(\xi_t)^n\in H_2(\rho,[0,T])$ with $\rho(t)\equiv 1$, then the following estimate is correct [83]

$$
\mathsf{M}\left\{\left|\int\limits_t^\tau \xi_s ds\right|^{2n}\right\} \le (\tau-t)^{2n-1}\int\limits_t^\tau \mathsf{M}\left\{|\xi_s|^{2n}\right\}ds,\quad 0\le t<\tau\le T. \tag{1.123}
$$

Let us consider the iterated stochastic integral with respect to martingales

$$
J[\psi^{(k)}]_{T,t}^{M}=\int\limits_{t}^{T}\psi_k(t_k)\ldots\int\limits_{t}^{t_2}\psi_1(t_1)dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}, \qquad (1.124)
$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$, $M_s^{(r,i)}$ $(r=1,\ldots,k,\ i=1,\ldots,m)$ are independent martingales for various $i=1,\ldots,m$, $M_s^{(r,0)}\stackrel{\rm def}{=}s$.

Now we can formulate the following theorem.

Theorem 1.8 [1]-[14], [39]. *Suppose that the following conditions are hold*:

1. *Every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1.1.7).

3. $M_s^{(l,i_l)}\in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ *with* $n=2^{k+1}$, $i_l=1,\ldots,m$, $l=1,\ldots,k$ $(k\in\mathbf{N})$.

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^{M}$ *with respect to martingales defined by* (1.124) *the following expansion*

$$
J[\psi^{(k)}]_{T,t}^{M}=\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)}-
$$

$$
-\underset{N\to\infty}{\rm l.i.m.}\sum_{(l_1,\ldots,l_k)\in{\rm G}_k}\phi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)
$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k=0,1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition similar to* (1.120), $\Delta M_{\tau_j}^{(r,i)}=M_{\tau_{j+1}}^{(r,i)}-M_{\tau_j}^{(r,i)}$ $(i=0,1,\ldots,m,\ r=1,\ldots,k)$,

$$
{\rm G}_k={\rm H}_k\backslash{\rm L}_k,\quad {\rm H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},
$$

$$
{\rm L}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\Bigr\},
$$

l.i.m. *is a limit in the mean-square sense*,

$$
\xi_j^{(l,i_l)}=\int\limits_{t}^{T}\phi_j(s)dM_s^{(l,i_l)}
$$

are independent for various i_l *(if* $i_l \ne 0$*) and uncorrelated for various* j *(if* $\rho(\tau)$ *is a constant,* $i_l \ne 0$*) random variables,*

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & otherwise \end{cases}, \quad t_1,\ldots,t_k \in [t,T], \quad k \ge 2,$$

and $K(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t,T]$.

Remark 1.4. *Note that from Theorem* 1.8 *for the case* $\rho(\tau) \equiv 1$ *we obtain the variant of Theorem* 1.1.

Proof. The proof of Theorem 1.8 is similar to the proof of Theorem 1.1. Some differences will take place in the proof of Lemmas 1.6, 1.7 (see below) and in the final part of the proof of Theorem 1.8.

Lemma 1.6. *Assume that* $M_s^{(r,i)} \in \mathrm{M}_2(\rho,[t,T])$ $(i = 1,\ldots,m)$, $M_s^{(r,0)} = s$ $(r = 1,\ldots,k)$, *and every* $\psi_l(\tau)$ $(l = 1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$. *Then*

$$J[\psi^{(k)}]_{T,t}^{M} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta M_{\tau_{j_l}}^{(l,i_l)} \quad w.\ p.\ 1, \tag{1.125}$$

where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition similar to* (1.120), $i_l = 0,1,\ldots,m$, $l = 1,\ldots,k$.

Proof. According to properties of stochastic integral with respect to martingale, we have [83]

$$\mathsf{M}\left\{\left(\int\limits_t^{\tau}\xi_s dM_s^{(l,i_l)}\right)^2\right\} = \int\limits_t^{\tau}\mathsf{M}\left\{|\xi_s|^2\right\}\rho(s)ds, \tag{1.126}$$

$$\mathsf{M}\left\{\left(\int\limits_t^{\tau}\xi_s ds\right)^2\right\} \le (\tau - t)\int\limits_t^{\tau}\mathsf{M}\left\{|\xi_s|^2\right\}ds, \tag{1.127}$$

where $\xi_s \in H_2(\rho, [0,T])$, $0 \le t < \tau \le T$, $i_l = 1,\ldots,m$, $l = 1,\ldots,k$. Then the integral sum for the integral $J[\psi^{(k)}]_{T,t}^M$ under the conditions of Lemma 1.6 can be represented as a sum of the prelimit expression from the right-hand side of (1.125) and the value, which converges to zero in the mean-square sense if $N \to \infty$. More detailed proof of the similar lemma for the case $\rho(\tau) \equiv 1$ can be found in Sect. 1.1.3 (see Lemma 1.1).

In the case when the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_j, $j = 0,1,\ldots,N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$. So, we can apply the argumentation as in Sect. 1.1.7.

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \Delta M_{\tau_{j_l}}^{(l,i_l)} \stackrel{\text{def}}{=} I[\Phi]_{T,t}^{(k)}, \qquad (1.128)$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition similar to (1.120) and $\Phi(t_1,\ldots,t_k) : [t,T]^k \to \mathbf{R}^1$ is a bounded nonrandom function.

Lemma 1.7. *Let $M_s^{(l,i_l)} \in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ with $n = 2^{k+1}$, $k \in \mathbf{N}$ $(i_l = 1,\ldots,m,\ l = 1,\ldots,k)$ and the functions $g_1(s),\ldots,g_k(s)$ satisfy the condition $(\star)$ (see Sect. 1.1.7). Then*

$$\prod_{l=1}^{k} \int_t^T g_l(s) dM_s^{(l,i_l)} = I[\Phi]_{T,t}^{(k)} \quad w.\ p.\ 1,$$

where $i_l = 0,1,\ldots,m$, $l = 1,\ldots,k$,

$$\Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} g_l(t_l).$$

Proof. Let us denote

$$J[g_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} g_l(\tau_j) \Delta M_{\tau_j}^{(l,i_l)}, \quad J[g_l]_{T,t} \stackrel{\text{def}}{=} \int_t^T g_l(s) dM_s^{(l,i_l)},$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition similar to (1.120).

Note that

$$\prod_{l=1}^{k} J[g_l]_N - \prod_{l=1}^{k} J[g_l]_{T,t} =$$

$$= \sum_{l=1}^{k} \left(\prod_{q=1}^{l-1} J[g_q]_{T,t} \right) \left(J[g_l]_N - J[g_l]_{T,t} \right) \left(\prod_{q=l+1}^{k} J[g_q]_N \right).$$

Using the Minkowski inequality and the inequality of Cauchy-Bunyakovsky as well as the conditions of Lemma 1.7, we obtain

$$\left(\mathsf{M} \left\{ \left| \prod_{l=1}^{k} J[g_l]_N - \prod_{l=1}^{k} J[g_l]_{T,t} \right|^2 \right\} \right)^{1/2} \le$$

$$\le C_k \sum_{l=1}^{k} \left(\mathsf{M} \left\{ \left| J[g_l]_N - J[g_l]_{T,t} \right|^4 \right\} \right)^{1/4}, \qquad (1.129)$$

where $C_k < \infty$ is a constant.

We have

$$J[g_l]_N - J[g_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta g_l]_{\tau_{q+1},\tau_q},$$

$$J[\Delta g_l]_{\tau_{q+1},\tau_q} = \int\limits_{\tau_q}^{\tau_{q+1}} \left(g_l(\tau_q) - g_l(s) \right) dM_s^{(l,i_l)}.$$

Let us introduce the notation

$$g_l^{(N)}(s) = g_l(\tau_q), \quad s \in [\tau_q, \tau_{q+1}), \quad q = 0, 1, \ldots, N-1.$$

Then

$$J[g_l]_N - J[g_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta g_l]_{\tau_{q+1},\tau_q} =$$

$$= \int\limits_{t}^{T} \left(g_l^{(N)}(s) - g_l(s) \right) dM_s^{(l,i_l)}.$$

Applying the estimate (1.122), we obtain

$$
\mathsf{M}\left\{\left|\int\limits_t^T \left(g_l^{(N)}(s)-g_l(s)\right)dM_s^{(l,i_l)}\right|^4\right\} \le
$$

$$
\le K_4\int\limits_t^T \left|g_l^{(N)}(s)-g_l(s)\right|^{\alpha} ds =
$$

$$
= K_4\sum_{q=0}^{N-1}\int\limits_{\tau_q}^{\tau_{q+1}} |g_l(\tau_q)-g_l(s)|^{\alpha} ds <
$$

$$
< K_4\varepsilon^{\alpha}\sum_{q=0}^{N-1}(\tau_{q+1}-\tau_q) = K_4\varepsilon^{\alpha}(T-t). \tag{1.130}
$$

Note that we used the estimate

$$
|g_l(\tau_q)-g_l(s)|<\varepsilon, \quad s\in[\tau_q,\tau_{q+1}], \quad q=0,1,\ldots,N-1 \tag{1.131}
$$

to derive (1.130), where $|\tau_{q+1}-\tau_q|<\delta(\varepsilon)$ and ε is an arbitrary small positive real number.

The inequality (1.131) is valid if the functions $g_l(s)$ are continuous at the interval $[t,T]$, i.e. these functions are uniformly continuous at this interval. So, $|g_l(\tau_q)-g_l(s)|<\varepsilon$ if $s\in[\tau_q,\tau_{q+1}]$, where $|\tau_{q+1}-\tau_q|<\delta(\varepsilon)$, $q=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on points of the interval $[t,T]$).

Thus, taking into account (1.130), we obtain that the right-hand side of (1.129) converges to zero when $N\to\infty$. Hence, we come to the affirmation of Lemma 1.7.

In the case when the functions $g_l(s)$ $(l=1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1.1.7) we can suppose that among the points τ_q, $q=0,1,\ldots,N$ there are all points of jumps of the functions $g_l(s)$ $(l=1,\ldots,k)$. So, we can apply the argumentation as in Sect. 1.1.7.

Obviously if $i_l=0$ for some $l=1,\ldots,k$, then we also come to the affirmation of Lemma 1.7. Lemma 1.7 is proved.

Proving Theorem 1.8 similar to the proof of Theorem 1.1 using Lemmas 1.6, 1.7 and estimates for moments of stochastic integrals with respect to martingales (see (1.126), (1.127)), we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times
$$

$$
\times\,\tilde\rho_1(t_1)dt_1\ldots\tilde\rho_k(t_k)dt_k\le \tag{1.132}
$$

$$
\le \bar C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=
$$

$$
=\bar C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k\to 0
$$

when $p_1,\ldots,p_k\to\infty$, where constant $\bar C_k$ depends only on k (k is the multiplicity of the iterated stochastic integral with respect to martingales) and $\tilde\rho_l(s)\equiv\rho(s)$ or $\tilde\rho_l(s)\equiv 1$ $(l=1,\ldots,k)$. Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$
R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times
$$

$$
\times\, dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}, \tag{1.133}
$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.133) are performed only in the values $dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover, r swapped with q in the permutation $(1,\ldots,k)$. Theorem 1.8 is proved.

1.6 One Modification of Theorems 1.5 and 1.8

1.6.1 Expansion of Iterated Stochastic Integrals with Respect to Martingales Based on Generalized Multiple Fourier Series. The Case $\rho(x)/r(x) < \infty$

Let us compare the expressions (1.104) and (1.132). If we suppose that $r(x) \ge 0$ and

$$\frac{\rho(x)}{r(x)} \le C < \infty,$$

then

$$\int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \rho(t_1)dt_1\ldots\rho(t_k)dt_k =$$

$$= \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \frac{\rho(t_1)}{r(t_1)}r(t_1)dt_1\ldots\frac{\rho(t_k)}{r(t_k)}r(t_k)dt_k \le$$

$$\le C_k' \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde C_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2 \times$$

$$\times \left(\prod_{l=1}^{k} r(t_l)\right) dt_1\ldots dt_k \to 0$$

if $p_1,\ldots,p_k \to \infty$ (see (1.105)), where C_k' is a constant, $\{\Psi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$, and the Fourier coefficient $\tilde C_{j_k\ldots j_1}$ has the form (1.103).

So, we obtain the following modification of Theorems 1.5 and 1.8.

Theorem 1.9 [13], [14], [39]. *Suppose that the following conditions are fulfilled*:

1. *Every* $\psi_l(\tau)$ $(l = 1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$.

2. $\{\Psi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal with weight* $r(x)\ \ge 0$ *system of functions in the space* $L_2([t,T])$, *each function* $\Psi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1.1.7*). Moreover,*

$$
\frac{\rho(x)}{r(x)} \le C < \infty.
$$

3. $M_s^{(l,i_l)} \in Q_4(\rho,[t,T]),\ G_n(\rho,[t,T])$ *with* $n = 2^{k+1},\ i_l = 1,\ldots,m,\ l = 1,\ldots,k$ $(k \in \mathbf{N})$.

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^{M}$ *with respect to martingales defined by* (1.124) *the following expansion*

$$
J[\psi^{(k)}]_{T,t}^{M} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)} -
$$

$$
-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k} \Psi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)
$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k = 1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *satisfying the condition similar to* (1.120), $\Delta M_{\tau_j}^{(r,i)} = M_{\tau_{j+1}}^{(r,i)} - M_{\tau_j}^{(r,i)}$ $(i = 1,\ldots,m,\ r = 1,\ldots,k)$,

$$
\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},
$$

$$
\mathrm{L}_k = \Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g\ne r);\ g,r = 1,\ldots,k\Bigr\},
$$

l.i.m. *is a limit in the mean-square sense,*

$$
\xi_j^{(l,i_l)} = \int\limits_t^T \Psi_j(s)dM_s^{(l,i_l)}
$$

are independent for various $i_l = 1,\ldots,m$ $(l = 1,\ldots,k)$ *and uncorrelated for various* j *(if* $\rho(x)\equiv r(x)$*) random variables,*

$$
\tilde{C}_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k
$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Remark 1.5. *Note that if* $\rho(x), r(x)\equiv 1$ *in Theorem* 1.9*, then we obtain the variant of Theorem* 1.1.

1.6.2 Example on Application of Theorem 1.9 and the System of Bessel Functions

Let us consider the following boundary-value problem

$$(p(x)\Phi'(x))'+q(x)\Phi(x)=-\lambda r(x)\Phi(x),$$
$$\alpha\Phi(a)+\beta\Phi'(a)=0,\quad \gamma\Phi(a)+\delta\Phi'(a)=0,$$

where the functions $p(x)$, $q(x)$, $r(x)$ satisfy the well known conditions and α, β, γ, δ, λ are real numbers.

It is well known (Steklov V.A.) that the eigenfunctions $\Phi_0(x)$, $\Phi_1(x)$, ... of this boundary-value problem form a complete orthonormal with weight $r(x)$ system of functions in the space $L_2([a,b])$ as well as the Fourier series of the function $\sqrt{r(x)}f(x)\in L_2([a,b])$ with respect to the system of functions $\sqrt{r(x)}\Phi_0(x)$, $\sqrt{r(x)}\Phi_1(x),\ldots$ converges in the mean-square sense to the function $\sqrt{r(x)}f(x)$ at the interval $[a,b]$. Moreover, the Fourier coefficients are defined by the formula

$$\tilde C_j=\int\limits_a^b f(x)\Phi_j(x)r(x)dx. \tag{1.134}$$

It is known that in the problem on fluctuations of circular membrane (common case) the boundary-value problem appears for the Euler–Bessel equation with the real parameter λ and an integer value n

$$r^2R''(r)+rR'(r)+\left(\lambda^2r^2-n^2\right)R(r)=0. \tag{1.135}$$

The eigenfunctions of this problem considering specific boundary conditions are the following functions

$$J_n\left(\mu_j\frac{\tau}{L}\right), \tag{1.136}$$

where $\tau \in [0, L]$ and μ_j $(j = 0, 1, 2, \ldots)$ are ordered in the ascending order positive roots of the Bessel function $J_n(\mu)$ $(n = 0, 1, 2, \ldots)$.

The problem on radial fluctuations of the circular membrane leads to the boundary-value problem for the equation (1.135) for $n = 0$, the eigenfunctions of which are the functions (1.136) when $n = 0$.

Let us consider the system of functions

$$\Psi_j(\tau) = \frac{\sqrt{2}}{TJ_{n+1}(\mu_j)} J_n\left(\frac{\mu_j}{T}\tau\right), \quad j = 0, 1, 2, \ldots, \tag{1.137}$$

where

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{n+2m} \frac{1}{\Gamma(m+1)\Gamma(m+n+1)}$$

is the Bessel function of the first kind and

$$\Gamma(z) = \int\limits_0^{\infty} e^{-x} x^{z-1} dx$$

is the gamma-function, μ_j are numbered in the ascending order positive roots of the function $J_n(x)$, and n is a natural number or zero.

Due to the well known properties of the Bessel functions, the system $\{\Psi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal with weight τ system of continuous functions in the space $L_2([0, T])$.

Let us use the system of functions (1.137) in Theorem 1.9.

Consider the following iterated stochastic integral with respect to martingales

$$\int\limits_0^T \int\limits_0^s dM_\tau^{(1)} dM_s^{(2)},$$

where

$$M_s^{(i)} = \int\limits_0^s \sqrt{\tau} d\mathbf{f}_\tau^{(i)} \quad (i = 1, 2),$$

$\mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes, $M_s^{(i)}$ is a martingale (here $\rho(\tau) \equiv \tau$), $0 \le s \le T$.

In addition, $M_s^{(i)}$ has a Gaussian distribution. It is obvious that the conditions of Theorem 1.9 for $k = 2$ are fulfilled.

Using Theorem 1.9, we obtain

$$\int\limits_0^T\int\limits_0^s dM_\tau^{(1)}dM_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\tilde C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)}=\int\limits_0^T\Psi_j(\tau)dM_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i=1,2,$ $j=0,1,2,\ldots$),

$$\tilde C_{j_2j_1}=\int\limits_0^T s\Psi_{j_2}(s)\int\limits_0^s\tau\Psi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient.

It is obvious that we can get the same result using the another approach: we can use Theorem 1.1 for the iterated Itô stochastic integral

$$\int\limits_0^T\sqrt{s}\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)},$$

and as a system of functions $\{\phi_j(s)\}_{j=0}^\infty$ in Theorem 1.1 we can take

$$\phi_j(s)=\frac{\sqrt{2s}}{TJ_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T}s\right),\quad j=0,1,2,\ldots$$

As a result, we obtain

$$\int\limits_0^T\sqrt{s}\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)}=\int\limits_0^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j ($i=1,2,$ $j=0,1,2,\ldots$),

$$C_{j_2j_1}=\int\limits_0^T\sqrt{s}\phi_{j_2}(s)\int\limits_0^s\sqrt{\tau}\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient. Obviously that $C_{j_2j_1}=\tilde{C}_{j_2j_1}$.

Easy calculation demonstrates that

$$\tilde{\phi}_j(s)=\frac{\sqrt{2(s-t)}}{(T-t)J_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T-t}(s-t)\right),\quad j=0,1,2,\ldots$$

is a complete orthonormal system of functions in the space $L_2([t,T])$.

Then, using Theorem 1.1, we obtain

$$\int\limits_t^T\sqrt{s-t}\int\limits_t^s\sqrt{\tau-t}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)},$$

where

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\tilde{\phi}_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,2,$ $j=0,1,2,\ldots)$,

$$C_{j_2j_1}=\int\limits_t^T\sqrt{s-t}\tilde{\phi}_{j_2}(s)\int\limits_t^s\sqrt{\tau-t}\tilde{\phi}_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient.

1.7 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals in Theorem 1.1

1.7.1 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicities 1 and 2

Let us address now to the convergence with probability 1 (w. p. 1). Consider in detail the iterated Itô stochastic integral (1.84) and its expansion, which is corresponds to (1.85) for the case $i_1\ne i_2$

$$I_{(00)T,t}^{(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right).\qquad(1.138)$$

First, note the well known fact [100].

Lemma 1.8. *If for the sequence of random variables ξ_p and for some $\alpha>0$ the number series*

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

In our specific case $(i_1\ne i_2)$

$$I_{(00)T,t}^{(i_1i_2)}=I_{(00)T,t}^{(i_1i_2)p}+\xi_p,\quad \xi_p=\frac{T-t}{2}\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right),$$

where

$$I_{(00)T,t}^{(i_1i_2)p}=\frac{T-t}{2}\left(\zeta_{0}^{(i_1)}\zeta_{0}^{(i_2)}+\sum_{i=1}^{p}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \tag{1.139}$$

Furthermore,

$${\sf M}\left\{|\xi_p|^2\right\}=\frac{(T-t)^2}{2}\sum_{i=p+1}^{\infty}\frac{1}{4i^2-1}\le\frac{(T-t)^2}{2}\int\limits_{p}^{\infty}\frac{1}{4x^2-1}dx=$$

$$=-\frac{(T-t)^2}{2}\frac{1}{4}\ln\left|1-\frac{2}{2p+1}\right|\le\frac{C}{p}, \tag{1.140}$$

where constant C is independent of p.

Therefore, taking $\alpha=2$ in Lemma 1.8, we cannot prove the convergence of ξ_p to zero w. p. 1, since the series

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^{2}\right\}$$

will be majorized by the divergent Dirichlet series with the index 1. Let us take $\alpha=4$ and estimate the value ${\sf M}\left\{|\xi_p|^4\right\}$.

From (1.66) for $k=2,$ $n=2$ and (1.140) we obtain

$${\sf M}\{|\xi_p|^4\}\le\frac{K}{p^2} \tag{1.141}$$

and

$$\sum_{p=1}^{\infty}{\sf M}\left\{|\xi_p|^{4}\right\}\le K\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \tag{1.142}$$

where constant K is independent of p.

Since the series on the right-hand side of (1.142) converges, then according to Lemma 1.8, we obtain that $\xi_p\to 0$ when $p\to\infty$ w. p. 1. Then

$$
I_{(00)T,t}^{(i_1i_2)p}\to I_{(00)T,t}^{(i_1i_2)}\ \ \hbox{when}\ \ p\to\infty\ \ \hbox{w. p. 1.}
$$

Let us analyze the following iterated Itô stochastic integrals

$$
I_{(01)T,t}^{(i_1i_2)}=\int\limits_t^T(t-s)\int\limits_t^s d{\bf f}_{\tau}^{(i_1)}d{\bf f}_{s}^{(i_2)},\quad I_{(10)T,t}^{(i_1i_2)}=\int\limits_t^T\int\limits_t^s(t-\tau)d{\bf f}_{\tau}^{(i_1)}d{\bf f}_{s}^{(i_2)},
$$

whose expansions based on Theorem 1.1 and Legendre polynomials have the following form (also see Sect. 5.1)

$$
I_{(01)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)p}-\frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_1)}\zeta_1^{(i_2)}}{\sqrt{3}}+\right.
$$

$$
\left.+\sum_{i=0}^{p}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_{i}^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right)+\xi_p^{(01)},
$$

$$
I_{(10)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)p}-\frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_2)}\zeta_1^{(i_1)}}{\sqrt{3}}+\right.
$$

$$
\left.+\sum_{i=0}^{p}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_{i}^{(i_1)}-(i+2)\zeta_{i}^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right)+\xi_p^{(10)},
$$

where

$$
\xi_p^{(01)}=-\frac{(T-t)^2}{4}\left(\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right)+\right.
$$

$$
\left.+\sum_{i=p+1}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_{i}^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),
$$

$$
\xi_p^{(10)}=-\frac{(T-t)^2}{4}\left(\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right)+\right.
$$

$$+\sum_{i=p+1}^{\infty}\left(\frac{(i+1)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+2)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr).$$

Then for the case $i_1\ne i_2$ we obtain

$$\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^2\right\}=\frac{(T-t)^4}{16}\times$$

$$\times\sum_{i=p+1}^{\infty}\left(\frac{2}{4i^2-1}+\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}+\frac{1}{(2i-1)^2(2i+3)^2}\right)\le$$

$$\le K\sum_{i=p+1}^{\infty}\frac{1}{i^2}\le\frac{K}{p},\tag{1.143}$$

where constant K is independent of p.

Analogously, we get

$$\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^2\right\}\le\frac{K}{p},\tag{1.144}$$

where constant K does not depend on p.

From (1.66) for $k=2$, $n=2$ and (1.143), (1.144) we have

$$\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\}+\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\}\le\frac{K_1}{p^2}$$

and

$$\sum_{p=1}^{\infty}\left(\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\}+\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\}\right)\le K_1\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty,\tag{1.145}$$

where constant K_1 is independent of p.

According to (1.145) and Lemma 1.8, we obtain that $\xi_p^{(01)},\ \xi_p^{(10)}\to 0$ when $p\to\infty$ w. p. 1. Then

$$I_{(01)T,t}^{(i_1i_2)p}\to I_{(01)T,t}^{(i_1i_2)},\quad I_{(10)T,t}^{(i_1i_2)p}\to I_{(10)T,t}^{(i_1i_2)}\quad\text{when}\quad p\to\infty\quad\text{w. p. 1},$$

where $i_1\ne i_2$.

Let us consider the case $i_1=i_2$

$$I_{(01)T,t}^{(i_1i_1)}=\frac{(T-t)^2}{4}-\frac{(T-t)^2}{4}\Biggl(\left(\zeta_0^{(i_1)}\right)^2+\frac{\zeta_0^{(i_1)}\zeta_1^{(i_1)}}{\sqrt{3}}+$$

$$
+\sum_{i=0}^{p}\left(\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right)\Biggr)+\mu_p^{(01)},
$$

$$
I_{(10)T,t}^{(i_1i_1)}=\frac{(T-t)^2}{4}-\frac{(T-t)^2}{4}\Biggl(\left(\zeta_0^{(i_1)}\right)^2+\frac{\zeta_0^{(i_1)}\zeta_1^{(i_1)}}{\sqrt{3}}+
$$

$$
+\sum_{i=0}^{p}\left(-\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right)\Biggr)+\mu_p^{(10)},
$$

where

$$
\mu_p^{(01)}=-\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right),
$$

$$
\mu_p^{(10)}=-\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(-\frac{\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_1)}}{(2i-1)(2i+3)}\right).
$$

Then

$$
\mathsf{M}\left\{\left(\mu_p^{(01)}\right)^2\right\}=\mathsf{M}\left\{\left(\mu_p^{(10)}\right)^2\right\}=\frac{(T-t)^4}{16}\times
$$

$$
\times\Biggl(\sum_{i=p+1}^{\infty}\frac{1}{(2i+1)(2i+5)(2i+3)^2}+\sum_{i=p+1}^{\infty}\frac{2}{(2i-1)^2(2i+3)^2}+
$$

$$
+\left(\sum_{i=p+1}^{\infty}\frac{1}{(2i-1)(2i+3)}\right)^2\Biggr)\le\frac{K}{p^2}
$$

and

$$
\sum_{p=1}^{\infty}\left(\mathsf{M}\left\{\left|\mu_p^{(01)}\right|^2\right\}+\mathsf{M}\left\{\left|\mu_p^{(10)}\right|^2\right\}\right)\le K\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \tag{1.146}
$$

where constant K is independent of p.

According to Lemma 1.8 and (1.146), we obtain that $\mu_p^{(01)}$, $\mu_p^{(10)}\to 0$ when $p\to\infty$ w. p. 1. Then

$$
I_{(01)T,t}^{(i_1i_1)p}\to I_{(01)T,t}^{(i_1i_1)},\quad I_{(10)T,t}^{(i_1i_1)p}\to I_{(10)T,t}^{(i_1i_1)}\quad\text{when}\quad p\to\infty\quad\text{w. p. 1}.
$$

Analogously, we have

$$
I_{(02)T,t}^{(i_1i_2)p}\to I_{(02)T,t}^{(i_1i_2)},\quad I_{(11)T,t}^{(i_1i_2)p}\to I_{(11)T,t}^{(i_1i_2)},\quad I_{(20)T,t}^{(i_1i_2)p}\to I_{(20)T,t}^{(i_1i_2)}\quad \hbox{when}\quad p\to\infty\quad \hbox{w. p. 1},
$$

where

$$
I_{(02)T,t}^{(i_1i_2)}=\int\limits_t^T(t-s)^2\int\limits_t^s d{\bf f}_{\tau}^{(i_1)}d{\bf f}_{s}^{(i_2)},\quad I_{(20)T,t}^{(i_1i_2)}=\int\limits_t^T\int\limits_t^s(t-\tau)^2 d{\bf f}_{\tau}^{(i_1)}d{\bf f}_{s}^{(i_2)},
$$

$$
I_{(11)T,t}^{(i_1i_2)}=\int\limits_t^T(t-s)\int\limits_t^s(t-\tau)d{\bf f}_{\tau}^{(i_1)}d{\bf f}_{s}^{(i_2)},
$$

$i_1, i_2=1,\ldots,m$. This result is based on the expansions of stochastic integrals $I_{(02)T,t}^{(i_1i_2)}$, $I_{(20)T,t}^{(i_1i_2)}$, $I_{(11)T,t}^{(i_1i_2)}$ (see the formulas (5.24)–(5.26) in Chapter 5).

Let us denote

$$
I_{(l)T,t}^{(i_1)}=\int\limits_t^T(t-s)^l d{\bf f}_{s}^{(i_1)},
$$

where $l=0,1,2\ldots$

The expansions (5.6)–(5.8), (5.27), (5.34) (see Chapter 5) for stochastic integrals $I_{(0)T,t}^{(i_1)}$, $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$, $I_{(3)T,t}^{(i_1)}$, $I_{(l)T,t}^{(i_1)}$ are correct w. p. 1 (they include $1, 2, 3, 4,$ and $l+1$ members of expansion, correspondently).

1.7.2 Convergence with Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicity k ($k\in{\bf N}$)

In this section, we formulate and prove the theorem on convergence with probability 1 (w. p. 1) of expansions of iterated Itô stochastic integrals in Theorem 1.1 for the case of multiplicity k ($k\in{\bf N}$). This section is written on the base of Sect. 1.7.2 from [14] as well as on Sect. 6 from [29] and Sect. 9 from [27].

Let us remind the well known fact from the mathematical analysis, which is connected to existence of iterated limits.

Proposition 1.1. *Let* $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$
\lim\limits_{n,m\to\infty}x_{n,m}=a<\infty.
$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \textit{for any } m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \textit{for any } n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Theorem 1.10 [14], [25], [27], [29], [30]. *Let $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are continuously differentiable nonrandom functions on the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \textit{if} \quad p \to \infty$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is the expression on the right-hand side of (1.10) *before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1 = \ldots = p_k = p$, i.e. (see Theorem* 1.1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \Biggr),$$

where $i_1, \ldots, i_k = 1, \ldots, m$, rest notations are the same as in Theorem 1.1.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2, \tag{1.147}$$

where

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}}, \tag{1.148}$$

where $t_1,\ldots,t_k\in[t,T]$ for $k\ge 2$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, $\mathbf{1}_A$ denotes the indicator of the set A,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{1.149}$$

is the Fourier coefficient.

Using (1.148), we obtain

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k.$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathsf{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

If $p_1=\ldots=p_k=p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}\stackrel{\mathsf{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathsf{def}}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathsf{def}}{=}\sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

and so on.

Let us consider the following lemma.

Lemma 1.9. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1} \tag{1.150}$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Proof. Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{1.151}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Obviously, the expression (1.151) defines the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le\sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2<\infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{1.152}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{1.153}$$

and equality of the limits (1.152) and (1.153).

Taking into account the above reasoning, we have

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C^2_{j_k\ldots j_1}. \tag{1.154}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

exists (see the Parseval equality (1.147)), then from Proposition 1.1 we have

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1}=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}. \tag{1.155}$$

Using (1.154) and Proposition 1.1, we get

$$\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1}=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}=\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1}. \tag{1.156}$$

Combining (1.156) and (1.155), we obtain

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

Repeating the above steps, we complete the proof of Lemma 1.9.

Further, let us show that for $s=1,\ldots,k$

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \qquad (1.157)
$$

Using the arguments which we used in the proof of Lemma 1.9, we have

$$
\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=0}^{p}\sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2 \qquad (1.158)
$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\}=\{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we obtain

$$
\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\ldots=
$$

$$
=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}\sum_{j_s=0}^{p}C_{j_k\ldots j_1}^2. \qquad (1.159)
$$

Using (1.158), (1.159) and Lemma 1.9, we get

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-
$$

$$
-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.
$$

So, the equality (1.7.2) is proved.

Using the Parseval equality and Lemma 1.9, we obtain

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} +$$

$$+ \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 = \ldots =$$

$$= \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{p}\ldots\sum_{j_{k-1}=0}^{p}\sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 \le$$

$$\le \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{s=1}^{k}\left(\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2\right). \qquad (1.160)$$

Note that we use the following

$$\sum_{j_1=0}^{p}\ldots\sum_{j_{s-1}=0}^{p}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\lim_{m_{s-1}\to\infty}\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\ldots\le$$

$$\le\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2$$

to derive (1.160), where $m_1,\ldots,m_{s-1}>p$.

Denote

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,$$

where $s=1,\ldots,k-1$.

Let us remind the Dini Theorem, which we will use further.

Theorem (Dini). *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a,b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a,b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a,b]$.*

For $s<k$ due to the Parseval equality and Dini Theorem as well as (1.7.2) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T \psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T \psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T \psi_k^2(t_k)\int\limits_t^{t_k} \psi_{k-1}^2(t_{k-1})\left(C_{j_{k-2}\ldots j_1}(t_{k-1})\right)^2 \times$$

$$\times dt_{k-1}dt_k \le$$

$$\le C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T \left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =$$

$$= C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T \sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =$$

$$= C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T \int\limits_t^{\tau} \psi_{k-2}^2(\theta)\left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le K\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T \left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le$$

$$\le \ldots \le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\int\limits_t^T \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$= C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T \sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \qquad (1.161)$$

where constants C, K depend on $T-t$ and constant C_k depends on k and $T-t$.

Let us explane more precisely how we obtain (1.161). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$
\sum_{j=0}^{\infty}\left(\int\limits_t^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2=
$$

$$
=\int\limits_t^{T}\left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s)ds=\int\limits_t^{\tau}g^2(s)ds. \tag{1.162}
$$

The equality (1.162) has been applied repeatedly when we obtaining (1.161). Using the replacement of integration order in Riemann integrals, we have

$$
C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s=
$$

$$
=\int\limits_t^{\tau}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{\tau}\phi_{j_2}(t_2)\psi_2(t_2)\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1\stackrel{\sf def}{=}
$$

$$
\stackrel{\sf def}{=}\tilde C_{j_s\ldots j_1}(\tau).
$$

For $l=1,\ldots,s$ we will use the following notation

$$
\tilde C_{j_s\ldots j_l}(\tau,\theta)=
$$

$$
=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.
$$

Using the Parseval equality and Dini Theorem, from (1.161) we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le
$$

$$
\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^{T}\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
=C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^{T}\sum_{j_1=0}^{\infty}\left(\tilde C_{j_s\ldots j_1}(\tau)\right)^2d\tau=
$$

$$= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \left(\tilde{C}_{j_s\ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = \qquad (1.163)$$

$$= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s\ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = \qquad (1.164)$$

$$= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t_1}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau, t_2)\right)^2 dt_2 dt_1 d\tau \le$$

$$\le C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau, t_2)\right)^2 dt_2 dt_1 d\tau \le$$

$$\le C_k' \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau, t_2)\right)^2 dt_2 d\tau \le$$

$$\le \ldots \le$$

$$\le C_k'' \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_{s-1}^2(t_{s-1}) \left(\tilde{C}_{j_s}(\tau, t_{s-1})\right)^2 dt_{s-1} d\tau \le$$

$$\le \tilde{C}_k \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \left(\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta\right)^2 du d\tau, \qquad (1.165)$$

where constants C_k', C_k'', $\tilde{C}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (1.165). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty} \left(\int\limits_{\theta}^{\tau} \phi_j(s) g(s) ds\right)^2 = \sum_{j=0}^{\infty} \left(\int\limits_t^T \mathbf{1}_{\{\theta<s<\tau\}} \phi_j(s) g(s) ds\right)^2 =$$

$$= \int\limits_t^T \left(\mathbf{1}_{\{\theta<s<\tau\}}\right)^2 g^2(s) ds = \int\limits_{\theta}^{\tau} g^2(s) ds. \qquad (1.166)$$

The equality (1.166) has been applied repeatedly when we obtaining (1.165).

Let us explane more precisely the passing from (1.163) to (1.164) (the same steps have been used when we derive (1.165)).

We have

$$
\int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\Bigr)^2 dt_1 d\tau -
$$

$$
- \sum_{j_2=0}^{n} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\Bigr)^2 dt_1 d\tau =
$$

$$
= \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\Bigr)^2 dt_1 d\tau =
$$

$$
= \lim_{N\to\infty} \sum_{j=0}^{N-1} \int\limits_t^{\tau_j} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\Bigr)^2 dt_1 \Delta\tau_j, \tag{1.167}
$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t, T]$ satisfying the condition (1.9).

Since the non-decreasing functional sequence $u_n(\tau_j, t_1)$ and its limit function $u(\tau_j, t_1)$ are continuous on the interval $[t, \tau_j] \subseteq [t, T]$ with respect to t_1, where

$$
u_n(\tau_j, t_1) = \sum_{j_2=0}^{n} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\Bigr)^2,
$$

$$
u(\tau_j, t_1) = \sum_{j_2=0}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\Bigr)^2 = \int\limits_{t_1}^{\tau_j} \psi_2^2(t_2) \Bigl(\tilde{C}_{j_s \ldots j_3}(\tau_j, t_2)\Bigr)^2 dt_2,
$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j, t_1)$ to $u(\tau_j, t_1)$ at the interval $[t, \tau_j] \subseteq [t, T]$ with respect to t_1. As a result, we obtain

$$
\sum_{j_2=n+1}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\Bigr)^2 < \varepsilon, \quad t_1 \in [t, \tau_j] \tag{1.168}
$$

for $n > N(\varepsilon) \in \mathbf{N}$ ($N(\varepsilon)$ exists for any $\varepsilon > 0$ and it does not depend on t_1).

From (1.167) and (1.168) we obtain

$$
\lim_{N\to\infty} \sum_{j=0}^{N-1} \int\limits_t^{\tau_j} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \Bigl(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\Bigr)^2 dt_1 \Delta\tau_j \le
$$

$$\leq \varepsilon \lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)dt_1\Delta\tau_j = \varepsilon \int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)dt_1 d\tau. \tag{1.169}$$

From (1.169) we get

$$\lim_{n\to\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau = 0.$$

This fact completes the proof of passing from (1.163) to (1.164).

Let us estimate the integral

$$\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta \tag{1.170}$$

from (1.165) for the cases when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the estimates for the integral

$$\int\limits_t^{\tau}\phi_j(\theta)\psi(\theta)d\theta, \quad j\geq p+1, \tag{1.171}$$

where $\psi(\theta)$ is a continuously differentiable function on the interval $[t,T]$, have been obtained in [6]-[14], [20], [31] (also see Sect. 2.2.5).

Let us estimate the integral (1.170) using the approach from [20], [31].

First, consider the case of Legendre polynomials. Then $\phi_j(s)$ is defined as follows

$$\phi_j(\theta)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(\theta-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j\geq 0,$$

where $P_j(x)$ $(j=0,1,2\ldots)$ is the Legendre polynomial.

Further, we have

$$\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta = \frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy =$$

$$
=\frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x))-P_{j-1}(z(x)))\psi(x)-(P_{j+1}(z(v))-P_{j-1}(z(v)))\psi(v)-
$$

$$
-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y)-P_{j-1}(y))\psi'(u(y))dy\Biggr), \tag{1.172}
$$

where $x, v\in (t,T)$, $j\ge p+1$, $u(y)$ and $z(x)$ are defined by the following relations

$$
u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t},
$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (1.172) we used the following well known property of the Legendre polynomials

$$
\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),\quad j=1,2,\ldots
$$

From (1.172) and the well known estimate for the Legendre polynomials [87] (also see [88])

$$
|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbf{N},
$$

where constant K does not depend on y and j, it follows that

$$
\left|\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+C_1\right), \tag{1.173}
$$

where $z(x), z(v)\in(-1,1)$, $x,v\in(t,T)$ and constants C, C_1 do not depend on j.

From (1.173) we obtain

$$
\left(\int\limits_v^x\phi_j(\theta)\psi(\theta)d\theta\right)^2<\frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}}+\frac{1}{(1-(z(v))^2)^{1/2}}+C_3\right), \tag{1.174}
$$

where constants C_2, C_3 do not depend on j.

Let us apply (1.174) for the estimate of the right-hand side of (1.165). We have

$$\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^1 \frac{dy}{\left(1-y^2\right)^{1/2}}+\int\limits_{-1}^1\int\limits_{-1}^x \frac{dy}{\left(1-y^2\right)^{1/2}}dx+K_2\right)\le$$

$$\le \frac{K_3}{j_s^2}, \tag{1.175}$$

where constants K_1, K_2, K_3 are independent of j_s.

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta)=\frac{1}{\sqrt{T-t}}\begin{cases} 1, & j=0\\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r\end{cases} \tag{1.176}$$

where $r=1,2,\ldots$

Using the system of functions (1.176), we have

$$\int\limits_v^x \phi_{2r-1}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=-\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\cos\frac{2\pi r(x-t)}{T-t}-\psi(v)\cos\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{1.177}$$

$$\int\limits_v^x \phi_{2r}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$= \sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\sin\frac{2\pi r(x-t)}{T-t} - \psi(v)\sin\frac{2\pi r(v-t)}{T-t} - \right.$$

$$\left. - \int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{1.178}$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (1.177) and (1.178), we obtain for the trigonometric case

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 \le \frac{C_4}{j^2}, \tag{1.179}$$

where constant C_4 is independent of j.

From (1.179) we finally have

$$\int\limits_t^T \int\limits_t^\tau \left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le \frac{K_4}{j_s^2}, \tag{1.180}$$

where constant K_4 is independent of j_s.

Combining (1.165), (1.175), and (1.180), we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} \le$$

$$\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2} \le L_k\int\limits_p^{\infty}\frac{dx}{x^2} = \frac{L_k}{p}, \tag{1.181}$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (1.160) and (1.181) we obtain

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C^2_{j_k\ldots j_1} \le \frac{G_k}{p}, \tag{1.182}$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use the estimate (1.66). Using (1.182) and the estimate (1.66) for the case $p_1 = \ldots = p_k = p$ and $n = 2$, we obtain

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le
$$

$$
\le C_{2,k}\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2\right)^2 \le
$$

$$
\le \frac{H_{2,k}}{p^2}, \tag{1.183}
$$

where

$$
C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!
$$

and $H_{2,k} = G_k^2 C_{2,k}$.

Let α and ξ_p in Lemma 1.8 be chosen as follows

$$
\alpha = 4, \quad \xi_p = \left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|.
$$

From (1.183) we obtain

$$
\sum_{p=1}^{\infty}\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty. \tag{1.184}
$$

Using Lemma 1.8 and the estimate (1.184), we have

$$
J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p \to \infty
$$

w. p. 1, where (see Theorem 1.1)

$$
J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - \right.
$$

$$
\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right) \tag{1.185}
$$

or (see Theorem 1.2)

$$
J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \tag{1.186}
$$

where $i_1, \ldots, i_k = 1, \ldots, m$ in (1.185) and (1.186).

Remark 1.6. *From Theorem* 1.4 *and Lemma* 1.9 *we obtain*

$$
\lim_{p_{q_1}\to 0} \ldots \lim_{p_{q_k}\to\infty} \mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} \right)^2 \right\} \le
$$

$$
\le k! \cdot \lim_{p_{q_1}\to 0} \ldots \lim_{p_{q_k}\to\infty} \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2 \right) =
$$

$$
= k! \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 \right) = 0
$$

for the following cases:

1. $i_1, \ldots, i_k = 1, \ldots, m$ *and* $0 < T - t < \infty$,
2. $i_1, \ldots, i_k = 0, 1, \ldots, m,\ \ i_1^2 + \ldots + i_k^2 > 0,$ *and* $0 < T - t < 1$.

At that, $(q_1, \ldots, q_k)$ *is any permutation such that* $\{q_1, \ldots, q_k\} = \{1, \ldots, k\}$, $J[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1.5), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (1.10) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ *; another notations are the same as in Theorem* 1.1.

Remark 1.7. *Taking into account Theorem* 1.4 *and the estimate* (1.182), *we obtain the following inequality*

$$
\mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right)^2 \right\} \le \frac{k!P_k(T-t)^k}{p}, \tag{1.187}
$$

where constant P_k *depends only on* k.

Remark 1.8. *The estimates* (1.66) *and* (1.182) *imply the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{\left(P_k\right)^n\left(T-t\right)^{nk}}{p^n},$$

where $n\in\mathbf{N}$ and constant P_k depends only on k.

1.8 Modification of Theorem 1.1 for the Case of the Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Itô Stochastic Integrals

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Define the following function on the hypercube $[t,T]^k$

$$\bar K(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k),$$

where the function $K(t_1,\ldots,t_k)$ is defined by (1.6), $s\in(t,T]$ (s is fixed), and $\mathbf{1}_A$ is the indicator of the set A. So we have

$$\bar K(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_1<\ldots<t_k<s\}}\psi_1(t_1)\ldots\psi_k(t_k)=$$

$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s\\ & \\ 0, & \text{otherwise}\end{cases},\qquad (1.188)$$

where $k\ge 1$, $t_1,\ldots,t_k\in[t,T]$, and $s\in(t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $\bar K(t_1,\ldots,t_k,s)$ defined by (1.188) is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $\bar K(t_1,\ldots,t_k,s)\in L_2([t,T]^k)$ is converging to this function in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|\bar{K}(t_1,\ldots,t_k,s)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{1.189}$$

where

$$C_{j_k\ldots j_1}(s)=\int\limits_{[t,T]^k}\bar{K}(t_1,\ldots,t_k,s)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k=$$

$$=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{1.190}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Note that

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{1.191}$$

$$=\int\limits_t^T\mathbf{1}_{\{t_k<s\}}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},$$

where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=0,1,\ldots,m.$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{1.192}$$

Theorem 1.11 [27]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1.1.7). Then*

$$
J[\psi^{(k)}]_{s,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -
$$

$$
-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),
$$

where $J[\psi^{(k)}]_{s,t}$ *is defined by* (1.191), $s\in(t,T]$ (s is fixed),

$$
\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},
$$

$$
\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},
$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$
\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$), $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (1.190), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ ($i=0,\ 1,\ldots,m$), $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$, *which satisfies the condition* (1.192).

Proof. By analogy with (1.23), we have

$$
J'[\Phi]_{s,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\mathbf{1}_{\{t_k<s\}}\sum_{(t_1,\ldots,t_k)}\Bigl(\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr)\quad \text{w. p. 1,} \tag{1.193}
$$

where $J'[\Phi]_{s,t}^{(k)}$ is defined by (1.22) and

$$
\sum_{(t_1,\ldots,t_k)}
$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$. At the same time the summation with respect to permutations $(t_1,\ldots,t_k)$ is performed in (1.193) only in the expression, which is enclosed in parentheses. Moreover, $\Phi(t_1,\ldots,t_k)\in C(D_k(s))$, where (see (1.17))

$$
D_k(s)=\{(t_1,\ldots,t_k):\ t\le t_1\le\ldots\le t_k\le s\le T\}.
$$

Let us write (1.193) as

$$J'[\Phi]_{s,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum\limits_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,} \tag{1.194}$$

where summation with respect to permutations $(t_1,\ldots,t_k)$ is performed in (1.194) only in the expression $\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$.

It is not difficult to notice that (1.193), (1.194) can be rewritten in the form (see (1.24))

$$J'[\Phi]_{s,t}^{(k)}=\sum\limits_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1,} \tag{1.195}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

According to Lemma 1.1, we have

$$J[\psi^{(k)}]_{s,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{l_k=0}^{N-1}\ldots\sum\limits_{l_1=0}^{l_2-1}\mathbf{1}_{\{\tau_{l_k}<s\}}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{l_k=0}^{N-1}\ldots\sum\limits_{l_1=0}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum\limits_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,} \tag{1.196}$$

where $K(t_1,\ldots,t_k)$ is defined by (1.6) and permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression $K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$.

According to Lemmas 1.1–1.3 and (1.24), (1.195), (1.196), we get the following representation

$$J[\psi^{(k)}]_{s,t}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+R_{T,t,s}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\times$$

$$\times\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+R_{T,t,s}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t,s}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\times$$

$$\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t,s}^{p_1,\ldots,p_k}\quad \text{w. p. 1},$$

where

$$R_{T,t,s}^{p_1,\ldots,p_k} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(\mathbf{1}_{\{t_k<s\}} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) \right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} K(t_1,\ldots,t_k) \mathbf{1}_{\{t_k<s\}} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} - \qquad (1.197)$$

$$- \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \qquad (1.198)$$

w. p. 1, where permutations $(t_1,\ldots,t_k)$ when summing in (1.197) are performed only in the values $\mathbf{1}_{\{t_k<s\}} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time permutations $(t_1,\ldots,t_k)$ when summing in (1.198) are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. Moreover, the indices near upper limits of integration in the iterated stochastic integrals in (1.197), (1.198) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t,s}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 1.2, we have

$$\mathsf{M}\left\{ \left(R_{T,t,s}^{p_1,\ldots,p_k} \right)^2 \right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) \mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times$$

$$\times\, dt_1 \ldots dt_k, \qquad (1.199)$$

where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{s,t}$ and permutations $(t_1,\ldots,t_k)$ when summing in (1.199) are performed only in the values $\mathbf{1}_{\{t_k<s\}}$ and $dt_1 \ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (1.199) are changed correspondently.

Since $K(t_1,\ldots,t_k)\equiv 0$ if the condition $t_1<\ldots<t_k$ is not fulfilled, then

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\prod_{l=1}^k \phi_{j_l}(t_l)\right)^2 \times$$

$$\times\, dt_1\ldots dt_k, \tag{1.200}$$

where permutations $(t_1,\ldots,t_k)$ when summing in (1.200) are performed only in the values $dt_1\ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (1.200) are changed correspondently.

Then from (1.36), (1.189), and (1.200) we obtain

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\prod_{l=1}^k \phi_{j_l}(t_l)\right)^2 \times$$

$$\times dt_1\ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(\bar{K}(t_1,\ldots,t_k,s) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\prod_{l=1}^k \phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{s,t}$. Theorem 1.11 is proved.

Remark 1.9. *Obviously from Theorem* 1.11 *for the case* $s=T$ *we obtain the variant of Theorem* 1.1.

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 1.11 we obtain

$$J[\psi^{(k)}]_{s,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s)\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.$$

Consider particular cases of Theorem 1.11 for $k=1,\ldots,5$

$$J[\psi^{(1)}]_{s,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}(s)\zeta_{j_1}^{(i_1)}, \tag{1.201}$$

$$J[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{1.202}$$

$$J[\psi^{(3)}]_{s,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}(s)\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{1.203}$$

$$J[\psi^{(4)}]_{s,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}(s)\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{1.204}$$

$$J[\psi^{(5)}]_{s,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}(s)\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$
-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right),
$$

where $\mathbf{1}_A$ is the indicator of the set A, $C_{j_k\ldots j_1}(s)$ $(k=1,\ldots,5)$ has the form (1.190), $s\in(t,T]$ (s is fixed).

Chapter 2

Expansions of Iterated Stratonovich Stochastic Integrals Based on Generalized Multiple and Iterated Fourier Series

This chapter is devoted to the expansions of iterated Startonovich stochastic integrals. We adapt the results of Chapter 1 (Theorem 1.1) for iterated Stratonovich stochastic integrals of multiplicities 1 to 5. The mean-square convergence of the mentioned expansions for the case of multiple Fourier–Legendre series as well as for the case of multiple trigonometric Fourier series is proved. The considered expansions contain only one operation of the limit transition in contrast to its existing analogues. This property is very important for the mean-square approximation of iterated stochastic integrals. Also, we consider a different approach to expansion of iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbf{N}$) based on generalized iterated Fourier series converging pointwise (Sect. 2.5).

2.1 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicity 2 Based on Theorem 1.1. The case $p_1, p_2 \to \infty$ and Smooth Weight Functions

2.1.1 Approach Based on Theorem 1.1 and Integration by Parts

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) \stackrel{\mathsf{def}}{=} f_t : [0,T] \times \Omega \to \mathbf{R}^1$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t \in [0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s < t$.

2. The Wiener process f_t is F_t-measurable for all $t \in [0, T]$.

3. The process $f_{t+\Delta} - f_t$ for all $t \geq 0$, $\Delta > 0$ is independent with the events of σ-algebra F_t.

Let $\mathrm{M}_2([0, T])$ be the class of random functions $\xi(t, \omega) \stackrel{\sf def}{=} \xi_t : [0, T] \times \Omega \rightarrow \mathbf{R}^1$ defined as in Sect. 1.1.2.

We introduce the class $\mathrm{Q}_m([0, T])$ of Itô processes η_τ, $\tau \in [0, T]$ of the form

$$\eta_\tau = \eta_0 + \int\limits_0^\tau a_s ds + \int\limits_0^\tau b_s df_s, \tag{2.1}$$

where $(a_\tau)^m, (b_\tau)^m \in \mathrm{M}_2([0, T])$ and

$$\mathsf{M}\left\{|b_s - b_\tau|^4\right\} \leq C|s - \tau|^\gamma$$

for all $s, \tau \in [0, T]$ and for some $C, \gamma \in (0, \infty)$.

The second integral on the right-hand side of (2.1) is the Itô stochastic integral (see Sect. 1.1.2).

Consider a function $F(x, \tau) : \mathbf{R}^1 \times [0, T] \rightarrow \mathbf{R}^1$ for fixed τ from the class $C_2(-\infty, \infty)$ consisting of twice continuously differentiable in x functions on the interval $(-\infty, \infty)$ such that the first two derivatives are bounded.

Let $\tau_j^{(N)}$, $j = 0, 1, \ldots, N$ be a partition of the interval $[t, T]$, $t \geq 0$ such that

$$t = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \qquad \max_{0 \leq j \leq N-1} \left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \rightarrow 0 \;\; \text{if} \;\; N \rightarrow \infty. \tag{2.2}$$

The mean-square limit

$$\underset{N \to \infty}{\text{l.i.m}} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j^{(N)}} + \eta_{\tau_{j+1}^{(N)}}\right), \tau_j^{(N)}\right)\left(f_{\tau_{j+1}^{(N)}} - f_{\tau_j^{(N)}}\right) \stackrel{\sf def}{=} \int\limits_t^{*T} F(\eta_\tau, \tau) df_\tau \tag{2.3}$$

is called [86] the Stratonovich stochastic integral of the process $F(\eta_\tau, \tau)$, $\tau \in [t, T]$, where $\tau_j^{(N)}$, $j = 0, 1, \ldots, N$ is a partition of the interval $[t, T]$ satisfying the condition (2.2).

It is known [86] (also see [67]) that under proper conditions, the following relation between Stratonovich and Itô stochastic integrals holds

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^{T} F(\eta_\tau,\tau)df_\tau + \frac{1}{2}\int\limits_t^{T} \frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau \quad \text{w. p. 1.} \tag{2.4}$$

If the Wiener processes in (2.1) and (2.3) are independent, then

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^{T} F(\eta_\tau,\tau)df_\tau \quad \text{w. p. 1.} \tag{2.5}$$

A possible variant of conditions under which the formulas (2.4) and (2.5) are correct, for example, consisits of the conditions

$$\eta_\tau \in \mathrm{Q}_4([t,T]), \quad F(\eta_\tau,\tau) \in \mathrm{M}_2([t,T]), \quad \text{and} \quad F(x,\tau) \in C_2(-\infty,\infty).$$

Further in Chapter 2, we will denote as $\{\phi_j(x)\}_{j=0}^{\infty}$ the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Also we will pay attention on the following well known facts about these two systems of functions [87].

Suppose that the function $f(x)$ is bounded at the interval $[t,T]$. Moreover, its derivative $f'(x)$ is continuous function at the interval $[t,T]$ except may be the finite number of points of the finite discontinuity.

Then the Fourier series

$$\sum_{j=0}^{\infty} C_j\phi_j(x), \quad C_j = \int\limits_t^T f(x)\phi_j(x)dx$$

converges at any internal point x of the interval $[t,T]$ to the value $(f(x+0)+f(x-0))/2$ and converges uniformly to $f(x)$ on any closed interval of continuity of the function $f(x)$ laying inside $[t,T]$. At the same time the Fourier–Legendre series converges if $x=t$ and $x=T$ to $f(t+0)$ and $f(T-0)$ correspondently, and the trigonometric Fourier series converges if $x=t$ and $x=T$ to $(f(t+0)+f(T-0))/2$ in the case of periodic continuation of the function $f(x)$.

In Sect. 2.1 we consider the case $k=2$ of the following iterated Stratonovich and Itô stochastic integrals

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.6}$$

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.7}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes.

Let us formulate and prove the following theorem on expansion of iterated Stratonovich stochastic integrals of multiplicity 2.

Theorem 2.1 [8] (2011), [10]-[20], [31]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *At the same time* $\psi_2(s)$ *is a continuously differentiable nonrandom function on* $[t, T]$ *and* $\psi_1(s)$ *is twice continuously differentiable nonrandom function on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense is valid, where

$$C_{j_2 j_1} = \int\limits_t^T \psi_2(s_2) \phi_{j_2}(s_2) \int\limits_t^{s_2} \psi_1(s_1) \phi_{j_1}(s_1) ds_1 ds_2$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. In accordance to the standard relations between Stratonovich and Itô stochastic integrals (see (2.4) and (2.5)) we have w. p. 1

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \mathbf{1}_{\{i_1 = i_2\}} \int\limits_t^T \psi_1(t_1) \psi_2(t_1) dt_1, \tag{2.8}$$

where here and further $\mathbf{1}_A$ is the indicator of the set A.

From the other side according to (1.42), we have

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{2.9}$$

From (2.8) and (2.9) it follows that Theorem 2.1 will be proved if

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{2.10}$$

Let us prove (2.10). Consider the function

$$K^*(t_1,t_2) = K(t_1,t_2) + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_1), \tag{2.11}$$

where $t_1,t_2\in[t,T]$ and $K(t_1,t_2)$ is defined by (1.6) for $k=2$.

Let us expand the function $K^*(t_1,t_2)$ defined by (2.11) using the variable t_1, when t_2 is fixed, into the generalized Fourier series at the interval (t,T)

$$K^*(t_1,t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2)\phi_{j_1}(t_1) \quad (t_1\ne t,T), \tag{2.12}$$

where

$$C_{j_1}(t_2) = \int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1 = \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1. \tag{2.13}$$

The equality (2.12) is satisfied pointwise in each point of the interval (t,T) with respect to the variable t_1, when $t_2\in[t,T]$ is fixed, due to a piecewise smoothness of the function $K^*(t_1,t_2)$ with respect to the variable $t_1\in[t,T]$ (t_2 is fixed).

Note also that due to well known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (2.12) converges when $t_1=t,T$.

Obtaining (2.12) we also used the fact that the right-hand side of (2.12) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K(t_2 - 0, t_2) + K(t_2 + 0, t_2)\right) = \frac{1}{2}\psi_1(t_2)\psi_2(t_2) = K^*(t_2, t_2).$$

The function $C_{j_1}(t_2)$ is a continuously differentiable one at the interval $[t, T]$. Let us expand it into the generalized Fourier series at the interval (t, T)

$$C_{j_1}(t_2) = \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_2) \quad (t_2 \neq t, T), \tag{2.14}$$

where

$$C_{j_2 j_1} = \int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2 = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2,$$

and the equality (2.14) is satisfied pointwise at any point of the interval (t, T) (the right-hand side of (2.14) converges when $t_2 = t, T$).

Let us substitute (2.14) into (2.12)

$$K^*(t_1, t_2) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad (t_1, t_2) \in (t, T)^2. \tag{2.15}$$

Note that the series on the right-hand side of (2.15) converges at the boundary of the square $[t, T]^2$.

It is easy to see that substituting $t_1 = t_2$ in (2.15), we obtain

$$\frac{1}{2}\psi_1(t_1)\psi_2(t_1) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1). \tag{2.16}$$

From (2.16) we formally have

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \int\limits_t^T \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\int\limits_t^T C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$
= \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =
$$

$$
= \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\mathbf{1}_{\{j_1=j_2\}} =
$$

$$
= \lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_2 j_1} = \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{2.17}
$$

Let us explain the second step in (2.17) (the fourth step in (2.17) follows from the orthonormality of functions $\phi_j(s)$ at the interval $[t,T]$).

We have

$$
\left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| \le
$$

$$
\le \int\limits_t^T \left|\psi_2(t_1)G_{p_1}(t_1)\right| dt_1 \le C\int\limits_t^T \left|G_{p_1}(t_1)\right| dt_1, \tag{2.18}
$$

where $C<\infty$ and

$$
\sum_{j=p+1}^{\infty}\int\limits_t^{\tau}\psi_1(s)\phi_j(s)ds\phi_j(\tau) \stackrel{\sf def}{=} G_p(\tau).
$$

Let us consider the case of Legendre polynomials. Then

$$
\left|G_{p_1}(t_1)\right| = \frac{1}{2}\left|\sum_{j_1=p_1+1}^{\infty}(2j_1+1)\int\limits_{-1}^{z(t_1)}\psi_1(u(y))P_{j_1}(y)dyP_{j_1}(z(t_1))\right|, \tag{2.19}
$$

where

$$
u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}, \tag{2.20}
$$

and $P_j(s)$ is the Legendre polynomial.

From (2.19) and the well known formula

$$
\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j=1,2,\ldots \tag{2.21}
$$

we obtain

$$|G_{p_1}(t_1)| = \frac{1}{2}\Biggl|\sum_{j_1=p_1+1}^{\infty}\Biggl\{\left(P_{j_1+1}(z(t_1)) - P_{j_1-1}(z(t_1))\right)\psi_1(t_1) -$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}\left(P_{j_1+1}(y) - P_{j_1-1}(y)\right)\psi_1'(u(y))dy\Biggr\}P_{j_1}(z(t_1))\Biggr| \le$$

$$\le C_0\Biggl|\sum_{j_1=p_1+1}^{\infty}\left(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1)) - P_{j_1-1}(z(t_1))P_{j_1}(z(t_1))\right)\Biggr| +$$

$$+\frac{T-t}{4}\Biggl|\sum_{j_1=p_1+1}^{\infty}\Biggl\{\psi_1'(t_1)\Biggl(\frac{1}{2j_1+3}\left(P_{j_1+2}(z(t_1)) - P_{j_1}(z(t_1))\right) -$$

$$-\frac{1}{2j_1-1}\left(P_{j_1}(z(t_1)) - P_{j_1-2}(z(t_1))\right)\Biggr) -$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}\Biggl(\frac{1}{2j_1+3}\left(P_{j_1+2}(y) - P_{j_1}(y)\right) -$$

$$-\frac{1}{2j_1-1}\left(P_{j_1}(y) - P_{j_1-2}(y)\right)\Biggr)\psi_1''(u(y))dy\Biggr\}P_{j_1}(z(t_1))\Biggr|, \qquad (2.22)$$

where C_0 is a constant, ψ_1' and ψ_1'' are derivatives of the function $\psi_1(s)$ with respect to the variable $u(y)$.

From (2.22) and the well known estimate for Legendre polynomials [87]

$$|P_n(y)| < \frac{K}{\sqrt{n+1}(1-y^2)^{1/4}}, \quad y \in (-1,1), \quad n \in \mathbf{N}, \qquad (2.23)$$

where constant K does not depend on y and n, we have

$$|G_{p_1}(t_1)| <$$

$$< C_0\Biggl|\lim_{n\to\infty}\sum_{j_1=p_1+1}^{n}\left(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1)) - P_{j_1-1}(z(t_1))P_{j_1}(z(t_1))\right)\Biggr| +$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}} + \int\limits_{-1}^{z(t_1)}\frac{dy}{(1-y^2)^{1/4}}\frac{1}{(1-(z(t_1))^2)^{1/4}}\right) <$$

$$< C_0\left|\lim_{n\to\infty}\left(P_{n+1}(z(t_1))P_n(z(t_1)) - P_{p_1}(z(t_1))P_{p_1+1}(z(t_1))\right)\right| +$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + C_2\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) <$$

$$< C_3\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{p_1}\right)\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} +$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + C_2\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) \le$$

$$\le C_4\left(\left(\frac{1}{p_1}+\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + \sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right) \le$$

$$\le \frac{K}{p_1}\left(\frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \qquad (2.24)$$

where $C_0, C_1, \ldots, C_4, K$ are constants, $t_1 \in (t, T)$, and

$$\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2} \le \int\limits_{p_1}^{\infty}\frac{dx}{x^2} = \frac{1}{p_1}. \qquad (2.25)$$

From (2.18) and (2.24) we get

$$\left|\int\limits_t^T\sum_{j_1=0}^{\infty}C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| <$$

$$< \frac{K}{p_1}\left(\int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/2}} + \int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/4}}\right) \to 0$$

if $p_1 \to \infty$. So, we obtain

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1 = \int\limits_t^T\sum_{j_1=0}^{\infty}C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 =$$

$$= \sum_{j_1=0}^{\infty}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty}\int\limits_t^T\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 =$$

$$
=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\int\limits_t^T C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1}. \qquad (2.26)
$$

In (2.26) we used the fact that the Fourier–Legendre series

$$
\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_1)
$$

of the smooth function $C_{j_1}(t_1)$ converges uniformly to this function at the interval $[t+\varepsilon,T-\varepsilon]$ for any $\varepsilon>0$, converges to this function at the any point $t_1\in(t,T)$, and converges to $C_{j_1}(t+0)$ and $C_{j_1}(T-0)$ when $t_1=t,\ T$.

The relation (2.10) is proved for the case of Legendre polynomials.

Let us consider the trigonometric case and suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in $L_2([t,T])$.

We have

$$
\left|\int\limits_t^T\sum_{j_1=0}^{\infty}C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1-\sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right|=
$$

$$
=\left|\int\limits_t^T\sum_{j_1=p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|=
$$

$$
=\frac{2}{T-t}\left|\int\limits_t^T\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^{t_1}\psi_1(s)\sin\frac{2\pi j_1(s-t)}{T-t}ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.
$$

$$
\left.\left.+\int\limits_t^{t_1}\psi_1(s)\cos\frac{2\pi j_1(s-t)}{T-t}ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\right)dt_1\right|=
$$

$$
=\frac{1}{\pi}\left|\int\limits_t^T\left(\psi_1(t)\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.
$$

$$
+\frac{T-t}{2\pi}\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\psi_1'(t_1)-\psi_1'(t)\cos\frac{2\pi j_1(t_1-t)}{T-t}-\right.
$$

$$
-\int\limits_t^{t_1}\sin\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}-
$$

$$
-\int\limits_{t}^{t_1}\cos\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\Biggr)\Biggr)dt_1\Biggr| \le
$$

$$
\le C_1\left|\int\limits_{t}^{T}\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right| + \frac{C_2}{p_1} =
$$

$$
= C_1\left|\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\int\limits_{t}^{T}\psi_2(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right| + \frac{C_2}{p_1}, \tag{2.27}
$$

where constants C_1, C_2 do not depend on p_1.

Here we used the fact that the functional series

$$
\sum_{j_1=1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t} \tag{2.28}
$$

converges uniformly at the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon > 0$ due to Dirichlet–Abel Theorem, and converges to zero at the points t and T. Moreover, the series (2.28) (with accuracy to a linear transformation) is the trigonometric Fourier series of the smooth function $K(t_1) = t_1 - t$, $t_1 \in [t, T]$. Thus, (2.28) converges at the any point $t_1 \in (t, T)$.

From (2.27) we obtain

$$
\left|\int\limits_{t}^{T}\sum_{j_1=p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_{t}^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right| \le
$$

$$
\le C_3\left|\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\psi_2(T)-\psi_2(t)-\int\limits_{t}^{T}\cos\frac{2\pi j_1(s-t)}{T-t}\psi_2'(s)ds\right)\right| + \frac{C_2}{p_1} \le \frac{K}{p_1} \ \to\ 0
$$

if $p_1 \to \infty$, where constants C_3, K do not depend on p_1.

Note that we obtained the estimate, which we will use further

$$
\left|\int\limits_{t}^{T}\sum_{j_1=p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_{t}^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right| \le \frac{K}{p_1}, \tag{2.29}
$$

where constant K does not depend on p_1.

Further steps are similar to the proof of (2.10) for the case of Legendre polynomials. Theorem 2.1 is proved.

2.1.2 Approach Based on Theorem 1.1 and Double Fourier–Legendre Series Summarized by Pringsheim Method

In Sect. 2.1.1 we considered the proof of Theorem 2.1 based on Theorem 1.1 and double integration by parts (this procedure leads to the requirement of double continuous differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$). In this section, we formulate and prove the analogue of Theorem 2.1 but under the weakened conditions: the functions $\psi_1(\tau)$, $\psi_2(\tau)$ only one time continuously differentiable at the interval $[t,T]$. At that we will use the double Fourier–Legendre series summarized by Pringsheim method.

Theorem 2.2 [13], [14], [26], [45]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$*. Moreover,* $\psi_1(s)$, $\psi_2(s)$ *are continuously differentiable functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \tag{2.30}$$

that converges in the mean-square sense is valid, where

$$C_{j_2j_1} = \int\limits_t^T \psi_2(s_2)\phi_{j_2}(s_2) \int\limits_t^{s_2} \psi_1(s_1)\phi_{j_1}(s_1) ds_1 ds_2 \tag{2.31}$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

Proof. Theorem 2.2 will be proved if we prove the equality (see the proof of Theorem 2.1)

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1},$$

where $C_{j_1j_1}$ is defined by the formula (1.8) for $k = 2$ and $j_1 = j_2$. At that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

Firstly, consider the sufficient conditions of convergence of double Fourier–Legendre series summarized by Pringsheim method.

Let $P_j(x)$ $(j = 0, 1, 2, \ldots)$ be the Legendre polynomial. Consider the function $f(x, y)$ defined for $(x, y) \in [-1, 1]^2$. Furthermore, consider the double Fourier–Legendre series summarized by Pringsheim method and corresponding to the function $f(x, y)$

$$\lim_{n,m\to\infty}\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{1}{2}\sqrt{(2j+1)(2i+1)}C^*_{ij}P_i(x)P_j(y) \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\sum_{i,j=0}^{\infty}\frac{1}{2}\sqrt{(2j+1)(2i+1)}C^*_{ij}P_i(x)P_j(y), \qquad (2.32)$$

where

$$C^*_{ij}=\frac{1}{2}\sqrt{(2j+1)(2i+1)}\int\limits_{[-1,1]^2} f(x,y)P_i(x)P_j(y)dxdy. \qquad (2.33)$$

Consider the generalization for the case of two variables [89] of the theorem on equiconvergence for the Fourier–Legendre series [88].

Proposition 2.1 [89]. *Let $f(x, y) \in L_2([-1, 1]^2)$ and the function*

$$f(x,y)\left(1-x^2\right)^{-1/4}\left(1-y^2\right)^{-1/4}$$

is integrable on $[-1, 1]^2$. Moreover, let

$$|f(x,y)-f(u,v)|\le G(y)|x-u|+H(x)|y-v|,$$

where $G(y), H(x)$ are bounded functions on $[-1, 1]^2$. Then for all $(x, y) \in (-1, 1)^2$ the following equality is satisfied

$$\lim_{n,m\to\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{1}{2}\sqrt{(2j+1)(2i+1)}C^*_{ij}P_i(x)P_j(y)-\right.$$

$$\left.-(1-x^2)^{-1/4}(1-y^2)^{-1/4}S_{nm}(\arccos x,\arccos y,F)\right)=0. \qquad (2.34)$$

At that, the convergence in (2.34) *is uniform on the rectangle*

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \textit{for any} \quad \varepsilon, \delta > 0,$$

$S_{nm}(\theta, \varphi, F)$ *is a partial sum of the double trigonometric Fourier series of the auxiliary function*

$$F(\theta, \varphi) = \sqrt{|\text{sin}\theta|}\sqrt{|\text{sin}\varphi|} f(\text{cos}\theta, \text{cos}\varphi), \quad \theta, \varphi \in [0, \pi],$$

and the Fourier coefficient C^*_{ij} *is defined by* (2.33).

Proposition 2.1 implies that the following equality

$$\lim_{n,m\to\infty} \left(\sum_{j=0}^{n} \sum_{i=0}^{m} \frac{1}{2}\sqrt{(2j+1)(2i+1)} C^*_{ij} P_i(x) P_j(y) - f(x,y) \right) = 0 \qquad (2.35)$$

is fulfilled for all $(x, y) \in (-1, 1)^2$, and convergence in (2.35) is uniform on the rectangle

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \text{for any} \quad \varepsilon, \delta > 0$$

if the corresponding conditions of convergence of the double trigonometric Fourier series of the auxiliary function

$$g(x,y) = f(x,y) \left(1 - x^2\right)^{1/4} \left(1 - y^2\right)^{1/4} \qquad (2.36)$$

are satisfied.

Note also that Proposition 2.1 does not imply any conclusions on the behavior of the double Fourier–Legendre series on the boundary of the square $[-1, 1]^2$.

For each $\delta > 0$ *let us call the exact upper edge of difference* $|f(\mathbf{t}') - f(\mathbf{t}'')|$ *in the set of all points* $\mathbf{t}'$, $\mathbf{t}''$ *which belong to the domain* D *as the module of continuity of the function* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *in the* k*-dimentional domain* D ($k \ge 1$) *if the distance between* $\mathbf{t}', \mathbf{t}''$ *satisfies the condition* $\rho(\mathbf{t}', \mathbf{t}'') < \delta$.

We will say that the function of k ($k \ge 1$) *variables* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *belongs to the Hölder class with the parameter* 1 ($f(\mathbf{t}) \in C^1(D)$) *in the domain* D *if the module of continuity of the function* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *in the domain* D *has the order* $O(\delta)$.

In 1967, Zhizhiashvili L.V. proved that the rectangular sums of multiple trigonometric Fourier series of the function of k variables in the hypercube $[t, T]^k$ converge uniformly to this function in the hypercube $[t, T]^k$ if the function belongs to $C^{\alpha}([t,T]^k)$, $\alpha>0$ (definition of the Hölder class with the parameter $\alpha>0$ can be found in the well known mathematical analysis tutorials [90]).

For example, the following statement is correct.

Proposition 2.2 [90]. *If the function $f(x_1,\ldots,x_n)$ is periodic with period 2π with respect to each variable and belongs in $\mathbf{R}^n$ to the Hölder class $C^{\alpha}(\mathbf{R}^n)$ for any $\alpha>0$, then the rectangular partial sums of multiple trigonometric Fourier series of the function $f(x_1,\ldots,x_n)$ converge to this function uniformly in $\mathbf{R}^n$.*

Let us back to the proof of Theorem 2.2 and consider the following Lemma.

Lemma 2.1. *Let the function $f(x,y)$ satisfies to the following condition*

$$|f(x,y)-f(x_1,y_1)|\le C_1|x-x_1|+C_2|y-y_1|,$$

where $C_1,C_2<\infty$ and (x,y), $(x_1,y_1)\in[-1,1]^2$. Then the following inequality is fulfilled

$$|g(x,y)-g(x_1,y_1)|\le K\rho^{1/4}, \tag{2.37}$$

where $g(x,y)$ in defined by (2.36),

$$\rho=\sqrt{(x-x_1)^2+(y-y_1)^2},$$

(x,y) and $(x_1,y_1)\in[-1,1]^2$, $K<\infty$.

Proof. First, we assume that $x\ne x_1$, $y\ne y_1$. In this case we have

$$|g(x,y)-g(x_1,y_1)|=$$

$$=\Bigl|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}\left(f(x,y)-f(x_1,y_1)\right)+$$

$$+f(x_1,y_1)\left(\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right)\Bigr|\le$$

$$\le C_1\left|x-x_1\right|+C_2\left|y-y_1\right|+$$

$$+C_3\left|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right|, \tag{2.38}$$

where $C_3<\infty$.

Moreover,

$$\left|\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\left(1-y_1^2\right)^{1/4}\right|=$$

$$=\Bigl|\left(1-x^2\right)^{1/4}\left(\left(1-y^2\right)^{1/4}-\left(1-y_1^2\right)^{1/4}\right)+$$

$$+\left(1-y_1^2\right)^{1/4}\left(\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right)\Bigr|\le$$

$$\le\left|\left(1-y^2\right)^{1/4}-\left(1-y_1^2\right)^{1/4}\right|+\left|\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right|, \qquad (2.39)$$

$$\left|\left(1-x^2\right)^{1/4}-\left(1-x_1^2\right)^{1/4}\right|=$$

$$=\Bigl|\left((1-x)^{1/4}-(1-x_1)^{1/4}\right)(1+x)^{1/4}+$$

$$+(1-x_1)^{1/4}\left((1+x)^{1/4}-(1+x_1)^{1/4}\right)\Bigr|\le$$

$$\le K_1\left(\left|(1-x)^{1/4}-(1-x_1)^{1/4}\right|+\left|(1+x)^{1/4}-(1+x_1)^{1/4}\right|\right), \qquad (2.40)$$

where $K_1<\infty$.

It is not difficult to see that

$$\left|(1\pm x)^{1/4}-(1\pm x_1)^{1/4}\right|=$$

$$=\frac{|(1\pm x)-(1\pm x_1)|}{\left((1\pm x)^{1/2}+(1\pm x_1)^{1/2}\right)\left((1\pm x)^{1/4}+(1\pm x_1)^{1/4}\right)}=$$

$$=|x_1-x|^{1/4}\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}}\cdot\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}}\le$$

$$\le|x_1-x|^{1/4}. \qquad (2.41)$$

The last inequality follows from the obvious inequalities

$$\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}}\le 1,$$

$$\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}}\le 1.$$

From (2.38)–(2.41) we obtain

$$|g(x,y)-g(x_1,y_1)|\le$$

$$\le C_1|x-x_1|+C_2|y-y_1|+C_4\left(|x_1-x|^{1/4}+|y_1-y|^{1/4}\right)\le$$

$$\le C_5\rho+C_6\rho^{1/4}\le K\rho^{1/4},$$

where $C_5,C_6,K<\infty$.

The cases $x=x_1,\ y\ne y_1$ and $x\ne x_1,\ y=y_1$ can be considered analogously to the case $x\ne x_1,\ y\ne y_1$. At that, the consideration begins from the inequalities

$$
|g(x,y)-g(x_1,y_1)|\le K_2\left|\left(1-y^2\right)^{1/4}f(x,y)-\left(1-y_1^2\right)^{1/4}f(x_1,y_1)\right|
$$

$(x=x_1,\ y\ne y_1)$ and

$$
|g(x,y)-g(x_1,y_1)|\le K_2\left|\left(1-x^2\right)^{1/4}f(x,y)-\left(1-x_1^2\right)^{1/4}f(x_1,y_1)\right|
$$

$(x\ne x_1,\ y=y_1)$, where $K_2<\infty$. Lemma 2.1 is proved.

Lemma 2.1 and Proposition 2.2 imply that rectangular sums of double trigonometric Fourier series of the function $g(x,y)$ converge uniformly to the function $g(x,y)$ in the square $[-1,1]^2$. This means that the equality (2.35) holds.

Consider the auxiliary function

$$
K'(t_1,t_2)=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases},\quad t_1,t_2\in[t,T]
$$

and prove that

$$
|K'(t_1,t_2)-K'(t_1^*,t_2^*)|\le L\left(|t_1-t_1^*|+|t_2-t_2^*|\right), \tag{2.42}
$$

where $L<\infty$ and $(t_1,t_2),\ (t_1^*,t_2^*)\in[t,T]^2$.

By the Lagrange formula for the functions $\psi_1(t_1^*),\ \psi_2(t_1^*)$ at the interval

$$
\left[\min\left\{t_1,t_1^*\right\},\max\left\{t_1,t_1^*\right\}\right]
$$

and for the functions $\psi_1(t_2^*),\ \psi_2(t_2^*)$ at the interval

$$
\left[\min\left\{t_2,t_2^*\right\},\max\left\{t_2,t_2^*\right\}\right]
$$

we obtain

$$
|K'(t_1,t_2)-K'(t_1^*,t_2^*)|\le
$$

$$
\le\left|\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}\right|+
$$

$$
+L_1\left|t_1-t_1^*\right|+L_2\left|t_2-t_2^*\right|, \quad L_1, L_2<\infty. \tag{2.43}
$$

We have

$$
\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases} - \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases} =
$$

$$
=\begin{cases}0, \quad t_1\ge t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\le t_2,\ t_1^*\le t_2^*\\ \\ \psi_2(t_1)\psi_1(t_2)-\psi_1(t_1)\psi_2(t_2), \quad t_1\ge t_2,\ t_1^*\le t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2)-\psi_2(t_1)\psi_1(t_2), \quad t_1\le t_2,\ t_1^*\ge t_2^*\end{cases}. \tag{2.44}
$$

By Lagrange formula for the functions $\psi_1(t_2)$, $\psi_2(t_2)$ at the interval

$$
[\min\{t_1,t_2\},\max\{t_1,t_2\}]
$$

we obtain the estimate

$$
\left|\ \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases} - \begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}\right| \le
$$

$$
\le L_3|t_2-t_1|\begin{cases}0, \quad t_1\ge t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\le t_2,\ t_1^*\le t_2^*\\ \\ 1, \quad t_1\le t_2,\ t_1^*\ge t_2^* \quad \text{or} \quad t_1\ge t_2,\ t_1^*\le t_2^*\end{cases}, \tag{2.45}
$$

where $L_3<\infty$.

Let us show that if $t_1\le t_2,\ t_1^*\ge t_2^*$ or $t_1\ge t_2,\ t_1^*\le t_2^*$, then the following inequality is satisfied

$$
|t_2-t_1|\le|t_1^*-t_1|+|t_2^*-t_2|. \tag{2.46}
$$

First, consider the case $t_1\ge t_2,\ t_1^*\le t_2^*$. For this case

$$
t_2+(t_1^*-t_2^*)\le t_2\le t_1.
$$

Then

$$
(t_1^*-t_1)-(t_2^*-t_2)\le t_2-t_1\le 0
$$

and (2.46) is satisfied.

For the case $t_1 \le t_2,\ t_1^* \ge t_2^*$ we obtain

$$t_1 + (t_2^* - t_1^*) \le t_1 \le t_2.$$

Then

$$(t_1 - t_1^*) - (t_2 - t_2^*) \le t_1 - t_2 \le 0$$

and also (2.46) is satisfied.

From (2.45) and (2.46) we have

$$\left| \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \ge t_2 \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1 \le t_2 \end{cases} - \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \ge t_2^* \\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^* \le t_2^* \end{cases} \right| \le$$

$$\le L_3 \left(|t_1^* - t_1| + |t_2^* - t_2|\right) \begin{cases} 0, & t_1 \ge t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \le t_2,\ t_1^* \le t_2^* \\ \\ 1, & t_1 \le t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \ge t_2,\ t_1^* \le t_2^* \end{cases} \le$$

$$\le L_3 \left(|t_1^* - t_1| + |t_2^* - t_2|\right) \begin{cases} 1, & t_1 \ge t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \le t_2,\ t_1^* \le t_2^* \\ \\ 1, & t_1 \le t_2,\ t_1^* \ge t_2^* \quad \text{or} \quad t_1 \ge t_2,\ t_1^* \le t_2^* \end{cases} =$$

$$= L_3 \left(|t_1^* - t_1| + |t_2^* - t_2|\right). \tag{2.47}$$

From (2.43), (2.47) we obtain (2.42). Let

$$t_1 = \frac{T-t}{2}x + \frac{T+t}{2}, \quad t_2 = \frac{T-t}{2}y + \frac{T+t}{2},$$

where $x, y \in [-1, 1]$. Then

$$K'(t_1, t_2) \equiv K''(x, y) = \begin{cases} \psi_2\left(h(x)\right)\psi_1\left(h(y)\right), & x \ge y \\ \\ \psi_1\left(h(x)\right)\psi_2\left(h(y)\right), & x \le y \end{cases},$$

where $x, y \in [-1, 1]$ and

$$h(x) = \frac{T-t}{2}x + \frac{T+t}{2}. \tag{2.48}$$

The inequality (2.42) can be rewritten in the form

$$|K''(x,y)-K''(x^*,y^*)|\le L^*\left(|x-x^*|+|y-y^*|\right), \tag{2.49}$$

where $L^*<\infty$ and (x,y), $(x^*,y^*)\in[-1,1]^2$.

Thus, the function $K''(x,y)$ satisfies the conditions of Lemma 2.1. Hence, for the function

$$K''(x,y)\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4}$$

the inequality (2.37) is correct.

Due to the continuous differentiability of the functions $\psi_1(h(x))$ and $\psi_2(h(x))$ at the interval $[-1,1]$ we have $K''(x,y)\in L_2([-1,1]^2)$. In addition

$$\int\limits_{[-1,1]^2}\frac{K''(x,y)dxdy}{(1-x^2)^{1/4}(1-y^2)^{1/4}}\le C\left(\int\limits_{-1}^{1}\frac{1}{(1-x^2)^{1/4}}\int\limits_{-1}^{x}\frac{1}{(1-y^2)^{1/4}}dydx+\right.$$

$$\left.+\int\limits_{-1}^{1}\frac{1}{(1-x^2)^{1/4}}\int\limits_{x}^{1}\frac{1}{(1-y^2)^{1/4}}dydx\right)<\infty,\quad C<\infty.$$

Thus, the conditions of Proposition 2.1 are fulfilled for the function $K''(x,y)$. Note that the mentioned properties of the function $K''(x,y)$, $x,y\in[-1,1]$ also correct for the function $K'(t_1,t_2)$, $t_1,t_2\in[t,T]$.

Let us expand the function $K'(t_1,t_2)$ into a multiple (double) Fourier–Legendre series in the square $[t,T]^2$. This series is summable by the method of rectangular sums (Pringsheim method), i.e.

$$K'(t_1,t_2)=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T\psi_1(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^T\psi_2(t_1)\phi_{j_1}(t_1)dt_1\right)dt_2\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\left(C_{j_2j_1}+C_{j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{2.50}$$

where $(t_1, t_2) \in (t, T)^2$. At that, the convergence of the series (2.50) is uniform on the rectangle

$[t+\varepsilon, T-\varepsilon] \times [t+\delta, T-\delta]$ for any $\varepsilon, \delta > 0$ (in particular, we can choose $\varepsilon = \delta$).

In addition, the series (2.50) converges to $K'(t_1, t_2)$ at any inner point of the square $[t, T]^2$.

Note that Proposition 2.1 does not answer the question of convergence of the series (2.50) on the boundary of the square $[t, T]^2$.

In obtaining (2.50) we replaced the order of integration in the second iterated integral.

Let us substitute $t_1 = t_2$ in (2.50). After that, let us rewrite the limit on the right-hand side of (2.50) as two limits. Let us replace j_1 with j_2, j_2 with j_1, n_1 with n_2, and n_2 with n_1 in the second limit. Thus, we get

$$\lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} C_{j_2 j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1), \quad t_1 \in (t, T). \tag{2.51}$$

According to the above reasoning, the convergence in (2.51) is uniform on the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon > 0$. Additionally, (2.51) holds at each interior point of the interval $[t, T]$.

Lemma 2.2. *Under the conditions of Theorem* 2.2 *the following limit*

$$\lim_{n\to\infty} \sum_{j_1=0}^{n} C_{j_1 j_1}$$

exists, where $C_{j_1 j_1}$ *is defined by* (2.31) *if* $j_1 = j_2$, *i.e.*

$$C_{j_1 j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2.$$

The proof of Lemma 2.2 will be given further in this section.

Let us fix $\varepsilon > 0$ and integrate the equality (2.51) at the interval $[t+\varepsilon, T-\varepsilon]$. Due to the uniform convergence of the series (2.51) we can swap the series and the integral

$$\lim_{n_1,n_2\to\infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} C_{j_2 j_1} \int\limits_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1) dt_1 = \frac{1}{2} \int\limits_{t+\varepsilon}^{T-\varepsilon} \psi_1(t_1)\psi_2(t_1) dt_1. \tag{2.52}$$

Taking into account Lemma 2.2, consider the equality (2.52) for $n_1 = n_2 = n$

$$\frac{1}{2}\int\limits_{t+\varepsilon}^{T-\varepsilon}\psi_1(t_1)\psi_2(t_1)dt_1 = \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$= \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\left(\int\limits_{t}^{T}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \int\limits_{t}^{t+\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \right.$$

$$\left. - \int\limits_{T-\varepsilon}^{T}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1\right) =$$

$$= \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\left(\mathbf{1}_{\{j_1=j_2\}} - \Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)\varepsilon\right) =$$

$$= \sum_{j_1=0}^{\infty} C_{j_1j_1} - \varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr), \qquad (2.53)$$

where $\theta \in [t, t+\varepsilon]$, $\lambda \in [T-\varepsilon, T]$. In obtaining (2.53) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j = 0, 1, 2\ldots$

Performing the passage to the limit $\lim\limits_{\varepsilon\to+0}$ in the equality (2.53), we get

$$\frac{1}{2}\int\limits_{t}^{T}\psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}.$$

Thus, to complete the proof of Theorem 2.2, it is necessary to prove Lemma 2.2.

Remark 2.1. *On the basis of* (2.42) *it can be argued that the function* $K'(t_1, t_2)$ *belongs to the Hölder class with parameter* 1 *in* $[t, T]^2$. *Hence by Proposition* 2.2 *this function can be expanded into the uniformly convergent double trigonometric Fourier series on the square* $[t, T]^2$, *which summarized by Pringsheim method. This means that Theorem* 2.2 *will remain valid if in this theorem instead of the double Fourier–Legendre series we consider the double trigonometric Fourier series. However, the expansions of iterated stochastic integrals obtained by using the system of Legendre polynomials are essentially simpler* *than their analogues obtained by using the trigonometric system of functions (see Chapter* 5).

For proof of Lemma 2.2 as well as for further consideration we will need some well known properties of Legendre polynomials [87], [88].

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \tag{2.54}$$

where $P_j(x)$ is the Legendre polynomial.

It is known that the Legendre polynomial $P_j(x)$ is represented as

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

At the boundary points of the orthogonality interval the Legendre polynomials satisfy the following relations

$$P_j(1) = 1, \quad P_j(-1) = (-1)^j,$$

where $j = 0, 1, 2, \ldots$

Relation of the Legendre polynomial $P_j(x)$ with derivatives of the Legendre polynomials $P_{j+1}(x)$ and $P_{j-1}(x)$ is expressed by the following equality

$$P_j(x) = \frac{1}{2j+1}\left(P'_{j+1}(x) - P'_{j-1}(x)\right), \quad j = 1, 2, \ldots$$

The recurrent relation has the form

$$xP_j(x) = \frac{(j+1)P_{j+1}(x) + jP_{j-1}(x)}{2j+1}, \quad j = 1, 2, \ldots$$

Orthogonality of Legendre polynomial $P_j(x)$ to any polynomial $Q_k(x)$ of lesser degree k we write in the following form

$$\int\limits_{-1}^{1} Q_k(x)P_j(x)dx = 0, \quad k = 0, 1, 2, \ldots, j-1.$$

From the property

$$\int\limits_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if } k \neq j \\ \\ 2/(2j+1) & \text{if } k = j \end{cases}$$

it follows that the orthonormal on the interval $[-1,1]$ Legendre polynomials determined by the relation

$$P_j^*(x) = \sqrt{\frac{2j+1}{2}}P_j(x), \quad j = 0, 1, 2, \ldots$$

Remind that there is the following estimate [87]

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1,1), \quad j = 1, 2, \ldots, \tag{2.55}$$

where constant K does not depend on y and j.

Moreover,

$$|P_j(x)| \le 1, \quad x \in [-1,1], \quad j = 0,\ 1, \ldots \tag{2.56}$$

The Christoffel–Darboux formula has the form

$$\sum_{j=0}^{n}(2j+1)P_j(x)P_j(y) = (n+1)\frac{P_n(x)P_{n+1}(y) - P_{n+1}(x)P_n(y)}{y-x}. \tag{2.57}$$

To complete the proof of Theorem 2.2 let us prove Lemma 2.2.

Fix $n > m$ $(n, m \in \mathbf{N})$. We have

$$\sum_{j_1=m+1}^{n} C_{j_1 j_1} = \sum_{j_1=m+1}^{n} \int\limits_t^T \psi_2(s)\phi_{j_1}(s) \int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau ds =$$

$$= \frac{T-t}{4} \sum_{j_1=m+1}^{n} (2j_1+1) \int\limits_{-1}^{1} \psi_2(h(x))P_{j_1}(x) \int\limits_{-1}^{x} \psi_1(h(y))P_{j_1}(y)dydx =$$

$$
=\frac{T-t}{4}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-
$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_2(h(x))P_{j_1}(x)\int\limits_{-1}^{x}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))dydx=
$$

$$
=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\sum_{j_1=m+1}^{n}\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-
$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\int\limits_{y}^{1}P_{j_1}(x)\psi_2(h(x))dxdy=
$$

$$
=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx+
$$

$$
+\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\frac{1}{2j_1+1}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\times
$$

$$
\times\Biggl(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_2(h(y))+
$$

$$
+\frac{T-t}{2}\int\limits_{y}^{1}\left(P_{j_1+1}(x)-P_{j_1-1}(x)\right)\psi_2'(h(x))dx\Biggr)dy, \tag{2.58}
$$

where ψ_1', ψ_2' are derivatives of the functions $\psi_1(s)$, $\psi_2(s)$ with respect to the variable $h(y)$ (see (2.48)).

Applying the estimate (2.55) and taking into account the boundedness of the functions $\psi_1(s)$, $\psi_2(s)$ and their derivatives, we finally obtain

$$
\left|\sum_{j_1=m+1}^{n}C_{j_1j_1}\right|\le C_1\left(\frac{1}{n}+\frac{1}{m}\right)\int\limits_{-1}^{1}\frac{dx}{\left(1-x^2\right)^{1/2}}+
$$

$$
+C_2 \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \left(\int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^{1} \frac{1}{(1-y^2)^{1/4}} \int\limits_{y}^{1} \frac{dx}{(1-x^2)^{1/4}} dy \right) \le
$$

$$
\le C_3 \left(\frac{1}{n} + \frac{1}{m} + \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \right) \to 0 \qquad (2.59)
$$

if $n, m \to \infty$, where constants C_1, C_2, C_3 do not depend on n and m.

The relation (2.59) completes the proof of Lemma 2.2. Theorem 2.2 is proved.

2.1.3 Approach Based on Generalized Double Multiple and Iterated Fourier Series

This section is devoted to the proof of Theorem 2.2 but by a simpler method [13], [14], [35] than in Sect. 2.1.1 and 2.1.2. We will consider two different parts of the expansion of iterated Stratonovich stochastic integrals of second multiplicity. The mean-square convergence of the first part will be proved on the base of generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^2)$. The mean-square convergence of the second part will be proved on the base of generalized iterated (double) Fourier series converging pointwise. At that, we will prove the iterated limit transition for the second part on the base of the classical theorems of mathematical analysis.

Thus, let us prove Theorem 2.2 by a simpler method than in Sect. 2.1.1 and 2.1.2.

Proof. Let us consider Lemma 1.1, definition of the multiple stochastic integral (1.16) together with the formula (1.18) when the function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary as well as Lemma 1.3 for the case $k=2$ (see Sect. 1.1.3).

In accordance to the standard relation between Stratonovich and Itô stochastic integrals (see (2.8)) we have w. p. 1

$$
J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_{t}^{T} \psi_1(t_1)\psi_2(t_1)dt_1. \qquad (2.60)
$$

Let us consider the function $K^*(t_1,t_2)$ defined by (2.11)

$$K^*(t_1,t_2)=\psi_1(t_1)\psi_2(t_2)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)=$$

$$=K(t_1,t_2)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_2), \tag{2.61}$$

where

$$K(t_1,t_2)=\begin{cases}\psi_1(t_1)\psi_2(t_2), & t_1<t_2\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,t_2\in[t,T].$$

Lemma 2.3. *In the conditions of Theorem* 2.2 *the following relation*

$$J[K^*]_{T,t}^{(2)}=J^*[\psi^{(2)}]_{T,t} \tag{2.62}$$

is valid w. p. 1, *where* $J[K^*]_{T,t}^{(2)}$ *is defined by the equality* (1.16).

Proof. Substituting (2.61) into (1.16) (the case $k=2$) and using Lemma 1.1 together with (1.18) (the case $k=2$) it is easy to see that w. p. 1

$$J[K^*]_{T,t}^{(2)}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=J^*[\psi^{(2)}]_{T,t}. \tag{2.63}$$

Let us consider the following generalized double Fourier sum

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),$$

where $C_{j_2j_1}$ is the Fourier coefficient defined as follows

$$C_{j_2j_1}=\int\limits_{[t,T]^2}K^*(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2. \tag{2.64}$$

Further, subsitute the relation

$$K^*(t_1,t_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)+K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)$$

into $J[K^*]^{(2)}_{T,t}$. At that we suppose that $p_1, p_2 < \infty$.

Then using Lemma 1.3 (the case $k=2$), we obtain

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + J[R_{p_1p_2}]^{(2)}_{T,t} \quad \text{w. p. 1}, \tag{2.65}$$

where the stochastic integral $J[R_{p_1p_2}]^{(2)}_{T,t}$ is defined in accordance with (1.16) and

$$R_{p_1p_2}(t_1,t_2) = K^*(t_1,t_2) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{2.66}$$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)},$$

$$J[R_{p_1p_2}]^{(2)}_{T,t} = \int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1.$$

Using standard moment properties of stochastic integrals [83] (see (1.25), (1.26)), we get

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]^{(2)}_{T,t}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)}\right)^2\right\} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2 \le$$

$$\le 2\left(\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2 + \int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_2dt_1\right) +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2 =$$

$$=2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \qquad (2.67)$$

We have

$$\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2 dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2 dt_1dt_2.$$

The function $K(t_1,t_2)$ is piecewise continuous in the square $[t,T]^2$. At this situation it is well known that the generalized multiple Fourier series of the function $K(t_1,t_2)\in L_2([t,T]^2)$ is converging to this function in the square $[t,T]^2$ in the mean-square sense, i.e.

$$\lim_{p_1,p_2\to\infty}\left\|K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^2)}=0,$$

where

$$\|f\|_{L_2([t,T]^2)}=\left(\int\limits_{[t,T]^2}f^2(t_1,t_2)dt_1dt_2\right)^{1/2}.$$

So, we obtain

$$\lim_{p_1,p_2\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2=0. \qquad (2.68)$$

Note that

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\psi_2(t_1)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)dt_1=$$

$$
=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}. \tag{2.69}
$$

From (2.69) we get

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1-\lim_{p_1\to\infty}\sum_{j_1=0}^{p_1}C_{j_1j_1}=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\infty}C_{j_1j_1}=
$$

$$
=\lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1, \tag{2.70}
$$

where $\overline{\lim}$ means lim sup.

If we prove the following relation

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=0, \tag{2.71}
$$

then from (2.70) we obtain

$$
\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1}, \tag{2.72}
$$

$$\lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=0. \qquad (2.73)$$

From (2.67), (2.68), and (2.73) we get

$$\lim_{p_1,p_2\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\}=0$$

and Theorem 2.2 will be proved.

Let us prove (2.71). From (2.66) and (2.15) we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}R_{p_1p_2}(t_1,t_1)=0,\quad t_1\in(t,T). \qquad (2.74)$$

Let $\varepsilon>0$ be a sufficiently small positive number. For fixed p_1,p_2 we have

$$0\le\left|\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right|=$$

$$=\left|\int\limits_t^{t+\varepsilon}R_{p_1p_2}(t_1,t_1)dt_1+\int\limits_{t+\varepsilon}^{T-\varepsilon}R_{p_1p_2}(t_1,t_1)dt_1+\int\limits_{T-\varepsilon}^{T}R_{p_1p_2}(t_1,t_1)dt_1\right|\le$$

$$\le\int\limits_t^{t+\varepsilon}\left|R_{p_1p_2}(t_1,t_1)\right|dt_1+\int\limits_{t+\varepsilon}^{T-\varepsilon}\left|R_{p_1p_2}(t_1,t_1)\right|dt_1+\int\limits_{T-\varepsilon}^{T}\left|R_{p_1p_2}(t_1,t_1)\right|\le$$

$$\le\int\limits_{t+\varepsilon}^{T-\varepsilon}\left|R_{p_1p_2}(t_1,t_1)\right|dt_1+2C\varepsilon, \qquad (2.75)$$

where $|R_{p_1p_2}(t_1,t_1)|\le C$ and C is a constant.

Let us consider the partition $\{\tau_j\}_{j=0}^N$ $(N>1)$ of the interval $[t+\varepsilon,T-\varepsilon]$ such that

$$\tau_0=t+\varepsilon,\quad \tau_N=T-\varepsilon,\quad \tau_j=\tau_0+j\Delta,\quad \Delta=\frac{1}{N}(T-t-2\varepsilon).$$

We have

$$\int\limits_{t+\varepsilon}^{T-\varepsilon}\left|R_{p_1p_2}(t_1,t_1)\right|dt_1\le\sum_{i=0}^{N-1}\max_{s\in[\tau_i,\tau_{i+1}]}\left|R_{p_1p_2}(s,s)\right|\Delta=$$

$$
= \sum_{i=0}^{N-1} \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \Delta + \sum_{i=0}^{N-1} \left(\max_{s\in[\tau_i,\tau_{i+1}]} \left| R_{p_1p_2}(s,s) \right| - \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \right) \Delta \le
$$

$$
\le \sum_{i=0}^{N-1} \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \Delta + \sum_{i=0}^{N-1} \left| \max_{s\in[\tau_i,\tau_{i+1}]} \left| R_{p_1p_2}(s,s) \right| - \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \right| \Delta =
$$

$$
= \sum_{i=0}^{N-1} \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \Delta + \sum_{i=0}^{N-1} \left| \left| R_{p_1p_2}(t_i^{(p_1p_2)},t_i^{(p_1p_2)}) \right| - \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \right| \Delta <
$$

$$
< \sum_{i=0}^{N-1} \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \Delta + \varepsilon_1 (T-t-2\varepsilon), \qquad (2.76)
$$

where $(t_i^{(p_1p_2)},t_i^{(p_1p_2)})$ is a point of maximum of the function $|R_{p_1p_2}(s,s)|$ at the interval $[\tau_i,\tau_{i+1}]$.

Getting (2.75) and (2.76), we used the well known properties of integrals, the first and the second Weierstrass Theorems for the function of one variable as well as continuity (which means uniform continuity) of the function $|R_{p_1p_2}(s,s)|$ at the interval $[t+\varepsilon,T-\varepsilon]$, i.e. $\forall \varepsilon_1>0 \ \exists \delta(\varepsilon_1)>0$, which does not depend on t_1, p_1, p_2 and if $\Delta<\delta(\varepsilon_1)$, then the following inequality takes place

$$
\left| \left| R_{p_1p_2}(t_i^{(p_1p_2)},t_i^{(p_1p_2)}) \right| - \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \right| < \varepsilon_1.
$$

From (2.75) and (2.76) we obtain

$$
0 \le \left| \int\limits_t^T R_{p_1p_2}(t_1,t_1) dt_1 \right| < \sum_{i=0}^{N-1} \left| R_{p_1p_2}(\tau_i,\tau_i) \right| \Delta + \varepsilon_1 (T-t-2\varepsilon) + 2C\varepsilon. \qquad (2.77)
$$

Let us implement the iterated passage to the limit $\lim\limits_{\varepsilon\to+0} \lim\limits_{p_1\to\infty} \overline{\lim\limits_{p_2\to\infty}}$ in the inequality (2.77) taking into account (2.69), (2.70), and (2.74). So, we get

$$
0 \le \lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \left| \int\limits_t^T R_{p_1p_2}(t_1,t_1) dt_1 \right| = \lim_{p_1\to\infty} \lim_{p_2\to\infty} \left| \int\limits_t^T R_{p_1p_2}(t_1,t_1) dt_1 \right| \le \varepsilon_1 (T-t).
$$

(2.78)

Then from (2.78) (according to arbitrariness of $\varepsilon_1 > 0$) we have (2.71). Theorem 2.2 is proved.

Note that (2.71) can be obtained by a more simple way.

Since the integral

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1$$

exists as Riemann integral, then it is equal to the corresponding Lebesgue integral. Moreover, the following equality

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty} R_{p_1p_2}(t_1,t_1)=0 \quad \hbox{when} \quad t_1\in [t, T]$$

holds with accuracy up to sets of measure zero.

According to (2.66), we have

$$R_{p_1p_2}(t_1,t_2)=\left(K^{*}(t_1,t_2)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2)-\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right), \qquad (2.79)$$

where the Fourier coefficient $C_{j_1}(t_2)$ is defined by (2.13).

Applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem and taking into account (2.12), (2.14), and (2.79), we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=0.$$

2.2 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3 Based on Theorem 1.1

This section is devoted to the development of the method of expansion and mean-square approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series converging in the mean (Theorem 1.1). We adapt this method for the iterated Stratonovich stochastic integrals of multiplicity 3. The main results of this section have been derived with using triple Fourier–Legendre series as well as triple trigonometric Fourier series for different cases of series summation and different cases of weight functions of iterated Stratonovich stochastic integrals.

2.2.1 The Case $p_1, p_2, p_3 \to \infty$ and Constant Weight Functions (The Case of Legendre Polynomials)

Theorem 2.3 [6]-[14], [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials in the space* $L_2([t, T])$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (2.80)$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(s)\int\limits_t^s \phi_{j_2}(s_1)\int\limits_t^{s_1} \phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

Proof. If we prove w. p. 1 the following equalities

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right), \quad (2.81)$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \quad (2.82)$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0, \quad (2.83)$$

then in accordance with the formulas (2.81)–(2.83), Theorem 1.1 (see (1.43)), standard relations between iterated Itô and Stratonovich stochastic integrals as well as in accordance with the formulas (they also follow from Theorem 1.1)

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right) \quad \text{w. p. 1},$$

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right) \quad \text{w. p. 1}$$

we will have

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau \quad \text{w. p. 1}.$$

It means that the expansion (2.80) will be proved.

Let us at first prove that

$$\sum_{j_1=0}^{\infty} C_{0 j_1 j_1} = \frac{1}{4}(T-t)^{3/2}, \tag{2.84}$$

$$\sum_{j_1=0}^{\infty} C_{1 j_1 j_1} = \frac{1}{4\sqrt{3}}(T-t)^{3/2}. \tag{2.85}$$

We have

$$C_{000} = \frac{(T-t)^{3/2}}{6},$$

$$C_{0 j_1 j_1} = \int\limits_t^T \phi_0(s)\int\limits_t^s \phi_{j_1}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2 ds_1 ds =$$

$$= \frac{1}{2}\int\limits_t^T \phi_0(s)\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 ds, \quad j_1 \ge 1, \tag{2.86}$$

where $\phi_j(s)$ looks as follows

$$\phi_j(s) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j \ge 0, \tag{2.87}$$

where $P_j(x)$ is the Legendre polynomial.

Let us substitute (2.87) into (2.86) and calculate $C_{0j_1j_1}$ $(j_1 \ge 1)$

$$C_{0j_1j_1} = \frac{2j_1+1}{2(T-t)^{3/2}} \int\limits_t^T \left(\int\limits_{-1}^{z(s)} P_{j_1}(y) \frac{T-t}{2} dy \right)^2 ds =$$

$$= \frac{(2j_1+1)\sqrt{T-t}}{8} \int\limits_t^T \left(\int\limits_{-1}^{z(s)} \frac{1}{2j_1+1} \left(P_{j_1+1}^{'}(y) - P_{j_1-1}^{'}(y) \right) dy \right)^2 ds =$$

$$= \frac{\sqrt{T-t}}{8(2j_1+1)} \int\limits_t^T \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right)^2 ds, \tag{2.88}$$

where here and further

$$z(s) = \left(s - \frac{T+t}{2} \right) \frac{2}{T-t}.$$

In (2.88) we used the following well known properties of the Legendre polynomials

$$P_j(y) = \frac{1}{2j+1} \left(P_{j+1}^{'}(y) - P_{j-1}^{'}(y) \right), \quad P_j(-1) = (-1)^j, \quad j \ge 1.$$

Also, we denote

$$\frac{dP_j}{dy}(y) \stackrel{\text{def}}{=} P_j^{'}(y).$$

From (2.88) using the property of orthogonality of the Legendre polynomials, we get the following relation

$$C_{0j_1j_1} = \frac{(T-t)^{3/2}}{16(2j_1+1)} \int\limits_{-1}^{1} \left(P_{j_1+1}^2(y) + P_{j_1-1}^2(y) \right) dy =$$

$$= \frac{(T-t)^{3/2}}{8(2j_1+1)} \left(\frac{1}{2j_1+3} + \frac{1}{2j_1-1} \right),$$

where we used the property

$$\int\limits_{-1}^{1} P_j^2(y) dy = \frac{2}{2j+1}, \quad j \ge 0.$$

Then

$$\sum_{j_1=0}^{\infty} C_{0j_1j_1} = \frac{(T-t)^{3/2}}{6} +$$

$$+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{(2j_1+1)(2j_1+3)} + \sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1} - \frac{1}{3} + \sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right) = \frac{(T-t)^{3/2}}{4}.$$

The relation (2.84) is proved.

Let us check the correctness of (2.85). Let us represent $C_{1j_1j_1}$ in the form

$$C_{1j_1j_1} = \frac{1}{2}\int\limits_t^T \phi_1(s)\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 ds =$$

$$= \frac{(T-t)^{3/2}(2j_1+1)\sqrt{3}}{16}\int\limits_{-1}^{1} P_1(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1 \ge 1.$$

Since the functions

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2, \quad j_1 \ge 1$$

are even, then the functions

$$P_1(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1 \ge 1$$

are uneven. It means that $C_{1j_1j_1} = 0$ $(j_1 \ge 1)$. From the other side

$$C_{100} = \frac{\sqrt{3}(T-t)^{3/2}}{16}\int\limits_{-1}^{1} y(y+1)^2 dy = \frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_1=0}^{\infty} C_{1j_1j_1} = C_{100} + \sum_{j_1=1}^{\infty} C_{1j_1j_1} = \frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (2.85) is proved.

Let us prove the equality (2.81). Using (2.85), we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1}\sum_{j_3=2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1}\sum_{j_3=2, j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.89)$$

Since

$$C_{j_3j_1j_1} = \frac{(T-t)^{3/2}(2j_1+1)\sqrt{2j_3+1}}{16}\int_{-1}^{1} P_{j_3}(y)\left(\int_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy$$

and degree of the polynomial

$$\left(\int_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

equals to $2j_1+2$, then $C_{j_3j_1j_1}=0$ for $j_3 > 2j_1+2$. It explains that we put $2j_1+2$ instead of p_3 on the right-hand side of the formula (2.89).

Moreover, the function

$$\left(\int_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_3}(y)\left(\int_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is uneven for uneven j_3. It means that $C_{j_3j_1j_1}=0$ for uneven j_3. That is why we summarize using even j_3 on the right-hand side of the formula (2.89).

Then we have

$$\sum_{j_1=0}^{p_1}\ \sum_{j_3=2,j_3-\mathrm{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\ \sum_{j_1=(j_3-2)/2}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=$$

$$=\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.90)$$

We replaced $(j_3-2)/2$ by zero on the right-hand side of the formula (2.90), since $C_{j_3j_1j_1}=0$ for $0\le j_1<(j_3-2)/2$.

Let us substitute (2.90) into (2.89)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)}+\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)}+$$

$$+\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.91)$$

It is easy to see that the right-hand side of the formula (2.91) does not depend on p_3.

If we prove that

$$\lim_{p_1\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\right)^2\right\}=0, \qquad (2.92)$$

then the relaion (2.81) will be proved.

Using (2.91) and (2.84), we can rewrite the left-hand side of (2.92) in the following form

$$\lim_{p_1\to\infty}\mathsf{M}\left\{\left(\left(\sum_{j_1=0}^{p_1} C_{0j_1j_1}-\frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_3)}+\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1} C_{0j_1j_1}-\frac{(T-t)^{3/2}}{4}\right)^2+\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\mathrm{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2. \qquad (2.93)$$

If we prove that

$$\lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=0, \tag{2.94}$$

then the relation (2.81) will be proved.

We have

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\left((s-t)-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2\right)ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2\le$$

$$\le\frac{1}{4}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2. \tag{2.95}$$

Obtaining (2.95), we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2=\int\limits_t^T\left(\mathbf{1}_{\{s_1<s\}}\right)^2ds_1=s-t \tag{2.96}$$

and the orthogonality property of the Legendre polynomials

$$\int\limits_t^T\phi_{j_3}(s)(s-t)ds=0,\quad j_3\ge 2. \tag{2.97}$$

Then we have

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 = \frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)dy\right)^2 =
$$

$$
= \frac{T-t}{4(2j_1+1)}\left(\int\limits_{-1}^{z(s)} \left(P_{j_1+1}^{'}(y) - P_{j_1-1}^{'}(y)\right)dy\right)^2 =
$$

$$
= \frac{T-t}{4(2j_1+1)}\left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s))\right)^2 \le
$$

$$
\le \frac{T-t}{2(2j_1+1)}\left(P_{j_1+1}^2(z(s)) + P_{j_1-1}^2(z(s))\right). \tag{2.98}
$$

Remind that for the Legendre polynomials the following estimate is correct

$$
|P_n(y)| < \frac{K}{\sqrt{n+1}(1-y^2)^{1/4}}, \quad y \in (-1,1), \quad n \in \mathbf{N}, \tag{2.99}
$$

where constant K does not depend on y and n.

The estimate (2.99) can be rewritten for the function $\phi_n(s)$ in the following form

$$
|\phi_n(s)| < \sqrt{\frac{2n+1}{n+1}}\frac{K}{\sqrt{T-t}}\frac{1}{\left(1-z^2(s)\right)^{1/4}} <
$$

$$
< \frac{K_1}{\sqrt{T-t}}\frac{1}{\left(1-z^2(s)\right)^{1/4}}, \tag{2.100}
$$

where $K_1 = K\sqrt{2}$, $s \in (t,T)$.

Let us estimate the right-hand side of (2.98) using the estimate (2.99)

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 < \frac{T-t}{2(2j_1+1)}\left(\frac{K^2}{j_1+2} + \frac{K^2}{j_1}\right)\frac{1}{(1-(z(s))^2)^{1/2}} <
$$

$$
< \frac{(T-t)K^2}{2j_1^2}\frac{1}{(1-(z(s))^2)^{1/2}}, \tag{2.101}
$$

where $s \in (t,T)$.

Substituting the estimate (2.101) into the relation (2.95) and using in (2.95) the estimate (2.100) for $|\phi_{j_3}(s)|$, we obtain

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} \right)^2 <$$

$$< \frac{(T-t)K^4 K_1^2}{16} \sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\int\limits_t^T \frac{ds}{\left(1-(z(s))^2\right)^{3/4}} \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \right)^2 =$$

$$= \frac{(T-t)^3 K^4 K_1^2 (p_1+1)}{64} \left(\int\limits_{-1}^{1} \frac{dy}{\left(1-y^2\right)^{3/4}} \right)^2 \left(\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \right)^2 . \qquad (2.102)$$

Since

$$\int\limits_{-1}^{1} \frac{dy}{\left(1-y^2\right)^{3/4}} < \infty \qquad (2.103)$$

and

$$\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \le \int\limits_{p_1}^{\infty} \frac{dx}{x^2} = \frac{1}{p_1}, \qquad (2.104)$$

then from (2.102) we find

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} \right)^2 < \frac{C(T-t)^3 (p_1+1)}{p_1^2} \quad \to 0 \quad \text{if} \quad p_1 \to \infty, \qquad (2.105)$$

where constant C does not depend on p_1 and $T-t$. The relation (2.105) implies (2.94), and the relation (2.94) implies the correctness of the formula (2.81).

Let us prove the equaity (2.82). Let us at first prove that

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = \frac{1}{4}(T-t)^{3/2}, \qquad (2.106)$$

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 j_1} = -\frac{1}{4\sqrt{3}}(T-t)^{3/2}. \qquad (2.107)$$

We have

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = C_{000} + \sum_{j_3=1}^{\infty} C_{j_3 j_3 0},$$

$$C_{000} = \frac{(T-t)^{3/2}}{6},$$

$$C_{j_3 j_3 0} = \frac{(T-t)^{3/2}}{16(2j_3+1)} \int\limits_{-1}^{1} \left(P_{j_3+1}^2(y) + P_{j_3-1}^2(y)\right) dy =$$

$$= \frac{(T-t)^{3/2}}{8(2j_3+1)} \left(\frac{1}{2j_3+3} + \frac{1}{2j_3-1}\right), \quad j_3 \ge 1.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = \frac{(T-t)^{3/2}}{6} +$$

$$+ \frac{(T-t)^{3/2}}{8} \left(\sum_{j_3=1}^{\infty} \frac{1}{(2j_3+1)(2j_3+3)} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8} \left(\sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1} - \frac{1}{3} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right) = \frac{(T-t)^{3/2}}{4}.$$

The relation (2.106) is proved. Let us check the equality (2.107). We have

$$C_{j_3 j_3 j_1} = \int\limits_t^T \phi_{j_3}(s) \int\limits_t^s \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds =$$

$$= \int\limits_t^T \phi_{j_1}(s_2) ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \int\limits_{s_1}^T \phi_{j_3}(s) ds =$$

$$= \frac{1}{2} \int\limits_t^T \phi_{j_1}(s_2) \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1\right)^2 ds_2 =$$

$$= \frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16} \int\limits_{-1}^{1} P_{j_1}(y) \left(\int\limits_y^1 P_{j_3}(y_1) dy_1\right)^2 dy, \quad j_3 \ge 1. \tag{2.108}$$

Since the functions

$$\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2, \quad j_3 \ge 1$$

are even, then the functions

$$P_1(y)\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2 dy, \quad j_3 \ge 1$$

are uneven. It means that $C_{j_3j_31} = 0$ $(j_3 \ge 1)$.

Moreover,

$$C_{001} = \frac{\sqrt{3}(T-t)^{3/2}}{16}\int\limits_{-1}^{1} y(1-y)^2 dy = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3j_31} = C_{001} + \sum_{j_3=1}^{\infty} C_{j_3j_31} = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (2.107) is proved. Using the obtained results, we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\sum_{j_1=2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} =$$

$$= \sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\ \sum_{j_1=2, j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.109)$$

Since

$$C_{j_3j_3j_1} = \frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1} P_{j_1}(y)\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2 dy, \quad j_3 \ge 1,$$

and degree of the polynomial

$$\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$$

equals to $2j_3+2$, then $C_{j_3j_3j_1}=0$ for $j_1>2j_3+2$. It explains that we put $2j_3+2$ instead of p_1 on the right-hand side of the formula (2.109).

Moreover, the function

$$\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_1}(y)\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$$

is uneven for uneven j_1. It means that $C_{j_3j_3j_1}=0$ for uneven j_1. It explains the summation with respect to even j_1 on the right-hand side of (2.109).

Then we have

$$\sum_{j_3=0}^{p_3}\ \sum_{j_1=2,j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=(j_1-2)/2}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{2.110}$$

We replaced $(j_1-2)/2$ by zero on the right-hand side of (2.110), since $C_{j_3j_3j_1}=0$ for $0\le j_3<(j_1-2)/2$.

Let us substitute (2.110) into (2.109)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_{0}^{(i_1)}-\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_{1}^{(i_1)}+$$

$$+\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{2.111}$$

It is easy to see that the right-hand side of the formula (2.111) does not depend on p_1.

If we prove that

$$\lim_{p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\right)^2\right\}=0, \tag{2.112}$$

then (2.82) will be proved.

Using (2.111) and (2.106), (2.107), we can rewrite the left-hand side of the formula (2.112) in the following form

$$\lim_{p_3\to\infty}\mathsf{M}\left\{\left(\left(\sum_{j_3=0}^{p_3}C_{j_3j_30}-\frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_1)}+\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_1}C_{j_3j_30}-\frac{(T-t)^{3/2}}{4}\right)^2+\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=$$

$$=\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2.$$

If we prove that

$$\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=0, \tag{2.113}$$

then the relation (2.82) will be proved.

From (2.108) we obtain

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\left((T-s_2)-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2\right)ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\phi_{j_1}(s_2)\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2ds_2\right)^2\le$$

$$\le\frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T|\phi_{j_1}(s_2)|\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)ds_1\right)^2ds_2\right)^2. \tag{2.114}$$

In order to get (2.114) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int_s^T \phi_{j_1}(s_1)ds_1\right)^2=\int_t^T\left(\mathbf{1}_{\{s<s_1\}}\right)^2 ds_1=T-s \qquad (2.115)$$

and the orthogonality property of the Legendre polynomials

$$\int_t^T \phi_{j_3}(s)(T-s)ds=0, \quad j_3\ge 2. \qquad (2.116)$$

Then we have

$$\left(\int_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2=\frac{(T-t)}{4(2j_3+1)}\left(P_{j_3+1}\left(z(s_2)\right)-P_{j_3-1}\left(z(s_2)\right)\right)^2\le$$

$$\le\frac{T-t}{2(2j_3+1)}\left(P_{j_3+1}^2\left(z(s_2)\right)+P_{j_3-1}^2\left(z(s_2)\right)\right)<$$

$$<\frac{T-t}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\frac{1}{(1-(z(s_2))^2)^{1/2}}<$$

$$<\frac{(T-t)K^2}{2j_3^2}\frac{1}{(1-(z(s_2))^2)^{1/2}}, \quad s\in(t,T). \qquad (2.117)$$

In order to get (2.117) we used the estimate (2.99).

Substituting the estimate (2.117) into the relation (2.114) and using in (2.114) the estimate (2.100) for $|\phi_{j_1}(s_2)|$, we obtain

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2<$$

$$<\frac{(T-t)K^4K_1^2}{16}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int_t^T\frac{ds_2}{(1-z^2(s_2))^{3/4}}\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2=$$

$$=\frac{(T-t)^3K^4K_1^2(p_3+1)}{64}\left(\int_{-1}^1\frac{dy}{\left(1-y^2\right)^{3/4}}\right)^2\left(\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2. \qquad (2.118)$$

Using (2.103) and (2.104) in (2.118), we get

$$\sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1}\right)^2 < \frac{C(T-t)^3(p_3+1)}{p_3^2} \ \to 0 \quad \text{with} \ \ p_3\to\infty, \tag{2.119}$$

where constant C does not depend on p_3 and $T-t$.

The relation (2.119) implies (2.113), and the relation (2.113) implies the correctness of the formula (2.82). The relation (2.82) is proved.

Let us prove the equality (2.83). Since $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)\equiv 1$, then the following relation for the Fourier coefficients is correct

$$C_{j_1 j_1 j_3}+C_{j_1 j_3 j_1}+C_{j_3 j_1 j_1}=\frac{1}{2}C_{j_1}^2 C_{j_3},$$

where $C_j=0$ for $j\ge 1$ and $C_0=\sqrt{T-t}$. Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2 C_{j_3}-C_{j_1 j_1 j_3}-C_{j_3 j_1 j_1}\right)\zeta_{j_3}^{(i_2)}. \tag{2.120}$$

Therefore, considering (2.81) and (2.82), we can write w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}=\frac{1}{2}C_0^3\zeta_0^{(i_2)}-$$

$$-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1 j_1 j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)-$$

$$-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)=0. \tag{2.121}$$

The relation (2.83) is proved. Theorem 2.3 is proved.

It is easy to see that the formula (2.80) can be proved for the case $i_1=i_2=i_3$ using the Itô formula

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}=\frac{1}{6}\left(\int\limits_t^T d\mathbf{f}_s^{(i_1)}\right)^3=\frac{1}{6}\left(C_0\zeta_0^{(i_1)}\right)^3=C_{000}\zeta_0^{(i_1)}\zeta_0^{(i_1)}\zeta_0^{(i_1)},$$

where the equality is fulfilled w. p. 1.

2.2.2 The Case $p_1, p_2, p_3 \to \infty$, Binomial Weight Functions, and Additional Restrictive Conditions (The Case of Legendre Polynomials)

Let us consider the following generalization of Theorem 2.3.

Theorem 2.4 [6]-[14], [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I^{*(i_1 i_2 i_3)}_{l_1 l_2 l_3 T,t} = {\int\limits_t^{*}}^{T} (t-t_3)^{l_3} {\int\limits_t^{*}}^{t_3} (t-t_2)^{l_2} {\int\limits_t^{*}}^{t_2} (t-t_1)^{l_1} d\mathbf{f}^{(i_1)}_{t_1} d\mathbf{f}^{(i_2)}_{t_2} d\mathbf{f}^{(i_3)}_{t_3}$$

the following expansion

$$I^{*(i_1 i_2 i_3)}_{l_1 l_2 l_3 T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3} \tag{2.122}$$

that converges in the mean-square sense is valid for each of the following cases

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
2. $i_1 = i_2 \ne i_3$ *and* $l_1 = l_2 \ne l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
3. $i_1 \ne i_2 = i_3$ *and* $l_1 \ne l_2 = l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
4. $i_1, i_2, i_3 = 1, \ldots, m;\ l_1 = l_2 = l_3 = l$ *and* $l = 0, 1, 2, \ldots,$

where $i_1, i_2, i_3 = 1, \ldots, m,$

$$C_{j_3 j_2 j_1} = \int\limits_t^T (t-s)^{l_3}\phi_{j_3}(s) \int\limits_t^s (t-s_1)^{l_2}\phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^{l_1}\phi_{j_1}(s_2) ds_2 ds_1 ds,$$

and

$$\zeta^{(i)}_j = \int\limits_t^T \phi_j(s) d\mathbf{f}^{(i)}_s$$

are independent standard Gaussian random variables for various i or j.

Proof. Case 1 directly follows from (1.43). Let us consider Case 2, i.e. $i_1 = i_2 \ne i_3$, $l_1 = l_2 = l \ne l_3$, and $l_1, l_3 = 0, 1, 2, \ldots$ So, we prove the following expansion

$$I^{*(i_1 i_1 i_3)}_{l_1 l_1 l_3 T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1}\zeta^{(i_1)}_{j_2}\zeta^{(i_3)}_{j_3} \quad (i_1, i_2, i_3 = 1, \ldots, m), \tag{2.123}$$

where $l_1, l_3 = 0, 1, 2, \ldots$ $(l_1 = l)$ and

$$C_{j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3} \int\limits_t^s (t-s_1)^l \phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^l \phi_{j_1}(s_2) ds_2 ds_1 ds. \quad (2.124)$$

If we prove w. p. 1 the formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)}, \quad (2.125)$$

where coefficients $C_{j_3 j_1 j_1}$ are defined by (2.124), then using Theorem 1.1 and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.123).

Using Theorem 1.1, we obtain

$$\frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\sum_{j_3=0}^{2l+l_3+1} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)} \quad \text{w. p. 1},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 ds.$$

Then

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)} + \sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}.$$

Therefore,

$$\lim_{p_1,p_3\to\infty} \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\int\limits_t^T (t-s)^{l_3}\int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_3)}\right)^2\right\} =$$

$$= \lim_{p_1\to\infty}\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 +$$

$$
+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \qquad (2.126)
$$

Let us prove that

$$
\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde C_{j_3}\right)^2=0. \qquad (2.127)
$$

We have

$$
\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde C_{j_3}\right)^2=
$$

$$
=\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds-\right.
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_t^s(t-s_1)^{2l}ds_1\right)ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s(t-s_1)^{2l}ds_1-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_t^s(t-s_1)^{2l}ds_1\right)ds\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \qquad (2.128)
$$

In order to get (2.128) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1, \qquad (2.129)$$

where

$$K(s,s_1)=(t-s_1)^l\ \mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

Taking into account the nondecreasing of the functional sequence

$$u_n(s)=\sum_{j_1=0}^{n}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2,$$

continuity of its members and continuity of the limit function

$$u(s)=\int\limits_t^s (t-s_1)^{2l}ds_1$$

at the interval $[t,T]$ in accordance with the Dini Theorem we have uniform convergence of the functional sequences $u_n(s)$ to the limit function $u(s)$ at the interval $[t,T]$.

From (2.128) using the inequality of Cauchy–Bunyakovsky, we obtain

$$\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2\le$$

$$\le\frac{1}{4}\int\limits_t^T \phi_{j_3}^2(s)(t-s)^{2l_3}ds\int\limits_t^T\left(\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right)^2 ds\le$$

$$\le\frac{1}{4}\varepsilon^2(T-t)^{2l_3}\int\limits_t^T \phi_{j_3}^2(s)ds(T-t)=\frac{1}{4}(T-t)^{2l_3+1}\varepsilon^2 \qquad (2.130)$$

when $p_1>N(\varepsilon)$, where $N(\varepsilon)\in\mathbf{N}$ exists for any $\varepsilon>0$. The relation (2.130) implies (2.127).

Further,

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.131)$$

We put $2(j_1+l+1)+l_3$ instead of p_3, since $C_{j_3j_1j_1} = 0$ for $j_3 > 2(j_1+l+1)+l_3$. This conclusion follows from the relation

$$C_{j_3j_1j_1} = \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)Q_{2(j_1+l+1)+l_3}(s)ds,$$

where $Q_{2(j_1+l+1)+l_3}(s)$ is a polynomial of the degree $2(j_1+l+1)+l_3$.

It is easy to see that

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.132)$$

Note that we included some zero coefficients $C_{j_3j_1j_1}$ into the sum $\sum\limits_{j_1=0}^{p_1}$. From (2.131) and (2.132) we have

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} = \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 =$$

$$= \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=0}^{p_1}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s (t-s_1)^{2l}ds_1 - \right.\right.$$

$$
\left.\left.-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right) ds\right)^2 =
$$

$$
=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \tag{2.133}
$$

In order to get (2.133) we used the Parseval equality (2.129) and the following relation

$$
\int\limits_t^T \phi_{j_3}(s)Q_{2l+1+l_3}(s)ds=0,\quad j_3>2l+1+l_3,
$$

where $Q_{2l+1+l_3}(s)$ is a polynomial of degree $2l+1+l_3$.

Further, we have

$$
\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}(2j_1+1)}{2^{2l+2}}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)(1+y)^l dy\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}}{2^{2l+2}(2j_1+1)}\times
$$

$$
\times\left((1+z(s))^l R_{j_1}(s)-l\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2 \le
$$

$$
\le\frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_1+1)}\times
$$

$$
\times\left(\left(\frac{2(s-t)}{T-t}\right)^{2l} R_{j_1}^2(s)+l^2\left(\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2\right) \le
$$

$$
\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\times
$$

$$
\times\left(2^{2l+1}Z_{j_1}(s)+l^2\int\limits_{-1}^{z(s)}(1+y)^{2l-2}dy\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)^2dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\times
$$

$$
\times\left(2^{2l+1}Z_{j_1}(s)+\frac{2l^2}{2l-1}\left(\frac{2(s-t)}{T-t}\right)^{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(2Z_{j_1}(s)+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right), \qquad (2.134)
$$

where

$$
R_{j_1}(s)=P_{j_1+1}(z(s))-P_{j_1-1}(z(s)),
$$
$$
Z_{j_1}(s)=P_{j_1+1}^2(z(s))+P_{j_1-1}^2(z(s)).
$$

Let us estimate the right-hand side of (2.134) using (2.99)

$$
\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2<
$$

$$
<\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\left(\frac{2}{\left(1-(z(s))^2\right)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/2}}\right)<
$$

$$
<\frac{(T-t)^{2l+1}K^2}{2j_1^2}\left(\frac{2}{\left(1-(z(s))^2\right)^{1/2}}+\frac{l^2\pi}{2l-1}\right),\quad s\in(t,T). \qquad (2.135)
$$

From (2.133) and (2.135) we obtain

$$
\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2\le
$$

$$\le \frac{1}{4}(T-t)^{2l_3}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2<$$

$$<\frac{(T-t)^{4l+2l_3+1}K^4K_1^2}{16}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\left(\int\limits_t^T\frac{2ds}{\left(1-(z(s))^2\right)^{3/4}}+\right.\right.$$

$$\left.\left.+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds}{\left(1-(z(s))^2\right)^{1/4}}\right)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2\le$$

$$\le\frac{(T-t)^{4l+2l_3+3}K^4K_1^2}{64}\ \frac{2p_1+1}{p_1^2}\left(\int\limits_{-1}^{1}\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2\le$$

$$\le C(T-t)^{4l+2l_3+3}\ \frac{2p_1+1}{p_1^2}\to 0\quad\text{when}\ \ p_1\ \to\ \infty, \tag{2.136}$$

where constant C does not depend on p_1 and $T-t$.

The relations (2.2.2), (2.127), and (2.136) imply (2.125), and the relation (2.125) implies the correctness of the formula (2.123).

Let us consider Case 3, i.e. $i_2=i_3\ne i_1$, $l_2=l_3=l\ne l_1$, and $l_1,l_3=0,1,2,\ldots$ So, we prove the following expansion

$$I_{l_1l_3l_3 T,t}^{*(i_1i_3i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_3)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{2.137}$$

where $l_1,l_3=0,1,2,\ldots$ $(l_3=l)$ and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.138}$$

If we prove w. p. 1 the formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds, \tag{2.139}$$

where the coefficients $C_{j_3j_3j_1}$ are defined by (2.138), then using Theorem 1.1 and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.137).

Using Theorem 1.1 and the Itô formula, we have

$$\frac{1}{2}\int\limits_t^T (t-s)^{2l}\int\limits_t^s (t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T (t-s_1)^{l_1}\int\limits_{s_1}^T (t-s)^{2l}dsd\mathbf{f}_{s_1}^{(i_1)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde C_{j_1}\zeta_{j_1}^{(i_1)}\quad \text{w. p. 1},$$

where

$$\tilde C_{j_1}=\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T (t-s)^{2l}dsds_1.$$

Then

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde C_{j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde C_{j_1}\right)\zeta_{j_1}^{(i_1)}+\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}.$$

Therefore,

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\int\limits_t^T (t-s)^{2l}\int\limits_t^s (t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds\right)^2\right\}=$$

$$=\lim_{p_3\to\infty}\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde C_{j_1}\right)^2+$$

$$+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}. \tag{2.140}$$

Let us prove that

$$\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde C_{j_1}\right)^2=0. \tag{2.141}$$

We have

$$\left(\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)^2 =$$

$$= \left(\sum_{j_3=0}^{p_3} \int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1} ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^{l} ds_1 \int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^{l} ds - \right.$$

$$\left. - \frac{1}{2}\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \int\limits_{s_1}^T (t-s)^{2l} ds ds_1\right)^2 =$$

$$= \left(\frac{1}{2}\sum_{j_3=0}^{p_3} \int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1} \left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^{l} ds_1\right)^2 ds_2 - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \int\limits_{s_1}^T (t-s)^{2l} ds ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \left(\sum_{j_3=0}^{p_3}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^{l} ds\right)^2 - \right.\right.$$

$$\left.\left. - \int\limits_{s_1}^T (t-s)^{2l} ds\right) ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \left(\int\limits_{s_1}^T (t-s)^{2l} ds - \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^{l} ds\right)^2 - \right.\right.$$

$$\left.\left. - \int\limits_{s_1}^T (t-s)^{2l} ds\right) ds_1\right)^2 =$$

$$= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1} \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^{l} ds\right)^2 ds_1\right)^2. \qquad (2.142)$$

In order to get (2.142) we used the Parseval equality

$$\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2 = \int\limits_{t}^{T} K^2(s,s_1)ds, \qquad (2.143)$$

where

$$K(s,s_1)=(t-s)^l\ \mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

Taking into account the nondecreasing of the functional sequence

$$u_n(s_1)=\sum_{j_3=0}^{n}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2,$$

continuity of its members and continuity of the limit function

$$u(s_1)=\int\limits_{s_1}^{T}(t-s)^{2l}ds$$

at the interval $[t,T]$ in accordance with the Dini Theorem we have uniform convergence of the functional sequence $u_n(s_1)$ to the limit function $u(s_1)$ at the interval $[t,T]$.

From (2.142) using the inequality of Cauchy–Bunyakovsky, we obtain

$$\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2\le$$

$$\le\frac{1}{4}\int\limits_{t}^{T}\phi_{j_1}^2(s_1)(t-s_1)^{2l_1}ds_1\int\limits_{t}^{T}\left(\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)(t-s)^l ds\right)^2\right)^2 ds_1\le$$

$$\le\frac{1}{4}\varepsilon^2(T-t)^{2l_1}\int\limits_{t}^{T}\phi_{j_1}^2(s_1)ds_1(T-t)=\frac{1}{4}(T-t)^{2l_1+1}\varepsilon^2 \qquad (2.144)$$

when $p_3>N(\varepsilon)$, where $N(\varepsilon)\in\mathbf{N}$ exists for any $\varepsilon>0$. The relation (2.141) follows from (2.144).

We have

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.145)$$

We put $2(j_3+l+1)+l_1$ instead of p_1, since $C_{j_3j_3j_1}=0$ when $j_1>2(j_3+l+1)+l_1$. This conclusion follows from the relation

$$
C_{j_3j_3j_1}=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)Q_{2(j_3+l+1)+l_1}(s_2)ds_2,
$$

where $Q_{2(j_3+l+1)+l_1}(s)$ is a polynomial of degree $2(j_3+l+1)+l_1$.

It is easy to see that

$$
\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{2.146}
$$

Note that we included some zero coefficients $C_{j_3j_3j_1}$ into the sum $\sum\limits_{j_3=0}^{p_3}$.

From (2.145) and (2.146) we have

$$
\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=
$$

$$
=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T(t-s_1)^{2l}ds_1-\right.\right.
$$

$$
\left.\left.-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2\right) ds_2\right)^2 =
$$

$$
=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_{t}^{T}\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2. \tag{2.147}
$$

In order to get (2.147) we used the Parseval equality (2.143) and the following relation

$$
\int\limits_{t}^{T}\phi_{j_1}(s)Q_{2l+1+l_1}(s)ds=0,\quad j_1>2l+1+l_1,
$$

where $Q_{2l+1+l_1}(s)$ is a polynomial of degree $2l+1+l_1$.

Further, we have

$$
\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}(2j_3+1)}{2^{2l+2}}\left(\int\limits_{z(s_2)}^{1}P_{j_3}(y)(1+y)^l dy\right)^2 =
$$

$$
=\frac{(T-t)^{2l+1}}{2^{2l+2}(2j_3+1)}\times
$$

$$
\times\left((1+z(s_2))^l\,Q_{j_3}(s_2)-l\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\right)^2\le
$$

$$
\le\frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_3+1)}\times
$$

$$
\times\left(\left(\frac{2(s_2-t)}{T-t}\right)^{2l}Q_{j_3}^2(s_2)+l^2\left(\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\right)^2\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\times
$$

$$
\times\left(2^{2l+1}H_{j_3}(s_2)+l^2\int\limits_{z(s_2)}^{1}(1+y)^{2l-2}dy\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)^2dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\times
$$

$$
\times\left(2^{2l+1}H_{j_3}(s_2)+\frac{2^{2l}l^2}{2l-1}\left(1-\left(\frac{s_2-t}{T-t}\right)^{2l-1}\right)\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right)\le
$$

$$
\le\frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(2H_{j_3}(s_2)+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right),\qquad (2.148)
$$

where

$$
Q_{j_3}(s_2)=P_{j_3-1}(z(s_2))-P_{j_3+1}(z(s_2)),
$$

$$
H_{j_3}(s_2)=P_{j_3-1}^2(z(s_2))+P_{j_3+1}^2(z(s_2)).
$$

Let us estimate the right-hand side of (2.148) using (2.99)

$$
\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2<
$$

$$
<\frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\frac{dy}{(1-y^2)^{1/2}}\right)<
$$

$$
<\frac{(T-t)^{2l+1}K^2}{2j_3^2}\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2\pi}{2l-1}\right),\quad s\in(t,T).\qquad (2.149)
$$

From (2.147) and (2.149) we obtain

$$
\mathrm{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\le
$$

$$
\le\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_{t}^{T}|\phi_{j_1}(s_2)|(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2ds_2\right)^2\le
$$

$$\le \frac{1}{4}(T-t)^{2l_1}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T|\phi_{j_1}(s_2)|\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2 <$$

$$< \frac{(T-t)^{4l+2l_1+1}K^4K_1^2}{16}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\left(\int\limits_t^T\frac{2ds_2}{\left(1-(z(s_2))^2\right)^{3/4}}+\right.\right.$$

$$\left.\left.+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds_2}{\left(1-(z(s_2))^2\right)^{1/4}}\right)\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2 \le$$

$$\le \frac{(T-t)^{4l+2l_1+3}K^4K_1^2}{64}\ \frac{2p_3+1}{p_3^2}\left(\int\limits_{-1}^1\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^1\frac{dy}{(1-y^2)^{1/4}}\right)^2 \le$$

$$\le C(T-t)^{4l+2l_1+3}\ \frac{2p_3+1}{p_3^2}\to 0 \quad \text{when} \ \ p_3 \ \to \ \infty, \tag{2.150}$$

where constant C does not depend on p_3 and $T-t$.

The relations (2.2.2), (2.141), and (2.150) imply (2.139), and the relation (2.139) implies the correctness of the expansion (2.137).

Let us consider Case 4, i.e. $l_1=l_2=l_3=l=0,1,2,\ldots$ and $i_1,i_2,i_3=1,\ldots,m$. So, we will prove the following expansion for iterated Stratonovich stochastic integral of third multiplicity

$$I_{lll_{T,t}}^{*(i_1i_2i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{2.151}$$

where the series converges in the mean-square sense, $l=0,1,2,\ldots,$ and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^l\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.152}$$

If we prove w. p. 1 the following formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{2.153}$$

where the coefficients $C_{j_3j_2j_1}$ are defined by (2.152), then using the formulas (2.125), (2.139) when $l_1 = l_3 = l$, Theorem 1.1, and standard relations between iterated Itô and Stratonovich stochastic integrals, we obtain the expansion (2.151).

Since $\psi_1(s),\ \psi_2(s),\ \psi_3(s) \equiv (t-s)^l$, then the following equality for the Fourier coefficients takes place

$$
C_{j_1j_1j_3} + C_{j_1j_3j_1} + C_{j_3j_1j_1} = \frac{1}{2}C_{j_1}^2 C_{j_3},
$$

where the coefficients $C_{j_3j_2j_1}$ are defined by (2.152) and

$$
C_{j_1} = \int\limits_t^T \phi_{j_1}(s)(t-s)^l ds.
$$

Then w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} =
$$

$$
= \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} \left(\frac{1}{2}C_{j_1}^2 C_{j_3} - C_{j_1j_1j_3} - C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \tag{2.154}
$$

Taking into account (2.125) and (2.139) when $l_3 = l_1 = l$ and the Itô formula, we have w. p. 1

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = \frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \sum_{j_3=0}^{l} C_{j_3}\zeta_{j_3}^{(i_2)} -
$$

$$
- \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)} - \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)} =
$$

$$
= \frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} - \frac{1}{2}\int\limits_t^T (t-s)^l \int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_2)} -
$$

$$
-\frac{1}{2}\int\limits_t^T (t-s)^{2l} \int\limits_t^s (t-s_1)^l d\mathbf{f}_{s_1}^{(i_2)} ds =
$$

$$
= \frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} + \frac{1}{2(2l+1)}\int\limits_t^T (t-s)^{3l+1} d\mathbf{f}_s^{(i_2)} -
$$

$$
-\frac{1}{2}\int\limits_t^T (t-s_1)^l \int\limits_{s_1}^T (t-s)^{2l} ds d\mathbf{f}_{s_1}^{(i_2)} =
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} + \frac{1}{2(2l+1)}\int\limits_t^T (t-s)^{3l+1} d\mathbf{f}_s^{(i_2)} -
$$

$$
-\frac{1}{2(2l+1)}\left((T-t)^{2l+1}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} + \int\limits_t^T (t-s)^{3l+1} d\mathbf{f}_s^{(i_2)}\right) =
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} - \frac{(T-t)^{2l+1}}{2(2l+1)}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} =
$$

$$
=\frac{1}{2}\left(\sum_{j_1=0}^{l} C_{j_1}^2 - \int\limits_t^T (t-s)^{2l} ds\right)\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} = 0.
$$

Here the Parseval equality looks as follows

$$
\sum_{j_1=0}^{\infty} C_{j_1}^2 = \sum_{j_1=0}^{l} C_{j_1}^2 = \int\limits_t^T (t-s)^{2l} ds = \frac{(T-t)^{2l+1}}{2l+1}
$$

and

$$
\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} = \sum_{j_3=0}^{l} C_{j_3}\zeta_{j_3}^{(i_2)} \quad \text{w. p. 1.}
$$

The expansion (2.151) is proved. Theorem 2.4 is proved.

It is easy to see that using the Itô formula if $i_1 = i_2 = i_3$ we obtain (see (1.55))

$$
\int\limits_t^{*T} (t-s)^l \int\limits_t^{*s} (t-s_1)^l \int\limits_t^{*s_1} (t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)} d\mathbf{f}_{s_1}^{(i_1)} d\mathbf{f}_s^{(i_1)} =
$$

$$
=\frac{1}{6}\left(\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_1)}\right)^3 = \frac{1}{6}\left(\sum_{j_1=0}^{l} C_{j_1}\zeta_{j_1}^{(i_1)}\right)^3 =
$$

$$
=\sum_{j_1,j_2,j_3=0}^{l} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)} \quad \text{w. p. 1.} \tag{2.155}
$$

2.2.3 The Case $p_1, p_2, p_3 \to \infty$ and Constant Weight Functions (The Case of Trigonometric Functions)

In this section, we will prove the following theorem.

Theorem 2.5 [6]-[14], [33]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \mathop{\hbox{\vtop{\offinterlineskip\hbox{l.i.m.}\vskip1pt\hbox{$\scriptstyle{p_1,p_2,p_3\to\infty}$}}}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \qquad (2.156)$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. If we prove w. p. 1 the following formulas

$$\mathop{\hbox{\vtop{\offinterlineskip\hbox{l.i.m.}\vskip1pt\hbox{$\scriptstyle{p_1,p_3\to\infty}$}}}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}, \qquad (2.157)$$

$$\mathop{\hbox{\vtop{\offinterlineskip\hbox{l.i.m.}\vskip1pt\hbox{$\scriptstyle{p_1,p_3\to\infty}$}}}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau, \qquad (2.158)$$

$$\mathop{\hbox{\vtop{\offinterlineskip\hbox{l.i.m.}\vskip1pt\hbox{$\scriptstyle{p_1,p_3\to\infty}$}}}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0, \qquad (2.159)$$

then from the equalities (2.157)–(2.159), Theorem 1.1, and standard relations between iterated Itô and Stratonovich stochastic integrals we will obtain the expansion (2.156).

We have

$$
\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{(T-t)^{3/2}}{6}+
$$

$$
+\sum_{j_1=1}^{p_1} C_{0,2j_1,2j_1}\zeta_0^{(i_3)}+\sum_{j_1=1}^{p_1} C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{p_1} C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)}+
$$

$$
+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3} C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)}+
$$

$$
+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)}, \tag{2.160}
$$

where the summation is stopped, when $2j_1,\ 2j_1-1>p_1$ or $2j_3,\ 2j_3-1>p_3$ and

$$
C_{0,2l,2l}=\frac{(T-t)^{3/2}}{8\pi^2l^2},\quad C_{0,2l-1,2l-1}=\frac{3(T-t)^{3/2}}{8\pi^2l^2},\quad C_{2l,0,0}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2l^2}, \tag{2.161}
$$

$$
C_{2r-1,2l,2l}=0,\quad C_{2l-1,0,0}=-\frac{\sqrt{2}(T-t)^{3/2}}{4\pi l},\quad C_{2r-1,2l-1,2l-1}=0, \tag{2.162}
$$

$$
C_{2r,2l,2l}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ \\ 0, & r\ne 2l\end{cases}, \tag{2.163}
$$

$$
C_{2r,2l-1,2l-1}=\begin{cases}\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ \\ -\sqrt{2}(T-t)^{3/2}/(4\pi^2l^2), & r=l\\ \\ 0, & r\ne l,\ r\ne 2l\end{cases}. \tag{2.164}
$$

After substituting (2.161)–(2.164) into (2.160), we obtain

$$
\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=(T-t)^{3/2}\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\right)\zeta_0^{(i_3)}-\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\right). \tag{2.165}
$$

Using Theorem 1.1 and the system of trigonometric functions, we get w. p. 1

$$\frac{1}{2}\int\limits_t^T\int\limits_t^s ds_1 d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\int\limits_t^T (s-t)d\mathbf{f}_s^{(i_3)}=\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_3)}\right). \tag{2.166}$$

From (2.165) and (2.166) it follows that

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\int\limits_t^T\int\limits_t^s ds_1 d\mathbf{f}_s^{(i_3)}\right)^2\right\}=$$

$$=\lim_{p_1,p_3\to\infty}(T-t)^3\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}-\frac{1}{4}\right)^2+\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\right)\right)=0.$$

So, the relation (2.157) is proved for the case of trigonometric system of functions.

Let us prove the relation (2.158). We have

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{(T-t)^{3/2}}{6}+$$

$$+\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,0}\zeta_0^{(i_1)}+\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)}+\sum_{j_1=1}^{p_1}C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}, \tag{2.167}$$

where the summation is stopped, when $2j_3,\ 2j_3-1>p_3$ or $2j_1,\ 2j_1-1>p_1$ and

$$C_{2l,2l,0}=\frac{(T-t)^{3/2}}{8\pi^2l^2},\quad C_{2l-1,2l-1,0}=\frac{3(T-t)^{3/2}}{8\pi^2l^2},\quad C_{0,0,2r}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2r^2}, \tag{2.168}$$

$$C_{2l-1,2l-1,2r-1}=0,\quad C_{0,0,2r-1}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi r},\quad C_{2l,2l,2r-1}=0, \tag{2.169}$$

$$C_{2l,2l,2r}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ & \\ 0, & r\ne 2l\end{cases}, \tag{2.170}$$

$$C_{2l-1,2l-1,2r}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ & \\ \sqrt{2}(T-t)^{3/2}/(4\pi^2l^2), & r=l\\ & \\ 0, & r\ne l,\ r\ne 2l\end{cases}. \tag{2.171}$$

After substituting (2.168)–(2.171) into (2.167), we obtain

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=(T-t)^{3/2}\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\right)\zeta_0^{(i_1)}+\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\right). \tag{2.172}$$

Using the Itô formula and Theorem 1.1 for the case of trigonometric system of functions, we have w. p. 1

$$\frac{1}{2}\int\limits_t^T\int\limits_t^s d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\left((T-t)\int\limits_t^T d\mathbf{f}_s^{(i_1)}+\int\limits_t^T(t-s)d\mathbf{f}_s^{(i_1)}\right)=$$

$$=\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right). \tag{2.173}$$

From (2.172) and (2.173) it follows that

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\int\limits_t^T\int\limits_t^s d\mathbf{f}_{s_1}^{(i_1)}ds\right)^2\right\}=$$

$$=\lim_{p_1,p_3\to\infty}(T-t)^3\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}-\frac{1}{4}\right)^2+\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\right)\right)=0.$$

So, the relation (2.158) is proved for the case of trigonometric system of functions.

Let us prove the equality (2.159). Since $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau) \equiv 1$, then the following relation for the Fourier coefficients is correct

$$C_{j_1j_1j_3} + C_{j_1j_3j_1} + C_{j_3j_1j_1} = \frac{1}{2}C_{j_1}^2 C_{j_3}.$$

Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} =$$

$$= \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} \left(\frac{1}{2}C_{j_1}^2 C_{j_3} - C_{j_1j_1j_3} - C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \qquad (2.174)$$

Taking into account (2.157) and (2.158), we can write w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} =$$

$$= \frac{1}{2}C_0^3\zeta_0^{(i_2)} - \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)} - \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)} =$$

$$= \frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)} - \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)} + \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right) -$$

$$-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right) = 0.$$

From Theorem 1.1 and (2.157)–(2.159) we obtain the expansion (2.156). Theorem 2.5 is proved.

2.2.4 The Case $p_1 = p_2 = p_3 \to \infty$, Smooth Weight Functions, and Additional Restrictive Conditions (The Cases of Legendre Polynomials and Trigonometric Functions)

Let us consider the following modification of Theorem 2.4.

Theorem 2.6 [10]-[14], [33]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ *are continuously differentiable functions* *at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{2.175}$$

that converges in the mean-square sense is valid for each of the following cases
1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$,
2. $i_1 = i_2 \ne i_3$ *and* $\psi_1(s) \equiv \psi_2(s)$,
3. $i_1 \ne i_2 = i_3$ *and* $\psi_2(s) \equiv \psi_3(s)$,
4. $i_1, i_2, i_3 = 1, \ldots, m$ *and* $\psi_1(s) \equiv \psi_2(s) \equiv \psi_3(s)$,

where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s) \int\limits_t^s \psi_2(s_1)\phi_{j_2}(s_1) \int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. Let us consider at first the polynomial case. Case 1 directly follows from Theorem 1.1. Further, consider Case 2. We will prove the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_3j_1j_1} \zeta_{j_3}^{(i_3)} = \frac{1}{2} \int\limits_t^T \psi_3(s) \int\limits_t^s \psi^2(s_1) ds_1 d\mathbf{f}_s^{(i_3)} \quad \text{w. p. } 1,$$

where

$$C_{j_3j_1j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s) \int\limits_t^s \psi(s_1)\phi_{j_1}(s_1) \int\limits_t^{s_1} \psi(s_2)\phi_{j_1}(s_2) ds_2 ds_1 ds.$$

Using Theorem 1.1, we can write w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\ \underset{p_3\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{p_3} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 ds.$$

We have

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} = \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\frac{1}{2}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\sum_{j_1=0}^{p}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 - \int\limits_t^s \psi^2(s_1)ds_1\right)ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds\right)^2. \qquad (2.176)$$

In order to get (2.176) we used the Parseval equality

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1 = \int\limits_t^s \psi^2(s_1)ds_1,$$

where

$$K(s,s_1) = \psi(s_1)\mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

We have

$$\left(\int\limits_t^s \psi(s_1)\phi_{j_1}(s_1)ds_1\right)^2 =$$

$$= \frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)\psi\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2 =$$

$$= \frac{T-t}{4(2j_1+1)}\Biggl(\left(P_{j_1+1}(z(s))-P_{j_1-1}(z(s))\right)\psi(s)-$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(s)}\left(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi'\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2, \qquad (2.177)$$

where

$$z(s) = \left(s-\frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ' is a derivative of the function $\psi(s)$ with respect to the variable

$$\frac{T-t}{2}y+\frac{T+t}{2}.$$

Further consideration is similar to the proof of Case 2 from Theorem 2.4. Finally, from (2.176) and (2.177) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} <$$

$$< K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2 \le$$

$$\le \frac{K_1}{p}\to 0 \quad \text{if} \quad p\to\infty,$$

where constants K, K_1 do not depend on p. Case 2 is proved.

Let us consider Case 3. In this case we will prove the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T\psi^2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds \quad \text{w. p. 1},$$

where

$$C_{j_3j_3j_1}=\int\limits_t^T\psi(s)\phi_{j_3}(s)\int\limits_t^s\psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.$$

Using the Itô formula, we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi^2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)}. \tag{2.178}$$

Moreover, using Theorem 1.1, we have w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)}=\frac{1}{2}\ \underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}^*\zeta_{j_1}^{(i_1)}, \tag{2.179}$$

where

$$C_{j_1}^*=\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\int\limits_{s_1}^T\psi^2(s)dsds_1.$$

Further,

$$C_{j_3j_3j_1}=\int\limits_t^T\psi(s)\phi_{j_3}(s)\int\limits_t^s\psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\psi(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi(s)\phi_{j_3}(s)dsds_1ds_2=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\left(\int\limits_{s_2}^T\psi(s_1)\phi_{j_3}(s_1)ds_1\right)^2ds_2. \tag{2.180}$$

From (2.178)–(2.180) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^p\left(\sum_{j_3=0}^p C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^*\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}=\sum_{j_1=0}^p\left(\sum_{j_3=0}^p C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^*\right)^2=$$

$$
=\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\left(\sum_{j_3=0}^{p}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds_1\right)^2-\right.\right.
$$

$$
\left.\left.-\int\limits_{s_1}^T\psi^2(s)ds\right)ds_1\right)^2=
$$

$$
=\frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T\phi_{j_1}(s_1)\psi_1(s_1)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds\right)^2ds_1\right)^2. \qquad (2.181)
$$

In order to get (2.181) we used the Parseval equality

$$
\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)\psi(s)ds\right)^2=\int\limits_t^T K^2(s,s_1)ds=\int\limits_{s_1}^T\psi^2(s)ds,
$$

where

$$
K(s,s_1)=\psi(s)\mathbf{1}_{\{s>s_1\}},\quad s,s_1\in[t,T].
$$

Further consideration is similar to the proof of Case 3 from Theorem 2.4. Finally, from (2.181) we get

$$
\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C_{j_1}^*\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}<
$$

$$
<K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2\le
$$

$$
\le\frac{K_1}{p}\to 0\quad\text{if}\quad p\to\infty,
$$

where constants K, K_1 do not depend on p. Case 3 is proved.

Let us consider Case 4. We will prove w. p. 1 the following relation

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0\quad(\psi_1(s),\psi_2(s),\psi_3(s)\equiv\psi(s)).
$$

In Case 4 we obtain w. p. 1

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\sum_{j_3=0}^{p}C_{j_3}\zeta_{j_3}^{(i_2)}-\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-
$$

$$
-\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{\infty}C_{j_1}^2\int\limits_t^T\psi(s)d\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_t^T\psi^2(s)\int\limits_t^s\psi(s_1)d\mathbf{f}_{s_1}^{(i_2)}ds-
$$

$$
-\frac{1}{2}\int\limits_t^T\psi(s)\int\limits_t^s\psi^2(s_1)ds_1d\mathbf{f}_s^{(i_2)}=\frac{1}{2}\int\limits_t^T\psi^2(s)ds\int\limits_t^T\psi(s)d\mathbf{f}_s^{(i_2)}-
$$

$$
-\frac{1}{2}\int\limits_t^T\psi(s_1)\int\limits_{s_1}^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}-\frac{1}{2}\int\limits_t^T\psi(s_1)\int\limits_t^{s_1}\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\psi^2(s)ds\int\limits_t^T\psi(s)d\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_t^T\psi(s_1)\int\limits_t^T\psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}=0,
$$

where we used the Parseval equality

$$
\sum_{j_1=0}^{\infty}C_j^2=\sum_{j=0}^{\infty}\left(\int\limits_t^T\psi(s)\phi_j(s)ds\right)^2=\int\limits_t^T\psi^2(s)ds.
$$

Case 4 and Theorem 2.6 are proved for the case of Legendre polynomials.

Let us consider the case of trigonometric functions. We have

$$
\int\limits_t^s\phi_{j_1}(\theta)\psi(\theta)d\theta=\frac{\sqrt{2}}{\sqrt{T-t}}\int\limits_t^s\begin{cases}\psi(\theta)\ \sin\frac{2\pi j_1(\theta-t)}{T-t}d\theta\\ \\ \psi(\theta)\ \cos\frac{2\pi j_1(\theta-t)}{T-t}d\theta\end{cases}=
$$

$$
=\sqrt{\frac{T-t}{2}}\frac{1}{\pi j_1}\left(\begin{cases}-\psi(s)\ \cos\frac{2\pi j_1(s-t)}{T-t}+\psi(t)\\ \\ \psi(s)\ \sin\frac{2\pi j_1(s-t)}{T-t}\end{cases}+\int\limits_t^s\begin{cases}\psi'(\theta)\ \cos\frac{2\pi j_1(\theta-t)}{T-t}d\theta\\ \\ -\psi'(\theta)\ \sin\frac{2\pi j_1(\theta-t)}{T-t}d\theta\end{cases}\right),
$$

where $j_1\ne 0$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in $L_2([t,T])$.

Then

$$
\left|\int\limits_t^s\phi_{j_1}(\theta)\psi(\theta)d\theta\right|\le\frac{K}{j_1}\quad(j_1\ne 0). \tag{2.182}
$$

Analogously, we obtain

$$
\left|\int\limits_s^T\phi_{j_1}(\theta)\psi(\theta)d\theta\right|\le\frac{K}{j_1}\quad(j_1\ne 0). \tag{2.183}
$$

Using (2.176), (2.181)–(2.183), we get

$$
\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}-\frac{1}{2}\tilde C_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{K_1}{p}\to 0\quad\text{if}\quad p\to\infty,
$$

$$
\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C^*_{j_1}\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}\le\frac{K_1}{p}\to 0\quad\text{if}\quad p\to\infty,
$$

where constant K_1 does not depend on p.

The consideration of Case 4 is similar to the case of Legendre polynomials. Theorem 2.6 is proved.

In the next section the analogue of Theorem 2.6 will be proved without the restrictions 1–4 (see the formulation of Theorem 2.6).

2.2.5 The Case $p_1=p_2=p_3\to\infty$, Smooth Weight Functions, and without Additional Restrictive Conditions (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 2.7 [10]-[14], [20], [31]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(s)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(s)$, $\psi_3(s)$ are twice continuously differentiable* *nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3)\int\limits_t^{*t_3} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{2.184}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^{T} \psi_3(s)\phi_{j_3}(s)\int\limits_t^{s} \psi_2(s_1)\phi_{j_2}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. Let us consider the case of Legendre polynomials.

From (1.43) for the case $p_1=p_2=p_3=p$ and standard relations between Itô and Stratonovich stochastic integrals we conclude that Theorem 2.7 will be proved if w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^{T} \psi_3(s)\int\limits_t^{s} \psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_s^{(i_3)}, \tag{2.185}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^{T} \psi_3(s)\psi_2(s)\int\limits_t^{s} \psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds, \tag{2.186}$$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0. \tag{2.187}$$

Let us prove (2.185). Using Theorem 1.1 for $k=1$ (also see (1.41)), we can write w. p. 1

$$\frac{1}{2}\int\limits_t^{T} \psi_3(s)\int\limits_t^{s} \psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{p} \tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s) \int\limits_t^s \psi_2(s_1)\psi_1(s_1)ds_1 ds.$$

We have

$$E_p \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\sum_{j_3=0}^{p}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 ds - \right.$$

$$\left. -\frac{1}{2}\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_1(s_1)\psi_2(s_1)ds_1 ds\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s\left(\sum_{j_1=0}^{p}\psi_2(s_1)\phi_{j_1}(s_1)\times\right.\right.$$

$$\left.\left.\times\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 - \frac{1}{2}\psi_1(s_1)\psi_2(s_1)\right)ds_1 ds\right)^2. \qquad (2.188)$$

Let us substitute $t_1 = t_2 = s_1$ into (2.12). Then for all $s_1 \in (t, T)$

$$\sum_{j_1=0}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 = \frac{1}{2}\psi_1(s_1)\psi_2(s_1). \qquad (2.189)$$

From (2.188) and (2.189) it follows that

$$E_p = \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \sum_{j_1=p+1}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 ds\right)^2. \qquad (2.190)$$

From (2.190) and (2.24) we obtain

$$E_p < C_1 \sum_{j_3=0}^{p} \left(\int\limits_t^T |\phi_{j_3}(s)| \frac{1}{p} \left(\int\limits_{-1}^{z(s)} \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^{z(s)} \frac{dy}{(1-y^2)^{1/4}} \right) ds \right)^2 <$$

$$< \frac{C_2}{p^2} \sum_{j_3=0}^{p} \left(\int\limits_t^T |\phi_{j_3}(s)| ds \right)^2 \le \frac{C_2(T-t)}{p^2} \sum_{j_3=0}^{p} \int\limits_t^T \phi_{j_3}^2(s) ds = \frac{C_3 p}{p^2} \to 0$$

if $p \to \infty$, where constants C_1, C_2, C_3 do not depend on p. The equality (2.185) is proved.

Let us prove (2.186). Using the Itô formula, we have

$$\frac{1}{2} \int\limits_t^T \psi_3(s)\psi_2(s) \int\limits_t^s \psi_1(s_1) d\mathbf{f}_{s_1}^{(i_1)} ds = \frac{1}{2} \int\limits_t^T \psi_1(s_1) \int\limits_{s_1}^T \psi_3(s)\psi_2(s) ds d\mathbf{f}_{s_1}^{(i_1)} \quad \text{w. p. 1.}$$

Moreover, using Theorem 1.1 for $k = 1$ (also see (1.41)), we obtain w. p. 1

$$\frac{1}{2} \int\limits_t^T \psi_1(s) \int\limits_s^T \psi_3(s_1)\psi_2(s_1) ds_1 d\mathbf{f}_s^{(i_1)} = \frac{1}{2} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} C_{j_1}^* \zeta_{j_1}^{(i_1)},$$

where

$$C_{j_1}^* = \int\limits_t^T \psi_1(s)\phi_{j_1}(s) \int\limits_s^T \psi_3(s_1)\psi_2(s_1) ds_1 ds. \tag{2.191}$$

We have

$$E_p' \stackrel{\text{def}}{=} \mathsf{M}\left\{ \left(\sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} - \frac{1}{2} \sum_{j_1=0}^{p} C_{j_1}^* \zeta_{j_1}^{(i_1)} \right)^2 \right\} =$$

$$= \mathsf{M}\left\{ \left(\sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^* \right) \zeta_{j_1}^{(i_1)} \right)^2 \right\} =$$

$$= \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^* \right)^2, \tag{2.192}$$

$$C_{j_3j_3j_1}=\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)dsds_1ds_2. \qquad (2.193)$$

From (2.191)–(2.193) we obtain

$$E_p'=\sum_{j_1=0}^p\left(\int\limits_t^T\psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T\left(\sum_{j_3=0}^p\psi_2(s_1)\phi_{j_3}(s_1)\times\right.\right.$$

$$\left.\left.\times\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)ds-\frac{1}{2}\psi_3(s_1)\psi_2(s_1)\right)ds_1ds_2\right)^2. \qquad (2.194)$$

We will prove the following equality for all $s_1\in(t,T)$

$$\sum_{j_3=0}^{\infty}\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T\psi_3(s)\phi_{j_3}(s)ds=\frac{1}{2}\psi_2(s_1)\psi_3(s_1). \qquad (2.195)$$

Let us denote

$$K_1^*(t_1,t_2)=K_1(t_1,t_2)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_2(t_1)\psi_3(t_1), \qquad (2.196)$$

where

$$K_1(t_1,t_2)=\psi_2(t_1)\psi_3(t_2)\mathbf{1}_{\{t_1<t_2\}},\quad t_1,t_2\in[t,T].$$

Let us expand the function $K_1^*(t_1,t_2)$ using the variable t_2, when t_1 is fixed, into the Fourier–Legendre series at the interval (t,T)

$$K_1^*(t_1,t_2)=\sum_{j_3=0}^{\infty}\psi_2(t_1)\int\limits_{t_1}^T\psi_3(t_2)\phi_{j_3}(t_2)dt_2\cdot\phi_{j_3}(t_2)\quad(t_2\ne t,T). \qquad (2.197)$$

The equality (2.197) is fulfilled in each point of the interval (t,T) with respect to the variable t_2, when $t_1\in[t,T]$ is fixed, due to piecewise smoothness of the function $K_1^*(t_1,t_2)$ with respect to the variable $t_2\in[t,T]$ (t_1 is fixed).

Obtaining (2.197), we also used the fact that the right-hand side of (2.197) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K_1(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K_1(t_1, t_1 - 0) + K_1(t_1, t_1 + 0)\right) = \frac{1}{2}\psi_2(t_1)\psi_3(t_1) = K_1^*(t_1, t_1).$$

Let us substitute $t_1 = t_2$ into (2.197). Then we have (2.195).

From (2.194) and (2.195) we get

$$E_p' = \sum_{j_1=0}^{p}\left(\int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T \sum_{j_3=p+1}^{\infty}\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds ds_1 ds_2\right)^2. \tag{2.198}$$

Analogously with (2.24) we obtain for the twice continuously differentiable function $\psi_3(s)$ the following estimate

$$\left|\sum_{j_3=p+1}^{\infty}\phi_{j_3}(s_1)\int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds\right| <$$

$$< \frac{C}{p}\left(\frac{1}{(1-(z(s_1))^2)^{1/2}} + \frac{1}{(1-(z(s_1))^2)^{1/4}}\right), \tag{2.199}$$

where $s_1 \in (t, T)$, $z(s_1)$ is defined by (2.20), and constant C does not depend on p.

Further consideration is analogously to the proof of (2.185). The relation (2.186) is proved.

Let us prove (2.187). We have

$$E_p'' \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\} = \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1 j_3 j_1}\right)^2, \tag{2.200}$$

$$C_{j_1 j_3 j_1} = \int\limits_t^T \psi_3(s)\phi_{j_1}(s)\int\limits_t^s \psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 ds =$$

$$= \int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2 \int\limits_{s_1}^T \psi_3(s)\phi_{j_1}(s)ds ds_1. \tag{2.201}$$

After substituting (2.201) into (2.200), we obtain

$$E_p'' = \sum_{j_3=0}^{p} \left(\int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1) \sum_{j_1=0}^{p} \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \int\limits_{s_1}^{T} \psi_3(s)\phi_{j_1}(s)dsds_1 \right)^2 . \tag{2.202}$$

Introduce the auxiliary function

$$\tilde{K}(t_1,t_2) = \psi_1(t_1)\mathbf{1}_{\{t_1<t_2\}}, \quad t_1,t_2 \in [t,T].$$

Let us expand the function $\tilde{K}(t_1,t_2)$ using the variable t_1, when t_2 is fixed, into the Fourier–Legendre series at the interval (t,T)

$$\tilde{K}(t_1,t_2) = \sum_{j_1=0}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \cdot \phi_{j_1}(t_1) \quad (t_1 \neq t_2). \tag{2.203}$$

Using (2.203), we have

$$\sum_{j_1=0}^{p} \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \int\limits_{s_1}^{T} \psi_3(s)\phi_{j_1}(s)ds =$$

$$= \int\limits_{s_1}^{T} \psi_3(s) \left(\sum_{j_1=0}^{p} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \right) ds =$$

$$= \int\limits_{s_1}^{T} \psi_3(s) \left(\sum_{j_1=0}^{\infty} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta - \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \right) ds =$$

$$= \int\limits_{s_1}^{T} \psi_3(s)\psi_1(s)\mathbf{1}_{\{s<s_1\}}ds - \int\limits_{s_1}^{T} \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds =$$

$$= - \int\limits_{s_1}^{T} \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds. \tag{2.204}$$

After substituting (2.204) into (2.202), we get

$$E_p'' = \sum_{j_3=0}^{p} \left(\int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_{s_1}^{T} \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int\limits_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds ds_1 \right)^2 =$$

$$
=\sum_{j_3=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}\psi_2(u_l^*)\phi_{j_3}(u_l^*)\int\limits_{u_l^*}^{T}\psi_3(s)\sum_{j_1=p+1}^{\infty}\phi_{j_1}(s)\int\limits_{t}^{u_l^*}\psi_1(\theta)\phi_{j_1}(\theta)d\theta ds\Delta u_l\right)^2=
$$

$$
=\sum_{j_3=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}\psi_2(u_l^*)\phi_{j_3}(u_l^*)\sum_{j_1=p+1}^{\infty}\int\limits_{u_l^*}^{T}\psi_3(s)\phi_{j_1}(s)ds\int\limits_{t}^{u_l^*}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\Delta u_l\right)^2, \tag{2.205}
$$

where $t=u_0<u_1<\ldots<u_N=T$, $\Delta u_l=u_{l+1}-u_l$, u_l^* is a point of minimum of the function $(1-(z(s))^2)^{-\alpha}$ $(0<\alpha<1)$ at the interval $[u_l,u_{l+1}]$, $l=0,1,\ldots,N-1$,

$$
\max_{0\le l\le N-1}\Delta u_l\to 0\quad\text{when}\quad N\to\infty.
$$

The last step in (2.205) is correct due to uniform convergence of the Fourier–Legendre series of the piecewise smooth function $\tilde{K}(s,u_l^*)$ at the interval $[u_l^*+\varepsilon,T-\varepsilon]$ for any $\varepsilon>0$ (the function $\tilde{K}(s,u_l^*)$ is continuous at the interval $[u_l^*,T]$).

Let us write the following relation

$$
\int\limits_{t}^{x}\psi_1(s)\phi_{j_1}(s)ds=\frac{\sqrt{T-t}\sqrt{2j_1+1}}{2}\int\limits_{-1}^{z(x)}P_{j_1}(y)\psi(u(y))dy=
$$

$$
=\frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\Biggl((P_{j_1+1}(z(x))-P_{j_1-1}(z(x)))\psi_1(x)-
$$

$$
-\frac{T-t}{2}\int\limits_{-1}^{z(x)}((P_{j_1+1}(y)-P_{j_1-1}(y))\psi_1{}'(u(y))dy\Biggr), \tag{2.206}
$$

where $x\in(t,T)$, $j_1\ge p+1$, $z(x)$ and $u(y)$ are defined by (2.20), $\psi_1{}'$ is a derivative of the function $\psi_1(s)$ with respect to the variable $u(y)$.

Note that in (2.206) we used the following well known property of the Legendre polynomials [88]

$$
P_{j+1}(-1)=-P_j(-1),\quad j=0,1,2,\ldots
$$

and (2.21).

From (2.99) and (2.206) we obtain

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds\right| < \frac{C}{j_1}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + C_1\right), \quad x \in (t,T), \tag{2.207}$$

where constants C, C_1 do not depend on j_1.

Similarly to (2.207) and due to

$$P_j(1) = 1, \quad j = 0, 1, 2, \ldots$$

we obtain an analogue of (2.207) for the integral, which is similar to the integral on the left-hand side of (2.207), but with integration limits x and T.

From the formula (2.207) and its analogue for the integral with integration limits x and T we obtain

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds \int\limits_x^T \psi_3(s)\phi_{j_1}(s)ds\right| < \frac{K}{j_1^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + K_1\right), \tag{2.208}$$

where $x \in (t,T)$ and constants K, K_1 do not depend on j_1.

Let us estimate the right-hand side of (2.205) using (2.208)

$$E_p'' \le L\sum_{j_3=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}|\phi_{j_3}(u_l^*)|\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(u_l^*))^2)^{1/2}} + K_1\right)\Delta u_l\right)^2 \le$$

$$\le \frac{L_1}{p^2}\sum_{j_3=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}\left(\frac{1}{(1-(z(u_l^*))^2)^{3/4}} + \frac{K_1}{(1-(z(u_l^*))^2)^{1/4}}\right)\Delta u_l\right)^2 \le$$

$$\le \frac{L_1}{p^2}\sum_{j_3=0}^{p}\left(\lim_{N\to\infty}\left(\int\limits_t^T \frac{ds}{(1-(z(s))^2)^{3/4}} + K_1\int\limits_t^T \frac{ds}{(1-(z(s))^2)^{1/4}}\right)\right)^2 =$$

$$= \frac{L_1}{p^2}\sum_{j_3=0}^{p}\left(\int\limits_t^T \frac{ds}{(1-(z(s))^2)^{3/4}} + K_1\int\limits_t^T \frac{ds}{(1-(z(s))^2)^{1/4}}\right)^2 =$$

$$= \frac{L_1(T-t)^2}{4p^2}\sum_{j_3=0}^{p}\left(\int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} + K_1\int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/4}}\right)^2 \le$$

$$\le \frac{L_2 p}{p^2} = \frac{L_2}{p} \to 0 \tag{2.209}$$

if $p \to \infty$, where constants L, L_1, L_2 do not depend on p and we used (2.25) and (2.99) in (2.209). The relation (2.187) is proved. Theorem 2.7 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. Analogously to (2.29) we obtain

$$\left| \int\limits_{s_2}^{T} \sum_{j_3=p+1}^{\infty} \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_{s_1}^{T} \psi_3(s)\phi_{j_3}(s) ds ds_1 \right| \le \frac{K_1}{p}, \tag{2.210}$$

where s_2 is fixed and constant K_1 does not depend on p.

Using (2.29) and (2.190), we obtain

$$E_p \le K \sum_{j_3=0}^{p} \left(\int\limits_{t}^{T} \left| \int\limits_{t}^{s} \sum_{j_1=p+1}^{\infty} \psi_2(s_1)\phi_{j_1}(s_1) \int\limits_{t}^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 \right| ds \right)^2 =$$

$$= K \sum_{j_3=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} \left| \int\limits_{t}^{u_l^*} \sum_{j_1=p+1}^{\infty} \psi_2(s_1)\phi_{j_1}(s_1) \int\limits_{t}^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 \right| \Delta u_l \right)^2 \le$$

$$\le K \sum_{j_3=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} \frac{K_1}{p} \Delta u_l \right)^2 \le \frac{K_2}{p^2} \sum_{j_3=0}^{p} (T-t)^2 \le \frac{L}{p} \to 0 \tag{2.211}$$

if $p \to \infty$, where constants K, K_1, K_2, L do not depend on p, $\Delta u_l = u_{l+1} - u_l$, $u_l^* \in [u_l, u_{l+1}]$, $l = 0, 1, \ldots, N-1$, $t = u_0 < u_1 < \ldots < u_N = T$,

$$\max_{0\le l\le N-1} \Delta u_l \to 0 \quad \text{when} \quad N \to \infty.$$

Analogously, using (2.210) and (2.198), we obtain that $E_p' \to 0$ if $p \to \infty$.

It is not difficult to see that in our case we have

$$\left| \int\limits_{t}^{x} \psi_1(s)\phi_{j_1}(s) ds \int\limits_{x}^{T} \psi_3(s)\phi_{j_1}(s) ds \right| < \frac{K}{j_1^2} \quad (j_1 \ne 0), \tag{2.212}$$

where constant K does not depend on j_1.

Using (2.205) and (2.212), we get

$$E_p'' \le K_1 \sum_{j_3=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} \sum_{j_1=p+1}^{\infty} \left| \int\limits_{u_l^*}^{T} \psi_3(s)\phi_{j_1}(s)ds \int\limits_{t}^{u_l^*} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \right| \Delta u_l \right)^2 \le$$

$$\le K_2 \sum_{j_3=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2}\Delta u_l \right)^2 \le \frac{K_2}{p^2}\sum_{j_3=0}^{p}(T-t)^2 \le \frac{L}{p} \to 0$$

if $p\to\infty$, where constants K_1, K_2, L do not depend on p, another notations are the same as in (2.211).

Theorem 2.7 is proved for the trigonometric case. Theorem 2.7 is proved.

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4 Based on Theorem 1.1. The Case $p_1 = \ldots = p_4 \to \infty$ (Cases of Legendre Polynomials and Trigonometric Functions)

In this section, we will develop the approach to expansion of iterated Stratonovich stochatic integrals based on Theorem 1.1 for the stochastic integrals of multiplicity 4.

Theorem 2.8 [8]-[14], [20], [31]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \tag{2.213}$$

that converges in the mean-square sense is valid, where

$$C_{j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(s_4) \int\limits_t^{s_4} \phi_{j_3}(s_3) \int\limits_t^{s_3} \phi_{j_2}(s_2) \int\limits_t^{s_2} \phi_{j_1}(s_1) ds_1 ds_2 ds_3 ds_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Proof. The relation (1.44) (in the case when $p_1 = \ldots = p_4 = p \to \infty$) implies that

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{T,t} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}} A_1^{(i_3 i_4)} + \mathbf{1}_{\{i_1=i_3\neq 0\}} A_2^{(i_2 i_4)} + \mathbf{1}_{\{i_1=i_4\neq 0\}} A_3^{(i_2 i_3)} + \mathbf{1}_{\{i_2=i_3\neq 0\}} A_4^{(i_1 i_4)} +$$

$$+\mathbf{1}_{\{i_2=i_4\neq 0\}} A_5^{(i_1 i_3)} + \mathbf{1}_{\{i_3=i_4\neq 0\}} A_6^{(i_1 i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}} \mathbf{1}_{\{i_3=i_4\neq 0\}} B_1 -$$

$$- \mathbf{1}_{\{i_1=i_3\neq 0\}} \mathbf{1}_{\{i_2=i_4\neq 0\}} B_2 - \mathbf{1}_{\{i_1=i_4\neq 0\}} \mathbf{1}_{\{i_2=i_3\neq 0\}} B_3, \qquad (2.214)$$

where $J[\psi^{(4)}]_{T,t}$ has the form (2.7) for $\psi_1(s), \ldots, \psi_4(s) \equiv 1$ and $i_1, \ldots, i_4 = 0, 1, \ldots, m$,

$$A_1^{(i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$A_2^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_3} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$A_3^{(i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_4} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$A_4^{(i_1 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)},$$

$$A_5^{(i_1 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_4 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)},$$

$$A_6^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2,j_1=0}^{p} C_{j_3 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$
B_1 = \lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_1}, \quad B_2 = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_3j_4j_3j_4},
$$

$$
B_3 = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_4j_3j_3j_4}.
$$

Using the integration order replacement in Riemann integrals, Theorem 1.1 for $k=2$ (see (1.42)) and (2.10), Parseval's equality and the integration order replacement technique for Itô stochastic integrals (see Chapter 3) [1]-[14], [60], [91], [92] or Itô's formula, we obtain

$$
A_1^{(i_3i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1) \left(\int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1) \sum_{j_1=0}^{p} \left(\int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1) \left((s_1 - t) - \sum_{j_1=p+1}^{\infty} \left(\int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 \right) ds_1 ds \times
$$

$$
\times \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1)(s_1 - t) ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \Delta_1^{(i_3i_4)} =
$$

$$
= \frac{1}{2} \int\limits_t^T \int\limits_t^s (s_1 - t) d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)} +
$$

$$
+ \frac{1}{2} \mathbf{1}_{\{i_3=i_4\neq 0\}} \lim_{p\to\infty} \sum_{j_3=0}^{p} \int\limits_t^T \phi_{j_3}(s) \int\limits_t^s \phi_{j_3}(s_1)(s_1 - t) ds_1 ds - \Delta_1^{(i_3i_4)} =
$$

$$
= \frac{1}{2} \int\limits_t^T \int\limits_t^s \int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)} + \frac{1}{4} \mathbf{1}_{\{i_3=i_4\neq 0\}} \int\limits_t^T (s_1 - t) ds_1 - \Delta_1^{(i_3i_4)} \quad \text{w. p. 1}, \quad (2.215)
$$

where

$$\Delta_1^{(i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$a_{j_4 j_3}^p = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1 ds. \qquad (2.216)$$

Let us consider $A_2^{(i_2 i_4)}$

$$A_2^{(i_2 i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\left(\int\limits_t^s \phi_{j_3}(s_1)ds_1\right)^2\int\limits_t^s \phi_{j_2}(s_1)ds_1 ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_1)\left(\int\limits_t^{s_1}\phi_{j_3}(s_2)ds_2\right)^2 ds_1 ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_3)\left(\int\limits_{s_3}^s \phi_{j_3}(s_1)ds_1\right)^2 ds_3 ds\right)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)(s-t)\int\limits_t^s \phi_{j_2}(s_1)ds_1 ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_1)(s_1-t)ds_1 ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_3)(s-t+t-s_3)ds_3 ds\right)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\Delta_2^{(i_2 i_4)} + \Delta_1^{(i_2 i_4)} + \Delta_3^{(i_2 i_4)} = -\Delta_2^{(i_2 i_4)} + \Delta_1^{(i_2 i_4)} + \Delta_3^{(i_2 i_4)} \ \text{w. p. 1}, \qquad (2.217)$$

where

$$\Delta_2^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} b_{j_4 j_2}^p \zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_2=0}^{p} c_{j_4 j_2}^{p} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$b_{j_4 j_2}^{p} = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s) \sum_{j_3=p+1}^{\infty} \left(\int\limits_t^s \phi_{j_3}(s_1)ds_1\right)^2 \int\limits_t^s \phi_{j_2}(s_1)ds_1 ds,$$

$$c_{j_4 j_2}^{p} = \frac{1}{2}\int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_2}(s_3) \sum_{j_3=p+1}^{\infty} \left(\int\limits_{s_3}^s \phi_{j_3}(s_1)ds_1\right)^2 ds_3 ds.$$

Let us consider $A_5^{(i_1 i_3)}$

$$A_5^{(i_1 i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_4}(s_2) \int\limits_{s_2}^T \phi_{j_3}(s_1) \int\limits_{s_1}^T \phi_{j_4}(s) ds ds_1 ds_2 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_4}(s) \int\limits_{s_3}^s \phi_{j_3}(s_1) \int\limits_{s_3}^{s_1} \phi_{j_4}(s_2) ds_2 ds_1 ds ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \Biggl(\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \left(\int\limits_{s_3}^T \phi_{j_4}(s)ds\right)^2 \int\limits_{s_3}^T \phi_{j_3}(s) ds ds_3 -$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s) \left(\int\limits_{s_3}^s \phi_{j_4}(s_1)ds_1\right)^2 ds ds_3 -$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s_2) \left(\int\limits_{s_2}^T \phi_{j_4}(s_1)ds_1\right)^2 ds_2 ds_3 \Biggr) \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} \Biggl(\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)(T-s_3) \int\limits_{s_3}^T \phi_{j_3}(s) ds ds_3 -$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s)(s-s_3) ds ds_3 -$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_2)(T-s_2)ds_2ds_3\Biggr)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}=-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}\quad \text{w. p. 1}, \qquad (2.218)
$$

where

$$
\Delta_4^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} d_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_5^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} e_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_6^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} f_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
d_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^T \phi_{j_4}(s)ds\right)^2\int\limits_{s_3}^T \phi_{j_3}(s)dsds_3,
$$

$$
e_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^s \phi_{j_4}(s_1)ds_1\right)^2 dsds_3,
$$

$$
f_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_4}(s_1)ds_1\right)^2 ds_2ds_3=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_4}(s_1)ds_1\right)^2\int\limits_t^{s_2} \phi_{j_1}(s_3)ds_3ds_2.
$$

Moreover,

$$
A_3^{(i_2i_3)}+A_5^{(i_2i_3)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\left(C_{j_4j_3j_2j_4}+C_{j_4j_3j_4j_2}\right)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_3}(s_1)\int\limits_t^{s_1} \phi_{j_2}(s_2)\int\limits_t^{s_1} \phi_{j_4}(s_3)ds_3ds_2ds_1ds\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 \int\limits_{s_1}^T \phi_{j_4}(s) ds ds_1 \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \left(\int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{T} \phi_{j_4}(s_3) ds_3 \int\limits_{s_1}^T \phi_{j_4}(s) ds ds_2 ds_1 - \right.
$$

$$
\left. - \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left(\int\limits_{s_1}^T \phi_{j_4}(s) ds \right)^2 ds_2 ds_1 \right) \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left((T-s_1) - \sum_{j_4=0}^{p} \left(\int\limits_{s_1}^T \phi_{j_4}(s_3) ds_3 \right)^2 \right) ds_2 ds_1 \times
$$

$$
\times \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} = 2\Delta_6^{(i_2 i_3)} \quad \text{w. p. 1.}
$$

Then

$$
A_3^{(i_2 i_3)} = 2\Delta_6^{(i_2 i_3)} - A_5^{(i_2 i_3)} = \Delta_4^{(i_2 i_3)} - \Delta_5^{(i_2 i_3)} + \Delta_6^{(i_2 i_3)} \quad \text{w. p. 1.} \tag{2.219}
$$

Let us consider $A_4^{(i_1 i_4)}$

$$
A_4^{(i_1 i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^{s} \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(s_2) \int\limits_{s_2}^{s} \phi_{j_3}(s_1) ds_1 ds_2 ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^{s} \phi_{j_1}(s_3) \sum_{j_3=0}^{p} \left(\int\limits_{s_3}^{s} \phi_{j_3}(s_2) ds_2 \right)^2 ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^{s} \phi_{j_1}(s_3) (s-s_3) ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \Delta_3^{(i_1 i_4)} =
$$

$$
= \frac{1}{2} \int\limits_t^T \int\limits_t^{s} (s-s_3) d\mathbf{w}_{s_3}^{(i_1)} d\mathbf{w}_s^{(i_4)} +
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\neq 0\}}\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)(s-s_3)ds_3ds-\Delta_3^{(i_1i_4)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\neq 0\}}\left(\sum_{j_4=0}^{\infty}\int\limits_t^T(s-t)\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)ds_3ds-\right.
$$

$$
\left.-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s(s_3-t)\phi_{j_4}(s_3)ds_3ds\right)-\Delta_3^{(i_1i_4)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-\Delta_3^{(i_1i_4)}\quad \text{w. p. 1.} \tag{2.220}
$$

Let us consider $A_6^{(i_1i_2)}$

$$
A_6^{(i_1i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2,j_1=0}^{p}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\int\limits_{s_2}^T\phi_{j_3}(s_1)\int\limits_{s_1}^T\phi_{j_3}(s)dsds_1ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\sum_{j_3=0}^{p}\left(\int\limits_{s_2}^T\phi_{j_3}(s)ds\right)^2ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)(T-s_2)ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_1}(s_3)ds_3ds_2\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T(T-s_2)\int\limits_t^{s_2}d\mathbf{w}_{s_3}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_2}(s_3)ds_3ds_2-\Delta_6^{(i_1i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T(T-s_2)ds_2-\Delta_6^{(i_1i_2)}\quad \text{w. p. 1.}
\tag{2.221}
$$

Let us consider B_1, B_2, B_3

$$
B_1=\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_1)\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2ds_1ds=
$$

$$
=\lim_{p\to\infty}\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_1)(s_1-t)ds_1ds-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^p=
$$

$$
=\frac{1}{4}\int\limits_t^T(s_1-t)ds_1-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^p,
\tag{2.222}
$$

$$
B_2=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_3)\left(\int\limits_t^{s_3}\phi_{j_4}(s)ds\right)^2\int\limits_t^{s_3}\phi_{j_3}(s_1)ds_1ds_3-\right.
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_3}(s_2)\left(\int\limits_t^{s_2}\phi_{j_4}(s_3)ds_3\right)^2ds_2ds_1-
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_3}(s)\left(\int\limits_s^{s_1}\phi_{j_4}(s_2)ds_2\right)^2dsds_1\right)=
$$

$$
=\sum_{j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_3)(s_3-t)\int\limits_t^{s_3}\phi_{j_3}(s_1)ds_1ds_3-\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p-
$$

$$
-\sum_{j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\int\limits_t^{s_1}(s_2-t)\phi_{j_3}(s_2)ds_2ds_1+
$$

$$
+\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}-\sum_{j_3=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_3}(s)(s_1-t+t-s)dsds_1+
$$

$$
+\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^{p}=
$$

$$
=\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}+\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^{p}-\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^{p}. \tag{2.223}
$$

Moreover,

$$
B_2+B_3=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(C_{j_3j_4j_3j_4}+C_{j_3j_4j_4j_3}\right)=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^{s}\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2ds_1ds=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2\int\limits_{s_1}^{T}\phi_{j_3}(s)dsds_1=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\int\limits_t^{T}\phi_{j_3}(s_2)ds_2\int\limits_{s_1}^{T}\phi_{j_3}(s)dsds_3ds_1-\right.
$$

$$
\left.-\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)ds\right)^2ds_3ds_1\right)=
$$

$$
=\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1-
$$

$$
-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1+2\lim_{p\to\infty}\sum_{j_4=0}^{p}f_{j_4j_4}^{p}=
$$

$$
=2\lim_{p\to\infty}\sum_{j_4=0}^{p}f_{j_4j_4}^{p}. \tag{2.224}
$$

Therefore,

$$B_3 = 2\lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3j_3}^{p} - \lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3j_3}^{p} - \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p} + \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3j_3}^{p}. \qquad (2.225)$$

After substituting the relations (2.215)–(2.225) into (2.214), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} =$$

$$= J[\psi^{(4)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_s^{(i_1)}ds_1 d\mathbf{w}_{s_2}^{(i_4)} + \frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1 +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2 ds_1 + R = J^*[\psi^{(4)}]_{T,t} +$$

$$+ R \quad \text{w. p. 1}, \qquad (2.226)$$

where

$$R = -\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2i_4)} + \Delta_1^{(i_2i_4)} + \Delta_3^{(i_2i_4)}\right) +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_4^{(i_2i_3)} - \Delta_5^{(i_2i_3)} + \Delta_6^{(i_2i_3)}\right) - \mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1i_4)} +$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1i_3)} + \Delta_5^{(i_1i_3)} + \Delta_6^{(i_1i_3)}\right) - \mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1i_2)} -$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3j_3}^{p} + \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p} - \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3j_3}^{p}\right) -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(2\lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3j_3}^{p} - \lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3j_3}^{p} -\right.$$

$$\left. - \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p} + \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3j_3}^{p}\right) +$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}. \qquad (2.227)
$$

From (2.226) and (2.227) it follows that Theorem 2.8 will be proved if

$$
\Delta_k^{(ij)}=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}=\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^{p}=\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^{p}=\lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^{p}=0, \qquad (2.228)
$$

where $k=1,2,\ldots,6,\ i,j=0,1,\ldots,m.$

Let us consider the case of Legendre polynomials. Let us prove that $\Delta_1^{(i_3i_4)}=0$ w. p. 1.

We have

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(2a_{j_3j_3}^{p}a_{j_3'j_3'}^{p}+\left(a_{j_3j_3'}^{p}\right)^2+2a_{j_3j_3'}^{p}a_{j_3'j_3}^{p}+\left(a_{j_3'j_3}^{p}\right)^2\right)+3\sum_{j_3'=0}^{p}\left(a_{j_3'j_3'}^{p}\right)^2=
$$

$$
=\left(\sum_{j_3=0}^{p}a_{j_3j_3}^{p}\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(a_{j_3j_3'}^{p}+a_{j_3'j_3}^{p}\right)^2+2\sum_{j_3'=0}^{p}\left(a_{j_3'j_3'}^{p}\right)^2\quad (i_3=i_4\neq 0), \qquad (2.229)
$$

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\sum_{j_3,j_4=0}^{p}\left(a_{j_4j_3}^{p}\right)^2\quad (i_3\neq i_4,\ i_3\neq 0,\ i_4\neq 0), \qquad (2.230)
$$

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\begin{cases}(T-t)\sum\limits_{j_4=0}^{p}\left(a_{j_4,0}^{p}\right)^2 & \text{if}\quad i_3=0,\ i_4\neq 0\\ \\ (T-t)\sum\limits_{j_3=0}^{p}\left(a_{0,j_3}^{p}\right)^2 & \text{if}\quad i_4=0,\ i_3\neq 0\\ \\ (T-t)^2\left(a_{00}^{p}\right)^2 & \text{if}\quad i_3=i_4=0\end{cases}. \qquad (2.231)
$$

Let us consider the case $i_3 = i_4 \ne 0$

$$a^p_{j_4 j_3} = \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_4}(y) \int\limits_{-1}^{y} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} (2j_1+1) \left(\int\limits_{-1}^{y_1} P_{j_1}(y_2)dy_2\right)^2 dy_1 dy =$$

$$= \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 \int\limits_{y_1}^{1} P_{j_4}(y)dydy_1 =$$

$$= \frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1) \left(P_{j_4-1}(y_1) - P_{j_4+1}(y_1)\right) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 \ne 0$ and

$$a^p_{j_4 j_3} = \frac{(T-t)^2\sqrt{2j_3+1}}{32}\times$$

$$\times \int\limits_{-1}^{1} P_{j_3}(y_1)(1-y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 = 0$.

From (2.99) and the estimate

$$|P_{j_4-1}(y) - P_{j_4+1}(y)| \le 2, \quad y \in [-1,1] \quad (|\, P_j(y)\, | \le 1, \quad y \in [-1,1])$$

we obtain

$$|a^p_{j_4 j_3}| \le \frac{C_0}{\sqrt{j_4}} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} \le \frac{C_1}{p\sqrt{j_4}} \quad (j_4 \ne 0), \tag{2.232}$$

$$|a^p_{0,j_3}| \le C_0 \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} \le \frac{C_1}{p}, \tag{2.233}$$

$$|a_{00}^p| \le C_0 \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/2}} \le \frac{C_1}{p}, \tag{2.234}$$

where constants C_0, C_1 do not depend on p.

Taking into account (2.229)–(2.234), we have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^p \zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} = \left(a_{00}^p + \sum_{j_3=1}^{p} a_{j_3j_3}^p\right)^2 + \sum_{j_3'=1}^{p}\left(a_{0,j_3'}^p + a_{j_3',0}^p\right)^2 +$$

$$+\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\left(a_{j_3j_3'}^p + a_{j_3'j_3}^p\right)^2 + 2\left(\sum_{j_3'=1}^{p}\left(a_{j_3'j_3'}^p\right)^2 + (a_{00})^2\right) \le$$

$$\le K_0\left(\frac{1}{p} + \frac{1}{p}\sum_{j_3=1}^{p}\frac{1}{\sqrt{j_3}}\right)^2 + \frac{K_1}{p} + K_2\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\frac{1}{p^2}\left(\frac{1}{\sqrt{j_3'}} + \frac{1}{\sqrt{j_3}}\right)^2 \le$$

$$\le K_0\left(\frac{1}{p} + \frac{1}{p}\int\limits_0^p \frac{dx}{\sqrt{x}}\right)^2 + \frac{K_1}{p} + \frac{K_3}{p}\sum_{j_3=1}^{p}\frac{1}{j_3} \le$$

$$\le K_0\left(\frac{1}{p} + \frac{2}{\sqrt{p}}\right)^2 + \frac{K_1}{p} + \frac{K_3}{p}\left(1 + \int\limits_1^p \frac{dx}{x}\right) \le$$

$$\le \frac{K_4}{p} + \frac{K_3\left(\ln p + 1\right)}{p} \to 0$$

if $p \to \infty$ $(i_3 = i_4 \ne 0)$.

The same result for the cases (2.230), (2.231) also follows from the estimates (2.232)–(2.234). Therefore,

$$\Delta_1^{(i_3i_4)} = 0 \quad \text{w. p. 1.} \tag{2.235}$$

It is not difficult to see that the formulas

$$\Delta_2^{(i_2i_4)} = 0, \quad \Delta_4^{(i_1i_3)} = 0, \quad \Delta_6^{(i_1i_3)} = 0 \quad \text{w. p. 1} \tag{2.236}$$

can be proved similarly with the proof of (2.235).

Moreover, from the estimates (2.232)–(2.234) we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3j_3}^p = 0. \tag{2.237}$$

The relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}=0 \quad \text{and} \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3}=0 \tag{2.238}$$

can also be proved analogously with (2.237).

Let us consider $\Delta_3^{(i_2i_4)}$

$$\Delta_3^{(i_2i_4)}=\Delta_4^{(i_2i_4)}+\Delta_6^{(i_2i_4)}-\Delta_7^{(i_2i_4)}=-\Delta_7^{(i_2i_4)} \quad \text{w. p. 1}, \tag{2.239}$$

where

$$\Delta_7^{(i_2i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_2,j_4=0}^{p} g^p_{j_4j_2}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$g^p_{j_4j_2}=\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_1}(s_2)ds_2\int\limits_s^T\phi_{j_1}(s_2)ds_2\right)ds_1ds=$$

$$=\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2\int\limits_t^s\phi_{j_2}(s_1)\int\limits_{s_1}^T\phi_{j_1}(s_2)ds_2ds_1ds. \tag{2.240}$$

The last step in (2.240) follows from the estimate

$$\left|g^p_{j_4j_2}\right|\le K\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{1}{(1-y^2)^{1/2}}\int\limits_{-1}^{y}\frac{1}{(1-x^2)^{1/2}}dxdy\ \le\frac{K_1}{p}.$$

Note that

$$g^p_{j_4j_4}=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\right)^2, \tag{2.241}$$

$$g^p_{j_4j_2}+g^p_{j_2j_4}=\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds, \tag{2.242}$$

and

$$g^p_{j_4j_2}=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_2+1)}}{16}\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)\times$$

$$\times\int\limits_{-1}^{y_1}P_{j_2}(y)\left(P_{j_1-1}(y)-P_{j_1+1}(y)\right)dydy_1,\quad j_4,j_2\le p.$$

Due to orthogonality of the Legendre polynomials we obtain

$$g_{j_4j_2}^{p}+g_{j_2j_4}^{p}=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_2+1)}}{16}\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)dy_1\times$$

$$\times\int\limits_{-1}^{1}P_{j_2}(y)\left(P_{j_1-1}(y)-P_{j_1+1}(y)\right)dy=$$

$$=\frac{(T-t)^2(2p+1)}{16}\frac{1}{2p+3}\left(\int\limits_{-1}^{1}P_p^2(y_1)dy_1\right)^2\cdot\begin{cases}1 & \text{if } j_2=j_4=p\\ \\ 0 & \text{otherwise}\end{cases}=$$

$$=\frac{(T-t)^2}{4(2p+3)(2p+1)}\cdot\begin{cases}1 & \text{if } j_2=j_4=p\\ \\ 0 & \text{otherwise}\end{cases},\qquad(2.243)$$

$$g_{j_4j_4}^{p}=\frac{(T-t)^2(2j_4+1)}{16}\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\cdot\frac{1}{2}\left(\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)dy_1\right)^2=$$

$$=\frac{(T-t)^2(2p+1)}{32}\frac{1}{2p+3}\left(\int\limits_{-1}^{1}P_p^2(y_1)dy_1\right)^2\cdot\begin{cases}1 & \text{if } j_4=p\\ \\ 0 & \text{otherwise}\end{cases}=$$

$$=\frac{(T-t)^2}{8(2p+3)(2p+1)}\cdot\begin{cases}1 & \text{if } j_4=p\\ \\ 0 & \text{otherwise}\end{cases}.\qquad(2.244)$$

From (2.229), (2.243), and (2.3) it follows that

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p} g_{j_4 j_2}^{p}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\left(\sum_{j_3=0}^{p} g_{j_3 j_3}^{p}\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(g_{j_3 j_3'}^{p}+g_{j_3' j_3}^{p}\right)^2+2\sum_{j_3'=0}^{p}\left(g_{j_3' j_3'}^{p}\right)^2=
$$

$$
=\left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2+\ 0\ +2\left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2\ \to 0
$$

if $p\to\infty$ $(i_2=i_4\ne 0)$.

Let us consider the case $i_2\ne i_4,\ \ i_2\ne 0,\ \ i_4\ne 0$ (see (2.230)). It is not difficult to see that

$$
g_{j_4 j_2}^{p}=\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_1)F_p(s,s_1)ds_1 ds=\int\limits_{[t,T]^2}K_p(s,s_1)\phi_{j_4}(s)\phi_{j_2}(s_1)ds_1 ds
$$

is a coefficient of the double Fourier–Legendre series of the function

$$
K_p(s,s_1)=\mathbf{1}_{\{s_1<s\}}F_p(s,s_1), \tag{2.245}
$$

where

$$
\sum_{j_1=p+1}^{\infty}\int\limits_{s_1}^{T}\phi_{j_1}(s_2)ds_2\int\limits_{s}^{T}\phi_{j_1}(s_2)ds_2\stackrel{\sf def}{=}F_p(s,s_1).
$$

The Parseval equality in this case looks as follows

$$
\lim_{p_1\to\infty}\sum_{j_4,j_2=0}^{p_1}\left(g_{j_4 j_2}^{p}\right)^2=\int\limits_{[t,T]^2}\left(K_p(s,s_1)\right)^2 ds_1 ds=\int\limits_t^T\int\limits_t^s\left(F_p(s,s_1)\right)^2 ds_1 ds. \tag{2.246}
$$

From (2.99) we obtain

$$
\left|\int\limits_{s_1}^{T}\phi_{j_1}(\theta)d\theta\right|=\frac{1}{2}\sqrt{2j_1+1}\sqrt{T-t}\left|\int\limits_{z(s_1)}^{1}P_{j_1}(y)dy\right|=
$$

$$
= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}} \left| P_{j_1-1}(z(s_1)) - P_{j_1+1}(z(s_1)) \right| \le \frac{K}{j_1} \frac{1}{\left(1-z^2(s_1)\right)^{1/4}}, \tag{2.247}
$$

where $z(s_1)$ is defined by (2.20), $s_1 \in (t, T)$.

From (2.247) we have

$$
\left(F_p(s, s_1)\right)^2 \le \frac{C^2}{p^2} \frac{1}{\left(1-z^2(s)\right)^{1/2}} \frac{1}{\left(1-z^2(s_1)\right)^{1/2}}, \quad s, s_1 \in (t, T). \tag{2.248}
$$

From (2.248) it follows that $|F_p(s, s_1)| \le M/p$ in the domain

$$
D_\varepsilon = \{(s, s_1): \ s \in [t+\varepsilon, T-\varepsilon], \ s_1 \in [t+\varepsilon, s]\} \quad \text{for any} \quad \varepsilon > 0,
$$

where constant M does not depend on s, s_1. Then we have the uniform convergence

$$
\sum_{j_1=0}^{p} \int\limits_{s}^{T} \phi_{j_1}(\theta) d\theta \int\limits_{s_1}^{T} \phi_{j_1}(\theta) d\theta \to \sum_{j_1=0}^{\infty} \int\limits_{s}^{T} \phi_{j_1}(\theta) d\theta \int\limits_{s_1}^{T} \phi_{j_1}(\theta) d\theta \tag{2.249}
$$

at the set D_ε if $p \to \infty$.

Because of continuity of the function on the left-hand side of (2.249) we obtain continuity of the limit function on the right-hand side of (2.249) at the set D_ε.

Using this fact and (2.248), we obtain

$$
\int\limits_{t}^{T} \int\limits_{t}^{s} \left(F_p(s, s_1)\right)^2 ds_1 ds = \lim_{\varepsilon \to +0} \int\limits_{t+\varepsilon}^{T-\varepsilon} \int\limits_{t+\varepsilon}^{s} \left(F_p(s, s_1)\right)^2 ds_1 ds \le
$$

$$
\le \frac{C^2}{p^2} \lim_{\varepsilon \to +0} \int\limits_{t+\varepsilon}^{T-\varepsilon} \int\limits_{t+\varepsilon}^{s} \frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}} \frac{ds}{\left(1-z^2(s)\right)^{1/2}} =
$$

$$
= \frac{C^2}{p^2} \int\limits_{t}^{T} \int\limits_{t}^{s} \frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}} \frac{ds}{\left(1-z^2(s)\right)^{1/2}} =
$$

$$
= \frac{K}{p^2} \int\limits_{-1}^{1} \int\limits_{-1}^{y} \frac{dy_1}{\left(1-y_1^2\right)^{1/2}} \frac{dy}{\left(1-y^2\right)^{1/2}} \le \frac{K_1}{p^2}, \tag{2.250}
$$

where constant K_1 does not depend on p.

From (2.250) and (2.246) we get

$$0 \le \sum_{j_2,j_4=0}^{p} \left(g_{j_4 j_2}^{p}\right)^2 \le \lim_{p_1\to\infty} \sum_{j_2,j_4=0}^{p_1} \left(g_{j_4 j_2}^{p}\right)^2 = \sum_{j_2,j_4=0}^{\infty} \left(g_{j_4 j_2}^{p}\right)^2 \le \frac{K_1}{p^2} \to 0 \qquad (2.251)$$

if $p\to\infty$. The case $i_2\ne i_4$, $i_2\ne 0$, $i_4\ne 0$ is proved.

The same result for the cases
1) $i_2=0$, $i_4\ne 0$,
2) $i_4=0$, $i_2\ne 0$,
3) $i_2=0$, $i_4=0$
can also be obtained. Then $\Delta_7^{(i_2 i_4)}=0$ and $\Delta_3^{(i_2 i_4)}=0$ w. p. 1.

Let us consider $\Delta_5^{(i_1 i_3)}$

$$\Delta_5^{(i_1 i_3)} = \Delta_4^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)} - \Delta_8^{(i_1 i_3)} \quad \text{w. p. 1},$$

where

$$\Delta_8^{(i_1 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} h_{j_3 j_1}^{p} \zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$h_{j_3 j_1}^{p} = \int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s)F_p(s_3,s)dsds_3.$$

Analogously, we obtain that $\Delta_8^{(i_1 i_3)}=0$ w. p. 1. Here we consider the function

$$K_p(s,s_3) = \mathbf{1}_{\{s_3<s\}}F_p(s_3,s)$$

and the relation

$$h_{j_3 j_1}^{p} = \int\limits_{[t,T]^2} K_p(s,s_3)\phi_{j_1}(s_3)\phi_{j_3}(s)dsds_3$$

for the case $i_1\ne i_3$, $i_1\ne 0$, $i_3\ne 0$.

For the case $i_1=i_3\ne 0$ we use (see (2.241), (2.242))

$$h_{j_1 j_1}^{p} = \sum_{j_4=p+1}^{\infty} \frac{1}{2}\left(\int\limits_t^T \phi_{j_1}(s)\int\limits_s^T \phi_{j_4}(s_1)ds_1ds\right)^2,$$

$$h_{j_3j_1}^p+h_{j_1j_3}^p=\sum_{j_4=p+1}^{\infty}\int\limits_t^T\phi_{j_1}(s)\int\limits_s^T\phi_{j_4}(s_2)ds_2ds\int\limits_t^T\phi_{j_3}(s)\int\limits_s^T\phi_{j_4}(s_2)ds_2ds.$$

Let us prove that

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p=0. \tag{2.252}$$

We have

$$c_{j_3j_3}^p=f_{j_3j_3}^p+d_{j_3j_3}^p-g_{j_3j_3}^p. \tag{2.253}$$

Moreover,

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^p=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}d_{j_3j_3}^p=0, \tag{2.254}$$

where the first equality in (2.254) has been proved earlier. Analogously, we can prove the second equality in (2.254).

From (2.3) we obtain

$$0\le\lim_{p\to\infty}\sum_{j_3=0}^{p}g_{j_3j_3}^p\le\lim_{p\to\infty}\frac{(T-t)^2}{8(2p+3)(2p+1)}=0.$$

So, (2.252) is proved. The relations (2.228) are proved for the polynomial case. Theorem 2.8 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. According to (2.216), in this case we have

$$a_{j_4j_3}^p=\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2\int\limits_{s_1}^T\phi_{j_4}(s)dsds_1.$$

Moreover,

$$\left|\int\limits_t^{s_1}\phi_j(s_2)ds_2\right|\le\frac{K_0}{j}\ (j\ne 0),\qquad \int\limits_{s_1}^T\phi_0(s)ds=\frac{T-s_1}{\sqrt{T-t}},$$

$$|a_{j_4j_3}^p|\le\frac{K_1}{j_4}\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\le\frac{K_1}{pj_4}\quad (j_4\ne 0),\qquad |a_{0,j_3}^p|\le\frac{K_1}{p}, \tag{2.255}$$

where K_0, K_1 are constants.

Taking into account (2.229)–(2.231) and (2.255), we obtain that $\Delta_1^{(i_3 i_4)} = 0$ w. p. 1. Analogously, $\Delta_2^{(i_2 i_4)} = 0$, $\Delta_4^{(i_1 i_3)} = 0$, $\Delta_6^{(i_1 i_3)} = 0$ w. p. 1 and

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3 j_3}^{p} = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3 j_3}^{p} = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3 j_3}^{p} = 0.$$

Let us consider $\Delta_3^{(i_2 i_4)}$. In this case for $i_2 = i_4 \neq 0$ we will use (2.239)–(2.242). We have

$$\int\limits_t^T \phi_{j_4}(s)\int\limits_s^T \phi_{j_1}(s_2)ds_2 ds = \frac{\sqrt{2}\sqrt{T-t}}{2\pi j_1}\int\limits_t^T \begin{cases} \phi_{j_4}(s)\left(1-\cos\frac{2\pi j_1(s-t)}{T-t}\right)ds \\ \\ \phi_{j_4}(s)\left(-\sin\frac{2\pi j_1(s-t)}{T-t}\right)ds \end{cases},$$

where $j_1 \geq p+1$, $j_4 = 0, 1, \ldots, p$.

Due to orthogonality of the trigonometric functions we have

$$\int\limits_t^T \phi_{j_4}(s)\int\limits_s^T \phi_{j_1}(s_2)ds_2 ds = \frac{\sqrt{2}(T-t)}{2\pi j_1}\cdot \begin{cases} 1 \ \text{ or } \ 0 & \text{if } j_4 = 0 \\ \\ 0 & \text{otherwise} \end{cases}, \quad j_1 \geq p+1. \tag{2.256}$$

From (2.256) and (2.240)–(2.242) we obtain

$$g_{j_4 j_2}^{p} + g_{j_2 j_4}^{p} = \sum_{j_1=p+1}^{\infty}\frac{(T-t)^2}{2\pi^2 j_1^2}\cdot \begin{cases} 1 \ \text{ or } \ 0 & \text{if } j_2 = j_4 = 0 \\ \\ 0 & \text{otherwise} \end{cases},$$

$$g_{j_4 j_4}^{p} = \sum_{j_1=p+1}^{\infty}\frac{(T-t)^2}{4\pi^2 j_1^2}\cdot \begin{cases} 1 \ \text{ or } \ 0 & \text{if } j_4 = 0 \\ \\ 0 & \text{otherwise} \end{cases}.$$

Therefore,

$$\begin{cases} \left|g_{j_4 j_2}^{p} + g_{j_2 j_4}^{p}\right| \leq \ K_1/p & \text{if } \ j_2 = j_4 = 0 \\ \\ g_{j_4 j_2}^{p} + g_{j_2 j_4}^{p} = 0 & \text{otherwise} \end{cases}, \qquad \begin{cases} \left|g_{j_4 j_4}^{p}\right| \leq \ K_1/p & \text{if } \ j_4 = 0 \\ \\ g_{j_4 j_4}^{p} = 0 & \text{otherwise} \end{cases}, \tag{2.257}$$

where constant K_1 does not depend on p.

From (2.257) and (2.229) it follows that $\Delta_7^{(i_2i_4)}=0$ and $\Delta_3^{(i_2i_4)}=0$ w. p. 1 for $i_2=i_4\ne 0$. Analogously to the polynomial case, we obtain $\Delta_7^{(i_2i_4)}=0$ and $\Delta_3^{(i_2i_4)}=0$ w. p. 1 for $i_2\ne i_4$, $i_2\ne 0$, $i_4\ne 0$. The similar arguments prove that $\Delta_5^{(i_1i_3)}=0$ w. p. 1.

Taking into account (2.253), (2.257) and the relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3j_3}^{p}=\lim_{p\to\infty}\sum_{j_3=0}^{p} d_{j_3j_3}^{p}=0,$$

which follows from the estimates

$$|f_{jj}^{p}|\le\frac{K_1}{pj},\quad |d_{jj}^{p}|\le\frac{K_1}{pj},\quad |f_{00}^{p}|\le\frac{K_1}{p},\quad |d_{00}^{p}|\le\frac{K_1}{p},$$

we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p}=-\lim_{p\to\infty}\sum_{j_3=0}^{p} g_{j_3j_3}^{p},$$

$$0\le\lim_{p\to\infty}\sum_{j_3=0}^{p} g_{j_3j_3}^{p}\le\lim_{p\to\infty}\frac{K_1}{p}=0.$$

Finally, we have

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3j_3}^{p}=0.$$

The relations (2.228) are proved for the trigonometric case. Theorem 2.8 is proved for the trigonometric case. Theorem 2.8 is proved.

2.4 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5 Based on Theorem 1.1. The Case $p_1=\ldots=p_5\to\infty$ and Constant Weight Functions (The Cases of Legendre Polynomials and Trigonometric Functions)

This section is devoted to the construction of expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on Theorem 1.1. The mentioned expansion converges in the mean-square sense and contains only one operation of the limit transition. As we saw in the previous sections, the expansions of iterated Stratonovich stochastic integrals turned out much simpler than the corresponding expansions of iterated Itô stochastic integrals (see Theorem 1.1). We use the expansions of the latter as a tool for the proof of expansions for the iterated Stratonovich stochastic integrals.

The following theorem adapt Theorem 1.1 for the iterated Stratonovich stochastic integrals (2.6) ($\psi_l(s)\equiv 1,\ i_l=0,1,\ldots,m,\ l=1,\ldots,5$).

Theorem 2.9 [14], [36]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}
$$

the following expansion

$$
J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where

$$
C_{j_5j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5,
$$

$i_1,i_2,i_3,i_4,i_5=0,1,\ldots,m,$ *and*

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j $(i\neq 0)$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. Note that we omit some details of the following proof, which can be done by analogy with the proof of Theorem 2.8. Let us denote

$$
A_1=\mathbf{1}_{\{i_1=i_2\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},
$$

$$A_2 = \mathbf{1}_{\{i_2=i_3\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_4,j_5=0}^{p} C_{j_5j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$A_3 = \mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_5=0}^{p} C_{j_5j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)},$$

$$A_4 = \mathbf{1}_{\{i_4=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$A_5 = \mathbf{1}_{\{i_1=i_3\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_4,j_5=0}^{p} C_{j_5j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$A_6 = \mathbf{1}_{\{i_1=i_4\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_5=0}^{p} C_{j_5j_1j_3j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)},$$

$$A_7 = \mathbf{1}_{\{i_1=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_1j_4j_3j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$A_8 = \mathbf{1}_{\{i_2=i_4\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_5=0}^{p} C_{j_5j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)},$$

$$A_9 = \mathbf{1}_{\{i_2=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_2j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$A_{10} = \mathbf{1}_{\{i_3=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_3j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$B_1 = \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3,j_5=0}^{p} C_{j_5j_3j_3j_1j_1}\zeta_{j_5}^{(i_5)},$$

$$B_2 = \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_4=0}^{p} C_{j_4j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)},$$

$$B_3 = \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3,j_4=0}^{p} C_{j_4j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)},$$

$$B_4 = \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_3,j_4=0}^{p} C_{j_3j_4j_3j_1j_1}\zeta_{j_4}^{(i_4)},$$

$$B_5 = \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p} C_{j_5j_2j_1j_2j_1}\zeta_{j_5}^{(i_5)},$$

$$B_6 = \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_4=0}^{p} C_{j_2j_4j_1j_2j_1}\zeta_{j_4}^{(i_4)},$$

$$B_7 = \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_4=0}^{p} C_{j_4j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)},$$

$$B_8 = \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p} C_{j_5j_1j_2j_2j_1}\zeta_{j_5}^{(i_5)},$$

$$B_9 = \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_2j_1}\zeta_{j_3}^{(i_3)},$$

$$B_{10} = \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_1}\zeta_{j_2}^{(i_2)},$$

$$B_{11} = \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_4=0}^{p} C_{j_1j_4j_2j_2j_1}\zeta_{j_4}^{(i_4)},$$

$$B_{12} = \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_2j_1}\zeta_{j_3}^{(i_3)},$$

$$B_{13} = \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_3j_2j_1}\zeta_{j_2}^{(i_2)},$$

$$B_{14} = \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)},$$

$$B_{15} = \mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)},$$

where $\mathbf{1}_A$ is the indicator of the set A.

From (1.45) for the case $p_1 = \ldots p_5 = p$ and $\psi_1(s),\ldots,\psi_5(s)\equiv 1$ we obtain

$$J[\psi^{(5)}]_{T,t} + \sum_{i=1}^{10} A_i - \sum_{i=1}^{15} B_i = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\ \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}, \tag{2.258}$$

where $J[\psi^{(5)}]_{T,t}$ is defined by (2.7) for $\psi_1(s),\ldots,\psi_5(s)\equiv 1$ and $i_1,\ldots,i_5=0,1,\ldots,m$.

Using the method of the proof of Theorems 2.1, 2.7, and 2.8, we obtain

$$
A_1=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_3)\right)^2\times
$$

$$
\times\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{\infty}\left(\int\limits_t^{t_3}\phi_{j_1}(t_3)\right)^2\times
$$

$$
\times\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=
$$

$$
=\frac{1}{2}\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1dt_3d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}dt_5\quad \text{w. p. 1},
$$

$$
A_2=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_4,j_5=0}^{p}C_{j_5j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \left(\int\limits_t^{t_4} \phi_{j_1}(t_1) \int\limits_{t_1}^{t_4} \phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \phi_{j_2}(t_3) dt_3 dt_2 dt_1 \right)\times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) \sum_{j_2=0}^{p} \left(\int\limits_{t_1}^{t_4} \phi_{j_2}(t_2) dt_2 \right)^2 dt_1 \times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) \sum_{j_2=0}^{\infty} \left(\int\limits_{t_1}^{t_4} \phi_{j_2}(t_2) dt_2 \right)^2 dt_1 \times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}} \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1)(t_4-t_1) dt_1 \times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}} \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5)(t_4-t) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) dt_1 \times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} +
$$

$$
+ \mathbf{1}_{\{i_2=i_3\ne 0\}} \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_4,j_5=0}^{p} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1)(t-t_1) dt_1 \times
$$

$$
\times dt_4 dt_5 \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{*T}\int\limits_t^{*t_5}(t_4-t)\int\limits_t^{*t_4}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}=
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{T}\int\limits_t^{t_5}(t_4-t)\int\limits_t^{t_4}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\int\limits_t^{T}\int\limits_t^{t_5}(t_1-t)dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^{T}(t_5-t)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{T}\int\limits_t^{t_5}\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\int\limits_t^{T}\int\limits_t^{t_5}(t-t_1)dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^{T}\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_4=
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{T}\int\limits_t^{t_5}(t_4-t)\int\limits_t^{t_4}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^{T}\int\limits_t^{t_5}\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^{T}(t_5-t)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5+
$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_4=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_4\quad \text{w. p. 1},$$

$$A_3=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_5=0}^{p}C_{j_5j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}=$$

$$=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\times$$

$$\times\left(\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}=$$

$$=\mathbf{1}_{\{i_3=i_4\ne 0\}}\times$$

$$\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\sum_{j_3=0}^{p}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2dt_2dt_1\times$$

$$\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}=$$

$$=\mathbf{1}_{\{i_3=i_4\ne 0\}}\times$$

$$\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)\sum_{j_3=0}^{\infty}\left(\int\limits_{t_2}^{t_5}\phi_{j_3}(t_3)dt_3\right)^2dt_2dt_1\times$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)(t_5-t_2)dt_2dt_1\times
$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)(t-t_2)dt_2dt_1\times
$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)(t_5-t)\int\limits_t^{t_5}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_5}\phi_{j_2}(t_2)dt_2dt_1\times
$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_2)(t-t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1dt_2\times
$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)(t_5-t)\int\limits_t^{t_5}\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1dt_2\times
$$

$$
\times dt_5\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\int\limits_t^{*T}\int\limits_t^{*t_5}(t-t_2)\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\int\limits_t^{*T}(t_5-t)\int\limits_t^{*t_5}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}d\mathbf{w}_{t_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\int\limits_t^{T}\int\limits_t^{t_5}(t-t_2)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}(t-t_1)dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\int\limits_t^T(t-t_5)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{2}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}(t_5-t_2)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}(t-t_1)dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}dt_1d\mathbf{w}_{t_5}^{(i_5)}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_5d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}(t-t_1)dt_1d\mathbf{w}_{t_5}^{(i_5)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}dt_1d\mathbf{w}_{t_5}^{(i_5)}\quad \text{w. p. 1},
$$

$$A_4 = \mathbf{1}_{\{i_4=i_5\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\int\limits_{t_3}^T \phi_{j_4}(t_4)\int\limits_{t_4}^T \phi_{j_4}(t_5)dt_5dt_4dt_3dt_2\times$$

$$\times dt_1\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\times$$

$$\times \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^T \phi_{j_4}(t_4)dt_4\right)^2 dt_3dt_2\times$$

$$\times dt_1\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\times$$

$$\times \frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)\sum_{j_4=0}^{\infty}\left(\int\limits_{t_3}^T \phi_{j_4}(t_4)dt_4\right)^2 dt_3dt_2\times$$

$$\times dt_1\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_3}(t_3)(T-t_3)dt_3dt_2\times$$

$$\times dt_1\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{2}\int\limits_t^{*T}(T-t_3)\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{2}\int\limits_t^{T}(T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} +$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_3}dt_1 d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_3 =
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}dt_4+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_3}dt_1 d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_3}d\mathbf{w}_{t_1}^{(i_1)}dt_3 \quad \text{w. p. 1},
$$

$$
A_5=A_6=A_7=A_8=A_9=A_{10}=0 \quad \text{w. p. 1},
$$

$$
B_1=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3,j_5=0}^{p}C_{j_5j_3j_3j_1j_1}\zeta_{j_5}^{(i_5)} =
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\times
$$

$$
\times dt_4 dt_5\zeta_{j_5}^{(i_5)} =
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{\infty}\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\times
$$

$$
\times dt_4 dt_5 \zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\sum_{j_3=0}^{p}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)dt_3dt_4dt_5\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\sum_{j_3=0}^{\infty}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)dt_3dt_4dt_5\zeta_{j_5}^{(i_5)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_5=0}^{p}\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}(t_4-t)dt_4dt_5\zeta_{j_5}^{(i_5)} =
$$

$$
= \frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}dt_1dt_2d\mathbf{w}_{t_5}^{(i_5)}\quad \text{w. p. 1},
$$

$$
B_2 = \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_2,j_4=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\int\limits_{t_3}^T\phi_{j_4}(t_4)\int\limits_{t_4}^T\phi_{j_4}(t_5)dt_5dt_4dt_3\times
$$

$$
\times dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^T\phi_{j_4}(t_4)dt_4\right)^2dt_3\times
$$

$$
\times dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\sum_{j_4=0}^{\infty}\left(\int\limits_{t_3}^T\phi_{j_4}(t_4)dt_4\right)^2dt_3\times
$$

$$
\times dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)(T-t_3)dt_3dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\sum_{j_2=0}^{p}\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)(T-t_3)dt_3dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\sum_{j_2=0}^{\infty}\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)(T-t_3)dt_3dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T(T-t_2)dt_2dt_1\zeta_{j_1}^{(i_1)} =
$$

$$
= \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_{t_1}^T(T-t_2)dt_2d\mathbf{w}_{t_1}^{(i_1)} =
$$

$$
= \frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2dt_5 \quad \text{w. p. 1},
$$

$$
B_3 = \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_3,j_4=0}^{p}C_{j_4j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}\int\limits_t^T\phi_{j_4}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{p}\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\times
$$

$$
\times dt_4dt_5\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\times
$$

$$
\times\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}\int\limits_t^T\phi_{j_4}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\sum_{j_1=0}^{\infty}\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\times
$$

$$
\times dt_4dt_5\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}\int\limits_t^T\phi_{j_4}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)dt_3\times
$$

$$
\times dt_4dt_5\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)(t_3-t)\int\limits_{t_3}^{T}\phi_{j_4}(t_4)\int\limits_{t_4}^{T}\phi_{j_4}(t_5)dt_5\times
$$

$$
\times dt_4dt_3\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)(t_3-t)\sum_{j_4=0}^{p}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2 dt_3\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)(t_3-t)\sum_{j_4=0}^{\infty}\left(\int\limits_{t_3}^{T}\phi_{j_4}(t_4)dt_4\right)^2 dt_3\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)(t_3-t)(T-t_3)dt_3\zeta_{j_3}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(t_3-t)(T-t_3)d\mathbf{w}_{t_3}^{(i_3)}=
$$

$$
=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1 d\mathbf{w}_{t_3}^{(i_3)}dt_5\quad \text{w. p. 1},
$$

$$
B_4=B_5=B_6=B_7=B_8=B_9=B_{10}=B_{11}=B_{12}=
$$
$$
=B_{13}=B_{14}=B_{15}=0\quad \text{w. p. 1.}
$$

Then

$$
J[\psi^{(5)}]_{T,t}+\sum_{i=1}^{10}A_i-\sum_{i=1}^{15}B_i=J[\psi^{(5)}]_{T,t}+
$$
$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1 d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$
$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}+
$$
$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3 d\mathbf{w}_{t_5}^{(i_5)}+
$$
$$
+\frac{1}{2}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}dt_4+
$$
$$
+S_1+S_2+S_3\quad \text{w. p. 1,} \tag{2.259}
$$

where

$$
S_1=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1 dt_3 d\mathbf{w}_{t_5}^{(i_5)}+
$$
$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}(t-t_1)dt_1 d\mathbf{w}_{t_5}^{(i_5)}+
$$
$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}dt_1 d\mathbf{w}_{t_5}^{(i_5)}-
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}dt_1dt_2d\mathbf{w}_{t_5}^{(i_5)},
$$

$$
S_2=\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(t_5-t)\int\limits_t^{t_5}d\mathbf{w}_{t_1}^{(i_1)}dt_5+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}(t-t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_4+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_3}d\mathbf{w}_{t_1}^{(i_1)}dt_3-
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2dt_5,
$$

$$
S_3=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}dt_5+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T(T-t_3)\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}-
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}dt_5.
$$

Usage of the theorem on integration order replacement for iterated Itô stochastic integrals (see Chapter 3) [1]-[14], [60], [91], [92] leads to

$$
S_1=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}dt_1dt_2d\mathbf{w}_{t_5}^{(i_5)}\quad \text{w. p. 1}, \tag{2.260}
$$

$$
S_2=\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2dt_5\quad \text{w. p. 1}, \tag{2.261}
$$

$$S_3 = \frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} dt_5 \quad \text{w. p. 1.} \tag{2.262}$$

Let us substitute (2.260)–(2.262) into (2.259)

$$J[\psi^{(5)}]_{T,t} + \sum_{i=1}^{10} A_i - \sum_{i=1}^{15} B_i = J[\psi^{(5)}]_{T,t} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 d\mathbf{w}_{t_5}^{(i_5)} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} dt_4 +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2} dt_1 dt_2 d\mathbf{w}_{t_5}^{(i_5)} +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 dt_5 +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} dt_5 \quad \text{w. p. 1.} \tag{2.263}$$

According to Theorem 2.12 (see Sect. 2.5.2) for the case $k=5$, the right-hand side of (2.263) is equal w. p. 1 to the following iterated Stratonovich stochastic integral of fifth multiplicity

$$\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)},$$

where $i_1, i_2, i_3, i_4, i_5 = 0, 1, \ldots, m$.

From the other hand, the left-hand side of (2.263) is represented (according to (2.258)) as the following mean-square limit

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\ \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}.$$

Thus, the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\ \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}$$

is proved, where $i_1, i_2, i_3, i_4, i_5 = 0, 1, \ldots, m$. Theorem 2.9 is proved.

2.5 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$) Based on Generalized Iterated Fourier Series Converging Pointwise

This section is devoted to the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) based on generalized iterated Fourier series. The case of Fourier–Legendre series and the case of trigonometric Fourier series are considered in detail. The obtained expansion provides a possibility to represent the iterated Stratonovich stochastic integral in the form of iterated series of products of standard Gaussian random variables. Convergence in the mean of degree $2n$ ($n \in \mathbf{N}$) of the expansion is proved.

2.5.1 Introduction

The idea of representing of iterated Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using generalized iterated Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of author's publications [59] (1997), [60] (1998) (also see [5]-[14], [32]). The results of this section convincingly testify that there is a doubtless relation between the multiplier factor 1/2, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Itô stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its generalized Fourier series converges to the value $(f(x+0)+f(x-0))/2$.

2.5.2 Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$

Consider the following iterated Stratonovich and Itô stochastic integrals

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.264}$$

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^{T} \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2.265}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Let us denote as $\{\phi_j(x)\}_{j=0}^{\infty}$ the complete orthonormal systems of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

In this section, we will pay attention on the well known facts about Fourier series with respect to these two systems of functions [87] (see Sect. 2.1.1).

Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} \tag{2.266}$$

for $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$, where $\mathbf{1}_A$ denotes the indicator of the set A.

Let us formulate the following theorem.

Theorem 2.10 [59] (1997), [60] (1998) (also see [5]-[14], [32]). *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuously differentiable function at the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then, the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ defined by (2.264) is expanded* *into the converging in the mean of degree* $2n$ $(n \in \mathbf{N})$ *iterated series*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{2.267}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \neq 0$*) and*

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \tag{2.268}$$

is the Fourier coefficient.

Note that (2.267) means the following

$$\lim_{p_1 \to \infty} \overline{\lim_{p_2 \to \infty}} \ldots \overline{\lim_{p_k \to \infty}} \mathsf{M} \left\{ \left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right)^{2n} \right\} = 0, \tag{2.269}$$

where $\overline{\lim}$ means lim sup.

Proof. The proof of Theorem 2.10 is based on Lemmas 1.1, 1.3 (see Sect. 1.1.3) and Theorems 2.11–2.13 (see below).

Define the function $K^*(t_1, \ldots, t_k)$ on the hypercube $[t, T]^k$ as follows

$$K^*(t_1, \ldots, t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \left(\mathbf{1}_{\{t_l < t_{l+1}\}} + \frac{1}{2} \mathbf{1}_{\{t_l = t_{l+1}\}} \right) =$$

$$= \prod_{l=1}^{k} \psi_l(t_l) \left(\prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} + \sum_{r=1}^{k-1} \frac{1}{2^r} \sum_{\substack{s_r, \ldots, s_1 = 1 \\ s_r > \ldots > s_1}}^{k-1} \prod_{l=1}^{r} \mathbf{1}_{\{t_{s_l} = t_{s_l+1}\}} \prod_{\substack{l=1 \\ l \neq s_1, \ldots, s_r}}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} \right) \tag{2.270}$$

for $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$ and $K^*(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$, where $\mathbf{1}_A$ is the indicator of the set A.

Theorem 2.11 [59] (1997). *Let the conditions of Theorem* 2.10 *be satisfied. Then, the function* $K^*(t_1, \ldots, t_k)$ *is represented in any internal point of the hypercube* $[t, T]^k$ *by the generalized iterated Fourier series*

$$K^*(t_1,\ldots,t_k)=\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{2.271}$$

where $(t_1,\ldots,t_k)\in(t,T)^k$ *and* $C_{j_k\ldots j_1}$ *is defined by* (2.268). *At that, the iterated series* (2.271) *converges at the boundary of the hypercube* $[t,T]^k$ *(not necessarily to the function* $K^*(t_1,\ldots,t_k)$*).*

Proof. We will perform the proof using induction. Consider the case $k=2$. Let us expand the function $K^*(t_1,t_2)$ using the variable t_1, when t_2 is fixed, into the generalized Fourier series with respect to the system $\{\phi_j(x)\}_{j=0}^{\infty}$ at the interval (t,T)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2)\phi_{j_1}(t_1)\quad (t_1\ne t,T), \tag{2.272}$$

where

$$C_{j_1}(t_2)=\int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1=\int\limits_t^T K(t_1,t_2)\phi_{j_1}(t_1)dt_1=$$

$$=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The equality (2.272) is satisfied pointwise at each point of the interval (t,T) with respect to the variable t_1, when $t_2\in[t,T]$ is fixed, due to a piecewise smoothness of the function $K^*(t_1,t_2)$ with respect to the variable $t_1\in[t,T]$ (t_2 is fixed).

Note also that due to the well known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (2.272) converges when $t_1=t,T$ (not necessarily to the function $K^*(t_1,t_2)$).

Obtaining (2.272), we also used the fact that the right-hand side of (2.272) converges when $t_1=t_2$ (point of a finite discontinuity of the function $K(t_1,t_2)$) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

The function $C_{j_1}(t_2)$ is continuously differentiable at the interval $[t,T]$. Let us expand it into the generalized Fourier series at the interval (t,T)

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\ne t,T), \tag{2.273}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=$$

$$=\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

and the equality (2.273) is satisfied pointwise at any point of the interval (t,T). Moreover, the right-hand side of (2.273) converges when $t_2=t,T$ (not necessarily to $C_{j_1}(t_2)$).

Let us substitute (2.273) into (2.272)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2. \tag{2.274}$$

Note that the series on the right-hand side of (2.274) converges at the boundary of the square $[t,T]^2$ (not necessarily to $K^*(t_1,t_2)$). Theorem 2.11 is proved for the case $k=2$.

Note that proving Theorem 2.11 for the case $k=2$ we obtained the following equality (see (2.272))

$$\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)=\sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\phi_{j_1}(t_1), \tag{2.275}$$

which is satisfied pointwise at the interval (t,T), besides the series on the right-hand side of (2.275) converges when $t_1=t,T$.

Let us introduce the induction assumption

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{2.276}$$

Then

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\psi_k(t_k)\times$$

$$\times\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}\prod_{l=1}^{k-1}\phi_{j_l}(t_l)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.277)$$

On the other hand, the left-hand side of (2.277) can be represented in the following form

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_k(t_k)\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}$$

into the generalized Fourier series at the interval (t,T) using the variable t_k. Theorem 2.11 is proved.

Let us introduce the following notations

$$
J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1}\ \stackrel{\rm def}{=}\ \prod_{p=1}^{l}{\bf 1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}}\ \times
$$

$$
\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times
$$

$$
\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times
$$

$$
\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d{\bf w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots
$$

$$
\ldots\ d{\bf w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d{\bf w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d{\bf w}_{t_k}^{(i_k)}, \tag{2.278}
$$

where

$$
{\rm A}_{k,l}=\bigl\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1,\ s_l,\ldots,s_1=1,\ldots,k-1\bigr\}, \tag{2.279}
$$

$$
(s_l,\ldots,s_1)\in{\rm A}_{k,l},\quad l=1,\ldots,[k/2],\quad i_s=0,1,\ldots,m,\quad s=1,\ldots,k,
$$

$[x]$ is an integer part of a real number x, and ${\bf 1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals $J^{*}[\psi^{(k)}]_{T,t}$, $J[\psi^{(k)}]_{T,t}$ of fixed multiplicity k, $k\in{\bf N}$ (see (2.264), (2.265)).

Theorem 2.12 [59] (1997). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuously differentiable function at the interval $[t,T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals*

$$
J^{*}[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in{\rm A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\hbox{w. p. 1} \tag{2.280}
$$

is correct, where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Proof. Let us prove the equality (2.280) using induction. The case $k=1$ is obvious. If $k=2$, then from (2.280) we get

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}J[\psi^{(2)}]_{T,t}^1 \quad \text{w. p. 1.} \tag{2.281}$$

Let us demonstrate that the equality (2.281) is correct w. p. 1. In order to do it let us consider the process $\eta_{t_2,t} = \psi_2(t_2)J[\psi^{(1)}]_{t_2,t}$, $t_2 \in [t,T]$ and find its stochastic differential using the Itô formula

$$d\eta_{t_2,t} = J[\psi^{(1)}]_{t_2,t}d\psi_2(t_2) + \psi_1(t_2)\psi_2(t_2)d\mathbf{w}_{t_2}^{(i_1)}. \tag{2.282}$$

From the equality (2.282) we obtain that the diffusion coefficient of the process $\eta_{t_2,t}$, $t_2 \in [t,T]$ equals to $\mathbf{1}_{\{i_1\ne 0\}}\psi_1(t_2)\psi_2(t_2)$.

Further, using the standard relations between Stratonovich and Itô stochastic integrals (see (2.4), (2.5)), we obtain the relation (2.281). Thus, the statement of Theorem 2.12 is proved for $k=1$ and $k=2$.

Assume that the statement of Theorem 2.12 is correct for some integer k ($k>2$). Let us prove its correctness when the value k is greater per unit. Using the induction assumption, we have w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t} =$$

$$= \int\limits_t^{*T} \psi_{k+1}(\tau)\left(J[\psi^k]_{\tau,t} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1}\right)d\mathbf{w}_\tau^{(i_{k+1})} =$$

$$= \int\limits_t^{*T} \psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_\tau^{(i_{k+1})} +$$

$$+ \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} \int\limits_t^{*T} \psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1}d\mathbf{w}_\tau^{(i_{k+1})}. \tag{2.283}$$

Using the Itô formula and standard relations between Stratonovich and Itô stochastic integrals, we get w. p. 1

$$\int\limits_t^{*T} \psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_\tau^{(i_{k+1})} = J[\psi^{(k+1)}]_{T,t} + \frac{1}{2}J[\psi^{(k+1)}]_{T,t}^k, \tag{2.284}$$

$$\int\limits_t^{*T} \psi_{k+1}(\tau) J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1} d\mathbf{w}_\tau^{(i_{k+1})} =$$

$$= \begin{cases} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} & \text{if} \ \ s_r = k-1 \\ \\ J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} + J[\psi^{(k+1)}]_{T,t}^{k,s_r,\ldots,s_1}/2 & \text{if} \ \ s_r < k-1 \end{cases}. \tag{2.285}$$

After substituting (2.284) and (2.285) into (2.283) and regrouping of summands, we pass to the following relations, which are valid w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} \tag{2.286}$$

when k is even and

$$J^*[\psi^{(k'+1)}]_{T,t} = J[\psi^{(k'+1)}]_{T,t} + \sum_{r=1}^{[k'/2]+1} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k'+1,r}} J[\psi^{(k'+1)}]_{T,t}^{s_r,\ldots,s_1} \tag{2.287}$$

when $k' = k+1$ is uneven.

From (2.286) and (2.287) we have w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{[(k+1)/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}. \tag{2.288}$$

Theorem 2.12 is proved.

For example, from Theorem 2.12 for $k = 1, 2, 3, 4$ we obtain the following well known equalities [67], which are fulfilled w. p. 1

$$\int\limits_t^{*T} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} = \int\limits_t^{T} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)},$$

$$\int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} = \int\limits_t^{T} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} +$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_2(t_2)\psi_1(t_2)dt_2,
$$

$$
\int\limits_t^{*T}\psi_3(t_3)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_3}^{(i_3)}=\int\limits_t^{T}\psi_3(t_3)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3, \tag{2.289}
$$

$$
\int\limits_t^{*T}\psi_4(t_4)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}=\int\limits_t^{T}\psi_4(t_4)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3d\mathbf{w}_{t_4}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_4+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2dt_4.
$$

Let us consider Lemma 1.1, definition of the multiple stochastic integral (1.16) together with the formula (1.18) when the function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary as well as Lemma 1.3 (see Sect. 1.1.3). Substituting (2.270) into (1.16) and using Lemma 1.1, (1.18), and Theorem 2.12 it is easy to see that w. p. 1

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J[K^*]_{T,t}^{(k)}, \qquad (2.290)$$

where the stochastic integral $J[K^*]_{T,t}^{(k)}$ is defined by (1.16).

Let us subsitute the relation

$$K^*(t_1,\ldots,t_k) = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) + K^*(t_1,\ldots,t_k) -$$

$$- \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)$$

into the right-hand side of (2.290) (here we suppose that $p_1,\ldots,p_k < \infty$). Then using Lemma 1.3 (see Sect. 1.1.3), we obtain

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + J[R_{p_1\ldots p_k}]_{T,t}^{(k)} \quad \text{w. p. 1}, \qquad (2.291)$$

where the stochastic integral $J[R_{p_1\ldots p_k}]_{T,t}^{(k)}$ is defined in accordance with (1.16) and

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) = K^*(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l), \qquad (2.292)$$

$$\zeta_{j_l}^{(i_l)} = \int\limits_t^T \phi_{j_l}(s) d\mathbf{w}_s^{(i_l)}.$$

At that, the following equality is satisfied pointwise in the open hypercube $(t,T)^k$ in accordance with Theorem 2.11

$$\lim_{p_1\to\infty} \ldots \lim_{p_k\to\infty} R_{p_1\ldots p_k}(t_1,\ldots,t_k) = 0. \qquad (2.293)$$

Theorem 2.13. *In the conditions of Theorem* 2.10

$$\lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} = 0, \quad n \in \mathbf{N}.$$

Proof. At first let us analize in detail the cases $k = 2, 3, 4$. Using (2.334) (see below) and (1.18), we have w. p. 1

$$
J[R_{p_1p_2}]_{T,t}^{(2)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1} + \sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\right) R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} +
$$

$$
+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1=0}^{N-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)} =
$$

$$
= \int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)} +
$$

$$
+ \mathbf{1}_{\{i_1=i_2\neq 0\}} \int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1,
$$

where we used the same notations as in the formulas (1.16), (1.18) and Lemma 1.1 (see Sect. 1.1.3). Moreover,

$$
R_{p_1p_2}(t_1,t_2) = K^*(t_1,t_2) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad p_1,\ p_2 < \infty. \tag{2.294}
$$

Let us consider the following well known estimates for moments of stochastic integrals [83]

$$
\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^{2n}\right\} \le (T-t)^{n-1}\left(n(2n-1)\right)^n \int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau, \tag{2.295}
$$

$$
\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau d\tau\right|^{2n}\right\} \le (T-t)^{2n-1} \int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau, \tag{2.296}
$$

where the process ξ_τ such that $(\xi_\tau)^n \in \text{M}_2([t,T])$ and f_τ is a scalar standard Wiener process, $n = 1, 2, \ldots$ (definition of the class $\text{M}_2([t,T])$ see in Sect. 1.1.2).

Using (2.295) and (2.296), we obtain

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} \le C_n \left(\int\limits_t^T\int\limits_t^{t_2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1dt_2 + \right.
$$

$$
\left. + \int\limits_t^T\int\limits_t^{t_1} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_2dt_1 + \mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \left(R_{p_1p_2}(t_1,t_1)\right)^{2n} dt_1\right), \qquad (2.297)
$$

where constant $C_n<\infty$ depends on n and $T-t$ $(n=1,2,\ldots)$.

Note that due to the above assumptions, the function $R_{p_1p_2}(t_1,t_2)$ is continuous in the domains of integration of integrals on the right-hand side of (2.297) and it is bounded at the boundary of the square $[t,T]^2$.

Let us estimate the first integral on the right-hand side of (2.297)

$$
0 \le \int\limits_t^T\int\limits_t^{t_2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1dt_2 = \left(\int\limits_{D_\varepsilon} + \int\limits_{\Gamma_\varepsilon}\right) \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1dt_2 \le
$$

$$
\le \sum_{i=0}^{N-1}\sum_{j=0}^{i} \max_{(t_1,t_2)\in[\tau_i,\tau_{i+1}]\times[\tau_j,\tau_{j+1}]} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n}\Delta\tau_i\Delta\tau_j + MS_{\Gamma_\varepsilon} \le
$$

$$
\le \sum_{i=0}^{N-1}\sum_{j=0}^{i} \left(R_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\Delta\tau_i\Delta\tau_j +
$$

$$
+ \sum_{i=0}^{N-1}\sum_{j=0}^{i} \left|\left(R_{p_1p_2}(t_i^{(p_1p_2)},t_j^{(p_1p_2)})\right)^{2n} - \left(R_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\right|\Delta\tau_i\Delta\tau_j + MS_{\Gamma_\varepsilon} \le
$$

$$
\le \sum_{i=0}^{N-1}\sum_{j=0}^{i} \left(R_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\Delta\tau_i\Delta\tau_j + \frac{\varepsilon_1}{2}(T-t-3\varepsilon)^2\left(1+\frac{1}{N}\right) + MS_{\Gamma_\varepsilon}, \quad (2.298)
$$

where

$$
D_\varepsilon = \{(t_1,t_2):\ t_2\in[t+2\varepsilon,T-\varepsilon],\ t_1\in[t+\varepsilon,t_2-\varepsilon]\}, \quad \Gamma_\varepsilon = D\backslash D_\varepsilon,
$$

$$
D = \{(t_1,t_2):\ t_2\in[t,T],\ t_1\in[t,t_2]\},
$$

ε is a sufficiently small positive number, S_{Γ_ε} is the area of Γ_ε, $M>0$ is a positive constant bounding the function $\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}$, $(t_i^{(p_1p_2)},t_j^{(p_1p_2)})$ is a point of maximum of this function when $(t_1,t_2)\in[\tau_i,\tau_{i+1}]\times[\tau_j,\tau_{j+1}]$,

$$
\tau_i = t+2\varepsilon+i\Delta \quad (i=0,1,\ldots,N), \quad \tau_N = T-\varepsilon, \quad \Delta = \frac{T-t-3\varepsilon}{N},
$$

$\Delta < \varepsilon$, $\varepsilon_1 > 0$ is any sufficiently small positive number.

Getting (2.298), we used the well known properties of Riemann integrals, the first and the second Weierstrass Theorems for the function of two variables as well as the continuity (which means uniform continuity) of the function $\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}$ in the domain D_ε, i.e. $\forall\ \varepsilon_1 > 0\ \ \exists\ \delta(\varepsilon_1) > 0$ which does not depend on t_1, t_2, p_1, p_2 and if $\sqrt{2}\Delta < \delta(\varepsilon_1)$, then the following inequality takes place

$$\left|\left(R_{p_1p_2}(t_i^{(p_1p_2)},t_j^{(p_1p_2)})\right)^{2n} - \left(R_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\right| < \varepsilon_1.$$

Considering (2.274), let us write

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n} = 0 \quad \text{when} \quad (t_1,t_2)\in D_\varepsilon$$

and perform the iterated passages to the limit $\lim\limits_{\varepsilon\to+0}\overline{\lim\limits_{p_1\to\infty}}\ \overline{\lim\limits_{p_2\to\infty}}$, $\lim\limits_{\varepsilon\to+0}\underline{\lim\limits_{p_1\to\infty}}\ \overline{\lim\limits_{p_2\to\infty}}$ (we use the property $\underline{\lim\limits_{p_1\to\infty}} \le \overline{\lim\limits_{p_1\to\infty}}$ in the second case; here $\underline{\lim\limits_{p_1\to\infty}}$ means $\liminf\limits_{p_1\to\infty}$) in the inequality (2.298). Then, according to arbitrariness of $\varepsilon_1 > 0$, we have

$$\underline{\lim_{p_1\to\infty}}\ \overline{\lim_{p_2\to\infty}}\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 = \overline{\lim_{p_1\to\infty}}\ \overline{\lim_{p_2\to\infty}}\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 =$$

$$= \lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 = 0. \tag{2.299}$$

Similarly to arguments given above we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1 = 0, \tag{2.300}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T\left(R_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1 = 0. \tag{2.301}$$

From (2.297), (2.299)–(2.301) we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} = 0, \quad n\in\mathbf{N}. \tag{2.302}$$

Note that (2.302) can be obtained by a more simple way. We have

$$\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2+\int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1=$$

$$=\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2+\int\limits_t^T\int\limits_{t_2}^{T}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2. \qquad (2.303)$$

Combining (2.297) and (2.303), we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\}\le$$

$$\le C_n\left(\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(R_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1\right), \qquad (2.304)$$

where constant $C_n<\infty$ depends on n and $T-t$ $(n=1,2,\ldots)$.

From the one hand, we can use the above reasoning to the integrals on the right-hand side of (2.304) instead of integrals on the right-hand side of (2.297). However, we can get the desired result even easier.

Since the integrals on the right-hand side of (2.304) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover, the following equality

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}=0$$

holds for all $(t_1,t_2)\in(t,T)^2$.

According to (2.294), we have

$$R_{p_1p_2}(t_1,t_2)=\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2)-\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right). \qquad (2.305)$$

Then, applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem and taking into account (2.272), (2.273), and (2.305), we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2=0, \tag{2.306}$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T\left(R_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1=0. \tag{2.307}$$

From (2.304), (2.306), and (2.307) we get (2.302). Recall that (2.307) for $2n=1$ has also been proved in Sect. 2.1.3.

Let us consider the case $k=3$. Using (2.335) (see below) and (1.18), we have w. p. 1

$$J[R_{p_1p_2p_3}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+$$

$$+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\Biggr)+$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}\Biggr)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{p_1p_2p_3}(t_2,t_2,t_3)dt_2d\mathbf{w}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{p_1p_2p_3}(t_2,t_3,t_2)dt_2d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_2,t_2)dt_2d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_1,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{p_1p_2p_3}(t_3,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}dt_3, \tag{2.308}
$$

where we used the same notations as in the formulas (1.16), (1.18) and Lemma 1.1 (see Sect. 1.1.3). Using (2.295) and (2.296), from (2.308) we obtain

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}\le
$$

$$
\le C_n\Biggl(\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+
$$

$$
+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Biggr)dt_1dt_2dt_3+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\Biggl(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\Biggr)dt_2dt_3\Biggr),\quad C_n<\infty. \tag{2.309}
$$

Due to (2.292) the function $R_{p_1,p_2,p_3}(t_1,t_2,t_3)$ is continuous in the domains of integration of iterated integrals on the right-hand side of (2.309) and it is bounded at the boundaries of these domains. Moreover, everywhere in $(t,T)^3$ the following formula takes place

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty} R_{p_1p_2p_3}(t_1,t_2,t_3)=0. \qquad (2.310)$$

Further, similarly to the estimate (2.298) (see 2-dimensional case) we perform the iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}$ under the integral signs on the right-hand side of (2.309) and we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}. \qquad (2.311)$$

From the other hand

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+$$

$$+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Biggr)dt_1dt_2dt_3=$$

$$=\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3, \qquad (2.312)$$

$$\int\limits_t^T\int\limits_t^{t_3}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Biggr)dt_2dt_3=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=$$

$$=\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3, \qquad (2.313)$$

$$\int\limits_t^T\int\limits_t^{t_3}\Biggl(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\Biggr)dt_2dt_3=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3=$$

$$
= \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2dt_3, \tag{2.314}
$$

$$
\int\limits_t^T \int\limits_t^{t_3} \left(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} + \left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\right) dt_2dt_3 =
$$

$$
= \int\limits_t^T \int\limits_t^{t_3} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2dt_3 + \int\limits_t^T \int\limits_{t_3}^{T} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2dt_3 =
$$

$$
= \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2dt_3. \tag{2.315}
$$

Combining (2.309) and (2.312)–(2.315), we have

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\} \le C_n \left(\int\limits_{[t,T]^3} \left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n} dt_1dt_2dt_3 + \right.
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n} dt_2dt_3 +
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2dt_3 +
$$

$$
\left. +\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2dt_3 \right). \tag{2.316}
$$

Since the integrals on the right-hand side of (2.316) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover, the following equality

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty} R_{p_1p_2p_3}(t_1,t_2,t_3) = 0
$$

holds for all $(t_1,t_2,t_3) \in (t,T)^3$.

According to the proof of Theorem 2.11 and (2.292) for $k=3$, we have

$$
R_{p_1p_2p_3}(t_1,t_2,t_3) = \left(K^*(t_1,t_2,t_3) - \sum_{j_1=0}^{p_1} C_{j_1}(t_2,t_3)\phi_{j_1}(t_1)\right) +
$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,t_3)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}(t_3)-\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\phi_{j_3}(t_3)\right)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right),$$

where

$$C_{j_1}(t_2,t_3)=\int\limits_t^T K^*(t_1,t_2,t_3)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3)=\int\limits_{[t,T]^2} K^*(t_1,t_2,t_3)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2.$$

Then, applying three times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3=0, \tag{2.317}$$

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=0, \tag{2.318}$$

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3=0, \tag{2.319}$$

$$\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3=0. \tag{2.320}$$

From (2.316) and (2.317)–(2.320) we get (2.311).

Let us consider the case $k=4$. Using (2.336) (see below) and (1.18), we have w. p. 1

$$J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\Biggl(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\times$$

$$
\left.\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{(l_2,l_2,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_1=0}^{l_3-1}\sum_{(l_1,l_3,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{(l_3,l_3,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_3},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{(l_2,l_2,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{l_4-1}\sum_{(l_1,l_4,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_4},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_4}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\sum_{(t_1,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_1,t_3,t_4)dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_1,t_4)dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_1)dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_2,t_4)d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_3}^{(i_3)}\right)+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)dt_2dt_4+\right.
$$

$$
\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_4,t_2,t_2)dt_2dt_4\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)dt_2dt_4+\right.
$$

$$
\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_2,t_4,t_2)dt_2dt_4\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)dt_2dt_4+\right.
$$

$$
\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_2,t_2,t_4)dt_2dt_4\right), \qquad (2.321)
$$

where the expression

$$
\sum\limits_{(a_1,\ldots,a_k)}
$$

means the sum with respect to all possible permutations $(a_1,\ldots,a_k)$. Moreover, we used in (2.321) the same notations as in the proof of Theorem 1.1 (see Sect. 1.1.3). Note that the analogue of (2.321) will be obtained in Sect. 2.7 (also see [10]-[14], [34]) with using the another approach.

By analogy with (2.316) we obtain

$$
\begin{aligned}
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\} \le C_n \Biggl(\int\limits_{[t,T]^4} \left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n} dt_1dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_2,t_3,t_4)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_1=i_3\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_3,t_2,t_4)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_1=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_2=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_3=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n} dt_2dt_3dt_4+ \\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n} dt_2dt_4+ \\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n} dt_2dt_4+ \\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n} dt_2dt_4 \Biggr), \quad C_n<\infty.
\end{aligned}
\tag{2.322}
$$

Since the integrals on the right-hand side of (2.322) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover, the following equality

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty} R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)=0
$$

holds for all $(t_1, t_2, t_3, t_4) \in (t, T)^4$.

According to the proof of Theorem 2.11 and (2.292) for $k = 4$, we have

$$R_{p_1p_2p_3p_4}(t_1, t_2, t_3, t_4) = \left(K^*(t_1, t_2, t_3, t_4) - \sum_{j_1=0}^{p_1} C_{j_1}(t_2, t_3, t_4)\phi_{j_1}(t_1) \right) +$$

$$+ \left(\sum_{j_1=0}^{p_1} \left(C_{j_1}(t_2, t_3, t_4) - \sum_{j_2=0}^{p_2} C_{j_2j_1}(t_3, t_4)\phi_{j_2}(t_2) \right) \phi_{j_1}(t_1) \right) +$$

$$+ \left(\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left(C_{j_2j_1}(t_3, t_4) - \sum_{j_3=0}^{p_3} C_{j_3j_2j_1}(t_4)\phi_{j_3}(t_3) \right) \phi_{j_2}(t_2)\phi_{j_1}(t_1) \right) +$$

$$+ \left(\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \left(C_{j_3j_2j_1}(t_4) - \sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\phi_{j_4}(t_4) \right) \phi_{j_3}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1) \right),$$

where

$$C_{j_1}(t_2, t_3, t_4) = \int\limits_t^T K^*(t_1, t_2, t_3, t_4)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3, t_4) = \int\limits_{[t,T]^2} K^*(t_1, t_2, t_3, t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$C_{j_3j_2j_1}(t_4) = \int\limits_{[t,T]^3} K^*(t_1, t_2, t_3, t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)dt_1dt_2dt_3.$$

Then, applying four times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty} \lim\limits_{p_2\to\infty} \lim\limits_{p_3\to\infty} \lim\limits_{p_4\to\infty}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty} \lim_{p_2\to\infty} \lim_{p_3\to\infty} \lim_{p_4\to\infty} \int\limits_{[t,T]^4} \left(R_{p_1p_2p_3p_4}(t_1, t_2, t_3, t_4)\right)^{2n} dt_1dt_2dt_3dt_4 = 0, \qquad (2.323)$$

$$\lim_{p_1\to\infty} \lim_{p_2\to\infty} \lim_{p_3\to\infty} \lim_{p_4\to\infty} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2, t_2, t_3, t_4)\right)^{2n} dt_2dt_3dt_4 = 0, \qquad (2.324)$$

$$\lim_{p_1\to\infty} \lim_{p_2\to\infty} \lim_{p_3\to\infty} \lim_{p_4\to\infty} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2, t_3, t_2, t_4)\right)^{2n} dt_2dt_3dt_4 = 0, \qquad (2.325)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \qquad (2.326)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n}dt_2dt_3dt_4=0, \qquad (2.327)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \qquad (2.328)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n}dt_2dt_3dt_4=0, \qquad (2.329)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n}dt_2dt_4=0, \qquad (2.330)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n}dt_2dt_4=0, \qquad (2.331)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n}dt_2dt_4=0. \qquad (2.332)$$

Combaining (2.322) with (2.323)–(2.332), we obtain that

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}.$$

Theorem 2.13 is proved for $k=4$.

Let us consider the case of arbitrary k, $k\in\mathbf{N}$. Let us analyze the stochastic integral defined by (1.16) and find its representation convenient for the following consideration. In order to do it we introduce several notations. Suppose that

$$S_N^{(k)}(a)=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\sum_{(j_1,\ldots,j_k)}a_{(j_1,\ldots,j_k)},$$

$$\mathrm{C}_{s_r}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=$$

$$=\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}a_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)},$$

where

$$\prod_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)\ \stackrel{\rm def}{=}\ \mathbf{I}_{j_{s_r},j_{s_r+1}}\ldots\mathbf{I}_{j_{s_1},j_{s_1+1}}(j_1,\ldots,j_k),$$

$$\mathrm{C}_{s_0}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=S_N^{(k)}(a),\quad \prod_{l=1}^{0}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)=(j_1,\ldots,j_k),$$

$$\mathbf{I}_{j_l,j_{l+1}}(j_{q_1},\ldots,j_{q_2},j_l,j_{q_3},\ldots,j_{q_{k-2}},j_l,j_{q_{k-1}},\ldots,j_{q_k})\stackrel{\rm def}{=}$$
$$\stackrel{\rm def}{=}(j_{q_1},\ldots,j_{q_2},j_{l+1},j_{q_3},\ldots,j_{q_{k-2}},j_{l+1},j_{q_{k-1}},\ldots,j_{g_k}),$$

where $l\ne q_1,\ldots,q_2,q_3,\ldots,q_{k-2},q_{k-1},\ldots,q_k$, $l\in\mathbf{N}$, $a_{(j_{q_1},\ldots,j_{q_k})}$ is a scalar value, $s_1,\ldots,s_r=1,\ldots,k-1$, $s_r>\ldots>s_1$, $q_1,\ldots,q_k=1,\ldots,k$, the expression

$$\sum_{(j_{q_1},\ldots,j_{q_k})}$$

means the sum with respect to all possible permutations $(j_{q_1},\ldots,j_{q_k})$.

Using induction it is possible to prove the following equality

$$\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{N-1}a_{(j_1,\ldots,j_k)}=\sum_{r=0}^{k-1}\ \sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\mathrm{C}_{s_r}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a),\tag{2.333}$$

where $k=2,3,\ldots$ and the sum with respect to the empty set is assumed as equals to 1.

Hereinafter in this section, we will identify the following records

$$a_{(j_1,\ldots,j_k)}=a_{(j_1\ldots j_k)}=a_{j_1\ldots j_k}.$$

In particular, from (2.333) for $k=2,3,4$ we get the following formulas

$$\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1}a_{(j_1,j_2)}=S_N^{(2)}(a)+\mathrm{C}_1S_N^{(2)}(a)=$$

$$=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2)}a_{(j_1j_2)}+\sum_{j_2=0}^{N-1}a_{(j_2j_2)}=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2}+a_{j_2j_1})+$$
$$+\sum_{j_2=0}^{N-1}a_{j_2j_2},\tag{2.334}$$

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2,j_3)} = S_N^{(3)}(a) + \mathrm{C}_1 S_N^{(3)}(a) + \mathrm{C}_2 S_N^{(3)}(a) + \mathrm{C}_2\mathrm{C}_1 S_N^{(3)}(a) =$$

$$= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3)} a_{(j_1 j_2 j_3)} + \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3)} a_{(j_2 j_2 j_3)} +$$

$$+ \sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3)} a_{(j_1 j_3 j_3)} + \sum_{j_3=0}^{N-1} a_{(j_3 j_3 j_3)} =$$

$$= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \left(a_{j_1 j_2 j_3} + a_{j_1 j_3 j_2} + a_{j_2 j_1 j_3} + a_{j_2 j_3 j_1} + a_{j_3 j_2 j_1} + a_{j_3 j_1 j_2}\right) +$$

$$+ \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \left(a_{j_2 j_2 j_3} + a_{j_2 j_3 j_2} + a_{j_3 j_2 j_2}\right) + \sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1} \left(a_{j_1 j_3 j_3} + a_{j_3 j_1 j_3} + a_{j_3 j_3 j_1}\right) +$$

$$+ \sum_{j_3=0}^{N-1} a_{j_3 j_3 j_3}, \qquad (2.335)$$

$$\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2,j_3,j_4)} = S_N^{(4)}(a) + \mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_2 S_N^{(4)}(a) +$$

$$+ \mathrm{C}_3 S_N^{(4)}(a) + \mathrm{C}_2\mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_1 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_2 S_N^{(4)}(a) + \mathrm{C}_3\mathrm{C}_2\mathrm{C}_1 S_N^{(4)}(a) =$$

$$= \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3,j_4)} a_{(j_1 j_2 j_3 j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3,j_4)} a_{(j_2 j_2 j_3 j_4)}$$

$$+ \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3,j_4)} a_{(j_1 j_3 j_3 j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_4,j_4)} a_{(j_1 j_2 j_4 j_4)} +$$

$$+ \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{(j_3,j_3,j_3,j_4)} a_{(j_3 j_3 j_3 j_4)} + \sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)} a_{(j_2 j_2 j_4 j_4)} +$$

$$+ \sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_4,j_4,j_4)} a_{(j_1 j_4 j_4 j_4)} + \sum_{j_4=0}^{N-1} a_{j_4 j_4 j_4 j_4} =$$

$$= \sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \left(a_{j_1 j_2 j_3 j_4} + a_{j_1 j_2 j_4 j_3} + a_{j_1 j_3 j_2 j_4} + a_{j_1 j_3 j_4 j_2} + \right.$$

$$+a_{j_1j_4j_3j_2}+a_{j_1j_4j_2j_3}+a_{j_2j_1j_3j_4}+a_{j_2j_1j_4j_3}+a_{j_2j_4j_1j_3}+a_{j_2j_4j_3j_1}+a_{j_2j_3j_1j_4}+$$

$$+a_{j_2j_3j_4j_1}+a_{j_3j_1j_2j_4}+a_{j_3j_1j_4j_2}+a_{j_3j_2j_1j_4}+a_{j_3j_2j_4j_1}+a_{j_3j_4j_1j_2}+a_{j_3j_4j_2j_1}+$$

$$+a_{j_4j_1j_2j_3}+a_{j_4j_1j_3j_2}+a_{j_4j_2j_1j_3}+a_{j_4j_2j_3j_1}+a_{j_4j_3j_1j_2}+a_{j_4j_3j_2j_1}\Bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\bigl(a_{j_2j_2j_3j_4}+a_{j_2j_2j_4j_3}+a_{j_2j_3j_2j_4}+a_{j_2j_4j_2j_3}+a_{j_2j_3j_4j_2}+a_{j_2j_4j_3j_2}+$$

$$+a_{j_3j_2j_2j_4}+a_{j_4j_2j_2j_3}+a_{j_3j_2j_4j_2}+a_{j_4j_2j_3j_2}+a_{j_4j_3j_2j_2}+a_{j_3j_4j_2j_2}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\bigl(a_{j_3j_3j_1j_4}+a_{j_3j_3j_4j_1}+a_{j_3j_1j_3j_4}+a_{j_3j_4j_3j_1}+a_{j_3j_4j_1j_3}+a_{j_3j_1j_4j_3}+$$

$$+a_{j_1j_3j_3j_4}+a_{j_4j_3j_3j_1}+a_{j_4j_3j_1j_3}+a_{j_1j_3j_4j_3}+a_{j_1j_4j_3j_3}+a_{j_4j_1j_3j_3}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\bigl(a_{j_4j_4j_1j_2}+a_{j_4j_4j_2j_1}+a_{j_4j_1j_4j_2}+a_{j_4j_2j_4j_1}+a_{j_4j_2j_1j_4}+a_{j_4j_1j_2j_4}+$$

$$+a_{j_1j_4j_4j_2}+a_{j_2j_4j_4j_1}+a_{j_2j_4j_1j_4}+a_{j_1j_4j_2j_4}+a_{j_1j_2j_4j_4}+a_{j_2j_1j_4j_4}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\bigl(a_{j_3j_3j_3j_4}+a_{j_3j_3j_4j_3}+a_{j_3j_4j_3j_3}+a_{j_4j_3j_3j_3}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\bigl(a_{j_2j_2j_4j_4}+a_{j_2j_4j_2j_4}+a_{j_2j_4j_4j_2}+a_{j_4j_2j_2j_4}+a_{j_4j_2j_4j_2}+a_{j_4j_4j_2j_2}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\bigl(a_{j_1j_4j_4j_4}+a_{j_4j_1j_4j_4}+a_{j_4j_4j_1j_4}+a_{j_4j_4j_4j_1}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}a_{j_4j_4j_4j_4}. \tag{2.336}$$

Perhaps, the formula (2.333) for any k ($k\in\mathbf{N}$) was found by the author for the first time [59] (1997).

Assume that

$$a_{(j_1,\ldots,j_k)}=\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)},$$

where $\Phi\left(t_1,\ldots,t_k\right)$ is a nonrandom function of k variables. Then from (1.16) and (2.333) we have

$$
J[\Phi]_{T,t}^{(k)}=\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\times
$$

$$
\times\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}\times
$$

$$
\times\Biggl[\Phi\Biggl(\tau_{j_1},\ldots,\tau_{j_{s_1-1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+2}},\ldots,\tau_{j_{s_r-1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+2}},\ldots,\tau_{j_k}\Biggr)\times
$$

$$
\times\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_{s_1-1}}}^{(i_{s_1-1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1+1})}\Delta\mathbf{w}_{\tau_{j_{s_1+2}}}^{(i_{s_1+2})}\ldots
$$

$$
\ldots\Delta\mathbf{w}_{\tau_{j_{s_r-1}}}^{(i_{s_r-1})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r+1})}\Delta\mathbf{w}_{\tau_{j_{s_r+2}}}^{(i_{s_r+2})}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_k)}\Biggr]=
$$

$$
=\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\quad\text{w. p. 1,}\tag{2.337}
$$

where

$$
I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}=\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times
$$

$$
\times\Biggl[\Phi\Biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Biggr)\times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1+1})}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots
$$

$$
\ldots d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r+1})}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Biggr],\tag{2.338}
$$

where $k\ge 2$, the set $\mathrm{A}_{k,r}$ is defined by the relation (2.279). We suppose that the right-hand side of (2.338) exists as the Itô stochastic integral.

Remark 2.1. *The summands on the right-hand side of* (2.338) *should be understood as follows: for each permutation from the set*

$$\prod_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)=$$

$$=\Bigl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Bigr)$$

it is necessary to perform replacement on the right-hand side of (2.338) *of all pairs* (*their number is equal to* r) *of differentials* $d\mathbf{w}_{t_p}^{(i)}d\mathbf{w}_{t_p}^{(j)}$ *with similar lower indices by the values* $\mathbf{1}_{\{i=j\ne 0\}}dt_p$.

Note that the term in (2.337) for $r=0$ should be understood as follows

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Bigl(\Phi\left(t_1,\ldots,t_k\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr),$$

where the notations are the same as in (1.23).

Using (2.295), (2.296), (2.337), and (2.338), we get

$$\mathsf{M}\left\{\left|J[\Phi]_{T,t}^{(k)}\right|^{2n}\right\}\le$$

$$\le C_{nk}\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\},\qquad(2.339)$$

where

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}\le$$

$$\le C_{nk}^{s_1\ldots s_r}\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times$$

$$\times\Phi^{2n}\Bigl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Bigr)\times$$

$$\times\,dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k,\qquad(2.340)$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing are performed in (2.340) only in the values

$$
\Phi^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr).
$$

Consider (2.339) and (2.340) for $\Phi(t_1,\ldots,t_k)\equiv R_{p_1\ldots p_k}(t_1,\ldots,t_k)$

$$
\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}\le
$$

$$
\le C_{nk}\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}, \qquad (2.341)
$$

where

$$
\mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}\le
$$

$$
\le C_{nk}^{s_1\ldots s_r}\int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)}\times
$$

$$
\times R_{p_1\ldots p_k}^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr)\times
$$

$$
\times\, dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k, \qquad (2.342)
$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing are performed in (2.342) only in the values

$$
R_{p_1\ldots p_k}^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr).
$$

From the other hand, we can consider the generalization of the formulas (2.304), (2.316), (2.322) for the case of arbitrary k ($k\in\mathbf{N}$). In order to do this, let us consider the sum with respect to all possible partitions defined by (1.49)

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}.
$$

Now we can generalize the formulas (2.304), (2.316), (2.322) for the case of arbitrary k ($k \in \mathbf{N}$)

$$\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} \le C_{nk}\left(\int\limits_{[t,T]^k}\left(R_{p_1\ldots p_k}(t_1,\ldots,t_k)\right)^{2n}dt_1\ldots dt_k+\right.$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\ldots\mathbf{1}_{\{i_{g_{2r-1}}=i_{g_{2r}}\ne 0\}}\times$$

$$\times\int\limits_{[t,T]^{k-r}}\left(R_{p_1\ldots p_k}\left(t_1,\ldots,t_k\right)\Bigg|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}\right)^{2n}\times$$

$$\left.\times\left(dt_1\ldots dt_k\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}\right), \qquad (2.343)$$

where C_{nk} is a constant,

$$\left(t_1,\ldots,t_k\right)\Bigg|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}$$

means the ordered set $(t_1,\ldots,t_k)$, where we put $t_{g_1}=t_{g_2},\ \ldots,\ t_{g_{2r-1}}=t_{g_{2r}}$.

Moreover,

$$\left(dt_1\ldots dt_k\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}$$

means the product $dt_1\ldots dt_k$, where we replace all pairs $dt_{g_1}dt_{g_2},\ \ldots,\ dt_{g_{2r-1}}dt_{g_{2r}}$ by $dt_{g_1},\ \ldots,\ dt_{g_{2r-1}}$ correspondingly.

Note that the estimate like (2.343), where all indicators $\mathbf{1}_{\{\cdot\}}$ must be replaced with 1, can be obtained from the estimates (2.341), (2.342).

The comparison of (2.343) with the formula (1.50) (see Theorem 1.2) shows their similar structure.

Let us consider the particular case of (2.343) for $k=4$

$$
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\} \le C_{n4}\left(\int\limits_{[t,T]^4} \left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n} dt_1dt_2dt_3dt_4+\right.
$$

$$
+\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} \mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigr|_{t_{g_1}=t_{g_2}}\right)^{2n}\times
$$

$$
\times\left(dt_1dt_2dt_3dt_4\right)\Bigr|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1}}+
$$

$$
+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} \mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\mathbf{1}_{\{i_{g_3}=i_{g_4}\ne 0\}}\times
$$

$$
\times\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigr|_{t_{g_1}=t_{g_2},t_{g_3}=t_{g_4}}\right)^{2n}\times
$$

$$
\left.\times\left(dt_1dt_2dt_3dt_4\right)\Bigr|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\left(dt_{g_3}dt_{g_4}\right)\curvearrowright dt_{g_3}}\right). \qquad (2.344)
$$

It is not difficult to notice that (2.344) is consistent with (2.322).

According to (2.270) and (2.292), we have the following expression

$$
R_{p_1\ldots p_k}(t_1,\ldots,t_k)=
$$

$$
=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right)-
$$

$$
-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l). \qquad (2.345)
$$

Due to (2.345) the function $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ is continuous in the domains of integration of integrals on the right-hand side of (2.342) and it is bounded at the boundaries of these domains (let us remind that the iterated series

$$
\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)
$$

converges at the boundary of the hypercube $[t,T]^k$).

Let us perform the iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ldots\overline{\lim\limits_{p_k\to\infty}}$ under the integral signs in the estimates (2.341), (2.342) (it was similarly performed for the 2-dimensional case (see above)). Then, taking into account (2.293), we get the required result.

From the other hand, we can perform the iterated passage to the limit $\lim\limits_{p_1\to\infty}\ldots\lim\limits_{p_k\to\infty}$ under the integral signs on the right-hand side of the estimate (2.343) (it was similarly performed for the 2-dimensional, 3-dimentional, and 4-dimensional cases (see above)). Then, taking into account (2.293), we obtain the required result. More precisely, since the integrals on the right-hand side of (2.343) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover, the following equality

$$
\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}R_{p_1\ldots p_k}(t_1,\ldots,t_k)=0
$$

holds for all $(t_1,\ldots,t_k)\in(t,T)^k$.

According to the proof of Theorem 2.11 and (2.292), we have

$$
R_{p_1\ldots p_k}(t_1,\ldots,t_k)=\left(K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right)+
$$

$$
+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,\ldots,t_k)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+
$$

$$
\ldots
$$

$$
+\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}}\left(C_{j_{k-1}\ldots j_1}(t_k)-\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)\right)\phi_{j_{k-1}}(t_{k-1})\ldots\phi_{j_1}(t_1)\right),
$$

where

$$
C_{j_1}(t_2,\ldots,t_k)=\int\limits_t^T K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)dt_1,
$$

$$C_{j_2 j_1}(t_3,\ldots,t_k)=\int\limits_{[t,T]^2} K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$\ldots$$

$$C_{j_{k-1}\ldots j_1}(t_k)=\int\limits_{[t,T]^{k-1}} K^*(t_1,\ldots,t_k)\prod_{l=1}^{k-1}\phi_{j_l}(t_l)dt_1\ldots dt_{k-1}.$$

Then, applying k times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\ldots\lim\limits_{p_k\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (2.343), we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}=0,\quad n\in\mathbf{N}.$$

Theorems 2.13 and 2.10 are proved.

It easy to notice that if we expand the function $K^*(t_1,\ldots,t_k)$ into the generalized Fourier series at the interval (t,T) at first with respect to the variable t_k, after that with respect to the variable t_{k-1}, etc., then we will have the expansion

$$K^*(t_1,\ldots,t_k)=\lim_{p_k\to\infty}\ldots\lim_{p_1\to\infty}\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) \qquad (2.346)$$

instead of the expansion (2.271).

Let us prove the expansion (2.346). Similarly with (2.275) we have the following equality

$$\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)=\sum_{j_k=0}^{\infty}\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\phi_{j_k}(t_k), \qquad (2.347)$$

which is satisfied pointwise at the interval (t,T), besides the series on the right-hand side of (2.347) converges when $t_1=t,T$.

Let us introduce the induction assumption

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.348)$$

Then

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\sum_{j_2=0}^{\infty}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2\prod_{l=2}^{k}\phi_{j_l}(t_l)=$$

$$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.349)$$

From the other hand, the left-hand side of (2.349) can be represented in the following form

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2$$

into the generalized Fourier series at the interval (t,T) using the variable t_1. Here we applied the following replacement of integration order

$$\int\limits_{t}^{T}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2dt_1=$$

$$= \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\ldots dt_k =$$

$$= C_{j_k\ldots j_1}.$$

The expansion (2.346) is proved. So, we can formulate the following theorem.

Theorem 2.14 [10] (2013) (also see [11]-[14], [32]). *Suppose that the conditions of Theorem* 2.10 *are fulfilled. Then*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \qquad (2.350)$$

where notations are the same as in Theorem 2.10.

Note that (2.350) means the following

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\ \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^{2n}\right\} = 0,$$

where $n \in \mathbf{N}$.

2.5.3 Further Remarks

In this section, we consider some approaches on the base of Theorem 2.10 for the case $k = 2$. Moreover, we explain the potential difficulties associated with the use of generalized multiple Fourier series converging almost everywhere or converging pointwise in the hypercube $[t, T]^k$ in the proof of Theorem 2.10.

First, we show how iterated series can be replaced by multiple one in Theorem 2.10 for $k = 2$ and $n = 1$ (the case of mean-square convergence).

Using Theorem 2.10 for $k = 2$ and $n = 1$, we obtain

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \lim_{p\to\infty}\overline{\lim_{q\to\infty}}\ \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le$$

$$
\le \lim_{p\to\infty}\overline{\lim_{q\to\infty}}\left(2\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}+\right.
$$

$$
\left.+2\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}\right)=
$$

$$
=2\lim_{p\to\infty}\overline{\lim_{q\to\infty}}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}=
$$

$$
=2\lim_{p\to\infty}\overline{\lim_{q\to\infty}}\sum_{j_1=0}^{p}\sum_{j_1'=0}^{p}\sum_{j_2=p+1}^{q}\sum_{j_2'=p+1}^{q}C_{j_2j_1}C_{j_2'j_1'}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_1'}^{(i_1)}\right\}\mathsf{M}\left\{\zeta_{j_2}^{(i_2)}\zeta_{j_2'}^{(i_2)}\right\}=
$$

$$
=2\lim_{p\to\infty}\lim_{q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q}C_{j_2j_1}^2=
$$

$$
=2\lim_{p\to\infty}\lim_{q\to\infty}\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}^2\right)= \qquad (2.351)
$$

$$
=2\left(\lim_{p,q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{q}C_{j_2j_1}^2-\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}^2\right)= \qquad (2.352)
$$

$$
=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2=0, \qquad (2.353)
$$

where the function $K(t_1,t_2)$ is defined by (1.6) for $k=2$.

Note that the transition from (2.351) to (2.352) is based on the theorem on reducing the limit to the iterated one. Moreover, the transition from (2.352) to (2.353) is based on the Parseval equality.

Thus, we obtain the following Theorem.

Theorem 2.15 [14], [32]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l=1,2)$ is a continuously differentiable nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

(2.264) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1, i_2 = 0, 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense is valid, where

$$C_{j_2 j_1} = \int\limits_t^{T} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 \qquad (2.354)$$

is the Fourier coefficient and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i = 1, \ldots, m)$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Note that Theorem 2.15 is a modification (for the case $p_1 = p_2 = p$ of series summation) of Theorem 2.2.

Consider the proof of Theorem 2.2 based on Theorems 1.1 and 2.10. Using Theorem 2.10, we have

$$0 \le \left| \lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \mathsf{M}\left\{ \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - J^*[\psi^{(k)}]_{T,t} \right\} \right| \le$$

$$\le \lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \left| \mathsf{M}\left\{ \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - J^*[\psi^{(k)}]_{T,t} \right\} \right| \le$$

$$\le \lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \mathsf{M}\left\{ \left| J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right| \right\} \le$$

$$\le \lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\right)^{1/2}=0. \tag{2.355}$$

From the other hand,

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}\right)=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}. \tag{2.356}$$

Combining (2.355) and (2.356), we obtain

$$\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}. \tag{2.357}$$

The relation (2.357) with $k=2$ implies the following

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(s)\psi_2(s)ds=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}, \tag{2.358}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Since

$$\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}},$$

then from (2.358) we obtain

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}, \tag{2.359}$$

where $C_{j_1j_1}$ is defined by (2.268) for $k=2$ and $j_1=j_2$, i.e.

$$C_{j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$

From (2.358) and (2.359) we obtain the following relation

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds. \tag{2.360}$$

Combining (1.42) and (2.360), we have

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right)=$$

$$=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(s)\psi_2(s)ds. \tag{2.361}$$

Since

$$J^{*}[\psi^{(2)}]_{T,t}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(s)\psi_2(s)ds\quad \text{w. p. 1}, \tag{2.362}$$

then from (2.361) we finally get the following expansion

$$J^{*}[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}.$$

Thus, Theorem 2.2 is proved.

Now we turn to multiple trigonometric Fourier series converging almost everywhere. Let us formulate the well known result from the theory of multiple trigonometric Fourier series.

Theorem 2.16 [93]. *Suppose that*

$$\int\limits_{[0,2\pi]^k} |f(x_1,\ldots,x_k)| \left(\log^+|f(x_1,\ldots,x_k)|\right)^k \log^+\log^+|f(x_1,\ldots,x_k)|\times$$

$$\times\, dx_1\ldots dx_k < \infty. \qquad (2.363)$$

Then, for the square partial sums

$$\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(x_l)$$

of the multiple trigonometric Fourier series we have

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(x_l) = f(x_1,\ldots,x_k)$$

almost everywhere in $[0,2\pi]^k$, *where* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of trigonometric functions in the space* $L_2([0,2\pi])$, $\log^+ x = \log\max\{1,\ x\}$,

$$C_{j_k\ldots j_1} = \int\limits_{[0,2\pi]^k} f(x_1,\ldots,x_k)\prod_{l=1}^{k}\phi_{j_l}(x_l)dx_1\ldots dx_k$$

is the Fourier coefficient of the function $f(x_1,\ldots,x_k)$.

Note that Theorem 2.16 can be reformulated for the hypercube $[t,T]^k$ instead of $[0,2\pi]^k$.

If we tried to apply Theorem 2.16 in the proof of Theorem 2.10, then we would encounter the following difficulties. Note that the right-hand side of (2.343) contains multiple integrals over hypercubes of various dimensions, namely over hypercubes $[t,T]^k$, $[t,T]^{k-1}$, etc. Obviously, the convergence almost everywhere in $[t,T]^k$ does not means the convergence almost everywhere in $[t,T]^{k-1}$, $[t,T]^{k-2}$, etc. This means that we could not apply the Lebesgue's Dominated Convergence Theorem in the proof of Theorem 2.13 and thus we could not complete the proof of Theorem 2.10. Although multiple series are more convenient in terms of approximation than iterated series as in Theorem 2.10. The exception is the case $k=2$.

Suppose that $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. In the proof of Theorem 2.2 (see (2.51)) we deduced that

$$\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)=\frac{1}{2}\psi_1(t_1)\psi_2(t_1)=K^*(t_1,t_1),\quad t_1\in(t,T), \tag{2.364}$$

where $C_{j_2j_1}$ is defined by (2.354).

This means that we can repeat the proof of Theorem 2.10 for the case $k=2$ and apply the Lebesgue's Dominated Convergence Theorem in the formula (2.343), since Theorem 2.16 and (2.364) imply the convergence almost everywhere in $[t,T]^2$ and $[t,T]$ $(t_1=t_2\in[t,T])$ of the multiple trigonometric Fourier series

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad t_1,t_2\in[t,T]^2.$$

So, we obtain the following theorem.

Theorem 2.17 [14], [32]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l=1,2)$ is a continuously differentiable nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral* (2.264) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad(i_1,i_2=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean of degree $2n$ $(n\in\mathbf{N})$, i.e.

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^{2n}\right\}=0$$

is valid, where the Fourier coefficient $C_{j_2j_1}$ is defined by (2.354) *and*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ *are independent standard Wiener processes* $(i = 1, \ldots, m)$ *and* $\mathbf{w}_{\tau}^{(0)} = \tau$*.*

Let us consider the another approach. The following fact is well known [90].

Proposition 2.3. *Let* $\left\{x_{n_1,\ldots,n_k}\right\}_{n_1,\ldots,n_k=1}^{\infty}$ *be a multi-index sequence and let there exists the limit*

$$\lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k} < \infty.$$

Moreover, let there exists the limit

$$\lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = y_{n_1,\ldots,n_{k-1}} < \infty \quad \textit{for any} \quad n_1, \ldots, n_{k-1}.$$

Then there exists the iterated limit

$$\lim_{n_1,\ldots,n_{k-1}\to\infty} \lim_{n_k\to\infty} x_{n_1,\ldots,n_k}$$

and moreover,

$$\lim_{n_1,\ldots,n_{k-1}\to\infty} \lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = \lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k}.$$

Denote

$$C_{j_s\ldots j_1}(t_{s+1}, \ldots, t_k) = \int\limits_{[t,T]^s} K(t_1, \ldots, t_k) \prod_{l=1}^{s} \phi_{j_l}(t_l) dt_1 \ldots dt_s,$$

where $s = 1, \ldots, k-1$ and $K(t_1, \ldots, t_k)$ is defined by (1.6). For $s = k$ we suppose that $C_{j_k\ldots j_1}$ is defined by (1.8).

Consider the following Fourier series

$$\lim_{p_1,p_2\to\infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1}(t_3, \ldots, t_k) \phi_{j_1}(t_1) \phi_{j_2}(t_2), \tag{2.365}$$

$$\lim_{p_1,p_2,p_3\to\infty} \sum_{j_1=0}^{p_1} \sum_{j_1=0}^{p_1} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}(t_4, \ldots, t_k) \phi_{j_1}(t_1) \phi_{j_2}(t_2) \phi_{j_3}(t_3), \tag{2.366}$$

$$\ldots$$

$$\lim_{p_1,\ldots,p_{k-1}\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_{k-1}=0}^{p_{k-1}} C_{j_{k-1}\ldots j_1}(t_k) \phi_{j_1}(t_1) \ldots \phi_{j_{k-1}}(t_{k-1}), \tag{2.367}$$

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k), \tag{2.368}$$

where $t_1,\ldots,t_k\in[t,T]$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

The author does not know the answer to the question on the existence of limits (2.365)–(2.368) even for the case $p_1=\ldots=p_k$ and trigonometric Fourier series. Obviously, at least for the case $k=2$ and $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ the answere to the above question is positive for the Fourier–Legendre series as well as for the trigonometric Fourier series.

If we suppose that the limits (2.365)–(2.368) exist, then combining Proposition 2.3 and the proof of Theorem 2.11, we obtain

$$K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)= \tag{2.369}$$

$$=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)=$$

$$=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{2.370}$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{2.371}$$

$$=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{\infty}C_{j_4\ldots j_1}(t_5,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_3}(t_4)= \tag{2.372}$$

$$=\ldots=$$

$$
= \lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k). \tag{2.373}
$$

Note that the transition from (2.370) to (2.371) is based on (2.369) and the proof of Theorem 2.11. The transition from (2.371) to (2.372) is based on (2.370) and the proof of Theorem 2.11.

Using (2.373), we could get the version of Theorem 2.10 with multiple series instead of iterated ones.

2.6 The Hypotheses on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$) Based on Theorem 1.1

2.6.1 Formulation of Hypotheses 2.1-2.3

In this section, on the base of the presented theorems (see Sect. 2.1–2.5) we formulate 3 hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbf{N}$) based on generalized multiple Fourier series converging in $L_2([t,T]^k)$. The considered expansions contain only one operation of the limit transition and substantially simpler than their analogues for iterated Itô stochastic integrals (Theorem 1.1).

Taking into account Theorems 2.1–2.10 and 2.14, let us formulate the following hypotheses on expansions of iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbf{N}$).

Hypothesis 2.1 [8]-[14], [37]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of multiplicity k*

$$
I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)} = {\int\limits_t^{*T}} \ldots {\int\limits_t^{*t_2}} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{2.374}
$$

the following expansion

$$
I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \tag{2.375}
$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau,$ $\lambda_l=0$ *if* $i_l=0$ *and* $\lambda_l=1$ *if* $i_l=1,\ldots,m$ $(l=1,\ldots,k).$

Hypothesis 2.1 allows to approximate the iterated Stratonovich stochastic integral $I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}$ by the sum

$$I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)p}=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{2.376}$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}-I_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)p}\right)^2\right\}=0.$$

The integrals (2.374) will be used in the Taylor–Stratonovich expansion (see Chapter 4). It means that the approximations (2.376) may be very useful for the construction of high-order strong numerical methods for Itô SDEs (see Chapter 4 for detail).

The expansion (2.375) contains only one operation of the limit transition and by this reason is convenient for approximation of iterated Stratonovich stochastic integrals.

Let us consider the more general hypothesis than Hypothesis 2.1.

Hypothesis 2.2 [14], [37]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of* *multiplicity* k

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \qquad (2.377)$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0, 1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Hypothesis 2.2 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \qquad (2.378)$$

where

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\} = 0.$$

Let us consider the more general hypothesis than Hypotheses 2.1 and 2.2.

Hypothesis 2.3 [14], [37]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space*

$L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of multiplicity k

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{2.379}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau.$

Let us consider the idea of the proof of Hypotheses 2.1–2.3.

According to (1.10), we have

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)}=J[\psi^{(k)}]_{T,t}+$$

$$+\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \text{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}\quad \text{w. p. 1}, \tag{2.380}$$

where the notations are the same as in (1.10).

From (2.380) and Theorem 2.12 it follows that

$$
J^*[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)} \qquad (2.381)
$$

if

$$
\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =
$$

$$
= \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)} \quad \text{w. p. 1},
$$

where the notations are the same as in Theorems 1.1 and 2.12.

In the case $p_1=\ldots=p_k=p$ and $\psi_l(s)\equiv 1$ $(l=1,\ldots,k)$ we obtain from (2.381) the statement of Hypothesis 2.1 (see (2.375)).

If $p_1=\ldots=p_k=p$ and every $\psi_l(s)$ $(l=1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$, then we obtain from (2.381) the statement of Hypothesis 2.2 (see (2.377)).

In the case when every $\psi_l(s)$ $(l=1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$ we obtain from (2.381) the statement of Hypothesis 2.3 (see (2.379)).

2.6.2 Hypotheses 2.1 and 2.2 from Point of View of the Wong–Zakai Approximation

The iterated Itô stochastic integrals and solutions of Itô SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbf{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [56], [57], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Itô stochastic integrals and solutions of Itô SDEs. The piecewise linear approximation as well as the regularization by convolution [56]-[58] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s \in [0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i = 1,\ldots,m$. It is well known that the following representation takes place [97], [128] (also see Sect. 6.1 of this book for detail)

$$\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^\tau \phi_j(s)ds\ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}, \tag{2.382}$$

where $\tau \in [t,T]$, $t \ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (2.382) converges for any $\tau \in [t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p} = \sum_{j=0}^{p} \int\limits_t^\tau \phi_j(s)ds\ \zeta_j^{(i)}. \tag{2.383}$$

From (2.383) we obtain

$$d\mathbf{f}_\tau^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{2.384}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{2.385}$$

where $p_1,\ldots,p_k \in \mathbf{N}$, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$
d\mathbf{w}_{\tau}^{(i)p} = \begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for} \quad i = 1, \ldots, m \\ & \\ d\tau & \text{for} \quad i = 0 \end{cases}, \qquad p \in \mathbf{N}, \tag{2.386}
$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (2.384).

Let us substitute (2.384) into (2.385)

$$
\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \quad (2.387)
$$

where $p_1, \ldots, p_k \in \mathbf{N}$,

$$
\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_s^{(0)} = s$,

$$
C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k
$$

is the Fourier coefficient.

To best of our knowledge [56]-[58] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [58] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (2.383) were not considered in [56], [57] (also see [58], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [58] for approximations of the Wiener process based on its series expansion (2.382) (also see (6.16)) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (2.387) to the iterated Stratonovich stochastic integral (2.6) does not follow from the results of the papers [56], [57] (also see [58], Theorems 7.1, 7.2) even for the case $p_1 = \ldots = p_k = p$.

From the other hand, Theorems 1.1, 2.1–2.9 from this monograph can be considered as the proof of the Wong–Zakai approximation based on the iterated Riemann–Stieltjes integrals (2.385) of multiplicities 1 to 5 and the Wiener process approximation (2.383) on the base of its series expansion. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 2.1–2.9) to the appropriate Stratonovich stochastic integrals (2.6). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (2.382), (2.383), and Theorems 2.1–2.9) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s),\ \psi_2(s)\equiv 1;\ i_1,i_2=1,\ldots,m.$

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [56]-[58]).

Let $\mathbf{b}_{\Delta}^{(i)}(t),\ t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t,\ t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)},\ i=1,\ldots,m,$ i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where $\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\ \ t\in[k\Delta,(k+1)\Delta),\ \ k=0,1,\ldots,N-1.$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{2.388}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (2.388) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=
$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \qquad (2.389)
$$

Using (2.389), it is not difficult to show (see Lemma 1.1, Remark 1.2, and (2.8)) that

$$
\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=
$$

$$
=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \qquad (2.390)
$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (2.390) agrees with Theorem 7.1 (see [58], p. 486).

The next example relates to the approximation (2.383) of the Wiener process based on its series expansion (2.382), where $t=0$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \qquad (2.391)
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (2.384).

Let us substitute (2.384) into (2.391)

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \qquad (2.392)
$$

where

$$
C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (2.387).

As we noted above, approximations of the Wiener process that are similar to (2.383) were not considered in [56], [57] (also see Theorems 7.1, 7.2 in [58]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [58] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of this book. More precisely, using Theorems 2.1, 2.2, 2.15 for the case $k=2$, we obtain from (2.392) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{2.393}$$

From the other hand, by Theorem 1.1 (see (1.42)) for the case $k=2$ we obtain from (2.392) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{2.394}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from (2.8) and (2.394) we obtain (2.393).

2.6.3 Wong–Zakai Type Theorems for Iterated Stratonovich Stochastic Integrals. The Case of Approximation of the Multidimensional Wiener Process Based on its Series Expansion Using Legendre Polynomials and Trigonometric Functions

As we mentioned above, there exists a lot of publications on the subject of Wong–Zakai approximation of stochastic integrals and SDEs [56]-[58] (also see [102]-[109]). However, these works did not consider the approximation of iterated stochastic integrals and SDEs for the case of approximation of the multidimensional Wiener process based on its series expansions. Usually, as an approximation of the Wiener process in the theorems of the Wong–Zakai type, the authors [56]-[58] (also see [102]-[109]) choose a piecewise linear approximation or an approximation based on the regularization by convolution.

The Wong–Zakai approximation is widely used to approximate stochastic integrals and SDEs. In particular, the Wong–Zakai approximation can be used to approximate the iterated Stratonovich stochastic integrals in the context of numerical integration of Itô SDEs in the framework of the approach based on the Taylor–Stratonovich expansion [67], [68] (see Chapter 4). It should be noted that the authors of the works [66] (pp. 438-439), [67] (Sect. 5.8, pp. 202–204), [68] (pp. 82-84), [76] (pp. 263-264) mention the Wong–Zakai approximation [56]-[58] within the frames of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process (see Sect. 6.2). However, in these works there is no rigorous proof of convergence for approximations of the mentioned stochastic integrals of miltiplicity 3 and higher (see discussion in Sect. 6.2).

From the other hand, the theory constructed in Chapters 1 and 2 of this monograph (also see [14], [15]) can be considered as the proof of the Wong–Zakai approximation for iterated Stratonovich stochastic integrals of multiplicities 1 to 5 based on the Wiener process series expansion using Legendre polynomials and trigonometric functions.

The subject of this section is to reformulate the main results of Chapter 2 of this book in the form of theorems on convergence of iterated Riemann–Stiltjes integrals to iterated Stratonovich stochastic integrals.

Let us reformulate Theorems 2.2–2.10, 2.14 and Hypotheses 2.1–2.3 of this monograph as statements on the convergence of the iterated Riemann–Stiltjes integrals (2.385) to the iterated Stratonovich stochastic integrals (2.264).

Theorem 2.18 [37] (reformulation of Theorem 2.2). *Suppose that the fol-*

lowing conditions are fulfilled:

1. *Every* $\psi_l(\tau)$ $(l = 1, 2)$ *is a continuously differentiable function at the interval* $[t, T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$.

Then, for the iterated Stratonovich stochastic integral of second multiplicity

$$
J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1, i_2 = 0, 1, \ldots, m)
$$

the following formula

$$
J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} d\mathbf{w}_{t_2}^{(i_2)p_2}
$$

is valid.

Theorem 2.19 [37] (reformulation of Theorems 2.3 and 2.5). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
\int\limits_t^{*T} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)
$$

the following formula

$$
\int\limits_t^{*T} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)p_1} d\mathbf{f}_{t_2}^{(i_2)p_2} d\mathbf{f}_{t_3}^{(i_3)p_3}
$$

is valid.

Theorem 2.20 [37] (reformulation of Theorem 2.4). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials in the space* $L_2([t, T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \int\limits_t^{*T} (t - t_3)^{l_3} \int\limits_t^{*t_3} (t - t_2)^{l_2} \int\limits_t^{*t_2} (t - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}
$$

the following formula

$$I^{*(i_1 i_2 i_3)}_{l_1 l_2 l_3 T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \int\limits_t^T (t-t_3)^{l_3} \int\limits_t^{t_3} (t-t_2)^{l_2} \int\limits_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)p_1} d\mathbf{f}_{t_2}^{(i_2)p_2} d\mathbf{f}_{t_3}^{(i_3)p_3},$$

where $i_1, i_2, i_3 = 1, \ldots, m,$ *is valid for each of the following cases*

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
2. $i_1 = i_2 \ne i_3$ *and* $l_1 = l_2 \ne l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
3. $i_1 \ne i_2 = i_3$ *and* $l_1 \ne l_2 = l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
4. $i_1, i_2, i_3 = 1, \ldots, m;$ $l_1 = l_2 = l_3 = l$ *and* $l = 0, 1, 2, \ldots$

Theorem 2.21 [37] (reformulation of Theorem 2.6). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_l(\tau)$ $(l = 1, 2, 3)$ *are continuously differentiable functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)},$$

where $i_1, i_2, i_3 = 1, \ldots, m,$ *the following formula*

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p}$$

is valid for each of the following cases:

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3,$
2. $i_1 = i_2 \ne i_3$ *and* $\psi_1(\tau) \equiv \psi_2(\tau),$
3. $i_1 \ne i_2 = i_3$ *and* $\psi_2(\tau) \equiv \psi_3(\tau),$
4. $i_1, i_2, i_3 = 1, \ldots, m$ *and* $\psi_1(\tau) \equiv \psi_2(\tau) \equiv \psi_3(\tau).$

Theorem 2.22 [37] (reformulation of Theorem 2.7). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space* $L_2([t,T])$. *Furthermore, let the function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$. *Then, for the iterated Stratonovich* *stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)},
$$

where $i_1,i_2,i_3=1,\ldots,m,$ *the following formula*

$$
J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p}d\mathbf{f}_{t_2}^{(i_2)p}d\mathbf{f}_{t_3}^{(i_3)p}
$$

is valid.

Theorem 2.23 [37] (reformulation of Theorem 2.8). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
I_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{*(i_1i_2i_3i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)},
$$

where $i_1,i_2,i_3,i_4=0,1,\ldots,m,$ *the following formula*

$$
I_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}
$$

is valid, where $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ *are independent standard Wiener processes and* $\mathbf{w}_\tau^{(0)}=\tau,$ $\lambda_l=0$ *if* $i_l=0$ *and* $\lambda_l=1$ *if* $i_l=1,2,\ldots,m$ $(l=1,2,3,4).$

Theorem 2.24 [37] (reformulation of Theorem 2.9). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
I_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}^{*(i_1i_2i_3i_4i_5)}=\int\limits_t^{*T}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)},
$$

where $i_1,\ldots,i_5=0,1,\ldots,m,$ *the following formula*

$$
I^{*(i_1 i_2 i_3 i_4 i_5)}_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^T\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}d\mathbf{w}_{t_5}^{(i_5)p}
$$

is valid, where $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ *are independent standard Wiener processes and* $\mathbf{w}_\tau^{(0)}=\tau,$ $\lambda_l=0$ *if* $i_l=0$ *and* $\lambda_l=1$ *if* $i_l=1,2,\ldots,m$ $(l=1,2,3,4,5).$

Theorem 2.25 [37] (reformulation of Theorem 2.10). *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuously differentiable function at the interval* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral*

$$
J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where $i_1,\ldots,i_k=0,1,\ldots,m,$ *the following formula*

$$
\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\ \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}=0
$$

is valid, where

$$
J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k},
$$

$n\in\mathbf{N},$ *and* $\overline{\lim}$ *means* $\lim\sup.$

Theorem 2.26 [37] (reformulation of Theorem 2.14). *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuously differentiable function at the interval* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral*

$$
J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where $i_1,\ldots,i_k=0,1,\ldots,m,$ *the following formula*

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}=0$$

is valid, where

$$J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k},$$

$n\in\mathbf{N},$ *and* $\overline{\lim}$ *means* $\limsup$.

Let us reformulate Hypotheses 2.1–2.3 in terms of the convergence of iterated Riemann–Stiltjes integrals to iterated Stratonovich stochastic integrals.

Hypothesis 2.4 [37] (reformulation of Hypothesis 2.1). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of* k*th multiplicity*

$$\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad(i_1,\ldots,i_k=0,1,\ldots,m)$$

the following formula

$$\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

is valid, where l.i.m. *is a limit in the mean-square sense,* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i=1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Hypothesis 2.5 [37] (reformulation of Hypothesis 2.2). *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of* k*th multiplicity*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad(i_1,\ldots,i_k=0,1,\ldots,m)$$

the following formula

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

is valid, where l.i.m. *is a limit in the mean-square sense,* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i = 1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Hypothesis 2.6 [37] (reformulation of Hypothesis 2.3). *Let* $\{\phi_j(x)\}_{j=0}^\infty$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover, every* $\psi_l(\tau)$ $(l = 1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of* k*th multiplicity*

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1,\ldots,i_k = 0,1,\ldots,m)$$

the following formula

$$J^*[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}$$

is valid, where l.i.m. *is a limit in the mean-square sense,* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i = 1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

2.7 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 and 4. Combained Approach Based on Generalized Multiple and Iterated Fourier series. Another Proof of Theorems 2.7 and 2.8

2.7.1 Introduction

In this section, we develop the approach from Sect. 2.1.3 for iterated Stratonovich stochastic integrals of multiplicities 3 and 4. We call this approach the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series converging in the sense of norm in $L_2([t,T]^k)$, $k=3,4$. The mean-square convergence of the second part is proved on the base of generalized iterated Fourier series converging pointwise. At that, we do not use iterated Itô stochastic integrals as a tool of the proof and directly consider iterated Stratonovich stochastic integrals.

2.7.2 Another Proof of Theorem 2.7

Let us consider (2.291) for $k=3$, $p_1=p_2=p_3=p$, and $i_1=i_2=i_3=1,\ldots,m$

$$J^*[\psi^{(3)}]_{T,t}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+J[R_{ppp}]_{T,t}^{(3)}\quad \text{w. p. 1},\qquad (2.395)$$

where

$$J[R_{ppp}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{ppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)},$$

$$R_{ppp}(t_1,t_2,t_3)\stackrel{\mathsf{def}}{=}K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3),$$

$$K^*(t_1,t_2,t_3)=\prod_{l=1}^{3}\psi_l(t_l)\Biggl(\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2<t_3\}}+$$

$$+\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2=t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2=t_3\}}\Biggr).$$

Using (2.308), we obtain w. p. 1

$$J[R_{ppp}]_{T,t}^{(3)}=R_{T,t}^{(1)ppp}+R_{T,t}^{(2)ppp},$$

where

$$R_{T,t}^{(1)ppp}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_2,t_3)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_3,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_2,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_2,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_3,t_2,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} R_{ppp}(t_3,t_1,t_2)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_1)},$$

$$R_{T,t}^{(2)ppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_2,t_2,t_3)dt_2d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_2,t_3,t_2)dt_2d\mathbf{f}_{t_3}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_2,t_2)dt_2d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3.$$

We have

$$\mathsf{M}\left\{\left(J[R_{ppp}]_{T,t}^{(3)}\right)^2\right\}\le 2\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}+2\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\}. \qquad (2.396)$$

Now, using standard estimates for moments of stochastic integrals [83], we obtain the following inequality

$$
{\sf M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\} \le
$$

$$
\le 6\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^2+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^2+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^2+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^2\Biggr)dt_1dt_2dt_3=
$$

$$
=6\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2dt_1dt_2dt_3.
$$

We have

$$
\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2dt_1dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^3}\left(K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2dt_1dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^3}\left(K(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2dt_1dt_2dt_3,
$$

where

$$
K(t_1,t_2,t_3)=\begin{cases}\psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1<t_2<t_3\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,t_2,t_3\in[t,T].
$$

So, we get

$$
\lim_{p\to\infty}{\sf M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}=
$$

$$= \lim_{p\to\infty} \int\limits_{[t,T]^3} \left(K(t_1,t_2,t_3) - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1dt_2dt_3 =$$

$$= 0, \qquad (2.397)$$

where $K(t_1,t_2,t_3) \in L_2([t,T]^3)$.

After replacement of the integration order in the iterated Itô stochastic integrals from $R_{T,t}^{(2)ppp}$ [1]-[14], [60], [91], [92] (see Chapter 3) we obtain w. p. 1

$$R_{T,t}^{(2)ppp} =$$

$$= \mathbf{1}_{\{i_1=i_2\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_2,t_2,t_3)dt_2 d\mathbf{f}_{t_3}^{(i_3)} + \int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3\right) +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_3,t_2,t_2)dt_2 d\mathbf{f}_{t_3}^{(i_1)} + \int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3\right) +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_2,t_3,t_2)dt_2 d\mathbf{f}_{t_3}^{(i_2)} + \int\limits_t^T \int\limits_t^{t_3} R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3\right) =$$

$$= \mathbf{1}_{\{i_1=i_2\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_1} R_{ppp}(t_2,t_2,t_1)dt_2 d\mathbf{f}_{t_1}^{(i_3)} + \int\limits_t^T \int\limits_{t_1}^{T} R_{ppp}(t_2,t_2,t_1)dt_2 d\mathbf{f}_{t_1}^{(i_3)}\right) +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_1} R_{ppp}(t_1,t_2,t_2)dt_2 d\mathbf{f}_{t_1}^{(i_1)} + \int\limits_t^T \int\limits_{t_1}^{T} R_{ppp}(t_1,t_2,t_2)dt_2 d\mathbf{f}_{t_1}^{(i_1)}\right) +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}} \left(\int\limits_t^T \int\limits_t^{t_1} R_{ppp}(t_2,t_1,t_2)dt_2 d\mathbf{f}_{t_1}^{(i_2)} + \int\limits_t^T \int\limits_{t_1}^{T} R_{ppp}(t_2,t_1,t_2)dt_2 d\mathbf{f}_{t_1}^{(i_2)}\right) =$$

$$= \mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \left(\int\limits_t^T R_{ppp}(t_2,t_2,t_3)dt_2\right) d\mathbf{f}_{t_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_t^T \left(\int\limits_t^T R_{ppp}(t_1,t_2,t_2)dt_2\right) d\mathbf{f}_{t_1}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_3,t_2,t_3)dt_3\right)d\mathbf{f}_{t_2}^{(i_2)}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\left(\frac{1}{2}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\right)\psi_1(t_2)\psi_2(t_2)\psi_3(t_3)-$$

$$-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_2)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\Biggr)dt_2d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\left(\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_1(t_1)\psi_2(t_2)\psi_3(t_2)-$$

$$-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_2)\Biggr)dt_2d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\psi_1(t_3)\psi_2(t_2)\psi_3(t_3)-$$

$$-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\Biggr)dt_3d\mathbf{f}_{t_2}^{(i_2)}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\phi_{j_3}(t_3)\right)d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\phi_{j_1}(t_1)\right)d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T(-1)\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_1j_2j_1}\phi_{j_2}(t_2)d\mathbf{f}_{t_2}^{(i_2)}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)-$$

$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}.$$

From the proof of Theorem 2.7 we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\}\le 3\left(\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\right.\right.\right.$$

$$\left.\left.-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}-\right.\right.$$

$$\left.\left.\left.-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\right)\to 0 \qquad (2.398)$$

if $p\to\infty$. From (2.395)–(2.398) we obtain the expansion (2.184). Theorem 2.7 is proved.

2.7.3 Another Proof of Theorem 2.8

Let us consider (2.291) for $k=4$, $p_1=\ldots=p_4=p$, and $\psi_1(s),\ldots,\psi_4(s)\equiv 1$

$$\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+J[R_{pppp}]_{T,t}^{(4)}\quad \text{w. p. 1,} \qquad (2.399)$$

where

$$J[R_{pppp}]_{T,t}^{(4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$
R_{pppp}(t_1,t_2,t_3,t_4) \stackrel{\sf def}{=} K^*(t_1,t_2,t_3,t_4)-
$$

$$
-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\phi_{j_4}(t_4), \qquad (2.400)
$$

$$
K^*(t_1,t_2,t_3,t_4) \stackrel{\sf def}{=} \prod_{l=1}^{3}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=
$$

$$
=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2=t_3<t_4\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{t_1=t_2=t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2<t_3=t_4\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2<t_3=t_4\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{t_1<t_2=t_3=t_4\}}+\frac{1}{8}\mathbf{1}_{\{t_1=t_2=t_3=t_4\}}.
$$

We have

$$
J[R_{pppp}]_{T,t}^{(4)}=\sum_{i=0}^{7} R_{T,t}^{(i)pppp} \quad \hbox{w. p. 1}, \qquad (2.401)
$$

where

$$
R_{T,t}^{(0)pppp}=\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\Biggl(R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\times
$$

$$
\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr),
$$

where summation with respect to permutations (l_1,l_2,l_3,l_4) is performed only in the expression, which is enclosed in parentheses,

$$
R_{T,t}^{(1)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_1\ne l_3, l_1\ne l_4, l_3\ne l_4}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},
$$

$$
R_{T,t}^{(2)pppp}=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2, l_1\ne l_4, l_2\ne l_4}}^{N-1} R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},
$$

$$R_{T,t}^{(3)pppp}=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(4)pppp}=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(5)pppp}=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(6)pppp}=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3},$$

$$R_{T,t}^{(7)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}.$$

From (2.399) and (2.401) it follows that Theorem 2.8 will be proved if

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(i)pppp}\right)^2\right\}=0,\quad i=0,1,\ldots,7.$$

We have (see (1.18), (1.23))

$$R_{T,t}^{(0)pppp}=\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_3,t_4)}\left(R_{pppp}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right),$$

where summation with respect to permutations (t_1,t_2,t_3,t_4) is performed only in the expression, which is enclosed in parentheses.

From the other hand (see (1.23), (1.24))

$$
R_{T,t}^{(0)pppp}=\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}R_{pppp}(t_1,t_2,t_3,t_4)d{\bf w}_{t_1}^{(i_1)}d{\bf w}_{t_2}^{(i_2)}d{\bf w}_{t_3}^{(i_3)}d{\bf w}_{t_4}^{(i_4)},
$$

where permutations (t_1,t_2,t_3,t_4) when summing are performed only in the values $d{\bf w}_{t_1}^{(i_1)}d{\bf w}_{t_2}^{(i_2)}d{\bf w}_{t_3}^{(i_3)}d{\bf w}_{t_4}^{(i_4)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation (t_1,t_2,t_3,t_4), then i_r swapped with i_q in the permutation (i_1,i_2,i_3,i_4).

So, we obtain

$$
{\sf M}\left\{\left(R_{T,t}^{(0)pppp}\right)^2\right\}\le 24\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\left(R_{pppp}(t_1,t_2,t_3,t_4)\right)^2dt_1dt_2dt_3dt_4=
$$

$$
=24\int\limits_{[t,T]^4}\left(R_{pppp}(t_1,t_2,t_3,t_4))\right)^2dt_1dt_2dt_3dt_4\ \to 0
$$

if $p\to\infty$, $K^{*}(t_1,t_2,t_3,t_4)\in L_2([t,T]^4)$ (see (2.400)).

Let us consider $R_{T,t}^{(1)pppp}$

$$
R_{T,t}^{(1)pppp}={\bf 1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_1\ne l_3,l_1\ne l_4,l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta{\bf w}_{\tau_{l_3}}^{(i_3)}\Delta{\bf w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
={\bf 1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta{\bf w}_{\tau_{l_3}}^{(i_3)}\Delta{\bf w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
={\bf 1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}\left(\frac{1}{2}{\bf 1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}}+\right.
$$

$$
+\frac{1}{4}{\bf 1}_{\{\tau_{l_1}=\tau_{l_3}<\tau_{l_4}\}}+\frac{1}{4}{\bf 1}_{\{\tau_{l_1}<\tau_{l_3}=\tau_{l_4}\}}+\frac{1}{8}{\bf 1}_{\{\tau_{l_1}=\tau_{l_3}=\tau_{l_4}\}}-
$$

$$
\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\right)\Delta\tau_{l_1}\Delta{\bf w}_{\tau_{l_3}}^{(i_3)}\Delta{\bf w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}} -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1} \Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}} -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1} \Biggl(0 -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\tau_{l_4} =
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} - \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right) +
$$

$$
+ \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} \quad \text{w. p. 1.}
$$

When proving Theorem 2.8 we have proved that

$$
\lim_{p\to\infty} \sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} = \frac{1}{4}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} +
$$

$$
+ \mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2 \quad \text{w. p. 1.}
$$

Then

$$\lim_{p\to\infty}\ \mathsf{M}\left\{\left(R_{T,t}^{(1)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(2)pppp}$

$$R_{T,t}^{(2)pppp}=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}\left(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-\right.$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\right)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\tau_{l_4}=$$

$$=-\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p}C_{j_4j_1j_4j_1}\quad \text{w. p. 1.}$$

When proving Theorem 2.8 we have proved that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}=0\quad \text{w. p. 1,}$$

$$\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0.$$

Then

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(2)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(3)pppp}$

$$R_{T,t}^{(3)pppp}=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1}\left(\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-\right.$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\right)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_3})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\tau_{l_3}=$$

$$=-\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}\quad \text{w. p. 1.}$$

When proving Theorem 2.8 we have proved that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=0\quad \text{w. p. 1},$$

$$\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0.$$

Then

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(3)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(4)pppp}$

$$R_{T,t}^{(4)pppp}=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\left(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}+\right.$$

$$+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\right)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\left(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-\right.$$

$$\left.-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\right)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}\left(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-\right.$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_4})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}\quad \text{w. p. 1.}
$$

When proving Theorem 2.8 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(4)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(5)pppp}$

$$
R_{T,t}^{(5)pppp}=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_3}}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=-\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_3})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_3}=
$$

$$
=-\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}\quad \text{w. p. 1.}
$$

When proving Theorem 2.8 we have proved that

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=0\quad \text{w. p. 1,}
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0.
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(5)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(6)pppp}$

$$
R_{T,t}^{(6)pppp}=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_3}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Delta\tau_{l_1}\Delta\tau_{l_3} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1} =
$$

$$
= \mathbf{1}_{\{i_3=i_4\ne 0\}}\Biggl(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3} dt_1dt_3-
$$

$$
-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\right)\quad \text{w. p. 1.}
$$

When proving Theorem 2.8 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(6)pppp}\right)^2\right\}=0.
$$

Finally, let us consider $R_{T,t}^{(7)pppp}$

$$
R_{T,t}^{(7)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1} G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4} =
$$

$$
=\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4} =
$$

$$
=\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\left(\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4} dt_2dt_4-\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}.
$$

When proving Theorem 2.8 we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4} dt_2dt_4,
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0,
$$

$$\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0.$$

Then

$$\lim_{p\to\infty} R_{T,t}^{(7)pppp}=0.$$

Theorem 2.8 is proved.

2.8 Modification of Theorem 2.2 for the Case of the Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 2 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 2.27. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover,* $\psi_1(\tau)$, $\psi_2(\tau)$ *are continuously differentiable functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{s,t}=\int\limits_t^{*s}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{s,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\tag{2.402}$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ *(*s *is fixed),*

$$C_{j_2j_1}(s)=\int\limits_t^{s}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,\tag{2.403}$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. In accordance to the standard relations between Stratonovich and Itô stochastic integrals (see (2.4) and (2.5)) we have w. p. 1

$$J^*[\psi^{(2)}]_{s,t} = J[\psi^{(2)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1, \tag{2.404}$$

where $s \in (t, T]$ (s is fixed), $\mathbf{1}_A$ is the indicator of the set A.

From the other side according to (1.202), we have

$$J[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty} C_{j_1j_1}(s). \tag{2.405}$$

From (2.404) and (2.405) it follows that Theorem 2.27 will be proved if

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}(s). \tag{2.406}$$

To prove (2.406), we multiply the equality (2.12) by the function

$$\mathbf{1}_{\{t_2<s\}} + \frac{1}{2}\mathbf{1}_{\{t_2=s\}}, \quad t_2 \in [t, T],$$

where $s \in (t, T]$ (s is fixed). So we have

$$K^*(t_1,t_2)\left(\mathbf{1}_{\{t_2<s\}} + \frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2)\left(\mathbf{1}_{\{t_2<s\}} + \frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)\phi_{j_1}(t_1), \tag{2.407}$$

where $t_1 \neq t, T$,

$$K^*(t_1,t_2) = \psi_1(t_1)\psi_2(t_2)\left(\mathbf{1}_{\{t_1<t_2\}} + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right), \quad t_1, t_2 \in [t, T],$$

$$C_{j_1}(t_2) = \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The function

$$C_{j_1}(t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right),\quad t_2\in[t,T]$$

has the same structure as the function $K^*(t_1,t_2)$. Then, by analogy with (2.12), we get

$$C_{j_1}(t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)=\sum_{j_2=0}^{\infty}C_{j_2j_1}(s)\phi_{j_2}(t_2), \qquad (2.408)$$

where $t_2\ne t,T$ and the Fourier coefficient $C_{j_2j_1}(s)$ is defined by (2.403).

Let us substitute (2.408) into (2.407)

$$K^*(t_1,t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}(s)\phi_{j_1}(t_1)\phi_{j_2}(t_2), \qquad (2.409)$$

where $(t_1,t_2)\in(t,T)^2$.

Note that the series on the right-hand side of (2.409) converges at the boundary of the square $[t,T]^2$.

It is easy to see that substituting $t_1=t_2$ in (2.409), we obtain

$$\frac{1}{2}\psi_1(t_1)\psi_2(t_1)\left(\mathbf{1}_{\{t_1<s\}}+\frac{1}{2}\mathbf{1}_{\{t_1=s\}}\right)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}(s)\phi_{j_1}(t_1)\phi_{j_2}(t_1), \qquad (2.410)$$

where $t_1\ne t,T$.

Denote

$$R_{p_1p_2}(t_1,t_2,s)=K^*(t_1,t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)-$$

$$-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\phi_{j_1}(t_1)\phi_{j_2}(t_2), \qquad (2.411)$$

where $p_1,p_2<\infty$. Then

$$R_{p_1p_2}(t_1,t_1,s)=\frac{1}{2}\psi_1(t_1)\psi_2(t_1)\left(\mathbf{1}_{\{t_1<s\}}+\frac{1}{2}\mathbf{1}_{\{t_1=s\}}\right)-$$

$$-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\phi_{j_1}(t_1)\phi_{j_2}(t_1).$$

Note that

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1,s)dt_1=\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1-$$

$$-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\int\limits_t^T \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\mathbf{1}_{\{j_1=j_2\}}=$$

$$=\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}(s). \qquad (2.412)$$

Using (2.412), we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1,s)dt_1=\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\infty}C_{j_1j_1}(s). \qquad (2.413)$$

The equality (2.413) means that Theorem 2.27 will be proved if

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1,s)dt_1=0.$$

Since the integral

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1,s)dt_1$$

exists as Riemann integral, then it is equal to the corresponding Lebesgue integral. Moreover, the following equality

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}R_{p_1p_2}(t_1,t_1,s)=0 \quad \text{when} \quad t_1\in[t,T]$$

holds with accuracy up to sets of measure zero (see (2.410)).

We have

$$R_{p_1p_2}(t_1,t_2,s)=\Biggl(K^*(t_1,t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)-$$

$$-\sum_{j_1=0}^{p_1}C_{j_1}(t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)\phi_{j_1}(t_1)\Biggr)+$$

$$+\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2)\left(\mathbf{1}_{\{t_2<s\}}+\frac{1}{2}\mathbf{1}_{\{t_2=s\}}\right)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1). \qquad (2.414)$$

Let us substitute $t_1=t_2$ into (2.414)

$$R_{p_1p_2}(t_1,t_1,s)=\Biggl(\frac{1}{2}\psi_1(t_1)\psi_1(t_1)\left(\mathbf{1}_{\{t_1<s\}}+\frac{1}{2}\mathbf{1}_{\{t_1=s\}}\right)-$$

$$-\sum_{j_1=0}^{p_1}C_{j_1}(t_1)\left(\mathbf{1}_{\{t_1<s\}}+\frac{1}{2}\mathbf{1}_{\{t_1=s\}}\right)\phi_{j_1}(t_1)\Biggr)+$$

$$+\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_1)\left(\mathbf{1}_{\{t_1<s\}}+\frac{1}{2}\mathbf{1}_{\{t_1=s\}}\right)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\phi_{j_2}(t_1)\right)\phi_{j_1}(t_1). \qquad (2.415)$$

Applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem and taking into account (2.407), (2.408), and (2.411) for the case $t_1=t_2$, we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1,s)dt_1=0.$$

Theorem 2.27 is proved.

Let us reformulate Theorem 2.27 in terms on the convergence of the solution of the system of ordinary differential equations (ODEs) to the solution of the system of Stratonovich SDEs (the so-called Wong–Zakai type theorem).

From (2.387) for $k=2,$ $i_1,i_2=1,\ldots,m,$ and $s\in(t,T]$ (s is fixed) we obtain

$$\int\limits_t^s\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2}=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \qquad (2.416)$$

where $p_1, p_2 \in \mathbf{N}$ and $d\mathbf{f}_\tau^{(i)p}$ is defined by (2.384); another notations are the same as in Theorem 2.27.

The iterated Riemann–Stiltjes integrals

$$Y_{s,t}^{(i_1 i_2)p_1 p_2} = \int\limits_t^s \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p_1} d\mathbf{f}_{t_2}^{(i_2)p_2}, \quad X_{s,t}^{(i_1)p_1} = \int\limits_t^s \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p_1}$$

are the solution of the following system of ODEs

$$\begin{cases} dY_{s,t}^{(i_1 i_2)p_1 p_2} = \psi_2(s) X_{s,t}^{(i_1)p_1} d\mathbf{f}_s^{(i_2)p_2}, & Y_{t,t}^{(i_1 i_2)p_1 p_2} = 0 \\ \\ dX_{s,t}^{(i_1)p_1} = \psi_1(s) d\mathbf{f}_s^{(i_1)p_1}, & X_{t,t}^{(i_1)p_1} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$Y_{s,t}^{(i_1 i_2)} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)}, \quad X_{s,t}^{(i_1)} = \int\limits_t^{*s} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ \\ dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorem 2.27 and (1.201) we obtain the following theorem.

Theorem 2.28. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Moreover,* $\psi_1(\tau)$, $\psi_2(\tau)$ *are continuously differentiable functions on* $[t, T]$*. Then for any fixed* s $(s \in (t, T])$

$$\underset{p_1, p_2 \to \infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1 i_2)p_1 p_2} = Y_{s,t}^{(i_1 i_2)}, \quad X_{s,t}^{(i_1)p_1} = \underset{p_1 \to \infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)}.$$

2.9 Modification of Theorem 2.7 for the Case of the Integration Interval $[t, s]$ $(s \in (t, T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 3 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 2.29. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t, T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{s,t} = \int\limits_t^{*s} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{s,t} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $s \in (t, T]$ (s *is fixed*),

$$C_{j_3 j_2 j_1}(s) = \int\limits_t^{s} \psi_3(t_3) \phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2) \phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. Let us consider the case of Legendre polynomials. From (1.203) for the case $p_1 = p_2 = p_3 = p$ and standard relations between Itô and Stratonovich stochastic integrals we conclude that Theorem 2.29 will be proved if w. p. 1

$$\underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1 = 0}^{p} \sum_{j_3 = 0}^{p} C_{j_3 j_1 j_1}(s) \zeta_{j_3}^{(i_3)} = \frac{1}{2} \int\limits_t^{s} \psi_3(\tau) \int\limits_t^{\tau} \psi_2(s_1) \psi_1(s_1) ds_1 d\mathbf{f}_{\tau}^{(i_3)}, \qquad (2.417)$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^s \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau, \qquad (2.418)
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1j_3j_1}(s)\zeta_{j_3}^{(i_2)} = 0. \qquad (2.419)
$$

The proof of the formulas (2.417), (2.419) is absolutely similar to the proof of the formulas (2.185), (2.187). It is only necessary to replace the interval of integration $[t,T]$ by $[t,s]$ in the proof of the formulas (2.185), (2.187) and use Theorem 1.11 instead of Theorem 1.1. Also in the case (2.419) it is necessary to use the estimate (1.173).

Let us prove (2.418). Using the Itô formula, we have

$$
\frac{1}{2}\int\limits_t^s \psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau = \frac{1}{2}\int\limits_t^s \psi_1(s_1)\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau d\mathbf{f}_{s_1}^{(i_1)} \quad \text{w. p. 1.}
$$

Moreover, using Theorem 1.11 for $k=1$ (also see (1.201)), we obtain w. p. 1

$$
\frac{1}{2}\int\limits_t^s \psi_1(s_1)\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau d\mathbf{f}_{s_1}^{(i_1)} = \frac{1}{2}\ \underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p} C_{j_1}^{*}(s)\zeta_{j_1}^{(i_1)},
$$

where

$$
C_{j_1}^{*}(s) = \int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau ds_1. \qquad (2.420)
$$

We have

$$
E_p'(s) \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)} - \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^{*}(s)\zeta_{j_1}^{(i_1)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s) - \frac{1}{2}C_{j_1}^{*}(s)\right)\zeta_{j_1}^{(i_1)}\right)^2\right\} =
$$

$$
= \sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s) - \frac{1}{2}C_{j_1}^{*}(s)\right)^2, \qquad (2.421)
$$

$$C_{j_3j_3j_1}(s)=\int\limits_t^s \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_t^{\tau}\psi_1(s_1)\phi_{j_1}(s_1)ds_1d\tau d\theta =$$

$$=\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1. \qquad (2.422)$$

From (2.420)–(2.422) we obtain

$$E_p'(s)=\sum_{j_1=0}^{p}\left(\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^{s}\left(\sum_{j_3=0}^{p}\psi_2(\tau)\phi_{j_3}(\tau)\times\right.\right.$$

$$\left.\left.\times\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta-\frac{1}{2}\psi_3(\tau)\psi_2(\tau)\right)d\tau ds_1\right)^2. \qquad (2.423)$$

We will prove the following equality for all $\tau\in(t,s)$

$$\sum_{j_3=0}^{\infty}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta=\frac{1}{2}\psi_2(\tau)\psi_3(\tau). \qquad (2.424)$$

Let us denote

$$K_1^*(t_1,t_2,s)=K_1(t_1,t_2,s)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2<s\}}\psi_2(t_1)\psi_3(t_1), \qquad (2.425)$$

where

$$K_1(t_1,t_2,s)=\psi_2(t_1)\psi_3(t_2)\mathbf{1}_{\{t_1<t_2<s\}},$$

$t_1,t_2\in[t,T]$, $s\in(t,T]$ (s is fixed).

Let us expand the function $K_1^*(t_1,t_2,s)$ using the variable t_2, when t_1 is fixed, into the Fourier–Legendre series at the interval (t,T)

$$K_1^*(t_1,t_2,s)=\sum_{j_3=0}^{\infty}\psi_2(t_1)\int\limits_{t_1}^{s}\psi_3(t_2)\phi_{j_3}(t_2)dt_2\cdot\phi_{j_3}(t_2)\quad (t_2\ne t,s,T). \qquad (2.426)$$

The equality (2.426) is fulfilled in each point of the intervals (t,s), (s,T) with respect to the variable t_2, when $t_1\in[t,T]$ is fixed, due to piecewise smoothness of the function $K_1^*(t_1,t_2,s)$ with respect to the variable $t_2\in[t,T]$ (t_1 is fixed).

Obtaining (2.426), we also used the fact that the right-hand side of (2.426) converges when $t_1 = t_2 < s$ (t_1 is the point of a finite discontinuity of the function $K_1^*(t_1, t_2, s)$) to the value

$$\frac{1}{2}\left(K_1(t_1, t_1 - 0, s) + K_1(t_1, t_1 + 0, s)\right) = \frac{1}{2}\psi_2(t_1)\psi_3(t_1) = K_1^*(t_1, t_1, s),$$

where $t_1 < s$.

Let us substitute $t_1 = t_2$ into (2.426). Then we have (2.424).

From (2.423) and (2.424) we get

$$E_p'(s) = \sum_{j_1=0}^{p}\left(\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^s \sum_{j_3=p+1}^{\infty}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}\psi_1(u_l^*)\phi_{j_1}(u_l^*)\int\limits_{u_l^*}^s \sum_{j_3=p+1}^{\infty}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\Delta u_l\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\lim_{N\to\infty}\sum_{l=0}^{N-1}\psi_1(u_l^*)\phi_{j_1}(u_l^*)\sum_{j_3=p+1}^{\infty}\int\limits_{u_l^*}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\Delta u_l\right)^2, \tag{2.427}$$

where $t = u_0 < u_1 < \ldots < u_N = s$, $\Delta u_l = u_{l+1} - u_l$, u_l^* is a point of minimum of the function $(1 - (z(\tau))^2)^{-\alpha}$ $(0 < \alpha < 1)$ at the interval $[u_l, u_{l+1}]$, $l = 0, 1, \ldots, N-1$,

$$\max_{0\le l\le N-1}\Delta u_l \to 0 \quad \text{when} \quad N\to\infty.$$

The last step in (2.427) is correct due to uniform convergence of the Fourier–Legendre series of the piecewise smooth function $K_1^*(\tau, \tau, s)$ at the interval $[u_l^* + \varepsilon, s - \varepsilon]$ (with respect to τ) for any $\varepsilon > 0$ (the function $K_1^*(\tau, \tau, s)$ is continuous at the interval $[u_l^*, s]$).

Using the inequality of Cauchy–Bunyakovsky and the estimates (1.174), (2.100), we obtain

$$\left|\int\limits_{u_l^*}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| \le$$

$$\le \int\limits_{u_l^*}^{s} \left(\psi_2(\tau)\phi_{j_3}(\tau)\right)^2 d\tau \int\limits_{u_l^*}^{s} \left(\int\limits_{\tau}^{s} \psi_3(\theta)\phi_{j_3}(\theta)d\theta\right)^2 d\tau \le$$

$$\le \frac{C}{{j_3}^2} \int\limits_{u_l^*}^{s} \left(\frac{1}{(1-(z(s))^2)^{1/2}} + \frac{1}{(1-(z(\tau))^2)^{1/2}} + C_1\right) d\tau \le$$

$$\le \frac{C_2}{{j_3}^2} \left(\frac{1}{(1-(z(s))^2)^{1/2}} + C_3\right),$$

where constants C, C_1, C_2, C_3 do not depend on j_3.

We can assume that $s \in (t, T)$ $(z(s) \ne \pm 1)$ since the case $s = T$ has already been considered in Theorem 2.7. Then

$$\left|\int\limits_{u_l^*}^{s} \psi_2(\tau)\phi_{j_3}(\tau) \int\limits_{\tau}^{s} \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| \le \frac{C_4}{{j_3}^2}, \tag{2.428}$$

where constant C_4 is independent of j_3.

Let us estimate the right-hand side of (2.427) using (2.428)

$$E_p'(s) \le L \sum_{j_1=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} |\phi_{j_1}(u_l^*)| \sum_{j_3=p+1}^{\infty} \frac{1}{j_3^2} \Delta u_l\right)^2 \le$$

$$\le \frac{L_1}{p^2} \sum_{j_1=0}^{p} \left(\lim_{N\to\infty} \sum_{l=0}^{N-1} \frac{1}{(1-(z(u_l^*))^2)^{1/4}} \Delta u_l\right)^2 \le$$

$$\le \frac{L_1}{p^2} \sum_{j_1=0}^{p} \left(\lim_{N\to\infty} \int\limits_{t}^{s} \frac{d\tau}{(1-(z(\tau))^2)^{1/4}}\right)^2 =$$

$$= \frac{L_1}{p^2} \sum_{j_1=0}^{p} \left(\int\limits_{t}^{s} \frac{d\tau}{(1-(z(\tau))^2)^{1/4}}\right)^2 \le$$

$$\le \frac{L_2 p}{p^2} = \frac{L_2}{p} \to 0 \tag{2.429}$$

if $p \to \infty$, where constants L, L_1, L_2 do not depend on p and we used (2.25) and (2.100) in (2.429). The relation (2.418) is proved. Theorem 2.29 is proved for the case of Legendre polynomials.

For the trigonometric case, we can use the following estimates

$$\left| \int\limits_{s_1}^{s} \sum_{j=p+1}^{\infty} \psi_2(\tau)\phi_j(\tau) \int\limits_{\tau}^{s} \psi_l(\theta)\phi_j(\theta)d\theta d\tau \right| \le \frac{C_1}{p}, \tag{2.430}$$

$$\left| \int\limits_{s_1}^{s} \psi_l(\theta)\phi_j(\theta)d\theta \right| \le \frac{C_2}{j} \quad (j \ne 0), \tag{2.431}$$

where constant C_1 is independent of p and constant C_2 does not depend on j, $l = 1$ or $l = 3$, $t \le s_1 < s \le T$.

The estimate (2.431) is obvious and the estimate (2.430) can be obtained similarly to the estimate (2.29). Theorem 2.29 is proved for the trigonometric case. Theorem 2.29 is proved.

Let us reformulate Theorem 2.29 in terms on the convergence of the solution of the system of ODEs to the solution of the system of Stratonovich SDEs (the so-called Wong–Zakai type theorem).

From (2.387) for the case $k = 3$, $p_1 = p_2 = p_3 = p$, $i_1, i_2, i_3 = 1, \ldots, m$, and $s \in (t, T]$ (s is fixed) we obtain

$$\int\limits_t^s \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{2.432}$$

where $p \in \mathbf{N}$ and $d\mathbf{f}_{\tau}^{(i)p}$ is defined by (2.384); another notations are the same as in Theorem 2.29.

The iterated Riemann–Stiltjes integrals

$$Z_{s,t}^{(i_1i_2i_3)p} = \int\limits_t^s \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p},$$

$$Y_{s,t}^{(i_1i_2)p} = \int\limits_t^s \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p},$$

$$X_{s,t}^{(i_1)p} = \int\limits_t^s \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p}$$

are the solution of the following system of ODEs

$$\begin{cases} dZ_{s,t}^{(i_1 i_2 i_3)p} = \psi_3(s) Y_{s,t}^{(i_1 i_2)p} d\mathbf{f}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\ dY_{s,t}^{(i_1 i_2)p} = \psi_2(s) X_{s,t}^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\ dX_{s,t}^{(i_1)p} = \psi_1(s) d\mathbf{f}_s^{(i_1)p}, & X_{t,t}^{(i_1)p_1} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$Z_{s,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*s} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)},$$

$$Y_{s,t}^{(i_1 i_2)} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)},$$

$$X_{s,t}^{(i_1)} = \int\limits_t^{*s} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dZ_{s,t}^{(i_1 i_2 i_3)} = \psi_3(s) Y_{s,t}^{(i_1 i_2)} * d\mathbf{f}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\ dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 2.28 and 2.29 we obtain the following theorem.

Theorem 2.30. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *At* *the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$. *Then for any fixed* s $(s\in(t,T])$

$$\underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1i_2i_3)p}=Z_{s,t}^{(i_1i_2i_3)},\quad \underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1i_2)p}=Y_{s,t}^{(i_1i_2)},$$

$$X_{s,t}^{(i_1)p}=\underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)}.$$

2.10 Modification of Theorem 2.8 for the Case of the Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 4 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 2.31. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{s,t}=\int\limits_t^{*s}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(4)}]_{s,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ *(*s *is fixed),*

$$C_{j_4j_3j_2j_1}(s)=\int\limits_t^{s}\phi_{j_4}(s_4)\int\limits_t^{s_4}\phi_{j_3}(s_3)\int\limits_t^{s_3}\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_1}(s_1)ds_1ds_2ds_3ds_4$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. The relation (1.204) (in the case when $p_1 = \ldots = p_4 = p \to \infty$) implies that

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{s,t}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}A_1^{(i_3i_4)}(s)+\mathbf{1}_{\{i_1=i_3\ne 0\}}A_2^{(i_2i_4)}(s)+\mathbf{1}_{\{i_1=i_4\ne 0\}}A_3^{(i_2i_3)}(s)+\mathbf{1}_{\{i_2=i_3\ne 0\}}A_4^{(i_1i_4)}(s)+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}A_5^{(i_1i_3)}(s)+\mathbf{1}_{\{i_3=i_4\ne 0\}}A_6^{(i_1i_2)}(s)-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}B_1(s)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}B_2(s)-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}B_3(s), \tag{2.433}
$$

where $J[\psi^{(4)}]_{s,t}$ has the form (1.191) for $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$ and $i_1,\ldots,i_4 = 0,1,\ldots,m$,

$$
A_1^{(i_3i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_1j_1}(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},
$$

$$
A_2^{(i_2i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_3}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},
$$

$$
A_3^{(i_2i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},
$$

$$
A_4^{(i_1i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)},
$$

$$
A_5^{(i_1i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
A_6^{(i_1i_2)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
B_1(s) = \lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} C_{j_4j_4j_1j_1}(s), \quad B_2(s) = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_3j_4j_3j_4}(s),
$$

$$
B_3(s) = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_4j_3j_3j_4}(s).
$$

Using the integration order replacement in Riemann integrals, Theorem 1.11 for $k=2$ (see (1.202)) and (2.406), Parseval's equality and the integration order replacement technique for Itô stochastic integrals (see Chapter 3) [1]-[14], [60], [91], [92] or Itô's formula, we obtain (see the derivation of the formula (2.215))

$$A_1^{(i_3 i_4)}(s)=\frac{1}{2}\int\limits_t^s\int\limits_t^{\tau}\int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{\tau}^{(i_4)}+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^s (s_1-t)ds_1-\Delta_1^{(i_3 i_4)}(s)\quad \text{w. p. 1}, \tag{2.434}$$

where

$$\Delta_1^{(i_3 i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^p(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$a_{j_4 j_3}^p(s)=\frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2 ds_1 d\tau. \tag{2.435}$$

Let us consider $A_2^{(i_2 i_4)}(s)$ (see the derivation of the formula (2.217))

$$A_2^{(i_2 i_4)}(s)=-\Delta_2^{(i_2 i_4)}(s)+\Delta_1^{(i_2 i_4)}(s)+\Delta_3^{(i_2 i_4)}(s)\ \text{w. p. 1}, \tag{2.436}$$

where

$$\Delta_2^{(i_2 i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p} b_{j_4 j_2}^p(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2 i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p} c_{j_4 j_2}^p(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$b_{j_4 j_2}^p(s)=\frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau)\sum_{j_3=p+1}^{\infty}\left(\int\limits_t^{\tau}\phi_{j_3}(s_1)ds_1\right)^2\int\limits_t^{\tau}\phi_{j_2}(s_1)ds_1 d\tau,$$

$$c_{j_4 j_2}^p(s)=\frac{1}{2}\int\limits_t^s \phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_2}(s_3)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_3}^{\tau}\phi_{j_3}(s_1)ds_1\right)^2 ds_3 d\tau.$$

Let us consider $A_5^{(i_1 i_3)}(s)$ (see the derivation of the formula (2.218))

$$
A_5^{(i_1 i_3)}(s) = -\Delta_4^{(i_1 i_3)}(s) + \Delta_5^{(i_1 i_3)}(s) + \Delta_6^{(i_1 i_3)}(s) \quad \text{w. p. 1}, \tag{2.437}
$$

where

$$
\Delta_4^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} d_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_5^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} e_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_6^{(i_1 i_3)}(s) = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} f_{j_3 j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
d_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_3}^{s} \phi_{j_4}(\tau)d\tau\right)^2 \int\limits_{s_3}^{s} \phi_{j_3}(\tau)d\tau ds_3,
$$

$$
e_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(\tau) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_3}^{\tau} \phi_{j_4}(s_1)ds_1\right)^2 d\tau ds_3,
$$

$$
f_{j_3 j_1}^{p}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_2}^{s} \phi_{j_4}(s_1)ds_1\right)^2 ds_2 ds_3 =
$$

$$
= \frac{1}{2}\int\limits_t^s \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int\limits_{s_2}^{s} \phi_{j_4}(s_1)ds_1\right)^2 \int\limits_t^{s_2} \phi_{j_1}(s_3)ds_3 ds_2.
$$

Moreover (see the derivation of the formula (2.219)),

$$
A_3^{(i_2 i_3)}(s) = 2\Delta_6^{(i_2 i_3)}(s) - A_5^{(i_2 i_3)}(s) = \Delta_4^{(i_2 i_3)}(s) - \Delta_5^{(i_2 i_3)}(s) + \Delta_6^{(i_2 i_3)}(s) \quad \text{w. p. 1}. \tag{2.438}
$$

Let us consider $A_4^{(i_1 i_4)}(s)$ (see the derivation of the formula (2.220))

$$
A_4^{(i_1 i_4)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{s_2} \int\limits_t^{s_1} d\mathbf{w}_{\tau}^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} - \Delta_3^{(i_1 i_4)}(s) \quad \text{w. p. 1}. \tag{2.439}
$$

Let us consider $A_6^{(i_1 i_2)}(s)$ (see the derivation of the formula (2.221))

$$
A_6^{(i_1 i_2)}(s) = \frac{1}{2}\int\limits_t^s\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{\tau}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^s (s-s_2)ds_2 - \Delta_6^{(i_1 i_2)}(s)\quad \text{w. p. 1.} \tag{2.440}
$$

Let us consider $B_1(s), B_2(s), B_3(s)$ (see the derivation of the formulas (2.222), (2.223))

$$
B_1(s) = \frac{1}{4}\int\limits_t^s (s_1-t)ds_1 - \lim_{p\to\infty}\sum_{j_4=0}^{p} a_{j_4 j_4}^p(s), \tag{2.441}
$$

$$
B_2(s) = \lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3 j_3}^p(s) + \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3 j_3}^p(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3 j_3}^p(s). \tag{2.442}
$$

Moreover (see the derivation of the formula (2.224)),

$$
B_2(s)+B_3(s) = 2\lim_{p\to\infty}\sum_{j_4=0}^{p} f_{j_4 j_4}^p(s).
$$

Therefore (see the derivation of the formula (2.225),

$$
B_3(s) = 2\lim_{p\to\infty}\sum_{j_3=0}^{p} f_{j_3 j_3}^p(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} a_{j_3 j_3}^p(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} c_{j_3 j_3}^p(s) + \lim_{p\to\infty}\sum_{j_3=0}^{p} b_{j_3 j_3}^p(s). \tag{2.443}
$$

After substituting the relations (2.434), (2.436)–(2.443) into (2.433), we obtain

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} =
$$

$$
= J[\psi^{(4)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^s\int\limits_t^{\tau}\int\limits_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{\tau}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^s\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_{\tau}^{(i_1)}ds_1 d\mathbf{w}_{s_2}^{(i_4)} + \frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^s\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{\tau}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1+R(s)=J^*[\psi^{(4)}]_{s,t}+
$$

$$
+R(s)\quad \hbox{w. p. 1,} \tag{2.444}
$$

where

$$
R(s)=-\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3i_4)}(s)+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2i_4)}(s)+\Delta_1^{(i_2i_4)}(s)+\Delta_3^{(i_2i_4)}(s)\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_4^{(i_2i_3)}(s)-\Delta_5^{(i_2i_3)}(s)+\Delta_6^{(i_2i_3)}(s)\right)-\mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1i_4)}(s)+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1i_3)}(s)+\Delta_5^{(i_1i_3)}(s)+\Delta_6^{(i_1i_3)}(s)\right)-\mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1i_2)}(s)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\lim\limits_{p\to\infty}\sum_{j_3=0}^p a_{j_3j_3}^p(s)+\lim\limits_{p\to\infty}\sum_{j_3=0}^p c_{j_3j_3}^p(s)-\lim\limits_{p\to\infty}\sum_{j_3=0}^p b_{j_3j_3}^p(s)\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(2\lim\limits_{p\to\infty}\sum_{j_3=0}^p f_{j_3j_3}^p(s)-\lim\limits_{p\to\infty}\sum_{j_3=0}^p a_{j_3j_3}^p(s)-\right.
$$

$$
\left.-\lim\limits_{p\to\infty}\sum_{j_3=0}^p c_{j_3j_3}^p(s)+\lim\limits_{p\to\infty}\sum_{j_3=0}^p b_{j_3j_3}^p(s)\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim\limits_{p\to\infty}\sum_{j_3=0}^p a_{j_3j_3}^p(s). \tag{2.445}
$$

From (2.444) and (2.445) it follows that Theorem 2.31 will be proved if

$$
\Delta_k^{(ij)}(s)=0, \tag{2.446}
$$

$$
\lim\limits_{p\to\infty}\sum_{j_3=0}^p a_{j_3j_3}^p(s)=\lim\limits_{p\to\infty}\sum_{j_3=0}^p b_{j_3j_3}^p(s)=\lim\limits_{p\to\infty}\sum_{j_3=0}^p c_{j_3j_3}^p(s)=\lim\limits_{p\to\infty}\sum_{j_3=0}^p f_{j_3j_3}^p(s)=0, \tag{2.447}
$$

where $k=1,2,\ldots,6,$ $i,j=0,1,\ldots,m.$

Let us consider the case of Legendre polynomials. Let us prove that

$$
\Delta_1^{(i_3i_4)}(s)=0\quad \hbox{w. p. 1.} \tag{2.448}
$$

First consider the case $i_3 = i_4 \neq 0$

$$a^p_{j_4 j_3}(s) = \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{z(s)} P_{j_4}(y) \int\limits_{-1}^{y} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} (2j_1+1) \left(\int\limits_{-1}^{y_1} P_{j_1}(y_2)dy_2\right)^2 dy_1 dy =$$

$$= \frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times \int\limits_{-1}^{z(s)} P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 \int\limits_{y_1}^{z(s)} P_{j_4}(y) dy dy_1 =$$

$$= \frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times$$

$$\times \int\limits_{-1}^{z(s)} P_{j_3}(y_1) \left(\left(P_{j_4+1}(z(s)) - P_{j_4-1}(z(s))\right) - \left(P_{j_4+1}(y_1) - P_{j_4-1}(y_1)\right)\right)\times$$

$$\times \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 \neq 0$ and

$$a^p_{j_4 j_3}(s) = \frac{(T-t)^2\sqrt{2j_3+1}}{32}\times$$

$$\times \int\limits_{-1}^{z(s)} P_{j_3}(y_1)(z(s)-y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \left(P_{j_1+1}(y_1) - P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 = 0$, where $z(s)$ is defined by (2.20).

We can assume that $s \in (t, T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 2.8. Now the further proof of the equality (2.448) is completely analogous to the proof of the equality (2.235).

It is not difficult to see that the formulas

$$\Delta_2^{(i_2 i_4)}(s) = 0, \quad \Delta_4^{(i_1 i_3)}(s) = 0, \quad \Delta_6^{(i_1 i_3)}(s) = 0 \quad \text{w. p. 1} \tag{2.449}$$

can be proved similarly with the proof of (2.448).

Moreover, the relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}(s)=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}(s)=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3}(s)=0 \qquad (2.450)$$

can also be proved analogously with (2.237), (2.238).

Let us consider $\Delta_3^{(i_2i_4)}(s)$ and prove that

$$\Delta_3^{(i_2i_4)}(s)=0\quad \text{w. p. 1.} \qquad (2.451)$$

We have

$$\Delta_3^{(i_2i_4)}(s)=\Delta_4^{(i_2i_4)}(s)+\Delta_6^{(i_2i_4)}(s)-\Delta_7^{(i_2i_4)}(s)=$$

$$=-\Delta_7^{(i_2i_4)}(s)\quad \text{w. p. 1,} \qquad (2.452)$$

where

$$\Delta_7^{(i_2i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_2,j_4=0}^{p} g^p_{j_4j_2}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$g^p_{j_4j_2}(s)=\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_2}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_{s_1}^{s}\phi_{j_1}(s_2)ds_2\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2\right)ds_1d\tau=$$

Note that (see (2.241))

$$g^p_{j_4j_4}(s)=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2d\tau\right)^2. \qquad (2.453)$$

Let us consider the case $i_2=i_4\neq 0$. The proof of (2.451) for the case $i_2=i_4\neq 0$ differs from the proof of the equality

$$\Delta_3^{(i_2i_4)}=0\quad \text{w. p. 1}$$

for the case $i_2=i_4\neq 0$ (see the proof of Theorem 4.8). In our case we will use Parseval's equality instead of the orthogonality property of the Legendre polynomials.

Using the Parseval equality, we obtain

$$
\sum_{j_4=0}^{p} g_{j_4j_4}^{p}(s)=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_\tau^s\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2-\int\limits_t^\tau\phi_{j_1}(s_2)ds_2\right)d\tau\right)^2\le
$$

$$
\le\sum_{j_4=0}^{p}\left(\int\limits_t^s\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2\le
$$

$$
\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2=
$$

$$
=\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2d\tau\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+
$$

$$
+\sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2\left(\int\limits_t^\tau\phi_{j_1}(s_2)ds_2\right)^2d\tau=
$$

$$
=(s-t)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+\sum_{j_1=p+1}^{\infty}\int\limits_t^s\left(\int\limits_t^\tau\phi_{j_1}(s_2)ds_2\right)^2d\tau. \quad (2.454)
$$

We can assume that $s \in (t, T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 2.8. Then from (2.454) and (2.207) we obtain

$$0 \le \sum_{j_4=0}^{p} g^{p}_{j_4 j_4}(s) \le \frac{C}{p}, \tag{2.455}$$

where constant C is independent of p.

By analogy with (2.251), we can derive the following estimate

$$\sum_{j_2, j_4=0}^{p} \left(g^{p}_{j_4 j_2}(s)\right)^2 \le \frac{C_1}{p^2} \tag{2.456}$$

for the case $s \in (t, T)$ or $z(s) \in (-1, 1)$ (the case $s = T$ has already been considered in Theorem 2.8), where constant C_1 does not depend on p. For this we have to use that

$$g^{p}_{j_4 j_2}(s) = \int\limits_t^T \mathbf{1}_{\{\tau < s\}} \phi_{j_4}(\tau) \int\limits_t^{\tau} \phi_{j_2}(s_1) F_p(s, s_1, s) ds_1 d\tau =$$

$$= \int\limits_{[t,T]^2} K_p(\tau, s_1, s) \phi_{j_4}(\tau) \phi_{j_2}(s_1) ds_1 d\tau$$

is a coefficient of the double Fourier–Legendre series of the function

$$K_p(\tau, s_1, s) = \mathbf{1}_{\{s_1 < \tau < s\}} F_p(\tau, s_1, s),$$

where

$$\sum_{j_1=p+1}^{\infty} \int\limits_{s_1}^{s} \phi_{j_1}(s_2) ds_2 \int\limits_{\tau}^{s} \phi_{j_1}(s_2) ds_2 \stackrel{\sf def}{=} F_p(\tau, s_1, s). \tag{2.457}$$

Moreover, we have to use the Parseval equality

$$\lim_{p_1 \to \infty} \sum_{j_4, j_2=0}^{p_1} \left(g^{p}_{j_4 j_2}(s)\right)^2 = \int\limits_{[t,T]^2} \left(K_p(\tau, s_1, s)\right)^2 ds_1 d\tau =$$

$$= \int\limits_t^s \int\limits_t^{\tau} \left(F_p(\tau, s_1, s)\right)^2 ds_1 d\tau.$$

Then from (2.229), (2.455), and (2.456) we have

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p} g_{j_4j_2}^{p}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\left(\sum_{j_3=0}^{p} g_{j_3j_3}^{p}(s)\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(g_{j_3j_3'}^{p}(s)+g_{j_3'j_3}^{p}(s)\right)^2+2\sum_{j_3'=0}^{p}\left(g_{j_3'j_3'}^{p}(s)\right)^2\le
$$

$$
\le\left(\sum_{j_3=0}^{p} g_{j_3j_3}^{p}(s)\right)^2+4\sum_{j_3,j_3'=0}^{p}\left(g_{j_3j_3'}^{p}(s)\right)^2+2\sum_{j_3'=0}^{p}\left(g_{j_3'j_3'}^{p}(s)\right)^2\le
$$

$$
\le\frac{C_2}{p^2}\ \to 0
$$

if $p\to\infty$ $(i_2=i_4\ne 0)$, where constant C_2 does not depend on p.

From (2.230) and (2.456) we obtain the equality (2.451) for the case $i_2\ne i_4$, $i_2\ne 0$, $i_4\ne 0$.

The same result follows from (2.456) for the cases (see (2.231))
1) $i_2=0$, $i_4\ne 0$,
2) $i_4=0$, $i_2\ne 0$,
3) $i_2=0$, $i_4=0$.

The equality (2.451) is proved.

Let us consider $\Delta_5^{(i_1i_3)}(s)$

$$
\Delta_5^{(i_1i_3)}(s)=\Delta_4^{(i_1i_3)}(s)+\Delta_6^{(i_1i_3)}(s)-\Delta_8^{(i_1i_3)}(s)\quad \text{w. p. 1},
$$

where

$$
\Delta_8^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p} h_{j_3j_1}^{p}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
h_{j_3j_1}^{p}(s)=\int\limits_t^s\phi_{j_1}(s_3)\int\limits_{s_3}^s\phi_{j_3}(\tau)F_p(s_3,\tau,s)d\tau ds_3,
$$

where $F_p(s_3,\tau,s)$ is defined by (2.457).

Analogously to (2.451), we obtain that $\Delta_8^{(i_1i_3)}(s)=0$ w. p. 1. In this case we consider the function

$$
K_p(s_3,\tau,s)=\mathbf{1}_{\{s_3<s\}}\mathbf{1}_{\{s_3<\tau<s\}}F_p(s_3,\tau,s)
$$

and the relation

$$h^p_{j_3 j_1}(s) = \int\limits_{[t,T]^2} K_p(s_3,\tau,s)\phi_{j_1}(s_3)\phi_{j_3}(\tau)d\tau ds_3.$$

For the case $i_1 = i_3 \neq 0$ we use (see (2.453))

$$h^p_{j_1 j_1}(s) = \sum_{j_4=p+1}^{\infty} \frac{1}{2}\left(\int\limits_t^s \phi_{j_1}(\tau)\int\limits_\tau^s \phi_{j_4}(s_1)ds_1 d\tau\right)^2.$$

Let us prove that

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3 j_3}(s) = 0. \tag{2.458}$$

We have

$$c^p_{j_3 j_3}(s) = f^p_{j_3 j_3}(s) + d^p_{j_3 j_3}(s) - g^p_{j_3 j_3}(s). \tag{2.459}$$

Moreover,

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3 j_3} = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} d^p_{j_3 j_3} = 0, \tag{2.460}$$

where the first equality in (2.460) has been proved earlier. Analogously, we can prove the second equality in (2.460).

From (2.455) we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} g^p_{j_3 j_3}(s) = 0.$$

So, (2.458) is proved. The relations (2.446), (2.447) are proved for the polynomial case. Theorem 2.31 is proved for the case of Legendre polynomials.

It is easy to see that the trigonometric case is considered by analogy with the case of Legendre polynomials using the estimate

$$\left|\int\limits_\tau^s \phi_j(\theta)d\theta\right| \le \frac{C}{j} \quad (j \neq 0),$$

where constant C is independent of p, $t \le \tau < s \le T$, and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Theorem 2.31 is proved.

Let us reformulate Theorem 2.31 in terms on the convergence of the solution of the system of ODEs to the solution of the system of Stratonovich SDEs.

From (2.387) for the case $k = 4$, $p_1 = \ldots = p_4 = p$, $i_1, \ldots, i_4 = 0, 1, \ldots, m$, and $s \in (t, T]$ (s is fixed) we obtain

$$\int\limits_t^s \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p} = \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

where $p \in \mathbf{N}$ and $d\mathbf{w}_{\tau}^{(i)p}$ is defined by (2.386); another notations are the same as in Theorem 2.31.

The iterated Riemann–Stiltjes integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)p} = \int\limits_t^s \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p}, \tag{2.461}$$

$$Z_{s,t}^{(i_1 i_2 i_3)p} = \int\limits_t^s \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p}, \tag{2.462}$$

$$Y_{s,t}^{(i_1 i_2)p} = \int\limits_t^s \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p}, \tag{2.463}$$

$$X_{s,t}^{(i_1)p} = \int\limits_t^s d\mathbf{w}_{t_1}^{(i_1)p} \tag{2.464}$$

are the solution of the following system of ODEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)p} = Z_{s,t}^{(i_1 i_2 i_3)p} d\mathbf{w}_s^{(i_4)p}, & V_{t,t}^{(i_1 i_2 i_3 i_4)p} = 0 \\ dZ_{s,t}^{(i_1 i_2 i_3)p} = Y_{s,t}^{(i_1 i_2)p} d\mathbf{w}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\ dY_{s,t}^{(i_1 i_2)p} = X_{s,t}^{(i_1)p} d\mathbf{w}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\ dX_{s,t}^{(i_1)p} = 1 \cdot d\mathbf{w}_s^{(i_1)p}, & X_{t,t}^{(i_1)p} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_4} {\int\limits_t^{*}}^{t_3} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}, \tag{2.465}$$

$$Z_{s,t}^{(i_1 i_2 i_3)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_3} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}, \tag{2.466}$$

$$Y_{s,t}^{(i_1 i_2)} = {\int\limits_t^{*}}^{s} {\int\limits_t^{*}}^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \tag{2.467}$$

$$X_{s,t}^{(i_1)} = {\int\limits_t^{*}}^{s} d\mathbf{w}_{t_1}^{(i_1)} \tag{2.468}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)} = Z_{s,t}^{(i_1 i_2 i_3)} * d\mathbf{w}_s^{(i_4)}, & V_{t,t}^{(i_1 i_2 i_3 i_4)} = 0 \\ dZ_{s,t}^{(i_1 i_2 i_3)} = Y_{s,t}^{(i_1 i_2)} * d\mathbf{w}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\ dY_{s,t}^{(i_1 i_2)} = X_{s,t}^{(i_1)} * d\mathbf{w}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ dX_{s,t}^{(i_1)} = 1 * d\mathbf{w}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* \, d\mathbf{w}_s^{(i)}$, $i = 0, 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 2.28, 2.30, and 2.31 we obtain the following theorem.

Theorem 2.32. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then for any fixed s ($s \in (t, T]$)*

$$\underset{p\to\infty}{\text{l.i.m.}}\; V_{s,t}^{(i_1 i_2 i_3 i_4)p} = V_{s,t}^{(i_1 i_2 i_3 i_4)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\; Z_{s,t}^{(i_1 i_2 i_3)p} = Z_{s,t}^{(i_1 i_2 i_3)},$$

$$\underset{p\to\infty}{\text{l.i.m.}}\; Y_{s,t}^{(i_1 i_2)p} = Y_{s,t}^{(i_1 i_2)}, \quad X_{s,t}^{(i_1)p} = \underset{p\to\infty}{\text{l.i.m.}}\; X_{s,t}^{(i_1)}.$$

2.11 Modification of Theorem 2.9 for the Case of the Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 5 and Wong–Zakai Type Theorem

Let us formulate the following theorem.

Theorem 2.33. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{s,t}=\int\limits_t^{*s}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}
$$

the following expansion

$$
J^*[\psi^{(5)}]_{s,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_5=0,1,\ldots,m,$

$$
C_{j_5j_4j_3j_2j_1}(s)=\int\limits_t^{s}\phi_{j_5}(s_5)\int\limits_t^{s_5}\phi_{j_4}(s_4)\int\limits_t^{s_4}\phi_{j_3}(s_3)\int\limits_t^{s_3}\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_1}(s_1)ds_1ds_2ds_3ds_4ds_5,
$$

$$
\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau.$

The proof of Theorem 2.33 can be carried out on the basis of the proof of Theorem 2.9 and the method of proof of Theorem 2.31.

Let us reformulate Theorem 2.33 in terms on the convergence of the solution of the system of ODEs to the solution of the system of Stratonovich SDEs.

From (2.387) for the case $k=5$, $p_1=\ldots=p_5=p$, $i_1,\ldots,i_5=0,1,\ldots,m,$ and $s\in(t,T]$ (s is fixed) we obtain

$$
U_{s,t}^{(i_1i_2i_3i_4i_5)p}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},
$$

where

$$\int\limits_t^s\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}d\mathbf{w}_{t_5}^{(i_5)p} \stackrel{\sf def}{=} U_{s,t}^{(i_1 i_2 i_3 i_4 i_5)p},$$

$p \in \mathbf{N}$, and $d\mathbf{w}_{\tau}^{(i)p}$ is defined by (2.386); another notations are the same as in Theorem 2.33.

The iterated Riemann–Stiltjes integrals $X_{s,t}^{(i_1)p}$, $Y_{s,t}^{(i_1 i_2)p}$, $Z_{s,t}^{(i_1 i_2 i_3)p}$, $V_{s,t}^{(i_1 i_2 i_3 i_4)p}$ (see (2.461)–(2.464)), and $U_{s,t}^{(i_1 i_2 i_3 i_4 i_5)p}$ are the solution of the following system of ODEs

$$\begin{cases} dU_{s,t}^{(i_1 i_2 i_3 i_4 i_5)p} = V_{s,t}^{(i_1 i_2 i_3 i_4)p} d\mathbf{w}_s^{(i_5)p}, & U_{t,t}^{(i_1 i_2 i_3 i_4 i_5)p} = 0\\ dV_{s,t}^{(i_1 i_2 i_3 i_4)p} = Z_{s,t}^{(i_1 i_2 i_3)p} d\mathbf{w}_s^{(i_4)p}, & V_{t,t}^{(i_1 i_2 i_3 i_4)p} = 0\\ dZ_{s,t}^{(i_1 i_2 i_3)p} = Y_{s,t}^{(i_1 i_2)p} d\mathbf{w}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0\\ dY_{s,t}^{(i_1 i_2)p} = X_{s,t}^{(i_1)p} d\mathbf{w}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0\\ dX_{s,t}^{(i_1)p} = 1 \cdot d\mathbf{w}_s^{(i_1)p}, & X_{t,t}^{(i_1)p} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals $X_{s,t}^{(i_1)}$, $Y_{s,t}^{(i_1 i_2)}$, $Z_{s,t}^{(i_1 i_2 i_3)}$, $V_{s,t}^{(i_1 i_2 i_3 i_4)}$ (see (2.465)–(2.468)), and $U_{s,t}^{(i_1 i_2 i_3 i_4 i_5)}$ are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dU_{s,t}^{(i_1 i_2 i_3 i_4 i_5)} = V_{s,t}^{(i_1 i_2 i_3 i_4)} * d\mathbf{w}_s^{(i_5)}, & U_{t,t}^{(i_1 i_2 i_3 i_4 i_5)} = 0\\ dV_{s,t}^{(i_1 i_2 i_3 i_4)} = Z_{s,t}^{(i_1 i_2 i_3)} * d\mathbf{w}_s^{(i_4)}, & V_{t,t}^{(i_1 i_2 i_3 i_4)} = 0\\ dZ_{s,t}^{(i_1 i_2 i_3)} = Y_{s,t}^{(i_1 i_2)} * d\mathbf{w}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0\\ dY_{s,t}^{(i_1 i_2)} = X_{s,t}^{(i_1)} * d\mathbf{w}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0\\ dX_{s,t}^{(i_1)} = 1 * d\mathbf{w}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $*\, d\mathbf{w}_s^{(i)}$, $i = 0, 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 2.28, 2.30, 2.32, and 2.33 we obtain the following theorem.

Theorem 2.34. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then for any fixed* s $(s \in (t, T])$

$$
\underset{p\to\infty}{\text{l.i.m.}}\ U_{s,t}^{(i_1 i_2 i_3 i_4 i_5)p} = U_{s,t}^{(i_1 i_2 i_3 i_4 i_5)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ V_{s,t}^{(i_1 i_2 i_3 i_4)p} = V_{s,t}^{(i_1 i_2 i_3 i_4)},
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1 i_2 i_3)p} = Z_{s,t}^{(i_1 i_2 i_3)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1 i_2)p} = Y_{s,t}^{(i_1 i_2)},
$$

$$
X_{s,t}^{(i_1)p} = \underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)}.
$$

Chapter 3

Integration Order Replacement Technique for Iterated Itô Stochastic Integrals and Iterated Stochastic Integrals with Respect to Martingales

This chapter is devoted to the integration order replacement technique for iterated Itô stochastic integrals and iterated stochastic integrals with respect to martingales. We consider the class of iterated Itô stochastic integrals, for which with probability 1 the formulas on integration order replacement corresponding to the rules of classical integral calculus are correct. The theorems on integration order replacement for the class of iterated Itô stochastic integrals are proved. Many examples of these theorems usage have been considered. The mentioned results are generalized for the class of iterated stochastic integrals with respect to martingales.

3.1 Introduction

In this chapter we performed rather laborious work connected with the theorems on integration order replacement for iterated Itô stochastic integrals. However, there may appear a question about a practical usefulness of this theory, since the significant part of its conclusions directly arise from the Itô formula.

It is not difficult to see that to obtain various relations for iterated Itô stochastic integrals (see, for example, Sect. 3.6) using the Itô formula, first of all these relations should be guessed. Then it is necessary to introduce corresponding Itô processes and afterwards to use the Itô formula. It is clear that this process requires intellectual expenses and it is not always trivial.

On the other hand, the technique on integration order replacement introduced in this chapter is formally comply with the similar technique for Riemann integrals, although it is related to Itô integrals, and it provides a possibility to perform transformations naturally (as with Riemann integrals) with iterated Itô stochastic integrals and to obtain various relations for them.

So, in order to implementation of transformations of the specific class of Itô processes, which is represented by iterated Itô stochastic integrals, it is more naturally and easier to use the theorems on integration order replacement, than the Itô formula.

Many examples of these theorems usage are presented in Sect. 3.6.

Note that in Chapters 1, 2, and 4 the integration order replacement technique for iterated Itô stochastic integrals has been successfully applied for the proof and development of the method of approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series (see Chapters 1 and 2) as well as for the construction of the so-called unified Taylor–Itô and Taylor–Stratonovich expansions (see Chapter 4).

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega): [0,T]\times\Omega \to \mathbf{R}^1$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$. Further, we will use the following notation: $f(t,\omega) \stackrel{\rm def}{=} f_t$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s<t$.

2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.

3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let us recall that the class $\mathrm{M}_2([0,T])$ (see Sect. 1.1.2) consists of functions $\xi: [0,T]\times\Omega\to\mathbf{R}^1$, which satisfy the conditions:

1. The function $\xi(t,\omega)$ is measurable with respect to the pair of variables (t,ω).

2. The function $\xi(t,\omega)$ is F_t-measurable for all $t\in[0,T]$ and $\xi(\tau,\omega)$ is independent with increments $f_{t+\Delta}-f_t$ for $t\ge\tau$, $\Delta>0$.

3. The following relation is fulfilled

$$\int\limits_0^T \mathsf{M}\left\{(\xi(t,\omega))^2\right\} dt < \infty.$$

4. $\mathsf{M}\left\{(\xi(t,\omega))^2\right\} < \infty$ for all $t \in [0,T]$.

Let us recall (see Sect. 1.1.2) that the stochastic integrals

$$\int\limits_0^T \xi_\tau df_\tau \quad \text{and} \quad \int\limits_0^T \xi_\tau d\tau, \tag{3.1}$$

where $\xi_t \in \mathrm{M}_2([0,T])$ and the first integral in (3.1) is the Itô stochastic integral, can be defined in the mean-square sense by the relations (1.2) and (1.4).

We will introduce the class $\mathrm{S}_2([0,T])$ of functions $\xi : [0,T]\times\Omega \to \mathbf{R}^1$, which satisfy the conditions:

1. $\xi(\tau,\omega) \in \mathrm{M}_2([0,T])$.

2. $\xi(\tau,\omega)$ is the mean-square continuous random process at the interval $[0,T]$.

As we noted above, the Itô stochastic integral exists in the mean-square sense (see (1.2)), if the random process $\xi(\tau,\omega) \in \mathrm{M}_2([0,T])$, i.e., perhaps this process does not satisfy the property of the mean-square continuity on the interval $[0,T]$. In this chapter we will formulate and prove the theorems on integration order replacement for the special class of iterated Itô stochastic integrals. At the same time, the condition of the mean-square continuity of integrand in the innermost stochastic integral will be significant.

Let us introduce the following class of iterated stochastic integrals

$$J[\phi,\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_1(t_1)\ldots \int\limits_t^{t_{k-1}} \psi_k(t_k) \int\limits_t^{t_k} \phi_\tau dw_\tau^{(k+1)} dw_{t_k}^{(k)} \ldots dw_{t_1}^{(1)},$$

where $\phi(\tau,\omega) \stackrel{\text{def}}{=} \phi_\tau$, $\phi_\tau \in \mathrm{S}_2([t,T])$, every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continous nonrandom function at the interval $[t,T]$, here and further $w_\tau^{(l)} = f_\tau$ or $w_\tau^{(l)} = \tau$ for $\tau \in [t,T]$ $(l = 1,\ldots,k+1)$, $(\psi_1,\ldots,\psi_k) \stackrel{\text{def}}{=} \psi^{(k)}$, $\psi^{(1)} \stackrel{\text{def}}{=} \psi_1$.

We will call the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$ as the iterated Itô stochastic integral.

It is well known that for the iterated Riemann integral in the case of specific conditions the formula on integration order replacement is correct. In particular, if the nonrandom functions $f(x)$ and $g(x)$ are continuous at the interval $[a, b]$, then

$$\int\limits_a^b f(x)\int\limits_a^x g(y)dydx = \int\limits_a^b g(y)\int\limits_y^b f(x)dxdy. \tag{3.2}$$

If we suppose that for the Itô stochastic integral

$$J[\phi,\psi_1]_{T,t} = \int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)}$$

the formula on integration order replacement, which is similar to (3.2), is valid, then we will have

$$\int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)} = \int\limits_t^T \phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}dw_\tau^{(2)}. \tag{3.3}$$

If, in addition $w_s^{(1)},\ w_s^{(2)} = f_s\ (s\in[t,T])$ in (3.3), then the stochastic process

$$\eta_\tau = \phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}$$

does not belong to the class $\mathrm{M}_2([t,T])$, and, consequently, for the Itô stochastic integral

$$\int\limits_t^T \eta_\tau dw_\tau^{(2)}$$

on the right-hand side of (3.3) the conditions of its existence are not fulfilled.

At the same time

$$\int\limits_t^T df_s\int\limits_t^T ds = \int\limits_t^T (s-t)df_s + \int\limits_t^T (f_s-f_t)ds \quad \text{w. p. 1}, \tag{3.4}$$

and we can obtain this equality, for example, using the Itô formula, but (3.4) can be considered as a result of integration order replacement (see below).

Actually, we can demonstrate that

$$\int\limits_t^T (f_s - f_t)ds = \int\limits_t^T \int\limits_t^s df_\tau ds = \int\limits_t^T \int\limits_\tau^T ds df_\tau \quad \text{w. p. 1.}$$

Then

$$\int\limits_t^T (s-t)df_s + \int\limits_t^T (f_s - f_t)ds = \int\limits_t^T \int\limits_t^\tau ds df_\tau + \int\limits_t^T \int\limits_\tau^T ds df_\tau = \int\limits_t^T df_s \int\limits_t^T ds \quad \text{w. p. 1.}$$

The aim of this chapter is to establish the strict mathematical sense of the formula (3.3) for the case $w_s^{(1)},\ w_s^{(2)} = f_s$ $(s \in [t,T])$ as well as its analogue corresponding to the iterated Itô stochastic integral $J[\phi, \psi^{(k)}]_{T,t}$, $k \ge 2$. At that, we will use the definition of the Itô stochastic integral which is more general than (1.2).

Let us consider the partition $\tau_j^{(N)}$, $j = 0, 1, \ldots, N$ of the interval $[t,T]$ such that

$$t = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \qquad \max_{0 \le j \le N-1} \left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0 \ \text{ if } \ N \to \infty. \tag{3.5}$$

In [86] Stratonovich R.L. introduced the definition of the so-called combined stochastic integral for the specific class of integrated processes. Taking this definition as a foundation, let us consider the following construction of stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j} \left(f_{\tau_{j+1}} - f_{\tau_j}\right) \theta_{\tau_{j+1}} \stackrel{\text{def}}{=} \int\limits_t^T \phi_\tau df_\tau \theta_\tau, \tag{3.6}$$

where $\phi_\tau,\ \theta_\tau \in \text{S}_2([t,T])$, $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition (3.5) (here and sometimes further for simplicity we write τ_j instead of $\tau_j^{(N)}$).

Further, we will prove existence of the integral (3.6) for $\phi_\tau \in \text{S}_2([t,T])$ and θ_τ from a little bit narrower class of processes than $\text{S}_2([t,T])$. In addition, the integral defined by (3.6) will be used for the formulation and proof of the theorem on integration order replacement for the iterated Itô stochastic integrals $J[\phi, \psi^{(k)}]_{T,t}$, $k \ge 1$.

Note that under the appropriate conditions the following properties of stochastic integrals defined by the formula (3.6) can be proved

$$\int_t^T \phi_\tau df_\tau g(\tau) = \int_t^T \phi_\tau g(\tau) df_\tau \quad \text{w. p. 1},$$

where $g(\tau)$ is a continuous nonrandom function at the interval $[t, T]$,

$$\int_t^T (\alpha\phi_\tau + \beta\psi_\tau)\, df_\tau \theta_\tau = \alpha \int_t^T \phi_\tau df_\tau \theta_\tau + \beta \int_t^T \psi_\tau df_\tau \theta_\tau \quad \text{w. p. 1},$$

$$\int_t^T \phi_\tau df_\tau\, (\alpha\theta_\tau + \beta\psi_\tau) = \alpha \int_t^T \phi_\tau df_\tau \theta_\tau + \beta \int_t^T \phi_\tau df_\tau \psi_\tau \quad \text{w. p. 1},$$

where $\alpha,\ \beta \in \mathbf{R}^1$.

At that, we suppose that the stochastic processes ϕ_τ, θ_τ, and ψ_τ are such that the integrals included in the mentioned properties exist.

3.2 Formulation of the Theorem on Integration Order Replacement for Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$

Let us define the stochastic integrals $\hat{I}[\psi^{(k)}]_{T,s}$, $k \ge 1$ of the form

$$\hat{I}[\psi^{(k)}]_{T,s} = \int_s^T \psi_k(t_k) dw_{t_k}^{(k)} \int_{t_k}^T \psi_{k-1}(t_{k-1}) dw_{t_{k-1}}^{(k-1)} \ldots \int_{t_2}^T \psi_1(t_1) dw_{t_1}^{(1)}$$

in accordance with the definition (3.6) by the following recurrence relation

$$\hat{I}[\psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_k(\tau_l) \Delta w_{\tau_l}^{(k)} \hat{I}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \tag{3.7}$$

where $k \ge 1$, $\hat{I}[\psi^{(0)}]_{T,s} \stackrel{\text{def}}{=} 1$, $[s, T] \subseteq [t, T]$, here and further $\Delta w_{\tau_l}^{(i)} = w_{\tau_{l+1}}^{(i)} - w_{\tau_l}^{(i)}$, $i = 1, \ldots, k+1$, $l = 0, 1, \ldots, N-1$.

Then, we will define the iterated stochastic integral $\hat{J}[\phi, \psi^{(k)}]_{T,t}$, $k \ge 1$

$$\hat{J}[\phi, \psi^{(k)}]_{T,t} = \int_t^T \phi_s dw_s^{(k+1)} \hat{I}[\psi^{(k)}]_{T,s}$$

similarly in accordance with the definition (3.6)

$$\hat{J}[\phi,\psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}.$$

Let us formulate the theorem on integration order replacement for iterated Itô stochastic integrals.

Theorem 3.1 [91] (1997), [60] (1998) (also see [1]-[14], [60], [92]). *Suppose that* $\phi_\tau \in {\rm S}_2([t,T])$ *and every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$. *Then, the stochastic integral* $\hat{J}[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$ *exists and*

$$J[\phi,\psi^{(k)}]_{T,t} = \hat{J}[\phi,\psi^{(k)}]_{T,t} \quad w.\ p.\ 1.$$

3.3 Proof of Theorem 3.1 for the Case of Iterated Itô Stochastic Integrals of Multiplicity 2

At first, let us prove Theorem 3.1 for the case $k=1$. We have

$$J[\phi,\psi_1]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l)\Delta w_{\tau_l}^{(1)} \int\limits_t^{\tau_l} \phi_\tau dw_\tau^{(2)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l)\Delta w_{\tau_l}^{(1)} \sum_{j=0}^{l-1} \int\limits_{\tau_j}^{\tau_{j+1}} \phi_\tau dw_\tau^{(2)}, \tag{3.8}$$

$$\hat{J}[\phi,\psi_1]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)} \int\limits_{\tau_{j+1}}^{T} \psi_1(s) dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)} \sum_{l=j+1}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s) dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s) dw_s^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j}\Delta w_{\tau_j}^{(2)}. \tag{3.9}$$

It is clear that if the difference ε_N of prelimit expressions on the right-hand sides of (3.8) and (3.9) tends to zero when $N\to\infty$ in the mean-square sense, then the stochastic integral $\hat{J}[\phi,\psi_1]_{T,t}$ exists and

$$J[\phi,\psi_1]_{T,t}=\hat{J}[\phi,\psi_1]_{T,t}\quad \text{w. p. 1.}$$

The difference ε_N can be represented in the form $\varepsilon_N=\tilde{\varepsilon}_N+\hat{\varepsilon}_N$, where

$$\tilde{\varepsilon}_N=\sum_{l=0}^{N-1}\psi_1(\tau_l)\Delta w_{\tau_l}^{(1)}\sum_{j=0}^{l-1}\int\limits_{\tau_j}^{\tau_{j+1}}\left(\phi_\tau-\phi_{\tau_j}\right)dw_\tau^{(2)};$$

$$\hat{\varepsilon}_N=\sum_{l=0}^{N-1}\int\limits_{\tau_l}^{\tau_{l+1}}\left(\psi_1(\tau_l)-\psi_1(s)\right)dw_s^{(1)}\sum_{j=0}^{l-1}\phi_{\tau_j}\Delta w_{\tau_j}^{(2)}.$$

We will demonstrate that

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N=0.$$

In order to do it we will analyze four cases:

1. $w_\tau^{(2)}=f_\tau,\ \Delta w_{\tau_l}^{(1)}=\Delta f_{\tau_l}.$
2. $w_\tau^{(2)}=\tau,\ \Delta w_{\tau_l}^{(1)}=\Delta f_{\tau_l}.$
3. $w_\tau^{(2)}=f_\tau,\ \Delta w_{\tau_l}^{(1)}=\Delta \tau_l.$
4. $w_\tau^{(2)}=\tau,\ \Delta w_{\tau_l}^{(1)}=\Delta \tau_l.$

Let us recall the well known standard moment properties of stochastic integrals [83]

$$\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau df_\tau\right|^2\right\}=\int\limits_t^T\mathsf{M}\left\{|\xi_\tau|^2\right\}d\tau,$$

$$\mathsf{M}\left\{\left|\int\limits_t^T\xi_\tau d\tau\right|^2\right\}\le (T-t)\int\limits_t^T\mathsf{M}\left\{|\xi_\tau|^2\right\}d\tau, \tag{3.10}$$

where $\xi_\tau\in \text{M}_2([t,T])$.

For Case 1 using standard moment properties for the Itô stochastic integral as well as mean-square continuity (which means uniform mean-square continuity) of the process ϕ_τ on the interval $[t,T]$, we obtain

$$
\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\}=\sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}\mathsf{M}\left\{\left|\phi_\tau-\phi_{\tau_j}\right|^2\right\}d\tau<
$$

$$
< C^2\varepsilon\sum_{k=0}^{N-1}\Delta\tau_k\sum_{j=0}^{k-1}\Delta\tau_j < C^2\varepsilon\frac{(T-t)^2}{2},
$$

i.e. $\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\}\to 0$ when $N\to\infty$. Here $\Delta\tau_j<\delta(\varepsilon)$, $j=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on τ), $|\psi_1(\tau)|<C$.

Let us consider Case 2. Using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (3.10) for the stochastic integral, we have

$$
\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\}=\sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\mathsf{M}\left\{\left(\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau-\phi_{\tau_j})d\tau\right)^2\right\}\le
$$

$$
\le\sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\left(\sum_{j=0}^{k-1}\left(\mathsf{M}\left\{\left(\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau-\phi_{\tau_j})d\tau\right)^2\right\}\right)^{1/2}\right)^2<
$$

$$
< C^2\varepsilon\sum_{k=0}^{N-1}\Delta\tau_k\left(\sum_{j=0}^{k-1}\Delta\tau_j\right)^2 < C^2\varepsilon\frac{(T-t)^3}{3},
$$

i.e. $\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\}\to 0$ when $N\to\infty$. Here $\Delta\tau_j<\delta(\varepsilon)$, $j=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on τ), $|\psi_1(\tau)|<C$.

For Case 3 using the Minkowski inequality, standard moment properties for the Itô stochastic integral as well as uniform mean-square continuity of the process ϕ_τ, we find

$$
\mathsf{M}\left\{\left|\tilde{\varepsilon}_N\right|^2\right\}\le\left(\sum_{k=0}^{N-1}\left|\psi_1(\tau_k)\right|\Delta\tau_k\left(\mathsf{M}\left\{\left(\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau-\phi_{\tau_j})df_\tau\right)^2\right\}\right)^{1/2}\right)^2=
$$

$$
= \left(\sum_{k=0}^{N-1} |\psi_1(\tau_k)| \Delta\tau_k \left(\sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} \mathsf{M}\left\{ |\phi_\tau - \phi_{\tau_j}|^2 \right\} d\tau \right)^{1/2} \right)^2 <
$$

$$
< C^2 \varepsilon \left(\sum_{k=0}^{N-1} \Delta\tau_k \left(\sum_{j=0}^{k-1} \Delta\tau_j \right)^{1/2} \right)^2 < C^2 \varepsilon \frac{4(T-t)^3}{9},
$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Finally, for Case 4 using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (3.10) for the stochastic integral, we obtain

$$
\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \le \left(\sum_{k=0}^{N-1} \sum_{j=0}^{k-1} |\psi_1(\tau_k)| \Delta\tau_k \left(\mathsf{M}\left\{ \left(\int\limits_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_{\tau_j}) d\tau \right)^2 \right\} \right)^{1/2} \right)^2 <
$$

$$
< C^2 \varepsilon \left(\sum_{k=0}^{N-1} \Delta\tau_k \sum_{j=0}^{k-1} \Delta\tau_j \right)^2 < C^2 \varepsilon \frac{(T-t)^4}{4},
$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Thus, we have proved that

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \tilde{\varepsilon}_N = 0.
$$

Analogously, taking into account the uniform continuity of the function $\psi_1(\tau)$ on the interval $[t, T]$, we can demonstrate that

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \hat{\varepsilon}_N = 0.
$$

Consequently,

$$
\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0.
$$

Theorem 3.1 is proved for the case $k = 1$.

Remark 3.1. *Proving Theorem* 3.1, *we used the fact that if the stochastic process* ϕ_t *is mean-square continuous at the interval* $[t,T]$, *then it is uniformly mean-square continuous at this interval, i.e.* $\forall\ \varepsilon>0\ \exists\ \delta(\varepsilon)>0$ *such that for all* $t_1,t_2\in[t,T]$ *satisfying the condition* $|t_1-t_2|<\delta(\varepsilon)$ *the inequality*

$$\mathsf{M}\left\{|\phi_{t_1}-\phi_{t_2}|^2\right\}<\varepsilon$$

is fulfilled (here $\delta(\varepsilon)$ *does not depend on* t_1 *and* t_2*).*

Proof. *Suppose that the stochastic process* ϕ_t *is mean-square continuous at the interval* $[t,T]$, *but not uniformly mean-square continuous at this interval. Then for some* $\varepsilon>0$ *and* $\forall\ \delta(\varepsilon)>0\ \exists\ t_1,t_2\in[t,T]$ *such that* $|t_1-t_2|<\delta(\varepsilon)$, *but*

$$\mathsf{M}\left\{|\phi_{t_1}-\phi_{t_2}|^2\right\}\ge\varepsilon.$$

Consequently, for $\delta=\delta_n=1/n$ $(n\in\mathbf{N})$ $\exists\ t_1^{(n)},\ t_2^{(n)}\in[t,T]$ *such that*

$$\left|t_1^{(n)}-t_2^{(n)}\right|<\frac{1}{n},$$

but

$$\mathsf{M}\left\{\left|\phi_{t_1^{(n)}}-\phi_{t_2^{(n)}}\right|^2\right\}\ge\varepsilon.$$

The sequence $t_1^{(n)}$ $(n\in\mathbf{N})$ *is bounded, consequently, according to the Bolzano–Weierstrass Theorem, we can choose from it the subsequence* $t_1^{(k_n)}$ $(n\in\mathbf{N})$ *that converges to a certain number* $\tilde t$ *(it is simple to demonstrate that* $\tilde t\in[t,T]$*). Similarly to it and in virtue of the inequality*

$$\left|t_1^{(n)}-t_2^{(n)}\right|<\frac{1}{n}$$

we have $t_2^{(k_n)}\to\tilde t$ *when* $n\to\infty$.

According to the mean-square continuity of the process ϕ_t *at the moment* $\tilde t$ *and the elementary inequality* $(a+b)^2\le 2(a^2+b^2)$, *we obtain*

$$0\le\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{t_2^{(k_n)}}\right|^2\right\}\le$$

$$\le 2\left(\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{\tilde t}\right|^2\right\}+\mathsf{M}\left\{\left|\phi_{t_2^{(k_n)}}-\phi_{\tilde t}\right|^2\right\}\right)\to 0$$

when $n\to\infty$. *Then*

$$\lim_{n\to\infty}\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{t_2^{(k_n)}}\right|^2\right\}=0.$$

It is impossible by virtue of the fact that

$$\mathsf{M}\left\{\left|\phi_{t_1^{(kn)}} - \phi_{t_2^{(kn)}}\right|^2\right\} \ge \varepsilon > 0.$$

The obtained contradiction proves the required statement.

3.4 Proof of Theorem 3.1 for the Case of Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbf{N})$

Let us prove Theorem 3.1 for the case $k > 1$. In order to do it we will introduce the following notations

$$I[\psi_q^{(r+1)}]_{\theta,s} \stackrel{\rm def}{=} \int\limits_s^{\theta} \psi_q(t_1)\ldots \int\limits_s^{t_r} \psi_{q+r}(t_{r+1}) dw_{t_{r+1}}^{(q+r)}\ldots dw_{t_1}^{(q)},$$

$$J[\phi,\psi_q^{(r+1)}]_{\theta,s} \stackrel{\rm def}{=} \int\limits_s^{\theta} \psi_q(t_1)\ldots \int\limits_s^{t_r} \psi_{q+r}(t_{r+1}) \int\limits_s^{t_{r+1}} \phi_{\tau} dw_{\tau}^{(q+r+1)} dw_{t_{r+1}}^{(q+r)}\ldots dw_{t_1}^{(q)},$$

$$G[\psi_q^{(r+1)}]_{n,m} = \sum_{j_q=m}^{n-1}\sum_{j_{q+1}=m}^{j_q-1}\ldots \sum_{j_{q+r}=m}^{j_{q+r-1}-1}\prod_{l=q}^{r+q} I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$(\psi_q,\ldots,\psi_{q+r}) \stackrel{\rm def}{=} \psi_q^{(r+1)}, \quad \psi_q^{(1)} \stackrel{\rm def}{=} \psi_q,$$

$$(\psi_1,\ldots,\psi_{r+1}) \stackrel{\rm def}{=} \psi_1^{(r+1)}, \quad \psi_1^{(r+1)} \stackrel{\rm def}{=} \psi^{(r+1)}.$$

Note that according to notations introduced above, we have

$$I[\psi_l]_{s,\theta} = \int\limits_{\theta}^{s} \psi_l(\tau) dw_{\tau}^{(l)}.$$

To prove Theorem 3.1 for $k > 1$ it is enough to show that

$$J[\phi,\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\rm l.i.m.}\ S[\phi,\psi^{(k)}]_N = \hat J[\phi,\psi^{(k)}]_{T,t} \quad \text{w. p. 1}, \tag{3.11}$$

where

$$S[\phi,\psi^{(k)}]_N = G[\psi^{(k)}]_{N,0}\sum_{l=0}^{j_k-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)},$$

where $\Delta w_{\tau_l}^{(k+1)}=w_{\tau_{l+1}}^{(k+1)}-w_{\tau_l}^{(k+1)}$.

At first, let us prove the right equality in (3.11). We have

$$\hat{J}[\phi,\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}. \tag{3.12}$$

On the basis of the inductive hypothesis we obtain that

$$I[\psi^{(k)}]_{T,\tau_{l+1}}=\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}\quad \text{w. p. 1}, \tag{3.13}$$

where $\hat{I}[\psi^{(k)}]_{T,s}$ is defined in accordance with (3.7) and

$$I[\psi^{(k)}]_{T,s}=\int\limits_s^T\psi_1(t_1)\ldots\int\limits_s^{t_{k-2}}\psi_{k-1}(t_{k-1})\int\limits_s^{t_{k-1}}\psi_k(t_k)dw_{t_k}^{(k)}dw_{t_{k-1}}^{(k-1)}\ldots dw_{t_1}^{(1)}.$$

Let us note that when $k\ge 4$ (for $k=2,3$ the arguments are similar) due to additivity of the Itô stochastic integral the following equalities are correct

$$I[\psi^{(k)}]_{T,\tau_{l+1}}=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(t_1)\int\limits_{\tau_{l+1}}^{t_1}\psi_2(t_2)I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}}dw_{t_2}^{(2)}dw_{t_1}^{(1)}=$$

$$=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(t_1)\left(\sum_{j_2=l+1}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}+\int\limits_{\tau_{j_1}}^{t_1}\right)\psi_2(t_2)I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}}dw_{t_2}^{(2)}dw_{t_1}^{(1)}=$$

$$=\ldots=G[\psi^{(k)}]_{N,l+1}+H[\psi^{(k)}]_{N,l+1}\quad \text{w. p. 1}, \tag{3.14}$$

where

$$H[\psi^{(k)}]_{N,l+1}=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(s)\int\limits_{\tau_{j_1}}^{s}\psi_2(\tau)I[\psi_3^{(k-2)}]_{\tau,\tau_{l+1}}dw_{\tau}^{(2)}dw_s^{(1)}+$$

$$+\sum_{r=2}^{k-2}G[\psi^{(r-1)}]_{N,l+1}\sum_{j_r=l+1}^{j_{r-1}-1}\int\limits_{\tau_{j_r}}^{\tau_{j_r+1}}\psi_r(s)\int\limits_{\tau_{j_r}}^{s}\psi_{r+1}(\tau)I[\psi_{r+2}^{(k-r-1)}]_{\tau,\tau_{l+1}}dw_{\tau}^{(r+1)}dw_s^{(r)}+$$

$$+G[\psi^{(k-2)}]_{N,l+1}\sum_{j_{k-1}=l+1}^{j_{k-2}-1}I[\psi_{k-1}^{(2)}]_{\tau_{j_{k-1}+1},\tau_{j_{k-1}}}. \tag{3.15}$$

Next, substitute (3.14) into (3.13) and (3.13) into (3.12). Then w. p. 1

$$\hat{J}[\phi,\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\biggl(G[\psi^{(k)}]_{N,l+1} + H[\psi^{(k)}]_{N,l+1}\biggr). \tag{3.16}$$

Since

$$\sum_{j_1=0}^{N-1}\sum_{j_2=0}^{j_1-1}\ldots\sum_{j_k=0}^{j_{k-1}-1} a_{j_1\ldots j_k} = \sum_{j_k=0}^{N-1}\sum_{j_{k-1}=j_k+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1} a_{j_1\ldots j_k}, \tag{3.17}$$

where $a_{j_1\ldots j_k}$ are scalars, then

$$G[\psi^{(k)}]_{N,l+1} = \sum_{j_k=l+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1}\prod_{l=1}^{k} I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}. \tag{3.18}$$

Let us substitute (3.18) into

$$\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)} G[\psi^{(k)}]_{N,l+1}$$

and use again the formula (3.17). Then

$$\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)} G[\psi^{(k)}]_{N,l+1} = S[\phi,\psi^{(k)}]_N. \tag{3.19}$$

Suppose that the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ S[\phi,\psi^{(k)}]_N \tag{3.20}$$

exists (its existence will be proved further).

Then from (3.19) and (3.16) it follows that for proof of the right equality in (3.11) we have to demonstrate that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)} H[\psi^{(k)}]_{N,l+1} = 0. \tag{3.21}$$

Analyzing the second moment of the prelimit expression on the left-hand side of (3.21) and taking into account (3.15), the independence of ϕ_{τ_l}, $\Delta w_{\tau_l}^{(k+1)}$, and $H[\psi^{(k)}]_{N,l+1}$ as well as the standard estimates for second moments of stochastic integrals and the Minkowski inequality, we find that (3.21) is correct. Thus, by the assumption of existence of the limit (3.20) we obtain that the right equality in (3.11) is fulfilled.

Let us demonstrate that the left equality in (3.11) is also fulfilled.

We have

$$J[\phi,\psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l)\Delta w_{\tau_l}^{(1)} J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}. \tag{3.22}$$

Let us use for the integral $J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}$ in (3.22) the same arguments, which resulted to the relation (3.14) for the integral $I[\psi^{(k)}]_{T,\tau_{l+1}}$. After that let us substitute the expression obtained for the integral $J[\phi,\psi_2^{(k-1)}]_{\tau_l,t}$ into (3.22). Further, using the Minkowski inequality and standard estimates for second moments of stochastic integrals it is easy to obtain that

$$J[\phi,\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}}\ R[\phi,\psi^{(k)}]_N \quad \text{w. p. } 1, \tag{3.23}$$

where

$$R[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1} \psi_1(\tau_{j_1})\Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} \int\limits_{\tau_l}^{\tau_{l+1}} \phi_\tau dw_\tau^{(k+1)}.$$

We will demonstrate that

$$\underset{N\to\infty}{\text{l.i.m.}}\ R[\phi,\psi^{(k)}]_N = \underset{N\to\infty}{\text{l.i.m.}}\ S[\phi,\psi^{(k)}]_N \quad \text{w. p. } 1. \tag{3.24}$$

It is easy to see that

$$R[\phi,\psi^{(k)}]_N = U[\phi,\psi^{(k)}]_N + V[\phi,\psi^{(k)}]_N + S[\phi,\psi^{(k)}]_N \quad \text{w. p. } 1, \tag{3.25}$$

where

$$U[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1} \psi_1(\tau_{j_1})\Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} I[\Delta\phi]_{\tau_{l+1},\tau_l},$$

$$V[\phi,\psi^{(k)}]_N = \sum_{j_1=0}^{N-1} I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} \phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)},$$

$$I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}} = \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} (\psi_1(\tau_{j_1}) - \psi_1(\tau))dw_\tau^{(1)},$$

$$
I[\Delta\phi]_{\tau_{l+1},\tau_l} = \int\limits_{\tau_l}^{\tau_{l+1}} (\phi_\tau - \phi_{\tau_l}) dw_\tau^{(k+1)}.
$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals, the condition that the process ϕ_τ belongs to the class $\mathrm{S}_2([t,T])$ as well as continuity (which means uniform continuity) of the function $\psi_1(\tau)$, we obtain that

$$
\underset{N\to\infty}{\text{l.i.m.}}\ V[\phi,\psi^{(k)}]_N = \underset{N\to\infty}{\text{l.i.m.}}\ U[\phi,\psi^{(k)}]_N = 0 \quad \text{w. p. 1.}
$$

Then, considering (3.25), we obtain (3.24). From (3.24) and (3.23) it follows that the left equality in (3.11) is fulfilled.

Note that the limit (3.20) exists because it is equal to the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$, which exists in the conditions of Theorem 3.1. So, the chain of equalities (3.11) is proved. Theorem 3.1 is proved.

3.5 Corollaries and Generalizations of Theorem 3.1

Assume that $D_k = \{(t_1,\ldots,t_k) : t \le t_1 < \ldots < t_k \le T\}$ and the following conditions are fulfilled:

AI. $\xi_\tau \in \mathrm{S}_2([t,T])$.

AII. $\Phi(t_1,\ldots,t_{k-1})$ is a continuous nonrandom function on the closed set D_{k-1} (we use the same symbol D_{k-1} for closed and not closed set D_{k-1}).

Let us define the following stochastic integrals

$$
\hat{J}[\xi,\Phi]_{T,t}^{(k)} = \int\limits_t^T \xi_{t_k} d\mathbf{w}_{t_k}^{(i_k)} \ldots \int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) d\mathbf{w}_{t_1}^{(i_1)} \stackrel{\text{def}}{=}
$$

$$
\stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta\mathbf{w}_{\tau_l}^{(i_k)} \int\limits_{\tau_{l+1}}^T d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} \ldots \int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) d\mathbf{w}_{t_1}^{(i_1)}
$$

for $k \ge 3$ and

$$
\hat{J}[\xi,\Phi]_{T,t}^{(2)} = \int\limits_t^T \xi_{t_2} d\mathbf{w}_{t_2}^{(i_2)} \int\limits_{t_2}^T \Phi(t_1) d\mathbf{w}_{t_1}^{(i_1)} \stackrel{\text{def}}{=}
$$

$$
\stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta{\bf w}_{\tau_l}^{(i_2)} \int\limits_{\tau_{l+1}}^{T} \Phi(t_1) d{\bf w}_{t_1}^{(i_1)}
$$

for $k=2$. Here ${\bf w}_{\tau}^{(i)}={\bf f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and ${\bf w}_{\tau}^{(0)}=\tau,$ ${\bf f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are ${\rm F}_{\tau}$-measurable for all $\tau\in[0,T]$ independent standard Wiener processes, $0\le t<T,$ $i_1,\ldots,i_k=0,1,\ldots,m.$

Let us denote

$$
J[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-1}}\Phi(t_1,\ldots,t_{k-1})\xi_{t_k}d{\bf w}_{t_k}^{(i_k)}\ldots d{\bf w}_{t_1}^{(i_1)},\quad k\ge 2, \tag{3.26}
$$

where the right-hand side of (3.26) is the iterated Itô stochastic integral.

Let us introduce the following iterated stochastic integrals

$$
\tilde J[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T d{\bf w}_{t_{k-1}}^{(i_{k-1})}\ldots \int\limits_{t_3}^T d{\bf w}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})d{\bf w}_{t_1}^{(i_1)}\stackrel{\rm def}{=}
$$

$$
\stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.} \sum_{l=0}^{N-1} \Delta{\bf w}_{\tau_l}^{(i_{k-1})} \int\limits_{\tau_{l+1}}^{T} d{\bf w}_{t_{k-2}}^{(i_{k-2})}\ldots \int\limits_{t_3}^T d{\bf w}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})d{\bf w}_{t_1}^{(i_1)},
$$

$$
J'[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-2}}\Phi(t_1,\ldots,t_{k-1})d{\bf w}_{t_{k-1}}^{(i_{k-1})}\ldots d{\bf w}_{t_1}^{(i_1)},\quad k\ge 2.
$$

Similarly to the proof of Theorem 3.1 it is easy to demonstrate that under the condition AII the stochastic integral $\tilde J[\Phi]_{T,t}^{(k-1)}$ exists and

$$
J'[\Phi]_{T,t}^{(k-1)}=\tilde J[\Phi]_{T,t}^{(k-1)}\quad \hbox{w. p. 1}. \tag{3.27}
$$

Moreover, using (3.27) the following generalization of Theorem 3.1 can be proved similarly to the proof of Theorem 3.1.

Theorem 3.2 [91] (1997), [60] (1998) (also see [1]-[14], [60], [92]). *Suppose that the conditions* AI, AII *of this section are fulfilled. Then, the stochastic integral* $\hat J[\xi,\Phi]_{T,t}^{(k)}$ *exists and for* $k\ge 2$

$$J[\xi,\Phi]_{T,t}^{(k)} = \hat{J}[\xi,\Phi]_{T,t}^{(k)} \quad w.\ p.\ 1.$$

Let us consider the following stochastic integrals

$$I = \int\limits_t^T d\mathbf{f}_{t_2}^{(i_2)} \int\limits_{t_2}^T \Phi_1(t_1,t_2) d\mathbf{f}_{t_1}^{(i_1)}, \quad J = \int\limits_t^T \int\limits_t^{t_2} \Phi_2(t_1,t_2) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)}.$$

If we consider

$$\int\limits_{t_2}^T \Phi_1(t_1,t_2) d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of I and

$$\int\limits_t^{t_2} \Phi_2(t_1,t_2) d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of J, then, due to independence of these integrands we may mistakenly think that $\mathsf{M}\{IJ\} = 0$. But it is not the fact. Actually, using the integration order replacement technique in the stochastic integral I, we have w. p. 1

$$I = \int\limits_t^T \int\limits_t^{t_1} \Phi_1(t_1,t_2) d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_1}^{(i_1)} = \int\limits_t^T \int\limits_t^{t_2} \Phi_1(t_2,t_1) d\mathbf{f}_{t_1}^{(i_2)} d\mathbf{f}_{t_2}^{(i_1)}.$$

So, using the standard properties of the Itô stochastic integral [83], we get

$$\mathsf{M}\{IJ\} = \mathbf{1}_{\{i_1=i_2\}} \int\limits_t^T \int\limits_t^{t_2} \Phi_1(t_2,t_1)\Phi_2(t_1,t_2) dt_1 dt_2,$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the following statement.

Theorem 3.3 [91] (1997), [60] (1998) (also see [1]-[14], [60], [92]). *Let the conditions of Theorem* 3.1 *are fulfilled and* $h(\tau)$ *is a continuous nonrandom function at the interval* $[t,T]$. *Then*

$$\int\limits_t^T \phi_\tau dw_\tau^{(k+1)} h(\tau) \hat{I}[\psi^{(k)}]_{T,\tau} = \int\limits_t^T \phi_\tau h(\tau) dw_\tau^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau} \quad \text{w. p. } 1, \qquad (3.28)$$

where stochastic integrals on the left-hand side of (3.28) *as well as on the right-hand side of* (3.28) *exist.*

Proof. According to Theorem 3.1, the iterated stochastic integral on the right-hand side of (3.28) exists. In addition

$$\int\limits_t^T \phi_\tau h(\tau) dw_\tau^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau} = \int\limits_t^T \phi_\tau dw_\tau^{(k+1)} h(\tau) \hat{I}[\psi^{(k)}]_{T,\tau} -$$

$$-\underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta h(\tau_l) \Delta w_{\tau_l}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}} \quad \text{w. p. 1},$$

where $\Delta h(\tau_l) = h(\tau_{l+1}) - h(\tau_l)$.

Using the arguments which resulted to the right equality in (3.11), we obtain

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta h(\tau_l) \Delta w_{\tau_l}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}}\ G[\psi^{(k)}]_{N,0} \sum_{l=0}^{j_k-1} \phi_{\tau_l} \Delta h(\tau_l) \Delta w_{\tau_l}^{(k+1)} \quad \text{w. p. 1}. \tag{3.29}$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals as well as continuity of the function $h(\tau)$, we obtain that the second moment of the prelimit expression on the right-hand side of (3.29) tends to zero when $N \to \infty$. Theorem 3.3 is proved.

Let us consider one corollary of Theorem 3.1.

Theorem 3.4 [91] (1997), [60] (1998) (also see [1]-[14], [60], [92]). *In the conditions of Theorem* 3.3 *the following equality*

$$\int\limits_t^T h(t_1) \int\limits_t^{t_1} \phi_\tau dw_\tau^{(k+2)} dw_{t_1}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,t_1} =$$

$$= \int\limits_t^T \phi_\tau dw_\tau^{(k+2)} \int\limits_\tau^T h(t_1) dw_{t_1}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,t_1} \quad w.\ p.\ 1 \tag{3.30}$$

is fulfilled. Moreover, the stochastic integrals in (3.30) *exist.*

Proof. Using Theorem 3.1 two times, we obtain

$$\int\limits_t^T \phi_\tau dw_\tau^{(k+2)} \int\limits_\tau^T h(t_1) dw_{t_1}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,t_1} =$$

$$= \int\limits_t^T \psi_1(t_1) \ldots \int\limits_t^{t_{k-1}} \psi_k(t_k) \int\limits_t^{t_k} \rho_\tau dw_\tau^{(k+1)} dw_{t_k}^{(k)} \ldots dw_{t_1}^{(1)} =$$

$$= \int\limits_t^T \rho_\tau dw_\tau^{(k+1)} \int\limits_\tau^T \psi_k(t_k) dw_{t_k}^{(k)} \ldots \int\limits_{t_2}^T \psi_1(t_1) dw_{t_1}^{(1)} \quad \text{w. p. 1},$$

where

$$\rho_\tau \stackrel{\text{def}}{=} h(\tau) \int\limits_t^\tau \phi_s dw_s^{(k+2)}.$$

Theorem 3.4 is proved.

3.6 Examples of Integration Order Replacement Technique for the Concrete Iterated Itô Stochastic Integrals

As we mentioned above, the formulas from this section could be obtained using the Itô formula. However, the method based on Theorem 3.1 is more simple and familiar, since it deals with usual rules of the integration order replacement in Riemann integrals.

Using the integration order replacement technique for iterated Itô stochastic integrals (Theorem 3.1), we obtain the following equalities which are fulfilled w. p. 1

$$\int\limits_t^T \int\limits_t^{t_2} df_{t_1} dt_2 = \int\limits_t^T (T - t_1) df_{t_1},$$

$$\int\limits_t^T \cos(t_2 - T) \int\limits_t^{t_2} df_{t_1} dt_2 = \int\limits_t^T \sin(T - t_1) df_{t_1},$$

$$\int\limits_t^T \sin(t_2-T)\int\limits_t^{t_2} df_{t_1}dt_2=\int\limits_t^T\left(\cos(T-t_1)-1\right)df_{t_1},$$

$$\int\limits_t^T e^{\alpha(t_2-T)}\int\limits_t^{t_2} df_{t_1}dt_2=\frac{1}{\alpha}\int\limits_t^T\left(1-e^{\alpha(t_1-T)}\right)df_{t_1},\quad \alpha\ne 0,$$

$$\int\limits_t^T (t_2-T)^{\alpha}\int\limits_t^{t_2} df_{t_1}dt_2=-\frac{1}{\alpha+1}\int\limits_t^T(t_1-T)^{\alpha+1}df_{t_1},\quad \alpha\ne -1,$$

$$J_{(100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)^2df_{t_1},\quad J_{(010)T,t}=\int\limits_t^T(t_1-t)(T-t_1)df_{t_1},$$

$$J_{(110)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}, \tag{3.31}$$

$$J_{(101)T,t}=\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2},\quad J_{(1011)T,t}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1101)T,t}=\int\limits_t^T\int\limits_t^{t_3}(t_3-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1110)T,t}=\int\limits_t^T(T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},\quad J_{(1100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_2)^2\int\limits_t^{t_2}df_{t_1}df_{t_2},$$

$$J_{(1010)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}, \tag{3.32}$$

$$J_{(1001)T,t}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)^2df_{t_1}df_{t_2},\quad J_{(0110)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0101)T,t}=\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0010)T,t} = \frac{1}{2}\int\limits_t^T (T-t_1)(t_1-t)^2 df_{t_1}, \quad J_{(0100)T,t} = \frac{1}{2}\int\limits_t^T (T-t_1)^2(t_1-t) df_{t_1},$$

$$J_{(1000)T,t} = \frac{1}{3!}\int\limits_t^T (T-t_1)^3 df_{t_1},$$

$$J_{(1\underbrace{0\ldots0}_{k-1})T,t} = \frac{1}{(k-1)!}\int\limits_t^T (T-t_1)^{k-1} df_{t_1},$$

$$J_{(11\underbrace{0\ldots0}_{k-2})T,t} = \frac{1}{(k-2)!}\int\limits_t^T (T-t_2)^{k-2}\int\limits_t^{t_2} df_{t_1}df_{t_2},$$

$$J_{(\underbrace{1\ldots1}_{k-1}0)T,t} = \int\limits_t^T (T-t_1) J_{(\underbrace{1\ldots1}_{k-2})t_1,t} df_{t_1},$$

$$J_{(1\underbrace{0\ldots0}_{k-2}1)T,t} = \frac{1}{(k-2)!}\int\limits_t^T\int\limits_t^{t_2} (t_2-t_1)^{k-2} df_{t_1}df_{t_2},$$

$$J_{(10\underbrace{1\ldots1}_{k-2})T,t} = \int\limits_t^T \ldots \int\limits_t^{t_3}\int\limits_t^{t_2} (t_2-t_1) df_{t_1}df_{t_2}\ldots df_{t_{k-1}},$$

$$J_{(\underbrace{1\ldots1}_{k-2}01)T,t} = \int\limits_t^T\int\limits_t^{t_{k-1}} (t_{k-1}-t_{k-2})\int\limits_t^{t_{k-2}}\ldots\int\limits_t^{t_2} df_{t_1}\ldots df_{t_{k-3}}df_{t_{k-2}}df_{t_{k-1}},$$

$$J_{(10)T,t} + J_{(01)T,t} = (T-t)J_{(1)T,t},$$

$$J_{(110)T,t} + J_{(101)T,t} + J_{(011)T,t} = (T-t)J_{(11)T,t},$$

$$J_{(001)T,t} + J_{(010)T,t} + J_{(100)T,t} = \frac{(T-t)^2}{2}J_{(1)T,t},$$

$$J_{(1100)T,t} + J_{(1010)T,t} + J_{(1001)T,t} + J_{(0110)T,t} +$$

$$+J_{(0101)T,t} + J_{(0011)T,t} = \frac{(T-t)^2}{2}J_{(11)T,t},$$

$$
J_{(1000)T,t}+J_{(0100)T,t}+J_{(0010)T,t}+J_{(0001)T,t}=\frac{(T-t)^3}{3!}J_{(1)T,t},
$$

$$
J_{(1110)T,t}+J_{(1101)T,t}+J_{(1011)T,t}+J_{(0111)T,t}=(T-t)J_{(111)T,t},
$$

$$
\sum_{l=1}^{k} J_{(\underbrace{0\ldots0}_{l-1}\,1\,\underbrace{0\ldots0}_{k-l})T,t}=\frac{1}{(k-1)!}(T-t)^{k-1}J_{(1)T,t},
$$

$$
\sum_{l=1}^{k} J_{(\underbrace{1\ldots1}_{l-1}\,0\,\underbrace{1\ldots1}_{k-l})T,t}=(T-t)J_{(\underbrace{1\ldots1}_{k-1})T,t},
$$

$$
\sum_{\substack{l_1+\ldots+l_k=m\\ l_i\in\{0,\ 1\},\ i=1,\ldots,k}} J_{(l_1\ldots l_k)T,t}=\frac{(T-t)^{k-m}}{(k-m)!}J_{(\underbrace{1\ldots1}_{m})T,t},
$$

where

$$
J_{(l_1\ldots l_k)T,t}=\int\limits_t^T\ldots\int\limits_t^{t_2} dw_{t_1}^{(1)}\ldots dw_{t_k}^{(k)},
$$

$l_i=1$ when $w_{t_i}^{(i)}=f_{t_i}$ and $l_i=0$ when $w_{t_i}^{(i)}=t_i$ $(i=1,\ldots,k)$, f_τ is a standard Wiener process.

Let us consider two examples and show explicitly the technique on integration order replacement for iterated Itô stochastic integrals.

Example 3.1. *Let us prove the equality* (3.31). *Using Theorems* 3.1 *and* 3.3, *we obtain*:

$$
J_{(110)T,t}\stackrel{\rm def}{=}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}df_{t_2}dt_3=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}\int\limits_{t_2}^T dt_3=
$$

$$
=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}(T-t_2)=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T (T-t_2)df_{t_2}=
$$

$$
=\int\limits_t^T (T-t_2)\int\limits_t^{t_2} df_{t_1}df_{t_2}\quad \text{w. p. 1.}
$$

Example 3.2. *Let us prove the equality* (3.32). *Using Theorems* 3.1 *and* 3.3, *we obtain*

$$
J_{(1010)T,t} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} df_{t_1} dt_2 df_{t_3} dt_4 = \int\limits_t^T df_{t_1} \int\limits_{t_1}^T dt_2 \int\limits_{t_2}^T df_{t_3} \int\limits_{t_3}^T dt_4 =
$$

$$
= \int\limits_t^T df_{t_1} \int\limits_{t_1}^T dt_2 \int\limits_{t_2}^T df_{t_3} (T-t_3) = \int\limits_t^T df_{t_1} \int\limits_{t_1}^T dt_2 \int\limits_{t_2}^T (T-t_3) df_{t_3} =
$$

$$
= \int\limits_t^T (T-t_3) \int\limits_t^{t_3} \int\limits_t^{t_2} df_{t_1} dt_2 df_{t_3} = \int\limits_t^T (T-t_3) \left(\int\limits_t^{t_3} \int\limits_t^{t_2} df_{t_1} dt_2 \right) df_{t_3} =
$$

$$
= \int\limits_t^T (T-t_3) \left(\int\limits_t^{t_3} df_{t_1} \int\limits_{t_1}^{t_3} dt_2 \right) df_{t_3} =
$$

$$
= \int\limits_t^T (T-t_3) \left(\int\limits_t^{t_3} df_{t_1} (t_3-t_1) \right) df_{t_3} =
$$

$$
= \int\limits_t^T (T-t_3) \left(\int\limits_t^{t_3} (t_3-t_1) df_{t_1} \right) df_{t_3} =
$$

$$
= \int\limits_t^T (T-t_2) \int\limits_t^{t_2} (t_2-t_1) df_{t_1} df_{t_2} \quad \hbox{w. p. 1.}
$$

3.7 Integration Order Replacement Technique for Iterated Stochastic Integrals with Respect to Martingale

In this section, we will generalize the theorems on integration order replacement for iterated Itô stochastic integrals to the class of iterated stochastic integrals with respect to martingale.

Let $(\Omega, {\rm F}, {\sf P})$ be a complete probability space and let $\{{\rm F}_t, t \in [0,T]\}$ be a nondecreasing family of σ-algebras defined on the probability space $(\Omega, {\rm F}, {\sf P})$. Suppose that M_t, $t \in [0,T]$ is an ${\rm F}_t$-measurable martingale for all $t \in [0,T]$, which satisfies the condition ${\sf M}\left\{|M_t|\right\} < \infty$. Moreover, for all $t \in [0,T]$ there exists an F_t- measurable and nonnegative w. p. 1 stochastic process ρ_t, $t\in[0,T]$ such that

$$\mathsf{M}\left\{(M_s-M_t)^2\mid \mathrm{F}_t\right\}=\mathsf{M}\left\{\int\limits_t^s \rho_\tau d\tau\Bigg|\mathrm{F}_t\right\}\quad \text{w. p. 1},$$

where $0\le t<s\le T$.

Let us consider the class $H_2(\rho,[0,T])$ of stochastic processes φ_t, $t\in[0,T]$, which are F_t-measurable for all $t\in[0,T]$ and satisfy the condition

$$\mathsf{M}\left\{\int\limits_0^T \varphi_t^2\rho_t dt\right\}<\infty.$$

For any partition $\tau_j^{(N)}$, $j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty \tag{3.33}$$

we will define the sequence of step functions

$$\varphi^{(N)}(t,\omega)=\varphi\left(\tau_j^{(N)},\omega\right)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),$$

where $j=0,1,\ldots,N-1$, $N=1,2,\ldots$

Let us define the stochastic integral with respect to martingale for $\varphi(t,\omega)\in H_2(\rho,[0,T])$ as the following mean-square limit [83]

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\varphi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(M\left(\tau_{j+1}^{(N)},\omega\right)-M\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\varphi_\tau dM_\tau,$$

where $\varphi^{(N)}(t,\omega)$ is any step function from the class $H_2(\rho,[0,T])$, which converges to the function $\varphi(t,\omega)$ in the following sense

$$\lim_{N\to\infty}\int\limits_0^T\mathsf{M}\left\{\left|\varphi^{(N)}(t,\omega)-\varphi(t,\omega)\right|^2\right\}\rho_t dt=0.$$

It is well known [83] that the stochastic integral

$$\int\limits_0^T\varphi_\tau dM_\tau$$

exists and it does not depend on the selection of sequence $\varphi^{(N)}(t,\omega)$.

Let $\tilde{H}_2(\rho,[0,T])$ be the class of stochastic processes φ_τ, $\tau\in[0,T]$, which are mean-square continuous for all $\tau\in[0,T]$ and belong to the class $H_2(\rho,[0,T])$.

Let us consider the following iterated stochastic integrals

$$S[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\phi_\tau dM_\tau^{(k+1)}dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}, \qquad (3.34)$$

$$S[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}. \qquad (3.35)$$

Here $\phi_\tau\in\tilde{H}_2(\rho,[t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous nonrandom functions at the interval $[t,T]$, $M_\tau^{(l)}=M_\tau$ or $M_\tau^{(l)}=\tau$ if $\tau\in[t,T]$, $l=1,\ldots,k+1$, M_τ is the martingale defined above.

Let us define the iterated stochastic integral $\hat{S}[\psi^{(k)}]_{T,s}$, $0\le t\le s\le T$, $k\ge 1$ with respect to martingale

$$\hat{S}[\psi^{(k)}]_{T,s}=\int\limits_s^T\psi_k(t_k)dM_{t_k}^{(k)}\ldots\int\limits_{t_2}^T\psi_1(t_1)dM_{t_1}^{(1)}$$

by the following recurrence relation

$$\hat{S}[\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\psi_k(\tau_l)\Delta M_{\tau_l}^{(k)}\hat{S}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \qquad (3.36)$$

where $k\ge 1$, $\hat{S}[\psi^{(0)}]_{T,s}\stackrel{\rm def}{=}1$, $[s,T]\subseteq[t,T]$, here and further $\Delta M_{\tau_l}^{(i)}=M_{\tau_{l+1}}^{(i)}-M_{\tau_l}^{(i)}$, $i=1,\ldots,k+1$, $l=0,1,\ldots,N-1$, $\{\tau_l\}_{l=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition similar to (3.33), other notations are the same as in (3.34), (3.35).

Further, let us define the iterated stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$ of the form

$$\hat{S}[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\phi_s dM_s^{(k+1)}\hat{S}[\psi^{(k)}]_{T,s}$$

by the equality

$$\hat{S}[\phi,\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta M_{\tau_l}^{(k+1)}\hat{S}[\psi^{(k)}]_{T,\tau_{l+1}},$$

where the sense of notations included in (3.34)–(3.36) is saved.

Let us formulate the theorem on integration order replacement for the iterated stochastic integrals with respect to martingale, which is the generalization of Theorem 3.1.

Theorem 3.5 [94] (1999) (also see [1]-[14], [92]). *Let* $\phi_\tau \in \tilde{H}_2(\rho,[t,T])$, *every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function at the interval* $[t,T]$, *and* $|\rho_\tau| \le K < \infty$ *w. p.* 1 *for all* $\tau \in [t,T]$. *Then, the stochastic integral* $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ *exists and*

$$S[\phi,\psi^{(k)}]_{T,t} = \hat{S}[\phi,\psi^{(k)}]_{T,t} \quad w.\ p.\ 1.$$

The proof of Theorem 3.5 is similar to the proof of Theorem 3.1.

Remark 3.2. *Let us note that we can propose another variant of the conditions in Theorem* 3.5. *For example, if we not require the boundedness of the process* ρ_τ, *then it is necessary to require the execution of the following additional conditions*:

1. $\mathsf{M}\{|\rho_\tau|\} < \infty$ *for all* $\tau \in [t,T]$.
2. *The process* ρ_τ *is independent with the processes* ϕ_τ *and* M_τ.

Remark 3.3. *Note that it is well known the construction of stochastic integral with respect to the Wiener process with integrable process, which is not an* F_τ*-measurable stochastic process — the so-called Stratonovich stochastic integral* [86].

The stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ *is also the stochastic integral with integrable process, which is not an* F_τ*-measurable stochastic process. However, in the conditions of Theorem* 3.5

$$S[\phi,\psi^{(k)}]_{T,t} = \hat{S}[\phi,\psi^{(k)}]_{T,t} \quad w.\ p.\ 1,$$

where $S[\phi,\psi^{(k)}]_{T,t}$ *is a usual iterated stochastic integral with respect to martingale. If, for example,* $M_\tau, \tau \in [t,T]$ *is the Wiener process, then the question on connection between stochastic integral* $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ *and Stratonovich stochastic integral is solving as a standard question on connection between Stratonovich and Itô stochastic integrals* [86].

Let us consider several statements, which are the generalizations of theorems formulated in the previous sections.

Assume that $D_k = \{(t_1,\ldots,t_k):\ t \le t_1 < \ldots < t_k \le T\}$ and the following conditions are fulfilled:

BI. $\xi_\tau \in \tilde{H}_2(\rho, [t, T])$.

BII. $\Phi(t_1, \ldots, t_{k-1})$ is a continuous nonrandom function on the closed set D_{k-1} (recall that we use the same symbol D_{k-1} for closed and not closed set D_{k-1}).

Let us define the following stochastic integrals with respect to martingale

$$\hat{S}[\xi,\Phi]_{T,t}^{(k)} = \int\limits_t^T \xi_{t_k} dM_{t_k}^{(k)} \ldots \int\limits_{t_3}^T dM_{t_2}^{(2)} \int\limits_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) dM_{t_1}^{(1)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \mathop{\rm l.i.m.}_{N\to\infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta M_{\tau_l}^{(k)} \int\limits_{\tau_{l+1}}^T dM_{t_{k-1}}^{(k-1)} \ldots \int\limits_{t_3}^T dM_{t_2}^{(2)} \int\limits_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) dM_{t_1}^{(1)}$$

for $k \ge 3$ and

$$\hat{S}[\xi,\Phi]_{T,t}^{(2)} = \int\limits_t^T \xi_{t_2} dM_{t_2}^{(2)} \int\limits_{t_2}^T \Phi(t_1) dM_{t_1}^{(1)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \mathop{\rm l.i.m.}_{N\to\infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta M_{\tau_l}^{(2)} \int\limits_{\tau_{l+1}}^T \Phi(t_1) dM_{t_1}^{(1)}$$

for $k = 2$, where the sense of notations included in (3.34)–(3.36) is saved. Moreover, the stochastic process ξ_τ, $\tau \in [t, T]$ belongs to the class $\tilde{H}_2(\rho, [t, T])$.

In addition, let

$$S[\xi,\Phi]_{T,t}^{(k)} = \int\limits_t^T \ldots \int\limits_t^{t_{k-1}} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} dM_{t_k}^{(k)} \ldots dM_{t_1}^{(1)}, \quad k \ge 2, \tag{3.37}$$

where the right-hand side of (3.37) is the iterated stochastic integral with respect to martingale.

Let us introduce the following iterated stochastic integrals with respect to martingale

$$\tilde{S}[\Phi]_{T,t}^{(k-1)} = \int\limits_t^T dM_{t_{k-1}}^{(k-1)} \ldots \int\limits_{t_3}^T dM_{t_2}^{(2)} \int\limits_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) dM_{t_1}^{(1)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \Delta M_{\tau_l}^{(k-1)} \int\limits_{\tau_{l+1}}^{T} dM_{t_{k-2}}^{(k-2)} \ldots \int\limits_{t_3}^{T} dM_{t_2}^{(2)} \int\limits_{t_2}^{T} \Phi(t_1, t_2, \ldots, t_{k-1}) dM_{t_1}^{(1)},$$

$$S'[\Phi]_{T,t}^{(k-1)} = \int\limits_{t}^{T} \ldots \int\limits_{t}^{t_{k-2}} \Phi(t_1, \ldots, t_{k-1}) dM_{t_{k-1}}^{(k-1)} \ldots dM_{t_1}^{(1)}, \quad k \geq 2.$$

It is easy to demonstrate similarly to the proof of Theorem 3.5 that under the condition BII the stochastic integral $\tilde{S}[\Phi]_{T,t}^{(k-1)}$ exists and

$$S'[\Phi]_{T,t}^{(k-1)} = \tilde{S}[\Phi]_{T,t}^{(k-1)} \quad \text{w. p. 1.}$$

In its turn, using this fact we can prove the following theorem similarly to the proof of Theorem 3.5.

Theorem 3.6 [94] (1999) (also see [1]-[14], [92]). *Let the conditions* BI, BII *of this section are fulfilled and* $|\rho_\tau| \leq K < \infty$ *w. p.* 1 *for all* $\tau \in [t, T]$. *Then, the stochastic integral* $\hat{S}[\xi, \Phi]_{T,t}^{(k)}$ *exists and for* $k \geq 2$

$$S[\xi, \Phi]_{T,t}^{(k)} = \hat{S}[\xi, \Phi]_{T,t}^{(k)} \quad w.\ p.\ 1.$$

Theorem 3.6 is the generalization of Theorem 3.2 for the case of iterated stochastic integrals with respect to martingale.

Let us consider two statements.

Theorem 3.7 [94] (1999) (also see [1]-[14], [92]). *Let the conditions of Theorem* 3.5 *are fulfilled and* $h(\tau)$ *is a continuous nonrandom function at the interval* $[t, T]$. *Then*

$$\int\limits_{t}^{T} \phi_\tau dM_\tau^{(k+1)} h(\tau) \hat{S}[\psi^{(k)}]_{T,\tau} = \int\limits_{t}^{T} \phi_\tau h(\tau) dM_\tau^{(k+1)} \hat{S}[\psi^{(k)}]_{T,\tau} \quad w.\ p.\ 1, \qquad (3.38)$$

where the stochastic integrals in (3.38) *exist.*

Theorem 3.8 [94] (1999) (also see [1]-[14], [92]). *In the conditions of Theorem* 3.5

$$\int\limits_{t}^{T} h(t_1) \int\limits_{t}^{t_1} \phi_\tau dM_\tau^{(k+2)} dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} =$$

$$
= \int\limits_{t}^{T} \phi_\tau dM_\tau^{(k+2)} \int\limits_{\tau}^{T} h(t_1) dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} \ \text{w. p. 1}, \qquad (3.39)
$$

where the stochastic integrals in (3.39) *exist.*

The proofs of Theorems 3.7 and 3.8 are similar to the proofs of Theorems 3.3 and 3.4 correspondingly.

Remark 3.4. *The integration order replacement technique for iterated Itô stochastic integrals* (*Theorems* 3.1–3.4) *has been successfully applied for construction of the so-called unified Taylor–Itô and Taylor–Stratonovich expansions* (*see Chapter* 4) *as well as for proof and development of the mean-square approximation method for iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series* (*see Chapters* 1 *and* 2).

Chapter 4

Four New Forms of the Taylor–Itô and Taylor–Stratonovich Expansions and its Application to the High-Order Strong Numerical Methods for Itô Stochastic Differential Equations

The problem of the Taylor–Itô and Taylor–Stratonovich expansions of the Itô stochastic processes in a neighborhood of a fixed time moment is considered in this chapter. The classical forms of the Taylor–Itô and Taylor–Stratonovich expansions are transformed to four new representations, which include the minimal sets of different types of iterated Itô and Stratonovich stochastic integrals. Therefore, these representations (the so-called unified Taylor–Itô and Taylor–Stratonovich expansions) are more convenient for constructing of the high-order strong numerical methods for Itô SDEs. Explicit one-step strong numerical schemes with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0 based on the unified Taylor–Itô and Taylor–Stratonovich expansions are derived.

4.1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a non-decreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Consider an Itô SDE in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{4.1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the Itô SDE (4.1). The nonrandom functions $\mathbf{a} : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n$, $B : \mathbf{R}^n \times [0, T] \to \mathbf{R}^{n\times m}$ guarantee the existence and uniqueness (up to stochastic equivalence) of a solution to the equation (4.1) [83]. The second integral on the right-hand side of (4.1) is interpreted as an Itô stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$. Also we assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [67], [68], [76], [95], [96] (also see [13]) that Itô SDEs are adequate mathematical models of dynamic systems of different physical nature that are affected by random perturbations. For example, Itô SDEs are used as mathematical models in stochastic mathematical finance, hydrology, seismology, geophysics, chemical kinetics, population dynamics, electrodynamics, medicine and other fields [67], [68], [76], [95], [96] (also see [13]).

Numerical integration of Itô SDEs based on the strong convergence criterion of approximations [67] is widely used for the numerical simulation of sample trajectories of solutions to Itô SDEs (which is required for constructing new mathematical models on the basis of such equations and for the numerical solution of different mathematical problems connected with Itô SDEs). Among these problems, we note the following: filtering of signals under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis [65], [67], [68], [75], [76], [95]–[116].

Exact solutions of Itô SDEs are known in rather rare cases. For this reason it is necessary to construct numerical procedures for solving these equations.

In this chapter, a promising approach [65], [67], [68], [75], [76] to the numerical integration of Itô SDEs based on the stochastic analogues of the Taylor formula (Taylor–Itô and Taylor–Stratonovich expansions) [119], [123] (also see [50], [60], [120], [121]) is used. This approach uses a finite discretization of the time variable and the numerical simulation of the solution to the Itô SDE at discrete time moments using the stochastic analogues of the Taylor formula mentioned above. A number of works (e.g., [65]-[68], [75], [76]) describe numerical schemes with the strong convergence orders $1.5, 2.0$, 2.5, and 3.0 for Itô SDEs; however, they do not contain efficient procedures of the mean-square approximation of the iterated stochastic integrals for the case of multidimensional nonadditive noise.

In this chapter, we consider the unified Taylor–Itô and Taylor–Stratonovich expansions [120], [121] (also see [50], [60]) which makes it possible (in contrast with its classical analogues [67], [119]) to use the minimal sets of iterated Itô and Stratonovich stochastic integrals; this is a simplifying factor for the numerical methods implementation. We prove the unified Taylor–Itô expansion [120] with using of the slightly different approach (which is taken from [121]) in comparison with the approach from [120]. Moreover, we obtain another (second) version of the unified Taylor–Itô expansion [64], [122]. In addition we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions [121]).

It should be noted that in Chapter 5 on the base of the results of Chapters 1, 2 we study methods of numerical simulation of specific iterated Itô and Stratonovich stochastic integrals of multiplicities $1, 2, 3, 4, 5$, and 6 from the Taylor–Itô and Taylor–Stratonovich expansions. These stochastic integrals are used in the strong numerical methods for Itô SDEs [65], [67], [68], [75] (also see [13]). To approximate the iterated Itô and Stratonovich stochastic integrals appearing in the numerical schemes with the strong convergence orders $1.0, 1.5, 2.0, 2.5$, and 3.0, the method of generalized multiple Fourier series (see Chapter 1) and especially method of multiple Fourier–Legendre series will be applied in Chapter 5. It is important that the method of generalized multiple Fourier series (Theorem 1.1) does not lead to the partitioning of the integration interval of the iterated Itô and Stratonovich stochastic integrals under consideration; this interval length is the integration step of the numerical methods used to solve Itô SDEs; therefore, it is already fairly small and does not need to be partitioned. Computational experiments [1] show that the numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors. Also note that the Legendre polynomials have essential advantage over the trigonomentric functions (see Chapter 5) constructing the mean-square approximations of iterated Itô and Stratonovich stochastic integrals in the framework of the method of generalized multiple Fourier series (Theorem 1.1).

Let us consider the following iterated Itô and Stratonovich stochastic integrals:

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (4.2)$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (4.3)$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

It should be noted that one of the main problems when constructing the high-order strong numerical methods for Itô SDEs on the base of the Taylor–Itô and Taylor–Stratonovich expansions is the mean-square approximation of the iterated Itô and Stratonovich stochastic integrals (4.2) and (4.3). Obviously, in the absence of procedures for the numerical simulation of stochastic integrals, the mentioned numerical methods are unrealizable in practice. For this reason, in Chapter 5 we give the extensive practical material on expansions and mean-square approximations of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich expansions. In Chapter 5, the main focus is on approximations based on multiple Fourier–Legendre series. Such approximations is more effective in comparison with the trigonometric approximations (see Sect. 5.2) at least for the numerical methods with the strong convergence order 1.5 and higher [19], [38].

The rest of this Chapter is organized as follows. In Sect. 4.1 (below) we consider a brief review of publications on the problem of construction of the Taylor–Itô and Taylor–Stratonovich expansions for the solutions of Itô SDEs. Sect. 4.2 is devoted to some auxiliary lemmas. In Sect. 4.3 we consider the classical Taylor–Itô expansion while Sect. 4.4 and Sect. 4.5 are devoted to first and second forms of the so-called unified Taylor–Itô expansion correspondingly. The classical Taylor–Stratonovich expansion is considered in Sect. 4.6. First and second forms of the unified Taylor–Stratonovich expansion are derived in Sect. 4.7 and Sect. 4.8. In Sect. 4.9 we give a comparative analysis of the unified Taylor–Itô and Taylor–Stratonovich expansions with the classical Taylor–Itô and Taylor–Stratonovich expansions. Application of the first form of the unified Taylor–Itô expansion to the high-order strong numerical methods for Itô SDEs is considered in Sect. 4.10. In Sect. 4.11 we construct the high-order strong numerical methods for Itô SDEs on the base of the first form of the unified

Taylor–Stratonovich expansion.

Let us give a brief review of publications on the problem of construction of the Taylor–Itô and Taylor–Stratonovich expansions for the solutions of Itô SDEs. A few variants of a stochastic analog of the Taylor formula have been obtained in [119], [123] (also see [65], [67]) for the stochastic processes in the form $R(\mathbf{x}_s, s)$, $s \in [0, T]$, where $\mathbf{x}_s$ is a solution of the Itô SDE (4.1) and $R : \mathbf{R}^n \times [0, T] \to \mathbf{R}^1$ is a sufficiently smooth nonrandom function.

The first result in this direction called the Itô–Taylor expansion has been obtained in [123] (also see [119]). This result gives an expansion of the process $R(\mathbf{x}_s, s)$, $s \in [0, T]$ into a series such that every term (if $k > 0$) contains the iterated Itô stochastic integral

$$\int\limits_t^T \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4.4}$$

as a multiplier factor, where $i_1, \ldots, i_k = 0, 1, \ldots, m$. Obviously, the iterated Itô stochastic integral (4.4) is a particular case of (4.2) for $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$.

In [119] another expansion of the stochastic process $R(\mathbf{x}_s, s)$, $s \in [0, T]$ into a series has been derived. The iterated Stratonovich stochastic integrals

$$\int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4.5}$$

were used instead of the iterated Itô stochastic integrals; the corresponding expansion was called the Stratonovich–Taylor expansion. In the formula (4.5) the indices $i_1, \ldots, i_k$ take values $0, 1, \ldots, m$.

In [120] the Itô–Taylor expansion [119] is reduced to the interesting and unexpected form (the so-called unified Taylor–Itô expansion) by special transformations (see Chapter 3). Every term of this expansion (if $k > 0$) contains the iterated Itô stochastic integral

$$\int\limits_t^{s} (s - t_k)^{l_k} \ldots \int\limits_t^{t_2} (s - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{4.6}$$

where $l_1, \ldots, l_k = 0, 1, 2, \ldots$ and $i_1, \ldots, i_k = 1, \ldots, m$.

It is worth to mention another form of the unified Taylor–Itô expansion [64], [122] (also see [1]-[14]). Terms of the latter expansion contain iterated Itô stochastic integrals of the form

$$\int\limits_t^s (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (4.7)$$

where $l_1,\ldots,l_k=0,1,2,\ldots$ and $i_1,\ldots,i_k=1,\ldots,m.$

Obviously that some of the iterated Itô stochastic integrals (4.4) or (4.5) are connected by linear relations, while this is not the case for integrals defined by (4.6), (4.7). In this sense, the total quantity of stochastic integrals defined by (4.6) or (4.7) is minimal. Futhermore, in this chapter we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions) [124] (also see [121]) such that every term (if $k>0$) contains as a multiplier the iterated Stratonovich stochastic integral of one of two types

$$\int\limits_t^{*s} (t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (4.8)$$

$$\int\limits_t^{*s} (s-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(s-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (4.9)$$

where $l_1,\ldots,l_k=0,1,2,\ldots$, $i_1,\ldots,i_k=1,\ldots,m$, and $k=1,2,\ldots$

It is not difficult to see that for the sets of iterated Stratonovich stochastic integrals (4.8) and (4.9) the property of minimality (see above) also holds as for the sets of iterated Itô stochastic integrals (4.6), (4.7).

As we noted above, the main problem in implementation of high-order strong numerical methods for Itô SDEs is the mean-square approximation of the iterated stochastic integrals (4.4)–(4.9). Obviously, these stochastic integrals are particular cases of the stochastic integrals (4.2), (4.3).

Taking into account the results of Chapters 1, 2, 3, 5 and the minimality of the sets of stochastic integrals (4.6)–(4.9), we conclude that the unified Taylor–Itô and Taylor–Stratonovich expansions based on the iterated stochastic integrals (4.6)–(4.9) can be useful for constructing of high-order strong numerical methods with the convergence orders 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, ... for Itô SDEs.

4.2 Auxiliary Lemmas

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) \stackrel{\mathsf{def}}{=} f_t : [0,T] \times \Omega \rightarrow \mathbf{R}^1$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t \in [0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s < t$.

2. The Wiener process f_t is F_t-measurable for all $t \in [0,T]$.

3. The process $f_{t+\Delta} - f_t$ for all $t \geq 0$, $\Delta > 0$ is independent with the events of σ-algebra F_t.

Let us consider the class $\mathrm{M}_2([0,T])$ of random functions $\xi(t,\omega) \stackrel{\mathsf{def}}{=} \xi_t : [0,T] \times \Omega \rightarrow \mathbf{R}^1$ (see Sect. 1.1.2).

Let us recall (see Sect. 2.1.1) that the class $\mathrm{Q}_m([0,T])$ consists of Itô processes η_τ, $\tau \in [0,T]$ of the form

$$\eta_\tau = \eta_0 + \int\limits_0^\tau a_s ds + \int\limits_0^\tau b_s df_s, \tag{4.10}$$

where $(a_\tau)^m, (b_\tau)^m \in \mathrm{M}_2([0,T])$ and

$$\mathsf{M}\left\{|b_s - b_\tau|^4\right\} \leq C|s-\tau|^\gamma$$

for all $s, \tau \in [0,T]$ and for some $C, \gamma \in (0,\infty)$.

The second integral on the right-hand side of (4.10) is the Itô stochastic integral.

Consider a function $F(x,\tau) : \mathbf{R}^1 \times [0,T] \rightarrow \mathbf{R}^1$ for fixed τ from the class $C_2(-\infty,\infty)$ consisting of twice continuously differentiable in x functions on the interval $(-\infty,\infty)$ such that the first two derivatives are bounded.

Let us recall that the definition of the Stratonovich stochastic integral in the mean-square sense is given by (2.3) (Sect. 2.1.1) and the relation between Stratonovich and Itô stochastic integrals (see Sect. 2.1.1) has the following form [86] (also see [67])

$$\int\limits_t^{*T} F(\eta_\tau,\tau) df_\tau = \int\limits_t^T F(\eta_\tau,\tau) df_\tau + \frac{1}{2}\int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau,\tau) b_\tau d\tau \quad \text{w. p. 1.} \tag{4.11}$$

If the Wiener processes in (4.10) and (4.11) are independent, then

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^{T} F(\eta_\tau,\tau)df_\tau \quad \text{w. p. 1.} \tag{4.12}$$

Let us remind that a possible variant of conditions providing the correctness of the formulas (4.11) and (4.12) consists of the following conditions

$$\eta_\tau \in \mathrm{Q}_4([t,T]), \quad F(\eta_\tau,\tau) \in \mathrm{M}_2([t,T]), \quad \text{and} \quad F(x,\tau) \in C_2(-\infty,\infty).$$

Let us apply Theorem 3.1 (see Sect. 3.2) to derive one property for Itô stochastic integrals.

Recall that $\mathrm{S}_2([0,T])$ is a subset of $\mathrm{M}_2([0,T])$ and $\mathrm{S}_2([0,T])$ consists of the mean-square continuous random functions (see Sect. 3.1).

Lemma 4.1 [14], [50]. *Let $h(\tau), g(\tau), G(\tau) : [t,s] \to \mathbf{R}^1$ be continuous nonrandom functions at the interval $[t,s]$ and let $G(\tau)$ be a antiderivative of the function $g(\tau)$. Furthermore, let $\xi_\tau \in \mathrm{S}_2([t,s])$. Then*

$$\int\limits_t^s g(\tau)\int\limits_t^\tau h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau = \int\limits_t^s (G(s)-G(\theta))h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}$$

w. p. 1, where $i,j=1,2$ and $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ are independent standard Wiener processes that are F_τ–measurable for all $\tau \in [t,s]$.

Proof. Applying Theorem 3.1 two times and Theorem 3.3, we get the following relations

$$\int\limits_t^s g(\tau)\int\limits_t^\tau h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau = \int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau =$$

$$= G(s)\int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)} - \int\limits_t^s \xi_u d\mathbf{f}_u^{(i)}\int\limits_u^s G(\theta)h(\theta)d\mathbf{f}_\theta^{(j)} =$$

$$= G(s)\int\limits_t^s h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} - \int\limits_t^s G(\theta)h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} =$$

$$= \int\limits_t^s (G(s)-G(\theta))h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} \quad \text{w. p. 1.}$$

The proof of Lemma 4.1 is completed.

Let us consider an analogue of Lemma 4.1 for Stratonovich stochastic integrals.

Lemma 4.2 [121] (also see [1]-[5], [12]-[14], [50]). *Let $h(\tau), g(\tau), G(\tau) : [t,s] \to \mathbf{R}^1$ be continuous nonrandom functions at the interval $[t,s]$ and let $G(\tau)$ be a antiderivative of the function $g(\tau)$. Furthermore, let $\xi_\tau^{(l)} \in \mathrm{Q}_4([t,s])$ and*

$$\xi_\tau^{(l)} = \int\limits_t^\tau a_u du + \int\limits_t^\tau b_u d\mathbf{f}_u^{(l)}, \quad l = 1, 2.$$

Then

$$\int\limits_t^s g(\tau) \int\limits_t^{*\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} d\tau = \int\limits_t^{*s} (G(s) - G(\theta)) h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} \qquad (4.13)$$

w. p. 1, where $i, j, l = 1, 2$ and $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ are independent standard Wiener processes that are F_τ–measurable for all $\tau \in [t,s]$.

Proof. Under the conditions of Lemma 4.2, we can apply the equalities (4.11) and (4.12) with $F(x,\theta) \equiv x h(\theta)$ and

$$\eta_\theta = \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)},$$

since the following obvious inclusions hold: $\eta_\theta \in \mathrm{Q}_4([t,s])$, $x h(\theta) \in C(-\infty, \infty)$, and $\eta_\theta h(\theta) \in \mathrm{M}_2([t,s])$. Thus, we have the equalities

$$\int\limits_t^{*\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} = \int\limits_t^{\tau} h(\theta) \int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} + \frac{1}{2} \mathbf{1}_{\{i=j\}} \int\limits_t^{\tau} h(\theta) \xi_\theta^{(l)} d\theta, \qquad (4.14)$$

$$\int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} = \int\limits_t^{\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} + \frac{1}{2} \mathbf{1}_{\{l=i\}} \int\limits_t^{\theta} b_u du \qquad (4.15)$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of the set A.

Substituting the formulas (4.14) and (4.15) into the left-hand side of the equality (4.13) and applying Theorem 3.1 twice and Theorem 3.3, we get the following relations

$$\int\limits_t^s g(\tau)\int\limits_t^{*\tau} h(\theta)\int\limits_t^{*\theta} \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} d\tau =$$

$$= \int\limits_t^s \xi_u^{(l)} d\mathbf{f}_u^{(i)} \int\limits_u^s h(\theta) d\mathbf{f}_\theta^{(j)} \int\limits_\theta^s g(\tau) d\tau +$$

$$+\frac{1}{2}\mathbf{1}_{\{l=i\}} \int\limits_t^s b_u du \int\limits_u^s h(\theta) d\mathbf{f}_\theta^{(j)} \int\limits_\theta^s g(\tau) d\tau + \frac{1}{2}\mathbf{1}_{\{i=j\}} \int\limits_t^s h(\theta)\xi_\theta^{(l)} d\theta \int\limits_\theta^s g(\tau) d\tau =$$

$$= G(s)\left(\int\limits_t^s \xi_u^{(l)} d\mathbf{f}_u^{(i)} \int\limits_u^s h(\theta) d\mathbf{f}_\theta^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}} \int\limits_t^s h(\theta)\xi_\theta^{(l)} d\theta + \right.$$

$$\left. +\frac{1}{2}\mathbf{1}_{\{l=i\}} \int\limits_t^s b_u du \int\limits_u^s h(\theta) d\mathbf{f}_\theta^{(j)}\right) -$$

$$-\left(\int\limits_t^s \xi_u^{(l)} d\mathbf{f}_u^{(i)} \int\limits_u^s G(\theta)h(\theta) d\mathbf{f}_\theta^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}} \int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)} d\theta + \right.$$

$$\left. +\frac{1}{2}\mathbf{1}_{\{l=i\}} \int\limits_t^s b_u du \int\limits_u^s h(\theta)G(\theta) d\mathbf{f}_\theta^{(j)}\right) =$$

$$= G(s)\left(\int\limits_t^s h(\theta) \int\limits_t^\theta \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}} \int\limits_t^s h(\theta)\xi_\theta^{(l)} d\theta + \right.$$

$$\left. +\frac{1}{2}\mathbf{1}_{\{l=i\}} \int\limits_t^s h(\theta) \int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right) -$$

$$-\left(\int\limits_t^s G(\theta)h(\theta) \int\limits_t^\theta \xi_u^{(l)} d\mathbf{f}_u^{(i)} d\mathbf{f}_\theta^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}} \int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)} d\theta + \right.$$

$$\left. +\frac{1}{2}\mathbf{1}_{\{l=i\}} \int\limits_t^s h(\theta)G(\theta) \int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right) \qquad (4.16)$$

w. p. 1. Applying successively the formulas (4.14), (4.15) together with the formula (4.14) in which $h(\theta)$ replaced by $G(\theta)h(\theta)$ as well as the relation (4.16), we obtain the equality (4.13). The proof of Lemma 4.2 is completed.

4.3 The Taylor–Itô Expansion

In this section, we use the Taylor–Itô expansion [119] and introduce some necessary notations. At that we will use the original notations introduced by the author of this book.

Let $C^{2,1}(\mathbf{R}^n\times[0,T])\stackrel{\sf def}{=}\mathrm{L}$ be the space of functions $R(\mathbf{x},t):\mathbf{R}^n\times[0,T]\to\mathbf{R}^1$ with the following property: these functions are twice continuously differentiable in $\mathbf{x}$ and have one continuous derivative in t. We consider the following operators on the space L

$$
LR(\mathbf{x},t)=\frac{\partial R}{\partial t}(\mathbf{x},t)+\sum_{i=1}^{n}a^{(i)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x},t)+
$$

$$
+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2 R}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}}(\mathbf{x},t), \tag{4.17}
$$

$$
G_0^{(i)}R(\mathbf{x},t)=\sum_{j=1}^{n}B^{(ji)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x},t),\quad i=1,\ldots,m, \tag{4.18}
$$

where $\mathbf{x}^{(j)}$ is the jth component of $\mathbf{x}$, $a^{(j)}(\mathbf{x},t)$ is the jth component of $a(\mathbf{x},t)$, and $B^{(ij)}(\mathbf{x},t)$ is the ijth element of $B(\mathbf{x},t)$.

By the Itô formula, we have the equality

$$
R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\int\limits_t^s LR(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^{m}\int\limits_t^s G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)} \tag{4.19}
$$

w. p. 1, where $0\le t<s\le T$. In the formula (4.19) it is assumed that the functions $\mathbf{a}(\mathbf{x},t)$, $B(\mathbf{x},t)$, and $R(\mathbf{x},t)$ satisfy the following condition: $LR(\mathbf{x}_\tau,\tau)$, $G_0^{(i)}R(\mathbf{x}_\tau,\tau)\in \mathrm{M}_2([0,T])$ for $i=1,\ldots,m$.

Introduce the following notations

$$
{}^{(k)}A=\Biggl\Vert A^{(i_1\ldots i_k)}\Biggr\Vert_{i_1=1,\ldots,i_k=1}^{m_1\ \ldots\ \ m_k},\quad m_1,\ldots,m_k\ge 1, \tag{4.20}
$$

$$
{}^{(k+l)}A \stackrel{l}{\cdot} {}^{(l)}B^{(k)} = \begin{cases} \left\| \sum\limits_{i_1=1}^{m_1} \ldots \sum\limits_{i_l=1}^{m_l} A^{(i_1 \ldots i_{k+l})} B^{(i_1 \ldots i_l)} \right\|_{i_{l+1}=1,\ldots,i_{l+k}=1}^{m_{l+1} \ \ldots \ m_{l+k}} & \text{for} \quad k \ge 1 \\ \\ \sum\limits_{i_1=1}^{m_1} \ldots \sum\limits_{i_l=1}^{m_l} A^{(i_1 \ldots i_l)} B^{(i_1 \ldots i_l)} & \text{for} \quad k = 0 \end{cases},
$$

$$
\left\| A_{k+1} D_k^{(i_k)} A_k \ldots A_2 D_1^{(i_1)} A_1 R(\mathbf{x}, t) \right\|_{i_1=1,\ldots,i_k=1}^{m_1 \ \ldots \ m_k} = {}^{(k)}A_{k+1} D_k A_k \ldots A_2 D_1 A_1 R(\mathbf{x}, t), \tag{4.21}
$$

where A_p and $D_q^{(i_q)}$ are operators defined on the space L for $p = 1, \ldots, k+1$, $q = 1, \ldots, k$, and $i_q = 1, \ldots, m_q$. It is assumed that the left-hand side of (4.21) exists. The symbol $\stackrel{0}{\cdot}$ is treated as the usual multiplication. If $m_l = 0$ in (4.20) for some $l \in \{1, \ldots, k\}$, then the right-hand side of (4.20) is treated as

$$
\left\| A^{(i_1 \ldots i_{l-1} i_{l+1} \ldots i_k)} \right\|_{i_1=1,\ldots,i_{l-1}=1, i_{l+1}=1,\ldots,i_k=1}^{m_1 \ \ldots \ m_{l-1} \ \ m_{l+1} \ \ldots \ m_k},
$$

(shortly, ${}^{(k-1)}A$).

We also introduce the following notations

$$
\left\| Q_{\lambda_l}^{(i_l)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) \right\|_{i_1=\lambda_1,\ldots,i_l=\lambda_l}^{m\lambda_1 \ \ldots \ m\lambda_l} \stackrel{\text{def}}{=} {}^{(p_l)}Q_{\lambda_l} \ldots Q_{\lambda_1} R(\mathbf{x}, t),
$$

$$
{}^{(p_k)}J_{(\lambda_k \ldots \lambda_1)s,t} = \left\| J_{(\lambda_k \ldots \lambda_1)s,t}^{(i_k \ldots i_1)} \right\|_{i_1=\lambda_1,\ldots,i_k=\lambda_k}^{m\lambda_1 \ \ldots \ m\lambda_k},
$$

$$
M_k = \left\{ (\lambda_k, \ldots, \lambda_1) : \lambda_l = 1 \text{ or } \lambda_l = 0; \ l = 1, \ldots, k \right\}, \quad k \ge 1,
$$

$$
J_{(\lambda_k \ldots \lambda_1)s,t}^{(i_k \ldots i_1)} = \int\limits_t^s \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_k)} \ldots d\mathbf{w}_{t_k}^{(i_1)}, \quad k \ge 1,
$$

where $\lambda_l = 1$ or $\lambda_l = 0$, $Q_{\lambda_l}^{(i_l)} = L$ and $i_l = 0$ for $\lambda_l = 0$, $Q_{\lambda_l}^{(i_l)} = G_0^{(i_l)}$ and $i_l = 1, \ldots, m$ for $\lambda_l = 1$,

$$
p_l = \sum_{j=1}^{l} \lambda_j \quad \text{for} \quad l = 1, \ldots, r+1, \quad r \in \mathbf{N},
$$

$\mathbf{w}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are F_{τ}-measurable for all $\tau\in[0,T]$ independent standard Wiener processes and $\mathbf{w}_{\tau}^{(0)}=\tau.$

Applying (4.19) to the process $R(\mathbf{x}_s,s)$ repeatedly, we obtain the following Taylor–Itô expansion [119]

$$
R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}{}^{(p_k)}Q_{\lambda_k}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\overset{p_k}{\cdot}{}^{(p_k)}J_{(\lambda_k\ldots\lambda_1)s,t}+
$$

$$
+\left(D_{r+1}\right)_{s,t}, \tag{4.22}
$$

w. p. 1, where

$$
\left(D_{r+1}\right)_{s,t}=
$$

$$
=\sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}}\int\limits_t^s\ldots\left(\int\limits_t^{t_2}{}^{(p_{r+1})}Q_{\lambda_{r+1}}\ldots Q_{\lambda_1}R(\mathbf{x}_{t_1},t_1)\overset{\lambda_{r+1}}{\cdot}d\mathbf{w}_{t_1}\right)\ldots\overset{\lambda_1}{\cdot}d\mathbf{w}_{t_{r+1}}. \tag{4.23}
$$

It is assumed that the right-hand sides of (4.22), (4.23) exist.

A possible variant of the conditions, under which the right-hand sides of (4.22), (4.23) exist is as follows

(i) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\in \mathrm{L}$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r}M_g;$

(ii) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_{\tau},\tau)\in \mathrm{M}_2([0,T])$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r+1}M_g.$

Let us rewrite the expansion (4.22) in the another form

$$
R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}+
$$

$$
+\left(D_{r+1}\right)_{s,t}\quad \text{w. p. 1.}
$$

Denote

$$
G_{rk}=\biggl\{(\lambda_k,\ldots,\lambda_1):\ r+1\le 2k-\lambda_1-\ldots-\lambda_k\le 2r\biggr\},
$$

$$
E_{qk}=\biggl\{(\lambda_k,\ldots,\lambda_1):\ 2k-\lambda_1-\ldots-\lambda_k=q\biggr\},
$$

where $\lambda_l=1$ or $\lambda_l=0$ $(l=1,\ldots,k).$

The Taylor–Itô expansion ordered according to the order of smallness (in the mean-square sense when $s \downarrow t$) of its terms has the form

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \tag{4.24}$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} Q_{\lambda_k}^{(i_k)} \ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)} + \left(D_{r+1}\right)_{s,t}.$$

4.4 The First Form of the Unified Taylor–Itô Expansion

In this section, we transform the right-hand side of (4.22) by Theorem 3.1 and Lemma 4.1 to a representation including the iterated Itô stochastic integrals (4.7).

Denote

$$I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)} = \int\limits_t^s (t-t_k)^{l_k} \ldots \int\limits_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1 \tag{4.25}$$

and

$$I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)} = 1 \quad \text{for} \quad k = 0,$$

where $i_1, \ldots, i_k = 1, \ldots, m$. Moreover, let

$${}^{(k)}I_{l_1\ldots l_{k\,s,t}} = \left\| I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)} \right\|_{i_1,\ldots,i_k=1}^{m},$$

$$G_p^{(i)} \stackrel{\mathsf{def}}{=} \frac{1}{p}\left(G_{p-1}^{(i)} L - L G_{p-1}^{(i)}\right), \quad p = 1, 2, \ldots, \quad i = 1, \ldots, m, \tag{4.26}$$

where L and $G_0^{(i)}$, $i = 1, \ldots, m$ are determined by the equalities (4.17), (4.18). Denote

$$A_q \stackrel{\mathsf{def}}{=} \left\{ (k, j, l_1, \ldots, l_k) : \ k + j + \sum_{p=1}^{k} l_p = q; \ k, j, l_1, \ldots, l_k = 0, 1, \ldots \right\},$$

$$\left\|G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^j R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\sf def}{=} {}^{(k)}G_{l_1}\ldots G_{l_k}L^j R(\mathbf{x},t),$$

$$L^j R(\mathbf{x},t) \stackrel{\sf def}{=} \begin{cases} \underbrace{L\ldots L}_{j} R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases}.$$

Theorem 4.1. *Let conditions* (i), (ii) *be satisfied. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *and for any positive integer* r, *the following expansion takes place w. p.* 1

$$R(\mathbf{x}_s,s)=$$

$$=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\text{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)}+$$

$$+\left(D_{r+1}\right)_{s,t}, \tag{4.27}$$

where $\left(D_{r+1}\right)_{s,t}$ *is defined by* (4.23).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}{}^{(p_q)}Q_{\lambda_q}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J_{(\lambda_q\ldots\lambda_1)s,t}=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in\text{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)} \tag{4.28}$$

w. p. 1. The equality (4.28) is valid for $q=1$. Assume that (4.28) is valid for some $q>1$. In this case, using the induction hypothesis, we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}}{}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_{q+1}}{\cdot}{}^{(p_{q+1})}J_{(\lambda_1\ldots\lambda_{q+1})s,t}=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^s\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}\left({}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J_{(\lambda_1\ldots\lambda_q)\theta,t}\right)\stackrel{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=$$

$$=\sum_{\lambda_{q+1}\in\{1,\ 0\}}\int\limits_t^s\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\frac{(\theta-t)^j}{j!}\times$$

$$
\times\left({}^{(k+\lambda_{q+1})}G_{l_1}\ldots G_{l_k}L^jQ_{\lambda_{q+1}}R(\mathbf{x}_t,t)\stackrel{k}{\cdot}{}^{(k)}I_{l_1\ldots l_{k\,s,t}}\right)\stackrel{\lambda_{q+1}}{\cdot}d\mathbf{w}_\theta=
$$

$$
=\sum_{(k,j,l_1,\ldots,l_k)\in A_q}\left(\left({}^{(k)}G_{l_1}\ldots G_{l_k}L^{j+1}R(\mathbf{x}_t,t)\stackrel{k}{\cdot}\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta+\right.\right.
$$

$$
\left.+\left({}^{(k+1)}G_{l_1}\ldots G_{l_k}L^jG_0R(\mathbf{x}_t,t)\stackrel{k}{\cdot}\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}\right)\stackrel{1}{\cdot}d\mathbf{f}_\theta\right) \qquad (4.29)
$$

w. p. 1.

Using Lemma 4.1, we obtain

$$
\int\limits_t^s\frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta=
$$

$$
=\frac{1}{(j+1)!}\begin{cases}(s-t)^{j+1} & \text{for } k=0\\ \\ (s-t)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k\,s,t}}-(-1)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0\end{cases} \qquad (4.30)
$$

w. p. 1. In addition (see (4.25)) we get

$$
\int\limits_t^s\frac{(\theta-t)^j}{j!}I_{l_1\ldots l_{k\,\theta,t}}^{(i_1\ldots i_k)}d\mathbf{f}_\theta^{(i_{k+1})}=\frac{(-1)^j}{j!}I_{l_1\ldots l_kj_{s,t}}^{(i_1\ldots i_ki_{k+1})} \qquad (4.31)
$$

in the notations just introduced. Substitute (4.30) and (4.31) into the formula (4.29). Grouping summands in the obtained expression with equal lower indices at iterated Itô stochastic integrals and using (4.26) and the equality

$$
G_p^{(i)}R(\mathbf{x},t)=\frac{1}{p!}\sum_{q=0}^p(-1)^qC_p^qL^qG_0^{(i)}L^{p-q}R(\mathbf{x},t),\quad C_p^q=\frac{p!}{q!(p-q)!} \qquad (4.32)
$$

(this equality follows from (4.26)), we note that the obtained expression equals to

$$
\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}}\frac{(s-t)^j}{j!}{}^{(k)}G_{l_1}\ldots G_{l_k}L^j\{\eta_t\}\stackrel{k}{\cdot}{}^{(k)}I_{l_1\ldots l_{k\,s,t}}
$$

w. p. 1. Summing the equalities (4.28) for $q=1,2,\ldots,r$ and applying the formula (4.22), we obtain the expression (4.27). The proof is completed.

Let us order terms of the expansion (4.27) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \tag{4.33}$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} +$$

$$+ \left(D_{r+1}\right)_{s,t},$$

$$\mathrm{D}_q = \left\{ (k, j, l_1, \ldots, l_k) : k + 2\left(j + \sum_{p=1}^{k} l_p\right) = q;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots \right\}, \tag{4.34}$$

$$\mathrm{U}_r = \left\{ (k, j, l_1, \ldots, l_k) : k + j + \sum_{p=1}^{k} l_p \le r, \right.$$

$$\left. k + 2\left(j + \sum_{p=1}^{k} l_p\right) \ge r + 1;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots \right\}, \tag{4.35}$$

and $\left(D_{r+1}\right)_{s,t}$ is defined by (4.23). Note that the remainder term $\left(H_{r+1}\right)_{s,t}$ in (4.33) has a higher order of smallness in the mean-square sense as $s \downarrow t$ than the terms of the main part of the expansion (4.33).

4.5 The Second Form of the Unified Taylor–Itô Expansion

Consider iterated Itô stochastic integrals of the form

$$J_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} = \int\limits_t^s (s-t_k)^{l_k} \ldots \int\limits_t^{t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1$$

and

$$J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=1 \quad \text{for} \quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m.$

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=\sum_{j_1=0}^{l_1}\ldots\sum_{j_k=0}^{l_k}\prod_{g=1}^{k}C_{l_g}^{j_g}(t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\ J^{(i_1\ldots i_k)}_{j_1\ldots j_k s,t} \quad \text{w. p. 1}, \qquad (4.36)$$

where

$$C_l^k=\frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Itô expansion of the process $\eta_s=R(\mathbf{x}_s,s),\ s\in[0,T]$ can be constructed either using the iterated stochastic integrals $I^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}$ similarly to the previous section or using the iterated stochastic integrals $J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}$. This is the main subject of this section.

Denote

$$\left\|J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}J_{l_1\ldots l_k s,t},$$

$$\left\|L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}L^jG_{l_1}\ldots G_{l_k}R(\mathbf{x},t).$$

Theorem 4.2. *Let conditions* (i), (ii) *be satisfied. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *and for any positive integer* r*, the following expansion is valid w. p.* 1

$$R(\mathbf{x}_s,s)=$$

$$=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\ \sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x}_t,t)J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}+$$

$$+\left(D_{r+1}\right)_{s,t}, \qquad (4.37)$$

where $\left(D_{r+1}\right)_{s,t}$ *is defined by* (4.23)*.*

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG^{(i_1)}_{l_1}\ldots G^{(i_k)}_{l_k}R(\mathbf{x}_t,t)J^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t}=$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} \quad \text{w. p. 1} \quad (4.38)$$

for $q=1,2,\ldots,r$. To check (4.38), substitute the expression (4.36) into the right-hand side of (4.38) and then use the formulas (4.26), (4.32).

Let us order terms of the expansion (4.37) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s,s) =$$

$$= R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1},$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t,t) J_{l_1\ldots l_{k_{s,t}}}^{(i_1\ldots i_k)} +$$

$$+ \left(D_{r+1}\right)_{s,t}.$$

The remainder term $\left(D_{r+1}\right)_{s,t}$ is defined by (4.23); the sets D_q and U_r are defined by (4.34) and (4.4), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Itô expansion (4.22) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s,s)$ as $r\to\infty$ for all $s,t\in[0,T]$ such that $s>t$ and $T<\infty$ has been proved in [67] (Proposition 5.9.2). Since the expansions (4.27) and (4.37) are obtained from the Taylor–Itô expansion (4.22) without any additional conditions, the truncated expansions (4.27) and (4.37) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of Proposition 5.9.2 [67] converge to the process $R(\mathbf{x}_s,s)$ w. p. 1 as $r\to\infty$ for all $s,t\in[0,T]$ such that $s>t$ and $T<\infty$.

4.6 The Taylor–Stratonovich Expansion

In this section, we use the Taylor–Stratonovich expansion [119] and introduce some necessary notations. At that we will use the original notations introduced by the author of this book.

Assume that $LR(\mathbf{x}_\tau,\tau),\ G_0^{(i)}R(\mathbf{x}_\tau,\tau) \in \mathrm{M}_2([0,T])$ for $i=1,\ldots,m$ and consider the Itô formula (4.19).

In addition, we assume that $G_0^{(i)}R(\mathbf{x},t)\in C_2(-\infty,\infty)$ for $i=1,\ldots,m$ and $R(\mathbf{x}_\tau,\tau)\in \mathrm{Q}_4([0,T])$. In this case, the relations (4.11) and (4.12) imply that

$$\int\limits_t^s G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)}=\int\limits_t^{*s} G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)}-\frac{1}{2}\int\limits_t^s G_0^{(i)}G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\tau \qquad (4.39)$$

w. p. 1, where $i=1,\ldots,m$.

Using the relation (4.39), let us write (4.19) in the following form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\int\limits_t^s \bar{L}R(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^m\int\limits_t^{*s} G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)}\quad \text{w. p. 1}, \qquad (4.40)$$

where

$$\bar{L}R(\mathbf{x},t)=LR(\mathbf{x},t)-\frac{1}{2}\sum_{i=1}^m G_0^{(i)}G_0^{(i)}R(\mathbf{x},t). \qquad (4.41)$$

Introduce the following notations

$$\left\|D_{\lambda_l}^{(i_l)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\right\|_{i_1=\lambda_1,\ldots,i_l=\lambda_l}^{m\lambda_1\ \ldots\ m\lambda_l}\ \stackrel{\mathrm{def}}{=}\ {}^{(p_l)}D_{\lambda_l}\ldots D_{\lambda_1}R(\mathbf{x},t),$$

$$^{(p_k)}J^*_{(\lambda_k\ldots\lambda_1)s,t}=\left\|J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}\right\|_{i_1=\lambda_1,\ldots,i_k=\lambda_k}^{m\lambda_1\ \ldots\ m\lambda_k},$$

$$M_k=\left\{(\lambda_k,\ldots,\lambda_1):\lambda_l=1 \text{ or } \lambda_l=0;\ l=1,\ldots,k\right\},\quad k\ge 1,$$

$$J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}=\int\limits_t^{*s}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_k)}\ldots d\mathbf{w}_{t_k}^{(i_1)},\quad k\ge 1,$$

where $\lambda_l=1$ or $\lambda_l=0$, $D_{\lambda_l}^{(i_l)}=\bar{L}$ and $i_l=0$ for $\lambda_l=0$, $D_{\lambda_l}^{(i_l)}=G_0^{(i_l)}$ and $i_l=1,\ldots,m$ for $\lambda_l=1$,

$$p_l=\sum_{j=1}^l\lambda_j\quad \text{for}\quad l=1,\ldots,r+1,\quad r\in\mathbf{N},$$

$\mathbf{w}_{\tau}^{(i)}$ $(i = 1,\ldots,m)$ are F_{τ}-measurable for all $\tau \in [0,T]$ independent standard Wiener processes and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Applying the formula (4.40) to the process $R(\mathbf{x}_s, s)$ repeatedly, we obtain the following Taylor–Stratonovich expansion [119]

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} {}^{(p_k)}D_{\lambda_k}\ldots D_{\lambda_1} R(\mathbf{x}_t, t) \overset{p_k}{\cdot} {}^{(p_k)}J^{*}_{(\lambda_k\ldots\lambda_1)s,t} +$$

$$+ \left(D_{r+1}\right)_{s,t}, \tag{4.42}$$

w. p. 1, where

$$\left(D_{r+1}\right)_{s,t} =$$

$$= \sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}} {\int\limits_{t}^{*s}} \ldots \left({\int\limits_{t}^{*t_2}} {}^{(p_{r+1})}D_{\lambda_{r+1}}\ldots D_{\lambda_1} R(\mathbf{x}_{t_1}, t_1) \overset{\lambda_{r+1}}{\cdot} d\mathbf{w}_{t_1}\right) \ldots \overset{\lambda_1}{\cdot} d\mathbf{w}_{t_{r+1}}. \tag{4.43}$$

It is assumed that the right-hand sides of (4.42), (4.43) exist.

A possible variant of the conditions under which the right-hand sides of (4.42), (4.43) exist is as follows

(i*) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) \in \mathrm{L}$ for all $(\lambda_l,\ldots,\lambda_1) \in \bigcup\limits_{g=1}^{r} M_g$;

(ii*) for all $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$, $t, s \in [0,T]$, $(\lambda_l,\ldots,\lambda_1) \in \bigcup\limits_{g=1}^{r+1} M_g$, and for some $\nu > 0$

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) - Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{y}, t)\right| \le K|\mathbf{x}-\mathbf{y}|, \tag{4.44}$$

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t)\right| \le K(1+|\mathbf{x}|), \tag{4.45}$$

and

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, t) - Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)} R(\mathbf{x}, s)\right| \le K|t-s|^{\nu}(1+|\mathbf{x}|),$$

where $K < \infty$ is a constant, $Q_{\lambda_l}^{(i_l)} = L$ and $i_l = 0$ for $\lambda_l = 0$, $Q_{\lambda_l}^{(i_l)} = G_0^{(i_l)}$ and $i_l = 1,\ldots,m$ for $\lambda_l = 1$;

(iii*) the functions $\mathbf{a}(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ are measurable with respect to all variables and satisfy the conditions (4.44) and (4.45);

(iv*) $\mathbf{x}_0$ is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^8\right\} < \infty$.

Let us rewrite the expansion (4.42) in another form

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} +$$

$$+ \left(D_{r+1}\right)_{s,t} \quad \text{w. p. 1.}$$

Denote

$$G_{rk} = \left\{(\lambda_k, \ldots, \lambda_1) : \ r + 1 \le 2k - \lambda_1 - \ldots - \lambda_k \le 2r\right\},$$

$$E_{qk} = \left\{(\lambda_k, \ldots, \lambda_1) : \ 2k - \lambda_1 - \ldots - \lambda_k = q\right\},$$

where $\lambda_l = 1$ or $\lambda_l = 0$ $(l = 1, \ldots, k)$.

Let us order terms of the Taylor–Stratonovich expansion according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q,k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1,} \tag{4.46}$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{k=1}^{r} \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}} \sum_{i_1=\lambda_1}^{m\lambda_1} \ldots \sum_{i_k=\lambda_k}^{m\lambda_k} D_{\lambda_k}^{(i_k)} \ldots D_{\lambda_1}^{(i_1)} R(\mathbf{x}_t, t) J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)} +$$

$$+ \left(D_{r+1}\right)_{s,t}.$$

4.7 The First Form of the Unified Taylor–Stratonovich Expansion

In this section, we transform the right-hand side of (4.42) by Theorem 3.1 and Lemma 4.2 to a representation including the iterated Stratonovich stochastic integrals (4.8).

Denote

$$I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*s} (t-t_k)^{l_k}\ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k\ge 1 \tag{4.47}$$

and

$$I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} = 1 \quad \text{for} \quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m.$

Moreover, let

$${}^{(k)}I^{*}_{l_1\ldots l_k s,t} = \left\| I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} \right\|_{i_1,\ldots,i_k=1}^{m},$$

$$\bar G_p^{(i)} \stackrel{\text{def}}{=} \frac{1}{p}\left(\bar G_{p-1}^{(i)}\bar L - \bar L\bar G_{p-1}^{(i)}\right), \quad p=1,2,\ldots, \quad i=1,\ldots,m, \tag{4.48}$$

where $\bar G_0^{(i)} \stackrel{\text{def}}{=} G_0^{(i)}$, $i=1,\ldots,m$. The operators $\bar L$ and $G_0^{(i)}$, $i=1,\ldots,m$ are determined by the equalities (4.17), (4.18), and (4.41).

Denote

$$A_q \stackrel{\text{def}}{=} \left\{(k,j,l_1,\ldots,l_k):\ k+j+\sum_{p=1}^{k} l_p = q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\},$$

$$\left\| \bar G_{l_1}^{(i_1)}\ldots \bar G_{l_k}^{(i_k)}\bar L^j R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\text{def}}{=} {}^{(k)}\bar G_{l_1}\ldots \bar G_{l_k}\bar L^j R(\mathbf{x},t),$$

$$\bar L^j R(\mathbf{x},t) \stackrel{\text{def}}{=} \begin{cases} \underbrace{\bar L\ldots\bar L}_{j} R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases}.$$

Theorem 4.3 [121] (also see [1]-[14], [50], [124]). *Let conditions* (i*)-(iv*) *be satisfied. Then for any* $s,t\in[0,T]$ *such that* $s>t$ *and for any positive integer* r*, the following expansion takes place w. p. 1*

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} +$$

$$+ \left(D_{r+1}\right)_{s,t}, \tag{4.49}$$

where $\left(D_{r+1}\right)_{s,t}$ *is defined by* (4.43).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q} {}^{(p_q)}D_{\lambda_q} \ldots D_{\lambda_1} R(\mathbf{x}_t, t) \overset{p_q}{\cdot} {}^{(p_q)}J^*_{(\lambda_q\ldots\lambda_1)s,t} =$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} \tag{4.50}$$

w. p. 1. The equality (4.50) is valid for $q = 1$. Assume that (4.50) is valid for some $q > 1$. In this case using the induction hypothesis we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}} {}^{(p_{q+1})}D_{\lambda_1} \ldots D_{\lambda_{q+1}} R(\mathbf{x}_t, t) \overset{p_{q+1}}{\cdot} {}^{(p_{q+1})}J^*_{(\lambda_1\ldots\lambda_{q+1})s,t} =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^{*s} \sum_{(\lambda_q,\ldots,\lambda_1)\in M_q} \left({}^{(p_{q+1})}D_{\lambda_1} \ldots D_{\lambda_{q+1}} R(\mathbf{x}_t, t) \overset{p_q}{\cdot} {}^{(p_q)}J^*_{(\lambda_1\ldots\lambda_q)\theta,t} \right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^{*s} \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \frac{(\theta-t)^j}{j!} \times$$

$$\times \left({}^{(k+\lambda_{q+1})}\bar{G}_{l_1} \ldots \bar{G}_{l_k} \bar{L}^j D_{\lambda_{q+1}} R(\mathbf{x}_t, t) \overset{k}{\cdot} {}^{(k)}I^*_{l_1\ldots l_{k_{s,t}}} \right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \left({}^{(k)}\bar{G}_{l_1} \ldots \bar{G}_{l_k} \bar{L}^{j+1} R(\mathbf{x}_t, t) \overset{k}{\cdot} \int\limits_t^{s} \frac{(\theta-t)^j}{j!} {}^{(k)}I^*_{l_1\ldots l_{k_{\theta,t}}} d\theta + \right.$$

$$\left. + \left({}^{(k+1)}\bar{G}_{l_1} \ldots \bar{G}_{l_k} \bar{L}^j \bar{G}_0 R(\mathbf{x}_t, t) \overset{k}{\cdot} \int\limits_t^{*s} \frac{(\theta-t)^j}{j!} {}^{(k)}I^*_{l_1\ldots l_{k_{\theta,t}}} \right) \overset{1}{\cdot} d\mathbf{f}_\theta \right) \tag{4.51}$$

w. p. 1.

Using Lemma 4.2, we obtain

$$\int\limits_t^s \frac{(\theta-t)^j}{j!} {}^{(k)}I^*_{l_1\ldots l_{k\,\theta,t}} d\theta =$$

$$= \frac{1}{(j+1)!}\begin{cases} (s-t)^{j+1} & \text{for } k=0 \\ (s-t)^{j+1}\cdot {}^{(k)}I^*_{l_1\ldots l_{k\,s,t}} - (-1)^{j+1}\cdot {}^{(k)}I^*_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0 \end{cases} \tag{4.52}$$

w. p. 1. In addition (see (4.47)) we get

$$\int\limits_t^{*s} \frac{(\theta-t)^j}{j!} I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,\theta,t}} d\mathbf{f}^{(i_{k+1})}_\theta = \frac{(-1)^j}{j!} I^{*(i_1\ldots i_k i_{k+1})}_{l_1\ldots l_k j_{s,t}} \tag{4.53}$$

in the notations just introduced. Substitute (4.52) and (4.53) into the formula (4.51). Grouping summands in the obtained expression with equal lower indices at iterated Stratonovich stochastic integrals and using (4.48) and the equality

$$\bar G^{(i)}_p R(\mathbf{x},t) = \frac{1}{p!}\sum_{q=0}^{p}(-1)^q C_p^q \bar L^q \bar G^{(i)}_0 \bar L^{p-q} R(\mathbf{x},t), \quad C_p^q = \frac{p!}{q!(p-q)!} \tag{4.54}$$

(this equality follows from (4.48)), we note that the obtained expression equals to

$$\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}} \frac{(s-t)^j}{j!} {}^{(k)}\bar G_{l_1}\ldots \bar G_{l_k}\bar L^j\{\eta_t\} \stackrel{k}{\cdot} {}^{(k)}I^*_{l_1\ldots l_{k\,s,t}}$$

w. p. 1. Summing the equalities (4.50) for $q = 1, 2, \ldots, r$ and applying the formula (4.42), we obtain the expression (4.49). The proof is completed.

Let us order terms of the expansion (4.49) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t,t) + \sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m} \bar G^{(i_1)}_{l_1}\ldots \bar G^{(i_k)}_{l_k}\bar L^j R(\mathbf{x}_t,t) I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}} +$$

$$+ (H_{r+1})_{s,t} \quad \text{w. p. 1}, \tag{4.55}$$

where

$$
(H_{r+1})_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t) I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} +
$$

$$
+ (D_{r+1})_{s,t}\ ,
$$

$$
\mathrm{D}_q = \left\{ (k,j,l_1,\ldots,l_k) : k + 2\left(j + \sum_{p=1}^{k} l_p\right) = q;\ k,j,l_1,\ldots,l_k = 0,1,\ldots \right\}, \tag{4.56}
$$

$$
\mathrm{U}_r = \left\{ (k,j,l_1,\ldots,l_k) : k + j + \sum_{p=1}^{k} l_p \le r, \right.
$$

$$
\left. k + 2\left(j + \sum_{p=1}^{k} l_p\right) \ge r+1;\ k,j,l_1,\ldots,l_k = 0,1,\ldots \right\}, \tag{4.57}
$$

and $(D_{r+1})_{s,t}$ is defined by (4.43). Note that the remainder term $(H_{r+1})_{s,t}$ in (4.55) has a higher order of smallness in the mean-square sense as $s \downarrow t$ than the terms of the main part of the expansion (4.55).

4.8 The Second Form of the Unified Taylor–Stratonovich Expansion

Consider iterated Stratonovich stochastic integrals of the form

$$
J_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = {\int\limits_t^{*}}^{s} (s-t_k)^{l_k} \ldots {\int\limits_t^{*}}^{t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1
$$

and

$$
J_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = 1 \quad \text{for} \quad k = 0,
$$

where $i_1,\ldots,i_k = 1,\ldots,m.$

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$
I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = \sum_{j_1=0}^{l_1} \ldots \sum_{j_k=0}^{l_k} \prod_{g=1}^{k} C_{l_g}^{j_g} (t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\ J_{j_1\ldots j_{k_{s,t}}}^{*(i_1\ldots i_k)} \quad \text{w. p. 1,} \tag{4.58}
$$

where

$$C_l^k = \frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Stratonovich expansion of the process $\eta_s = R(\mathbf{x}_s, s)$, $s \in [0, T]$ can be constructed either using the iterated stochastic integrals $I_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)}$ similarly to the previous section or using the iterated stochastic integrals $J_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)}$. This is the main subject of this section.

Denote

$$\left\| J_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)} \right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\rm def}{=} {}^{(k)} J_{l_1 \ldots l_{k s,t}}^{*},$$

$$\left\| \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}, t) \right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\rm def}{=} {}^{(k)} \bar{L}^j \bar{G}_{l_1} \ldots \bar{G}_{l_k} R(\mathbf{x}, t).$$

Theorem 4.4 [121] (also see [1]-[14], [50], [124]). *Let conditions* (i*)-(iv*) *be satisfied. Then for any $s, t \in [0, T]$ such that $s > t$ and for any positive integer r, the following expansion is valid w. p.* 1

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)} +$$

$$+ \left(D_{r+1}\right)_{s,t}, \qquad (4.59)$$

where $\left(D_{r+1}\right)_{s,t}$ is defined by (4.43).

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)} =$$

$$\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I_{l_1 \ldots l_{k s,t}}^{*(i_1 \ldots i_k)} \quad \text{w. p. 1} \qquad (4.60)$$

for $q = 1, 2, \ldots, r$. To check (4.60), substitute the expression (4.58) into the right-hand side of (4.60) and then use the formulas (4.48), (4.54).

Let us order terms of the expansion (4.59) according to their smallness orders as $s \downarrow t$ in the mean-square sense

$$
R(\mathbf{x}_s, s) =
$$

$$
= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} +
$$

$$
+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1},
$$

where

$$
\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J^{*(i_1\ldots i_k)}_{l_1\ldots l_k s,t} +
$$

$$
+ \left(D_{r+1}\right)_{s,t}.
$$

The remainder term $\left(D_{r+1}\right)_{s,t}$ is defined by (4.43); the sets D_q and U_r are defined by (4.56) and (4.7), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Stratonovich expansion (4.42) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s, s)$ as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$ has been proved in [67] (Proposition 5.10.2). Since the expansions (4.49) and (4.59) are obtained from the Taylor–Stratonovich expansion (4.42) without any additional conditions, the truncated expansions (4.49) and (4.59) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of Proposition 5.10.2 [67] converge to the process $R(\mathbf{x}_s, s)$ w. p. 1 as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$.

4.9 Comparison of the Unified Taylor–Itô and Taylor–Stratonovich Expansions with the Classical Taylor–Itô and Taylor–Stratonovich Expansions

Note that the truncated unified Taylor–Itô and Taylor–Stratonovich expansions contain the less number of various iterated Itô and Stratonovich stochastic integrals (moreover, their major part will have less multiplicity) in comparison with the classical Taylor–Itô and Taylor–Stratonovich expansions [119].

It is easy to notice that the stochastic integrals from the sets (4.4), (4.5) are connected by linear relations. However, the stochastic integrals from the sets (4.6), (4.7) cannot be connected by linear relations. This also holds for the stochastic integrals from the sets (4.8), (4.9). Therefore, we will call the sets (4.6)–(4.9) as the *stochastic bases.*

Let us call the numbers $\text{rank}_{\text{A}}(r)$ and $\text{rank}_{\text{D}}(r)$ of various iterated Itô and Stratonovich stochastic integrals, which are included in the sets (4.6)–(4.9) as the *ranks of stochastic bases* when summation in the stochastic expansions is performed using the sets A_q $(q = 1, \ldots, r)$ and D_q $(q = 1, \ldots, r)$ correspondingly. Here r is a fixed natural number.

At the beginning, let us analyze several examples related to the Taylor–Itô expansions (obviously, the same conclusions will hold for the Taylor–Stratonovich expansions).

Assume that the summation in the unified Taylor–Itô expansions is performed using the sets D_q $(q = 1, \ldots, r)$. It is easy to see that the truncated unified Taylor–Itô expansion (4.33), where the summation is performed using the sets D_q when $r = 3$ includes 4 $(\text{rank}_{\text{D}}(3) = 4)$ various iterated Itô stochastic integrals

$$I_{0_{s,t}}^{(i_1)}, \quad I_{00_{s,t}}^{(i_1 i_2)}, \quad I_{1_{s,t}}^{(i_1)}, \quad I_{000_{s,t}}^{(i_1 i_2 i_3)}.$$

The same truncated classical Taylor–Itô expansion (4.3) [67] contains 5 various iterated Itô stochastic integrals

$$J_{(1)s,t}^{(i_1)}, \quad J_{(11)s,t}^{(i_1 i_2)}, \quad J_{(10)s,t}^{(i_1 0)}, \quad J_{(01)s,t}^{(0 i_1)}, \quad J_{(111)s,t}^{(i_1 i_2 i_3)}.$$

For $r = 4$ we have 7 $(\text{rank}_{\text{D}}(4) = 7)$ stochastic integrals

$$I_{0_{s,t}}^{(i_1)}, \quad I_{00_{s,t}}^{(i_1 i_2)}, \quad I_{1_{s,t}}^{(i_1)}, \quad I_{000_{s,t}}^{(i_1 i_2 i_3)}, \quad I_{01_{s,t}}^{(i_1 i_2)}, \quad I_{10_{s,t}}^{(i_1 i_2)}, \quad I_{0000_{s,t}}^{(i_1 i_2 i_3 i_4)}$$

against 9 stochastic integrals

$$J_{(1)s,t}^{(i_1)}, \ J_{(11)s,t}^{(i_1 i_2)}, \ J_{(10)s,t}^{(i_1 0)}, \ J_{(01)s,t}^{(0 i_1)}, \ J_{(111)s,t}^{(i_1 i_2 i_3)}, \ J_{(101)s,t}^{(i_1 0 i_3)}, \ J_{(110)s,t}^{(i_1 i_2 0)}, \ J_{(011)s,t}^{(0 i_1 i_2)}, \ J_{(1111)s,t}^{(i_1 i_2 i_3 i_4)}.$$

For $r = 5$ $(\text{rank}_{\text{D}}(5) = 12)$ we get 12 integrals against 17 integrals and for $r = 6$ and $r = 7$ we have 20 against 29 and 33 against 50 correspondingly.

We will obtain the same results when compare the unified Taylor–Stratonovich expansions [121] (also see [1]-[14], [50], [124]) with their classical analogues [67], [119] (see previous sections).

Note that the summation with respect to the sets D_q is usually used while constructing strong numerical methods (built according to the mean-square criterion of convergence) for Itô SDEs [65], [67] (also see [13]). The summation with respect to the sets A_q is usually used when building weak numerical methods (built in accordance with the weak criterion of convergence) for Itô SDEs [65], [67]. For example, $\text{rank}_{\text{A}}(4) = 15$ while the total number of various iterated Itô stochastic integrals (included in the classical Taylor–Itô expansion [67] when $r = 4$) equals to 26.

Let us show that [3]-[14], [50]

$$\mathrm{rank}_{\mathrm{A}}(r) = 2^r - 1.$$

Let $(l_1, \ldots, l_k)$ be an ordered set such that $l_1, \ldots, l_k = 0, 1, \ldots$ and $k = 1, 2, \ldots$ Consider $S(k) \stackrel{\text{def}}{=} l_1 + \ldots + l_k = p$ (p is a fixed natural number or zero). Let $N(k, p)$ be a number of all ordered combinations $(l_1, \ldots, l_k)$ such that $l_1, \ldots, l_k = 0, 1, \ldots,$ $k = 1, 2, \ldots,$ and $S(k) = p$. First, let us show that

$$N(k, p) = C_{p+k-1}^{k-1},$$

where

$$C_n^m = \frac{n!}{m!(n-m)!}$$

is a binomial coefficient.

It is not difficult to see that

$$N(1, p) = 1 = C_{p+1-1}^{1-1},$$

$$N(2, p) = p + 1 = C_{p+2-1}^{2-1},$$

$$N(3, p) = \frac{(p+1)(p+2)}{2} = C_{p+3-1}^{3-1}.$$

Moreover,

$$N(k+1, p) = \sum_{l=0}^{p} N(k, l) = \sum_{l=0}^{p} C_{l+k-1}^{k-1} = C_{p+k}^{k},$$

where we used the induction assumption and the well known property of binomial coefficients.

Then

$$\mathrm{rank}_A(r) =$$

$$= N(1, 0) + (N(1, 1) + N(2, 0)) + (N(1, 2) + N(2, 1) + N(3, 0)) + \ldots$$

$$\ldots + (N(1, r-1) + N(2, r-2) + \ldots + N(r, 0)) =$$

$$= C_0^0 + (C_1^0 + C_1^1) + (C_2^0 + C_2^1 + C_2^2) + \ldots$$

$$\ldots + (C_{r-1}^0 + C_{r-1}^1 + C_{r-1}^2 + \ldots + C_{r-1}^{r-1}) =$$

$$= 2^0 + 2^1 + 2^2 + \ldots + 2^{r-1} = 2^r - 1.$$

Let $n_{\rm M}(r)$ be the total number of various iterated stochastic integrals included in the classical Taylor–Itô expansion (4.22) [67], where summation is performed with respect to the set

$$\bigcup_{k=1}^{r} {\rm M}_k.$$

If we exclude from the consideration the integrals, which are equal to

$$\frac{(s-t)^j}{j!},$$

then

$$n_M(r)=$$
$$=(2^1-1)+(2^2-1)+(2^3-1)+\ldots+(2^r-1)=$$
$$=2(1+2+2^2+\ldots+2^{r-1})-r=2(2^r-1)-r.$$

It means that

$$\lim_{r\to\infty}\frac{n_M(r)}{{\rm rank}_A(r)}=2.$$

Numbers

$${\rm rank}_{\rm A}(r),\quad n_{\rm M}(r),\quad f(r)=n_{\rm M}(r)/{\rm rank}_{\rm A}(r)$$

for various values r are shown in Table 4.1.

Let us show that [3]-[14], [50]

$${\rm rank}_{\rm D}(r)=\begin{cases}\displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{(r-1)/2+[s/2]} C_l^s & \hbox{for}\quad r=1,\ 3,\ 5,\ldots\\ \\ \displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{r/2-1+[(s+1)/2]} C_l^s & \hbox{for}\quad r=2,\ 4,\ 6,\ldots\end{cases},\tag{4.61}$$

where $[x]$ is an integer part of a real number x and C_n^m is a binomial coefficient.

For the proof of (4.61) we rewrite the condition

$$k+2(j+S(k))\le r,$$

where $S(k)\stackrel{\sf def}{=}l_1+\ldots+l_k$ $(k,j,l_1,\ldots,l_k=0,1,\ldots)$ in the form

$$j+S(k)\le (r-k)/2$$

Table 4.1: Numbers $\text{rank}_{\text{A}}(r)$, $n_{\text{M}}(r)$, $f(r) = n_{\text{M}}(r)/\text{rank}_{\text{A}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{A}}(r)$	1	3	7	15	31	63	127	255	511	1023
$n_{\text{M}}(r)$	1	4	11	26	57	120	247	502	1013	2036
$f(r)$	1	1.3333	1.5714	1.7333	1.8387	1.9048	1.9449	1.9686	1.9824	1.9902

and perform the consideration of all possible combinations with respect to $k = 1, \ldots, r$. Moreover, we take into account the above reasoning.

Let us calculate the number $n_E(r)$ of all different iterated Itô stochastic integrals from the classical Taylor–Itô expansion (4.3) [67] if the summation in this expansion is performed with respect to the set

$$\bigcup_{q,k=1}^{r} E_{qk}.$$

The summation condition can be rewritten in this case in the form

$$0 \le p + 2q \le r,$$

where q is a total number of integrations with respect to time while p is a total number of integrations with respect to the Wiener processes in the selected iterated stochastic integral from the Taylor–Itô expansion (4.3) [67]. At that the multiplicity of the mentioned stochastic integral equals to $p+q$ and it is not more than r. Let us rewrite the above condition $(0 \le p + 2q \le r)$ in the form: $0 \le q \le (r-p)/2 \Leftrightarrow 0 \le q \le [(r-p)/2]$, where $[x]$ means an integer part of a real number x. Then, performing the consideration of all possible combinations with respect to $p = 1, \ldots, r$ and using the combinatorial reasoning, we come to the formula

$$n_E(r) = \sum_{s=1}^{r} \sum_{l=0}^{[(r-s)/2]} C^s_{[(r-s)/2]+s-l}, \tag{4.62}$$

where $[x]$ means an integer part of a real number x.

Numbers

$$\text{rank}_{\text{D}}(r), \quad n_{\text{E}}(r), \quad g(r) = n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$$

for various values r are shown in Table 4.2.

Table 4.2: Numbers $\text{rank}_{\text{D}}(r)$, $n_{\text{E}}(r)$, $g(r) = n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{D}}(r)$	1	2	4	7	12	20	33	54	88	143
$n_{\text{E}}(r)$	1	2	5	9	17	29	50	83	138	261
$g(r)$	1	1	1.2500	1.2857	1.4167	1.4500	1.5152	1.5370	1.5682	1.8252

4.10 Application of First Form of the Unified Taylor–Itô Expansion to the High-Order Strong Numerical Methods for Itô SDEs

Let us rewrite (4.33) for all $s, t \in [0, T]$ such that $s > t$ in the following form

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \text{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d\in\mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t) + \left(\bar{H}_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \qquad (4.63)$$

where

$$\left(\bar{H}_{r+1}\right)_{s,t} = \left(H_{r+1}\right)_{s,t} - \mathbf{1}_{\{r=2d-1, d\in\mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t).$$

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0\le j\le N-1} \left|\tau_{j+1} - \tau_j\right|.$$

From (4.63) for $s = \tau_{p+1}$, $t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (4.1), which is based on first form of the unified Taylor–Itô expansion

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \text{D}_q} \frac{(\tau_{p+1}-\tau_p)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{y}_p \; \hat{I}_{l_1\ldots l_k \tau_{p+1},\tau_p}^{(i_1\ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d\in\mathbf{N}\}} \frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \qquad (4.64)$$

where $\hat{I}^{(i_1\ldots i_k)}_{l_1\ldots l_k\tau_{p+1},\tau_p}$ is an approximation of iterated Itô stochastic integral $I^{(i_1\ldots i_k)}_{l_1\ldots l_k\tau_{p+1},\tau_p}$ of the form

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_k s,t} = \int\limits_t^s (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}.$$

Note that we understand the equality (4.64) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $T=\tau_N$, $p=0,1,\ldots,N$.

It is known [67] that under the appropriate conditions the numerical scheme (4.64) has strong order of convergence $r/2$ ($r\in\mathbf{N}$).

Let $B_j(\mathbf{x},t)$ is the jth column of the matrix function $B(\mathbf{x},t)$.

Below we consider particular cases of the numerical scheme (4.64) for $r=2,3,4,5,$ and 6, i.e. explicit one-step strong numerical schemes for the Itô SDE (4.1) with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. At that for simplicity we will write $\mathbf{a}$, $L\mathbf{a}$, B_i, $G_0^{(i)}B_j$, $\ldots$ instead of $\mathbf{a}(\mathbf{y}_p,\tau_p)$, $L\mathbf{a}(\mathbf{y}_p,\tau_p)$, $B_i(\mathbf{y}_p,\tau_p)$, $G_0^{(i)}B_j(\mathbf{y}_p,\tau_p)$, $\ldots$ correspondingly. Moreover, the operators L and $G_0^{(i)}$, $i=1,\ldots,m$ are determined by the equalities (4.17), (4.18).

Scheme with strong order 1.0 (Milstein Scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1}\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^m G_0^{(i_2)}B_{i_1}\hat{I}^{(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}. \qquad (4.65)$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1}\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^m G_0^{(i_2)}B_{i_1}\hat{I}^{(i_2i_1)}_{00_{\tau_{p+1},\tau_p}} +$$

$$+\sum_{i_1=1}^m\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}^{(i_1)}_{0_{\tau_{p+1},\tau_p}} + \hat{I}^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right) - LB_{i_1}\hat{I}^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right] +$$

$$+\sum_{i_1,i_2,i_3=1}^m G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}} +$$

$$
+\frac{\Delta^2}{2}L\mathbf{a}. \tag{4.66}
$$

Scheme with strong order 2.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4 i_3 i_2 i_1)}. \tag{4.67}
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+
$$

$$
+\frac{1}{2}LLB_{i_1}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\Bigr)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-
$$

$$
-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{00000_{\tau_{p+1},\tau_p}}^{(i_5i_4i_3i_2i_1)}+
$$

$$
+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{4.68}
$$

Scheme with strong order 3.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \qquad (4.69)
$$

where

$$
\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+
$$

$$
+\frac{1}{2}LLB_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\Bigr)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-
$$

$$
-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{00000_{\tau_{p+1},\tau_p}}^{(i_5i_4i_3i_2i_1)}+
$$

$$
+\frac{\Delta^3}{6}LL\mathbf{a},
$$

and

$$
\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{\Delta^2}{2}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+
$$

$$
+\frac{1}{2}LLG_0^{(i_2)}B_{i_1}\hat{I}_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+G_0^{(i_2)}LG_0^{(i_1)}\mathbf{a}\left(\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right)+
$$

$$
+LG_0^{(i_2)}LB_{i_1}\left(\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+
$$

$$
+G_0^{(i_2)}LLB_{i_1}\left(\frac{1}{2}\hat{I}_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{1}{2}\hat{I}_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-
$$

$$
\left.-LG_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\hat{I}_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}\left[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\hat{I}_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+\right.
$$

$$
+G_0^{(i_4)}G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat{I}_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)-
$$

$$
-LG_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+
$$

$$
+G_0^{(i_4)}LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+
$$

$$
\left.+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000000_{\tau_{p+1},\tau_p}}^{(i_6i_5i_4i_3i_2i_1)}.
$$

It is well known [67] that under the standard conditions the numerical schemes (4.65)–(4.69) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Itô stochastic integrals from the numerical schemes (4.65)–(4.69) [67] (also see [13])

$$
\mathsf{M}\left\{\left(I_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{(i_1\ldots i_k)}-\hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^{r+1},
$$

where constant C is independent of Δ and $r/2$ is the strong convergence orders for the numerical schemes (4.65)–(4.69), i.e. $r/2=1.0, 1.5, 2.0, 2.5,$ and 3.0.

As we mentioned above, the numerical schemes (4.65)–(4.69) are unrealizable in practice without procedures for the numerical simulation of iterated Itô stochastic integrals from (4.63).

In Chapter 5 we give an extensive material on the mean-square approximation of specific iterated Itô stochastic integrals from the numerical schemes (4.65)–(4.69). The mentioned material based on the results of Chapter 1.

4.11 Application of First Form of the Unified Taylor–Stratonovich Expansion to the High-Order Strong Numerical Methods for Itô SDEs

Let us rewrite (4.55) for all $s, t \in [0, T]$ such that $s > t$ in the following from

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d\in \mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t) + \left(\bar{H}_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \qquad (4.70)$$

where

$$\left(\bar{H}_{r+1}\right)_{s,t} = \left(H_{r+1}\right)_{s,t} - \mathbf{1}_{\{r=2d-1, d\in \mathbf{N}\}} \frac{(s-t)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} R(\mathbf{x}_t, t).$$

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0\le j\le N-1} |\tau_{j+1} - \tau_j|.$$

From (4.70) for $s = \tau_{p+1},\ t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (4.1), which is based on first form of the unified Taylor–Stratonovich expansion

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(\tau_{p+1}-\tau_p)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j \mathbf{y}_p \ \hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d\in \mathbf{N}\}} \frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \qquad (4.71)$$

where $\hat{I}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)}$ is an approximation of iterated Stratonovich stochastic integral $I_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}^{*(i_1\ldots i_k)}$ of the form

$$I_{l_1\ldots l_{k_{s,t}}}^{*(i_1\ldots i_k)} = \int\limits_t^{*s} (t-t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}.$$

Note that we understand the equality (4.71) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $T = \tau_N$, $p = 0, 1, \ldots, N$.

It is known [67] that under the appropriate conditions the numerical scheme (4.71) has strong order of convergence $r/2$ ($r \in \mathbf{N}$).

Denote

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t) - \frac{1}{2}\sum_{j=1}^{m} G_0^{(j)} B_j(\mathbf{x}, t),$$

where $B_j(\mathbf{x}, t)$ is the jth column of the matrix function $B(\mathbf{x}, t)$.

It is not difficult to show that (see (4.41))

$$\bar{L}R(\mathbf{x}, t) = \frac{\partial R}{\partial t}(\mathbf{x}, t) + \sum_{j=1}^{n} \bar{\mathbf{a}}^{(j)}(\mathbf{x}, t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x}, t), \tag{4.72}$$

where $\bar{\mathbf{a}}^{(j)}(\mathbf{x}, t)$ is the jth component of the vector function $\bar{\mathbf{a}}(\mathbf{x}, t)$.

Below we consider particular cases of the numerical scheme (4.71) for $r = 2, 3, 4, 5$, and 6, i.e. explicit one-step strong numerical schemes for the Itô SDE (4.1) with the convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. At that, for simplicity we will write $\bar{\mathbf{a}}$, $\bar{L}\bar{\mathbf{a}}$, $L\mathbf{a}$, B_i, $G_0^{(i)} B_j, \ldots$ instead of $\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $\bar{L}\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $L\mathbf{a}(\mathbf{y}_p, \tau_p)$, $B_i(\mathbf{y}_p, \tau_p)$, $G_0^{(i)} B_j(\mathbf{y}_p, \tau_p)$, $\ldots$ correspondingly. Moreover, the operators $\bar{L}$ and $G_0^{(i)}$, $i = 1, \ldots, m$ are determined by the equalities (4.17), (4.18), and (4.72).

Scheme with strong order 1.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} B_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)}. \tag{4.73}$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} B_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} +$$

$$+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)}\bar{\mathbf{a}} \left(\Delta \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} \right) - \bar{L} B_{i_1} \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} \right] +$$

$$
+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+
$$

$$
+\frac{\Delta^2}{2}L\mathbf{a}. \qquad (4.74)
$$

Scheme with strong order 2.0

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat I_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat I_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat I_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar L B_{i_1}\hat I_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar L\bar{\mathbf{a}}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_2)}\bar L B_{i_1}\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar L G_0^{(i_2)}B_{i_1}\hat I_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\right.
$$

$$
\left.+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat I_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat I_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\right]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}. \qquad (4.75)
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat I_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat I_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat I_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar L B_{i_1}\hat I_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar L\bar{\mathbf{a}}+
$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)-$$

$$-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{00000_{\tau_{p+1},\tau_p}}^{*(i_5i_4i_3i_2i_1)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{4.76}$$

Scheme with strong order 3.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \tag{4.77}$$

where

$$\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{00000_{\tau_{p+1},\tau_p}}+$$

$$+\frac{\Delta^3}{6}\bar{L}\bar{L}\bar{\mathbf{a}},$$

and

$$\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_2)}\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)\right)+$$

$$+\bar{L}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_2)}\bar{L}\bar{L}B_{i_1}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\frac{1}{2}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)-$$

$$\left.-\bar{L}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\left[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)+\right.$$

$$+G_0^{(i_4)}G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_4)}\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}\right)+$$

$$\left.+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_6i_5i_4i_3i_2i_1)}_{000000_{\tau_{p+1},\tau_p}}.$$

It is well known [67] that under the standard conditions the numerical schemes (4.73)–(4.77) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from the numerical schemes (4.73)–(4.77) [67] (also see [13])

$$\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}-\hat{I}^{*(i_1\ldots i_k)}_{l_1\ldots l_{k_{\tau_{p+1},\tau_p}}}\right)^2\right\}\le C\Delta^{r+1},$$

where constant C is independent of Δ and $r/2$ is the strong convergence orders for the numerical schemes (4.73)–(4.77), i.e. $r/2=1.0, 1.5, 2.0, 2.5,$ and 3.0.

As we mentioned above, the numerical schemes (4.73)–(4.77) are unrealizable in practice without procedures for the numerical simulation of iterated Stratonovich stochastic integrals from (4.70).

In Chapter 5 we give an extensive material on the mean-square approximation of specific iterated Itô and Stratonovich stochastic integrals from the numerical schemes (4.65)–(4.69), (4.73)–(4.77). The mentioned material based on the results of Chapters 1 and 2.

Chapter 5

Mean-Square Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of Multiplicities 1 to 6 from the Taylor–Itô and Taylor–Stratonovich Expansions Based on Theorems 1.1, 2.1–2.10

5.1 Mean-Square Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of multiplicities 1 to 6 Based on Legendre Polynomials

This section is devoted to the extensive practical material on expansions and mean-square approximations of specific iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 on the base of Theorems 1.1, 2.1–2.10 and multiple Fourier–Legendre series. The considered iterated Itô and Stratonovich stochastic integrals are part of the stochastic Taylor expansions (Taylor–Itô and Taylor–Stratonovich expansions). Therefore, the results of this section can be useful for the numerical solution of Itô SDEs with non-commutative noise.

Consider the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.1}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.2}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes; $i_1,\ldots,i_k=0,1,\ldots,m.$

As we saw in Chapter 4, $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ in (5.1), (5.2) if we consider the iterated stochastic integrals from the classical Taylor–Itô and Taylor–Stratonovich expansions [67]. At the same time $\psi_l(\tau)\equiv(t-\tau)^{q_l}$ $(l=1,\ldots,k,\ q_1,\ldots,q_k=0,1,2,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ for the iterated stochastic integrals from the unified Taylor–Itô and Taylor–Stratonovich expansions [1]-[14], [50], [120], [121].

Thus, in this section, we will consider the following collections of iterated Itô and Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{5.3}$$

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{5.4}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ l_1,\ldots,l_k=0,1,\ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{5.5}$$

where

$$P_j(x)=\frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j$$

is the Legendre polynomial.

Let us recall some properties of Legendre polynomials [88] (see Sect. 2.1.2)

$$P_j(1)=1,\quad P_{j+1}(-1)=-P_j(-1),\quad j=0,1,2,\ldots,$$

$$\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),$$

$$xP_j(x)=\frac{(j+1)P_{j+1}(x)+jP_{j-1}(x)}{2j+1},\quad j=1,2,\ldots,$$

$$\int\limits_{-1}^{1} x^k P_j(x)dx = 0, \quad k = 0, 1, \ldots, j-1,$$

$$\int\limits_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if} \ \ k \ne j \\ \\ 2/(2j+1) & \text{if} \ \ k = j \end{cases},$$

$$P_n(x)P_m(x) = \sum_{k=0}^{m} K_{m,n,k} P_{n+m-2k}(x),$$

where

$$K_{m,n,k} = \frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}} \cdot \frac{2n+2m-4k+1}{2n+2m-2k+1}, \quad a_k = \frac{(2k-1)!!}{k!}, \quad m \le n.$$

Using the above properties of the Legendre polynomial system (5.5), Theorems 1.1, 2.1–2.10, and Hypotheses 2.1–2.3, we obtain the following expansions of iterated Itô and Stratonovich stochastic integrals from the sets (5.3), (5.4)

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.6}$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.7}$$

$$I_{(2)T,t}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{5.8}$$

$$I_{(00)T,t}^{*(i_1i_2)} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{5.9}$$

$$I_{(00)T,t}^{(i_1i_2)} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.10}$$

$$I_{(01)T,t}^{*(i_1i_2)} = -\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)} - \frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)} +$$

$$+\sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr), \tag{5.11}$$

$$
I_{(10)T,t}^{*(i_1i_2)} = -\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)} - \frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+
$$

$$
+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr) \qquad (5.12)
$$

or

$$
I_{(01)T,t}^{*(i_1i_2)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p} C_{j_2j_1}^{01}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
I_{(10)T,t}^{*(i_1i_2)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p} C_{j_2j_1}^{10}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

where

$$
C_{j_2j_1}^{01} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2j_1}^{01},
$$

$$
C_{j_2j_1}^{10} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2j_1}^{10},
$$

$$
\bar{C}_{j_2j_1}^{01} = -\int_{-1}^{1}(1+y)P_{j_2}(y)\int_{-1}^{y}P_{j_1}(x)dxdy,
$$

$$
\bar{C}_{j_2j_1}^{10} = -\int_{-1}^{1}P_{j_2}(y)\int_{-1}^{y}(1+x)P_{j_1}(x)dxdy;
$$

$$
I_{(10)T,t}^{(i_1i_2)} = I_{(10)T,t}^{*(i_1i_2)} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2 \quad \text{w. p. 1},
$$

$$
I_{(01)T,t}^{(i_1i_2)} = I_{(01)T,t}^{*(i_1i_2)} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2 \quad \text{w. p. 1},
$$

$$
I_{(01)T,t}^{(i_1i_2)} = -\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)} - \frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+
$$

$$
+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr), \qquad (5.13)
$$

$$
I_{(10)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)} - \frac{(T-t)^2}{4} \Biggl(\frac{1}{\sqrt{3}} \zeta_0^{(i_2)} \zeta_1^{(i_1)} +
$$

$$
+ \sum_{i=0}^{\infty} \Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)} \zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)} \zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)} \zeta_i^{(i_2)}}{(2i-1)(2i+3)} \Biggr) \Biggr) \tag{5.14}
$$

or

$$
I_{(01)T,t}^{(i_1 i_2)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1}^{01} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2\}} \mathbf{1}_{\{j_1 = j_2\}} \Biggr),
$$

$$
I_{(10)T,t}^{(i_1 i_2)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1}^{10} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2\}} \mathbf{1}_{\{j_1 = j_2\}} \Biggr),
$$

$$
I_{(000)T,t}^{*(i_1 i_2 i_3)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{5.15}
$$

$$
I_{(000)T,t}^{(i_1 i_2 i_3)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1 = i_2\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} -
$$

$$
- \mathbf{1}_{\{i_2 = i_3\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{5.16}
$$

$$
I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6} (T-t)^{3/2} \Biggl(\left(\zeta_0^{(i_1)} \right)^3 - 3 \zeta_0^{(i_1)} \Biggr) \quad \text{w. p. 1},
$$

$$
I_{(000)T,t}^{*(i_1 i_1 i_1)} = \frac{1}{6} (T-t)^{3/2} \left(\zeta_0^{(i_1)} \right)^3 \quad \text{w. p. 1},
$$

where

$$
C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8} (T-t)^{3/2} \bar{C}_{j_3 j_2 j_1}, \tag{5.17}
$$

$$
\bar{C}_{j_3 j_2 j_1} = \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy dz; \tag{5.18}
$$

here and further in this section

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}\quad (i=1,\ldots,m,\ j=0,1,\ldots)$$

are independent standard Gaussian random variables for various i or j;

$$I_{(000)T,t}^{(i_1i_2i_3)}=I_{(000)T,t}^{*(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}I_{(1)T,t}^{(i_3)}-$$
$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left((T-t)I_{(0)T,t}^{(i_1)}+I_{(1)T,t}^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(02)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(01)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$
$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$
$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr],\tag{5.19}$$

$$I_{(20)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(10)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$
$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$
$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr],\tag{5.20}$$

$$I_{(11)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$
\left.\left.+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] \tag{5.21}
$$

or

$$
I_{(02)T,t}^{*(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{02}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
I_{(20)T,t}^{*(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{20}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

$$
I_{(11)T,t}^{*(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{11}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},
$$

where

$$
C_{j_2j_1}^{02}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{02},
$$

$$
C_{j_2j_1}^{20}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{20},
$$

$$
C_{j_2j_1}^{11}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{11},
$$

$$
\bar{C}_{j_2j_1}^{02}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)^2\int\limits_{-1}^{y}P_{j_1}(x)dxdy,
$$

$$
\bar{C}_{j_2j_1}^{20}=\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)^2dxdy,
$$

$$
\bar{C}_{j_2j_1}^{11}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdy,
$$

$$
I_{(11)T,t}^{*(i_1 i_1)}=\frac{1}{2}\left(I_{(1)T,t}^{(i_1)}\right)^2\quad\text{w. p. 1},
$$

$$
I_{(02)T,t}^{(i_1 i_2)}=I_{(02)T,t}^{*(i_1 i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3\quad\text{w. p. 1}, \tag{5.22}
$$

$$I_{(20)T,t}^{(i_1 i_2)} = I_{(20)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1}, \tag{5.23}$$

$$I_{(11)T,t}^{(i_1 i_2)} = I_{(11)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$I_{(02)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - (T-t) I_{01_{T,t}}^{(i_1 i_2)} + \frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} +$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr] -$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \tag{5.24}$$

$$I_{(20)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - (T-t) I_{(10)T,t}^{(i_1 i_2)} + \frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} +$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr] -$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \tag{5.25}$$

$$I_{(11)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - \frac{T-t}{2}\left(I_{(10)T,t}^{(i_1 i_2)} + I_{(01)T,t}^{(i_1 i_2)}\right) +$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$
+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_{i}^{(i_1)}-\zeta_{i}^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-
$$

$$
-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \tag{5.26}
$$

or

$$
I_{(02)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{02}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),
$$

$$
I_{(20)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{20}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),
$$

$$
I_{(11)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{11}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),
$$

$$
I_{(3)T,t}^{(i_1)}=-\frac{(T-t)^{7/2}}{4}\Biggl(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\Biggr), \tag{5.27}
$$

$$
I_{(0000)T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},
$$

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{5.28}
$$

$$I_{(0000)T,t}^{(i_1 i_1 i_1 i_1)} = \frac{1}{24}(T-t)^2 \left(\left(\zeta_0^{(i_1)}\right)^4 - 6\left(\zeta_0^{(i_1)}\right)^2 + 3 \right) \quad \text{w. p. 1},$$

$$I_{(0000)T,t}^{*(i_1 i_1 i_1 i_1)} = \frac{1}{24}(T-t)^2 \left(\zeta_0^{(i_1)}\right)^4 \quad \text{w. p. 1},$$

where

$$C_{j_4 j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2 \bar{C}_{j_4 j_3 j_2 j_1}, \tag{5.29}$$

$$\bar{C}_{j_4 j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du; \tag{5.30}$$

$$I_{(001)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{001} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$I_{(010)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{010} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$I_{(100)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{100} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$I_{(001)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{001} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{5.31}$$

$$I_{(010)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{5.32}$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{5.33}$$

where

$$C_{j_3 j_2 j_1}^{001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{001},$$

$$C_{j_3 j_2 j_1}^{010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{010},$$

$$C_{j_3 j_2 j_1}^{100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{100},$$

$$\bar{C}_{j_3 j_2 j_1}^{100} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)(x+1) dx dy dz,$$

$$\bar{C}_{j_3 j_2 j_1}^{010} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y)(y+1) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}_{j_3 j_2 j_1}^{001} = -\int\limits_{-1}^{1} P_{j_3}(z)(z+1) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz;$$

$$I_{(lll)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6} \left(\left(I_{(l)T,t}^{(i_1)} \right)^3 - 3 I_{(l)T,t}^{(i_1)} \Delta_{l(T,t)} \right) \quad \text{w. p. 1},$$

$$I_{(lll)T,t}^{*(i_1 i_1 i_1)} = \frac{1}{6} \left(I_{(l)T,t}^{(i_1)} \right)^3 \quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{(i_1 i_1 i_1 i_1)} = \frac{1}{24} \left(\left(I_{(l)T,t}^{(i_1)} \right)^4 - 6 \left(I_{(l)T,t}^{(i_1)} \right)^2 \Delta_{(l)T,t} + 3 \left(\Delta_{(l)T,t} \right)^2 \right) \quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{*(i_1 i_1 i_1 i_1)} = \frac{1}{24} \left(I_{(l)T,t}^{(i_1)} \right)^4 \quad \text{w. p. 1},$$

where

$$
I_{(l)T,t}^{(i_1)}=\sum_{j=0}^{l} C_j^l \zeta_j^{(i_1)}\quad \text{w. p. 1,} \tag{5.34}
$$

$$
\Delta_{l(T,t)}=\int\limits_t^T (t-s)^{2l}ds,\quad C_j^l=\int\limits_t^T (t-s)^l\phi_j(s)ds;
$$

$$
I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},
$$

$$
\begin{aligned}
I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{aligned} \tag{5.35}
$$

$$
I_{(00000)T,t}^{(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right)\quad \text{w. p. 1,}
$$

$$I_{(00000)T,t}^{*(i_1 i_1 i_1 i_1 i_1)} = \frac{1}{120}(T-t)^{5/2}\left(\zeta_0^{(i_1)}\right)^5 \quad \text{w. p. 1},$$

where

$$C_{j_5 j_4 j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar{C}_{j_5 j_4 j_3 j_2 j_1},$$

$$\bar{C}_{j_5 j_4 j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_5}(v)\int\limits_{-1}^{v} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdudv;$$

$$I_{(0001)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0010)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0100)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(1000)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$
I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{1000} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

where

$$C_{j_4j_3j_2j_1}^{0001}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar C_{j_4j_3j_2j_1}^{0001},$$

$$C_{j_3j_2j_1}^{0010}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar C_{j_4j_3j_2j_1}^{0010},$$

$$C_{j_4j_3j_2j_1}^{0100}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar C_{j_3j_2j_1}^{0100},$$

$$C_{j_4j_3j_2j_1}^{1000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar C_{j_4j_3j_2j_1}^{1000},$$

$$\bar C_{j_4j_3j_2j_1}^{1000}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydz,$$

$$\bar C_{j_4j_3j_2j_1}^{0100}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar C_{j_4j_3j_2j_1}^{0010}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar C_{j_4j_3j_2j_1}^{0001}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz;$$

$$I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)},$$

$$I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$

$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$

$$
\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),
$$

$$
I_{(000000)T,t}^{(i_1 i_1 i_1 i_1 i_1 i_1)}=\frac{1}{720}(T-t)^3\left(\left(\zeta_0^{(i_1)}\right)^6-15\left(\zeta_0^{(i_1)}\right)^4+45\left(\zeta_0^{(i_1)}\right)^2-15\right)\quad \text{w. p. 1},
$$

$$
I_{(000000)T,t}^{*(i_1 i_1 i_1 i_1 i_1 i_1)}=\frac{1}{720}(T-t)^3\left(\zeta_0^{(i_1)}\right)^3\quad \text{w. p. 1},
$$

where

$$
C_{j_6 j_5 j_4 j_3 j_2 j_1} =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3 \bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1},
$$

$$
\bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1} =
$$

$$
= \int\limits_{-1}^{1} P_{j_6}(w) \int\limits_{-1}^{w} P_{j_5}(v) \int\limits_{-1}^{v} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du dv dw.
$$

It should be noted that instead of the expansion (5.15) we can consider the following expansion, which is derived by direct calculation

$$
I_{(000)T,t}^{*(i_1 i_2 i_3)} = -\frac{1}{T-t}\left(I_{(0)T,t}^{(i_3)} I_{(10)T,t}^{*(i_2 i_1)} + I_{(0)T,t}^{(i_1)} I_{(10)T,t}^{*(i_2 i_3)}\right) + \frac{1}{2} I_{(0)T,t}^{(i_3)}\left(I_{(00)T,t}^{*(i_1 i_2)} - I_{(00)T,t}^{*(i_2 i_1)}\right) -
$$

$$
-(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)} + \sqrt{3}\zeta_1^{(i_2)} - \frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right) + \frac{1}{4} D_{T,t}^{(i_1 i_2 i_3)}\right), \qquad (5.36)
$$

where

$$
D_{T,t}^{(i_1 i_2 i_3)} = \sum_{\substack{i=1,\ j=0,\ k=i \\ 2i \ge k+i-j \ge -2;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{i+1,k+1,\frac{k+i-j}{2}+1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} +
$$

$$
+ \sum_{\substack{i=1,\ j=0,\ 1 \le k \le i-1 \\ 2k \ge k+i-j \ge -2;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{k+1,i+1,\frac{k+i-j}{2}+1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -
$$

$$
- \sum_{\substack{i=1,\ j=0,\ k=i+2 \\ 2i+2 \ge k+i-j \ge 0;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{i+1,k-1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -
$$

$$
- \sum_{\substack{i=1,\ j=0,\ 1 \le k \le i+1 \\ 2k-2 \ge k+i-j \ge 0;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{k-1,i+1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -
$$

$$
- \sum_{\substack{i=1,\ j=0,\ k=i-2, k \ge 1 \\ 2i-2 \ge k+i-j \ge 0;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{i-1,k+1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -
$$

$$
- \sum_{\substack{i=1,\ j=0,\ 1 \le k \le i-3 \\ 2k+2 \ge k+i-j \ge 0;\ k+i-j\ \text{--even}}}^{\infty} N_{ijk} K_{k+1,i-1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} +
$$

$$
+\sum_{\substack{i=1,\ j=0,\ k=i\\ 2i\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty} N_{ijk}K_{i-1,k-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+
$$

$$
+\sum_{\substack{i=1,\ j=0\ 1\le k\le i-1\\ 2k\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty} N_{ijk}K_{k-1,i-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)},
$$

where

$$
N_{ijk}=\sqrt{\frac{1}{(2k+1)(2j+1)(2i+1)}}\ ,
$$

$$
K_{m,n,k}=\frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1},\quad a_k=\frac{(2k-1)!!}{k!},\quad m\le n.
$$

However, as we will see further the expansion (5.16) is more convenient for the practical implementation then (5.36).

Also note the following relation between iterated Itô and Stratonovich stochastic integrals

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)}=I_{(0000)T,t}^{*(i_1i_2i_3i_4)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}I_{(10)T,t}^{*(i_3i_4)}-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I_{(10)T,t}^{*(i_1i_4)}-I_{(01)T,t}^{*(i_1i_4)}\right)-
$$

$$
-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left((T-t)I_{(00)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+\frac{1}{8}(T-t)^2\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}}\quad \text{w. p. 1.}
$$

Let us denote as

$$
I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)q},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}
$$

the approximations of iterated Itô and Stratonovich stochastic integrals

$$
I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)},\quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}
$$

defined by (5.3), (5.4), i.e. we replace ∞ on q in the expansions of these stochastic integrals. For example, $I_{(00)T,t}^{*(i_1i_2)q}$ is the approximation of the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1i_2)}$ obtained from (5.9) by replacing ∞ on q, etc.

It is easy to prove that

$$
\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2).\quad (5.37)
$$

Moreover, using Theorem 1.3, we obtain for $i_1 \ne i_2$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\right.
$$

$$
\left.-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \qquad (5.38)
$$

For the case $i_1 = i_2$ using Theorem 1.3, we have

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_1)}-I_{(10)T,t}^{(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_1)}-I_{(01)T,t}^{(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \qquad (5.39)
$$

In Tables 5.1–5.3 we have calculations according to the formulas (5.37)–(5.39) for various values of q. In the given tables ε means the right-hand sides of these formulas. Obviously, these results are consistent with the estimate (1.187).

Let us consider (5.11), (5.12) for $i_1 = i_2$

$$
I_{(01)T,t}^{*(i_1i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.
$$

$$
\left.+\sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}-\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right), \qquad (5.40)
$$

$$
I_{(10)T,t}^{*(i_1i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.
$$

$$
\left.+\sum_{i=0}^{\infty}\left(-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}+\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right). \qquad (5.41)
$$

Table 5.1: Confirmation of the formula (5.37)

$2\varepsilon/(T-t)^2$	0.1667	0.0238	0.0025	$2.4988 \cdot 10^{-4}$	$2.4999 \cdot 10^{-5}$
q	1	10	100	1000	10000

Table 5.2: Confirmation of the formula (5.38)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450 \cdot 10^{-4}$	$6.2495 \cdot 10^{-5}$
q	1	10	100	1000	10000

From (5.40), (5.41), considering (5.6) and (5.7), we obtain

$$I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{2}\left(\left(\zeta_0^{(i_1)}\right)^2 + \frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}\right) = I_{(0)T,t}^{(i_1)} I_{(1)T,t}^{(i_1)} \quad \text{w. p. 1.} \tag{5.42}$$

Obtaining (5.42) we supposed that the formulas (5.11), (5.12) are valid w. p. 1. The complete proof of this fact is given in Sect. 1.7.2 (Theorem 1.10).

Note that it is easy to obtain the equality (5.42) using the Itô formula and standard relations between iterated Itô and Stratonovich stochastic integrals.

Using the Itô formula, we obtain

$$I_{(11)T,t}^{*(i_1 i_1)} = \frac{\left(I_{(1)T,t}^{(i_1)}\right)^2}{2} \quad \text{w. p. 1.} \tag{5.43}$$

In addition, using the Itô formula, we have

$$I_{(20)T,t}^{(i_1 i_1)} + I_{(02)T,t}^{(i_1 i_1)} = I_{(0)T,t}^{(i_1)} I_{(2)T,t}^{(i_1)} - \frac{(T-t)^3}{3} \quad \text{w. p. 1.} \tag{5.44}$$

From (5.44), considering the formulas (5.22), (5.23), we obtain

$$I_{(20)T,t}^{*(i_1 i_1)} + I_{(02)T,t}^{*(i_1 i_1)} = I_{(0)T,t}^{(i_1)} I_{(2)T,t}^{(i_1)} \quad \text{w. p. 1.} \tag{5.45}$$

Let us check whether the formulas (5.43), (5.45) follow from (5.19)–(5.21),

Table 5.3: Confirmation of the formula (5.39)

$16\varepsilon/(T-t)^4$	0.0070	$4.3551 \cdot 10^{-5}$	$6.0076 \cdot 10^{-8}$	$6.2251 \cdot 10^{-11}$	$6.3178 \cdot 10^{-14}$
q	1	10	100	1000	10000

if we suppose $i_1 = i_2$ in the last ones. From (5.19)–(5.21) for $i_1 = i_2$ we get

$$I_{(20)T,t}^{*(i_1 i_1)} + I_{(02)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{2} I_{(00)T,t}^{*(i_1 i_1)} - (T-t)\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) +$$
$$+ \frac{(T-t)^3}{4}\left(\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2 + \frac{2}{3\sqrt{5}}\zeta_2^{(i_1)}\zeta_0^{(i_1)}\right), \tag{5.46}$$

$$I_{(11)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{*(i_1 i_1)} - \frac{T-t}{2}\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) + \frac{(T-t)^3}{24}\left(\zeta_1^{(i_1)}\right)^2. \tag{5.47}$$

It is easy to see that considering (5.42) and (5.6)–(5.9), we actually obtain the equalities (5.43) and (5.45) from (5.46) and (5.47). This fact indirectly confirms the correctness of the formulas (5.19)–(5.21).

Obtaining (5.43), (5.45) we supposed that the formulas (5.19)–(5.21) are valid w. p. 1. The complete proof of this fact is given in Sect. 1.7.2 (Theorem 1.10).

On the basis of the presented expansions of iterated stochastic integrals we can see that increasing of multiplicities of these integrals or degree indices of their weight functions leads to noticeable complication of formulas for the mentioned expansions.

However, increasing of the mentioned parameters leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated stochastic integrals. This leads to sharp decrease of member quantities in expansions of iterated stochastic integrals, which are required for achieving the acceptable accuracy of approximation. In this context, let us consider the approach to the approximation of iterated stochastic integrals, which provides a possibility to obtain the mean-square approximations of the required accuracy without using the complex expansions like (5.36).

Let us analyze the following approximation of iterated Itô stochastic integral of multiplicity 3 using (5.16)

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.48}$$

where $C_{j_3 j_2 j_1}$ is defined by (5.17), (5.18).

In particular, from (5.48) for $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we obtain

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}. \tag{5.49}$$

Furthermore, using Theorem 1.3 for $k=3$, we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 \quad (i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3), \tag{5.50}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3), \tag{5.51}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \quad (i_1 = i_3 \neq i_2), \tag{5.52}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3). \tag{5.53}$$

From the other hand, from Theorem 1.4 for $k=3$ we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2\right), \tag{5.54}$$

where $i_1, i_2, i_3 = 1, \ldots, m$.

We can act similarly with more complicated iterated stochastic integrals. For example, for the approximation of stochastic integral $I_{(0000)T,t}^{(i_1i_2i_3i_4)}$ we can write (see (5.28))

$$
\begin{gathered}
I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}=\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
\end{gathered}
\tag{5.55}
$$

where $C_{j_4j_3j_2j_1}$ is defined by (5.29), (5.30).

Moreover, according to Theorem 1.4 for $k=4$, we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}\le 24\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}^2\right),
$$

where $i_1,i_2,i_3,i_4=1,\ldots,m$.

For pairwise different $i_1,i_2,i_3,i_4=1,\ldots,m$ from Theorem 1.3 we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}^2. \tag{5.56}
$$

Using Theorem 1.3, we can calculate exactly the left-hand side of (5.56) for any possible combinations of i_1,i_2,i_3,i_4. These relations were obtained in Sect. 1.2. For example

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)}C_{j_4j_3j_2j_1}\right)\right),
$$

where $i_1=i_2\ne i_3=i_4$ and

$$
\sum_{(j_1,j_2)}
$$

means the sum with respect to permutations (j_1,j_2).

Table 5.4: Coefficients $\bar{C}_{0j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{4}{3}$	$\frac{-2}{3}$	$\frac{2}{15}$	0	0	0	0
$j_2=1$	0	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{4}{105}$	0	0	0
$j_2=2$	$\frac{-4}{15}$	$\frac{2}{15}$	$\frac{2}{105}$	$\frac{-2}{35}$	$\frac{2}{105}$	0	0
$j_2=3$	0	$\frac{-2}{35}$	$\frac{2}{35}$	$\frac{2}{315}$	$\frac{-2}{63}$	$\frac{8}{693}$	0
$j_2=4$	0	0	$\frac{-8}{315}$	$\frac{2}{63}$	$\frac{2}{693}$	$\frac{-2}{99}$	$\frac{10}{1287}$
$j_2=5$	0	0	0	$\frac{-10}{693}$	$\frac{2}{99}$	$\frac{2}{1287}$	$\frac{-2}{143}$
$j_2=6$	0	0	0	0	$\frac{-4}{429}$	$\frac{2}{143}$	$\frac{2}{2145}$

Table 5.5: Coefficients $\bar{C}_{1j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{2}{3}$	$\frac{-4}{15}$	0	$\frac{2}{105}$	0	0	0
$j_2=1$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2=2$	$\frac{-2}{15}$	$\frac{8}{105}$	0	$\frac{-2}{105}$	0	$\frac{4}{1155}$	0
$j_2=3$	$\frac{-2}{35}$	0	$\frac{8}{315}$	0	$\frac{-38}{3465}$	0	$\frac{20}{9009}$
$j_2=4$	0	$\frac{-4}{315}$	0	$\frac{46}{3465}$	0	$\frac{-64}{9009}$	0
$j_2=5$	0	0	$\frac{-4}{693}$	0	$\frac{74}{9009}$	0	$\frac{-32}{6435}$
$j_2=6$	0	0	0	$\frac{-10}{3003}$	0	$\frac{4}{715}$	0

Table 5.6: Coefficients $\bar{C}_{2j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2=1$	$\frac{2}{15}$	$\frac{-4}{105}$	0	$\frac{-2}{315}$	0	$\frac{8}{3465}$	0
$j_2=2$	$\frac{2}{105}$	0	0	0	$\frac{-2}{495}$	0	$\frac{4}{3003}$
$j_2=3$	$\frac{-2}{35}$	$\frac{8}{315}$	0	$\frac{-2}{3465}$	0	$\frac{-116}{45045}$	0
$j_2=4$	$\frac{-8}{315}$	0	$\frac{4}{495}$	0	$\frac{-2}{6435}$	0	$\frac{-16}{9009}$
$j_2=5$	0	$\frac{-4}{693}$	0	$\frac{38}{9009}$	0	$\frac{-8}{45045}$	0
$j_2=6$	0	0	$\frac{-8}{3003}$	0	$\frac{118}{45045}$	0	$\frac{-4}{36465}$

Table 5.7: Coefficients $\bar{C}_{3j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	$\frac{2}{105}$	0	$\frac{-4}{315}$	0	$\frac{2}{693}$	0
$j_2=1$	$\frac{4}{105}$	0	$\frac{-2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{10}{9009}$
$j_2=2$	$\frac{2}{35}$	$\frac{-2}{105}$	0	$\frac{4}{3465}$	0	$\frac{-74}{45045}$	0
$j_2=3$	$\frac{2}{315}$	0	$\frac{-2}{3465}$	0	$\frac{16}{45045}$	0	$\frac{-10}{9009}$
$j_2=4$	$\frac{-2}{63}$	$\frac{46}{3465}$	0	$\frac{-32}{45045}$	0	$\frac{2}{9009}$	0
$j_2=5$	$\frac{-10}{693}$	0	$\frac{38}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{122}{765765}$
$j_2=6$	0	$\frac{-10}{3003}$	0	$\frac{20}{9009}$	0	$\frac{-226}{765765}$	0

Table 5.8: Coefficients $\bar{C}_{4j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	$\frac{2}{315}$	0	$\frac{-4}{693}$	0	$\frac{2}{1287}$
$j_2=1$	0	$\frac{2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{-10}{9009}$	0
$j_2=2$	$\frac{2}{105}$	0	$\frac{-2}{495}$	0	$\frac{4}{6435}$	0	$\frac{-38}{45045}$
$j_2=3$	$\frac{2}{63}$	$\frac{-38}{3465}$	0	$\frac{16}{45045}$	0	$\frac{2}{9009}$	0
$j_2=4$	$\frac{2}{693}$	0	$\frac{-2}{6435}$	0	0	0	$\frac{2}{13923}$
$j_2=5$	$\frac{-2}{99}$	$\frac{74}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{-2}{153153}$	0
$j_2=6$	$\frac{-4}{429}$	0	$\frac{118}{45045}$	0	$\frac{-4}{13923}$	0	$\frac{-2}{188955}$

Table 5.9: Coefficients $\bar{C}_{5j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	$\frac{2}{693}$	0	$\frac{-4}{1287}$	0
$j_2=1$	0	0	$\frac{8}{3465}$	0	$\frac{-10}{9009}$	0	$\frac{-4}{6435}$
$j_2=2$	0	$\frac{4}{1155}$	0	$\frac{-74}{45045}$	0	$\frac{16}{45045}$	0
$j_2=3$	$\frac{8}{693}$	0	$\frac{-116}{45045}$	0	$\frac{2}{9009}$	0	$\frac{8}{58905}$
$j_2=4$	$\frac{2}{99}$	$\frac{-64}{9009}$	0	$\frac{2}{9009}$	0	$\frac{4}{153153}$	0
$j_2=5$	$\frac{2}{1287}$	0	$\frac{-8}{45045}$	0	$\frac{-2}{153153}$	0	$\frac{4}{415701}$
$j_2=6$	$\frac{-2}{143}$	$\frac{4}{715}$	0	$\frac{-226}{765765}$	0	$\frac{-8}{415701}$	0

Table 5.10: Coefficients $\bar{C}_{6j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	0	$\frac{2}{1287}$	0	$\frac{-4}{2145}$
$j_2=1$	0	0	0	$\frac{10}{9009}$	0	$\frac{-4}{6435}$	0
$j_2=2$	0	0	$\frac{4}{3003}$	0	$\frac{-38}{45045}$	0	$\frac{8}{36465}$
$j_2=3$	0	$\frac{20}{9009}$	0	$\frac{-10}{9009}$	0	$\frac{8}{58905}$	0
$j_2=4$	$\frac{10}{1287}$	0	$\frac{-16}{9009}$	0	$\frac{2}{13923}$	0	$\frac{4}{188955}$
$j_2=5$	$\frac{2}{143}$	$\frac{-32}{6435}$	0	$\frac{122}{765765}$	0	$\frac{4}{415701}$	0
$j_2=6$	$\frac{2}{2145}$	0	$\frac{-4}{36465}$	0	$\frac{-2}{188955}$	0	0

Table 5.11: Coefficients $\bar{C}_{00j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{3}$	$\frac{-2}{5}$	$\frac{2}{15}$
$j_2=1$	$\frac{-2}{15}$	$\frac{2}{15}$	$\frac{-2}{21}$
$j_2=2$	$\frac{-2}{15}$	$\frac{2}{35}$	$\frac{2}{105}$

Table 5.12: Coefficients $\bar{C}_{10j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{5}$	$\frac{-2}{9}$	$\frac{2}{35}$
$j_2=1$	$\frac{-2}{45}$	$\frac{2}{35}$	$\frac{-2}{45}$
$j_2=2$	$\frac{-2}{21}$	$\frac{2}{45}$	$\frac{2}{315}$

Table 5.13: Coefficients $\bar{C}_{02j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{15}$	$\frac{2}{21}$	$\frac{-4}{105}$
$j_2=1$	$\frac{2}{35}$	$\frac{-4}{105}$	$\frac{2}{105}$
$j_2=2$	$\frac{4}{105}$	$\frac{-2}{105}$	0

Table 5.14: Coefficients $\bar{C}_{01j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{45}$	$\frac{-2}{105}$	$\frac{2}{315}$
$j_2=2$	$\frac{-2}{35}$	$\frac{2}{63}$	$\frac{-2}{315}$

Table 5.15: Coefficients $\bar{C}_{11j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Assume that $q_1 = 6$. In Tables 5.4–5.10 we have the exact values of coefficients $\bar{C}_{j_3j_2j_1}$ $(j_1, j_2, j_3 = 0, 1, \ldots, 6)$. Here and further in this section the Fourier–Legendre coefficients have been calculated exactly using computer algebra system Derive. Note that in [51], [52] the database with 270,000 exactly calculated Fourier–Legendre coefficients was described. This database was used in the software package, which is written in Python programming language for the implementation of the numerical schemes (4.65)-(4.69), (4.73)-(4.77).

Calculating the value on the right-hand side of (5.50) for $q_1 = 6$ $(i_1 \neq i_2$, $i_1 \neq i_3$, $i_3 \neq i_2)$, we obtain the following approximate equality

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\} \approx 0.01956(T-t)^3.$$

Let us choose, for example, $q_2 = 2$. In Tables 5.11–5.19 we have the exact values of coefficients $\bar{C}_{j_4j_3j_2j_1}$ $(j_1, j_2, j_3, j_4 = 0, 1, 2)$. In the case of pairwise

Table 5.16: Coefficients $\bar{C}_{20j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 5.17: Coefficients $\bar{C}_{21j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{21}$	$\frac{-2}{45}$	$\frac{2}{315}$
$j_2=1$	$\frac{2}{315}$	$\frac{2}{315}$	$\frac{-2}{225}$
$j_2=2$	$\frac{-2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 5.18: Coefficients $\bar{C}_{12j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{35}$	$\frac{2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{63}$	$\frac{-2}{105}$	$\frac{2}{225}$
$j_2=2$	$\frac{2}{105}$	$\frac{-2}{225}$	$\frac{-2}{3465}$

different i_1, i_2, i_3, i_4 we obtain from (5.56) the following approximate equality

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}\approx 0.0236084(T-t)^4. \tag{5.57}$$

Let us analyze the following four approximations of the iterated Itô stochastic integrals (see (5.31)–(5.35))

$$I_{(001)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.58}$$

$$I_{(010)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{010}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.59}$$

$$I_{(100)T,t}^{(i_1i_2i_3)q_3}=\sum_{j_1,j_2,j_3=0}^{q_3}C_{j_3j_2j_1}^{100}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.60}$$

Table 5.19: Coefficients $\bar{C}_{22j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{105}$	$\frac{-2}{315}$	0
$j_2=1$	$\frac{2}{315}$	0	$\frac{-2}{1155}$
$j_2=2$	0	$\frac{2}{3465}$	0

$$
I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4} C_{j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$
$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$
$$
+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr). \tag{5.61}
$$

Assume that $q_3=2$, $q_4=1$. In Tables 5.20–5.36 we have the exact values of Fourier–Legendre coefficients $\bar{C}_{j_3j_2j_1}^{001}$, $\bar{C}_{j_3j_2j_1}^{010}$, $\bar{C}_{j_3j_2j_1}^{100}$ $(j_1,j_2,j_3=0,1,2)$, $\bar{C}_{j_5j_4j_3j_2j_1}$ $(j_1,\ldots,j_5=0,1)$.

In the case of pairwise different $i_1,\ldots,i_5$ from Tables 5.20–5.36 we obtain

Table 5.20: Coefficients $\bar{C}_{0j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	-2	$\frac{14}{15}$	$\frac{-2}{15}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{15}$	$\frac{6}{35}$
$j_2 = 2$	$\frac{2}{5}$	$\frac{-22}{105}$	$\frac{-2}{105}$

Table 5.21: Coefficients $\bar{C}_{1j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-6}{5}$	$\frac{22}{45}$	$\frac{-2}{105}$
$j_2 = 1$	$\frac{-2}{9}$	$\frac{-2}{105}$	$\frac{26}{315}$
$j_2 = 2$	$\frac{22}{105}$	$\frac{-38}{315}$	$\frac{-2}{315}$

Table 5.22: Coefficients $\bar{C}_{2j_2j_1}^{001}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{5}$	$\frac{2}{21}$	$\frac{4}{105}$
$j_2 = 1$	$\frac{-22}{105}$	$\frac{4}{105}$	$\frac{2}{105}$
$j_2 = 2$	0	$\frac{-2}{105}$	0

Table 5.23: Coefficients $\bar{C}_{0j_2j_1}^{100}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{3}$	$\frac{2}{15}$	$\frac{2}{15}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{45}$	$\frac{2}{35}$
$j_2 = 2$	$\frac{2}{15}$	$\frac{-2}{35}$	$\frac{-4}{105}$

Table 5.24: Coefficients $\bar{C}_{1j_2j_1}^{100}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{5}$	$\frac{2}{45}$	$\frac{2}{21}$
$j_2 = 1$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2 = 2$	$\frac{2}{35}$	$\frac{-2}{63}$	$\frac{-2}{105}$

Table 5.25: Coefficients $\bar{C}_{2j_2j_1}^{100}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2 = 1$	$\frac{-2}{21}$	$\frac{-2}{315}$	$\frac{2}{105}$
$j_2 = 2$	$\frac{-2}{105}$	$\frac{-2}{315}$	0

Table 5.26: Coefficients $\bar{C}_{0j_2j_1}^{010}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-4}{3}$	$\frac{8}{15}$	0
$j_2 = 1$	$\frac{-4}{15}$	0	$\frac{8}{105}$
$j_2 = 2$	$\frac{4}{15}$	$\frac{-16}{105}$	0

Table 5.27: Coefficients $\bar{C}_{1j_2j_1}^{010}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-4}{5}$	$\frac{4}{15}$	$\frac{4}{105}$
$j_2 = 1$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2 = 2$	$\frac{4}{35}$	$\frac{-8}{105}$	0

Table 5.28: Coefficients $\bar{C}_{2j_2j_1}^{010}$

	$j_1 = 0$	$j_1 = 1$	$j_1 = 2$
$j_2 = 0$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2 = 1$	$\frac{-4}{21}$	$\frac{4}{105}$	$\frac{4}{315}$
$j_2 = 2$	$\frac{-4}{105}$	0	0

Table 5.29: Coefficients $\bar{C}_{000j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{4}{15}$	$\frac{-8}{45}$
$j_2 = 1$	$\frac{-4}{45}$	$\frac{8}{105}$

Table 5.30: Coefficients $\bar{C}_{010j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{4}{45}$	$\frac{-16}{315}$
$j_2 = 1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.31: Coefficients $\bar{C}_{110j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{8}{105}$	$\frac{-2}{45}$
$j_2 = 1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.32: Coefficients $\bar{C}_{011j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{8}{315}$	$\frac{-4}{315}$
$j_2 = 1$	0	$\frac{2}{945}$

Table 5.33: Coefficients $\bar{C}_{001j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	0	$\frac{4}{315}$
$j_2 = 1$	$\frac{8}{315}$	$\frac{-2}{105}$

Table 5.34: Coefficients $\bar{C}_{100j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{8}{45}$	$\frac{-4}{35}$
$j_2 = 1$	$\frac{-16}{315}$	$\frac{2}{45}$

Table 5.35: Coefficients $\bar{C}_{101j_2j_1}$

	$j_1 = 0$	$j_1 = 1$
$j_2 = 0$	$\frac{4}{315}$	0
$j_2 = 1$	$\frac{4}{315}$	$\frac{-8}{945}$

Table 5.36: Coefficients $\bar{C}_{111j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{2}{105}$	$\frac{-8}{945}$
$j_2=1$	$\frac{2}{945}$	0

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{100}\right)^2\approx 0.00815429(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{010}\right)^2\approx 0.0173903(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{001}\right)^2\approx 0.0252801(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.$$

Note that from Theorem 1.4 for $k=5$ we have

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^2\right),$$

where $i_1,\ldots,i_5=1,\ldots,m$.

Moreover, from Theorem 1.4 we obtain the following useful estimates

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(20))T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(1000)T,t}^{(i_1i_2i_3i_4)}-I_{(1000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0100)T,t}^{(i_1i_2i_3i_4)}-I_{(0100)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0010)T,t}^{(i_1i_2i_3i_4)}-I_{(0010)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0001)T,t}^{(i_1i_2i_3i_4)}-I_{(0001)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right),$$

$$
\mathsf{M}\left\{\left(I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)}-I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)q}\right)^2\right\}\le
$$

$$
\le 720\left(\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6 j_5 j_4 j_3 j_2 j_1}^2\right).
$$

In addition, from Theorem 1.3 for $k=2$ we get

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_2)}-I_{(10)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{10}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{10}C_{j_1 j_2}^{10}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_2)}-I_{(10)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{10}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_2)}-I_{(01)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{01}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{01}C_{j_1 j_2}^{01}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_2)}-I_{(01)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{01}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1 i_2)}-I_{(20)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{20}C_{j_1 j_2}^{20}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1 i_2)}-I_{(20)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)}-I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
= \frac{(T-t)^6}{18} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2 j_1}^{11}\right)^2 - \sum_{j_1,j_2=0}^{q} C_{j_2 j_1}^{11} C_{j_1 j_2}^{11} \quad (i_1 = i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)} - I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^6}{18} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2 j_1}^{11}\right)^2 \quad (i_1 \ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(02)}^{(i_1 i_2)} - I_{(02)T,t}^{(i_1 i_2)q}\right)^2\right\} =
$$

$$
= \frac{(T-t)^6}{6} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2 j_1}^{02}\right)^2 - \sum_{j_1,j_2=0}^{q} C_{j_2 j_1}^{02} C_{j_1 j_2}^{02} \quad (i_1 = i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1 i_2)} - I_{(02)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^6}{6} - \sum_{j_1,j_2=0}^{q} \left(C_{j_2 j_1}^{02}\right)^2 \quad (i_1 \ne i_2).
$$

Clearly, expansions for iterated Stratonovich stochastic integrals (see Theorems 2.1–2.10 and Hypotheses 2.1–2.3) are simpler than expansions for iterated Itô stochastic integrals (see Theorems 1.1, 1.2 and (1.41)–(1.47)). However, the calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Itô stochastic integrals. Below we consider how we can estimate or calculate exactly (for some particular cases) the mean-square approximation error for iterated Stratonovich stochastic integrals.

Consider the iterated Stratonovich stochastic integral of multiplicity 2

$$
J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_1)} \quad (i_1 = 1, \ldots, m),
$$

where $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions on $[t, T]$.

By Theorem 2.2 we have

$$
J^*[\psi^{(2)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_1)}.
$$

Consider the following approximation of the stochastic integral $J^*[\psi^{(2)}]_{T,t}$

$$
J^*[\psi^{(2)}]_{T,t}^q = \sum_{j_1,j_2=0}^{q} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_1)}.
$$

According to the standard relation between Stratonovich and Itô stochastic integrals (see (2.362)) and (1.83), we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-J^*[\psi^{(2)}]_{T,t}^q\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}+\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q+\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q\right)^2\right\}+\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2=
$$

$$
=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}C_{j_1j_2}+
$$

$$
+\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{q}C_{j_1j_1}\right)^2,
$$

where

$$
J[\psi^{(2)}]_{T,t}^q=\sum_{j_1,j_2=0}^{q}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\sum_{j_1=0}^{q}C_{j_1j_1}
$$

is the approximation (see (1.42)) of the iterated Itô stochastic integral

$$
J[\psi^{(2)}]_{T,t}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}\quad(i_1=1,\ldots,m).
$$

It is not difficult to see that the value

$$
\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-J^*[\psi^{(2)}]_{T,t}^q\right)^2\right\}
$$

is greater than the value

$$
\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q\right)^2\right\}
$$

by

$$E_q^{(i_1)} = \left(\frac{1}{2}\int\limits_t^T \psi_1(s)\psi_2(s)ds - \sum_{j_1=0}^{q} C_{j_1j_1}\right)^2.$$

For some particular cases $E_q^{(i_1)} = 0$. For example, for the case $\psi_1(\tau),\ \psi_2(\tau) \equiv 1$ ($\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$) we have

$$\sum_{j_1=0}^{q} C_{j_1j_1} = \frac{1}{2}\sum_{j_1=0}^{q}\left(C_{j_1}\right)^2 = \frac{1}{2}\left(C_0\right)^2 = \frac{1}{2}(T-t) = \frac{1}{2}\int\limits_t^T ds.$$

However, $E_q^{(i_1)} \ne 0$ in a general case.

Consider the following iterated Stratonovich stochastic integral of multiplicity 3

$$I_{(000)T,t}^{*(i_1i_2i_3)} = {\int\limits_t^{*}}^{T}{\int\limits_t^{*}}^{t_3}{\int\limits_t^{*}}^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m).$$

Taking into account the standard relations between Itô and Stratonovich stochastic integrals (see (2.289)) and Theorem 1.1 (the case $k=3$) together with Theorem 2.7, we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)} - I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau - I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)q} + I_{(000)T,t}^{(i_1i_2i_3)q} + \mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \right.\right.$$

$$\left.\left. + \mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau - I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}, \tag{5.62}$$

where the approximations $I_{(000)T,t}^{*(i_1i_2i_3)q}$, $I_{(000)T,t}^{(i_1i_2i_3)q}$ are defined by the relations (see (5.15), (5.16))

$$I_{(000)T,t}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{5.63}$$

$$I_{(000)T,t}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{5.64}$$

Substituting (5.63) and (5.64) into (5.62) yields

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)+\right.\right.$$

$$\left.\left.+\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)-\mathbf{1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}\le$$

$$\le 4\left(\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+\mathbf{1}_{\{i_1=i_2\}}F_q^{(i_3)}+\mathbf{1}_{\{i_2=i_3\}}G_q^{(i_1)}+\mathbf{1}_{\{i_1=i_3\}}H_q^{(i_2)}\right), \tag{5.65}$$

where

$$F_q^{(i_3)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}, \tag{5.66}$$

$$G_q^{(i_1)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}, \tag{5.67}$$

$$H_q^{(i_2)}=\mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \tag{5.68}$$

In the cases of Legendre polynomials or trigonometric functions, we have (see Theorem 2.7) the equalities

$$\lim_{q\to\infty}F_q^{(i_3)}=0,\quad \lim_{q\to\infty}G_q^{(i_1)}=0,\quad \lim_{q\to\infty}H_q^{(i_2)}=0.$$

However, in accordance with (5.65) the value

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}
$$

with a finite q can be estimated by terms of a rather complex structure (see (5.66)-(5.68)).

As is easily observed, this peculiarity will also apply to the iterated Stratonovich stochastic integrals of multiplicities $k\ge 4$ with the only difference that the number of additional terms like (5.66)-(5.68) will be considerably higher and their structure will be more complicated.

Therefore, the payment for a relatively simple approximation of the iterated Stratonovich stochastic integrals (Theorems 2.1–2.9) in comparison with the iterated Itô stochastic integrals (Theorems 1.1, 1.2) is a much more difficult calculation or estimation procedure of their mean-square approximation errors.

As we mentioned above, on the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated Stratonovich stochastic integrals ($T-t\ll 1$ because the length $T-t$ of integration interval $[t,T]$ of the iterated Stratonovich stochastic integrals plays the role of integration step for the numerical methods for Itô SDEs, i.e. $T-t$ is already fairly small). This leads to sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals which are required for achieving the acceptable accuracy of approximation.

From (5.37) ($i_1\ne i_2$) we obtain

$$
\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le
$$

$$
\le\frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx=-\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{(T-t)^2}{q},\quad (5.69)
$$

where C_1 is a constant.

It is easy to notice that for a sufficiently small $T-t$ (recall that $T-t\ll 1$ since it is a step of integration for the numerical schemes for Itô SDEs) there exists a constant C_2 such that

$$
\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C_2\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\},\qquad (5.70)
$$

where $I^{*(i_1\ldots i_k)q}_{(l_1\ldots l_k)T,t}$ is an approximation of the iterated Stratonovich stochastic integral $I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)T,t}$.

From (5.69) and (5.70) we finally obtain

$$
\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)T,t}-I^{*(i_1\ldots i_k)q}_{(l_1\ldots l_k)T,t}\right)^2\right\}\le C\frac{(T-t)^2}{q}, \tag{5.71}
$$

where constant C is independent of $T-t$.

The same idea can be found in [67] in the framework of the method of approximation of iterated Stratonovich stochastic integrals based on the trigonometric expansion of the Brownian bridge process.

We can get more information about the numbers q (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since for pairwise different $i_1,\ldots,i_k=1,\ldots,m$

$$
J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}\quad \text{w. p. 1},
$$

where $J[\psi^{(k)}]_{T,t}$, $J^*[\psi^{(k)}]_{T,t}$ are defined by (5.1) and (5.2) correspondingly, then for pairwise different $i_1,\ldots,i_6=1,\ldots,m$ from Theorem 1.3 we obtain

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(01)T,t}-I^{*(i_1i_2)q}_{(01)T,t}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C^{01}_{j_2j_1}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(10)T,t}-I^{*(i_1i_2)q}_{(10)T,t}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C^{10}_{j_2j_1}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(000)T,t}-I^{*(i_1i_2i_3)q}_{(000)T,t}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C^2_{j_3j_2j_1},
$$

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0000)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C^2_{j_4j_3j_2j_1},
$$

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(100)T,t}-I^{*(i_1i_2i_3)q}_{(100)T,t}\right)^2\right\}=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{100}_{j_3j_2j_1}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(010))T,t}-I^{*(i_1i_2i_3)q}_{(010)T,t}\right)^2\right\}=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{010}_{j_3j_2j_1}\right)^2,
$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{*(i_1i_2i_3)}-I_{(001)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)q}\right)^2\right\}=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(20)T,t}^{*(i_1i_2)}-I_{(20)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{*(i_1i_2)}-I_{(11)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{*(i_1i_2)}-I_{(02)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(1000)T,t}^{*(i_1i_2i_3i_4)}-I_{(1000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0100)T,t}^{*(i_1i_2i_3i_4)}-I_{(0100)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0010)T,t}^{*(i_1i_2i_3i_4)}-I_{(0010)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0001)T,t}^{*(i_1i_2i_3i_4)}-I_{(0001)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}-I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)q}\right)^2\right\}=\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6j_5j_4j_3j_2j_1}^2.$$

Recall that the systems of iterated stochastic integrals (5.1)–(5.4) are part of the Taylor–Itô and Taylor–Stratonovich expansions (see Chapter 4).

The function $K(t_1,\ldots,t_k)$ from Theorem 1.1 for the set (5.3) is defined by

$$K(t_1,\ldots,t_k)=(t-t_k)^{l_k}\ldots(t-t_1)^{l_1}\ \mathbf{1}_{\{t_1<\ldots<t_k\}},\quad t_1,\ldots,t_k\in[t,T],\quad (5.72)$$

where $\mathbf{1}_A$ is the indicator of the set A.

In particular, for the stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$, $I_{(00)T,t}^{(i_1i_2)}$, $I_{(000)T,t}^{(i_1i_2i_3)}$, $I_{(01)T,t}^{(i_1i_2)}$, $I_{(10)T,t}^{(i_1i_2)}$, $I_{(0000)T,t}^{(i_1\ldots i_4)}$, $I_{(20)T,t}^{(i_1i_2)}$, $I_{(11)T,t}^{(i_1i_2)}$, $I_{(02)T,t}^{(i_1i_2)}$ $(i_1,\ldots,i_4=1,\ldots,m)$ the functions $K(t_1,\ldots,t_k)$ defined by (5.72) correspondently look as follows

$$K_1(t_1)=t-t_1,\quad K_2(t_1)=(t-t_1)^2,\quad K_{00}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2\}}, \tag{5.73}$$

$$K_{000}(t_1,t_2,t_3)=\mathbf{1}_{\{t_1<t_2<t_3\}},\quad K_{01}(t_1,t_2)=(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{5.74}$$

$$K_{10}(t_1,t_2)=(t-t_1)\mathbf{1}_{\{t_1<t_2\}},\quad K_{0000}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}, \tag{5.75}$$

$$K_{20}(t_1,t_2)=(t-t_1)^2\mathbf{1}_{\{t_1<t_2\}},\quad K_{11}(t_1,t_2)=(t-t_1)(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{5.76}$$

$$K_{02}(t_1,t_2)=(t-t_2)^2\mathbf{1}_{\{t_1<t_2\}}, \tag{5.77}$$

where $t_1,\ldots,t_4\in[t,T]$.

It is obviously that the most simple expansion for the polynomial of a finite degree into the Fourier series using the complete orthonormal system of functions in the space $L_2([t,T])$ will be its Fourier–Legendre expansion (finite sum). The polynomial functions are included in the functions (5.73)–(5.77) as their components if $l_1^2+\ldots+l_k^2>0$. So, it is logical to expect that the most simple expansions for the functions (5.73)–(5.77) into generalized multiple Fourier series will be Fourier–Legendre expansions of these functions when $l_1^2+\ldots+l_k^2>0$.

Note that the given assumption is confirmed completely (compare the formulas (5.7), (5.8) with the formulas (5.78), (5.83) (see below) correspondently). So, usage of Legendre polynomials for the approximation of iterated Itô and Stratonovich stochastic integrals is a step forward.

5.2 Mean-Square Approximation of Specific Iterated Stratonovich Stochastic Integrals of multiplicities 1 to 3 Based on Trigonometric System of Functions

In [1]-[14], [48] on the base of Theorems 1.1, 2.2, 2.5, and 2.7 the author obtained (also see early publications [59] (1997), [60] (1998), [63] (1994), [64] (1996)) the following expansions of the iterated Stratonovich stochastic integrals (5.4) (independently from the papers [65]-[70], [75] excepting the method, in which the additional random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ are introduced)

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{*(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.78}$$

$$I_{(00)T,t}^{*(i_1 i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$

$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr), \tag{5.79}$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\right.$$

$$\left.\left.+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\right)+D_{T,t}^{(i_1 i_2 i_3)q}\Biggr), \tag{5.80}$$

where

$$D_{T,t}^{(i_1 i_2 i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\Biggl(\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+$$

$$
+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{0}^{(i_3)}-\frac{l}{r}\zeta_{0}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_{0}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{4\sqrt{2}\pi^2}\Biggl(\sum_{r,m=1}^{q}\Biggl(\frac{2}{rm}\left(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\right.
$$

$$
\left.+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{m(r+m)}\left(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)\Biggr)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl(\frac{1}{m(l-m)}\left(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{l(l-m)}\left(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-\right.
$$

$$
\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)\Biggr)\Biggr),
$$

$$
I_{(10)T,t}^{*(i_1i_2)q}=-(T-t)^2\Biggl(\frac{1}{6}\zeta_{0}^{(i_1)}\zeta_{0}^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_{0}^{(i_1)}+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_2)}\zeta_{0}^{(i_1)}-2\mu_q^{(i_1)}\zeta_{0}^{(i_2)}\Biggr)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_{0}^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_{0}^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{0}^{(i_2)}\right)\Biggr)-
$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\Biggr)+
$$

$$
+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr),\quad (5.81)
$$

$$I_{(01)T,t}^{*(i_1i_2)q}=(T-t)^2\Biggl(-\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-2\xi_q^{(i_2)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)-$$

$$-\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}\right)-\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$

$$+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr)-$$

$$-\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)-\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),\tag{5.82}$$

$$I_{(2)T,t}^{*(i_1)q}=(T-t)^{5/2}\Biggl(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\Biggr)-$$

$$-\frac{1}{\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr),\tag{5.83}$$

where

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},\quad \mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},$$

$$\beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},\quad \zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)},$$

where $\phi_j(s)$ is defined by (1.61) and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,$ $i=1,\ldots,m)$ are independent standard Gaussian random variables $(i_1,i_2,i_3=1,\ldots,m)$.

Note that (5.81), (5.82) imply the following

$$\sum_{j=0}^{\infty}C_{jj}^{10}=\sum_{j=0}^{\infty}C_{jj}^{01}=-\frac{(T-t)^2}{4},\tag{5.84}$$

where

$$C_{jj}^{10} = \int\limits_t^T \phi_j(x) \int\limits_t^x \phi_j(y)(t-y)dydx,$$

$$C_{jj}^{01} = \int\limits_t^T \phi_j(x)(t-x) \int\limits_t^x \phi_j(y)dydx.$$

Note that the formulas (5.84) are particular cases of the more general relation (2.10), which has been applied for the proof of Theorem 2.1.

Let us consider the mean-square errors of approximations (5.79)–(5.82). From the relations (5.79)–(5.82) when $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ by direct calculation we obtain

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)} - I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{5.85}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = (T-t)^3\left(\frac{1}{4\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{55}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{r,l=1\\ r\neq l}}^{\infty} - \sum_{\substack{r,l=1\\ r\neq l}}^{q}\right)\frac{5l^4+4r^4-3l^2r^2}{r^2l^2(r^2-l^2)^2}\right), \tag{5.86}$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)} - I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\} = (T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{5}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{\infty} - \sum_{\substack{k,l=1\\ k\neq l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\right), \tag{5.87}$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)} - I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\} = (T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right) + \right.$$

$$\left. + \frac{5}{32\pi^4}\left(\frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}\right) + \frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{\infty} - \sum_{\substack{k,l=1\\ k\neq l}}^{q}\right)\frac{l^2+k^2}{l^2(l^2-k^2)^2}\right). \tag{5.88}$$

It is easy to demonstrate that the relations (5.86), (5.87), and (5.88) can be represented using Theorem 1.3 in the following form

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right), \tag{5.89}
$$

$$
{\sf M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\right), \tag{5.90}
$$

$$
{\sf M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}\right). \tag{5.91}
$$

Comparing (5.89)–(5.91) and (5.86)–(5.88), we note that

$$
\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}=\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}\frac{l^2+k^2}{l^2\left(l^2-k^2\right)^2}=\frac{\pi^4}{48}, \tag{5.92}
$$

$$
\sum_{\substack{r,l=1\\ r\ne l}}^{\infty}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}=\frac{9\pi^4}{80}. \tag{5.93}
$$

Let us consider approximations of stochastic integrals $I_{(10)T,t}^{*(i_1i_1)}$, $I_{(01)T,t}^{*(i_1i_1)}$ and conditions for selecting number q using the trigonometric system of functions

$$
I_{(10)T,t}^{*(i_1i_1)q}=-(T-t)^2\left(\frac{1}{6}\left(\zeta_0^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-\right.
$$

$$
-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)-
$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr),
$$

$$
I_{(01)T,t}^{*(i_1i_1)q}=(T-t)^2\Biggl(-\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-
$$

$$
-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{\pi^2r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr).
$$

Furthermore, we have

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_1)}-I_{(01)T,t}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_1)}-I_{(10)T,t}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{2}{\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{\pi^4}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^2+
$$

$$
+\frac{1}{\pi^4}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{\infty}-\sum_{\substack{k,l=1\\ k\neq l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr). \tag{5.94}
$$

Considering (5.92), we can rewrite the relation (5.94) in the following form

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_1)}-I_{(01)T,t}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_1)}-I_{(10)T,t}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{17}{240}-\frac{1}{3\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\frac{2}{\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}+
$$

Table 5.37: Confirmation of the formula (5.89)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.38: Confirmation of the formulas (5.90), (5.91)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

$$+\frac{1}{\pi^4}\left(\sum_{r=1}^{q}\frac{1}{r^2}\right)^2-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr). \tag{5.95}$$

In Tables 5.37–5.39 we confirm numerically the formulas (5.89)–(5.91), (5.95) for various values of q. In Tables 5.37–5.39 the number ε means right-hand sides of the mentioned formulas. Obviously, these results are consistent with the estimate (1.187).

The formulas (5.92), (5.93) appear to be interesting. Let us confirm numerically their correctness in Tables 5.40 and 5.41 (the number ε_q is an absolute deviation of multiple partial sums with the upper limit of summation q for the series (5.92), (5.93) from the right-hand sides of the formulas (5.92), (5.93); convergence of multiple series is regarded here when $p_1=p_2=q\to\infty$, which is acceptable according to Theorems 1.1, 2.2, 2.5, and 2.7).

Using the trigonometric system of functions, let us consider approximations of iterated stochastic integrals of the following form

$$J^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)T,t}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\lambda_l=1$ if $i_l=1,\ldots,m$ and $\lambda_l=0$ if $i_l=0$, $l=1,\ldots,k$ ($\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$).

Table 5.39: Confirmation of the formula (5.95)

$4\varepsilon/(T-t)^4$	0.0268	0.0034	$3.3955\cdot 10^{-4}$	$3.3804\cdot 10^{-5}$	$3.3778\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.40: Confirmation of the formula (5.92)

ε_q	2.0294	0.3241	0.0330	0.0033	$3.2902 \cdot 10^{-4}$
q	1	10	100	1000	10000

Table 5.41: Confirmation of the formula (5.93)

ε_q	10.9585	1.8836	0.1968	0.0197	0.0020
q	1	10	100	1000	10000

It is easy to see that the approximations

$$J^{*(i_1 i_2)q}_{(\lambda_1\lambda_2)T,t}, \quad J^{*(i_1 i_2 i_3)q}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

of the stochastic integrals

$$J^{*(i_1 i_2)}_{(\lambda_1\lambda_2)T,t}, \quad J^{*(i_1 i_2 i_3)}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

are defined by the right-hand sides of the formulas (5.79), (5.80), where it is necessary to take

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{5.96}$$

and $i_1, i_2, i_3 = 0, 1, \ldots, m$.

Since

$$\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(0)} = \begin{cases} \sqrt{T-t} & \text{if } j = 0 \\ \\ 0 & \text{if } j \neq 0 \end{cases},$$

then it is easy to get from (5.79) and (5.80), considering that in these equalities $\zeta_j^{(i)}$ is defined by (5.96) and $i_1, i_2, i_3 = 0, 1, \ldots, m$, the following family of formulas

$$J^{*(i_1 0)q}_{(10)T,t} = \frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_1)} + \frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.97}$$

$$J^{*(0 i_2)q}_{(01)T,t} = \frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_2)} + \sqrt{\alpha_q}\xi_q^{(i_2)}\right)\right), \tag{5.98}$$

$$J^{*(00i_3)q}_{(001)T,t}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_3)}+\sqrt{\beta_q}\mu_q^{(i_3)}\right)-\right.$$

$$\left.-\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_3)}+\sqrt{\alpha_q}\xi_q^{(i_3)}\right)\right),$$

$$J^{*(0i_20)q}_{(010)T,t}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_2)}-\frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_2)}+\sqrt{\beta_q}\mu_q^{(i_2)}\right)\right),$$

$$J^{*(i_100)q}_{(100)T,t}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)+\right.$$

$$\left.+\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),$$

$$J^{*(0i_2i_3)q}_{(011)T,t}=(T-t)^{2}\left(\frac{1}{6}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_3)}\zeta_0^{(i_2)}+\right.$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_3)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_3)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\right)\right)-$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_2)}\zeta_{2l}^{(i_3)}+\frac{l}{r}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)+$$

$$+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_2)}\zeta_{2r-1}^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\right)+\right.$$

$$\left.\left.+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\right)\right)\right),\qquad(5.99)$$

$$J^{*(i_1i_20)q}_{(110)T,t}=(T-t)^{2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)}+\right.$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr)+
$$

$$
+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_1)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+
$$

$$
+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),
$$

$$
J_{(101)T,t}^{*(i_1 0 i_3)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}\right)\Biggr)-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_3)}-
$$

$$
-\sum_{r=1}^{q}\frac{1}{4\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\right)\Biggr).
$$

5.3 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Numerical Solution of Itô SDEs

The section is devoted to comparative analysis of efficiency of application the Legendre polynomials and trigonometric functions for the numerical integration of Itô SDEs in the framework of the method of approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series (Theorems 1.1, 2.1–2.9). This section is written on the base of the papers [19], [38], and [30] (Sect. 4).

Using the iterated Itô stochastic integrals of multiplicities 1 to 3 appearing in the Taylor–Itô expansion as an example, it is shown that their expansions obtained using multiple Fourier–Legendre series are significantly simpler and less computationally costly than their analogues obtained on the basis of multiple trigonometric Fourier series.

Let us consider the following set of iterated Itô and Stratonovich stochastic integrals from the classical Taylor–Itô and Taylor–Stratonovich expansions [67]

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.100}$$

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5.101}$$

where $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$, $\lambda_l=0$ for $i_l=0$ and $\lambda_l=1$ for $i_l=1,\ldots,m$ $(l=1,\ldots,k)$.

In [65] Milstein G.N. obtained the following expansion of $J_{(11)T,t}^{(i_1i_2)}$ on the base of the Karhunen–Loève expansion of the Brownian bridge process (we will discuss the method [65] in detail in Sect. 6.2)

$$J_{(11)T,t}^{(i_1i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.$$
$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{5.102}$$

where the series converges in the mean-square sense, $i_1\ne i_2$, $i_1,i_2=1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,\ldots,m,\ j=0,1,\ldots)$,

$$\phi_j(s) = \frac{1}{\sqrt{T-t}} \begin{cases} 1 & \text{for } j = 0 \\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)) & \text{for } j = 2r-1, \\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)) & \text{for } j = 2r \end{cases} \tag{5.103}$$

where $r = 1, 2, \ldots$

Moreover,

$$J_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

where $i_1 = 1, \ldots, m$.

In principle, for implementing the strong numerical method with the convergence order 1.0 (Milstein method [65], see Sect. 4.10) for Itô SDEs we can take the following approximations

$$J_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.104}$$

$$J_{(11)T,t}^{(i_1 i_2)q} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.$$

$$\left.\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{5.105}$$

where $i_1 \ne i_2$, $i_1, i_2 = 1, \ldots, m$.

It is not difficult to show that

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1 i_2)} - J_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{5.106}$$

However, this approach has an obvious drawback. Indeed, we have too complex formulas for the stochastic integrals with Gaussian distribution

$$J_{(01)T,t}^{(0i_1)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{5.107}$$

$$J_{(001)T,t}^{(00i_1)} = (T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)} + \frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)} - \frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(01)T,t}^{(0i_1)q} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

where the meaning of notations used in (5.105) is retained.

In [65] Milstein G.N. proposed the following mean-square approximations on the base of (5.102), (5.107)

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),\tag{5.108}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\right),\tag{5.109}$$

where $i_1\ne i_2$ in (5.109), and

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},\tag{5.110}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $r=1,\ldots,q$, $i=1,\ldots,m$ are independent standard Gaussian random variables.

Obviously, for the approximations (5.108) and (5.109) we obtain [65]

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,$$

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right).\tag{5.111}$$

This idea has been developed in [66]-[68]. For example, the approximation $J_{(001)T,t}^{(00i_1)q}$, which corresponds to (5.108), (5.109) is defined by [66]-[68]

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.$$

$$\left.-\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),\tag{5.112}$$

where $\xi_q^{(i)}$, α_q have the form (5.110),

$$\mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}} \sum_{r=q+1}^{\infty} \frac{1}{r^2}\, \zeta_{2r}^{(i)}, \quad \beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^4},$$

$\phi_j(s)$ is defined by (5.103), and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r = 1, \ldots, q,\ i = 1, \ldots, m)$ are independent standard Gaussian random variables.

Moreover,

$$\mathsf{M}\left\{\left(J_{(001)T,t}^{(00i_1)} - J_{(001)T,t}^{(00i_1)q}\right)^2\right\} = 0.$$

Nevetheless, the expansions (5.108), (5.112) are too complex for the approximation of two Gaussian random variables $J_{(01)T,t}^{(0i_1)}$, $J_{(001)T,t}^{(00i_1)}$.

Further, we will see that the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of stochastic integral $J_{(111)T,t}^{(i_1 i_2 i_3)}$ $(i_1, i_2, i_3 = 1, \ldots, m)$. This is due to the fact that the number q is fixed for stochastic integrals included into the considered collection. However, it is clear that due to the smallness of $T - t$, the number q for $J_{(111)T,t}^{(i_1 i_2 i_3)}$ could be taken significantly less than in the formula (5.109). This feature is also valid for the formulas (5.108), (5.112).

On the other hand, the following very simple formulas are well known (see (5.6)–(5.8))

$$J_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.113}$$

$$J_{(01)T,t}^{(0i_1)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.114}$$

$$J_{(001)T,t}^{(00i_1)} = \frac{(T-t)^{5/2}}{6}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{5.115}$$

where $\zeta_0^{(i)}$, $\zeta_1^{(i)}$, $\zeta_2^{(i)}$ $(i = 1, \ldots, m)$ are indepentent standard Gaussian random variables. Obviously, that the formulas (5.113)-(5.115) are part of the method based on Theorem 1.1 (also see Sect. 5.1).

To obtain the Milstein expansion for the stochastic integral (5.2) the truncated expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals in (5.2), and the integrals must be calculated starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (5.2) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple integrals (5.2) were obtained [65]-[68], [75], [76] by the Milstein approach [65] based on the Karhunen–Loève expansion of the Brownian bridge process.

At that, in [65], [75] the case $\psi_1(s),\ \psi_2(s)\equiv 1$ and $i_1,i_2=0,1,\ldots,m$ $(i_1\ne i_2)$ is considered. In [66]-[68], [76] the attempt to consider the case $\psi_1(s)$, $\psi_2(s),\ \psi_3(s)\equiv 1$ and $i_1,i_2,i_3=0,1,\ldots,m$ is realized. Note that, generally speaking, the mean-square convergence of $J^{*(i_1i_2i_3)q}_{(111)T,t}$ to $J^{*(i_1i_2i_3)}_{(111)T,t}$ if $q\to\infty$ was not proved rigorously in [66]-[68], [76] within the frames of the Milstein approach [65] together with the Wong–Zakai approximation [56]-[58] (see discussions in Sect. 2.6.2, 6.2).

5.3.1 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integral $J^{(i_1i_2)}_{(11)T,t}$

Using Theorem 1.1 and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, we have (see (5.10))

$$
J^{(i_1i_2)}_{(11)T,t}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.116}
$$

where series converges in the mean-square sense, $i_1,i_2=1,\ldots,m$,

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j,

$$
\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{5.117}
$$

where $P_j(x)$ is the Legendre polynomial.

The formula (5.116) has been derived for the first time in [59] (1997) with using Theorem 2.10.

Remind the formula (5.37) [59] (1997)

$$
\mathsf{M}\left\{\left(J^{(i_1i_2)}_{(11)T,t}-J^{(i_1i_2)q}_{(11)T,t}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{5.118}
$$

Table 5.42: Numbers $q_{\rm trig}$, $q_{\rm trig}^{*}$, $q_{\rm pol}$

$T-t$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}	2^{-11}	2^{-12}
$q_{\rm trig}$	3	4	7	14	27	53	105	209
$q_{\rm trig}^{*}$	6	11	20	40	79	157	312	624
$q_{\rm pol}$	5	9	17	33	65	129	257	513

where

$$J_{(11)T,t}^{(i_1 i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right). \tag{5.119}$$

Let us compare (5.119) with (5.109) and (5.118) with (5.111). Consider minimal natural numbers $q_{\rm trig}$ and $q_{\rm pol}$, which satisfy to (see Table 5.42)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q_{\rm pol}}\frac{1}{4i^2-1}\right)\le (T-t)^3, \tag{5.120}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^3.$$

Thus, we have

$$\frac{q_{\rm pol}}{q_{\rm trig}}\approx\ 1.67,\ \ 2.22,\ \ 2.43,\ \ 2.36,\ \ 2.41,\ \ 2.43,\ \ 2.45,\ \ 2.45.$$

From the other hand, the formula (5.109) includes $(4q+4)m$ independent standard Gaussian random variables. At the same time the folmula (5.119) includes only $(2q+2)m$ independent standard Gaussian random variables. Moreover, the formula (5.119) is simpler than the formula (5.109). Thus, in this case we can talk about approximately equal computational costs for the formulas (5.109) and (5.119).

There is one important feature. As we mentioned above, further we will see that the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of stochastic integral $J_{(111)T,t}^{(i_1 i_2 i_3)}$ $(i_1,i_2,i_3=1,\ldots,m)$. This is due to the fact that the number q is fixed for all stochastic integrals, which included into the considered collection. However, it is clear that due to the smallness of $T-t$, the number q for $J_{(111)T,t}^{(i_1 i_2 i_3)}$ could be chosen significantly less than in the formula (5.109). This feature is also valid for the formulas (5.108), (5.112). However, for the case of Legendre polynomials we can choose different numbers q for different stochastic integrals.

From the other hand, if we will not introduce the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1 i_2)}$ will be three times larger (see (5.106)). Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0i_1)}$, $J_{(001)T,t}^{(00i_1)}$ (with Gaussian distribution) will be approximated worse.

Consider minimal natural numbers $q_{\rm trig}^*$, which satisfy to (see Table 5.42)

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}^*}\frac{1}{r^2}\right)\le (T-t)^3.$$

In this situation we can talk about the advantage of Legendre polynomials ($q_{\rm trig}^* > q_{pol}$ and (5.109) is more complex than (5.119)).

5.3.2 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integrals $J_{(1)T,t}^{(i_1)}$, $J_{(11)T,t}^{(i_1 i_2)}$, $J_{(01)T,t}^{(0i_1)}$, $J_{(10)T,t}^{(i_1 0)}$, $J_{(111)T,t}^{(i_1 i_2 i_3)}$

It is well known [65]-[68], [75] (also see [14]) that for the numerical realization of strong Taylor–Itô numerical methods with the convergence order 1.5 for Itô SDEs we need to approximate the following collection of iterated Itô stochastic integrals (see Sect. 4.10)

$$J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1 i_2)},\quad J_{(01)T,t}^{(0i_1)},\quad J_{(10)T,t}^{(i_1 0)},\quad J_{(111)T,t}^{(i_1 i_2 i_3)}.$$

Using Theorem 1.1 for the system of trigonometric functions, we have (see Sect. 5.2)

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{5.121}$$

$$J_{(11)T,t}^{(i_1 i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$
$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+$$
$$+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\Biggr), \tag{5.122}$$

$$J_{(01)T,t}^{(0i_1)q} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.123}$$

$$J_{(10)T,t}^{(i_10)q} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{5.124}$$

$$J_{(111)T,t}^{(i_1i_2i_3)q} = (T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)} + \frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)} - \xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right) +$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)} - 2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)} + \mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right) +$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)} - \zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right) +$$

$$+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)} - 2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)} + \zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr) +$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)} - \zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)} + \zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right) +$$

$$+\frac{1}{8\pi^2 r^2}\Bigl(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)} + \zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)} - 6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)} +$$

$$+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)} - 2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)} + \zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\Bigr)\Biggr) + D_{T,t}^{(i_1i_2i_3)q}\Biggr), \tag{5.125}$$

where in (5.125) we suppose that $i_1 \ne i_2$, $i_1 \ne i_3$, $i_2 \ne i_3$,

$$D_{T,t}^{(i_1i_2i_3)q} = \frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\Biggl(\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)} - \zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)} +$$

$$+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)} - \frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr) - \frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr) +$$

$$+\frac{1}{4\sqrt{2}\pi^2}\Biggl(\sum_{r,m=1}^{q}\Biggl(\frac{2}{rm}\Bigl(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)} + \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)} +$$

$$
+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{m(r+m)}\Biggl(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)\Biggr)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl(\frac{1}{m(l-m)}\Biggl(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{l(l-m)}\Biggl(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)\Biggr)\Biggr),
$$

where

$$
\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},
$$

$$
\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},
$$

and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,\ i=1,\ldots,m)$ are independent standard Gaussian random variables.

The mean-square errors of approximations (5.122)–(5.125) are represented by the formulas

$$
\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,
$$

$$
\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=0,
$$

$$
\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\Biggl(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\Biggr), \tag{5.126}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q}\right)^2\right\}=(T-t)^3\Biggl(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

Table 5.43: Confirmation of the formula (5.127)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

$$
-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\Biggr), \qquad (5.127)
$$

where $i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3$.

In Table 5.43 we can see the numerical confirmation of the formula (5.127) (ε means the right-hand side of (5.127)).

Note that the formulas (5.121), (5.122) have been obtained for the first time in [65]. Using (5.121), (5.122), we can realize numerically an explicit one-step strong numerical method with the convergence order 1.0 for Itô SDEs (Milstein method [65]; also see Sect. 4.10).

An analogue of the formula (5.125) has been obtained for the first time in [66], [67].

As we mentioned above, the Milstein expansion (i.e. expansion based on the Karhunen–Loève expansion of the Brownian bridge process) for iterated stochastic integrals leads to iterated application of the operation of limit transition. The analogue of (5.125) for iterated Stratonovich stochastic integrals has been derived in [66], [67] on the base of the Milstein expansion together with the Wong–Zakai approximation [56]-[58] (without rigorous proof). It means that the authors in [66], [67] formally could not use the double sum with the upper limit q in the analogue of (5.125). From the other hand, the correctness of (5.125) follows directly from Theorem 1.1. Note that (5.125) has been obtained reasonably for the first time in [1]. The version of (5.125) but without the introducing of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ can be found in [59] (1997).

Note that the formula (5.126) appears for the first time in [65]. The mean-square error (5.127) has been obtained for the first time in [64] (1996) on the base of the simplified variant of Theorem 1.1 (the case of pairwise different $i_1,\ldots,i_k$).

The number q as we noted above must be the same in (5.122)–(5.125). This is the main drawback of this approach, because really the number q in (5.125) can be chosen essentially smaller than in (5.122).

Note that in (5.125) we can replace $J_{(111)T,t}^{(i_1i_2i_3)q}$ with $J_{(111)T,t}^{*(i_1i_2i_3)q}$ and (5.125) then will be valid for any $i_1, i_2, i_3 = 0, 1, \ldots, m$ (see Theorems 2.5–2.7).

Consider now approximations of iterated stochastic integrals

$$J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1i_2)},\quad J_{(01)T,t}^{(0i_1)},\quad J_{(10)T,t}^{(i_10)},\quad J_{(111)T,t}^{(i_1i_2i_3)}\quad (i_1, i_2, i_3 = 1, \ldots, m)$$

on the base of Theorem 1.1 (the case of Legendre polynomials) [1]-[14], [30]

$$J_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.128}$$

$$J_{(11)T,t}^{(i_1i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{5.129}$$

$$J_{(01)T,t}^{(0i_1)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.130}$$

$$J_{(10)T,t}^{(i_10)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.131}$$

$$J_{(111)T,t}^{(i_1i_2i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad q_1 \ll q, \tag{5.132}$$

$$J_{(111)T,t}^{(i_1i_1i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right),$$

where

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(z)\int\limits_t^z \phi_{j_2}(y)\int\limits_t^y \phi_{j_1}(x)dxdydz =$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz, \tag{5.133}$$

$\phi_j(x)$ is defined by (5.117) and $P_i(x)$ is the Legendre polynomial ($i = 0, 1, 2, \ldots$).

The mean-square errors of approximations (5.129), (5.132) are represented by the formulas (see Theorems 1.3 and 1.4; also see Sect. 5.1)

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1 i_2)} - J_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \quad (i_1 \ne i_2), \quad (5.134)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 \quad (i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3), \quad (5.135)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \ (i_1 \ne i_2 = i_3), \quad (5.136)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \quad (i_1 = i_3 \ne i_2), \quad (5.137)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 -$$

$$- \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \ne i_3), \quad (5.138)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2\right), \quad (5.139)$$

where $i_1, i_2, i_3 = 1, \ldots, m$ in (5.139).

Let us compare the efficiency of application of Legendre polynomials and trigonometric functions for the approximation of iterated stochastic integrals $J_{(11)T,t}^{(i_1 i_2)}$, $J_{(111)T,t}^{(i_1 i_2 i_3)}$.

Consider the following conditions $(i_1 \neq i_2,\ i_1 \neq i_3,\ i_2 \neq i_3)$

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le (T-t)^4, \tag{5.140}$$

$$(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}\frac{\left(C_{j_3j_2j_1}\right)^2}{(T-t)^3}\right) \le (T-t)^4, \tag{5.141}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p}\frac{1}{r^2}\right) \le (T-t)^4, \tag{5.142}$$

$$(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{p_1}\frac{1}{r^2}-\frac{55}{32\pi^4}\sum_{r=1}^{p_1}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\neq l}}^{p_1}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right) \le (T-t)^4, \tag{5.143}$$

where

$$C_{j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_i(x)$ is the Legendre polynomial.

In Tables 5.44 and 5.45 we can see the minimal numbers q, q_1, p, p_1, which satisfy the conditions (5.140)–(5.143). As we mentioned above, the numbers q, q_1 are different. At that $q_1 \ll q$ (the case of Legendre polynomials). As we saw in the previous sections, we cannot take different numbers p, p_1 for the case of trigonometric functions. Thus, we should choose $q = p$ in (5.122)–(5.125). This leads to huge computational costs (see the fairly complicated formula (5.125)).

From the other hand, we can take different numbers q in (5.122)–(5.125). At that we should exclude random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (5.122)–(5.125). At this situation for the case $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we have

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p^*}\frac{1}{r^2}\right) \le (T-t)^4, \tag{5.144}$$

Table 5.44: Numbers q, q_1

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 5.45: Numbers p, p_1, p^*, p_1^*

$T-t$	0.08222	0.05020	0.02310	0.01956
p	8	21	96	133
p_1	1	1	3	4
p^*	23	61	286	398
p_1^*	1	2	4	5

$$
(T-t)^3\left(\frac{5}{36}-\frac{1}{2\pi^2}\sum_{r=1}^{p_1^*}\frac{1}{r^2}-\frac{79}{32\pi^4}\sum_{r=1}^{p_1^*}\frac{1}{r^4}-\right.
$$

$$
\left.-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\neq l}}^{p_1^*}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \qquad (5.145)
$$

where the left-hand sides of (5.144), (5.145) correspond to (5.122), (5.125) but without $\xi_q^{(i)}$, $\mu_q^{(i)}$. In Table 5.45 we can see minimal numbers p^*, p_1^*, which satisfy the conditions (5.144), (5.145).

Moreover,

$$
\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)\neq 0, \qquad (5.146)
$$

Table 5.46: Confirmation of the formula (5.145)

$\varepsilon/(T-t)^3$	0.0629	0.0097	0.0010	$1.0129\cdot 10^{-4}$	$1.0132\cdot 10^{-5}$
q	1	10	100	1000	10000

where $J_{(01)T,t}^{(0i_1)q}$, $J_{(10)T,t}^{(i_10)q}$ are defined by (5.123), (5.124) but without $\xi_q^{(i)}$.

It is not difficult to see that the numbers $q_{\rm trig}$ in Table 5.42 correspond to minimal numbers $q_{\rm trig}$, which satisfy the condition (compare with (5.146))

$$\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^4.$$

From the other hand, the right-hand sides of (5.130), (5.131) include only 2 random variables. In this situation we again can talk about the advantage of Ledendre polynomials.

In Table 5.46 we can see the numerical confirmation of the formula (5.145) (ε means the left-hand side of (5.145)).

5.3.3 A Comparative Analysis of Efficiency of Using the Legendre Polynomials and Trigonometric Functions for the Integral $J_{(011)T,t}^{*(0i_1i_2)}$

In this section, we compare computational costs for approximation of the iterated Stratonovich stochastic integral $J_{(011)T,t}^{*(0i_1i_2)}$ $(i_1,i_2=1,\ldots,m)$ within the framework of the method of generalized multiple Fourier series for the Legendre polynomial system and the system of trigomomenric functions.

Using Theorem 2.1 for the case of trigonometric system of functions, we obtain [6]-[14], [38]

$$J_{(011)T,t}^{*(0i_1i_2)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right)-$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+$$

Table 5.47: Confirmation of the formula (5.148)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.48: Confirmation of the formula (5.150)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450\cdot 10^{-4}$	$6.2495\cdot 10^{-5}$
q	1	10	100	1000	10000

$$
+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr). \tag{5.147}
$$

For the case $i_1\ne i_2$ from Theorem 1.3 we get [6]-[16], [29], [38]

$$
\mathsf{M}\left\{\left(J_{(011)T,t}^{*(0i_1i_2)}-J_{(011)T,t}^{*(0i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\Biggl(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-
$$

$$
-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\Biggr). \tag{5.148}
$$

Analogues of the formulas (5.147), (5.148) for the case of Legendre polynomials will look as follows [6]-[16], [29], [38]

$$
J_{(011)T,t}^{*(0i_1i_2)q}=\frac{T-t}{2}J_{(11)T,t}^{*(i_1i_2)q}+\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+
$$

$$
+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr), \tag{5.149}
$$

where

$$
J_{(11)T,t}^{*(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),
$$

$$
\mathsf{M}\left\{\left(J_{(011)T,t}^{*(0i_1i_2)}-J_{(011)T,t}^{*(0i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\Biggl(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-
$$

$$
-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\Biggr), \qquad (5.150)
$$

where $i_1 \ne i_2$.

In Tables 5.47 and 5.48 we can see the numerical confirmation of the formulas (5.148) and (5.150) (ε means the right-hand side of (5.148) or (5.150)).

Let us compare the complexity of the formulas (5.147) and (5.149). The formula (5.147) includes the double sum

$$
\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right).
$$

Thus, the formula (5.147) is more complex, than the formula (5.149) even if we take identical numbers q in these formulas. As we noted above, the number q in (5.147) must be equal to the number q from the formula (5.122), so it is much larger than the number q from the formula (5.149). As a result we have obvious advantage of the formula (5.149) in computational costs.

As we mentioned above, if we will not introduce the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the number q in (5.147) can be chosen smaller, but the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1i_2)}$ will be three times larger (see (5.106)). Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0i_1)}$, $J_{(10)T,t}^{(i_10)}$, $J_{(001)T,t}^{(00i_1)}$ (with Gaussian distribution) will be approximated worse. In this situation, we can again talk about the advantage of Ledendre polynomials.

5.3.4 Conclusions

Summing up the results of previous sections we can come to the following conclusions.

1. We can talk about approximately equal computational costs for the formulas (5.122) and (5.129). This means that computational costs for realizing the Milstein scheme (explicit one-step strong numerical method with the convergence order $\gamma = 1.0$ for Itô SDEs; see Sect. 4.10) for the case of Legendre polynomials and for the case of trigonometric functions are approximately the same.

2. If we will not introduce the random variables $\xi_q^{(i)}$ (see (5.122)), then

the mean-square error of approximation of the stochastic integral $J^{(i_1 i_2)}_{(11)T,t}$ will be three times larger (see (5.106)). In this situation, we can talk about the advantage of Ledendre polynomials in the Milstein method. Moreover, in this case the stochastic integrals $J^{(0i_1)}_{(01)T,t}$, $J^{(i_1 0)}_{(10)T,t}$, $J^{(00i_1)}_{(001)T,t}$ (with Gaussian distribution) will be approximated worse.

3. If we talk about the explicit one-step strong numerical scheme with the convergence order $\gamma = 1.5$ for Itô SDEs (see Sect. 4.10), then the numbers q, q_1 (see (5.129), (5.132)) are different. At that $q_1 \ll q$ (the case of Legendre polynomials). The number q must be the same in (5.122)–(5.125) (the case of trigonometric functions). This leads to huge computational costs (see the fairly complicated formula (5.125)). From the other hand, we can take different numbers q in (5.122)–(5.125). At that we should exclude the random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (5.122)–(5.125). This leads to another problems which we discussed above (see Conclusion 2).

4. In addition, the author supposes that effect described in Conclusion 3 will be more impressive when analyzing more complex sets of iterated Itô and Stratonovich stochastic integrals (when $\gamma = 2.0, 2.5, 3.0, \ldots$). This supposition is based on the fact that the polynomial system of functions has the significant advantage (in comparison with the trigonometric system) when approximating the iterated stochastic integrals for which not all weight functions are equal to 1 (see Sect 5.4.3 and conclusion at the end of Sect. 5.1).

Chapter 6

Other Methods of Approximation of Specific Iterated Itô and Stratonovich Stochastic Integrals of Multiplicities 1 to 4

6.1 New Simple Method for Obtainment an Expansion of Iterated Itô Stochastic integrals of Multiplicity 2 Based on the Wiener Process Expansion Using Legendre Polynomials and Trigonometric Functions

This section is devoted to the expansion of iterated Itô stochastic integrals of multiplicity 2 based on the Wiener process expansion using complete orthonormal systems of functions in $L_2([t,T])$. The expansions of these stochastic integrals using Legendre polynomials and trigonometric functions are considered. In contrast to the method of expansion of iterated Itô stochastic integrals based on the Karhunen–Loève expansion of the Brownian bridge process [65]-[67], this method allows the use of different systems of basis functions, not only the trigonometric system of functions. The proposed method makes it possible to obtain expansions of iterated Itô stochastic integrals of multiplicity 2 much easier than the method based on generalized multiple Fourier series (see Chapters 1 and 2). The latter involve the calculation of coefficients of multiple Fourier series, which is a time-consuming task. However, the proposed method can be applied only to iterated Itô stochastic integrals of multiplicity 2.

It is well known that the idea of representing of the Wiener process as a functional series with random coefficients (that are independent standard Gaussian random variables) with using the complete orthonormal system of trigonometric functions in $L_2([0,T])$ goes back to the works of Wiener [125] (1924) and Lévy [126] (1951). The specified series was used in [125] and [126] for construction of the Brownian motion process (Wiener process). A little later, Itô and McKean in [127] (1965) used for this purpose the complete orthonormal system of Haar functions in $L_2([0,T])$.

Let $\mathbf{f}_\tau$, $\tau\in[0,T]$ be an m-dimestional standard Wiener process with independent components $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m.$

We have

$$\mathbf{f}_s^{(i)}-\mathbf{f}_t^{(i)}=\int\limits_t^s d\mathbf{f}_\tau^{(i)}=\int\limits_t^T \mathbf{1}_{\{\tau<s\}}d\mathbf{f}_\tau^{(i)},$$

where

$$\int\limits_t^T \mathbf{1}_{\{\tau<s\}}d\mathbf{f}_\tau^{(i)}$$

is the Itô stochastic integral, $t\ge 0$, and

$$\mathbf{1}_{\{\tau<s\}}=\begin{cases}1, & \tau<s\\ & \\ 0, & \text{otherwise}\end{cases},\quad \tau,s\in[t,T].$$

Consider the Fourier expansion of $\mathbf{1}_{\{\tau<s\}}\in L_2([t,T])$ at the interval $[t,T]$ (see, for example, [128])

$$\sum_{j=0}^{\infty}\int\limits_t^T \mathbf{1}_{\{\tau<s\}}\phi_j(\tau)d\tau\phi_j(\tau)=\sum_{j=0}^{\infty}\int\limits_t^s \phi_j(\tau)d\tau\phi_j(\tau), \tag{6.1}$$

where $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$ and the series (6.1) converges in the mean-square sense, i.e.

$$\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\phi_j(\tau)\right)^2 d\tau\to 0\quad \text{if}\quad q\to\infty.$$

Let $\mathbf{f}_{s,t}^{(i)q}$ be the mean-square approximation of the process $\mathbf{f}_s^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_{s,t}^{(i)q}=\int\limits_t^T\left(\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\phi_j(\tau)\right)d\mathbf{f}_\tau^{(i)}=\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}. \tag{6.2}$$

Moreover,

$$\mathsf{M}\left\{\left(\mathbf{f}_s^{(i)}-\mathbf{f}_t^{(i)}-\mathbf{f}_{s,t}^{(i)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^q\int\limits_t^s\phi_j(\tau)d\tau\phi_j(\tau)\right)d\mathbf{f}_\tau^{(i)}\right)^2\right\}=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^q\int\limits_t^s\phi_j(\tau)d\tau\phi_j(\tau)\right)^2 d\tau\to 0\quad \text{if } q\to\infty. \tag{6.3}$$

In [70] it was proposed to use an expansion similar to (6.2) for the expansion of iterated Itô stochastic integrals

$$I_{(00)T,t}^{(i_1 i_2)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m). \tag{6.4}$$

At that, to obtain the mentioned expansion of (6.4), the truncated expansions (6.2) of components of the Wiener process $\mathbf{f}_s$ have been iteratively substituted in the single integrals [70]. This procedure leads to the calculation of coefficients of the double Fourier series, which is a time-consuming task for not too complex problem of expansion of the iterated Itô stochastic integral (6.4). In [70] the expansions on the base of Haar functions and trigonometric functions have been considered.

In contrast to [70] we subsitute the expansion (6.2) only one time and only into the innermost integral in (6.4). This procedure leads to the simple calculation of the coefficients

$$\int\limits_t^s\phi_j(\tau)d\tau\quad (j=0,1,2,\ldots)$$

of the usual (not double) Fourier series.

Moreover, we use the Legendre polynomials [49], [129] for the construction of the expansion of (6.4). For the first time the Legendre polynomials have been applied in the framework of the mentioned problem in the author's papers [59] (1997), [60] (1998), [61] (2000), [62] (2001) (also see [1]-[54]) while in the papers of other author's these polynomials have not been considered as the basis functions for the construction of expansions of iterated Itô or Stratonovich stochastic integrals.

Theorem 5.1 [14], [49], [129]. *Let $\phi_j(\tau)$ $(j=0,1,\ldots)$ be a complete orthonormal system of functions in the space $L_2([t,T])$. Let*

$$\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \sum_{j=0}^q \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i_1)} \int\limits_t^T \int\limits_t^s \phi_j(\tau) d\tau d\mathbf{f}_\tau^{(i_2)} \tag{6.5}$$

be an approximation of the iterated Itô stochastic integral (6.4) *for $i_1\ne i_2$. Then*

$$I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)},$$

where $i_1, i_2 = 1,\ldots,m$.

Proof. Using the standard properties of the Itô stochastic integral as well as (6.3) and the property of orthonormality of functions $\phi_j(\tau)$ $(j=0,1,\ldots)$ at the interval $[t,T]$, we obtain

$$\mathsf{M}\left\{\left(\int\limits_t^T \int\limits_t^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} - \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)}\right)^2\right\} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left(\mathbf{f}_s^{(i_1)} - \mathbf{f}_t^{(i_1)} - \mathbf{f}_{s,t}^{(i_1)q}\right)^2\right\} ds =$$

$$= \int\limits_t^T \int\limits_t^T \left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^q \int\limits_t^s \phi_j(\tau) d\tau \phi_j(\tau)\right)^2 d\tau ds =$$

$$= \int\limits_t^T \left((s-t) - \sum_{j=0}^q \left(\int\limits_t^s \phi_j(\tau) d\tau\right)^2\right) ds. \tag{6.6}$$

Because of continuity and nondecreasing of members of the functional sequence

$$u_q(s) = \sum_{j=0}^q \left(\int\limits_t^s \phi_j(\tau) d\tau\right)^2$$

and because of the property of continuity of the limit function $u(s) = s - t$ according to Dini Theorem, we have the uniform convergence $u_q(s)$ to $u(s)$ at the interval $[t, T]$.

Then from this fact as well as from (6.6) we obtain

$$I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)}. \tag{6.7}$$

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$, which has the form (5.5). Then

$$\int\limits_t^s \phi_j(\tau) d\tau = \frac{T-t}{2}\left(\frac{\phi_{j+1}(s)}{\sqrt{(2j+1)(2j+3)}} - \frac{\phi_{j-1}(s)}{\sqrt{4j^2-1}}\right) \quad \text{for} \quad j \geq 1. \tag{6.8}$$

Let us denote

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)} \quad (i = 1, \ldots, m).$$

From (6.5) and (6.8) we obtain

$$\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \frac{1}{\sqrt{T-t}} \zeta_0^{(i_1)} \int\limits_t^T (s-t) \mathbf{f}_s^{(i_2)} +$$

$$+ \frac{T-t}{2} \sum_{j=1}^{q} \zeta_j^{(i_1)} \left(\frac{1}{\sqrt{(2j+1)(2j+3)}} \zeta_{j+1}^{(i_2)} - \frac{1}{\sqrt{4j^2-1}} \zeta_{j-1}^{(i_2)} \right) =$$

$$= \frac{T-t}{2} \zeta_0^{(i_1)} \left(\zeta_0^{(i_2)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_2)} \right) +$$

$$+ \frac{T-t}{2} \sum_{j=1}^{q} \zeta_j^{(i_1)} \left(\frac{1}{\sqrt{(2j+1)(2j+3)}} \zeta_{j+1}^{(i_2)} - \frac{1}{\sqrt{4j^2-1}} \zeta_{j-1}^{(i_2)} \right) =$$

$$= \frac{T-t}{2} \left(\zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{j=1}^{q} \frac{1}{\sqrt{4j^2-1}} \left(\zeta_{j-1}^{(i_1)} \zeta_j^{(i_2)} - \zeta_j^{(i_1)} \zeta_{j-1}^{(i_2)} \right) \right) +$$

$$+ \frac{T-t}{2} \zeta_q^{(i_1)} \zeta_{q+1}^{(i_2)} \frac{1}{\sqrt{(2q+1)(2q+3)}}. \tag{6.9}$$

Then from (6.7) and (6.9) we get

$$
I_{(00)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} =
$$

$$
= \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{j=1}^{\infty}\frac{1}{\sqrt{4j^2-1}}\left(\zeta_{j-1}^{(i_1)}\zeta_j^{(i_2)} - \zeta_j^{(i_1)}\zeta_{j-1}^{(i_2)}\right)\right). \qquad (6.10)
$$

It is not difficult to see that the relation (6.10) has been obtained in Sect. 5.1 (see (5.10)).

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be a complete orthonormal system of trigonomertic functions in the space $L_2([t,T])$, which has the form (5.103).

We have

$$
\int\limits_t^s \phi_j(\tau)d\tau = \frac{T-t}{2\pi r}\begin{cases} \phi_{2r-1}(s), & j = 2r \\ \\ \sqrt{2}\phi_0(s) - \phi_{2r}(s), & j = 2r-1 \end{cases}, \qquad (6.11)
$$

where $j \ge 1$ and $r = 1, 2, \ldots$

From (6.5) and (6.11) we obtain

$$
\int\limits_t^T \mathbf{f}_{s,t}^{(i_1)q} d\mathbf{f}_s^{(i_2)} = \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\int\limits_t^T (s-t)\mathbf{f}_s^{(i_2)} +
$$

$$
+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right) =
$$

$$
= \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right) +
$$

$$
+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right) =
$$

$$
= \frac{1}{2}(T-t)\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.
$$

$$
\left.\left. +\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\right) -
$$

$$
-\frac{T-t}{\pi\sqrt{2}}\zeta_0^{(i_1)}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}. \tag{6.12}
$$

From (6.12) and (6.7) we get

$$
I_{(00)T,t}^{(i_1 i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+
$$

$$
+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr)\Biggr), \tag{6.13}
$$

where $i_1\ne i_2$.

It is obvious that (6.13) is consistent with (5.79) for $i_1\ne i_2$ (we consider here (5.79) without the random variables $\xi_q^{(i)}$).

6.2 Milstein method of Expansion of Iterated Itô and Stratonovich Stochastic Integrals

The method that is considered in this section was proposed by Milstein G.N. [65] (1988) and probably until the mid-2000s remained one of the most famous methods for strong approximation of iterated stochastic integrals (also see [66]-[68], [74]-[76], [79], [80]). However, in light of the results of Chapters 1 and 2 as well as Sect. 5.1 and 5.3, it can be argued that the method based on Theorem 1.1 is more general and effective.

The mentioned Milstein method [65] is based on the expansion of the Brownian bridge process into the trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loève expansion).

Let us consider the Brownian bridge process

$$
\mathbf{f}_t-\frac{t}{\Delta}\mathbf{f}_\Delta,\quad t\in[0,\Delta],\quad \Delta>0, \tag{6.14}
$$

where $\mathbf{f}_t$ is a standard Wiener process with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$.

The componentwise Karhunen–Loève expansion of the process (6.14) has the following form

$$
\mathbf{f}_t^{(i)}-\frac{t}{\Delta}\mathbf{f}_\Delta^{(i)}=\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{6.15}
$$

where the series converges in the mean-square sense and

$$a_{i,r} = \frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)} - \frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,$$

$$b_{i,r} = \frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)} - \frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds,$$

$r = 0, 1, \ldots, \ i = 1, \ldots, m.$

It is easy to demonstrate [65] that the random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\} = \mathsf{M}\left\{a_{i,r}b_{i,k}\right\} = 0, \quad \mathsf{M}\left\{a_{i,r}a_{i,k}\right\} = \mathsf{M}\left\{b_{i,r}b_{i,k}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\} = \mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\} = 0, \quad \mathsf{M}\left\{a_{i,r}^2\right\} = \mathsf{M}\left\{b_{i,r}^2\right\} = \frac{\Delta}{2\pi^2 r^2},$$

where $i, i_1, i_2 = 1, \ldots, m, \ r \neq k, \ i_1 \neq i_2$.

According to (6.15), we have

$$\mathbf{f}_t^{(i)} = \mathbf{f}_{\Delta}^{(i)}\frac{t}{\Delta} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{6.16}$$

where the series converges in the mean-square sense.

Note that the trigonometric functions are the eigenfunctions of the covariance operator of the Brownian bridge process. That is why the basis functions are the trigonometric functions in the considered approach.

Using the relation (6.16), it is easy to get the following expansions [65]-[67]

$$\int\limits_0^t d\mathbf{f}_{\tau}^{(i)} = \frac{t}{\Delta}\mathbf{f}_{\Delta}^{(i)} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{6.17}$$

$$\int\limits_0^t\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i)}d\tau = \frac{t^2}{2\Delta}\mathbf{f}_{\Delta}^{(i)} + \frac{t}{2}a_{i,0} +$$

$$+\frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(a_{i,r}\sin\frac{2\pi rt}{\Delta} - b_{i,r}\left(\cos\frac{2\pi rt}{\Delta} - 1\right)\right), \tag{6.18}$$

$$
\int\limits_0^t\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i)} = t\int\limits_0^t d\mathbf{f}_t^{(i)} - \int\limits_0^t\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \frac{t^2}{2\Delta}\mathbf{f}_{\Delta}^{(i)} +
$$

$$
+t\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi r t}{\Delta} + b_{i,r}\sin\frac{2\pi r t}{\Delta}\right) -
$$

$$
-\frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(a_{i,r}\sin\frac{2\pi r t}{\Delta} - b_{i,r}\left(\cos\frac{2\pi r t}{\Delta} - 1\right)\right), \qquad (6.19)
$$

$$
\int\limits_0^t\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{1}{\Delta}\mathbf{f}_{\Delta}^{(i_1)}\int\limits_0^t\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i_2)} + \frac{1}{2}a_{i_1,0}\int\limits_0^t d\mathbf{f}_{\tau}^{(i_2)} +
$$

$$
+\frac{t\pi}{\Delta}\sum_{r=1}^{\infty} r\left(a_{i_1,r}b_{i_2,r} - b_{i_1,r}a_{i_2,r}\right) +
$$

$$
+\frac{1}{4}\sum_{r=1}^{\infty}\Biggl(\left(a_{i_1,r}a_{i_2,r} - b_{i_1,r}b_{i_2,r}\right)\left(1 - \cos\frac{4\pi r t}{\Delta}\right) +
$$

$$
+\left(a_{i_1,r}b_{i_2,r} + b_{i_1,r}a_{i_2,r}\right)\sin\frac{4\pi r t}{\Delta} +
$$

$$
+\frac{2}{\pi r}\mathbf{f}_{\Delta}^{(i_2)}\left(a_{i_1,r}\sin\frac{2\pi r t}{\Delta} + b_{i_1,r}\left(\cos\frac{2\pi r t}{\Delta} - 1\right)\right)\Biggr) +
$$

$$
+\sum_{k=1}^{\infty}\sum_{r=1 (r\ne k)}^{\infty} k\Biggl(a_{i_1,r}a_{i_2,k}\left(\frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} + \frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)} - \frac{k}{k^2-r^2}\right) +
$$

$$
+a_{i_1,r}b_{i_2,k}\left(\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} + \frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right) +
$$

$$
+b_{i_1,r}b_{i_2,k}\left(\frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)} - \frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} - \frac{r}{k^2-r^2}\right) +
$$

$$
+\frac{\Delta}{2\pi}b_{i_1,r}a_{i_2,k}\left(\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} - \frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right)\Biggr) \qquad (6.20)
$$

converging in the mean-square sense, where we suppose that $i_1 \neq i_2$ in (6.20).

It is necessary to pay a special attention to the fact that the double series in (6.20) should be understood as the iterated one, and not as a multiple series (as in Theorem 1.1), i.e. as the iterated passage to the limit for the sequence of double partial sums. So, the Milstein method of approximation of iterated stochastic integrals [65] leads to iterated application of the limit transition (in contrast with the method of generalized multiple Fourier series (Theorem 1.1), for which the limit transition is implemented only once) starting at least from the second or third multiplicity of iterated stochastic integrals (we mean at least double or triple integration with respect to components of the Wiener process). Multiple series are more preferential for approximation than the iterated ones, since the partial sums of multiple series converge for any possible case of joint converging to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, as we will see further in this section in [66] (pp. 438-439), [67] (Sect. 5.8, pp. 202–204), [68] (pp. 82-84), [76] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the Milstein method [65] together with the Wong–Zakai approximation [56]-[58] (also see discussion in Sect. 2.6.2). Furthermore, in order to obtain the Milstein expansion for iterated stochastic integral, the truncated expansions (6.16) of components of the Wiener process $\mathbf{f}_t$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to the expansion of iterated stochastic integral of multiplicity k ($k \in \mathbf{N}$).

Assume that $t = \Delta$ in the relations (6.17)–(6.20) (at that double partial sums of iterated series in (6.20) will become zero). As a result, we get the following expansions

$$\int\limits_0^{\Delta} d\mathbf{f}_{\tau}^{(i)} = \mathbf{f}_{\Delta}^{(i)}, \tag{6.21}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \frac{1}{2}\Delta\left(\mathbf{f}_{\Delta}^{(i)} + a_{i,0}\right), \tag{6.22}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i)} = \frac{1}{2}\Delta\left(\mathbf{f}_{\Delta}^{(i)} - a_{i,0}\right), \tag{6.23}$$

$$
\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{1}{2}\mathbf{f}_{\Delta}^{(i_1)}\mathbf{f}_{\Delta}^{(i_2)} - \frac{1}{2}\left(a_{i_2,0}\mathbf{f}_{\Delta}^{(i_1)} - a_{i_1,0}\mathbf{f}_{\Delta}^{(i_2)}\right) +
$$

$$
+\pi\sum_{r=1}^{\infty} r\left(a_{i_1,r}b_{i_2,r} - b_{i_1,r}a_{i_2,r}\right) \qquad (6.24)
$$

converging in the mean-square sense, where we suppose that $i_1 \ne i_2$ in (6.24).

Deriving (6.21)–(6.24), we used the relation

$$
a_{i,0} = -2\sum_{r=1}^{\infty} a_{i,r}, \qquad (6.25)
$$

which results from (6.15) when $t = \Delta$.

Let us compare expansions of some iterated stochastic integrals of first and second multiplicity obtained by Milstein method [65] and method based on generalized multiple Fourier series (Theorem 1.1).

Let us denote

$$
\zeta_{2r-1}^{(i)} = \sqrt{\frac{2}{\Delta}}\int\limits_0^{\Delta} \sin\frac{2\pi r s}{\Delta} d\mathbf{f}_s^{(i)}, \quad \zeta_{2r}^{(i)} = \sqrt{\frac{2}{\Delta}}\int\limits_0^{\Delta} \cos\frac{2\pi r s}{\Delta} d\mathbf{f}_s^{(i)}, \qquad (6.26)
$$

$$
\zeta_0^{(i)} = \frac{1}{\sqrt{\Delta}}\int\limits_0^{\Delta} d\mathbf{f}_s^{(i)}, \qquad (6.27)
$$

where $r = 1, 2, \ldots,\ i = 1, \ldots, m$.

Using the Itô formula, it is not difficult to show that

$$
a_{i,r} = -\frac{1}{\pi r}\sqrt{\frac{\Delta}{2}}\zeta_{2r-1}^{(i)}, \quad b_{i,r} = \frac{1}{\pi r}\sqrt{\frac{\Delta}{2}}\zeta_{2r}^{(i)} \quad \text{w. p. 1.} \qquad (6.28)
$$

From (6.25) we get

$$
a_{i,0} = \frac{\sqrt{2\Delta}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)}. \qquad (6.29)
$$

After substituting (6.28), (6.29) into (6.21)–(6.24) and taking into account (6.26), (6.27), we have

$$
\int\limits_0^{\Delta} d\mathbf{f}_{\tau}^{(i_1)} = \sqrt{\Delta}\zeta_0^{(i_1)}, \qquad (6.30)
$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i_1)} = \frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{6.31}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\tau = \frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{6.32}$$

$$\int\limits_0^{\Delta}\int\limits_0^{\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.$$

$$\left.\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\right). \tag{6.33}$$

Obviously, the formulas (6.30)–(6.33) are consistent with the formulas (5.6), (5.79), (5.97), (5.98). It testifies that at least for the considered iterated stochastic integrals and trigonometric system of functions, the Milstein method and the method based on generalized multiple Fourier series (Theorem 1.1) give the same result (it is an interesting fact, although it is rather expectable).

Further, we will discuss the usage of Milstein method for the iterated stochastic integrals of third multiplicity.

First, we note that the authors of the monograph [67] based on the results of Wong E. and Zakai M. [56], [57] (also see [58]) concluded (without rigorous proof) that the expansions of iterated stochastic integrals on the basis of (6.16) (the case $i_1, i_2, i_3 = 1,\ldots,m$) converge to the iterated Stratonovich stochastic integrals (see discussion in Sect. 2.6.2). It is obvious that this conclusion is consistent with the results given above in this section for the case $i_1 \neq i_2$.

As we mentioned before, the technical peculiarities of the Milstein method [65] may result to the iterated series of products of standard Gaussian random variables (in contradiction to multiple series as in Theorem 1.1). In the case of simplest stochastic integral of second multiplicity this problem was avoided as we saw above. However, the situation is not the same for the simplest stochastic integrals of third multiplicity.

Let us denote

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\lambda_l = 1$ if $i_l = 1, \ldots, m$ and $\lambda_l = 0$ if $i_l = 0$, $l = 1, \ldots, k$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Let us consider the expansion of iterated Stratonovich stochastic integral of third multiplicity obtained in [66]-[68], [76] by the Milstein method [65]

$$J^{*(i_1 i_2 i_3)}_{(111)\Delta,0} = \frac{1}{\Delta} J^{*(i_1)}_{(1)\Delta,0} J^{*(0 i_2 i_3)}_{(011)\Delta,0} +$$

$$+\frac{1}{2} a_{i_1,0} J^{*(i_2 i_3)}_{(11)\Delta,0} + \frac{1}{2\pi} b_{i_1} J^{(i_2)}_{(1)\Delta,0} J^{*(i_3)}_{(1)\Delta,0} - \Delta J^{*(i_2)}_{(1)\Delta,0} B_{i_1 i_3} +$$

$$+ \Delta J^{*(i_3)}_{(1)\Delta,0} \left(\frac{1}{2} A_{i_1 i_2} - C_{i_2 i_1}\right) + \Delta^{3/2} D_{i_1 i_2 i_3}, \qquad (6.34)$$

where

$$J^{*(0 i_2 i_3)}_{(011)\Delta,0} = \frac{1}{6} J^{*(i_2)}_{(1)\Delta,0} J^{*(i_3)}_{(1)\Delta,0} - \frac{1}{\pi} \Delta J^{*(i_3)}_{(1)\Delta,0} b_{i_2} +$$

$$+\Delta^2 B_{i_2 i_3} - \frac{1}{4} \Delta a_{i_3,0} J^{*(i_2)}_{(1)\Delta,0} + \frac{1}{2\pi} \Delta b_{i_3} J^{*(i_2)}_{(1)\Delta,0} + \Delta^2 C_{i_2 i_3} + \frac{1}{2} \Delta^2 A_{i_2 i_3},$$

$$A_{i_2 i_3} = \frac{\pi}{\Delta} \sum_{r=1}^{\infty} r \left(a_{i_2,r} b_{i_3,r} - b_{i_2,r} a_{i_3,r}\right),$$

$$C_{i_2 i_3} = -\frac{1}{\Delta} \sum_{l=1}^{\infty} \sum_{r=1 (r \ne l)}^{\infty} \frac{r}{r^2 - l^2} \left(r a_{i_2,r} a_{i_3,l} + l b_{i_2,r} b_{i_3,l}\right),$$

$$B_{i_2 i_3} = \frac{1}{2\Delta} \sum_{r=1}^{\infty} \left(a_{i_2,r} a_{i_3,r} + b_{i_2,r} b_{i_3,r}\right), \ \ b_i = \sum_{r=1}^{\infty} \frac{1}{r} b_{i,r},$$

$$D_{i_1 i_2 i_3} = -\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} l \Biggl(a_{i_2,l} \left(a_{i_3,l+r} b_{i_1,r} - a_{i_1,r} b_{i_3,l+r}\right) +$$

$$+ b_{i_2,l} \left(a_{i_1,r} a_{i_3,r+l} + b_{i_1,r} b_{i_3,l+r}\right) \Biggr) +$$

$$+\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{l-1} l \Biggl(a_{i_2,l} \left(a_{i_1,r} b_{i_3,l-r} + a_{i_3,l-r} b_{i_1,r}\right) -$$

$$
\left.-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\right)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{\infty}\sum_{r=l+1}^{\infty} l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+
$$

$$
\left.+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\right).
$$

From the expansion (6.34) and expansion of the stochastic integral $J^{*(0i_2i_3)}_{(011)\Delta,0}$ we can conclude that they include iterated (double) series. Moreover, for approximation of the stochastic integral $J^{*(i_1i_2i_3)}_{(111)\Delta,0}$ in the works [66] (pp. 438-439), [67] (Sect. 5.8, pp. 202–204), [68] (pp. 82-84), [76] (pp. 263-264) it is proposed to put upper limits of summation by equal q (on the base of the Wong–Zakai approximation [56]-[58] but without rigorous proof; also see discussion in Sect. 2.6.2).

For example, the value $D_{i_1i_2i_3}$ is approximated in [66]-[68], [76] by the double sums of the form

$$
D^q_{i_1i_2i_3}=-\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{q} l\Biggl(a_{i_2,l}\left(a_{i_3,l+r}b_{i_1,r}-a_{i_1,r}b_{i_3,l+r}\right)+
$$

$$
\left.+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r+l}+b_{i_1,r}b_{i_3,l+r}\right)\right)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{l-1} l\Biggl(a_{i_2,l}\left(a_{i_1,r}b_{i_3,l-r}+a_{i_3,l-r}b_{i_1,r}\right)-
$$

$$
\left.-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\right)+
$$

$$
+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=l+1}^{2q} l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+
$$

$$
\left.+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\right).
$$

We can avoid the mentioned problem (iterated application of the operation of limit transition) using the method based on Theorems 1.1, 2.1–2.9.

From the other hand, if we prove that the members of the expansion (6.34) coincide with the members of its analogue obtained using Theorem 1.1, then we can replace the iterated series in (6.34) by the multiple series (see Theorems 1.1, 2.1–2.9) as was made formally in [66]-[68], [76]. However, it requires the separate argumentation.

6.3 Usage of Integral Sums for Approximation of Iterated Itô Stochastic Integrals

It should be noted that there is an approach to the mean-square approximation of iterated stochastic integrals based on multiple integral sums (see, for example, [65], [75], [77], [130]). This method implies the partitioning of the integration interval $[t,T]$ of the iterated stochastic integral under consideration; this interval is the integration step of the numerical methods used to solve Itô SDEs (see Chapter 4); therefore, it is already fairly small and does not need to be partitioned. Computational experiments [1] (also see below in this section) show that the application of the method [65], [75], [77], [130] to stochastic integrals with multiplicities $k\ge 2$ leads to unacceptably high computational cost and accumulation of computation errors.

As we noted in the introduction to this book, considering the modern state of question on the approximation of iterated stochastic integrals, the method analyzed in this section is hardly important for practice. However, we will consider this method in order to get the overall view. In this section, we will analyze one of the simplest modifications of the mentioned method.

Let the functions $\psi_l(\tau)$, $l=1,\ldots,k$ satisfy the Lipschitz condition at the interval $[t,T]$ with constants C_l

$$|\psi_l(\tau_1)-\psi_l(\tau_2)|\le C_l|\tau_1-\tau_2|\quad \hbox{for all}\quad \tau_1,\tau_2\in[t,T]. \tag{6.35}$$

Then, according to Lemma 1.1 (see Sect. 1.1.3), the following equality is correct

$$J[\psi^{(k)}]_{T,t}=\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr l.i.m.\cr}}}\lim_{N\to\infty}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^k\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\quad \hbox{w. p. 1},$$

where notations are the same as in (1.12).

Let us consider the following approximation

$$J[\psi^{(k)}]_{T,t}^N = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \qquad (6.36)$$

of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}$. The relation (6.36) can be rewritten as

$$J[\psi^{(k)}]_{T,t}^N = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \sqrt{\Delta \tau_{j_l}} \psi_l(\tau_{j_l}) \mathbf{u}_{j_l}^{(i_l)}, \qquad (6.37)$$

where $\mathbf{u}_j^{(i)} \stackrel{\text{def}}{=} \left(\mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)} \right) / \sqrt{\Delta \tau_j}$, $i = 1, \ldots, m$ are independent standard Gaussian random variables for various i or j, $\mathbf{u}_j^{(0)} = \sqrt{\Delta \tau_j}$.

Assume that

$$\tau_j = t + j\Delta, \quad j = 0, 1, \ldots, N, \quad \tau_N = T, \quad \Delta > 0. \qquad (6.38)$$

Then

$$J[\psi^{(k)}]_{T,t}^N = \Delta^{k/2} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(t + j_l \Delta) \mathbf{u}_{j_l}^{(i_l)}, \qquad (6.39)$$

where $\mathbf{u}_j^{(i)} \stackrel{\text{def}}{=} \left(\mathbf{w}_{t+(j+1)\Delta}^{(i)} - \mathbf{w}_{t+j\Delta}^{(i)} \right) / \sqrt{\Delta}$, $i = 1, \ldots, m$, $\mathbf{u}_j^{(0)} = \sqrt{\Delta}$.

Lemma 6.1. *Suppose that the functions $\psi_l(\tau)$, $l = 1, \ldots k$ satisfy the Lipschitz condition* (6.35) *and $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition* (6.38). *Then for a sufficiently small value $T - t$ there exists a constant $H_k < \infty$ such that*

$$\mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^N \right)^2 \right\} \le \frac{H_k (T-t)^2}{N}.$$

Proof. It is easy to see that in the case of a sufficiently small value $T - t$ there exists a constant L_k such that

$$\mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^N \right)^2 \right\} \le L_k \mathsf{M}\left\{ \left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^N \right)^2 \right\},$$

where

$$J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^N = \sum_{j=1}^{3} S_j^N,$$

$$S_1^N=\sum_{j_1=0}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_2(t_2)\int\limits_{\tau_{j_1}}^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)},$$

$$S_2^N=\sum_{j_1=0}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\psi_2(t_2)-\psi_2(\tau_{j_1})\right)d\mathbf{w}_{t_2}^{(i_2)}\sum_{j_2=0}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)},$$

$$S_3^N=\sum_{j_1=0}^{N-1}\psi_2(\tau_{j_1})\Delta\mathbf{w}_{\tau_{j_1}}^{(i_2)}\sum_{j_2=0}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\left(\psi_1(t_1)-\psi_1(\tau_{j_2})\right)d\mathbf{w}_{t_1}^{(i_1)}.$$

Therefore, according to the Minkowski inequality, we have

$$\left(\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^N\right)^2\right\}\right)^{1/2}\le\sum_{j=1}^{3}\left(\mathsf{M}\left\{\left(S_j^N\right)^2\right\}\right)^{1/2}.$$

Using standard moment properties of stochastic integrals (see (1.25), (1.26)), let us estimate the values $\mathsf{M}\left\{\left(S_j^N\right)^2\right\}$, $j=1,2,3$.

Let us consider four cases.

Case 1. $i_1,i_2\ne 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\}\le\frac{\Delta}{2}(T-t)\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\}\le\frac{\Delta^2}{6}(T-t)^2\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\}\le\frac{\Delta^2}{6}(T-t)^2\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

Case 2. $i_1\ne 0,\ i_2=0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\}\le\frac{\Delta}{2}(T-t)^2\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\}\le\frac{\Delta^2}{3}(T-t)^3\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\}\le\frac{\Delta^2}{3}(T-t)^3\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

Case 3. $i_2 \ne 0,\ i_1 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)^3\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{8}(T-t)^3\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

Case 4. $i_1 = i_2 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^2\max_{s\in[t,T]}\psi_2^2(s)\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^4\left(C_2\right)^2\max_{s\in[t,T]}\psi_1^2(s),$$

$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{16}(T-t)^4\left(C_1\right)^2\max_{s\in[t,T]}\psi_2^2(s).$$

According to the obtained estimates, we have

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^N\right)^2\right\} \le H_k(T-t)\Delta = \frac{H_k(T-t)^2}{N},$$

where $H_k < \infty$. Lemma 6.1 is proved.

It is easy to check that the following relation is correct

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)N}\right)^2\right\} = \frac{(T-t)^2}{2N}, \tag{6.40}$$

where $i_1, i_2 = 1,\ldots,m$ and $I_{(00)T,t}^{(i_1 i_2)N}$ is the approximation of the iterated stochastic integral $I_{(00)T,t}^{(i_1 i_2)}$ (see (6.4)) obtained according to the formula (6.39).

Finally, we will demonstrate that the method based on generalized multiple Fourier series (Theorem 1.1) is signficiantly better, than the method based on multiple integral sums in the sense of computational costs on modeling of iterated stochastic integrals.

Let us consider the approximations of iterated Itô stochastic integrals obtained using the method based on multiple integral sums

$$I_{(0)T,t}^{(1)q} = \sqrt{\Delta}\sum_{j=0}^{q-1}\xi_j^{(1)}, \tag{6.41}$$

Table 6.1: Values $T_{\rm sum}/T_{\rm pol}$.

$T-t$	2^{-5}	2^{-6}	2^{-7}
$T_{\rm sum}/T_{\rm pol}$	8.67	23.25	55.86

$$
I_{(00)T,t}^{(12)q}=\Delta\sum_{j=0}^{q-1}\left(\sum_{i=0}^{j-1}\xi_i^{(1)}\right)\xi_j^{(2)}, \tag{6.42}
$$

where

$$
\xi_j^{(i)}=\left({\bf f}_{t+(j+1)\Delta}^{(i)}-{\bf f}_{t+j\Delta}^{(i)}\right)/\sqrt{\Delta},\quad i=1,2
$$

are independent standard Gaussian random variables, $\Delta=(T-t)/q$, $I_{(00)T,t}^{(12)q}$, $I_{(0)T,t}^{(1)q}$ are approximations of the iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$ (see (6.4)), $I_{(0)T,t}^{(1)}={\bf f}_T^{(1)}-{\bf f}_t^{(1)}$.

Let us choose the number q (see (6.41), (6.42)) from the condition

$$
{\sf M}\left\{\left(I_{(00)T,t}^{(12)}-I_{(00)T,t}^{(12)q}\right)^2\right\}=\frac{(T-t)^2}{2q}\le (T-t)^3.
$$

Let us implement 200 independent numerical modelings of the collection of iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$, $I_{(0)T,t}^{(1)}$ using the formulas (6.41), (6.42) for $T-t=2^{-j}$, $j=5,6,7$. We denote by $T_{\rm sum}$ the computer time which is necessary for performing this task.

Let us repeat the above experiment for the case when the approximations of iterated Itô stochastic integrals $I_{(00)T,t}^{(12)}$, $I_{(0)T,t}^{(1)}$ are defined by (5.128), (5.129) and the number q is chosen from the condition (5.120) (method based on Theorem 1.1, the case of Legendre polynomials). Let $T_{\rm pol}$ be the computer time which is necessary for performing this task.

Considering the results from Table 6.1, we come to conclusion that the method based on multiple integral sums even when $T-t=2^{-7}$ is more than 50 times worse in terms of computer time for modeling the collection of iterated Itô stochastic integrals $I_{00_{T,t}}^{(12)}$, $I_{0_{T,t}}^{(1)}$, than the method based on generalized multiple Fourier series.

It is not difficult to see that this effect will be more essential if we consider iterated stochastic integrals of multiplicities $3,4,\ldots$ or choose value $T-t$ smaller than 2^{-7}.

6.4 Iterated Itô Stochastic Integrals as Solutions of Systems of Linear Itô SDEs

Milstein G.N. [65] (also see [82])) proposed an approach to numerical modeling of iterated Itô stochastic integrals based on their representation in the form of systems of linear Itô SDEs. Let us consider this approach using the following set of iterated Itô stochastic integrals

$$I_{(0)s,t}^{(i_1)}=\int\limits_t^s d\mathbf{f}_{t_1}^{(i_1)},\quad I_{(00)s,t}^{(i_1 i_2)}=\int\limits_t^s\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}, \tag{6.43}$$

where $i_1,i_2=1,\ldots,m,\ 0\le t<s\le T,\ \mathbf{f}_s^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes.

Obviously, we have the following representation

$$d\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}d\mathbf{f}_s^{(i_2)}+\begin{pmatrix}1&0\\0&0\end{pmatrix}d\begin{pmatrix}\mathbf{f}_s^{(i_1)}\\ \mathbf{f}_s^{(i_2)}\end{pmatrix}. \tag{6.44}$$

It is well known [65], [67] that the solution of system (6.44) has the following integral form

$$\begin{pmatrix} I_{(0)s,t}^{(i_1)}\\ \\ I_{(00)s,t}^{(i_1 i_2)}\end{pmatrix}=\int\limits_t^s e^{\begin{pmatrix}0&0\\1&0\end{pmatrix}\left(\mathbf{f}_s^{(i_2)}-\mathbf{f}_\theta^{(i_2)}\right)}\begin{pmatrix}1&0\\0&0\end{pmatrix}d\begin{pmatrix}\mathbf{f}_\theta^{(i_1)}\\ \mathbf{f}_\theta^{(i_2)}\end{pmatrix}, \tag{6.45}$$

where e^A is a matrix exponent

$$e^A\stackrel{\rm def}{=}\sum_{k=0}^{\infty}\frac{A^k}{k!},$$

A is a square matrix, and $A^0\stackrel{\rm def}{=}I$ is a unity matrix.

Numerical modeling of the right-hand side of (6.45) is unlikely simpler task than the jointly numerical modeling of the collection of stochastic integrals (6.43). We have to perform numerical modeling of (6.43) within the frames of the considered approach by numerical integration of the system of linear Itô SDEs (6.44). This procedure can be realized using the Euler (Euler–Maruyama) method [65]. Note that the expressions of more accurate numerical methods for the system (6.44) (see Chapter 4) contain the iterated Itô stochastic integrals (6.43) and therefore they useless in our situation.

Let $\{\tau_j\}_{j=0}^N$ be the partition of $[t,s]$ such that

$$\tau_j = t + j\Delta, \quad j = 0, 1, \ldots, N, \quad \tau_N = s.$$

Let us consider the Euler method for the system of linear Itô SDEs (6.44)

$$\begin{pmatrix} \mathbf{y}_{p+1}^{(i_1)} \\ \\ \mathbf{y}_{p+1}^{(i_1 i_2)} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_{p}^{(i_1)} \\ \\ \mathbf{y}_{p}^{(i_1 i_2)} \end{pmatrix} + \begin{pmatrix} \Delta\mathbf{f}_{\tau_p}^{(i_1)} \\ \\ \mathbf{y}_{p}^{(i_1)}\Delta\mathbf{f}_{\tau_p}^{(i_2)} \end{pmatrix}, \quad \mathbf{y}_0^{(i_1)} = 0, \quad \mathbf{y}_0^{(i_1 i_2)} = 0, \tag{6.46}$$

where

$$\mathbf{y}_{\tau_p}^{(i_1)} \stackrel{\text{def}}{=} \mathbf{y}_p^{(i_1)}, \quad \mathbf{y}_{\tau_p}^{(i_1 i_2)} \stackrel{\text{def}}{=} \mathbf{y}_p^{(i_1 i_2)}$$

are approximations of the iterated Itô stochastic integrals $I_{(0)\tau_p,t}^{(i_1)}$, $I_{(00)\tau_p,t}^{(i_1 i_2)}$ obtained using the numerical scheme (6.46), $\Delta\mathbf{f}_{\tau_p}^{(i)} = \mathbf{f}_{\tau_{p+1}}^{(i)} - \mathbf{f}_{\tau_p}^{(i)}$, $i = 1, \ldots, m$.

Iterating the expression (6.46), we have

$$\mathbf{y}_N^{(i_1)} = \sum_{l=0}^{N-1} \Delta\mathbf{f}_{\tau_l}^{(i_1)}, \quad \mathbf{y}_N^{(i_1 i_2)} = \sum_{q=0}^{N-1}\sum_{l=0}^{q-1} \Delta\mathbf{f}_{\tau_l}^{(i_1)}\Delta\mathbf{f}_{\tau_q}^{(i_2)}, \tag{6.47}$$

where $\sum\limits_{\emptyset} \stackrel{\text{def}}{=} 0$.

Obviously, the formulas (6.47) are formulas for approximations of the iterated Itô stochastic integrals (6.43) obtained using the method based on multiple integral sums (see (6.41), (6.42)).

Consequently, the efficiency of methods for the approximation of iterated Itô stochastic integrals based on multiple integral sums and numerical integration of systems of linear Itô SDEs on the base of the Euler method turns out to be equivalent.

6.5 Combined Method of the Mean-Square Approximation of Iterated Itô Stochastic Integrals

This section is written of the base of the work [131] (also see [14]) and devoted to the combined method of approximation of iterated Itô stochastic integrals based on Theorem 1.1 and the method of multiple integral sums (see Sect. 6.3).

The combined method of approximation of iterated Itô stochastic integrals provides a possibility to minimize significantly the total number of the Fourier–Legendre coefficients which are necessary for the approximation of iterated Itô stochastic integrals. However, in this connection the computational costs for approximation of the mentioned stochastic integrals are become bigger.

Using the additive property of the Itô stochastic integral, we have

$$I_{(0)T,t}^{(i_1)}=\sqrt{\Delta}\sum_{k=0}^{N-1}\zeta_{0,k}^{(i_1)}\quad \text{w. p. 1}, \tag{6.48}$$

$$I_{(1)T,t}^{(i_1)}=\sum_{k=0}^{N-1}\left(I_{(1)\tau_{k+1},\tau_k}^{(i_1)}-\Delta^{3/2}k\zeta_{0,k}^{(i_1)}\right)\quad \text{w. p. 1}, \tag{6.49}$$

$$I_{(00)T,t}^{(i_1i_2)}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\zeta_{0,l}^{(i_1)}\zeta_{0,k}^{(i_2)}+\sum_{k=0}^{N-1}I_{(00)\tau_{k+1},\tau_k}^{(i_1i_2)}\quad \text{w. p. 1}, \tag{6.50}$$

$$I_{(000)T,t}^{(i_1i_2i_3)}=\Delta^{3/2}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\sum_{q=0}^{l-1}\zeta_{0,q}^{(i_1)}\zeta_{0,l}^{(i_2)}\zeta_{0,k}^{(i_3)}+$$

$$+\sqrt{\Delta}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1i_2)}\zeta_{0,k}^{(i_3)}+\zeta_{0,l}^{(i_1)}I_{(00)\tau_{k+1},\tau_k}^{(i_2i_3)}\right)+\sum_{k=0}^{N-1}I_{(000)\tau_{k+1},\tau_k}^{(i_1i_2i_3)}\quad \text{w. p. 1}, \tag{6.51}$$

where stochastic integrals

$$I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1i_2)},\quad I_{(000)T,t}^{(i_1i_2i_3)}$$

have the form (5.3), $i_1,\ldots,i_k=1,\ldots,m$, $T-t=N\Delta$, $\tau_k=t+k\Delta$,

$$\zeta_{0,k}^{(i)}\stackrel{\text{def}}{=}\frac{1}{\sqrt{\Delta}}\int\limits_{\tau_k}^{\tau_{k+1}}d\mathbf{f}_s^{(i)},$$

$k=0,1,\ldots,N-1$, the sum with respect to the empty set is equal to zero.

Substituting the relation

$$I_{(1)\tau_{k+1},\tau_k}^{(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_{0,k}^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\quad \text{w. p. 1}$$

into (6.49), where $\zeta_{0,k}^{(i_1)}$, $\zeta_{1,k}^{(i_1)}$ are independent standard Gaussian random variables, we get

$$I_{(1)T,t}^{(i_1)}=-\Delta^{3/2}\sum_{k=0}^{N-1}\left(\left(\frac{1}{2}+k\right)\zeta_{0,k}^{(i_1)}+\frac{1}{2\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\quad \text{w. p. 1}. \tag{6.52}$$

Consider approximations of the following iterated Itô stochastic integrals using the method based on multiple Fourier–Legendre series (Theorem 1.1)

$$I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)}, \quad I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)}, \quad I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)}.$$

As a result we get

$$I_{(00)T,t}^{(i_1 i_2)N,q} = \Delta \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \zeta_{0,l}^{(i_1)} \zeta_{0,k}^{(i_2)} + \sum_{k=0}^{N-1} I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q}, \tag{6.53}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)N,q_1,q_2} = \Delta^{3/2} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \sum_{q=0}^{l-1} \zeta_{0,q}^{(i_1)} \zeta_{0,l}^{(i_2)} \zeta_{0,k}^{(i_3)} +$$

$$+ \sqrt{\Delta} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1} \zeta_{0,k}^{(i_3)} + \zeta_{0,l}^{(i_1)} I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1} \right) + \sum_{k=0}^{N-1} I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}, \tag{6.54}$$

where we suppose that the approximations

$$I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q}, \quad I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q_1}, \quad I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}$$

are obtained using Theorem 1.1 (the case of Legendre polynomials).

In particular, when $N = 2$, the formulas (6.48), (6.52)-(6.54) will look as follows

$$I_{(0)T,t}^{(i_1)} = \sqrt{\Delta} \left(\zeta_{0,0}^{(i_1)} + \zeta_{0,1}^{(i_1)} \right) \quad \text{w. p. 1}, \tag{6.55}$$

$$I_{(1)T,t}^{(i_1)} = -\Delta^{3/2} \left(\frac{1}{2} \zeta_{0,0}^{(i_1)} + \frac{3}{2} \zeta_{0,1}^{(i_1)} + \frac{1}{2\sqrt{3}} \left(\zeta_{1,0}^{(i_1)} + \zeta_{1,1}^{(i_1)} \right) \right) \quad \text{w. p. 1}, \tag{6.56}$$

$$I_{(00)T,t}^{(i_1 i_2)2,q} = \Delta \left(\zeta_{0,0}^{(i_1)} \zeta_{0,1}^{(i_2)} + I_{(00)\tau_1,\tau_0}^{(i_1 i_2)q} + I_{(00)\tau_2,\tau_1}^{(i_1 i_2)q} \right), \tag{6.57}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)2,q_1,q_2} = \sqrt{\Delta} \left(I_{(00)\tau_1,\tau_0}^{(i_1 i_2)q_1} \zeta_{0,1}^{(i_3)} + \zeta_{0,0}^{(i_1)} I_{(00)\tau_2,\tau_1}^{(i_2 i_3)q_1} \right) +$$

$$+ I_{(000)\tau_1,\tau_0}^{(i_1 i_2 i_3)q_2} + I_{(000)\tau_2,\tau_1}^{(i_1 i_2 i_3)q_2}, \tag{6.58}$$

where $\Delta = (T - t)/2$, $\tau_k = t + k\Delta$, $k = 0, 1, 2$.

Note that if $N = 1$, then (6.48), (6.52)-(6.54) are the formulas for numerical modeling of the mentioned stochastic integrals using the method based on Theorem 1.1.

Further, we will demonstrate that modeling of the iterated Itô stochastic integrals

$$
I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1 i_2)},\quad I_{(000)T,t}^{(i_1 i_2 i_3)}
$$

using the formulas (6.55)–(6.58) results in abrupt decrease of the total number of Fourier–Legendre coefficients, which are necessary for approximation of these stochastic integrals using the method based on Theorem 1.1.

From the other hand, the formulas (6.57), (6.58) include two approximations of iterated Itô stochastic integrals of second and third multiplicity, and each one of them should be obtained using the method based on Theorem 1.1. Obviously, this leads to an increase in computational costs for the approximation.

Let us calculate the mean-square approximation errors for the formulas (6.53), (6.54). We have

$$
E_N^q \stackrel{\rm def}{=} {\sf M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)N,q}\right)^2\right\} = \sum_{k=0}^{N-1} {\sf M}\left\{\left(I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)} - I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q}\right)^2\right\} =
$$

$$
= N\frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right) = \frac{(T-t)^2}{2N}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right), \tag{6.59}
$$

$$
E_N^{q_1,q_2} \stackrel{\rm def}{=} {\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)N,q_1,q_2}\right)^2\right\} =
$$

$$
= {\sf M}\Biggl\{\Biggl(\sum_{k=0}^{N-1}\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)} - I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1}\right) +
$$

$$
+\zeta_{0,l}^{(i_1)}\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)} - I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1}\right)\Biggr)\Biggr) + I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)} - I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}\Biggr)\Biggr)^2\Biggr\} =
$$

$$
= \sum_{k=0}^{N-1}{\sf M}\Biggl\{\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)} - I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1}\right) +
$$

$$
+\zeta_{0,l}^{(i_1)}\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)} - I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1}\right)\Biggr)\Biggr) + I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)} - I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2}\Biggr)^2\Biggr\} =
$$

$$
= \sum_{k=0}^{N-1} \left(\Delta \mathsf{M} \left\{ \left(\zeta_{0,k}^{(i_3)} \sum_{l=0}^{k-1} \left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)} - I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1} \right) \right)^2 \right\} + \right.
$$

$$
\left. + \Delta \mathsf{M} \left\{ \left(\left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)} - I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1} \right) \sum_{l=0}^{k-1} \zeta_{0,l}^{(i_1)} \right)^2 \right\} + H_{k,q_2}^{(i_1 i_2 i_3)} \right) =
$$

$$
= \sum_{k=0}^{N-1} \left(\Delta \sum_{l=0}^{k-1} \mathsf{M} \left\{ \left(I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)} - I_{(00)\tau_{l+1},\tau_l}^{(i_1 i_2)q_1} \right)^2 \right\} + \right.
$$

$$
\left. + k\Delta \mathsf{M} \left\{ \left(I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)} - I_{(00)\tau_{k+1},\tau_k}^{(i_2 i_3)q_1} \right)^2 \right\} + H_{k,q_2}^{(i_1 i_2 i_3)} \right) =
$$

$$
= \sum_{k=0}^{N-1} \left(2k\Delta \mathsf{M} \left\{ \left(I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)} - I_{(00)\tau_{k+1},\tau_k}^{(i_1 i_2)q_1} \right)^2 \right\} + H_{k,q_2}^{(i_1 i_2 i_3)} \right) =
$$

$$
= \sum_{k=0}^{N-1} \left(2k\Delta \frac{\Delta^2}{2} \left(\frac{1}{2} - \sum_{l=1}^{q_1} \frac{1}{4l^2 - 1} \right) + H_{k,q_2}^{(i_1 i_2 i_3)} \right) =
$$

$$
= \Delta^3 \frac{N(N-1)}{2} \left(\frac{1}{2} - \sum_{l=1}^{q_1} \frac{1}{4l^2 - 1} \right) + \sum_{k=0}^{N-1} H_{k,q_2}^{(i_1 i_2 i_3)} =
$$

$$
= \frac{1}{2}(T-t)^3 \left(\frac{1}{N} - \frac{1}{N^2} \right) \left(\frac{1}{2} - \sum_{l=1}^{q_1} \frac{1}{4l^2 - 1} \right) +
$$

$$
+ \sum_{k=0}^{N-1} H_{k,q_2}^{(i_1 i_2 i_3)}, \tag{6.60}
$$

where

$$
H_{k,q_2}^{(i_1 i_2 i_3)} = \mathsf{M} \left\{ \left(I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)} - I_{(000)\tau_{k+1},\tau_k}^{(i_1 i_2 i_3)q_2} \right)^2 \right\}.
$$

Moreover, we suppose that $i_1 \neq i_2$ in (6.59) and not all indices i_1, i_2, i_3 in (6.60) are equal. Otherwise there are simple relationships for modeling the integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$

$$
I_{(00)T,t}^{(i_1 i_1)} = \frac{1}{2}(T-t) \left(\left(\zeta_0^{(i_1)} \right)^2 - 1 \right) \quad \text{w. p. 1},
$$

$$I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right) \quad \text{w. p. 1},$$

where

$$\zeta_0^{(i_1)} = \frac{1}{\sqrt{T-t}}\int\limits_t^T d\mathbf{f}_s^{(i_1)}$$

is a standard Gaussian random variable.

For definiteness, assume that i_1, i_2, i_3 are pairwise different in (6.60) (other cases are represented by (5.51)–(5.54)). Then from Theorem 1.3 we have

$$H_{k,q_2}^{(i_1 i_2 i_3)} = \Delta^3\left(\frac{1}{6} - \sum_{j_1,j_2,j_3=0}^{q_2}\frac{C_{j_3 j_2 j_1}^2}{\Delta^3}\right), \tag{6.61}$$

where

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

and $P_i(x)$ $(i = 0, 1, 2, \ldots)$ is the Legendre polynomial.

Substituting (6.61) into (6.60), we obtain

$$E_N^{q_1,q_2} = \frac{1}{2}(T-t)^3\left(\frac{1}{N} - \frac{1}{N^2}\right)\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) +$$

$$+\frac{(T-t)^3}{N^2}\left(\frac{1}{6} - \sum_{j_1,j_2,j_3=0}^{q_2}\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{64}\bar{C}_{j_3 j_2 j_1}^2\right). \tag{6.62}$$

Note that for $N = 1$ the formulas (6.59), (6.62) pass into the corresponding formulas for the mean-square approximation errors of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ (see Theorem 1.3).

Let us consider modeling the integrals $I_{(0)T,t}^{(i_1)}$, $I_{(00)T,t}^{(i_1 i_2)}$. To do it we can use the relations (6.48), (6.53). At that, the mean-square approximation error for the integral $I_{(00)T,t}^{(i_1 i_2)}$ is defined by the formula (6.59) for the case of Legendre polynomials. Let us calculate the value E_N^q for various N and q

$$E_3^2 \approx 0.0167(T-t)^2, \quad E_2^3 \approx 0.0179(T-t)^2, \tag{6.63}$$

Table 6.2: $T-t=0.1$.

N	q	q_1	q_2	M
1	13	–	1	21
2	6	0	0	7
3	4	0	0	5

Table 6.3: $T-t=0.05$.

N	q	q_1	q_2	M
1	50	–	2	77
2	25	2	0	26
3	17	1	0	18

$$E_1^6 \approx 0.0192(T-t)^2. \tag{6.64}$$

Note that the combined method (see (6.63)) requires calculation of a significantly smaller number of the Fourier–Legendre coefficients than the method based on Theorem 1.1 (see (6.64)).

Assume that the mean-square approximation error of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ equals to $(T-t)^4$.

In Tables 6.2–6.4 we can see the values N, q, q_1, q_2, which satisfy the system of inequalities

$$\begin{cases} E_N^q \le (T-t)^4 \\ \\ E_N^{q_1,q_2} \le (T-t)^4 \end{cases} \tag{6.65}$$

as well as the total number M of the Fourier–Legendre coefficients, which are nesessary for approximation of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ when $T-t=0.1, 0.05, 0.02$ (the numbers q, q_1, q_2 were taken in such a manner that the number M was the smallest one).

From Tables 6.2–6.4 it is clear that the combined method with the small N

Table 6.4: $T-t=0.02$.

N	q	q_1	q_2	M
1	312	–	6	655
2	156	4	2	183
3	104	6	0	105

($N = 2$) provides a possibility to decrease significantly the total number of the Fourier–Legendre coefficients, which are necessary for the approximation of the iterated Itô stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$ in comparison with the method based on Theorem 1.1 ($N = 1$). However, as we noted before, as a result the computational costs for the approximation are increased. The approximation accuracy of iterated Itô stochastic integrals for the combined method and the method based on Theorem 1.1 was taken $(T-t)^4$.

6.6 Representation of Iterated Itô Stochastic Integrals Based on Hermite Polynomials

In Chapters 1, 2, and 5 we analyzed the general theory of the approximation of iterated Itô and Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. However, in some narrow special cases we can get exact expressions for iterated Itô and Stratonovich stochastic integrals in the form of polynomials of finite degrees from one standard Gaussian random variable. This and next sections will be devoted to this question. The results described in them can be found, for example, in [98] (also see [65], [67]).

Let us consider the set of polynomials $H_n(x,y)$, $n = 0, 1, \ldots$ defined by

$$H_n(x,y) = \left(\frac{d^n}{d\alpha^n} e^{\alpha x - \alpha^2 y/2}\right)\Bigg|_{\alpha=0}.$$

It is well known that polynomials $H_n(x,y)$ are connected with the Hermite polynomials $h_n(x)$ by the formula

$$H_n(x,y) = \left(\frac{y}{2}\right)^{n/2} h_n\left(\frac{x}{\sqrt{2y}}\right),$$

where $h_n(x)$ is the Hermite polynomial.

Using the recurrent formulas

$$\frac{dh_n}{dz}(z) = 2n h_{n-1}(z), \quad n = 1,\ 2, \ldots,$$

$$h_n(z) = 2z h_{n-1}(z) - 2(n-1) h_{n-2}(z), \quad n = 2,\ 3, \ldots,$$

it is easy to get the following recurrent relations for polynomials $H_n(x,y)$

$$\frac{\partial H_n}{\partial x}(x,y) = n H_{n-1}(x,y), \quad n = 1, 2, \ldots, \tag{6.66}$$

$$\frac{\partial H_n}{\partial y}(x,y)=\frac{n}{2y}H_n(x,y)-\frac{nx}{2y}H_{n-1}(x,y),\quad n=1,2,\ldots, \tag{6.67}$$

$$\frac{\partial H_n}{\partial y}(x,y)=-\frac{n(n-1)}{2}H_{n-2}(x,y),\quad n=2,3,\ldots \tag{6.68}$$

From (6.66) – (6.68) it follows that

$$\frac{\partial H_n}{\partial y}(x,y)+\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(x,y)=0,\quad n=2,3,\ldots \tag{6.69}$$

Using the Itô formula, we have

$$H_n(f_t,t)-H_n(0,0)=\int\limits_0^t \frac{\partial H_n}{\partial x}(f_s,s)df_s+$$

$$+\int\limits_0^t\left(\frac{\partial H_n}{\partial y}(f_s,s)+\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(f_s,s)\right)ds\quad \text{w. p. 1}, \tag{6.70}$$

where $t\in[0,T]$ and f_t is a scalar standard Wiener process.

Note that $H_n(0,0)=0$, $n=2,3,\ldots$ Then from (6.69) and (6.70) we get

$$H_n(f_t,t)=\int\limits_0^t nH_{n-1}(f_s,s)df_s,\quad \text{w. p. 1}\quad (n=2,3,\ldots).$$

Furthermore, by induction it is easy to get the following relation

$$\int\limits_0^t\ldots\int\limits_0^{t_2}df_{t_1}\ldots df_{t_n}=\frac{H_n(f_t,t)}{n!}\quad \text{w. p. 1}\quad (n=1,2,\ldots). \tag{6.71}$$

Let us consider one generalization of the formula (6.71) [98]

$$\int\limits_0^t\psi(t_n)\ldots\int\limits_0^{t_2}\psi(t_1)df_{t_1}\ldots df_{t_n}=\frac{H_n(\delta_t,\Delta_t)}{n!}\quad \text{w. p. 1}, \tag{6.72}$$

where $t\in[0,T]$, $n=1,2,\ldots,$ and

$$\delta_t\stackrel{\text{def}}{=}\int\limits_0^t\psi(\tau)df_\tau,\quad \Delta_t\stackrel{\text{def}}{=}\int\limits_0^t\psi^2(\tau)d\tau,$$

where $\psi(\tau)$ is a continuous nonrandom function at the interval $[0,T]$.

It is easy to check that first eight formulas from the set (6.72) have the following from

$$J_t^{(1)}=\frac{1}{1!}\delta_t,\quad J_t^{(2)}=\frac{1}{2!}\left(\delta_t^2-\Delta_t\right),$$

$$J_t^{(3)}=\frac{1}{3!}\left(\delta_t^3-3\delta_t\Delta_t\right),\quad J_t^{(4)}=\frac{1}{4!}\left(\delta_t^4-6\delta_t^2\Delta_t+3\Delta_t^2\right),$$

$$J_t^{(5)}=\frac{1}{5!}\left(\delta_t^5-10\delta_t^3\Delta_t+15\delta_t\Delta_t^2\right),$$

$$J_t^{(6)}=\frac{1}{6!}\left(\delta_t^6-15\delta_t^4\Delta_t+45\delta_t^2\Delta_t^2-15\Delta_t^3\right),$$

$$J_t^{(7)}=\frac{1}{7!}\left(\delta_t^7-21\delta_t^5\Delta_t+105\delta_t^3\Delta_t^2-105\delta_t\Delta_t^3\right),$$

$$J_t^{(8)}=\frac{1}{8!}\left(\delta_t^8-28\delta_t^6\Delta_t+210\delta_t^4\Delta_t^2-420\delta_t^2\Delta_t^3+105\Delta_t^4\right)$$

w. p. 1, where

$$J_t^{(n)}\stackrel{\sf def}{=}\int\limits_0^t\psi(t_n)\ldots\int\limits_0^{t_2}\psi(t_1)df_{t_1}\ldots df_{t_n}.$$

As follows from the results of Sect. 1.1.6, for the case $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv\psi(\tau)$ and $i_1=\ldots=i_k=1,\ldots,m$ the formula (1.50) transforms into (6.72).

6.7 One Formula for Iterated Stratonovich Stochastic Integrals of Multiplicity k

Let us prove the following relation for iterated Stratonovich stochastic integrals (see, for example, [67])

$$\int\limits_0^{*t}\ldots\int\limits_0^{*t_2}df_{t_1}\ldots df_{t_n}=\frac{\left(f_t\right)^n}{n!}\quad\text{w. p. 1,}\tag{6.73}$$

where $t\in[0,T]$.

At first, we will consider the case $n=2$. Using Theorem 2.12, we obtain

$$\int\limits_0^{*t}\int\limits_0^{*t_2}df_{t_1}df_{t_2}=\int\limits_0^{t}\int\limits_0^{t_2}df_{t_1}df_{t_2}+\frac{1}{2}\int\limits_0^{t}dt_1\quad\text{w. p. 1.}\tag{6.74}$$

From the relation (6.71) for $n = 2$ it follows that

$$\int\limits_0^t \int\limits_0^{t_2} df_{t_1} df_{t_2} = \frac{\left(f_t\right)^2}{2!} - \frac{1}{2}\int\limits_0^t dt_1 \quad \text{w. p. 1.} \tag{6.75}$$

Substituting (6.75) into (6.74), we have

$$\int\limits_0^{*t} \int\limits_0^{*t_2} df_{t_1} df_{t_2} = \frac{\left(f_t\right)^2}{2!} \quad \text{w. p. 1.}$$

So, the formula (6.73) is correct for $n = 2$. Using the induction assumption and (2.4), we obtain

$$\int\limits_0^{*t} \ldots \int\limits_0^{*t_2} df_{t_1} \ldots df_{t_{n+1}} = \int\limits_0^{*t} \frac{\left(f_\tau\right)^n}{n!} df_\tau =$$

$$= \int\limits_0^t \frac{\left(f_\tau\right)^n}{n!} df_\tau + \frac{1}{2}\int\limits_0^t \frac{\left(f_\tau\right)^{n-1}}{(n-1)!} d\tau \quad \text{w. p. 1.} \tag{6.76}$$

From the other hand, using the Itô formula, we get

$$\frac{\left(f_t\right)^{n+1}}{(n+1)!} = \int\limits_0^t \frac{\left(f_\tau\right)^{n-1}}{2(n-1)!} d\tau + \int\limits_0^t \frac{\left(f_\tau\right)^n}{n!} df_\tau \quad \text{w. p. 1.} \tag{6.77}$$

From (6.76) and (6.77) we obtain (6.73).

It is easy to see that the formula (6.73) admits the following generalization

$$\int\limits_0^{*t} \psi(t_n) \ldots \int\limits_0^{*t_2} \psi(t_1) df_{t_1} \ldots df_{t_n} = \frac{1}{n!}\left(\int\limits_0^t \psi(\tau) df_\tau\right)^n \quad \text{w. p. 1,}$$

where $t \in [0, T]$ and $\psi(\tau)$ is a continuously differentiable nonrandom function at the interval $[0, T]$.

6.8 Weak Approximation of Iterated Itô Stochastic Integrals of Multiplicity 1 to 4

In the previous chapters of the book and previous sections of this chapter we analyzed in detail the methods of so-called strong or mean-square approximation of iterated stochastic integrals. For numerical integration of Itô SDEs the so-called weak approximations of iterated Itô stochastic integrals from the Taylor–Itô expansions (see Chapter 4) are also interesting.

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener process, which is F_t-measurable for all $t\in[0,T]$. We suppose that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider an Itô SDE in the integral form

$$
\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau,\quad \mathbf{x}_0=\mathbf{x}(0,\omega), \tag{6.78}
$$

where $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the Itô SDE (6.78), the nonrandom functions $\mathbf{a}:\mathbf{R}^n\times[0,T]\to\mathbf{R}^n$, $B:\mathbf{R}^n\times[0,T]\to\mathbf{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (6.78) [83], $\mathbf{x}_0$ is an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\}<\infty$, $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ are independent for $t>0$.

Let us consider the iterated Itô stochastic integrals from the classical Taylor–Itô expansion (see Chapter 4)

$$
J_{(\lambda_1\ldots\lambda_k)s,t}^{(i_1\ldots i_k)}=\int\limits_t^s\ldots\int\limits_t^{\tau_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (k\ge 1),
$$

where $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_l=0$ if $\lambda_l=0$ and $i_l=1,\ldots,m$ if $\lambda_l=1$ $(l=1,\ldots,k)$. Moreover, let

$$
\mathrm{M}_k=\biggl\{(\lambda_1,\ldots,\lambda_k):\ \ \lambda_l=0\ \text{or}\ 1,\ l=1,\ldots,k\biggr\}.
$$

Weak approximations of iterated Itô stochastic stochastic integrals are formed or selected from the specific moment conditions [65], [67], [68], [75], [76] (see below) and they are significantly simpler than their mean-square analogues. However, weak approximations are focused on the numerical solution of other problems [65], [67], [68], [75], [76] connected with Itô SDEs in comparison with mean-square approximations.

We will say that the set of weak approximations

$$
\hat{J}_{(\lambda_1\ldots\lambda_k)s,t}^{(i_1\ldots i_k)}
$$

of the iterated Itô stochastic integrals

$$J^{(i_1\ldots,i_k)}_{(\lambda_1\ldots\lambda_k)s,t}$$

from the Taylor–Itô expansion (4.22) has the order r, if [65], [67] for $t\in[t_0,T]$ and $r\in\mathbf{N}$ there exists a constant $K\in(0,\infty)$ such that the condition

$$\left|\mathsf{M}\left\{\prod_{g=1}^{l} J^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t,t_0}-\prod_{g=1}^{l}\hat J^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t,t_0}\Bigg|\mathrm{F}_{t_0}\right\}\right|\le K(t-t_0)^{r+1}\ \ \text{w. p. 1}\ \ (6.79)$$

is satisfied for all $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)\in\mathrm{M}_{k_g},\ i_1^{(g)},\ldots,i_{k_g}^{(g)}=0,1,\ldots,m,\ k_g\le r,\ g=1,\ldots,l,\ l=1,2,\ldots,2r+1.$

If we talk about the unified Taylor–Itô expansion (4.27), then we will say that the set of weak approximations

$$\hat I^{(i_1\ldots i_k)}_{l_1\ldots l_{k s,t}}$$

of the iterated Itô stochastic integrals

$$I^{(i_1\ldots i_k)}_{l_1\ldots l_{k s,t}}=\int\limits_t^s(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}\quad(i_1,\ldots,i_k=1,\ldots,m)$$

has the order r, if for $t\in[t_0,T]$ and $r\in\mathbf{N}$ there exists a constant $K\in(0,\infty)$ such that the condition

$$\left|\mathsf{M}\left\{\prod_{g=1}^{l}\frac{(t-t_0)^{j_g}}{j_g!}\ I^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}\,t,t_0}-\prod_{g=1}^{l}\frac{(t-t_0)^{j_g}}{j_g!}\ \hat I^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}\,t,t_0}\Bigg|\mathrm{F}_{t_0}\right\}\right|\le$$

$$\le K(t-t_0)^{r+1}\quad\text{w. p. 1}\qquad(6.80)$$

is satisfied for all $\left(k_g,j_g,l_1^{(g)},\ldots,\ l_{k_g}^{(g)}\right)\in\mathrm{A}_{q_g},\ i_1^{(g)},\ldots,i_{k_g}^{(g)}=1,\ldots,m,\ q_g\le r,$ $g=1,\ldots,l,\ l=1,2,\ldots,2r+1$, where

$$\mathrm{A}_q=\left\{(k,j,l_1,\ldots,l_k):k+j+\sum_{p=1}^{k}l_p=q;\ \ k,j,l_1,\ldots,l_k=0,1,\ldots\right\}.$$

The theory of weak approximations of iterated Itô stochastic integrals is not so rich as the theory of mean-square approximations. On the one hand, it is connected with the sufficiency for practical needs of already found approximations [65], [67], [75], and on the other hand, it is connected with the complexity of their formation owing to the necessity to satisfy a lot of moment conditions.

Let us consider the basic results in this area.

In [67] (also see [65]) the authors found the weak approximations with the orders $r=1,2$ when $m,n\ge 1$ as well as with the order $r=3$ when $m=1$, $n\ge 1$ for iterated Itô stochastic integrals

$$J_{(\lambda_1\ldots\lambda_k)t,t_0}^{(i_1\ldots i_k)}.$$

Recall that n is a dimension of the Itô process $\mathbf{x}_t$, which is a solution of the Itô SDE (6.78) and m is a dimension of the Wiener process in (6.78).

Further, we will consider the mentioned weak approximations as well as weak approximations with the order $r=4$ when $m=1$, $n\ge 1$ [132] (2000) for iterated Itô stochastic integrals

$$I_{l_1\ldots l_{k t,t_0}}^{(i_1\ldots i_k)}.$$

In order to shorten the record let us write

$$\mathsf{M}\left\{\prod_{g=1}^{l} J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)t_0+\Delta,t_0}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\Bigg|\mathrm{F}_{t_0}\right\}\stackrel{\rm def}{=}\mathsf{M}'\left\{\prod_{g=1}^{l} J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}, \tag{6.81}$$

where $\Delta\in[0,T-t_0]$, $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)\in\mathrm{M}_{k_g}$, $k_g\le r$, $g=1,\ldots,l$.

Further in this section, equalities and inequalities for conditional expectations are understood w. p. 1. As before, $\mathbf{1}_A$ means the indicator of the set A.

Let us consider the exact values of conditional expectations (6.81) calculated in [65], [67] and necessary to form weak approximations

$$\hat J_{(\lambda_1\ldots\lambda_k)t_0+\Delta,t_0}^{(i_1\ldots i_k)}$$

of the orders $r=1,2$ when $m,n\ge 1$

$$\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}\right\}=\Delta\mathbf{1}_{\{i_1=i_2\}}, \tag{6.82}$$

$$\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(01)}^{(0i_2)}\right\}=\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(10)}^{(i_20)}\right\}=\frac{1}{2}\Delta^2\mathbf{1}_{\{i_1=i_2\}}, \tag{6.83}$$

$$\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}\right\}=\frac{1}{2}\Delta^2\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \tag{6.84}$$

$$
\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\begin{cases}\Delta^2 & \text{when } i_1=\ldots=i_4\\ \\ \Delta^2/2 & \begin{matrix}\text{when } i_3\ne i_4,\ i_1=i_3,\ i_2=i_4\\ \text{or } i_3\ne i_4,\ i_1=i_4,\ i_2=i_3\end{matrix},\\ \\ 0 & \text{otherwise}\end{cases} \tag{6.85}
$$

$$
\mathsf{M}'\left\{J_{(1)}^{(i_1)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}=\begin{cases}3\Delta^2 & \text{when } i_1=\ldots=i_4\\ \\ \Delta^2 & \begin{matrix}\text{if among } i_1,\ldots,i_4 \text{ there are}\\ \text{two pairs of identical numbers}\end{matrix},\\ \\ 0 & \text{otherwise}\end{cases} \tag{6.86}
$$

$$
\mathsf{M}'\left\{J_{(10)}^{(i_1 0)}J_{(01)}^{(0 i_2)}\right\}=\frac{1}{6}\Delta^3\mathbf{1}_{\{i_1=i_2\}}, \tag{6.87}
$$

$$
\mathsf{M}'\left\{J_{(10)}^{(i_1 0)}J_{(10)}^{(i_2 0)}\right\}=\mathsf{M}'\left\{J_{(01)}^{(0 i_1)}J_{(01)}^{(0 i_2)}\right\}=\frac{1}{3}\Delta^3\mathbf{1}_{\{i_1=i_2\}}, \tag{6.88}
$$

$$
\mathsf{M}'\left\{J_{(01)}^{(0 i_1)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}=\mathsf{M}'\left\{J_{(10)}^{(i_1 0)}J_{(1)}^{(i_2)}J_{(1)}^{(i_3)}J_{(1)}^{(i_4)}\right\}=
$$

$$
=\begin{cases}3\Delta^3/2 & \text{when } i_1=\ldots=i_4\\ \\ \Delta^3/2 & \begin{matrix}\text{if among } i_1,\ldots,i_4 \text{ there are}\\ \text{two pairs of identical numbers}\end{matrix},\\ \\ 0 & \text{otherwise}\end{cases} \tag{6.89}
$$

$$
\mathsf{M}'\left\{J_{(01)}^{(0 i_1)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\frac{1}{6}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \tag{6.90}
$$

$$
\mathsf{M}'\left\{J_{(10)}^{(i_1 0)}J_{(1)}^{(i_2)}J_{(11)}^{(i_3 i_4)}\right\}=\frac{1}{3}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}, \tag{6.91}
$$

$$
\mathsf{M}'\left\{J_{(1)}^{(i_1)}\ldots J_{(1)}^{(i_6)}\right\}=\begin{cases}15\Delta^3 & \text{when } i_1=\ldots=i_6\\ 3\Delta^3 & \text{if among } i_1,\ldots,i_6 \text{ there is a pair and a quad of identical numbers}\\ \Delta^3 & \text{if among } i_1,\ldots,i_6 \text{ there are three pairs of identical numbers}\\ 0 & \text{otherwise}\end{cases}, \qquad (6.92)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_5i_6)}\right\}=
$$

$$
=\frac{1}{6}\Delta^3\Biggl(\mathbf{1}_{\{i_2=i_4\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_3=i_6\}}+\mathbf{1}_{\{i_1=i_6\}}\mathbf{1}_{\{i_3=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_6\}}\left(\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_4=i_5\}}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i_3=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_4=i_6\}}\left(\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_5\}}+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{i_1=i_5\}}\right)\Biggr), \qquad (6.93)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}\right\}=
$$

$$
=\frac{1}{2}\Delta^3\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{i_5=i_6\}}+
$$

$$
+\frac{1}{6}\Delta^3\Biggl(2\cdot\mathbf{1}_{\{i_1=i_3\}}\left(\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{i_4=i_6\}}+\mathbf{1}_{\{i_2=i_6\}}\mathbf{1}_{\{i_4=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_4=i_6\}}+\mathbf{1}_{\{i_1=i_6\}}\mathbf{1}_{\{i_4=i_5\}}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\left(\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{i_2=i_6\}}+\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{i_2=i_5\}}\right)+
$$

$$
+2\cdot\mathbf{1}_{\{i_2=i_4\}}\left(\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i_3=i_6\}}+\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{i_1=i_6\}}\right)\Biggr), \qquad (6.94)
$$

$$
\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(1)}^{(i_3)}\ldots J_{(1)}^{(i_6)}\right\}=
$$

$$
=\frac{1}{2}\left(\mathsf{M}'\left\{J_{(1)}^{(i_1)}\ldots J_{(1)}^{(i_6)}\right\}-\Delta\mathbf{1}_{\{i_1=i_2\}}\mathsf{M}'\left\{J_{(1)}^{(i_3)}\ldots J_{(1)}^{(i_6)}\right\}\right). \qquad (6.95)
$$

Let us explain the formula (6.94). From the following equality

$$J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}=J_{(11)}^{(i_5i_6)}+J_{(11)}^{(i_6i_5)}+\Delta\mathbf{1}_{\{i_5=i_6\}}\quad \text{w. p. 1}$$

we obtain

$$\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(1)}^{(i_5)}J_{(1)}^{(i_6)}\right\}=\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_5i_6)}\right\}+$$

$$+\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}J_{(11)}^{(i_6i_5)}\right\}+\Delta\mathbf{1}_{\{i_5=i_6\}}\mathsf{M}'\left\{J_{(11)}^{(i_1i_2)}J_{(11)}^{(i_3i_4)}\right\}. \tag{6.96}$$

Applying (6.84), (6.93) to the right-hand side of (6.96) gives (6.94). It is necessary to note [65], [67] that

$$\mathsf{M}'\left\{\prod_{g=1}^{l}J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}=0$$

if the number of units included in all multi-indices $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)$ is odd ($k_g\le r$, $g=1,\ldots,l$). In addition [65], [67]

$$\left|\mathsf{M}'\left\{\prod_{g=1}^{l}J_{\left(\lambda_1^{(g)}\ldots\ \lambda_{k_g}^{(g)}\right)}^{\left(i_1^{(g)}\ldots\ i_{k_g}^{(g)}\right)}\right\}\right|\le K\Delta^{\gamma_l},$$

where $\gamma_l=\delta_l/2+\rho_l$, δ_l is a number of units and ρ_l is a number of zeros included in all multi-indices $\left(\lambda_1^{(g)}\ldots\lambda_{k_g}^{(g)}\right)$, $k_g\le r$, $g=1,\ldots,l$, $K\in(0,\infty)$ is a constant.

In the case $n,m\ge 1$ and $r=1$ we can put [65], [67]

$$\hat{J}_{(1)}^{(i)}=\Delta\tilde{\mathbf{f}}^{(i)}\quad (i=1,\ldots,m),$$

where $\Delta\tilde{\mathbf{f}}^{(i)}$, $i=1,\ldots,m$ are independent discrete random variables for which

$$\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)}=\pm\sqrt{\Delta}\right\}=\frac{1}{2}.$$

It is not difficult to see that the approximation

$$\hat{J}_{(1)}^{(i)}=\sqrt{\Delta}\zeta_0^{(i)}\quad (i=1,\ldots,m)$$

also satisfies the condition (6.79) when $r=1$. Here $\zeta_0^{(i)}$ are independent standard Gaussian random variables.

In the case $n, m \geq 1$ and $r = 2$ as the approximations $\hat{J}_{(1)}^{(i_1)}$, $\hat{J}_{(11)}^{(i_1 i_2)}$, $\hat{J}_{(10)}^{(i_1 0)}$, $\hat{J}_{(01)}^{(0 i_1)}$ are taken the following ones [65], [66]

$$
\hat{J}_{(1)}^{(i_1)} = \Delta\tilde{\mathbf{f}}^{(i_1)}, \quad \hat{J}_{(10)}^{(i_1 0)} = \hat{J}_{(01)}^{(0 i_1)} = \frac{1}{2}\Delta \cdot \Delta\tilde{\mathbf{f}}^{(i_1)}, \tag{6.97}
$$

$$
\hat{J}_{(11)}^{(i_1 i_2)} = \frac{1}{2}\left(\Delta\tilde{\mathbf{f}}^{(i_1)}\Delta\tilde{\mathbf{f}}^{(i_2)} + V^{(i_1 i_2)}\right), \tag{6.98}
$$

where $\Delta\tilde{\mathbf{f}}^{(i)}$ are independent Gaussian random variables with zero expectation and variance Δ or independent discrete random variables for which the following conditions are fulfilled

$$
\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)} = \pm\sqrt{3\Delta}\right\} = \frac{1}{6},
$$

$$
\mathsf{P}\left\{\Delta\tilde{\mathbf{f}}^{(i)} = 0\right\} = \frac{2}{3},
$$

$V^{(i_1 i_2)}$ are independent discrete random variables satisfying the conditions

$$
\mathsf{P}\left\{V^{(i_1 i_2)} = \pm\Delta\right\} = \frac{1}{2} \quad \text{when} \quad i_2 < i_1,
$$

$$
V^{(i_1 i_1)} = -\Delta, \quad V^{(i_1 i_2)} = -V^{(i_2 i_1)} \quad \text{when} \quad i_1 < i_2,
$$

where $i_1, i_2 = 1, \ldots, m$.

Let us consider the case $r = 3$ and $m = 1, n \geq 1$. In this situation in addition to the formulas (6.82)–(6.96) we need a number of formulas for the conditional expectations (6.81) when $m = 1$.

We have [65], [67]

$$
\mathsf{M}'\{J_{(1)}J_{(111)}\} = \mathsf{M}'\{J_{(01)}J_{(111)}\} = \mathsf{M}'\{J_{(10)}J_{(111)}\} = 0,
$$

$$
\mathsf{M}'\{J_{(011)}J_{(11)}\} = \mathsf{M}'\{J_{(101)}J_{(11)}\} = \mathsf{M}'\{J_{(110)}J_{(11)}\} = \frac{1}{6}\Delta^3,
$$

$$
\mathsf{M}'\{J_{(001)}J_{(1)}\} = \mathsf{M}'\{J_{(010)}J_{(1)}\} = \mathsf{M}'\{J_{(100)}J_{(1)}\} = \frac{1}{6}\Delta^3,
$$

$$
\mathsf{M}'\{J_{(100)}J_{(10)}\} = \mathsf{M}'\{J_{(001)}J_{(01)}\} = \frac{1}{8}\Delta^4, \quad \mathsf{M}'\{J_{(111)}J_{(11)}\} = 0,
$$

$$
\mathsf{M}'\{J_{(010)}J_{(10)}\} = \mathsf{M}'\{J_{(010)}J_{(01)}\} = \frac{1}{6}\Delta^4, \quad \mathsf{M}'\left\{\left(J_{(111)}\right)^2\right\} = \frac{1}{6}\Delta^3,
$$

$$
\mathsf{M}'\{J_{(100)}J_{(01)}\} = \mathsf{M}'\{J_{(001)}J_{(10)}\} = \frac{1}{24}\Delta^4,
$$

$$\mathsf{M}'\{J_{(110)}J_{(10)}\} = \mathsf{M}'\{J_{(110)}J_{(01)}\} = \mathsf{M}'\{J_{(101)}J_{(10)}\} = 0,$$

$$\mathsf{M}'\{J_{(101)}J_{(01)}\} = \mathsf{M}'\{J_{(011)}J_{(10)}\} = \mathsf{M}'\{J_{(011)}J_{(01)}\} = 0,$$

$$\mathsf{M}'\left\{J_{(011)}\left(J_{(1)}\right)^2\right\} = \mathsf{M}'\left\{J_{(101)}\left(J_{(1)}\right)^2\right\} = \mathsf{M}'\left\{J_{(110)}\left(J_{(1)}\right)^2\right\} = \frac{1}{6}\Delta^3,$$

$$\mathsf{M}'\left\{J_{(111)}\left(J_{(1)}\right)^3\right\} = \Delta^3, \quad \mathsf{M}'\left\{J_{(111)}J_{(11)}J_{(1)}\right\} = \frac{1}{2}\Delta^3,$$

where

$$J_{(\lambda_1\ldots\lambda_k)} \stackrel{\rm def}{=} \int\limits_{t_0}^{t_0+\Delta} \ldots \int\limits_{t_0}^{t_2} df_{t_1}^{(\lambda_1)} \ldots df_{t_k}^{(\lambda_k)},$$

$f_t^{(0)} \stackrel{\rm def}{=} t$, $f_t^{(1)} \stackrel{\rm def}{=} f_t$ is standard scalar Wiener process, $\lambda_l = 0$ or $\lambda_l = 1$, $l = 1, \ldots, k$.

In [65], [67] using the given moment relations the authors proposed the following weak approximations of iterated Itô stochastic integrals for $r = 3$ when $m = 1$, $n \geq 1$

$$\hat{J}_{(1)} = \Delta\tilde{f}, \tag{6.99}$$

$$\hat{J}_{(10)} = \Delta\hat{f}, \quad \hat{J}_{(01)} = \Delta \cdot \Delta\tilde{f} - \Delta\hat{f}, \tag{6.100}$$

$$\hat{J}_{(11)} = \frac{1}{2}\left(\left(\Delta\tilde{f}\right)^2 - \Delta\right), \quad \hat{J}_{(001)} = \hat{J}_{(010)} = \hat{J}_{(100)} = \frac{1}{6}\Delta^2 \cdot \Delta\tilde{f},$$

$$\hat{J}_{(110)} = \hat{J}_{(101)} = \hat{J}_{(011)} = \frac{1}{6}\Delta\left(\left(\Delta\tilde{f}\right)^2 - \Delta\right),$$

$$\hat{J}_{(111)} = \frac{1}{6}\Delta\tilde{f}\left(\left(\Delta\tilde{f}\right)^2 - 3\Delta\right),$$

where

$$\Delta\tilde{f} \sim {\rm N}(0, \Delta), \quad \Delta\hat{f} \sim {\rm N}\left(0, \frac{1}{3}\Delta^3\right), \quad \mathsf{M}\left\{\Delta\tilde{f}\Delta\hat{f}\right\} = \frac{1}{2}\Delta^2.$$

Here ${\rm N}(0, \sigma^2)$ is a Gaussian distribution with zero expectation and variance σ^2.

Finally, we will form the weak approximations of iterated Itô stochastic integrals for $r = 4$ when $m = 1$, $n \geq 1$ [1]-[14].

The truncated Taylor–Itô expansion (4.22) when $r = 4$ and $m = 1$ includes 26 various iterated Itô stochastic integrals. The formation of weak approximations for these stochastic integrals satisfying the condition (6.80) when $r = 4$ is extremely difficult due to the necessity to consider a lot of moment conditions. However, this problem can be simplified if we consider the truncated unified Taylor–Itô expansion (4.27) when $r = 4$ and $m = 1$, since this expansion includes only 15 various iterated Itô stochastic integrals

$$I_0,\ I_1,\ I_{00},\ I_{000},\ I_2,\ I_{10},\ I_{01},\ I_3,\ I_{11},\ I_{20},\ I_{02},\ I_{100},\ I_{010},\ I_{001},\ I_{0000},$$

where

$$I_{l_1\ldots l_k} \stackrel{\rm def}{=} \int\limits_{t_0}^{t_0+\Delta} (t_0-t_k)^{l_k}\ldots \int\limits_{t_0}^{t_2} (t_0-t_1)^{l_1} df_{t_1}\ldots df_{t_k} \quad (k\ge 1)$$

and f_t is standard scalar Wiener process.

It is not difficult to notice that the condition (6.80) will be satisfied for $r = 4$ and $i_1 = \ldots = i_4$ if the following more strong condition is fulfilled

$$\left|{\sf M}\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}} - \prod_{g=1}^{l} \hat I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\Biggl|{\rm F}_{t_0}\right\}\right| \le K(t-t_0)^5 \quad \hbox{w. p. 1} \tag{6.101}$$

for all $l_1^{(g)}\ldots\ l_{k_g}^{(g)} \in A,\ k_g\le 4,\ g=1,\ldots,l,\ l=1,2,\ldots,9,$ where $K\in(0,\infty)$ and

$$A=\Bigl\{0,1,00,000,2,10,01,3,11,20,02,100,010,001,0000\Bigr\}$$

is the set of multi-indices.

Let (see Sect. 5.1 and 6.6) [14], [132]

$$\hat I_0=\sqrt{\Delta}\zeta_0,\quad \hat I_{00}=\frac{1}{2}\Delta\left(\left(\zeta_0\right)^2-1\right), \tag{6.102}$$

$$\hat I_1=-\frac{\Delta^{3/2}}{2}\left(\zeta_0+\frac{1}{\sqrt{3}}\zeta_1\right),\quad \hat I_{000}=\frac{\Delta^{3/2}}{6}\left(\left(\zeta_0\right)^3-3\zeta_0\right), \tag{6.103}$$

$$\hat I_{0000}=\frac{\Delta^2}{24}\left(\left(\zeta_0\right)^4-6\left(\zeta_0\right)^2+3\right). \tag{6.104}$$

Here and further

$$\zeta_0 \stackrel{\rm def}{=} \frac{1}{\sqrt{\Delta}}\int\limits_{t_0}^{t_0+\Delta} df_s,\quad \zeta_1 \stackrel{\rm def}{=} \frac{2\sqrt{3}}{\Delta^{3/2}}\int\limits_{t_0}^{t_0+\Delta}\left(s-t_0-\frac{\Delta}{2}\right)df_s,$$

where f_s is scalar standard Wiener process.

It is not difficult to see that ζ_0, ζ_1 are independent standard Gaussian random variables. In addition, the approximations (6.102)–(6.104) equal w. p. 1

to the iterated Itô stochastic integrals corresponding to these approximations. This implies that all products

$$\prod_{g=1}^{l} \hat{I}_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}},$$

which contain only the approximations (6.102)–(6.104) will convert the left-hand side of (6.101) to zero w. p. 1, i.e. the condition (6.101) will be fulfilled automatically.

For forming the approximations

$$\hat{I}_{100},\ \hat{I}_{010},\ \hat{I}_{001},\ \hat{I}_{10},\ \hat{I}_{01},\ \hat{I}_{11},\ \hat{I}_{20},\ \hat{I}_{02},\ \hat{I}_{2},\ \hat{I}_{3}$$

it is necessary to calculate several conditional expectations

$$\mathsf{M}\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\Bigg|\mathrm{F}_{t_0}\right\}, \tag{6.105}$$

where $l_1^{(g)}\ \ldots\ l_{k_g}^{(g)} \in A.$

We will denote (6.105) (as before) as follows

$$\mathsf{M}'\left\{\prod_{g=1}^{l} I_{l_1^{(g)}\ldots\ l_{k_g}^{(g)}}\right\}.$$

We have

$$\mathsf{M}'\{I_3\} = \mathsf{M}'\{I_3(I_0)^2\} = \mathsf{M}'\{I_3I_{00}\} = 0, \quad \mathsf{M}'\{I_3I_0\} = -\frac{\Delta^4}{4},$$

$$\mathsf{M}'\{I_2(I_0)^2\} = \mathsf{M}'\{I_2I_{00}\} = \mathsf{M}'\{I_2I_{000}\} = \mathsf{M}'\{I_2I_{0000}\} = 0,$$

$$\mathsf{M}'\{I_2(I_{00})^2\} = \mathsf{M}'\{I_2(I_0)^4\} = \mathsf{M}'\{I_2I_{000}I_0\} = 0,$$

$$\mathsf{M}'\{I_2I_{00}(I_0)^2\} = \mathsf{M}'\{I_2I_{10}\} = \mathsf{M}'\{I_2I_{01}\} = \mathsf{M}'\{I_2I_1I_0\} = \mathsf{M}'\{I_2\} = 0,$$

$$\mathsf{M}'\{I_2I_0\} = \frac{\Delta^3}{3}, \quad \mathsf{M}'\{I_2(I_0)^3\} = \Delta^4, \quad \mathsf{M}'\{I_2I_{00}I_0\} = \frac{\Delta^4}{3},$$

$$\mathsf{M}'\{I_2I_1\} = -\frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_\mu\} = \mathsf{M}'\{I_\mu I_0\} = \mathsf{M}'\{I_\mu I_{000}\} = \mathsf{M}'\{I_\mu(I_0)^3\} = 0,$$

$$\mathsf{M}'\{I_\mu I_{00}I_0\} = \mathsf{M}'\{I_\mu I_1\} = 0, \quad \mathsf{M}'\{I_{20}(I_0)^2\} = \frac{\Delta^4}{6}, \quad \mathsf{M}'\{I_{20}I_{00}\} = \frac{\Delta^4}{12},$$

$$\mathsf{M}'\{I_{11}(I_0)^2\} = \frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_{11}I_{00}\} = \frac{\Delta^4}{8}, \quad \mathsf{M}'\{I_{02}(I_0)^2\} = \frac{\Delta^4}{2},$$

$$\mathsf{M}'\{I_{02}I_{00}\} = \frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_\lambda\} = \mathsf{M}'\{I_\lambda I_0\} = \mathsf{M}'\{I_\lambda(I_0)^2\} = \mathsf{M}'\{I_\lambda I_{00}\} = 0,$$

$$\mathsf{M}'\{I_\lambda I_1\} = \mathsf{M}'\{I_\lambda I_{0000}\} = \mathsf{M}'\{I_\lambda(I_{00})^2\} = \mathsf{M}'\{I_\lambda(I_0)^4\} = 0,$$

$$\mathsf{M}'\{I_\lambda I_{000}I_0\} = \mathsf{M}'\{I_\lambda I_{00}(I_0)^2\} = \mathsf{M}'\{I_\lambda I_{10}\} = 0,$$

$$\mathsf{M}'\{I_\lambda I_{01}\} = \mathsf{M}'\{I_\lambda I_1 I_0\} = 0,$$

$$\mathsf{M}'\{I_{100}I_{000}\} = -\frac{\Delta^4}{24}, \quad \mathsf{M}'\{I_{100}(I_0)^3\} = -\frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_{100}I_{00}I_0\} = -\frac{\Delta^4}{8},$$

$$\mathsf{M}'\{I_{010}I_{000}\} = -\frac{\Delta^4}{12}, \quad \mathsf{M}'\{I_{010}(I_0)^3\} = -\frac{\Delta^4}{2}, \quad \mathsf{M}'\{I_{010}I_{00}I_0\} = -\frac{\Delta^4}{4},$$

$$\mathsf{M}'\{I_{001}I_{000}\} = -\frac{\Delta^4}{8}, \quad \mathsf{M}'\{I_{001}(I_0)^3\} = -\frac{3\Delta^4}{4}, \quad \mathsf{M}'\{I_{001}I_{00}I_0\} = -\frac{3\Delta^4}{8},$$

$$\mathsf{M}'\{I_\rho I_0\} = \mathsf{M}'\{I_\rho I_{000}\} = \mathsf{M}'\{I_\rho(I_0)^3\} = \mathsf{M}'\{I_\rho I_{00}I_0\} = 0,$$

$$\mathsf{M}'\{I_\rho I_1\} = \mathsf{M}'\{I_\rho I_{0000}\} = \mathsf{M}'\{I_\rho(I_0)^5\} = \mathsf{M}'\{I_\rho(I_{00})^2 I_0\} = 0,$$

$$\mathsf{M}'\{I_\rho I_{00}(I_0)^3\} = \mathsf{M}'\{I_\rho I_{000}(I_0)^2\} = \mathsf{M}'\{I_\rho I_{0000}I_0\} = 0,$$

$$\mathsf{M}'\{I_\rho I_{000}I_{00}\} = \mathsf{M}'\{I_\rho I_{100}\} = \mathsf{M}'\{I_\rho I_{010}\} = 0,$$

$$\mathsf{M}'\{I_\rho I_{001}\} = \mathsf{M}'\{I_\rho I_2\} = \mathsf{M}'\{(I_\rho)^2 I_0\} = \mathsf{M}'\{I_\rho I_{00}I_1\} = 0,$$

$$\mathsf{M}'\{I_{10}I_{01}I_0\} = \mathsf{M}'\{I_\rho\} = \mathsf{M}'\{I_\rho I_1(I_0)^2\} = 0,$$

$$\mathsf{M}'\{I_{10}(I_0)^2\} = -\frac{\Delta^3}{3}, \quad \mathsf{M}'\{I_{10}I_{00}\} = -\frac{\Delta^3}{6}, \quad \mathsf{M}'\{I_{10}(I_{00})^2\} = -\frac{\Delta^4}{3},$$

$$\mathsf{M}'\{I_{10}(I_0)^4\} = -2\Delta^4, \quad \mathsf{M}'\{I_{10}I_{000}I_0\} = -\frac{\Delta^4}{6},$$

$$\mathsf{M}'\{I_{10}I_{00}(I_0)^2\} = -\frac{5\Delta^4}{6},$$

$$\mathsf{M}'\{(I_{10})^2\} = \frac{\Delta^4}{12}, \quad \mathsf{M}'\{I_{10}I_{01}\} = \frac{\Delta^4}{8}, \quad \mathsf{M}'\{I_{10}I_1 I_0\} = \frac{5\Delta^4}{24},$$

$$\mathsf{M}'\{I_{01}(I_0)^2\} = -\frac{2\Delta^3}{3}, \quad \mathsf{M}'\{I_{01}I_{00}\} = -\frac{\Delta^3}{3}, \quad \mathsf{M}'\{I_{01}(I_{00})^2\} = -\frac{2\Delta^4}{3},$$

$$\mathsf{M}'\{I_{01}(I_0)^4\} = -4\Delta^4, \quad \mathsf{M}'\{I_{01}I_{000}I_0\} = -\frac{\Delta^4}{3},$$

$$\mathsf{M}'\{I_{01}I_{00}(I_0)^2\} = -\frac{5\Delta^4}{3}, \quad \mathsf{M}'\{(I_{01})^2\} = \frac{\Delta^4}{4}, \quad \mathsf{M}'\{I_{01}I_1I_0\} = \frac{3\Delta^4}{8},$$

where

$$\mu = 02,\ 11,\ 20, \quad \lambda = 100,\ 010,\ 001, \quad \rho = 10,\ 01$$

(these recordings should be understood as sequences of digits).

The above relations are obtained using the standard properties of the Itô stochastic integral and the following equalities resulting from the Itô formula

$$(I_0)^4 = 24I_{0000} + 12\Delta I_{00} + 3\Delta^2, \quad (I_{00})^2 = 6I_{0000} + 2\Delta I_{00} + \frac{\Delta^2}{2},$$

$$I_{00}(I_0)^2 = 12I_{0000} + 5\Delta I_{00} + \Delta^2, \quad I_1I_0 = I_{10} + I_{01} - \frac{\Delta^2}{2},$$

$$I_{00}(I_0)^3 = 60I_{00000} + 27\Delta I_{000} + 6\Delta^2 I_0,$$

$$(I_0)^5 = 120I_{00000} + 60\Delta I_{000} + 15\Delta^2 I_0,$$

$$(I_{00})^2 I_0 = 30I_{00000} + 12\Delta I_{000} + \frac{10\Delta^2}{4} I_0,$$

$$I_{000}(I_0)^2 = 20I_{00000} + 7\Delta I_{000} + \Delta^2 I_0, \quad I_{0000}I_0 = 5I_{00000} + \Delta I_{000},$$

$$I_{000}I_{00} = 10I_{00000} + 3\Delta I_{000} + \frac{\Delta^2}{2} I_0, \quad I_{00}I_1 = I_{001} + I_{010} + I_{100} - \frac{\Delta^2}{2} I_0,$$

$$(I_0)^3 = 6I_{000} + 3\Delta I_0, \quad I_{00}I_0 = 3I_{000} + \Delta I_0,$$

$$I_{10}I_0 = I_{010} + I_{100} + \Delta I_1 + I_2, \quad I_{000}I_0 = 4I_{0000} + \Delta I_{00}, \quad (I_0)^2 = 2I_{00} + \Delta,$$

$$I_{01}I_0 = 2I_{001} + I_{010} - \frac{1}{2}\left(I_2 + \Delta^2 I_0\right)$$

w. p. 1.

Using the given before moment relations, we can form the weak approximations $\hat{I}_{100}$, $\hat{I}_{010}$, $\hat{I}_{001}$, $\hat{I}_{10}$, $\hat{I}_{01}$, $\hat{I}_{11}$, $\hat{I}_{20}$, $\hat{I}_{02}$, $\hat{I}_2$, $\hat{I}_3$ [14], [132]

$$\hat{I}_{100} = -\frac{\Delta^{5/2}}{24}\Bigl((\zeta_0)^3 - 3\zeta_0\Bigr), \quad \hat{I}_{010} = -\frac{\Delta^{5/2}}{12}\Bigl((\zeta_0)^3 - 3\zeta_0\Bigr), \tag{6.106}$$

$$\hat{I}_{001} = -\frac{\Delta^{5/2}}{8}\Bigl((\zeta_0)^3 - 3\zeta_0\Bigr), \quad \hat{I}_{11} = \frac{\Delta^3}{8}\Bigl((\zeta_0)^2 - 1\Bigr), \tag{6.107}$$

$$\hat{I}_{20} = \frac{\Delta^3}{12}\Bigl((\zeta_0)^2 - 1\Bigr), \quad \hat{I}_{02} = \frac{\Delta^3}{4}\Bigl((\zeta_0)^2 - 1\Bigr), \tag{6.108}$$

$$\hat{I}_3 = -\frac{\Delta^{7/2}}{4}\zeta_0, \quad \hat{I}_2 = \frac{\Delta^{5/2}}{3}\left(\zeta_0 + \frac{\sqrt{3}}{2}\zeta_1\right), \tag{6.109}$$

$$\hat{I}_{10} = \Delta^2\left(-\frac{1}{6}\left(\left(\zeta_0\right)^2 - 1\right) - \frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \pm \frac{1}{12\sqrt{2}}\left(\left(\zeta_1\right)^2 - 1\right)\right), \tag{6.110}$$

$$\hat{I}_{01} = \Delta^2\left(-\frac{1}{3}\left(\left(\zeta_0\right)^2 - 1\right) - \frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \mp \frac{1}{12\sqrt{2}}\left(\left(\zeta_1\right)^2 - 1\right)\right), \tag{6.111}$$

where ζ_0, ζ_1 are the same random variables as in (6.102)–(6.104).

It is easy to check that the approximations (6.102)–(6.104), (6.106)–(6.111) satisfy the condition (6.101) for $r = 4$ and $m = 1$, $n \geq 1$, i.e. they are weak approximations of the order $r = 4$ for the case $m = 1$, $n \geq 1$.

Chapter 7

Approximation of Iterated Stochastic Integrals with Respect to the Q-Wiener Process. Application to the High-Order Strong Numerical Methods for Non-Commutative Semilinear SPDEs with Nonliear Multiplicative Trace Class Noise

7.1 Introduction

There exists a lot of publications on the subject of numerical integration of stochastic partial differential equations (SPDEs) (see, for example [133]-[157]).

One of the perspective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for semilinear SPDEs is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [139], [141]-[144]. A significant step in this direction was made in [143] (2015), [144] (2016), where the exponential Milstein and Wagner–Platen methods for semilinear SPDEs with nonlinear multiplicative trace class noise were constructed. Under the appropriate conditions [143], [144] these methods have strong orders of convergence $1.0-\varepsilon$ and $1.5-\varepsilon$ correspondingly with respect to the temporal variable (where ε is an arbitrary small posilive real number). It should be noted that in [148] (2007) the convergence with strong order 1.0 of the exponential Milstein scheme for semilinear SPDEs was proved under additional smoothness assumptions.

An important feature of the mentioned numerical methods is the presence in them the so-called iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [150]. Approximation of these stochastic integrals is a complex problem. The problem of numerical modeling of these stochastic integrals with multiplicities 1 to 3 was solved in [143], [144] for the case when special commutativity conditions for semilinear SPDE with nonlinear multiplicative trace class noise are fulfilled.

If the mentioned commutativity conditions are not fulfilled, which often corresponds to SPDEs in numerous applications, the numerical modeling of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process becomes much more difficult. Note that the exponential Milstein scheme [143] contains the iterated stochastic integrals of multiplicities 1 and 2 with respect to the infinite-dimensional Q-Wiener process and the exponential Wagner–Platen scheme [144] contains the mentioned stochastic integrals of multiplicities 1 to 3 (see Sect. 7.2).

In [156] (2017), [157] (2018) two methods of the mean-square approximation of simplest iterated (double) stochastic integrals from the exponential Milstein scheme for semilinear SPDEs with nonlinear multiplicative trace class noise and without the commutativity conditions are considered and theorems on the convergence of these methods are given. At that, the basic idea (first of the mentioned methods [156], [157]) about the Karhunen–Loève expansion of the Brownian bridge process was taken from the monograph [65] (Milstein approach, see Sect. 6.2). The second of the mentioned methods [156], [157] is based on the results of Wiktorsson M. [71], [72] (2001).

Note that the mean-square error of approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process consists of two components [156], [157]. The first component is related with the finite-dimentional approximation of the infinite-dimentional Q-Wiener process while the second one is connected with the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions.

It is important to note that the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be reduced to the approximation of iterated Itô stochastic integrals with respect to the finite-dimensional Wiener process. In a lot of author's publications [1]-[54] (see Chapters 1, 2, and 5) an effective method of the mean-square approximation of iterated Itô (and Stratonovich) stochastic integrals with respect to the finite-dimensional Wiener process was proposed and developed. This method is based on the generalized multiple Fourier series, in particular, on the multiple Fourier-Legendre series (see Sect. 5.1).

The purpose of this chapter is an adaptation of the method [1]-[54] for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process. In the author's publications [22], [46] (see Sect. 7.3) the problem of the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process in the sense of the second component of approximation error (see above) has been solved for arbitraty multiplicity k ($k \in \mathbf{N}$) of stochastic integrals and without the assumptions of commutativity for SPDE. More precisely, in [22], [46] the method of generalized multiple Fourier series (Theorem 1.1) for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process (in the sense of the second component of approximation error).

In Sect. 7.4 (also see [23], [47]), we extend the method [156], [157] and estimate the first component of approximation error for iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. In addition, we combine the obtained results with the results from [22], [46] (see Sect. 7.3). Thus, the results of this chapter can be applied to the implementation of exponential Milstein and Wagner–Platen schemes for semilinear SPDEs with nonlinear multiplicative trace class noise and without the commutativity conditions.

Let U, H be separable $\mathbf{R}$-Hilbert spaces and $L_{HS}(U,H)$ be a space of Hilbert–Schmidt operators mapping from U to H. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$ [150], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here and further $L(U)$ denotes all bounded linear operators on U. Let U_0 be an $\mathbf{R}$-Hilbert space defined as $U_0 = Q^{1/2}(U)$. At that, a scalar product in U_0 is given by the relation [144]

$$\langle u, w \rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w \right\rangle_U$$

for all $u, w \in U_0$.

Consider the semilinear parabolic SPDE with nonlinear multiplicative trace class noise

$$dX_t = (AX_t + F(X_t))\, dt + B(X_t) d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}], \qquad (7.1)$$

where nonlinear operators F, B ($F: H \to H$, $B: H \to L_{HS}(U_0, H)$), the linear operator $A: D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the SPDE mild solution (see [144], Assumptions A1–A4).

It is well known [153] that Assumptions A1–A4 [144] guarantee the existence and uniqueness (up to modifications) of the mild solution $X_t: [0, \bar{T}] \times \Omega \to H$ of the SPDE (7.1)

$$X_t = \exp(At)\xi + \int\limits_0^t \exp(A(t-\tau))F(X_\tau)d\tau + \int\limits_0^t \exp(A(t-\tau))B(X_\tau)d\mathbf{W}_\tau \quad (7.2)$$

w. p. 1 for all $t \in [0, \bar{T}]$, where $\exp(At)$, $t \geq 0$ is the semigroup generated by the operator A.

As we mentioned earlier, numerical methods of high orders of accuracy (with respect to the temporal discretization) for approximating the mild solution of the SPDE (7.1), which are based on the Taylor formula for operators and an exponential formula for the mild solution of SPDEs, contain iterated stochastic integrals with respect to the Q-Wiener process [139], [141]-[144], [148].

Note that the exponential Milstein type numerical scheme [143] and the exponential Wagner-Platen type numerical scheme [144] contain, for example, the following iterated stochastic integrals (see Sect. 7.2)

$$\int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \quad \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \quad (7.3)$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}, \quad \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \quad (7.4)$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \quad (7.5)$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \quad (7.6)$$

where $0 \leq t < T \leq \bar{T}$, $Z: \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and F', B', B'' denote Frêchet derivatives. At that, the exponential Milstein type scheme [143] contains integrals (7.3) while the exponential Wagner–Platen type scheme [144] contains integrals (7.3)–(7.6) (see Sect. 7.2).

It is easy to notice that the numerical schemes for SPDEs with higher orders of convergence (with respect to the temporal discretization) in contrast with the numerical schemes from [143], [144] will include iterated stochastic integrals (with respect to the Q-Wiener process) with multiplicities $k > 3$ [142] (2011). So, this chapter is partially devoted to the approximation of iterated stochastic integrals of the form

$$I[\Phi^{(k)}(Z)]_{T,t} = \int\limits_t^T \Phi_k(Z) \left(\ldots \left(\int\limits_t^{t_3} \Phi_2(Z) \left(\int\limits_t^{t_2} \Phi_1(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2} \right) \ldots \right) d\mathbf{W}_{t_k}, \tag{7.7}$$

where $0 \le t < T \le \bar{T}$, $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and $\Phi_k(v)(\,\ldots(\Phi_2(v)(\Phi_1(v))\ldots\,))$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0 \times \ldots \times U_0}_{k \text{ times}}$ to H for all $v \in H$.

In Sect. 7.3.1 we consider the approximation of more general iterated stochastic integrals than (7.7). In Sect. 7.3.2 and 7.3.3 some other types of iterated stochastic integrals of multiplicities 2–4 with respect to the Q-Wiener process will be considered.

Note that the stochastic integral (7.6) is not a special case of the stochastic integral (7.7) for $k = 3$. Nevertheless, the extended representation for approximation of the stochastic integral (7.6) is similar to (7.12) (see below) for $k = 3$. Moreover, the mentioned representation for approximation of the stochastic integral (7.6) contains the same iterated Itô stochastic integrals of third multiplicity as in (7.12) for $k = 3$ (see Sect. 7.3.2). These conclusions mean that one of the main results of this chapter (Theorem 7.1, Sect. 7.3.1) for $k = 3$ can be reformulated naturally for the stochastic integral (7.6) (see Sect. 7.3.2).

It should be noted that by developing the approach from the work [144], which uses the Taylor formula for operators and a formula for the mild solution of the SPDE (7.1), we obviously obtain a number of other iterated stochastic integrals. For example, the following stochastic integrals

$$\int\limits_t^T B'''(Z) \left(\int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z) d\mathbf{W}_{t_1} \right) d\mathbf{W}_{t_2},$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3},$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3},$$

$$\int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)dt_3,$$

$$\int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2,$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}$$

will be considered in Sect. 7.3.3. Here $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and B', B'', B''', F', F'' are Frêchet derivatives.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i=(i_1,\ldots,i_d)\in J,\ \ x=(x_1,\ldots,x_d)\in U,$ and $J=\bigl\{i: i\in\mathbf{N}^d$ and $\lambda_i>0\bigr\}$.

The series representation of the Q-Wiener process has the form [150]

$$\mathbf{W}(t,x)=\sum_{i\in J} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar T]$$

or in the shorter notations

$$\mathbf{W}_t=\sum_{i\in J} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar T],$$

where $\mathbf{w}_t^{(i)}$, $i\in J$ are independent standard Wiener processes.

Note that eigenfunctions e_i, $i\in J$ form an orthonormal basis of U [150].

Consider the finite-dimensional approximation of $\mathbf{W}_t$ [150]

$$\mathbf{W}_t^M=\sum_{i\in J_M} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar T], \tag{7.8}$$

where

$$J_M=\bigl\{i:\ 1\le i_1,\ldots,i_d\le M \text{ and } \lambda_i>0\bigr\}. \tag{7.9}$$

Using (7.8) and the relation [150]

$$\mathbf{w}_t^{(i)} = \frac{1}{\sqrt{\lambda_i}}\langle e_i, \mathbf{W}_t\rangle_U, \quad i \in J, \tag{7.10}$$

we obtain

$$\mathbf{W}_t^M = \sum_{i\in J_M} e_i \ \langle e_i, \mathbf{W}_t\rangle_U, \quad t \in [0, \bar{T}], \tag{7.11}$$

where $\langle \cdot, \cdot\rangle_U$ is a scalar product in U.

Taking into account (7.10) and (7.11), we note that the approximation $I[\Phi^{(k)}(Z)]_{T,t}^M$ of the iterated stochastic integral $I[\Phi^{(k)}(Z)]_{T,t}$ (see (7.7)) can be rewritten w. p. 1 in the following form

$$I[\Phi^{(k)}(Z)]_{T,t}^M =$$

$$= \int\limits_t^T \Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\left(\int\limits_t^{t_2}\Phi_1(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)\ldots\right)d\mathbf{W}_{t_k}^M =$$

$$= \sum_{r_1,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\times$$

$$\times\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2} d\langle e_{r_1}, \mathbf{W}_{t_1}\rangle_U \ d\langle e_{r_2}, \mathbf{W}_{t_2}\rangle_U \ \ldots \ d\langle e_{r_k}, \mathbf{W}_{t_k}\rangle_U =$$

$$= \sum_{r_1,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\times$$

$$\times\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}, \tag{7.12}$$

where $0 \le t < T \le \bar{T}$.

Remark 7.1. *Obviously, without the loss of generality we can write* $J_M = \{1, 2, \ldots, M\}$.

As we mentioned before, when special conditions of commutativity for the SPDE (7.1) be fulfilled, it is proposed to simulate numerically the stochastic integrals (7.3)–(7.6) using the simple formulas [143], [144]. In this case, the numerical simulation of the mentioned stochastic integrals requires the use of increments of the Q-Wiener process only. However, if these commutativity conditions are not fulfilled (which often corresponds to SPDEs in numerous applications), the numerical simulation of the stochastic integrals (7.3)–(7.6) becomes much more difficult. Recall that in [156], [157] two methods for the mean-square approximation of simplest iterated (double) stochastic integrals defined by (7.3) are proposed. In this Chapter, we consider a substantially more general and effective method (based on the results of Chapters 1 and 5) for the mean-square approximation of iterated stochastic integrals of multiplicity k ($k \in \mathbf{N}$) with respect to the Q-Wiener process. The convergence analysis in the transition from J_M to J, i.e., from the finite-dimensional Wiener process to the infinite-dimensional one will be carried out in Sect. 7.4 for integrals of multiplicities 1 to 3 similar to the proof of Theorem 1 [157].

7.2 Exponential Milstein and Wagner–Platen Numerical Schemes for Non-Commutative Semilinear SPDEs

Let assumptions of Sect. 7.1 are fulfilled. Let $\Delta > 0$, $\tau_p = p\Delta$ $(p = 0, 1, \ldots, N)$, and $N\Delta = \bar{T}$. Consider the exponential Milstein numerical scheme [143]

$$Y_{p+1} = \exp\left(A\Delta\right)\left(Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$\left. + \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s\right) \qquad (7.13)$$

and the exponential Wagner–Platen numerical scheme [144]

$$Y_{p+1} = \exp\left(\frac{A\Delta}{2}\right)\left(\exp\left(\frac{A\Delta}{2}\right)Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$+ \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s +$$

$$
+\frac{\Delta^2}{2}F'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)+\int\limits_{\tau_p}^{\tau_{p+1}}F'(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}\right)ds+
$$

$$
+\frac{\Delta^2}{4}\sum_{i\in J}\lambda_i F''(Y_p)\Bigl(B(Y_p)e_i,B(Y_p)e_i\Bigr)+
$$

$$
+A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}ds-\frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}}B(Y_p)d\mathbf{W}_s\right)+
$$

$$
+\Delta\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_s-\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_{\tau}ds+
$$

$$
+\frac{1}{2}\int\limits_{\tau_p}^{\tau_{p+1}}B''(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau},\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}\right)d\mathbf{W}_s+
$$

$$
+\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\left(\int\limits_{\tau_p}^{s}B'(Y_p)\left(\int\limits_{\tau_p}^{\tau}B(Y_p)d\mathbf{W}_{\theta}\right)d\mathbf{W}_{\tau}\right)d\mathbf{W}_s\Biggr) \qquad (7.14)
$$

for the SPDE (7.1), where Y_p is an approximation of X_{τ_p} (mild solution (7.2) at the time moment τ_p), $p=0,1,\ldots,N$, and B', B'', F', F'' are Frêchet derivatives.

Note that in addition to the temporal discretization, the implementation of numerical schemes (7.13) and (7.14) also requires a discretization of the infinite-dimensional Hilbert space H (approximation with respect to the space domain) and a finite-dimensional approximation of the Q-Wiener process. Let us focus on the approximation connected with the Q-Wiener process.

Consider the following iterated Itô stochastic integrals

$$
J_{(1)T,t}^{(r_1)}=\int\limits_t^T d\mathbf{w}_{t_1}^{(r_1)},\quad J_{(10)T,t}^{(r_10)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}dt_2,\quad J_{(01)T,t}^{(0r_2)}=\int\limits_t^T\int\limits_t^{t_2}dt_1d\mathbf{w}_{t_2}^{(r_2)}, \qquad (7.15)
$$

$$
J_{(11)T,t}^{(r_1r_2)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)},\quad J_{(111)T,t}^{(r_1r_2r_3)}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}d\mathbf{w}_{t_3}^{(r_3)}, \qquad (7.16)
$$

where $r_1, r_2, r_3 \in J_M$, $0 \le t < T \le \bar{T}$, and J_M is defined by (7.9).

Let us replace the infinite-dimensional Q-Wiener process in the iterated stochastic integrals from (7.13), (7.14) by its finite-dimensional approximation (7.8). Then we have w. p. 1

$$\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s^M = \sum_{r_1\in J_M} B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}J_{(1)\tau_{p+1},\tau_p}^{(r_1)}, \tag{7.17}$$

$$A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M ds - \frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s^M\right) =$$

$$= A\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)\left(\frac{\tau_{p+1}}{2} - s + \frac{\tau_p}{2}\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1\in J_M} AB(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\frac{\Delta}{2}J_{(1)\tau_{p+1},\tau_p}^{(r_1)} - J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{7.18}$$

$$\Delta\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)d\mathbf{W}_s^M - \int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)d\mathbf{W}_\tau^M ds =$$

$$= \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\int\limits_{\tau_p}^{s}\Bigl(AY_p + F(Y_p)\Bigr)d\tau d\mathbf{W}_s^M =$$

$$= \sum_{r_1\in J_M} B'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}, \tag{7.19}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} F'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)ds =$$

$$= \sum_{r_1\in J_M} F'(Y_p)B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\Delta J_{(1)\tau_{p+1},\tau_p}^{(r_1)} - J_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{7.20}$$

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_{\tau}^{M}\right)d\mathbf{W}_{s}^{M} =
$$

$$
=\sum_{r_1,r_2\in J_M} B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)\tau_{p+1},\tau_p}^{(r_1r_2)}, \tag{7.21}
$$

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B'(Y_p)\left(\int\limits_{\tau_p}^{\tau} B(Y_p)d\mathbf{W}_{\theta}^{M}\right)d\mathbf{W}_{\tau}^{M}\right)d\mathbf{W}_{s}^{M} =
$$

$$
=\sum_{r_1,r_2,r_3\in J_M} B'(Y_p)\left(B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)}, \tag{7.22}
$$

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_{\tau}^{M},\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_{\tau}^{M}\right)d\mathbf{W}_{s}^{M} =
$$

$$
=\sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1},B(Y_p)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times
$$

$$
\times\int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)}\right)d\mathbf{w}_{s}^{(r_3)}. \tag{7.23}
$$

Note that in (7.18)–(7.20) we used the Itô formula. Moreover, using the Itô formula we obtain

$$
\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)} = J_{(11)s,\tau_p}^{(r_1r_2)} + J_{(11)s,\tau_p}^{(r_2r_1)} + \mathbf{1}_{\{r_1=r_2\}}(s-\tau_p)\quad \text{w. p. 1}, \tag{7.24}
$$

where $\mathbf{1}_A$ is the indicator of the set A. From (7.24) we have w. p. 1

$$
\int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)}\right)d\mathbf{w}_{s}^{(r_3)} = J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + J_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)\tau_{p+1},\tau_p}^{(0r_3)}. \tag{7.25}
$$

After substituting (7.25) into (7.23), we obtain w. p. 1

$$\int_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right) d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1}, B(Y_p)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + J_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)\tau_{p+1},\tau_p}^{(0r_3)}\right). \qquad (7.26)$$

Thus, for the implementation of numerical schemes (7.13) and (7.14) we need to approximate the following collection of iterated Itô stochastic integrals

$$J_{(1)T,t}^{(r_1)},\quad J_{(01)T,t}^{(0r_1)},\quad J_{(11)T,t}^{(r_1r_2)},\quad J_{(111)T,t}^{(r_1r_2r_3)}, \qquad (7.27)$$

where $r_1, r_2, r_3 \in J_M,\ \ 0 \le t < T \le \bar{T}$.

The problem of the mean-square approximation of iterated Itô stochastic integrals (7.27) is considered completely in Chapters 1 and 5.

7.3 Approximation of Iterated Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$) with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

In this section, we consider a method for the approximation of iterated stochastic integrals of multiplicity k ($k \in \mathbf{N}$) with respect to the finite-dimensional approximation $\mathbf{W}_t^M$ of the Q-Wiener process $\mathbf{W}_t$ using the mean-square approximation method of iterated Itô stochastic integrals based on Theorem 1.1.

7.3.1 Theorem on the Mean-Square Approximation of Iterated Stochastic Integrals of Multiplicity k ($k \in \mathbf{N}$) with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

Consider the iterated stochastic integral with respect to the Q-Wiener process in the following form

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\right)\psi_2(t_2)d\mathbf{W}_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k},\qquad(7.28)$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $\mathbf{W}_\tau$ is the Q-Wiener process, every nonrandom function $\psi_l(\tau)$ $(l=1,\ldots,k)$ is continuous on the interval $[t,T]$, and $\Phi_k(v)(\ldots(\Phi_2(v)(\Phi_1(v)))\ldots)$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v\in H$.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M$ be the approximation of the iterated stochastic integral (7.28)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}^M\right)\psi_2(t_2)d\mathbf{W}_{t_2}^M\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}^M=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^k\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)},\qquad(7.29)$$

where $0\le t<T\le\bar{T}$ and

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated Itô stochastic integral (1.5).

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}$ be the approximation of the iterated stochastic integral (7.29)

$$
I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}=
$$

$$
=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\times
$$

$$
\times\ J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}, \tag{7.30}
$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as a prelimit expression in (1.10)

$$
J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}-
$$

$$
-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(r_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(r_k)}\Biggr) \tag{7.31}
$$

or as a prelimit expression in (1.50)

$$
J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}+\sum_{m=1}^{[k/2]}(-1)^m\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{m}\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}\Biggr).
$$

$$
\tag{7.32}
$$

Let U, H be separable $\mathbf{R}$-Hilbert spaces, $U_0=Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let $L(U,H)_0=\left\{T|_{U_0}:\ T\in L(U,H)\right\}$ (here $T|_{U_0}$ is the restriction of operator T to the space U_0). It is known [150] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 7.1 [14], [22], [46]. *Let the conditions of Theorem* 1.1 *be fulfilled as well as the following conditions*:

1. $Q\in L(U)$ *is a nonnegative and symmetric trace class operator* (λ_i *and* e_i ($i\in J$) *are its eigenvalues and eigenfunctions* (*which form an orthonormal basis of* U) *correspondingly*) *and* $\mathbf{W}_\tau$, $\tau\in[0,\bar{T}]$ *is an* U*-valued* Q*-Wiener process.*

2. $Z:\Omega\to H$ *is an* $\mathbf{F}_t/\mathcal{B}(H)$*-measurable mapping.*

3. $\Phi_1\in L(U,H)_0,\ \Phi_2\in L(H,L(U,H)_0),$ *and* $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ *is a* k*-linear Hilbert–Schmidt operator mapping from* $\underbrace{U_0\times\ldots\times U_0}_{k\ \textit{times}}$ *to* H *for all* $v\in H$ *such that*

$$
\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H^2\le L_k<\infty
$$

w. p. 1 for all $r_1,r_2,\ldots,r_k\in J_M,\ M\in\mathbf{N}.$ *Then*

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}-I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}\right\|_H^2\right\}\le
$$

$$
\le L_k(k!)^2\left(\mathrm{tr}\ Q\right)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right), \qquad (7.33)
$$

where

$$
\mathrm{tr}\ Q=\sum_{i\in J}\lambda_i<\infty,
$$

$$
I_k=\|K\|^2_{L_2([t,T]^k)}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k,
$$

$$
C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k
$$

is the Fourier coefficient,

$$
K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\ =\ \prod_{l=1}^k\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}},
$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ *and* $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$ $(\mathbf{1}_A$ *denotes the indicator of the set* $A)$.

Remark 7.2. *It should be noted that the right-hand side of the inequality* (7.33) *is independent of M and tends to zero if $p_1,\ldots,p_k\to\infty$ due to the Parseval equality.*

Remark 7.3. *Recall the estimate* (1.92)*, which we will use in the proof of Theorem* 7.1

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)^2\right\}\le k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),
$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}$ is defined by (1.5) *and $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined by* (7.31) *or* (7.32).

Proof. Using (1.92), we obtain

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}-I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1,\ldots,p_k}\right\|_H^2\right\}=
$$

$$
=\mathsf{M}\left\{\left\|\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\times\right.\right.
$$

$$
\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\right\|_H^2\right\}= \qquad (7.34)
$$

$$
=\left|\mathsf{M}\left\{\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times\right.\right.
$$

$$
\times\Biggl\langle\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\ ,
$$

$$
\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{\prime}}\right)e_{r_2^{\prime}}\right)\ldots\right)e_{r_k^{\prime}}\Biggr\rangle_H\times
$$

$$
\times\ \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.
$$

$$
\left.\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right\}\right|\le \qquad (7.35)
$$

$$
\le \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{'}}\right)^{1/2}\times
$$

$$
\times{\sf M}\Biggl\{\Bigl\Vert\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\Bigr\Vert_H\times
$$

$$
\times\Bigl\Vert\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{'}}\right)e_{r_2^{'}}\right)\ldots\right)e_{r_k^{'}}\Bigr\Vert_H\times
$$

$$
\times\Biggl|{\sf M}\Biggl\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times
$$

$$
\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})}-J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})p_1,\ldots,p_k}\right)\Biggl|{\bf F}_t\Biggr\}\Biggr|\Biggr\}\le
$$

$$
\le L_k\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{'}}\right)^{1/2}\times
$$

$$
\times{\sf M}\Biggl\{\Biggl|\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times
$$

$$
\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})}-J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})p_1,\ldots,p_k}\right)\Biggr|\Biggr\}\le
$$

$$
\le L_k\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{'}}\right)^{1/2}\times
$$

$$
\times\left({\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\times
$$

$$
\times\left({\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})}-J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\left(\prod_{l=1}^{k}\lambda_{r_l^{\prime}}\right)^{1/2}\times
$$

$$
\times\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\le
$$

$$
\le L_k\sum_{r_1,r_2,\ldots,r_k\in J_M} k!\ \lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)=
$$

$$
=L_k\left(k!\right)^2\sum_{r_1,r_2,\ldots,r_k\in J_M}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\le
$$

$$
\le L_k\left(k!\right)^2\left(\mathrm{tr}\ Q\right)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),
$$

where $\langle\cdot,\cdot\rangle_H$ is a scalar product in H, and

$$
\sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}}
$$

means the sum with respect to all possible permutations $(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime})$ such that $\{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}$.

The transition from (7.34) to (7.35) is based on the following theorem.

Theorem 7.2 [14], [22], [46]. *The following equality is true*

$$
\mathsf{M}\Biggl\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}\right)\times
$$

$$
\times\left(J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}-J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\right)\Biggl|\mathbf{F}_t\Biggr\}=0 \qquad (7.36)
$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ *such that* $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Proof. Using the standard moment properties of the Itô stochastic integral, we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\Bigg|\mathbf{F}_t\right\}=0 \tag{7.37}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $(r_1,\ldots,r_k)\neq(m_1,\ldots,m_k)$.

Let us rewrite the formulas (7.31), (7.32) (also see (1.41)–(1.47)) in the form

$$J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(m_l)}-S_{j_1,\ldots,j_k}^{(m_1\ldots m_k)}\right). \tag{7.38}$$

From the proof of Theorem 1.1 (see (1.37) and (1.38)) it follows that

$$\prod_{l=1}^{k}\zeta_{j_l}^{(m_l)}-S_{j_1,\ldots,j_k}^{(m_1\ldots m_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\neq l_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(m_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(m_k)}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\quad \text{w. p. 1}, \tag{7.39}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then m_r swapped with m_q in the permutation $(m_1,\ldots,m_k)$; another notations are the same as in Theorem 1.1.

Then w. p. 1

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\Bigg|\mathbf{F}_t\right\}.$$

From the standard moment properties of the Itô stochastic integral it follows that

$$
\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\Bigg|\mathbf{F}_t\right\}=0
$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Then

$$
\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=0 \tag{7.40}
$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

Using (7.38), (7.39), we have

$$
\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Bigg|\mathbf{F}_t\right\}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{q_1=0}^{p_1}\ldots\sum_{q_k=0}^{p_k}C_{q_k\ldots q_1}\times
$$

$$
\times\mathsf{M}\left\{\left(\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\right)\times\right.
$$

$$
\left.\times\left(\sum_{(q_1,\ldots,q_k)}\int\limits_t^T\phi_{q_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{q_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\right)\Bigg|\mathbf{F}_t\right\}=0 \tag{7.41}
$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbf{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

From (7.37), (7.40), and (7.41) we obtain (7.36). Theorem 7.2 is proved.

Corollary 7.1 [14], [22], [46]. *The following equality is true*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}\right)\times\right.
$$

$$
\left.\times\left(J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)}-J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)q_1,\ldots,q_l}\right)\Bigg|\mathbf{F}_t\right\}=0
$$

w. p. 1 for all $l=1,2,\ldots,k-1$ *and* $r_1,\ldots,r_k,$ $m_1,\ldots,m_l\in J_M,$ $p_1,\ldots,p_k,$ $q_1,\ldots,q_l=0,1,2,\ldots$

7.3.2 Approximation of Some Iterated Stochastic Integrals of Miltiplicities 2 and 3 with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 7.1)

$$I_0[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2}F(Z)dt_1\right)d\mathbf{W}_{t_2}^M, \tag{7.42}$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)dt_2, \tag{7.43}$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M. \tag{7.44}$$

Let the conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B''(v)(B(v),B(v))$ be a 3-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0$ to H for all $v\in H$. Then we have w. p. 1 (see (7.29))

$$I_0[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M}B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}J_{(01)T,t}^{(0r_1)}, \tag{7.45}$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M}F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}J_{(10)T,t}^{(r_10)}, \tag{7.46}$$

$$I_2[B(Z)]_{T,t}^M=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}. \tag{7.47}$$

Using the Itô formula, we obtain

$$\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}=J_{(11)s,t}^{(r_1r_2)}+J_{(11)s,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}(s-t)\quad \text{w. p. 1.} \tag{7.48}$$

From (7.48) we have

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)} = J_{(111)T,t}^{(r_1r_2r_3)} + J_{(111)T,t}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)} \quad \text{w. p. 1.} \tag{7.49}$$

Note that in (7.45), (7.46), (7.48), and (7.49) we use the notations from Sect. 7.2 (see (7.15), (7.16)). After substituting (7.49) into (7.47), we have

$$I_2[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)} + J_{(111)T,t}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)}\right) \quad \text{w. p. 1.} \tag{7.50}$$

Taking into account (5.130), (5.131), we put for $q=1$

$$J_{(01)T,t}^{(0r_3)q} = J_{(01)T,t}^{(0r_3)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(r_3)}\right) \quad \text{w. p. 1,} \tag{7.51}$$

$$J_{(10)T,t}^{(r_10)q} = J_{(10)T,t}^{(r_10)} = \frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right) \quad \text{w. p. 1,} \tag{7.52}$$

where $J_{(01)T,t}^{(0r_3)q}$, $J_{(10)T,t}^{(r_10)q}$ denote the approximations of corresponding iterated Itô stochastic integrals.

Denote by $I_0[B(Z),F(Z)]_{T,t}^{M,q}$, $I_1[B(Z),F(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$ the approximations of iterated stochastic integrals (7.45), (7.46), (7.50)

$$I_0[B(Z),F(Z)]_{T,t}^{M,q} = \sum_{r_1\in J_M} B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}J_{(01)T,t}^{(0r_1)q}, \tag{7.53}$$

$$I_1[B(Z),F(Z)]_{T,t}^{M,q} = \sum_{r_1\in J_M} F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}J_{(10)T,t}^{(r_10)q}, \tag{7.54}$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)q} + J_{(111)T,t}^{(r_2r_1r_3)q} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)q}\right), \tag{7.55}$$

where $q = 1$ in (7.53), (7.54) and the approximations $J_{(111)T,t}^{(r_1 r_2 r_3)q}$, $J_{(111)T,t}^{(r_2 r_1 r_3)q}$ are defined by (1.90) for some $q \ge 1$.

From (7.45), (7.46), (7.50), (7.53)–(7.55) we have

$$I_0[B(Z),F(Z)]_{T,t}^{M} - I_0[B(Z),F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. } 1,$$

$$I_1[B(Z),F(Z)]_{T,t}^{M} - I_1[B(Z),F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. } 1,$$

$$I_2[B(Z)]_{T,t}^{M} - I_2[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3 \in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(\left(J_{(111)T,t}^{(r_1 r_2 r_3)} - J_{(111)T,t}^{(r_1 r_2 r_3)q}\right) + \left(J_{(111)T,t}^{(r_2 r_1 r_3)} - J_{(111)T,t}^{(r_2 r_1 r_3)q}\right)\right) \quad \text{w. p. } 1.$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k = 3$, we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M} - I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le 4C(3!)^2 \left(\text{tr } Q\right)^3 \left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2\right),$$

where here and further constant C has the same meaning as constant L_k in Theorem 7.1 (k is the multiplicity of the iterated stochastic integral), and

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

where $P_j(x)$ is the Legendre polynomial.

7.3.3 Approximation of Some Iterated Stochastic Integrals of Miltiplicities 3 and 4 with Respect to the Finite-Dimensional Approximation $\mathbf{W}_t^M$ of the Q-Wiener Process

In this section, we consider the approximation of iterated stochastic integrals of the following form (see Sect. 7.1)

$$I_3[B(Z)]_{T,t}^M = \int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M,$$

$$I_4[B(Z)]_{T,t}^M =$$

$$= \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M\right) d\mathbf{W}_{t_3}^M,$$

$$I_5[B(Z)]_{T,t}^M =$$

$$= \int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M\right) d\mathbf{W}_{t_3}^M,$$

$$I_6[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M\right) dt_3,$$

$$I_7[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) dt_2,$$

$$I_8[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M.$$

Consider the stochastic integral $I_3[B(Z)]_{T,t}^M$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B'''(v)(B(v),B(v),B(v))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times \int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)} \int\limits_t^s d\mathbf{w}_\tau^{(r_2)} \int\limits_t^s d\mathbf{w}_\tau^{(r_3)} \right) d\mathbf{w}_s^{(r_4)} \quad \text{w. p. 1.} \tag{7.56}$$

By analogy with (2.308) or using the Itô formula, we obtain

$$J_{(1)s,t}^{(r_1)} J_{(1)s,t}^{(r_2)} J_{(1)s,t}^{(r_3)} = J_{(111)s,t}^{(r_1 r_2 r_3)} + J_{(111)s,t}^{(r_1 r_3 r_2)} + J_{(111)s,t}^{(r_2 r_1 r_3)} + J_{(111)s,t}^{(r_2 r_3 r_1)} + J_{(111)s,t}^{(r_3 r_1 r_2)} + J_{(111)s,t}^{(r_3 r_2 r_1)} +$$

$$+ \mathbf{1}_{\{r_1 = r_2\}} \left(J_{(10)s,t}^{(r_3 0)} + J_{(01)s,t}^{(0 r_3)} \right) + \mathbf{1}_{\{r_1 = r_3\}} \left(J_{(10)s,t}^{(r_2 0)} + J_{(01)s,t}^{(0 r_2)} \right) +$$

$$+ \mathbf{1}_{\{r_2 = r_3\}} \left(J_{(10)s,t}^{(r_1 0)} + J_{(01)s,t}^{(0 r_1)} \right) =$$

$$= \sum_{(r_1, r_2, r_3)} J_{(111)s,t}^{(r_1 r_2 r_3)} + (s-t) \left(\mathbf{1}_{\{r_2 = r_3\}} J_{(1)s,t}^{(r_1)} + \mathbf{1}_{\{r_1 = r_3\}} J_{(1)s,t}^{(r_2)} + \mathbf{1}_{\{r_1 = r_2\}} J_{(1)s,t}^{(r_3)} \right) \tag{7.57}$$

w. p. 1, where

$$\sum_{(r_1, r_2, r_3)}$$

means the sum with respect to all possible permutations (r_1, r_2, r_3). We also use the notations from Sect. 7.2 (see (7.15), (7.16)).

After substituting (7.57) into (7.56), we obtain

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3, r_4 \in J_M} B'''(Z) \left(B(Z) e_{r_1}, B(Z) e_{r_2}, B(Z) e_{r_3} \right) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times$$

$$\times \left(\sum_{(r_1, r_2, r_3)} J_{(1111)T,t}^{(r_1 r_2 r_3 r_4)} - \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(r_3 r_4)} - \mathbf{1}_{\{r_1 = r_3\}} I_{(01)T,t}^{(r_2 r_4)} - \mathbf{1}_{\{r_2 = r_3\}} I_{(01)T,t}^{(r_1 r_4)} \right) \tag{7.58}$$

w. p. 1, where $J_{(1111)T,t}^{(r_1 r_2 r_3 r_4)}$ is defined by (5.100) and

$$I_{(01)T,t}^{(r_1 r_2)} = \int\limits_t^T (t-s) \int\limits_t^s d\mathbf{w}_\tau^{(r_1)} d\mathbf{w}_s^{(r_2)}. \tag{7.59}$$

Denote by $I_3[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.58), which has the following form

$$I_3[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)} J_{(1111)T,t}^{(r_1r_2r_3r_4)q} - \mathbf{1}_{\{r_1=r_2\}} I_{(01)T,t}^{(r_3r_4)q} - \mathbf{1}_{\{r_1=r_3\}} I_{(01)T,t}^{(r_2r_4)q} - \mathbf{1}_{\{r_2=r_3\}} I_{(01)T,t}^{(r_1r_4)q}\right), \tag{7.60}$$

where the approximations $J_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $I_{(01)T,t}^{(r_1r_2)q}$ are based on Theorem 1.1 and Legendre polynomials (see (5.13) and (5.55)).

For example, from (5.13) we have (here we use the notation $I_{(01)T,t}^{(r_1r_2)}$ from the formula (5.13))

$$I_{(01)T,t}^{(r_1r_2)q} = -\frac{T-t}{2}J_{(11)T,t}^{(r_1r_2)q} - \frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_1)}\zeta_1^{(r_2)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(r_1)}\zeta_{i+2}^{(r_2)} - (i+1)\zeta_{i+2}^{(r_1)}\zeta_i^{(r_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \tag{7.61}$$

$$J_{(11)T,t}^{(r_1r_2)q} = \frac{T-t}{2}\left(\zeta_0^{(r_1)}\zeta_0^{(r_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(r_1)}\zeta_i^{(r_2)} - \zeta_i^{(r_1)}\zeta_{i-1}^{(r_2)}\right) - \mathbf{1}_{\{r_1=r_2\}}\right), \tag{7.62}$$

where notations are the same as in Theorem 1.1. For $r_1\ne r_2$ we get (see (5.38))

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(r_1r_2)} - I_{(01)T,t}^{(r_1r_2)q}\right)^2\right\} = \frac{(T-t)^4}{16}\left(\frac{5}{9} - 2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2} - \sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \tag{7.63}$$

From (1.92) and (7.63) we obtain

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(r_1r_2)} - I_{(01)T,t}^{(r_1r_2)q}\right)^2\right\} \le \frac{(T-t)^4}{8}\left(\frac{5}{9} - 2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\right.$$

$$\left.-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2} - \sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right),$$

where $r_1,r_2=1,\ldots,M$.

From (7.58) and (7.60) it follows that

$$I_3[B(Z)]_{T,t}^{M} - I_3[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\sum_{(r_1,r_2,r_3)}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)} - J_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right) - \mathbf{1}_{\{r_1=r_2\}}\left(I_{(01)T,t}^{(r_3r_4)} - I_{(01)T,t}^{(r_3r_4)q}\right) -$$

$$-\mathbf{1}_{\{r_1=r_3\}}\left(I_{(01)T,t}^{(r_2r_4)} - I_{(01)T,t}^{(r_2r_4)q}\right) - \mathbf{1}_{\{r_2=r_3\}}\left(I_{(01)T,t}^{(r_1r_4)} - I_{(01)T,t}^{(r_1r_4)q}\right)\Biggr) \quad \text{w. p. 1.} \quad (7.64)$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$, we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M} - I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\text{tr}\ Q)^4\left(6^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + 3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.63) and

$$C_{j_4j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2\bar{C}_{j_4j_3j_2j_1}, \qquad (7.65)$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

where $P_j(x)$ is the Legendre polynomial.

Consider the stochastic integral $I_4[B(Z)]_{T,t}^{M}$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B'(v)(B''(v)(B(v),B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_4[B(Z)]_{T,t}^{M} = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\int\limits_t^T\int\limits_t^s\left(\int\limits_t^{\tau}d\mathbf{w}_u^{(r_1)}\int\limits_t^{\tau}d\mathbf{w}_u^{(r_2)}\right)d\mathbf{w}_{\tau}^{(r_3)}d\mathbf{w}_s^{(r_4)}\quad \hbox{w. p. 1.}\quad (7.66)$$

From (7.49) and (7.66) we obtain

$$I_4[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M}B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)}+J_{(1111)T,t}^{(r_2r_1r_3r_4)}-\mathbf{1}_{\{r_1=r_2\}}I_{(10)T,t}^{(r_3r_4)}\right)\quad \hbox{w. p. 1,}\quad (7.67)$$

where

$$I_{(10)T,t}^{(r_3r_4)}=\int\limits_t^T\int\limits_t^s(t-\tau)d\mathbf{w}_{\tau}^{(r_3)}d\mathbf{w}_s^{(r_4)}. \qquad (7.68)$$

Denote by $I_4[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.67), which has the following form

$$I_4[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M}B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)q}+J_{(1111)T,t}^{(r_2r_1r_3r_4)q}-\mathbf{1}_{\{r_1=r_2\}}I_{(10)T,t}^{(r_3r_4)q}\right)\quad \hbox{w. p. 1,}\quad (7.69)$$

where the approximations $J_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $I_{(10)T,t}^{(r_1r_2)q}$ are based on Theorem 1.1 and Legendre polynomials.

For example, from (5.14) we have (here we use the notation $I_{(10)T,t}^{(r_1r_2)}$ from the formula (5.14))

$$I_{(10)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}J_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_2)}\zeta_1^{(r_1)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(r_2)}\zeta_i^{(r_1)}-(i+2)\zeta_i^{(r_2)}\zeta_{i+2}^{(r_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \qquad (7.70)$$

where the approximation $J_{(11)T,t}^{(r_1r_2)q}$ is defined by (7.62).

Moreover,

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(r_1r_2)}-I_{(10)T,t}^{(r_1r_2)q}\right)^2\right\}=E_q\quad (r_1\ne r_2), \tag{7.71}$$

where E_q is the right-hand side of (7.63) (see (5.38)).

From (7.67), (7.69) we have

$$I_4[B(Z)]_{T,t}^{M}-I_4[B(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\left(J_{(1111)T,t}^{(r_1r_2r_3r_4)}-J_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right)+\left(J_{(1111)T,t}^{(r_2r_1r_3r_4)}-J_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right)-$$

$$-\mathbf{1}_{\{r_1=r_2\}}\left(I_{(10)T,t}^{(r_3r_4)}-I_{(10)T,t}^{(r_3r_4)q}\right)\Biggr)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$, we obtain

$$\mathsf{M}\left\{\left\|I_4[B(Z)]_{T,t}^{M}-I_4[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le C\left(\text{tr }Q\right)^4\left(2^2(4!)^2\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right)+(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.63) and $C_{j_4j_3j_2j_1}$ is defined by (7.65).

Consider the stochastic integral $I_5[B(Z)]_{T,t}^{M}$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled. Let $B''(v)(B(v),B'(v)(B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (7.29))

$$I_5[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3},B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\int_t^T\left(\int_t^s d\mathbf{w}_\tau^{(r_3)}\int_t^s\int_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)}\quad \text{w. p. 1.} \tag{7.72}$$

Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and Example 3.1) or the Itô formula, we obtain

$$
\int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)} \int\limits_t^s \int\limits_t^\tau d\mathbf{w}_u^{(r_2)} d\mathbf{w}_\tau^{(r_1)}\right) d\mathbf{w}_s^{(r_4)} =
$$

$$
= J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)} +
$$

$$
+ \mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)} - I_{(01)T,t}^{(r_2 r_4)}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)} \quad \text{w. p. 1}, \tag{7.73}
$$

where we use the notations from Sect. 7.2 (see (7.16)) and $I_{(01)T,t}^{(r_1 r_2)}$, $I_{(10)T,t}^{(r_1 r_2)}$ are defined by (7.59), (7.68).

After substituting (7.73) into (7.72), we obtain

$$
I_5[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4} \times
$$

$$
\times \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \Biggl(J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)} +
$$

$$
+ \mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)} - I_{(01)T,t}^{(r_2 r_4)}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)} \Biggr) \quad \text{w. p. 1}. \tag{7.74}
$$

Denote by $I_5[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.74), which has the following form

$$
I_5[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4} \times
$$

$$
\times \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \Biggl(J_{(1111)T,t}^{(r_2 r_1 r_3 r_4)q} + J_{(1111)T,t}^{(r_2 r_3 r_1 r_4)q} + J_{(1111)T,t}^{(r_3 r_2 r_1 r_4)q} +
$$

$$
+ \mathbf{1}_{\{r_1=r_3\}} \left(I_{(10)T,t}^{(r_2 r_4)q} - I_{(01)T,t}^{(r_2 r_4)q}\right) - \mathbf{1}_{\{r_2=r_3\}} I_{(10)T,t}^{(r_1 r_4)q} \Biggr) \quad \text{w. p. 1}, \tag{7.75}
$$

where the approximations $J_{(1111)T,t}^{(r_1 r_2 r_3 r_4)q}$, $I_{(01)T,t}^{(r_1 r_2)q}$, and $I_{(10)T,t}^{(r_1 r_2)q}$ are based on Theorem 1.1 and Legendre polynomials.

From (7.74), (7.75) it follows that

$$I_5[B(Z)]_{T,t}^{M} - I_5[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\left(J_{(1111)T,t}^{(r_2r_1r_3r_4)} - J_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right)+\left(J_{(1111)T,t}^{(r_2r_3r_1r_4)} - J_{(1111)T,t}^{(r_2r_3r_1r_4)q}\right)+\left(J_{(1111)T,t}^{(r_3r_2r_1r_4)} - J_{(1111)T,t}^{(r_3r_2r_1r_4)q}\right)+$$

$$+\mathbf{1}_{\{r_1=r_3\}}\left(\left(I_{(10)T,t}^{(r_2r_4)} - I_{(10)T,t}^{(r_2r_4)q}\right) - \left(I_{(01)T,t}^{(r_2r_4)} - I_{(01)T,t}^{(r_2r_4)q}\right)\right) -$$

$$-\mathbf{1}_{\{r_2=r_3\}}\left(I_{(10)T,t}^{(r_1r_4)} - I_{(10)T,t}^{(r_1r_4)q}\right)\Biggr) \quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the cases $k=2$ and $k=4$ and taking into account (7.71), we obtain

$$\mathsf{M}\left\{\left\|I_5[B(Z)]_{T,t}^{M} - I_5[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\mathrm{tr}\ Q)^4\left(3^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + 3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (7.63) and $C_{j_4j_3j_2j_1}$ is defined by (7.65).

Consider the stochastic integral $I_6[B(Z),F(Z)]_{T,t}^{M}$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled. We have (see (7.29))

$$I_6[B(Z),F(Z)]_{T,t}^{M} = \sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\int\limits_t^T\int\limits_t^s\int\limits_t^{\tau} d\mathbf{w}_u^{(r_1)}d\mathbf{w}_{\tau}^{(r_2)}ds \quad \text{w. p. 1.} \qquad (7.76)$$

Using the theorem on replacement of the integration order in iterated Itô stochastic integrals (see Theorem 3.1 and Example 3.1) or the Itô formula, we obtain

$$\int\limits_t^T\int\limits_t^s\int\limits_t^{\tau} d\mathbf{w}_u^{(r_1)}d\mathbf{w}_{\tau}^{(r_2)}ds = (T-t)J_{(11)T,t}^{(r_1r_2)} + I_{(01)T,t}^{(r_1r_2)} \quad \text{w. p. 1.} \qquad (7.77)$$

After substituting (7.77) into (7.76), we have

$$I_6[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(11)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_1r_2)}\right)\quad \text{w. p. 1.} \tag{7.78}$$

Denote by $I_6[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.78), which has the following form

$$I_6[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(11)T,t}^{(r_1r_2)q}+I_{(01)T,t}^{(r_1r_2)q}\right), \tag{7.79}$$

where the approximations $I_{(01)T,t}^{(r_1r_2)q}$, $J_{(11)T,t}^{(r_1r_2)q}$ are defined by (7.61), (7.62).

From (7.78), (7.79) we get

$$I_6[B(Z),F(Z)]_{T,t}^{M}-I_6[B(Z),F(Z)]_{T,t}^{M,q}=$$

$$=\sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)\left(J_{(11)T,t}^{(r_1r_2)}-J_{(11)T,t}^{(r_1r_2)q}\right)+\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)\right)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_6[B(Z),F(Z)]_{T,t}^{M}-I_6[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2C(2!)^2\left(\text{tr }Q\right)^2\left((T-t)^2G_q+E_q\right),$$

where G_q and E_q are the right-hand sides of (5.37) and (7.63) correspondingly.

Consider the stochastic integral $I_7[B(Z),F(Z)]_{T,t}^{M}$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled.

Then we have (see (7.29))

$$I_7[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds\quad \text{w. p. 1.}\tag{7.80}$$

From (7.48) and (7.77) we get w. p. 1

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)ds=$$

$$=\int\limits_t^T J_{(11)s,t}^{(r_1r_2)}ds+\int\limits_t^T J_{(11)s,t}^{(r_2r_1)}ds+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)\left(J_{(11)T,t}^{(r_1r_2)}+J_{(11)T,t}^{(r_2r_1)}\right)+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)\left(J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}-\mathbf{1}_{\{r_1=r_2\}}(T-t)\right)+$$

$$+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$=(T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}.\tag{7.81}$$

After substituting (7.81) into (7.80), we obtain

$$I_7[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)}+I_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right)\quad \text{w. p. 1.}\tag{7.82}$$

Denote by $I_7[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.82), which has the following form

$$I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left((T-t)J_{(1)T,t}^{(r_1)}J_{(1)T,t}^{(r_2)}+I_{(01)T,t}^{(r_1r_2)q}+I_{(01)T,t}^{(r_2r_1)q}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right), \quad (7.83)$$

where the approximation $I_{(01)T,t}^{(r_1r_2)q}$ is defined by (7.61).

From (7.82), (7.83) it follows that

$$I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}}\left(\left(I_{(01)T,t}^{(r_1r_2)}-I_{(01)T,t}^{(r_1r_2)q}\right)+\left(I_{(01)T,t}^{(r_2r_1)}-I_{(01)T,t}^{(r_2r_1)q}\right)\right)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 4C(2!)^2\left(\text{tr }Q\right)^2E_q,$$

where E_q is the right-hand side of (7.63).

Consider the stochastic integral $I_8[B(Z),F(Z)]_{T,t}^{M}$. Let the conditions 1, 2 of Theorem 7.1 be fulfilled.

Then we have w. p. 1 (see (7.29))

$$I_8[B(Z),F(Z)]_{T,t}^{M}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(01)T,t}^{(r_1r_2)}. \quad (7.84)$$

Denote by $I_8[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (7.84), which has the following form

$$I_8[B(Z),F(Z)]_{T,t}^{M,q}=-\sum_{r_1,r_2\in J_M}B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(01)T,t}^{(r_1r_2)q}, \quad (7.85)$$

where the approximation $I_{(01)T,t}^{(r_1r_2)q}$ is defined by (7.61).

From (7.84), (7.85) we get

$$I_8[B(Z),F(Z)]^{M}_{T,t}-I_8[B(Z),F(Z)]^{M,q}_{T,t}=$$

$$=-\sum_{r_1,r_2\in J_M} B''(Z)\left(F(Z),B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}\left(I^{(r_1r_2)}_{(01)T,t}-I^{(r_1r_2)q}_{(01)T,t}\right)\quad \hbox{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_8[B(Z),F(Z)]^{M}_{T,t}-I_8[B(Z),F(Z)]^{M,q}_{T,t}\right\|_H^2\right\}\le C(2!)^2\left(\hbox{tr}\ Q\right)^2E_q,$$

where E_q is the right-hand side of (7.63).

7.4 Approximation of Iterated Stochastic Integrals of Miltiplicities 1 to 3 with Respect to the Infinite-Dimensional Q-Wiener Process

This section is devoted to the application of Theorem 1.1 and multiple Fourier–Legendre series for the approximation of iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. These iterated stochastic integrals are part of the exponential Milstein and Wagner–Platen numerical methods for semilinear SPDEs with nonlinear multiplicative trace class noise (see Sect. 7.2). Theorem 7.3 (see below) on the mean-square convergence of approximations of iterated stochastic integrals of multiplicities 2 and 3 with respect to the infinite-dimensional Q-Wiener process is formulated and proved. The results of this section can be applied to the implementation of high-order strong numerical methods for non-commutative semilinear SPDEs with nonlinear multiplicative trace class noise.

7.4.1 Formulas for the Numerical Modeling of Iterated Stochastic Integrals of Miltiplicities 1 to 3 with Respect to the Infinite-Dimensional Q-Wiener Process Based on Theorem 1.1 and Legendre Polynomials

This section is devoted to the formulas for numerical modeling of iterated stochastic integrals from the Milstein type scheme (7.13) and the Wagner–Platen type scheme (7.14) for non-commutative semilinear SPDEs. These integrals have the following form (below we introduce new notations for the stochastic integrals (7.89)-(7.92) and their approximations)

$$J_1[B(Z)]_{T,t} = \int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \tag{7.86}$$

$$J_2[B(Z)]_{T,t} = A\left(\int\limits_t^T\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}dt_2 - \frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_{t_1}\right), \tag{7.87}$$

$$J_3[B(Z),F(Z)]_{T,t} = (T-t)\int\limits_t^T B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1} -$$

$$-\int\limits_t^T\int\limits_t^{t_2} B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}dt_2, \tag{7.88}$$

$$J_4[B(Z),F(Z)]_{T,t} = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \tag{7.89}$$

$$I_1[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{7.90}$$

$$I_2[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \tag{7.91}$$

$$I_3[B(Z)]_{T,t} = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{7.92}$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $0\le t<T\le\bar{T}$.

Note that according to (7.17)–(7.20), (5.6), (5.130), and (5.131), we can write the following relatively simple formulas for numerical modeling [23], [47]

$$J_1[B(Z)]_{T,t}^M = \int\limits_t^T B(Z)d\mathbf{W}_s^M =$$

$$
= (T-t)^{1/2} \sum_{r_1 \in J_M} B(Z) e_{r_1} \sqrt{\lambda_{r_1}} \zeta_0^{(r_1)},
$$

$$
J_2[B(Z)]_{T,t}^M = A \left(\int_t^T \int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M dt_2 - \frac{(T-t)}{2} \int_t^T B(Z) d\mathbf{W}_{t_1}^M \right) =
$$

$$
= -\frac{(T-t)^{3/2}}{2\sqrt{3}} \sum_{r_1 \in J_M} AB(Z) e_{r_1} \sqrt{\lambda_{r_1}} \zeta_1^{(r_1)}, \tag{7.93}
$$

$$
J_3[B(Z), F(Z)]_{T,t}^M = (T-t) \int_t^T B'(Z) \Bigl(AZ + F(Z) \Bigr) d\mathbf{W}_{t_1}^M -
$$

$$
- \int_t^T \int_t^{t_2} B'(Z) \Bigl(AZ + F(Z) \Bigr) d\mathbf{W}_{t_1}^M dt_2 =
$$

$$
= \frac{(T-t)^{3/2}}{2} \sum_{r_1 \in J_M} B'(Z) \Bigl(AZ + F(Z) \Bigr) e_{r_1} \sqrt{\lambda_{r_1}} \left(\zeta_0^{(r_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(r_1)} \right), \tag{7.94}
$$

$$
J_4[B(Z), F(Z)]_{T,t}^M = \int_t^T F'(Z) \left(\int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M \right) dt_2 =
$$

$$
= \frac{(T-t)^{3/2}}{2} \sum_{r_1 \in J_M} F'(Z) B(Z) e_{r_1} \sqrt{\lambda_{r_1}} \left(\zeta_0^{(r_1)} - \frac{1}{\sqrt{3}} \zeta_1^{(r_1)} \right), \tag{7.95}
$$

where $\zeta_0^{(r_1)}$, $\zeta_1^{(r_1)}$ $(r_1 \in J_M)$ are independent standard Gaussian random variables.

Further, consider the stochastic integrals (7.90)–(7.92), which are more complicate, in detail.

Let $I_1[B(Z)]_{T,t}^M$, $I_2[B(Z)]_{T,t}^M$, $I_3[B(Z)]_{T,t}^M$ be approximations of the stochastic integrals (7.90)–(7.92), which have the following form (see (7.21), (7.22), and (7.26))

$$
I_1[B(Z)]_{T,t}^M = \int_t^T B'(Z) \left(\int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M \right) d\mathbf{W}_{t_2}^M =
$$

$$= \sum_{r_1,r_2\in J_M} B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)T,t}^{(r_1r_2)}, \tag{7.96}$$

$$I_2[B(Z)]_{T,t}^M = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)T,t}^{(r_1r_2r_3)}, \tag{7.97}$$

$$I_3[B(Z)]_{T,t}^M = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)} + J_{(111)T,t}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)}\right). \tag{7.98}$$

Let $I_1[B(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M.q}$, $I_3[B(Z)]_{T,t}^{M,q}$ be approximations of the stochastic integrals (7.96)–(7.98), which look as follows

$$I_1[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2\in J_M} B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(11)T,t}^{(r_1r_2)q},$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3\in J_M} B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}J_{(111)T,t}^{(r_1r_2r_3)q}, \tag{7.99}$$

$$I_3[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(J_{(111)T,t}^{(r_1r_2r_3)q} + J_{(111)T,t}^{(r_2r_1r_3)q} + \mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(0r_3)}\right),$$

where the approximations $J_{(11)T,t}^{(r_1r_2)q}$, $J_{(111)T,t}^{(r_1r_2r_3)q}$, $J_{(111)T,t}^{(r_2r_1r_3)q}$ of the stochastic integrals (7.16) are defined by (5.129), (5.132) and $J_{(01)T,t}^{(0r_3)}$ has the form (5.130), $q\ge 1$.

7.4.2 Theorem on the Mean-Square Approximation of Iterated Stochastic Integrals of Multiplicities 2 and 3 with Respect to the Ininite-Dimensional Q-Wiener Process

Recall that $L_{HS}(U_0,H)$ is a space of Hilbert–Schmidt operators mapping from U_0 to H. Moreover, let $L_{HS}^{(2)}(U_0,H)$ and $L_{HS}^{(3)}(U_0,H)$ be spaces of bilinear and 3-linear Hilbert–Schmidt operators mapping from $U_0\times U_0$ to H and from $U_0\times U_0\times U_0$ to H correspondingly. Furthermore, let

$$
\|\cdot\|_{L_{HS}(U_0,H)},\quad \|\cdot\|_{L_{HS}^{(2)}(U_0,H)},\quad \|\cdot\|_{L_{HS}^{(3)}(U_0,H)}
$$

be operator norms in these spaces.

Theorem 7.3 [14], [23], [47], [54]. *Let the conditions* 1, 2 *of Theorem* 7.1 *as well as the conditions of Theorem* 1.1 *be fulfilled. Futhermore, let*

$$
B(v)\in L_{HS}(U_0,H),\quad B'(v)(B(v))\in L_{HS}^{(2)}(U_0,H),
$$

$$
B'(v)(B'(v)(B(v))),\quad B''(v)(B(v),B(v))\in L_{HS}^{(3)}(U_0,H)
$$

for all $v\in H$ *(we suppose that Frêchet derivatives* B', B'' *exist*; *see Sect.* 7.1*). Moreover, let there exists a constant* C *such that w. p.* 1

$$
\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}<C,\quad \left\|B'(Z)(B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)}<C,
$$

$$
\left\|B'(Z)(B'(Z)(B(Z)))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C,
$$

$$
\left\|B''(Z)(B(Z),B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C
$$

for some $\alpha>0$. *Then*

$$
\mathsf{M}\left\{\left\|I_1[B(Z)]_{T,t}-I_1[B(Z)]_{T,t}^{M,p}\right\|_H^2\right\}\le
$$

$$
\le (T-t)^2\left(C_0\left(\text{tr }Q\right)^2\left(\frac{1}{2}-\sum_{j=1}^{p}\frac{1}{4j^2-1}\right)+K_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right),\quad (7.100)
$$

$$
\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le
$$

$$
\le (T-t)^3\left(C_1\left(\text{tr }Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+L_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \quad (7.101)
$$

$$
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le
$$

$$
\le (T-t)^3\left(C_2\left(\text{tr }Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+M_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \quad (7.102)
$$

where $p,q\in\mathbf{N},\ C_0,C_1,C_2,K_Q,L_Q,M_Q<\infty,$ *and*

$$
\hat{C}_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\bar{C}_{j_3j_2j_1},
$$

$$
\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,
$$

where $P_j(x)$ $(j=0,1,2,\ldots)$ *is the Legendre polynomial.*

Remark 7.4. *Note that the estimate similar to* (7.100) *has been derived in* [156], [157] (*also see* [143]) *with the difference connected with the first term on the right-hand side of* (7.100). *In* [157] *the authors used the Karhunen–Loève expansion of the Brownian bridge process for the approximation of iterated Itô stochastic integrals with respect to the finite-dimensional Wiener process* (*Milstein approach, see Sect.* 6.2). *In this section, we apply Theorem* 1.1 *and the system of Legendre polynomials to obtain the first term on the right-hand side of* (7.100).

Remark 7.5. *If we assume that* $\lambda_i\le C'i^{-\gamma}$ $(\gamma>1, C'<\infty)$ *for* $i\in J$, *then the parameter* $\alpha>0$ *obviously increases with decreasing of* γ [156].

Proof. The estimate (7.100) follows directly from (7.33) for $k=2$ (the first term on the right-hand side of (7.100)) and Theorem 1 from [157] (the second term on the right-hand side of (7.100)). Further C_3, C_4, ... denote various constants.

Let us prove the estimates (7.101), (7.102). Using Theorem 7.1, we obtain

$$
\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+
$$

$$
+2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le
$$

$$
\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+
$$

$$
+C_3(T-t)^3\left(\operatorname{tr}\,Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \qquad (7.103)
$$

$$
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+
$$

$$
+2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}. \qquad (7.104)
$$

Repeating with an insignificant modification the proof of Theorem 7.1 for the case $k=3$, we have (also see Sect. 7.3.2)

$$
\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le
$$

$$
\le 4\tilde{C}(3!)^2\left(\operatorname{tr}\,Q\right)^3(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \qquad (7.105)
$$

where constant $\tilde{C}$ has the same meaning as constant L_k in Theorem 7.1 (k is the multiplicity of the iterated stochastic integral).

Combining (7.104) and (7.105), we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+$$

$$+\,C_4(T-t)^3\left(\text{tr}\ Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right). \qquad (7.106)$$

Let us estimate the right-hand sides of (7.103) and (7.106). Using the elementary inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$, we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(E_{T,t}^{1,M}+E_{T,t}^{2,M}+E_{T,t}^{3,M}\right), \qquad (7.107)$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(G_{T,t}^{1,M}+G_{T,t}^{2,M}+G_{T,t}^{3,M}\right), \qquad (7.108)$$

where

$$E_{T,t}^{1,M}=$$

$$=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{2,M}=$$

$$=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{3,M}=$$

$$=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\left(\mathbf{W}_{t_3}-\mathbf{W}_{t_3}^M\right)\right\|_H^2\right\},$$

$$
G_{T,t}^{1,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right) d\mathbf{W}_{t_2}\right\|_H^2\right\},
$$

$$
G_{T,t}^{2,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right), \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}\right\|_H^2\right\},
$$

$$
G_{T,t}^{3,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2} - \mathbf{W}_{t_2}^M\right)\right\|_H^2\right\}.
$$

We have

$$
E_{T,t}^{1,M} =
$$

$$
= \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)\right\|_{L_{HS}(U_0,H)}^2\right\} dt_3 \le
$$

$$
\le C_5 \int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right) d\mathbf{W}_{t_2}\right\|_H^2\right\} dt_3 =
$$

$$
= C_5 \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\} dt_2 dt_3 \le
$$

$$
\le C_6 \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\} dt_2 dt_3 \le \tag{7.109}
$$

$$
\le C_6 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_1 dt_2 dt_3 \le \tag{7.110}
$$

$$
\le C_7 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (7.111)
$$

Note that the transition from (7.109) to (7.110) was made by analogy with the proof of Theorem 1 in [157] (also see [143]). More precisely, taking into account the relation $Q^{\alpha}e_i = \lambda_i^{\alpha}e_i$, we have (see [157], Sect. 3.1)

$$
\mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\} =
$$

$$
= \mathsf{M}\left\{\left\|\sum_{i\in J\backslash J_M}\sqrt{\lambda_i}\int\limits_t^{t_2} B(Z)e_i d\mathbf{w}_{t_1}^{(i)}\right\|_H^2\right\} =
$$

$$
= \sum_{i\in J\backslash J_M}\lambda_i\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}Q^{\alpha}e_i\right\|_H^2\right\}dt_1 =
$$

$$
= \sum_{i\in J\backslash J_M}\lambda_i^{1+2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 =
$$

$$
= \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\sum_{i\in J\backslash J_M}\lambda_i\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 \le
$$

$$
\le \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\sum_{i\in J}\lambda_i\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 =
$$

$$
= \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1. \qquad (7.112)
$$

Further, we also will use the estimate similar to (7.112).

We have

$$
E_{T,t}^{2,M} =
$$

$$
= \int\limits_t^T \mathsf{M}\left\{\left\| B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3 \le
$$

$$
\le C_8 \int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right\|_H^2\right\} dt_3 \le
$$

$$
\le C_8 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\| B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_2 dt_3 \le
$$

$$
\le C_8 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\| B'(Z)\left(B(Z)\right) Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)}^2\right\} (t_2-t) dt_2 dt_3 \le
$$

$$
\le C_9 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (7.113)
$$

Moreover,

$$
E_{T,t}^{3,M} \le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times
$$

$$
\times \int\limits_t^T \mathsf{M}\left\{\left\| B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\mathbf{W}_{t_2}^M\right) Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_3 \le
$$

$$
\le C_{10} \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times
$$

$$
\times \int\limits_t^T \mathsf{M}\left\{\left\| B'(Z)\left(B'(Z)\left(B(Z)\right)\right) Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}^2\right\} \frac{(t_3-t)^2}{2} dt_3 \le
$$

$$
\le C_{11}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \tag{7.114}
$$

Combining (7.103), (7.107), (7.111), (7.113), and (7.114), we obtain (7.101).

We have

$$
G_{T,t}^{1,M}=
$$

$$
=\int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1},\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3\le
$$

$$
\le C_{12}\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^2\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_3\le
$$

$$
\le C_{12}\int\limits_t^T\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^4\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\le
$$

$$
\le C_{13}\int\limits_t^T\int\limits_t^{t_2}\left(\mathsf{M}\left\{\left\|B(Z)\right\|_{L_{HS}(U_0,H)}^4\right\}\right)^{1/2}dt_1\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\le
$$

$$
\le C_{14}\int\limits_t^T(t_2-t)\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3. \tag{7.115}
$$

Let us estimate the right-hand side of (7.115). If $s>t$, then for fixed $M\in\mathbf{N}$ and for some $N>M$ $(N\in\mathbf{N})$ we have

$$
\mathsf{M}\left\{\left\|\int\limits_t^s B(Z)d\left(\mathbf{W}_{t_1}^N-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}=
$$

$$
=\mathsf{M}\left\{\left\langle\sum_{j\in J_N\backslash J_M}\sqrt{\lambda_j}B(Z)e_j\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right),\right.\right.
$$

$$
\left.\left.\sum_{j'\in J_N\backslash J_M}\sqrt{\lambda_{j'}}B(Z)e_{j'}\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\right\rangle_H^2\right\}=
$$

$$
=\sum_{j,j',l,l'\in J_N\backslash J_M}\sqrt{\lambda_j\lambda_{j'}\lambda_l\lambda_{l'}}\ \ \mathsf{M}\left\{\left\langle B(Z)e_j,B(Z)e_{j'}\right\rangle_H\left\langle B(Z)e_l,B(Z)e_{l'}\right\rangle_H\times\right.
$$

$$
\left.\times\mathsf{M}\left\{\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right)\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\left(\mathbf{w}_s^{(l)}-\mathbf{w}_t^{(l)}\right)\left(\mathbf{w}_s^{(l')}-\mathbf{w}_t^{(l')}\right)\Bigl|\mathbf{F}_t\right\}\right\}=
$$

$$
=3(s-t)^2\sum_{j\in J_N\backslash J_M}\lambda_j^2\mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^4\right\}+
$$

$$
+(s-t)^2\sum_{j,j'\in J_N\backslash J_M(j\ne j')}\lambda_j\lambda_{j'}\left(\mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^2\left\|B(Z)e_{j'}\right\|_H^2\right\}+\right.
$$

$$
\left.+2\left\langle B(Z)e_j,B(Z)e_{j'}\right\rangle_H^2\right)\le
$$

$$
\le 3(s-t)^2\left(\sum_{j\in J_N\backslash J_M}\lambda_j^2\mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^4\right\}+\right.
$$

$$
\left.+\sum_{j,j'\in J_N\backslash J_M(j\ne j')}\lambda_j\lambda_{j'}\mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^2\left\|B(Z)e_{j'}\right\|_H^2\right\}\right)=
$$

$$
=3(s-t)^2\mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M}\lambda_j\left\|B(Z)e_j\right\|_H^2\right)^2\right\}\le
$$

$$
\le 3(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha}\mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M}\lambda_j\left\|B(Z)Q^{-\alpha}e_j\right\|_H^2\right)^2\right\}\le
$$

$$\le C_{15}(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|^4_{L_{HS}(U_0,H)}\right\}. \qquad (7.116)$$

Carrying out the passage to the limit $\lim\limits_{N\to\infty}$ in (7.116) and using (7.115), we obtain

$$G^{1,M}_{T,t}\le C_{16}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.117)$$

Absolutely analogously we get

$$G^{2,M}_{T,t}\le C_{17}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \qquad (7.118)$$

Let us estimate $G^{3,M}_{T,t}$. We have

$$G^{3,M}_{T,t}\le\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\times$$

$$\times\int\limits_t^T\mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}^M_{t_1},\int\limits_t^{t_2}B(Z)d\mathbf{W}^M_{t_1}\right)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}dt_2\le$$

$$\le\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\sum_{i\in J}\sum_{j,l\in J_M}\lambda_i\lambda_j\lambda_l\times$$

$$\times\int\limits_t^T(t_2-t)^2\left(\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j,B(Z)e_l)Q^{-\alpha}e_i\right\|^2_H\right\}+\right.$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j,B(Z)e_j)Q^{-\alpha}e_i\right\|_H\left\|B''(Z)(B(Z)e_l,B(Z)e_l)Q^{-\alpha}e_i\right\|_H\right\}+$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j,B(Z)e_l)Q^{-\alpha}e_i\right\|_H\times\right.$$

$$\left.\left.\times\left\|B''(Z)(B(Z)e_l,B(Z)e_j)Q^{-\alpha}e_i\right\|_H\right\}\right)dt_2\le$$

$$\leq C_{18} \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha} (T-t)^3. \tag{7.119}$$

Combining (7.106), (7.108), and (7.117)–(7.119), we obtain (7.102). Theorem 7.3 is proved.

Let us consider the convergence analysis for the stochastic integrals (7.87)–(7.89) (convergence of the stochastic integral (7.86) follows from (7.112) (see Theorem 1 in [157] or [143])).

Using the Itô formula, we obtain w. p. 1 [144]

$$J_2[B(Z)]_{T,t} = \int\limits_t^T \left(\frac{T}{2} - s + \frac{t}{2} \right) AB(Z) d\mathbf{W}_s,$$

$$J_3[B(Z), F(Z)]_{T,t} = \int\limits_t^T (s-t) B'(Z) \Bigl(AZ + F(Z) \Bigr) d\mathbf{W}_s.$$

Suppose that

$$\mathsf{M} \left\{ \left\| B'(Z) \Bigl(AZ + F(Z) \Bigr) Q^{-\alpha} \right\|^2_{L_{HS}(U_0, H)} \right\} < \infty,$$

$$\mathsf{M} \left\{ \left\| AB(Z) Q^{-\alpha} \right\|^2_{L_{HS}(U_0, H)} \right\} < \infty$$

for some $\alpha > 0$.

Then by analogy with (7.112) we get

$$\mathsf{M} \left\{ \left\| J_2[B(Z)]_{T,t} - J_2[B(Z)]^M_{T,t} \right\|^2_H \right\} \leq$$

$$\leq C_{19} (T-t)^3 \left(\sup_{i \in J \backslash J_M} \lambda_i \right)^{2\alpha},$$

$$\mathsf{M} \left\{ \left\| J_3[B(Z), F(Z)]_{T,t} - J_3[B(Z), F(Z)]^M_{T,t} \right\|^2_H \right\} \leq$$

$$
\leq C_{20}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},
$$

where $J_2[B(Z)]_{T,t}^M$, $J_3[B(Z),F(Z)]_{T,t}^M$ are defined by (7.93), (7.94).

Moreover, in the conditions of Theorem 7.3 we obtain for some $\alpha>0$

$$
\mathsf{M}\left\{\left\|J_4[B(Z),F(Z)]_{T,t}-J_4[B(Z),F(Z)]_{T,t}^M\right\|_H^2\right\}=
$$

$$
=\mathsf{M}\left\{\left\|\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)dt_2\right\|_H^2\right\}\leq
$$

$$
\leq (T-t)\int\limits_t^T\mathsf{M}\left\{\left\|F'(Z)\left(\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_H^2\right\}dt_2\leq
$$

$$
\leq C_{21}(T-t)\int\limits_t^T\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2\leq
$$

$$
\leq C_{21}(T-t)\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1dt_2\leq
$$

$$
\leq C_{22}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},
$$

where $J_4[B(Z),F(Z)]_{T,t}^M$ is defined by (7.95).

Epilogue

The results presented in this book were developed [52] in the form of a software package in the Python programming language that implements the numerical methods (4.65)-(4.69), (4.73)-(4.77) (see Chapter 4) with the orders 1.0, 1.5, 2.0, 2.5, and 3.0 of strong convergence based on the unified Taylor–Ito and Taylor–Stratonovich expansions. At that for the numerical simulation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 we used [52] the formulas from Sect. 5.1, i.e. method based on Theorem 1.1 and multiple Fourier–Legendre series. Note that in [52] we used the database with 270,000 exactly calculated Fourier–Legendre coefficients.

Using computational experiments it was shown in [53] that in most cases all the exact formulas from Sect. 1.2.3 for the mean-square approximation errors of iterated Itô stochastic integarls can be replaced by the formula (1.71) for $k = 1, \ldots, 5$. This allows us to neglect the multiplier factor $k!$ (see the formula (1.92)). As a result, the computational costs for the approximation of iterated Itô stochastic integrals are significantly reduced.

Iterated stochastic integrals are a fundamental tool for describing and studying the dynamics of various types of stochastic equations. In recent years and decades, numerical methods of high orders of accuracy have been constructed using iterated stochastic integrals not only for Itô SDEs, but also for SDEs with jumps [76], SPDEs with multiplicative trace class noise [143], [144], [148], McKean SDEs [158], SDEs with switchings [159], mean-field SDEs [160], Itô–Volterra stochastic integral equations [148], etc.

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