DMITRIY F. KUZNETSOV

New Theory of the Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals: Method of Generalized Multiple Fourier Series

Application to Numerical Integration of Ito SDEs and semilinear SPDEs

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CONTENTS

PREFACE

The book is a collection of 28 articles (arXiv.org), which are devoted to the problem of strong (mean-square) approximation of iterated Ito and Stratonovich stochastic integrals in the context of numerical integration of Ito stochastic differential equations (SDEs) and non-commutative semilinear stochastic partial differential equations (SPDEs) with nonlinear multiplicative trace class noise. The presented monograph open a new direction in researching of iterated stochastic integrals and summarizes the author's research on the mentioned problem carried out in the period 1994–2023 (also see the monograph: Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184v46 [math.PR], 2023, 998 pp.).

For the first time we successfully use the generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ for the expansion and strong approximation of iterated Ito stochastic integrals of arbitrary multiplicity k, $k \in \mathbb{N}$ (Chapter 1).

This result has been adapted for iterated Stratonovich stochastic integrals of multiplicities 1 to 6 for the Legendre polynomial system and the system of trigonometric functions (Chapter 2) as well as for some other types of iterated stochastic integrals (Chapter 1).

Two theorems on expansions of iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbb{N}$) based on generalized iterated Fourier series with the pointwise convergence are formulated and proved (Chapter 2).

The integration order replacement technique for the class of iterated Ito stochastic integrals has been introduced (Chapter 6). This result is generalized for the class of iterated stochastic integrals with respect to martingales.

We derived the exact and approximate expressions for the mean-square approximation error of iterated Ito stochastic integrals of multiplicity k, $k \in \mathbb{N}$ (Chapter 1). Furthermore, we provided a significant practical material (Chapter 3) devoted to the expansions and approximations of specific iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions (Chapter 4) using the system of Legendre polynomials and the system of trigonometric functions.

The methods formulated in this book have been compared with some existing methods of strong approximation of iterated Ito and Stratonovich stochastic integrals (Chapter 3).

The results of Chapter 1 were applied (Chapter 5) to the approximation of iterated stochastic integrals with respect to the finite-dimensional approximation $\mathbf{W}_t^M$ of the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of arbitrary multiplicity k, $k \in \mathbb{N}$) and to the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process $\mathbf{W}_t$ (for integrals of multiplicities 1 to 3).

Chapter 7 is devoted to the implementation of strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 (Chapter 4) for Ito SDEs with multidimensional non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions (Chapter 4) and multiple Fourier–Legendre series (Chapter 3). Algorithms for the implementation of these methods are constructed and a package of programs on the Python programming language is presented. An important part of this software package, concerning the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 with respect to components of the multidimensional Wiener process, is based on the method of generalized multiple Fourier series (Chapters 1–3). More precisely, we used the multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ ($k = 1\ldots,6$) for the mean-square approximation of iterated Ito and Stratonovich stochastic integrals.

The importance of the problem of numerical integration of Ito SDEs and semilinear SPDEs is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems of various physical nature under random disturbances and to the application of these equations for solving various mathematical problems, among which we mention signal filtering in the background of random noise, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems, etc.

It is well known that one of the effective and perspective approaches to the numerical integration of Ito SDEs and semilinear SPDEs is an approach based on the stochastic analogues of the Taylor formula for solutions of these equations. This approach uses the finite discretization of temporal variable and performs numerical modeling of solutions of Ito SDEs and semilinear SPDEs in discrete moments of time using stochastic analogues of the Taylor formula.

Speaking about Ito SDEs, note that the most important feature of the mentioned stochastic analogues of the Taylor formula for solutions of Ito SDEs is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which are the functionals of a complex structure with respect to components of the multidimensional Wiener process. These iterated stochastic integrals are subject for study in this book. The mentioned iterated Ito and Stratonovich stochastic integrals are defined by the following formulas

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{(Ito integrals)},$$

$$\int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{(Stratonovich integrals)},$$

where $\psi_l(\tau)$ $(l=1,\ldots,k)$ are nonrandom functions at the interval $[t,T]$ (as a rule, in the applications they are identically equal to 1 or have a binomial form (see Chapter 4)), $\mathbf{w}_\tau$ is a random vector with an $m+1$ components: $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes, $i_1,\ldots,i_k=0,1,\ldots,m.$

Apparently, one of the first who began the study of such stochastic integrals (the case $k=2$, $m=2$, $\psi_1(\tau),\psi_2(\tau)\equiv 1$, $i_1=1$, $i_2=2$) was Lévy, who introduced the concept of the so-called Lévy stochastic area and studied its properties.

The above iterated stochastic integrals are the specific objects of the theory of stochastic processes. From the one side, nonrandomness of weight functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ is the factor simplifying their structure. From the other side, nonscalarity of the Wiener process $\mathbf{f}_\tau$ with independent components $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ and the fact that the functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ are different for various l $(l=1,\ldots,k)$ are essential complicating factors of the structure of iterated stochastic integrals. Taking into account features mentioned above, we suppose that the systems of iterated Ito and Stratonovich stochastic integrals play the extraordinary and perhaps the key role for solving the problem of numerical integration of Ito SDEs.

A natural question arises: is it possible to construct a numerical scheme for Ito SDE that includes only increments of the Wiener processes $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$, but has a higher order of convergence than the Euler method? It is known that this is impossible for $m>1$ in the general case. This fact is called the "Clark–Cameron paradox" and explains the need to use iterated stochastic integrals for constructing high-order numerical methods for Ito SDEs.

We want to mention in short that there are two main criteria of numerical methods convergence for Ito SDEs: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution. Both mentioned criteria are independent, i.e. in general it is impossible to state that from the execution of strong criterion follows the execution of weak criterion and vice versa. Each of two convergence criteria is oriented on the solution of specific classes of mathematical problems connected with Ito SDEs.

Numerical integration of Ito SDEs based on the strong convergence criterion of approximation is widely used for the numerical simulation of sample trajectories of solutions to Ito SDEs (which is required for constructing new mathematical models based on such equations and for the numerical solution of different mathematical problems connected with Ito SDEs). Among these problems, we note the following: signal filtering under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing of estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis.

Exact solutions of Ito SDEs and semilinear SPDEs are known in rather rare cases. Therefore, the need arises to construct numerical procedures for solving these equations.

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Ito or Stratonovich stochastic integrals is very important and difficult from theoretical and computing point of view.

Seems that iterated stochastic integrals may be approximated by multiple integral sums. However, this approach implies the partitioning of the interval $[t, T]$ of integration for iterated stochastic integrals. The length $T-t$ of this interval is already fairly small (because it is a step of integration of numerical methods for Ito SDEs) and does not need to be partitioned. Computational experiments show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors.

The problem of effective decreasing of the mentioned cost (in several times or even in several orders) is very difficult and requires new complex investigations. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case allows the investigation with using of the Ito formula. In the more general case, when not all numbers $i_1, \ldots, i_k$ are equal, the mentioned problem turns out to be more complex (it cannot be solved using the Ito formula and requires more deep and complex investigation). Note that even for the case $i_1 = \ldots = i_k \neq 0$, but for different functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ the mentioned difficulties persist and simple sets of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (with respect to the mean-square approximation) using the system of standard Gaussian random variables. The Ito formula is also useless in this case and as a result we need to use more complex but effective expansions.

Why the problem of the mean-square approximation of iterated stochastic integrals is so complex?

Firstly, the mentioned stochastic integrals (in the case of fixed limits of integration) are the random variables, whose density functions are unknown in the general case. The exception is connected with the narrow particular case which is the simplest iterated Ito stochastic integral with multiplicity 2 and $\psi_1(\tau), \psi_2(\tau) \equiv 1$; $i_1, i_2 = 1, \ldots, m$. Nevertheless, the knowledge of this density function not gives a simple way for approximation of iterated Ito stochastic integral of multiplicity 2.

Secondly, we need to approximate not only one stochastic integral, but several iterated stochastic integrals that are complexly dependent in a probabilistic sense.

Often, the problem of combined mean-square approximation of iterated Ito and Stratonovich stochastic integrals occurs even in cases when the exact solution of Ito SDE is known. It means that even if you know the solution of Ito SDE exactly, you cannot model it numerically without the combined numerical modeling of iterated stochastic integrals.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated stochastic integrals may be simplified but cannot be solved. Equations with additive vector noise, with non-additive scalar noise, with additive scalar noise, with a small parameter are related to such types of equations. In these cases, simplifications are connected to the fact that some members from stochastic Taylor expansions are equal to zero or we may neglect some members from these expansions due to the presence of a small parameter.

Furthermore, the problem of combined numerical modeling (with respect to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals is rather new.

One of the main and unexpected achievements of this book is the successful usage of functional analysis methods (more concretely, we mean generalized multiple and iterated Fourier series (convergence in $L_2([t,T]^k)$ and pointwise correspondently) through various systems of basis functions) in this scientific field.

The problem of combined numerical modeling (with respect to the mean-square convergence criterion) of systems of iterated Ito and Stratonovich stochastic integrals was systematically analyzed in the context of the problem of numerical integration of Ito SDEs in the following monographs:

[I] Milstein G.N. Numerical integration of stochastic differential equations. Kluwer. 1995 (Russian Ed. 1988).

[II] Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. Springer-Verlag. Berlin. 1992 (2nd Ed. 1995, 3rd Ed. 1999).

[III] Milstein G.N., Tretyakov M. V. Stochastic numerics for mathematical physics. Springer-Verlag. Berlin. 2004.

[IV] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. Polytechnical University Publ. St.-Petersburg. 2007. DOI: http://doi.org/10.18720/SPBPU/2/s17-228

2nd Ed. 2007. DOI: http://doi.org/10.18720/SPBPU/2/s17-229

3rd Ed. 2009. DOI: http://doi.org/10.18720/SPBPU/2/s17-230

4th Ed. 2010. DOI: http://doi.org/10.18720/SPBPU/2/s17-231

5th Ed. 2017. Published online, El. J. Differential Equations and Control Processes, no. 2, 2017, P. A.1-A.1000. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

6th Ed. 2018. Published online, El. J. Differential Equations and Control Processes, no. 4, 2018, P. A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

Note that the initial version of the book [IV] has been published in 2006:

[IV*] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. Polytechnical University Publ. St.-Petersburg, 2006. DOI: http://doi.org/10.18720/SPBPU/2/s17-227

We also mention the monograph:

[V] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184v46 [math.PR], 2023, 998 pp.

The books [I] and [III] analyze the problem of the mean-square approximation of iterated stochastic integrals only for two simplest iterated Ito stochastic integrals of 1st and 2nd multiplicities ($k=1$ and 2, $\psi_1(\tau)$ and $\psi_2(\tau)\equiv 1$) for the multidimensional case: $i_1,i_2=0,1,\ldots,m$. In addition, the main idea is based on the expansion of the so-called Brownian bridge process into the trigonometric Fourier series (version of the so-called Karhunen–Loève expansion). This method is called in [I] and [III] as the Fourier method.

In [II] using the Fourier method [I], the attempt was made to obtain the mean-square approximation of elementary iterated stochastic integrals of multiplicities 1 to 3 ($k = 1, \ldots, 3$, $\psi_1(\tau), \ldots, \psi_3(\tau) \equiv 1$) for the multidimensional case: $i_1, \ldots, i_3 = 0, 1, \ldots, m$. However, as we can see in the presented book, the results of the monograph [II], related to the mean-square approximation of iterated stochastic integrals of 3rd multiplicity, cause a number of critical remarks (see discussion in Sect. 2.8).

The main purpose of this book is to construct and develop newer and more effective methods (than presented in the books [I]–[III]) of combined mean-square approximation of iterated Ito and Stratonovich stochastic integrals.

Talking about the history of solving the problem of combined mean-square approximation of iterated stochastic integrals, we note that the idea to find a basis of random variables using which we may represent iterated stochastic integrals turned out to be useful. This idea was transformed several times during last decades.

Attempts to approximate the iterated stochastic integrals using various integral sums were made until 1980s and later, i.e. the interval of integration $[t, T]$ of the stochastic integral was divided into n parts and the iterated stochastic integral was represented approximately by the multiple integral sum, which included the system of independent standard Gaussian random variables whose numerical modeling is not a problem.

However, as we noted above, it is obvious that the length $T - t$ of integration interval $[t, T]$ of the iterated stochastic integrals is a step of integration of numerical methods for Ito SDEs, which is already a rather small value even without the additional splitting. Numerical experiments demonstrate that such approach results in drastic increasing of computational costs accompanied by the growth of multiplicity of the stochastic integrals (beginning from 2nd and 3rd multiplicity) that is necessary for construction of high-order strong numerical methods for Ito SDEs or in the case of decrease of integration step of numerical methods, and thereby it is almost useless for practice.

The new step for solution of the problem of combined mean-square approximation of iterated stochastic integrals was made by Milstein G.N. in his monograph [I] (1988). He proposed to use the trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karhunen–Loève expansion) for expansion of iterated stochastic integrals. Using this method, the expansions of two simplest iterated Ito stochastic integrals of 1st and 2nd multiplicities into the series of products of standard Gaussian random variables were obtained and their mean-square convergence was proved [I].

As we noted above, the attempt to develop this idea together with the Wong–Zakai approximation was made in the monograph [II] (1992), where the expansions of simplest iterated Stratonovich stochastic integrals of multiplicities 1 to 3 were obtained. However, due to a number of limitations and technical difficulties which are typical for the method [I], in [II] and following publications this problem was not solved more completely. In addition, the author has reasonable doubts about application of the Wong–Zakai approximation for the iterated stochastic integrals of 3rd multiplicity in the monograph [II] (see discussion in Sect. 2.8).

It is necessary to note that the computational cost for the method [I] is significantly less than for the method of multiple integral sums.

Regardless of the method [I] positive features, the number of its limitations are also outlined. Among them let us mention the following.

1. The absence of explicit formula for calculation of expansion coefficients for iterated stochastic integrals.

2. The practical impossibility of exact calculation of the mean-square approximation error of iterated stochastic integrals with the exception of simplest integrals of 1st and 2nd multiplicity (as a result, it is necessary to consider redundant terms of expansions and it results to the growth of computational cost and complication of the numerical methods for Ito SDEs).

3. There is a hard limitation on the system of basis functions — it may be only the trigonometric functions.

4. There are some technical problems if we use this method for iterated stochastic integrals whose multiplicity is greater than 2nd.

Nevertheless, it should be noted that the analyzed method is a concrete step forward in this scientific field.

The author thinks that the method presented by him in [IV], [V] (for the first time this method is appeared in the final form in [IV*] (2006)) and in this book (hereafter this method is reffered to as the method of generalized multiple Fourier series) is a breakthrough in solution of the problem of combined mean-square approximation of iterated Ito stochastic integrals.

The idea of this method is as follows: the iterated Ito stochastic integral of multiplicity k ($k \in \mathbb{N}$) is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t, T]^k$, where $[t, T]$ is the interval of integration of the iterated Ito stochastic integral. Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t, T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t, T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Ito stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral.

As a result, we obtain the following new possibilities and advantages in comparison with the Fourier method [I].

1. There is an explicit formula for calculation of expansion coefficients of iterated Ito stochastic integral with any fixed multiplicity k. In other words, we can calculate (without any preliminary and additional work) the expansion coefficient with any fixed number in the expansion of iterated Ito stochastic integral of the preset fixed multiplicity. At that, we do not need any knowledge about coefficients with other numbers or about other iterated Ito stochastic integrals included in the considered set.

2. We have new possibilies for obtainment the exact and approximate expressions for the mean-square approximation errors of iterated Ito stochastic integrals. These possibilities are realized by the exact and estimate formulas for the mentioned mean-square approximation errors. As a result, we would not need to consider redundant terms of expansions that may complicate approximations of iterated Ito stochastic integrals.

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t, T]^k)$, we have new possibilities for approximation — we can use not only the trigonometric functions as in [I] but the Legendre polynomials as well as the systems of Haar and Rademacher–Walsh functions.

4. As it turned out, it is more convenient to work with Legendre polynomials for approximation of iterated Ito stochastic integrals. The approximations themselves are simpler than their analogues based on the system of trigonometric functions. For the systems of Haar and Rademacher–Walsh functions the expansions of iterated stochastic integrals become too complex and ineffective for practice [IV].

5. The question about what kind of functions (polynomial or trigonometric) is more convenient in the context of computational costs for approximation turns out to be nontrivial, since it is necessary to compare approximations not for one stochastic integral but for several stochastic integrals at the same time. At that there is a possibility that computational costs for some integrals will be smaller for the system of Legendre polynomials and for others — for the system of trigonometric functions. The author proved (see Sect. 3.2 in this book) that the computational costs are significantly less for the system of Legendre polynomials at least in the case of approximation of the special set of iterated Ito stochastic integrals, which are necessary for the implementation of strong numerical methods for Ito SDEs with the order of convergence $\gamma = 1.5$. In addition, the author supposes that this effect will be more impressive when analyzing more complex sets of iterated Ito stochastic integrals ($\gamma = 2.0$, 2.5, 3.0, ...). This supposition is based on the fact that the polynomial system of functions has the

significant advantage (in comparison with the trigonometric system of functions) in approximation of iterated Ito stochastic integrals for which not all weight functions are equal to 1.

6. The Milstein approach [I] for approximation of iterated Ito stochastic integrals leads to iterated applicaton of the operation of limit transition (in contrast with the method of generalized multiple Fourier series, for which the operation of limit transition is implemented only once) starting at least from the second or third multiplicity of iterated Ito stochastic integrals (we mean at least double or triple integration with respect to components of the multidimensional Wiener process). Multiple series are more preferential for approximation than the iterated ones, since the partial sums of multiple series converge for any possible case of joint converging to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, in [II] the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the Milstein approach [I] together with the Wong–Zakai approximation (see discussion in Sect. 2.8).

7. The convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) as well as the convergence with probability 1 of approximations from the method of generalized multiple Fourier series are proved.

8. The method of generalized multiple Fourier series has been applied for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson random measures and iterated stochastic integrals with respect to martingales) as well as for approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process.

9. Another modification of the method of generalized multiple Fourier series is connected with the application of complete orthonormal with weight $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in the space $L_2([t,T]^k)$.

10. As it turned out, the method of generalized multiple Fourier series can be adapted for iterated Stratonovich stochastic integrals of multiplicities 1 to 6 (see Chapter 2).

11. The results of Chapters 1 and 2 can be considered from the point of view of the Wong–Zakai approximation for the case of a multidimensional Wiener process and the Wiener process approximation based on its series expansion using Legendre polynomials and trigonometric functions (see discussion in Sect. 2.8). These results overcome a number of difficulties that were noted above and relate to the Fourier method [I].

Iterated stochastic integrals are a fundamental tool for describing and studying the dynamics of various types of stochastic equations. In recent years and decades, numerical methods of high orders of accuracy have been constructed using iterated stochastic integrals not only for Ito SDEs, but also for SDEs with jumps, SPDEs with multiplicative trace class noise, McKean SDEs, SDEs with switchings, mean-field SDEs, Ito–Volterra stochastic integral equations, etc.

Dmitriy F. Kuznetsov November, 2023

Basic Notations

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$, $\mathbb{R}^1$	set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$n!$	$1 \cdot 2 \cdot \ldots \cdot n$ for $n \in \mathbb{N}$ $(0! = 1)$
$(2n-1)!!$	$1 \cdot 3 \cdot \ldots \cdot (2n-1)$ for $n \in \mathbb{N}$
$\stackrel{\text{def}}{=}$	equal by definition
$\equiv$	identically equal to
C_n^m	binomial coefficient $n!/(m!(n-m)!)$
$\emptyset$	empty set
$\mathbf{1}_{\mathcal{A}}$	indicator of the set A
$x \in X$	x is an element of the set X
$X \bigcup Y$	union of sets X and Y
$X \times Y$	Cartesian product of sets X and Y
$\overline{\lim\limits_{n\to\infty}}$	$\limsup\limits_{n\to\infty}$
$\underline{\lim\limits_{n\to\infty}}$	$\liminf\limits_{n\to\infty}$
$x \ll y$	x much less than y
$[x]$	largest integer number not exceeding x
$\|x\|$	absolute value of the real number x
$F:\ X \to Y$	function F from X into Y
$A^{(ij)}$	ijth element of the matrix A
A_i	ith colomn of the matrix A
$\mathbf{x}^{(i)}$	ith component of the vector $\mathbf{x} \in \mathbb{R}^n$

$O(x)$	expression being divided by x remains bounded as $x \to 0$
$\sum\limits_{(i_1,\ldots,i_k)}$	sum with respect to all possible permutations $(i_1,\ldots,i_k)$
$\mathsf{M}\{\xi\}$	expectation of ξ
$\mathsf{M}\{\xi\vert\mathrm{F}\}$	conditional expectation of ξ with respect to F
$\xi \sim \mathrm{N}(m,\sigma^2)$	Gaussian random variable ξ with expectation m and variance σ^2
$\underset{n\to\infty}{\text{l.i.m.}}$	limit in the mean square sense
$\mathcal{B}(X)$	σ-algebra of Borel subsets of X
f_t	scalar standard Wiener process
$\mathbf{f}_t$	vector standard Wiener process with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$
w. p. 1	with probability 1
$\mathbf{w}_t$	vector with components $\mathbf{w}_t^{(i)}$, $i=0,1,\ldots,m$ and property $\mathbf{w}_t^{(i)}=\mathbf{f}_t^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_t^{(0)}=t$
$\dfrac{\partial F}{\partial \mathbf{x}^{(i)}}$	partial derivative of $F:\mathbb{R}^n\to\mathbb{R}$
$\dfrac{\partial^2 F}{\partial \mathbf{x}^{(i)}\partial \mathbf{x}^{(j)}}$	2nd order partial derivative of $F:\mathbb{R}^n\to\mathbb{R}$
$\int\limits_t^T \ldots d\mathbf{w}_\tau^{(i)}$	Ito stochastic integral
$\int\limits_t^{*T} \ldots d\mathbf{w}_\tau^{(i)}$	Stratonovich stochastic integral
$\mathbf{W}_t$	Q-Wiener process
$J[\psi^{(k)}]_{T,t}$, $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$	iterated Ito stochastic integrals
$J^*[\psi^{(k)}]_{T,t}$, $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$	iterated Stratonovich stochastic integrals
$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$, $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$	approximations of iterated Ito stochastic integrals
$J^*[\psi^{(k)}]_{T,t}^{p}$, $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}$	approximations of iterated Stratonovich stochastic integrals
$L_2(D)$	Hilbert space of square integrable functions on D

$\|\cdot\|_{L_2(D)}$	norm in the Hilbert space $L_2(D)$
$\mathrm{tr}\ A$	trace of the operator A
$\|\cdot\|_H$	norm in the Hilbert space H
$\langle u, v\rangle_H$	scalar product in the Hilbert space H
$L_{HS}(U, H)$	space of Hilbert–Schmidt operators from U to H
$\|\cdot\|_{L_{HS}(U,H)}$	operator norm in the space of Hilbert–Schmidt operators from U to H
$\int\limits_t^T \ldots d\mathbf{W}_\tau$	stochastic integral with respect to the Q-Wiener process
$J'[\Phi]_{T,t}^{(k)},\quad J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$	multiple Wiener stochastic integrals
$J[\Phi]_{T,t}^{(k)},\quad J[\Phi]_{T,t}^{(i_1\ldots i_k)}$	multiple Stratonovich stochastic integrals
$P_n(x)$	Legendre polynomials
$H_n(x),\ h_n(x)$	Hermite polynomials
$H_n(x, y)$	polynomials related to the Hermite polynomials
$(a_1, \ldots, a_n)$	ordered set with elements $a_1, \ldots, a_n$
$\{a_1, \ldots, a_n\}$	unordered set with elements $a_1, \ldots, a_n$

Chapter 1.

Method of Expansion and Approximation of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

EXPANSION OF ITERATED ITO STOCHASTIC INTEGRALS OF ARBITRARY MULTIPLICITY BASED ON GENERALIZED MULTIPLE FOURIER SERIES CONVERGING IN THE MEAN

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansions of iterated Ito stochastic integrals based on generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$, $k \in \mathbb{N}$. The method of generalized multiple Fourier series for expansion and mean-square approximation of iterated Ito stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) with respect to components of the multidimensional Wiener process is proposed and developed. The obtained expansions contain only one operation of the limit transition in contrast to its existing analogues. In the article it is also obtained the generalization of the proposed method for an arbitrary complete orthonormal systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbb{N}$ as well as for complete orthonormal with weight $r(t_1)\ldots r(t_k)$ systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbb{N}$. The comparison of the considered method with the well-known expansions of iterated Ito stochastic integrals based on the Ito formula and Hermite polynomials is given. The convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) and with probability 1 of the proposed method is proved.

Contents

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1. Introduction

The idea of representing of iterated Ito and Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using multiple Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of publications of the author [1]-[59] (also see related publications [60], [61]). Note that another approaches to expansions of iterated stochastic integrals can be found in [62]-[76]. Specifically, the approach [1]-[59] appeared for the first time in [1] (1994), [2] (1996). In these works the mentioned idea is formulated more likely at the level of guess (without any satisfactory grounding), and as a result the papers [1], [2] contain rather fuzzy formulations and a number of incorrect conclusions. Note that in [1], [2] we used the trigonometric multiple Fourier series converging in the sence of norm in the space $L_2([t,T]^k)$, $k=1,2,3$. In the final form the approach from [1], [2] has been formulated and proved for the first time in the monograph [7] (2006) (see Theorem 1 below). It should be noted that the results of [1], [2] are correct for a sufficiently narrow particular case when the numbers $i_1,...,i_k$ are pairwise different, $i_1,\ldots,i_k=1,\ldots,m$ (see Sect. 2 for detail).

Usage of Fourier series with respect to the system of Legendre polynomials for approximation of iterated stochastic integrals took place for the first time in [3] (1997) [4] (1998), [5] (2000) [6] (2001) (also see [7]-[61]).

The question about what integrals (Ito or Stratonovich) are more suitable for expansions within the frames of the considered direction of researches has turned out to be rather interesting and difficult.

On the one side, Theorem 1 (see Sect. 2) conclusively demonstrates that the structure of iterated Ito stochastic integrals is rather convenient for expansions into multiple series with respect to the system of standard Gaussian random variables regardless of the multiplicity k of iterated Ito stochastic integrals.

On the other side, the results of [3]-[6], [11]-[25], [27]-[33], [35], [37]-[39], [41]-[44], [50]-[55] convincingly testify that there is a doubtless relation between multiplier factor 1/2, which is typical for Stratonovich stochastic integral and included into the sum, connecting Stratonovich and Ito stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its Fourier series converges to the value $(f(x-0)+f(x+0))/2$. In addition, as it is demonstrated in [12]-[25], [27]-[33], [35], [37]-[39], [41]-[44], [50]-[55] the final formulas for expansions of iterated Stratonovich stochastic integrals (of mutiplicities 1 to 5 in the general case and of multiplicity 6 in particular case but for the case of a multidimentional Wiener process) are more compact than their analogues for iterated Ito stochastic integrals.

2. Theorem on Expansion of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$)

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider the following iterated Ito stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m.$

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Sect. 15).

Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases} = \prod_{l=1}^k\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{2}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{3}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{4}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{5}$$

Theorem 1 [7] (2006) [8]-[59]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{6}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{7}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (4), *$\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition* (5).

Proof. At first, let us prove preparatory lemmas.

Lemma 1. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1}, \tag{8}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t, T]$ *satisfying the condition* (5), *hereinafter w. p. 1 means "with probability 1".*

Proof. It is easy to notice that using the additive property of stochastic integrals, we can write

$$J[\psi^{(k)}]_{T,t} = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} + \varepsilon_N \qquad \text{w. p. 1}, \tag{9}$$

where

$$\varepsilon_N = \sum_{j_k=0}^{N-1} \int\limits_{\tau_{j_k}}^{\tau_{j_k+1}} \psi_k(s) \int\limits_{\tau_{j_k}}^{s} \psi_{k-1}(\tau) J[\psi^{(k-2)}]_{\tau,t} d\mathbf{w}_{\tau}^{(i_{k-1})} d\mathbf{w}_{s}^{(i_k)} +$$

$$+ \sum_{r=1}^{k-3} G[\psi_{k-r+1}^{(k)}]_N \sum_{j_{k-r}=0}^{j_{k-r+1}-1} \int\limits_{\tau_{j_{k-r}}}^{\tau_{j_{k-r}+1}} \psi_{k-r}(s) \int\limits_{\tau_{j_{k-r}}}^{s} \psi_{k-r-1}(\tau) J[\psi^{(k-r-2)}]_{\tau,t} d\mathbf{w}_{\tau}^{(i_{k-r-1})} d\mathbf{w}_{s}^{(i_{k-r})} +$$

$$+ G[\psi_3^{(k)}]_N \sum_{j_2=0}^{j_3-1} J[\psi^{(2)}]_{\tau_{j_2+1},\tau_{j_2}},$$

$$G[\psi_m^{(k)}]_N = \sum_{j_k=0}^{N-1} \sum_{j_{k-1}=0}^{j_k-1} \ldots \sum_{j_m=0}^{j_{m+1}-1} \prod_{l=m}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$J[\psi_l]_{s,\theta} = \int\limits_{\theta}^{s} \psi_l(\tau) d\mathbf{w}_{\tau}^{(i_l)},$$

$$(\psi_m, \psi_{m+1}, \ldots, \psi_k) \stackrel{\text{def}}{=} \psi_m^{(k)}, \quad (\psi_1, \ldots, \psi_k) \stackrel{\text{def}}{=} \psi_1^{(k)} = \psi^{(k)}.$$

Using the standard estimates (22) for moments of stochastic integrals, we obtain w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0. \tag{10}$$

Comparing (9) and (10), we get

$$J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} \quad \text{w. p. 1}. \tag{11}$$

Let us rewrite $J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}$ in the form

$$J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} = \psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} + \int\limits_{\tau_{j_l}}^{\tau_{j_l+1}} (\psi_l(\tau) - \psi_l(\tau_{j_l}))d\mathbf{w}_{\tau}^{(i_l)}$$

and substitute it into (11). Then, due to the moment properties of stochastic integrals and continuity (which means uniform continuity) of the functions $\psi_l(s)$ $(l = 1, \ldots, k)$ it is easy to see that the prelimit expression on the right-hand side of (11) is a sum of the prelimit expression on the right-hand side of (8) and the value which tends to zero in the mean-square sense if $N \to \infty$. Lemma 1 is proved.

Remark 1. *It is easy to see that if $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (8) for some $l \in \{1, \ldots, k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p = 2,\ i_l \neq 0)$, then the differential $d\mathbf{w}_{t_l}^{(i_l)}$ in the integral $J[\psi^{(k)}]_{T,t}$ will be replaced with dt_l. If $p = 3, 4, \ldots,$ then the right-hand side of the formula (8) will become zero w. p. 1. If we replace $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (8) for some $l \in \{1, \ldots, k\}$ with $(\Delta\tau_{j_l})^p$ $(p = 2, 3, \ldots)$, then the right-hand side of the formula (8) also will be equal to zero w. p. 1.*

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J[\Phi]_{T,t}^{(k)}. \tag{12}$$

Denote

$$D_k = \{(t_1, \ldots, t_k) :\ t \le t_1 < \ldots < t_k \le T\}. \tag{13}$$

We will use the same symbol D_k to denote the open and closed domains corresponding to the domain D_k defined by (13). However, we always specify what domain we consider (open or closed).

Also we will write $\Phi(t_1, \ldots, t_k) \in C(D_k)$ if $\Phi(t_1, \ldots, t_k)$ is a continuous nonrandom function of k variables in the closed domain D_k.

Let us consider the iterated Ito stochastic integral

$$I[\Phi]_{T,t}^{(k)} \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k)d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{14}$$

where $\Phi(t_1, \ldots, t_k) \in C(D_k)$.

Using the arguments which similar to the arguments used for the proof of Lemma 1 it is easy to demonstrate that if $\Phi(t_1, \ldots, t_k) \in C(D_k)$, then the following equality is fulfilled

$$I[\Phi]_{T,t}^{(k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1.} \tag{15}$$

In order to explain, let us check the correctness of the equality (15) when $k = 3$. For definiteness we will suggest that $i_1, i_2, i_3 = 1, \ldots, m$. We have

$$I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1, t_2, t_3)d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \int\limits_t^{\tau_{j_3}} \int\limits_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \left(\int\limits_t^{\tau_{j_2}} + \int\limits_{\tau_{j_2}}^{t_2} \right) \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +
$$

$$
+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \tag{16}
$$

Let us demonstrate that the second limit on the right-hand side of (16) equals to zero. Actually, the second moment of its prelimit expression equals to

$$
\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi^2(t_1,t_2,\tau_{j_3}) dt_1 dt_2 \Delta\tau_{j_3} \le M^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \frac{1}{2} \left(\Delta\tau_{j_2}\right)^2 \Delta\tau_{j_3} \to 0
$$

if $N \to \infty$. Here M is a constant, which restricts the module of function $\Phi(t_1,t_2,t_3)$ due to its continuity, $\Delta\tau_j = \tau_{j+1} - \tau_j$.

Considering the obtained conclusions, we have

$$
I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +
$$

$$
+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,\tau_{j_2},\tau_{j_3}) - \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +
$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3})\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \tag{17}$$

In order to get the sought result, we just have to demonstrate that the first two limits on the right-hand side of (17) equal to zero. Let us prove that the first one of them equals to zero (proof for the second limit is similar).

The second moment of a prelimit expression of the first limit on the right-hand side of (17) equals to the following expression

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2dt_1dt_2\Delta\tau_{j_3}. \tag{18}$$

Since the function $\Phi(t_1,t_2,t_3)$ is continuous in the closed bounded domain D_3, then it is uniformly continuous in this domain. Therefore, if the distance between two points of the domain D_3 is less than $\delta(\varepsilon)$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on mentioned points), then the corresponding oscillation of the function $\Phi(t_1,t_2,t_3)$ for these two points of the domain D_3 is less than ε.

If we assume that $\Delta\tau_j<\delta(\varepsilon)$ $(j=0,1,\ldots,N-1)$, then the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$ is obviously less than $\delta(\varepsilon)$. In this case

$$|\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})|<\varepsilon.$$

Consequently, when $\Delta\tau_j<\delta(\varepsilon)$ $(j=0,\ 1,\ldots,N-1)$ the expression (18) is evaluated by the following value

$$\varepsilon^2\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3}<\varepsilon^2\frac{(T-t)^3}{6}.$$

Therefore, the first limit on the right-hand side of (17) equals to zero. Similarly we can prove equality to zero of the second limit on the right-hand side of (17).

Consequently, the equality (15) is proved for $k=3$. The cases $k=2$ and $k>3$ are analyzed absolutely similarly.

It is necessary to note that the proof of correctness of (15) is similar when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary.

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J'[\Phi]_{T,t}^{(k)}. \tag{19}$$

Then, according to (15) we will get the following

$$J'[\Phi]_{T,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Bigl(\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr), \tag{20}$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$. At the same time permutations $(t_1,\ldots,t_k)$ when summing are performed in (20) only in the expression, which is enclosed in parentheses. Moreover, the nonrandom function $\Phi(t_1,\ldots,t_k)$ is assumed to be continuous in the corresponding closed domains of integration. The case when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domains of integration and bounded at their boundaries is also possible.

It is not difficult to see that (20) can be rewritten in the form

$$J'[\Phi]_{T,t}^{(k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{21}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us consider the class $\mathrm{M}_2([0,T])$ of functions $\xi : [0,T]\times\Omega \to \mathbb{R}$, which are measurable in accordance with the collection of variables (t,ω) and F_t-measurable for all $t\in[0,T]$. Moreover, $\xi(\tau,\omega)$ is independent with increments $\mathbf{f}_{t+\Delta}-\mathbf{f}_t$ for $t\ge\tau$ $(\Delta>0)$,

$$\int\limits_0^T \mathsf{M}\left\{\xi^2(t,\omega)\right\} dt < \infty,$$

and $\mathsf{M}\{\xi^2(t,\omega)\}<\infty$ for all $t\in[0,T]$.

It is well known [77] that the Ito stochastic integral exists in the mean-square sence for any $\xi\in\mathrm{M}_2([0,T])$. Further, we will denote $\xi(\tau,\omega)$ as ξ_τ.

Lemma 2. *Suppose that $\Phi(t_1,\ldots,t_k)\in C(D_k)$ or $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function in the open domain D_k and bounded at its boundary. Then*

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^2\right\} \le C_k \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k, \quad C_k<\infty,$$

where $I[\Phi]_{T,t}^{(k)}$ is defined by the formula (14).

Proof. Using standard properties and moments estimates of stochastic integrals, we have for $\xi_\tau\in\mathrm{M}_2([t,T])$ [77]

$$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^2\right\} = \int\limits_t^T \mathsf{M}\{|\xi_\tau|^2\}d\tau, \quad \mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau d\tau\right|^2\right\} \le (T-t)\int\limits_t^T \mathsf{M}\{|\xi_\tau|^2\}d\tau. \tag{22}$$

Let us denote

$$\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t}=\int\limits_t^{t_{l+1}}\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_l}^{(i_l)},$$

where $l=1,\ldots,\ k-1$ and

$$\xi[\Phi]^{(0)}_{t_1,\ldots,t_k,t}\stackrel{\rm def}{=}\Phi(t_1,\ldots,t_k).$$

In accordance with the induction it is easy to demonstrate that

$$\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t}\in {\rm M}_2([t,T])$$

with respect to the variable t_{l+1}. Further, using the estimates (22) repeatedly we obtain the statement of Lemma 2.

It is not difficult to see that in the case $i_1,\ldots,i_k=1,\ldots,m$ from the proof of Lemma 2 we obtain

$$\mathsf{M}\left\{\left|I[\Phi]^{(k)}_{T,t}\right|^2\right\}=\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{23}$$

Lemma 3. *Suppose that every $\varphi_l(s)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$. Then*

$$\prod_{l=1}^k J[\varphi_l]_{T,t}=J[\Phi]^{(k)}_{T,t}\quad \hbox{w. p. 1}, \tag{24}$$

where

$$J[\varphi_l]_{T,t}=\int\limits_t^T\varphi_l(s)d\mathbf{w}_s^{(i_l)},\quad \Phi(t_1,\ldots,t_k)=\prod_{l=1}^k\varphi_l(t_l)$$

and the integral $J[\Phi]^{(k)}_{T,t}$ is defined by the equality (12).

Proof. Let at first $i_l\ne 0,\ l=1,\ldots,k$. Denote

$$J[\varphi_l]_N\stackrel{\rm def}{=}\sum_{j=0}^{N-1}\varphi_l(\tau_j)\Delta\mathbf{w}_{\tau_j}^{(i_l)}.$$

Since

$$\prod_{l=1}^k J[\varphi_l]_N-\prod_{l=1}^k J[\varphi_l]_{T,t}=$$

$$=\sum_{l=1}^k\left(\prod_{g=1}^{l-1}J[\varphi_g]_{T,t}\right)\left(J[\varphi_l]_N-J[\varphi_l]_{T,t}\right)\left(\prod_{g=l+1}^k J[\varphi_g]_N\right), \tag{25}$$

then because of the Minkowski inequality and the inequality of Cauchy-Bunyakovsky we obtain

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k} J[\varphi_l]_N - \prod_{l=1}^{k} J[\varphi_l]_{T,t}\right|^2\right\}\right)^{1/2} \le C_k \sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|J[\varphi_l]_N - J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{26}$$

where C_k is a constant.

Note that

$$J[\varphi_l]_N - J[\varphi_l]_{T,t} = \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}, \quad J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))\, d\mathbf{w}_s^{(i_l)}.$$

Since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ are independent for various j, then [78]

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \tag{27}$$

It is obviously that $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ are Gaussian random variables. Then we have

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds,$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds\right)^2.$$

Using this relations and continuity (which means uniform continuity) of the functions $\varphi_l(s)$, we get

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4\left(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\right) <$$

$$< 3\varepsilon^4\left(\delta(\varepsilon)(T-t) + (T-t)^2\right),$$

where $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$). Then the right-hand side of the formula (27) tends to zero when $N \to \infty$.

Considering this fact as well as (26), we come to (24).

If $\mathbf{w}_{t_l}^{(i_l)} = t_l$ for some $l \in \{1, \ldots, k\}$, then the proof of Lemma 3 becomes obviously simpler and it is performed similarly. Lemma 3 is proved.

Remark 2. *It is easy to see that if* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (24) *for some* $l\in\{1,\ldots,k\}$ *is replaced with* $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p=2,\ i_l\ne 0)$, *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\Phi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p=3,4,\ldots,$ *then the right-hand side of the formula* (24) *will become zero w. p. 1.*

Let us consider the case $p=2$ in detail. Let $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (24) for some $l\in\{1,\ldots,k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^2$ $(i_l\ne 0)$ and

$$J[\varphi_l]_N\ \stackrel{\rm def}{=}\ \sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2,\quad J[\varphi_l]_{T,t}\ \stackrel{\rm def}{=}\ \int\limits_t^T\varphi_l(s)ds.$$

We have

$$\left(\mathsf{M}\left\{\left|J[\varphi_l]_N-J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4}=\left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\int\limits_t^T\varphi_l(s)ds\right|^4\right\}\right)^{1/4}=$$

$$=\left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\int\limits_{\tau_j}^{\tau_{j+1}}\varphi_l(s)ds\right)\right|^4\right\}\right)^{1/4}\le$$

$$\le\left(\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)\right|^4\right\}\right)^{1/4}+\left|\sum_{j=0}^{N-1}\int\limits_{\tau_j}^{\tau_{j+1}}\left(\varphi_l(\tau_j)-\varphi_l(s)\right)ds\right|. \tag{28}$$

From the relation, which is similar to (27), we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}\varphi_l(\tau_j)\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)\right|^4\right\}=\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^4\right\}+$$

$$+6\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_j}^{(i_l)}\right)^2-\Delta\tau_j\right)^2\right\}\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\mathsf{M}\left\{\left(\left(\Delta\mathbf{w}_{\tau_q}^{(i_l)}\right)^2-\Delta\tau_q\right)^2\right\}=$$

$$=60\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^4\left(\Delta\tau_j\right)^4+24\sum_{j=0}^{N-1}\left(\varphi_l(\tau_j)\right)^2\left(\Delta\tau_j\right)^2\sum_{q=0}^{j-1}\left(\varphi_l(\tau_q)\right)^2\left(\Delta\tau_q\right)^2\le C\left(\Delta_N\right)^2\ \to 0$$

if $N\to\infty$, where constant C does not depend on N.

The second term on the right-hand side of (28) tends to zero if $N\to\infty$ due to continuity (which means uniform continuity) of the function $\varphi_l(s)$ on the interval $[t,T]$. Then, taking into account (25), (26), we come to the affirmation of Remark 2.

Let us prove Theorem 1. According to Lemma 1, we have

$$J[\psi^{(k)}]_{T,t}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Bigl(K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr),\tag{29}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression, which is enclosed in parentheses.

It is easy to see that (29) can be rewritten in the form

$$J[\psi^{(k)}]_{T,t}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Since the integration of a bounded function with respect to the set of measure zero for Riemann or Lebesgue integrals gives zero result, then the following formula is correct for these integrals

$$\int\limits_{[t,T]^k}|G(t_1,\ldots,t_k)|dt_1\ldots dt_k=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}|G(t_1,\ldots,t_k)|dt_1\ldots dt_k,\tag{30}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $dt_1,\ldots,dt_k$. At the same time the indexes near upper limits of integration are changed correspondently and the function $|G(t_1,\ldots,t_k)|$ is assumed to be integrable in the hypercube $[t,T]^k$.

According to Lemmas 1, 3 and (20), (29), we get the following representation w. p. 1

$$J[\psi^{(k)}]_{T,t} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}\right) +$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} + \tag{31}$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \left(\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +$$

$$+R_{T,t}^{p_1,\ldots,p_k}, \tag{32}$$

where

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{33}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 2 and (30), we have

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k\to 0 \qquad (34)$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral. Theorem 1 is proved.

Note that (6) can be written as (see (31))

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \qquad (35)$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (19).

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 1 we obtain

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.$$

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases (see Remark 2) for $k=1,\ldots,7$ [7]-[59]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \qquad (36)$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \qquad (37)$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (38)$$

$$
\begin{aligned}
J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{aligned}
\tag{40}
$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{41}$$

$$
J[\psi^{(7)}]_{T,t}=\underset{p_1,\ldots,p_7\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_7=0}^{p_7}C_{j_7\ldots j_1}\Biggl(\prod_{l=1}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0,j_1=j_6\}}\prod_{\substack{l=1\\ l\ne 1,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6\}}\prod_{\substack{l=1\\ l\ne 2,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_3=i_6\ne 0,j_3=j_6\}}\prod_{\substack{l=1\\ l\ne 3,6}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_4=i_6\ne 0,j_4=j_6\}}\prod_{\substack{l=1\\ l\ne 4,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_5=i_6\ne 0,j_5=j_6\}}\prod_{\substack{l=1\\ l\ne 5,6}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2\}}\prod_{\substack{l=1\\ l\ne 1,2}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3\}}\prod_{\substack{l=1\\ l\ne 1,3}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4\}}\prod_{\substack{l=1\\ l\ne 1,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5\}}\prod_{\substack{l=1\\ l\ne 1,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3\}}\prod_{\substack{l=1\\ l\ne 2,3}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4\}}\prod_{\substack{l=1\\ l\ne 2,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5\}}\prod_{\substack{l=1\\ l\ne 2,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0,j_3=j_4\}}\prod_{\substack{l=1\\ l\ne 3,4}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_3=i_5\ne 0,j_3=j_5\}}\prod_{\substack{l=1\\ l\ne 3,5}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_4=i_5\ne 0,j_4=j_5\}}\prod_{\substack{l=1\\ l\ne 4,5}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1\}}\prod_{\substack{l=1\\ l\ne 1,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2\}}\prod_{\substack{l=1\\ l\ne 2,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3\}}\prod_{\substack{l=1\\ l\ne 3,7}}^{7}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4\}}\prod_{\substack{l=1\\ l\ne 4,7}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5\}}\prod_{\substack{l=1\\ l\ne 7,5}}^{7}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6\}}\prod_{\substack{l=1\\ l\ne 7,6}}^{7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=5,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=4,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,6,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+
$$

$$+\mathbf{1}_{\{i_6=i_1\neq 0, j_6=j_1, i_2=i_4\neq 0, j_2=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_1\neq 0, j_6=j_1, i_4=i_5\neq 0, j_4=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_1\neq 0, j_6=j_1, i_2=i_3\neq 0, j_2=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_3=i_5\neq 0, j_3=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_4=i_5\neq 0, j_4=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_3=i_4\neq 0, j_3=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_1=i_5\neq 0, j_1=j_5\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_1=i_4\neq 0, j_1=j_4\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_2\neq 0, j_6=j_2, i_1=i_3\neq 0, j_1=j_3\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_2=i_5\neq 0, j_2=j_5\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_4=i_5\neq 0, j_4=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_2=i_4\neq 0, j_2=j_4\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_1=i_5\neq 0, j_1=j_5\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_1=i_4\neq 0, j_1=j_4\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_3\neq 0, j_6=j_3, i_1=i_2\neq 0, j_1=j_2\}}\prod_{l=4,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_3=i_5\neq 0, j_3=j_5\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_2=i_5\neq 0, j_2=j_5\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_2=i_3\neq 0, j_2=j_3\}}\prod_{l=1,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_1=i_5\neq 0, j_1=j_5\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_1=i_3\neq 0, j_1=j_3\}}\prod_{l=2,5,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_4\neq 0, j_6=j_4, i_1=i_2\neq 0, j_1=j_2\}}\prod_{l=3,5,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_3=i_4\neq 0, j_3=j_4\}}\prod_{l=1,2,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_2=i_4\neq 0, j_2=j_4\}}\prod_{l=1,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_2=i_3\neq 0, j_2=j_3\}}\prod_{l=1,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_1=i_4\neq 0, j_1=j_4\}}\prod_{l=2,3,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_1=i_3\neq 0, j_1=j_3\}}\prod_{l=2,4,7}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_6=i_5\neq 0, j_6=j_5, i_1=i_2\neq 0, j_1=j_2\}}\prod_{l=3,4,7}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_2=i_3\neq 0, j_2=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_2=i_4\neq 0, j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_2=i_5\neq 0, j_2=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_2=i_6\neq 0, j_2=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_3=i_4\neq 0, j_3=j_4\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_3=i_5\neq 0, j_3=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_3=i_6\neq 0, j_3=j_6\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_4=i_5\neq 0, j_4=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_1\neq 0, j_7=j_1, i_4=i_6\neq 0, j_4=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0, j_7=j_1, i_7=i_1\neq 0, j_5=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\neq 0, j_7=j_2, i_1=i_3\neq 0, j_1=j_3\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\neq 0, j_7=j_2, i_1=i_4\neq 0, j_1=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\neq 0, j_7=j_2, i_1=i_5\neq 0, j_1=j_5\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_2\ne 0,j_7=j_2,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=4,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=4,2,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_3\ne 0,j_7=j_3,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,5,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,5,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_4\ne 0,j_7=j_4,i_5=i_6\ne 0,j_5=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_1=i_6\ne 0,j_1=j_6\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,6}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_2=i_6\ne 0,j_2=j_6\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,6}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_3=i_6\ne 0,j_3=j_6\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_5\ne 0,j_7=j_5,i_4=i_6\ne 0,j_4=j_6\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_2\ne 0,j_1=j_2\}}\prod_{l=3,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_3\ne 0,j_1=j_3\}}\prod_{l=2,4,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_4\ne 0,j_1=j_4\}}\prod_{l=2,3,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_1=i_5\ne 0,j_1=j_5\}}\prod_{l=2,3,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_3\ne 0,j_2=j_3\}}\prod_{l=1,4,5}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_4\ne 0,j_2=j_4\}}\prod_{l=1,3,5}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_2=i_5\ne 0,j_2=j_5\}}\prod_{l=1,3,4}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_5\ne 0,j_3=j_5\}}\prod_{l=1,2,4}\zeta_{j_l}^{(i_l)}+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_4=i_5\ne 0,j_4=j_5\}}\prod_{l=1,2,3}\zeta_{j_l}^{(i_l)}+$$

$$+\mathbf{1}_{\{i_7=i_6\ne 0,j_7=j_6,i_3=i_4\ne 0,j_3=j_4\}}\prod_{l=1,2,5}\zeta_{j_l}^{(i_l)}-$$

$$-\Biggl(\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_1}^{(i_1)}-$$

$$-\Biggl(\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_6\ne 0,j_1=j_6,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_4\ne 0,j_3=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_5\ne 0,j_3=j_5,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_3=i_6\ne 0,j_3=j_6,i_4=i_5\ne 0,j_4=j_5\}}\Biggr)\zeta_{j_2}^{(i_2)}-$$

$$-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_5\ne 0,j_4=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_6\ne 0,j_4=j_6,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_4=i_7\ne 0,j_4=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_5\ne 0,j_2=j_5,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$\left.+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_6\ne 0,j_2=j_6,i_4=i_5\ne 0,j_4=j_5\}}\right)\zeta_{j_3}^{(i_3)}-$$

$$-\left(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\right.$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_5=i_6\ne 0,j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_5=i_6\ne 0,j_5=j_6\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_5\ne 0,j_2=j_5,i_3=i_6\ne 0,j_3=j_6\}}+$$

$$\left.+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_5\ne 0,j_3=j_5\}}\right)\zeta_{j_4}^{(i_4)}-$$

$$-\left(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_6\ne 0,j_3=j_6,i_4=i_7\ne 0,j_4=j_7\}}+\right.$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_6=i_7\ne 0,j_6=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_6\ne 0,j_2=j_6,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_6\ne 0,j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_6=i_7\ne 0,j_6=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_6\ne 0,j_2=j_6,i_3=i_7\ne 0,j_3=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_6\ne 0,j_3=j_6\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_6=i_1\ne 0,j_6=j_1,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_3\ne 0,j_2=j_3,i_4=i_6\ne 0,j_4=j_6\}}+\mathbf{1}_{\{i_1=i_7\ne 0,j_1=j_7,i_2=i_4\ne 0,j_2=j_4,i_3=i_6\ne 0,j_3=j_6\}}+$$

$$\left.+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_6\ne 0,j_2=j_6,i_3=i_4\ne 0,j_3=j_4\}}\right)\zeta_{j_5}^{(i_5)}-$$

$$-\left(\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_4\ne 0,j_3=j_4,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_5\ne 0,j_3=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\right.$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0,j_1=j_2,i_3=i_7\ne 0,j_3=j_7,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_4\ne 0,j_2=j_4,i_5=i_7\ne 0,j_5=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_5\ne 0,j_2=j_5,i_4=i_7\ne 0,j_4=j_7\}}+\mathbf{1}_{\{i_1=i_3\ne 0,j_1=j_3,i_2=i_7\ne 0,j_2=j_7,i_4=i_5\ne 0,j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_3\ne 0,j_2=j_3,i_5=i_7\ne 0,j_5=j_7\}}+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_5\ne 0,j_2=j_5,i_3=i_7\ne 0,j_3=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0,j_1=j_4,i_2=i_7\ne 0,j_2=j_7,i_3=i_5\ne 0,j_3=j_5\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_3\ne 0,j_2=j_3,i_4=i_7\ne 0,j_4=j_7\}}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_4\ne 0,j_2=j_4,i_3=i_7\ne 0,j_3=j_7\}}+\mathbf{1}_{\{i_1=i_5\ne 0,j_1=j_5,i_2=i_7\ne 0,j_2=j_7,i_3=i_4\ne 0,j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_3\ne 0,j_2=j_3,i_4=i_5\ne 0,j_4=j_5\}}+\mathbf{1}_{\{i_7=i_1\ne 0,j_7=j_1,i_2=i_4\ne 0,j_2=j_4,i_3=i_5\ne 0,j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_7=i_1\ne 0, j_7=j_1, i_2=i_5\ne 0, j_2=j_5, i_3=i_4\ne 0, j_3=j_4\}}\Biggr)\zeta_{j_6}^{(i_6)}-$$

$$-\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_3=i_4\ne 0, j_3=j_4, i_5=i_6\ne 0, j_5=j_6\}}+\mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_3=i_5\ne 0, j_3=j_5, i_4=i_6\ne 0, j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0, j_1=j_2, i_3=i_6\ne 0, j_3=j_6, i_4=i_5\ne 0, j_4=j_5\}}+\mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_2=i_4\ne 0, j_2=j_4, i_5=i_6\ne 0, j_5=j_6\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_2=i_5\ne 0, j_2=j_5, i_4=i_6\ne 0, j_4=j_6\}}+\mathbf{1}_{\{i_1=i_3\ne 0, j_1=j_3, i_2=i_6\ne 0, j_2=j_6, i_4=i_5\ne 0, j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_4=i_1\ne 0, j_4=j_1, i_2=i_3\ne 0, j_2=j_3, i_5=i_6\ne 0, j_5=j_6\}}+\mathbf{1}_{\{i_4=i_1\ne 0, j_4=j_1, i_2=i_5\ne 0, j_2=j_5, i_3=i_6\ne 0, j_3=j_6\}}+$$

$$+\mathbf{1}_{\{i_4=i_1\ne 0, j_4=j_1, i_2=i_6\ne 0, j_2=j_6, i_3=i_5\ne 0, j_3=j_5\}}+\mathbf{1}_{\{i_5=i_1\ne 0, j_5=j_1, i_2=i_3\ne 0, j_2=j_3, i_4=i_6\ne 0, j_4=j_6\}}+$$

$$+\mathbf{1}_{\{i_5=i_1\ne 0, j_5=j_1, i_2=i_4\ne 0, j_2=j_4, i_3=i_6\ne 0, j_3=j_6\}}+\mathbf{1}_{\{i_5=i_1\ne 0, j_5=j_1, i_2=i_6\ne 0, j_2=j_6, i_3=i_4\ne 0, j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0, j_6=j_1, i_2=i_3\ne 0, j_2=j_3, i_4=i_5\ne 0, j_4=j_5\}}+\mathbf{1}_{\{i_6=i_1\ne 0, j_6=j_1, i_2=i_4\ne 0, j_2=j_4, i_3=i_5\ne 0, j_3=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0, j_6=j_1, i_2=i_5\ne 0, j_2=j_5, i_3=i_4\ne 0, j_3=j_4\}}\Biggr)\zeta_{j_7}^{(i_7)}\Biggr), \tag{42}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider the generalization of the formulas (36)–(42) for the case of arbitrary multiplicity k of $J[\psi^{(k)}]_{T,t}$. In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{43}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (43) is the partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{44}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (44)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135}+$$

$$+a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4}+$$

$$+a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2}+$$

$$+a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now, we can formulate Theorem 1 (see (6)) using the alternative form.

Theorem 2 [10] (2009) (also see [11]-[17], [20]-[26], [34], [46]-[51]). *Under the conditions of Theorem* 1 *the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (45)$$

converging in the mean-square sense is valid, where $i_1,\ldots,i_k = 0,1,\ldots,m$, $[x]$ *is an integer part of a real number* x, $\prod_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum_{\emptyset} \stackrel{\sf def}{=} 0$; *another notations are the same as in Theorem* 1.

Proof. The equality (45) will be proved by induction in Sect. 18 (see the proof of Theorem 21).

In particular, from (45) for $k = 5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$
-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+
$$

$$
\left.+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\right).
$$

The last equality obviously agrees with (40).

3. Comparison of Theorem 2 With Representations of Iterated Ito Stochastic Integrals Based on Hermite Polynomials

Note that the correctness of the formulas (36)–(42) can be verified by the fact that if $i_1=\ldots=i_7=i=1,\ldots,m$ and $\psi_1(s),\ldots,\psi_7(s)\equiv\psi(s)$, then we can derive from (36)–(42) [9] (2007) (also see [10]-[17], [20]-[26]) the well-known equalities

$$
J[\psi^{(1)}]_{T,t}=\frac{1}{1!}\delta_{T,t},
$$

$$
J[\psi^{(2)}]_{T,t}=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right),
$$

$$
J[\psi^{(3)}]_{T,t}=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right),
$$

$$
J[\psi^{(4)}]_{T,t}=\frac{1}{4!}\left(\delta_{T,t}^4-6\delta_{T,t}^2\Delta_{T,t}+3\Delta_{T,t}^2\right),
$$

$$
J[\psi^{(5)}]_{T,t}=\frac{1}{5!}\left(\delta_{T,t}^5-10\delta_{T,t}^3\Delta_{T,t}+15\delta_{T,t}\Delta_{T,t}^2\right),
$$

$$
J[\psi^{(6)}]_{T,t}=\frac{1}{6!}\left(\delta_{T,t}^6-15\delta_{T,t}^4\Delta_{T,t}+45\delta_{T,t}^2\Delta_{T,t}^2-15\Delta_{T,t}^3\right),
$$

$$
J[\psi^{(7)}]_{T,t}=\frac{1}{7!}\left(\delta_{T,t}^7-21\delta_{T,t}^5\Delta_{T,t}+105\delta_{T,t}^3\Delta_{T,t}^2-105\delta_{T,t}\Delta_{T,t}^3\right),
$$

which fulfilled w. p. 1, where

$$
\delta_{T,t}=\int\limits_t^T\psi(s)d\mathbf{f}_s^{(i)},\qquad \Delta_{T,t}=\int\limits_t^T\psi^2(s)ds.
$$

The above equalities can be independently obtained using the Ito formula and Hermite polynomials.

When $k=1$ everything is evident. Let us consider the cases $k=2,\ 3$. When $k=2$ for the case $p_1=p_2=p$ we obtain

$$
J[\psi^{(2)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}-\sum_{j_1=0}^{p}C_{j_1j_1}\right)=
$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\Bigl(C_{j_2j_1}+C_{j_1j_2}\Bigr)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\sum_{j_1=0}^{p}C_{j_1j_1}\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\sum_{\substack{j_1,j_2=0\\ j_1\ne j_2}}^{p}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^2-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\right)=$$

$$= \frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right). \tag{46}$$

Let us explain the last step in (46). For the Ito stochastic integrals the following estimate is valid [77]

$$\mathsf{M}\left\{\left|\int_t^T \xi_\tau df_\tau\right|^q\right\}\le K_q\mathsf{M}\left\{\left(\int_t^T|\xi_\tau|^2d\tau\right)^{q/2}\right\}, \tag{47}$$

where $q>0$ is a fixed number, f_τ is a scalar standard Wiener process, $\xi_\tau\in \text{M}_2([t,T])$, K_q is a constant depending only on q,

$$\int_t^T|\xi_\tau|^2d\tau<\infty\quad \text{w. p. 1},$$

$$\mathsf{M}\left\{\left(\int_t^T|\xi_\tau|^2d\tau\right)^{q/2}\right\}<\infty.$$

Since

$$\delta_{T,t}-\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}=\int_t^T\Bigl(\psi(s)-\sum_{j_1=0}^{p}C_{j_1}\phi_{j_1}(s)\Bigr)d\mathbf{f}_s^{(i)},$$

then using the estimate (47) to the right-hand side of this expression and considering that

$$\int_t^T\Bigl(\psi(s)-\sum_{j_1=0}^{p}C_{j_1}\phi_{j_1}(s)\Bigr)^2ds\ \to 0$$

if $p\to\infty$, we obtain

$$\int\limits_t^T \psi(s)d\mathbf{f}_s^{(i)} = q\text{ - }\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}, \quad q>0, \tag{48}$$

where $q\text{ - }\underset{p\to\infty}{\text{l.i.m.}}$ is a limit in the mean of degree q. Hence, if $q=4$, then it is easy to conclude that

$$\underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^2 = \delta_{T,t}^2.$$

This equality as well as Parseval's equality were used in the last transition of the formula (46).
If $k=3$ for the case $p_1=p_2=p_3=p$ we have

$$J[\psi^{(3)}]_{T,t} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i)} - \sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1}\zeta_{j_1}^{(i)} - \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}\zeta_{j_2}^{(i)}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} - \sum_{j_1,j_3=0}^{p} \Bigl(C_{j_3j_1j_1} + C_{j_1j_1j_3} + C_{j_1j_3j_1}\Bigr)\zeta_{j_3}^{(i)}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1} \Bigl(C_{j_3j_2j_1} + C_{j_3j_1j_2} + C_{j_2j_1j_3} + C_{j_2j_3j_1} + C_{j_1j_2j_3} + C_{j_1j_3j_2}\Bigr)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} + \right.$$

$$+ \sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} \Bigl(C_{j_3j_1j_3} + C_{j_1j_3j_3} + C_{j_3j_3j_1}\Bigr)\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)} +$$

$$+ \sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1} \Bigl(C_{j_3j_1j_1} + C_{j_1j_1j_3} + C_{j_1j_3j_1}\Bigr)\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)} +$$

$$\left. + \sum_{j_1=0}^{p} C_{j_1j_1j_1}\left(\zeta_{j_1}^{(i)}\right)^3 - \sum_{j_1,j_3=0}^{p} \left(C_{j_3j_1j_1} + C_{j_1j_1j_3} + C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1} C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)} + \right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{\substack{j_1,j_2,j_3=0\\ j_1\ne j_2,j_2\ne j_3,j_1\ne j_3}}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\right.$$

$$-\frac{1}{6}\left(3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\right)+$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2 C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2 C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2 C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\sum_{j_3=0}^{p}C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right). \qquad (49)$$

The last step in (49) follows from the Parseval equality, Theorem 1 for $k=1$, and the equality

$$\underset{p\to\infty}{\text{l.i.m.}}\ \left(\sum_{j_1=0}^{p} C_{j_1}\zeta_{j_1}^{(i)}\right)^3=\delta_{T,t}^3,$$

which can be obtained easily when $q=8$ (see (48)).

In addition, we used the following relations between Fourier coefficients for the considered case

$$C_{j_1j_2}+C_{j_2j_1}=C_{j_1}C_{j_2},\quad 2C_{j_1j_1}=C_{j_1}^2,$$

$$C_{j_1j_2j_3}+C_{j_1j_3j_2}+C_{j_2j_3j_1}+C_{j_2j_1j_3}+C_{j_3j_2j_1}+C_{j_3j_1j_2}=C_{j_1}C_{j_2}C_{j_3},$$

$$2\left(C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}\right)=C_{j_1}^2C_{j_3},$$

$$6C_{j_1j_1j_1}=C_{j_1}^3.$$

4. On Usage of Discontinuous Complete Orthonormal Systems of Functions in Theorem 1

Analyzing the proof of Theorem 1, we can ask a natural question: can we weaken the condition of continuity of the functions $\phi_j(x)$, $j=1,2,\ldots$?

We will say that the function $f(x):[t,T]\to\mathbb{R}$ satisfies the condition $(\star)$ if it is continuous on the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity as well as it is right-continuous on the interval $[t,T]$.

Furthermore, let us suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

It is easy to see that continuity of the functions $\phi_j(x)$ was used substantially in the proof of Theorem 1 in two places: Lemma 3 and the formula (15). It is clear that without the loss of generality the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ in Lemma 3 and in the formula (15) can be taken so "dense" that among the points τ_j of this partition will be all points of jumps of the functions $\varphi_1(\tau)=\phi_{j_1}(\tau)$, $\ldots$, $\varphi_k(\tau)=\phi_{j_k}(\tau)$ $(j_1,\ldots,j_k<\infty)$ and among the points $(\tau_{j_1},\ldots,\tau_{j_k})$ for which $0\le j_1<\ldots<j_k\le N-1$ there will be all points of jumps of the function $\Phi(t_1,\ldots,t_k)$.

Let us demonstrate how to modify the proofs of Lemma 3 and the formula (15) in the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

At first, consider Lemma 3. In the proof of this lemma we obtained the following relations

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1}J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}=\sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}+$$

$$+6\sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\},\qquad(50)$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds,$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds\right)^2.$$

Suppose that the functions $\varphi_l(s)$ $(l = 1, \ldots, k)$ satisfy the condition $(\star)$ and the partition $\{\tau_j\}_{j=0}^N$ includes all points of jumps of the functions $\varphi_l(s)$ $(l = 1, \ldots, k)$. It means that for the integral

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds$$

the integrand function is continuous at the interval $[\tau_j, \tau_{j+1}]$, except possibly the point τ_{j+1} of finite discontinuity.

Let $\mu \in (0, \Delta\tau_j)$ be fixed. Then, due to continuity (which means uniform continuity) of the functions $\varphi_l(s)$ $(l = 1, \ldots, k)$ on the interval $[\tau_j, \tau_{j+1} - \mu]$ we have

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds =$$

$$= \int\limits_{\tau_j}^{\tau_{j+1}-\mu} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds + \int\limits_{\tau_{j+1}-\mu}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds < \varepsilon^2(\Delta\tau_j - \mu) + M^2\mu. \tag{51}$$

Obtaining the inequality (51), we proposed that $\Delta\tau_j < \delta(\varepsilon)$ for $j = 0,\ 1, \ldots, N-1$ $(\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on $s)$,

$$|\varphi_l(\tau_j) - \varphi_l(s)| < \varepsilon$$

if $s \in [\tau_j, \tau_{j+1} - \mu]$ (due to uniform continuity of the functions $\varphi_l(s)$ $(l = 1, \ldots, k)$),

$$|\varphi_l(\tau_j) - \varphi_l(s)| < M$$

if $s \in [\tau_{j+1} - \mu, \tau_{j+1}]$, M is a constant (potential point of discontinuity of the function $\varphi_l(s)$ is supposed in the point τ_{j+1}).

Performing the passage to the limit in the inequality (51) when $\mu \to +0$, we get

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds \le \varepsilon^2 \Delta\tau_j.$$

Using this estimate for the right-hand side of (50), we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4\left(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2+6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\right) <$$

$$< 3\varepsilon^4\left(\delta(\varepsilon)(T-t)+(T-t)^2\right). \tag{52}$$

This implies that

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1}\tau_j}\right|^4\right\} \to 0$$

if $N\to\infty$. So, Lemma 3 remains valid.

Now, let us present explanations concerning the correctness of the formula (15) when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for $j<\infty$ satisfies the condition $(\star)$.

Let us consider the case $k=3$ and the representation (17). We can demonstrate that in the studied case the first limit on the right-hand side of (17) equals to zero (similarly we demonstrate that the second limit on the right-hand side of (17) equals to zero; proof of the second limit equality to zero on the right-hand side of the formula (16) is the same as for the case of continuous functions $\phi_j(x)$, $j=0,1,\ldots$).

The second moment of the prelimit expression of first limit on the right-hand side of (17) looks as follows

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2\Delta\tau_{j_3}.$$

Further, for the fixed $\mu\in(0,\Delta\tau_{j_2})$ and $\rho\in(0,\Delta\tau_{j_1})$ we have

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2 =$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\right)\left(\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2 =$$

$$=\left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho}+\int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\right)\times$$

$$\times\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2 <$$

$$<\varepsilon^2\left(\Delta\tau_{j_2}-\mu\right)\left(\Delta\tau_{j_1}-\rho\right)+M^2\rho\left(\Delta\tau_{j_2}-\mu\right)+M^2\mu\left(\Delta\tau_{j_1}-\rho\right)+M^2\mu\rho, \tag{53}$$

where M is a constant, $\Delta\tau_j < \delta(\varepsilon)$ for $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points (t_1, t_2, τ_{j_3}), $(t_1, \tau_{j_2}, \tau_{j_3})$). We suppose here that the partition $\{\tau_j\}_{j=0}^N$ contains all discontinuity points of the function $\Phi(t_1, t_2, t_3)$ as points τ_j (for every variable). When obtaining (53), we also supposed that potential discontinuity points of this function (for every variable) are contained among the points $\tau_{j_1+1}, \tau_{j_2+1}, \tau_{j_3+1}$.

Let us explain in detail how we obtained the inequality (53). Since the function $\Phi(t_1, t_2, t_3)$ is continuous on the closed bounded set

$$Q_3 = \left\{ (t_1, t_2, t_3) : t_1 \in [\tau_{j_1}, \tau_{j_1+1} - \rho], t_2 \in [\tau_{j_2}, \tau_{j_2+1} - \mu], t_3 \in [\tau_{j_3}, \tau_{j_3+1} - \nu] \right\},$$

where ρ, μ, ν are fixed small positive numbers such that

$$\nu \in (0, \Delta\tau_{j_3}), \quad \mu \in (0, \Delta\tau_{j_2}), \quad \rho \in (0, \Delta\tau_{j_1}),$$

then this function is also uniformly continous on this set and bounded on the closed set D_3.

Since the distance between the points (t_1, t_2, τ_{j_3}), $(t_1, \tau_{j_2}, \tau_{j_3}) \in Q_3$ is obviously less than $\delta(\varepsilon)$ ($\Delta\tau_j < \delta(\varepsilon)$ for $j = 0, 1, \ldots, N-1$), then

$$|\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})| < \varepsilon.$$

This inequality was used to estimate the first double integral in (53). Estimating the three remaining double integrals, we used the property of boundedness of the function $\Phi(t_1, t_2, t_3)$ in form of inequality

$$|\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})| < M.$$

Performing the passage to the limit in the inequality (53) if $\mu, \rho \to +0$, we obtain the estimate

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right)^2 dt_1 dt_2 \le \varepsilon^2 \Delta\tau_{j_2} \Delta\tau_{j_1}.$$

Usage of this estimate provides

$$\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1, t_2, \tau_{j_3}) - \Phi(t_1, \tau_{j_2}, \tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3} \le$$

$$\le \varepsilon^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1} \Delta\tau_{j_2} \Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.$$

The last estimate means that in the considered case the first limit on the right-hand side of (17) equals to zero (similarly we can demonstrate that the second limit on the right-hand side of (17) equals to zero).

Consequently, the formula (15) is correct when $k = 3$ in the considered case. Similarly, we perform argumentation for the cases $k = 2$ and $k > 3$.

Consequently, in Theorem 1 we can use complete orthonormal systems of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t, T])$, each function $\phi_j(x)$ of which for $j < \infty$ satisfies the condition $(\star)$.

One of the examples of such systems of functions is a complete orthonormal system of Haar functions in the space $L_2([t, T])$

$$\phi_0(x)=\frac{1}{\sqrt{T-t}},\qquad \phi_{nj}(x)=\frac{1}{\sqrt{T-t}}\varphi_{nj}\left(\frac{x-t}{T-t}\right),$$

where $n=0,1,\ldots,$ $j=1,2,\ldots,2^n$, and the functions $\varphi_{nj}(x)$ have the following form

$$\varphi_{nj}(x)=\begin{cases}2^{n/2}, & x\in[(j-1)/2^n,\ (j-1)/2^n+1/2^{n+1})\\ \\ -2^{n/2}, & x\in[(j-1)/2^n+1/2^{n+1},\ j/2^n)\\ \\ 0, & \text{otherwise}\end{cases},$$

where $n=0,1,\ldots,$ $j=1,2,\ldots,2^n$ (we choose the values of Haar functions in the points of discontinuity in such a way that these functions will be right-continuous).

The other example of similar system of functions is a complete orthonormal system of Rademacher–Walsh functions in the space $L_2([t,T])$

$$\phi_0(x)=\frac{1}{\sqrt{T-t}},$$

$$\phi_{m_1\ldots m_k}(x)=\frac{1}{\sqrt{T-t}}\varphi_{m_1}\left(\frac{x-t}{T-t}\right)\ldots\varphi_{m_k}\left(\frac{x-t}{T-t}\right),$$

where $0<m_1<\ldots<m_k$, $m_1,\ldots,m_k=1,2,\ldots,$ $k=1,2,\ldots,$

$$\varphi_m(x)=(-1)^{[2^m x]},$$

$x\in[0,1]$, $m=1,2,\ldots,$ $[y]$ is an integer part of a real number y.

5. Remark on Usage of Complete Orthonormal Systems of Functions in Theorem 1

Note that actually the functions $\phi_j(s)$ of complete orthonormal system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ depend not only on s, but on t and T.

For example, the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ have the following form

$$\phi_j(s,t,T)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}\right),$$

where $P_j(s)$ $(j=0,1,2,\ldots)$ is the Legendre polynomial,

$$\phi_j(s,t,T)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(s-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(s-t)/(T-t)\right), & j=2r\end{cases}$$

where $r=1,2,\ldots$

Note that the specified systems of functions are assumed to be used in the context of implementing of numerical methods for Ito stochastic differential equations for the sequences of time intervals

$$[T_0,T_1],\ [T_1,T_2],\ [T_2,T_3],\ \ldots\ ,$$

and spaces

$$L_2([T_0,T_1]),\ L_2([T_1,T_2]),\ L_2([T_2,T_3]),\ \ldots$$

We can explain that the dependence of functions $\phi_j(s,t,T)$ from t and T (hereinafter these constants will mean fixed moments of time) will not affect the main properties of independence of the random variables

$$\zeta_{(j)T,t}^{(i)}=\int\limits_t^T\phi_j(s,t,T)d\mathbf{w}_s^{(i)}, \tag{54}$$

where $i=1,\ldots,m$ and $j=0,1,2,\ldots$

Indeed, for fixed t and T due to orthonormality of the mentioned systems of functions we have

$$\mathsf{M}\left\{\zeta_{(j)T,t}^{(i)}\zeta_{(g)T,t}^{(r)}\right\}=\mathbf{1}_{\{i=r\}}\mathbf{1}_{\{j=g\}}, \tag{55}$$

where

$$\zeta_{(j)T,t}^{(i)}=\int\limits_t^T\phi_j(s,t,T)d\mathbf{w}_s^{(i)},\quad i,r=1,\ldots,m,\quad j,g=0,1,2,\ldots$$

Note that (55) means the independence of random variables (54) for various i or j.

On the other side, the random variables

$$\zeta_{(j)T_1,t_1}^{(i)}=\int\limits_{t_1}^{T_1}\phi_j(s,t_1,T_1)d\mathbf{w}_s^{(i)},\quad \zeta_{(j)T_2,t_2}^{(i)}=\int\limits_{t_2}^{T_2}\phi_j(s,t_2,T_2)d\mathbf{w}_s^{(i)}$$

are independent if $[t_1,T_1]\cap[t_2,T_2]=\emptyset$ (the case $T_1=t_2$ is possible) according to the property of the Ito stochastic integral.

Therefore, two important characteristics of random variables $\zeta_{(j)T,t}^{(i)}$, which are the basic motive of their usage, are saved.

6. Convergence in the Mean of Degree $2n$ ($n \in \mathbb{N}$) of Expansion of Iterated Ito Stochastic Integrals From Theorem 1

This section is written on the base of the following paper [8] (2007) (also see [9]-[17], [20]-[26]).

Constructing the expansions of iterated Ito stochastic integrals from Theorem 1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will be convergent not only in the mean-square sense but in the stronger probabilistic senses.

We will obtain the general estimate which proves the convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) of expansions from Theorem 1.

According to the notations of Theorem 1 (see (33)), we have

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{56}$$

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) \stackrel{\rm def}{=} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l),$$

where $i_1,\ldots,i_k = 1,\ldots,m$.

For definiteness we will consider in this section the case $i_1,\ldots,i_k = 1,\ldots,m$ (it is obviously quite enough for the unified Taylor–Ito expansions [38] (also see [7]-[25]) of solutions of Ito stochastic differential equations). Another notations from this section are the same as in the formulation and proof of Theorem 1.

Proving Theorem 1, we obtained that (see (33))

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k) dt_1\ldots dt_k,$$

where C_k is a constant.

In the next section we will show that

$$C_k = k!$$

for the following cases

1. $i_1,\ldots,i_k = 1,\ldots,m$ and $0 < T-t < \infty$,
2. $i_1,\ldots,i_k = 0,1,\ldots,m$, $i_1^2+\ldots+i_k^2 > 0$, and $0 < T-t < 1$.

Assume that

$$\eta_{t_l,t}^{(l-1)} \stackrel{\rm def}{=} \int\limits_t^{t_l} \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_{l-1}}^{(i_{l-1})}, \quad l = 2,\ 3,\ldots,k+1,$$

$$\eta_{t_{k+1},t}^{(k)} \stackrel{\rm def}{=} \eta_{T,t}^{(k)},$$

$$\eta_{T,t}^{(k)} \stackrel{\rm def}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}.$$

Using the Ito formula it is easy to demonstrate that [77]

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\} = n(2n-1)\int\limits_{t_0}^t \mathsf{M}\left\{\left(\int\limits_{t_0}^s \xi_u df_u\right)^{2n-2}\xi_s^2\right\}ds.$$

Using the Holder inequality (under the sign of integration on the right-hand side of the last equality) for $p = n/(n-1)$, $q = n$ and using the increasing of the value

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\}$$

with the growth of t, we get

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\} \le n(2n-1)\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\}\right)^{(n-1)/n}\int\limits_{t_0}^t \left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds.$$

Raising to power n the obtained inequality and dividing it on

$$\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\}\right)^{n-1},$$

we get the following estimate

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\} \le (n(2n-1))^n\left(\int\limits_{t_0}^t \left(\mathsf{M}\left\{\xi_s^{2n}\right\}\right)^{1/n}ds\right)^n. \tag{57}$$

Using the estimate (57), we have

$$\mathsf{M}\left\{\left(\eta_{T,t}^{(k)}\right)^{2n}\right\} \le (n(2n-1))^n\left(\int\limits_t^T \left(\mathsf{M}\left\{\left(\eta_{t_k,t}^{(k-1)}\right)^{2n}\right\}\right)^{1/n}dt_k\right)^n \le$$

$$
\le (n(2n-1))^n \left(\int\limits_t^T \left((n(2n-1))^n \left(\int\limits_t^{t_k} \left(\mathsf{M}\left\{ \left(\eta_{t_{k-1},t}^{(k-2)} \right)^{2n} \right\} \right)^{1/n} dt_{k-1} \right)^n \right)^{1/n} dt_k \right)^n =
$$

$$
= (n(2n-1))^{2n} \left(\int\limits_t^T \int\limits_t^{t_k} \left(\mathsf{M}\left\{ \left(\eta_{t_{k-1},t}^{(k-2)} \right)^{2n} \right\} \right)^{1/n} dt_{k-1} dt_k \right)^n \le \ldots
$$

$$
\ldots \le (n(2n-1))^{n(k-1)} \left(\int\limits_t^T \int\limits_t^{t_k} \ldots \int\limits_t^{t_3} \left(\mathsf{M}\left\{ \left(\eta_{t_2,t}^{(1)} \right)^{2n} \right\} \right)^{1/n} dt_3 \ldots dt_{k-1} dt_k \right)^n =
$$

$$
= (n(2n-1))^{n(k-1)} (2n-1)!! \left(\int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}^2 (t_1,\ldots,t_k) dt_1 \ldots dt_k \right)^n \le
$$

$$
\le (n(2n-1))^{n(k-1)} (2n-1)!! \left(\int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2 (t_1,\ldots,t_k) dt_1 \ldots dt_k \right)^n .
$$

The penultimate step was obtained using the formula

$$
\mathsf{M}\left\{ \left(\eta_{t_2,t}^{(1)} \right)^{2n} \right\} = (2n-1)!! \left(\int\limits_t^{t_2} R_{p_1\ldots p_k}^2 (t_1,\ldots,t_k) dt_1 \right)^n ,
$$

which follows from the Gaussianity of

$$
\eta_{t_2,t}^{(1)} = \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)}.
$$

Similarly, we estimate the each summand on the right-hand side of (56). Then, from (56) using the Minkowski inequality we finally get

$$
\mathsf{M}\left\{ \left(R_{T,t}^{p_1,\ldots,p_k} \right)^{2n} \right\} \le
$$

$$
\le \left(k! \left((n(2n-1))^{n(k-1)} (2n-1)!! \left(\int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2 (t_1,\ldots,t_k) dt_1 \ldots dt_k \right)^n \right)^{1/2n} \right)^{2n} =
$$

$$= (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!! \left(\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^n. \tag{58}$$

Due to orthonormality of the functions $\phi_j(s)$ $(j = 0, 1, 2, \ldots)$ we obtain

$$\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k =$$

$$= \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}. \tag{59}$$

Let us substitute (59) into (58)

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\} \le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!! \times$$

$$\times \left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \right)^n. \tag{60}$$

The inequality (58) (or (60)) means that the expansions of iterated Ito stochastic integrals, obtained using Theorem 1, converge in the mean of degree $2n$ $(n \in \mathbb{N})$, as according to the Parseval equality

$$\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k =$$

$$= \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \ \to\ 0$$

if $p_1,\ldots,p_k \to \infty$.

7. Estimate for the Mean-Square Error of Approximation of Iterated Ito Stochastic Integrals Based on Theorem 1

In this section, we prove the useful estimate for the mean-square error of approximation in Theorem 1.

Theorem 3 [20]-[25], [34]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function of which for finite j satisfies the condition $(\star)$ (see Sect. 4). Then the estimate*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right) \tag{61}
$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{T,t}$ is the stochastic integral (1), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (6) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; another notations are the same as in Theorem 1.

Proof. Prooving Theorem 1, we obtained w. p. 1 the following representation (see (32), (33))

$$
J[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}+R_{T,t}^{p_1,\ldots,p_k},
$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (6) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ and

$$
R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{62}
$$

where

$$
\sum_{(t_1,\ldots,t_k)}
$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$, which are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

In the case of any fixed k and numbers $i_1,\ldots,i_k=1,\ldots,m$ the integrals on the right-hand side of (62) will be dependent in a stochastic sense. Let us estimate the second moment of $R_{T,t}^{p_1,\ldots,p_k}$. From (22), (62) and elementary inequality

$$(a_1+a_2+\ldots+a_p)^2\le p\left(a_1^2+a_2^2+\ldots+a_p^2\right),\quad p\in\mathbb{N}, \tag{63}$$

we obtain the following estimate for the case $i_1,\ldots,i_k=1,\ldots,m$ $(0<T-t<\infty)$

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k\right)=$$

$$=k!\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right). \tag{64}$$

For the case of any fixed k and numbers $i_1,\ldots,i_k=0,1,\ldots,m$ $(i_1^2+\ldots+i_k^2>0)$ from (22), (62), (63) we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=C_k\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),$$

where C_k is a constant.

It is not difficult to see that the constant C_k depends on k (k is the multiplicity of the iterated Ito stochastic integral) and $T-t$ ($T-t$ is the length of integration interval of the iterated Ito stochastic integral). Moreover, C_k has the following form

$$C_k = k! \cdot \max\Bigl\{(T-t)^{\alpha_1},\ (T-t)^{\alpha_2},\ \ldots,\ (T-t)^{\alpha_{k!}}\Bigr\},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{k!} = 0, 1, \ldots, k-1$.

However, $T-t$ is supposed as an integration step of numerical procedures for Ito stochastic differential equations, which is a rather small value. For example $0 < T-t < 1$. Then $C_k \le k!$

It means, that for the case of any fixed k and $i_1, \ldots, i_k = 0, 1, \ldots, m$, $i_1^2 + \ldots + i_k^2 > 0$ $(0 < T-t < 1)$ we can write (61). Theorem 3 is proved.

8. Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Complete Orthonormal With Weight $r(t_1)\ldots r(t_k) \ge 0$ Systems of Functions in the Space $L_2([t,T]^k)$

In this section, we consider the modification of Theorem 1 for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t,T]^k)$ $(k \in \mathbb{N})$.

Let $\{\Psi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal with weight $r(x) \ge 0$ system of functions in the space $L_2([t,T])$. It is well known that the Fourier series with respect to the system $\{\Psi_j(x)\}_{j=0}^{\infty}$ of function

$$f(x) \quad \Bigl(f(x)\sqrt{r(x)} \in L_2([t,T])\Bigr)$$

converges to the function $f(x)$ in the mean-square sense with weight $r(x)$, i.e.

$$\lim_{p\to\infty} \int\limits_t^T \Biggl(f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x)\Biggr)^2 r(x)dx = 0, \tag{65}$$

where

$$\tilde{C}_j = \int\limits_t^T f(x)\Psi_j(x)r(x)dx \tag{66}$$

is the Fourier coefficient.

Obviously, the relation (65) can be obtained if we will expand the function $f(x)\sqrt{r(x)} \in L_2([t,T])$ into a usual Fourier series with respect to the complete orthonormal with weight 1 system of functions

$$\Bigl\{\Psi_j(x)\sqrt{r(x)}\Bigr\}_{j=0}^{\infty}$$

in the space $L_2([t,T])$. Then

$$\lim_{p\to\infty} \int\limits_t^T \Biggl(f(x)\sqrt{r(x)} - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x)\sqrt{r(x)}\Biggr)^2 dx =$$

$$= \lim_{p\to\infty} \int\limits_t^T \left(f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \right)^2 r(x) dx = 0, \tag{67}$$

where $\tilde{C}_j$ has the form (66).

Let us consider an obvious generalization of this approach to the case of several variables. Let us expand the function $K(t_1, \ldots, t_k)$ such that

$$K(t_1, \ldots, t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} \in L_2([t, T]^k)$$

using the complete orthonormal system of functions

$$\prod_{l=1}^{k} \Psi_{j_l}(t_l) \sqrt{r(t_l)}, \quad j_l = 0, 1, 2, \ldots, \quad l = 1, \ldots, k$$

in the space $L_2([t, T]^k)$ into the generalized multiple Fourier series.

It is well known that the mentioned generalized multiple Fourier series converges in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \int\limits_{[t,T]^k} \left(K(t_1, \ldots, t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \sqrt{r(t_l)} \right)^2 dt_1 \ldots dt_k =$$

(68)

$$= \lim_{p_1,\ldots,p_k\to\infty} \int\limits_{[t,T]^k} \left(K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \right)^2 \left(\prod_{l=1}^{k} r(t_l) \right) dt_1 \ldots dt_k = 0,$$

where

$$\tilde{C}_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \Bigl(\Psi_{j_l}(t_l) r(t_l) \Bigr) dt_1 \ldots dt_k.$$

Let us consider the following iterated Ito stochastic integrals

$$\tilde{J}[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \sqrt{r(t_k)} \ldots \int\limits_t^{t_2} \psi_1(t_1) \sqrt{r(t_1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{69}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a nonrandom function on $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

So, we obtain the following version of Theorem 1.

Theorem 4 [22]-[25] (also see [21], [36]). *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on the interval $[t, T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x) \geq 0)$ is a complete* *orthonormal system of functions in the space* $L_2([t,T])$*, each function* $\Psi_j(x)\sqrt{r(x)}$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect. 4). Then*

$$\tilde{J}[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{70}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m$*,*

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$*,* $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$*, which satisfies the condition* (5)*,*

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$*.*

Proof. According to Lemmas 1, 3 and (20), (29), we get the following representation w. p. 1

$$\tilde{J}[\psi^{(k)}]_{T,t}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\prod_{l=1}^k\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\tilde{R}_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+
$$

$$
+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.
$$

$$
\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=
$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times
$$

$$
\times\left(\prod_{l=1}^k\tilde{\zeta}_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+\tilde{R}_{T,t}^{p_1,\ldots,p_k}, \tag{71}
$$

where

$$
\tilde{R}_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)\prod_{l=1}^k\sqrt{r(t_l)}-\right.
$$

$$
\left.-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^k\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{72}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us evaluate the remainder $\tilde{R}_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 2 and (30), we have

$$\mathsf{M}\left\{\left(\tilde{R}_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k)\prod_{l=1}^k \sqrt{r(t_l)} - \right.$$

$$\left. - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\prod_{l=1}^k \left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)^2 dt_1\ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\prod_{l=1}^k \Psi_{j_l}(t_l)\right)^2 \left(\prod_{l=1}^k r(t_l)\right) dt_1\ldots dt_k \to 0 \tag{73}$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral (69). Theorem 4 is proved.

Let us formulate the version of Theorem 3.

Theorem 5 [22]-[25], [36]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Moreover, let $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect. 4). Then the estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t} - \tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^k r(t_l)\right)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}^2\right) \tag{74}$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ is the stochastic integral (69), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (70) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; another notations are the same as in Theorem 4.

9. Convergence With Probability 1 of Expansion of Iterated Ito Stochastic Integrals in Theorem 1 for the Case of Multiplicity k $(k\in\mathbb{N})$

This section is written on the base of Sect. 1.7.2 from [22]-[25] and Sect. 6 from [28]. Remind that in a lot of author's publications [7]-[17], [20]-[25] the convergence in Theorem 1 has been considered in different probabilistic senses. For example, the mean-square convergence [7] (2006) (also see [8]-[17], [20]-[25]) and convergence in the mean of degree $2n$ $(n\in\mathbb{N})$ [8] (2007) (also see [9]-[17], [20]-[25]) have been proved. On the examples of specific iterated Ito stochastic integrals of mutiplicities 1 and 2 the convergence with probability 1 has been considered in [8] (2007) (also see [9]-[17], [20]-[25]). However, these examples are narrow particular cases of the iterated Ito stochastic integrals (1).

In this section, we formulate and prove the theorem [22]-[25], [28] on convergence with probability 1 (w. p. 1) of the expansions of iterated Ito stochastic integrals from Theorem 1.

Let us remind the well-known fact from the mathematical analysis, which is connected to existence of iterated limits.

Proposition 1. *Let* $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$\lim_{n,m\to\infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \textit{for any} \quad m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \textit{for any} \quad n.$$

Then, there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Theorem 6 [22]-[25], [28]. *Let* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *are continuously differentiable nonrandom functions on the interval* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \textit{if} \quad p\to\infty$$

w. p. 1, *where* $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ *is the expression on the right-hand side of* (6) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ *for the case* $p_1=\ldots=p_k=p$, *i.e.* (*see Theorem* 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),$$

where $i_1,\ldots,i_k=1,\ldots,m$.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2, \tag{75}$$

where

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\ =\ \prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{76}$$

where $t_1,\ldots,t_k\in[t,T]$ for $k\ge 2$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, $\mathbf{1}_A$ denotes the indicator of the set A,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{77}$$

is the Fourier coefficient.

Using (76), we obtain

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k.$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

If $p_1=\ldots=p_k=p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

and so on.

Let us consider the following lemma.

Lemma 4. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1} \tag{78}$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Proof. Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{79}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Obviously, (79) is the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le\sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2<\infty.$$

Then, the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{80}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{81}$$

and equality of the limits (80) and (81).

Taking into account the above reasoning, we have

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{82}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}$$

exists (see the Parseval equality (75)), then from Proposition 1 we have

$$\sum_{j_{q_1}=0}^{\infty}\ \sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} =$$

$$= \lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} = \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{83}$$

Using (82) and Proposition 1, we obtain

$$\sum_{j_{q_2}=0}^{\infty}\ \sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} =$$

$$= \lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} = \sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{84}$$

Combining (84) and (83), we get

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\ \sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

Repeating the above steps, we complete the proof of Lemma 4.
Further, let us show that for $s=1,\ldots,k$

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{85}$$

Using the arguments which we used when proving Lemma 4, we have

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=0}^{p}\ \sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} \tag{86}$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\}=\{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we have

$$\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\ldots=$$

$$=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}\sum_{j_s=0}^{p}C_{j_k\ldots j_1}^2. \tag{87}$$

Using (86), (87), and Lemma 4, we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-$$

$$-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.$$

The equality (85) is proved.

Using the Parseval equality and Lemma 4, we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$= \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 + \sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} \sum_{j_3=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} +$$

$$+ \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1}^2 = \ldots =$$

$$= \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 + \sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \sum_{j_4=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 + \ldots + \sum_{j_1=0}^{p} \ldots \sum_{j_{k-1}=0}^{p} \sum_{j_k=p+1}^{\infty} C_{j_k \ldots j_1}^2 \le$$

$$\le \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 + \sum_{j_1=0}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=p+1}^{\infty} \sum_{j_4=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 + \ldots + \sum_{j_1=0}^{\infty} \ldots \sum_{j_{k-1}=0}^{\infty} \sum_{j_k=p+1}^{\infty} C_{j_k \ldots j_1}^2 =$$

$$= \sum_{s=1}^{k} \left(\sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 \right). \tag{88}$$

Note that deriving (88) we use the following

$$\sum_{j_1=0}^{p} \ldots \sum_{j_{s-1}=0}^{p} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 \le$$

$$\le \sum_{j_1=0}^{m_1} \ldots \sum_{j_{s-1}=0}^{m_{s-1}} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 \le$$

$$\le \lim_{m_{s-1} \to \infty} \sum_{j_1=0}^{m_1} \ldots \sum_{j_{s-1}=0}^{m_{s-1}} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{m_1} \ldots \sum_{j_{s-2}=0}^{m_{s-2}} \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 \le$$

$$\le \ldots \le$$

$$\leq \sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2,$$

where $m_1,\ldots,m_{s-1}>p$.

Denote

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,$$

where $s=1,\ldots,k-1$.

Let us remind the Dini Theorem, which we will use further.

Theorem (Dini). *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a,b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a,b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a,b]$.*

For $s<k$ due to the Parseval equality and Dini Theorem as well as (85) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$\stackrel{(85)}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=$$

$$\stackrel{\text{(Dini Th.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\int\limits_t^{t_k}\psi_{k-1}^2(t_{k-1})\left(C_{j_{k-2}\ldots j_1}(t_{k-1})\right)^2\times$$

$$\times dt_{k-1}dt_k\leq$$

$$\leq C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=$$

$$\stackrel{\text{(Dini Th.)}}{=} C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =$$

$$\stackrel{\text{(Parseval Eq.)}}{=} C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_{k-2}^2(\theta)\left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le K\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le$$

$$\le \ldots \le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\int\limits_t^T\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$\stackrel{\text{(Dini Th.)}}{=} C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \qquad (89)$$

where constants C, K depend on $T-t$ and constant C_k depends on k and $T-t$.

Let us explane more precisely how we obtain (89). For any function $g(s)\in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty}\left(\int\limits_t^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s)ds=\int\limits_t^{\tau}g^2(s)ds. \qquad (90)$$

The equality (90) has been applied repeatedly when we obtaining (89).

Using the replacement of integrating order in Riemann integrals, we have

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s=$$

$$=\int\limits_t^{\tau}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{\tau}\phi_{j_2}(t_2)\psi_2(t_2)\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1\stackrel{\text{def}}{=}$$

$$\stackrel{\sf def}{=} \tilde{C}_{j_s\ldots j_1}(\tau).$$

For $l=1,\ldots,s$ we will use the following notation

$$\tilde{C}_{j_s\ldots j_l}(\tau,\theta)=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.$$

Using the Parseval equality and Dini Theorem, from (89) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=$$

$$=C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_1}(\tau)\right)^2d\tau=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \tag{91}$$

$$\stackrel{\text{(Dini Th.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \tag{92}$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t_1}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le C_k'\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2d\tau\le$$

$$\le \ldots \le$$

$$\le C_k^{''} \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_{s-1}^2(t_{s-1}) \left(\tilde{C}_{j_s}(\tau, t_{s-1})\right)^2 dt_{s-1} d\tau \le$$

$$\le \tilde{C}_k \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \left(\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta \right)^2 du d\tau, \tag{93}$$

where constants $C_k^{'}$, $C_k^{''}$, $\tilde{C}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (93). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty} \left(\int\limits_{\theta}^{\tau} \phi_j(s) g(s) ds \right)^2 = \sum_{j=0}^{\infty} \left(\int\limits_t^T \mathbf{1}_{\{\theta < s < \tau\}} \phi_j(s) g(s) ds \right)^2 =$$

$$= \int\limits_t^T \left(\mathbf{1}_{\{\theta < s < \tau\}}\right)^2 g^2(s) ds = \int\limits_{\theta}^{\tau} g^2(s) ds. \tag{94}$$

The equality (94) has been applied repeatedly when we obtaining (93).

Let us explane more precisely the passing from (91) to (92) (the same steps have been used when we deriving (93)).

We have

$$\int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau - \sum_{j_2=0}^{n} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau =$$

$$= \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau =$$

$$= \lim_{N \to \infty} \sum_{j=0}^{N-1} \int\limits_t^{\tau_j} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2 dt_1 \Delta\tau_j, \tag{95}$$

where $\{\tau_j\}_{j=0}^{N}$ is the partition of the interval $[t,T]$, which satisfies the condition (5).

Since the non-decreasing functional sequence $u_n(\tau_j, t_1)$ and its limit function $u(\tau_j, t_1)$ are continuous on the interval $[t, \tau_j] \subseteq [t,T]$ with respect to t_1, where

$$u_n(\tau_j, t_1) = \sum_{j_2=0}^{n} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2,$$

$$u(\tau_j,t_1)=\sum_{j_2=0}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2=\int\limits_{t_1}^{\tau_j}\psi_2^2(t_2)\left(\tilde C_{j_s\ldots j_3}(\tau_j,t_2)\right)^2dt_2,$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j,t_1)$ to $u(\tau_j,t_1)$ at the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1. As a result, we obtain

$$\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2<\varepsilon,\quad t_1\in[t,\tau_j] \tag{96}$$

for $n>N(\varepsilon)$ ($N(\varepsilon)$ exists for any $\varepsilon>0$ and it does not depend on t_1).

From (95) and (96) we obtain

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2dt_1\Delta\tau_j\le\varepsilon\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)dt_1\Delta\tau_j=$$

$$=\varepsilon\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)dt_1d\tau. \tag{97}$$

From (97) we get

$$\lim_{n\to\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau=0.$$

This fact completes the proof of passing from (91) to (92).

Let us estimate the integral

$$\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta \tag{98}$$

from (93) for the cases when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the estimates for the integral

$$\int\limits_t^{\tau}\phi_j(\theta)\psi(\theta)d\theta,\quad j\ge p+1, \tag{99}$$

where $\psi(\theta)$ is a continuously differentiable function on the interval $[t,T]$, have been obtained in [35] (see the formulas (54) (55), (60)) or in [27] (see the formulas (57), (58), (63)). The same estimates also can be found in early publications [16], [17], [20], [21] and in [22]-[25] (2020, 2021, 2023).

Let us estimate the integral (98) using the approach from [27], [35].

First, consider the case of Legendre polynomials. Then $\phi_j(s)$ looks as follows

$$\phi_j(\theta) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(\theta - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j \ge 0, \tag{100}$$

where $P_j(x)$ $(j = 0, 1, 2\ldots)$ is a complete orthonormal system of Legendre polynomials in the space $L_2([-1, 1])$.

Further, we have

$$\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta = \frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)} P_j(y)\psi(u(y))dy =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x)) - P_{j-1}(z(x)))\psi(x) - (P_{j+1}(z(v)) - P_{j-1}(z(v)))\psi(v) -$$

$$-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)} ((P_{j+1}(y) - P_{j-1}(y))\psi'(u(y))dy\Biggr), \tag{101}$$

where $x, v \in (t, T)$, $j \ge p + 1$, $u(y)$ and $z(x)$ are defined by the following relations

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(x) = \left(x - \frac{T+t}{2}\right)\frac{2}{T-t},$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (101) we used the following well-known property of the Legendre polynomials

$$\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j = 1, 2, \ldots$$

From (101) and the well-known estimate for the Legendre polynomials

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j \in \mathbb{N}, \tag{102}$$

where constant K does not depend on y and j, it follows that

$$\left|\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + \frac{1}{(1-(z(v))^2)^{1/4}} + C_1\right), \tag{103}$$

where $j \in \mathbb{N}$, $z(x), z(v) \in (-1, 1)$, $x, v \in (t, T)$, constants C, C_1 do not depend on j.

From (103) we obtain

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 < \frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + \frac{1}{(1-(z(v))^2)^{1/2}} + C_3\right), \tag{104}$$

where $j \in \mathbb{N}$, constants C_2, C_3 do not depend on j.

Let us apply (104) for the estimate of the right-hand side of (93). We have

$$\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^1 \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^1\int\limits_{-1}^x \frac{dy}{(1-y^2)^{1/2}}dx + K_2\right) \le$$

$$\le \frac{K_3}{j_s^2}, \tag{105}$$

where $j_s \in \mathbb{N}$, constants K_1, K_2, K_3 are independent of j_s.

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta) = \frac{1}{\sqrt{T-t}}\begin{cases} 1, & j=0\\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r \end{cases} \tag{106}$$

where $r = 1, 2, \ldots$

Using the system of functions (106), we have

$$\int\limits_v^x \phi_{2r-1}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}}\int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$= -\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\cos\frac{2\pi r(x-t)}{T-t} - \psi(v)\cos\frac{2\pi r(v-t)}{T-t} - \right.$$

$$\left. - \int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{107}$$

$$\int\limits_v^x \phi_{2r}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}}\int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$= \sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\sin\frac{2\pi r(x-t)}{T-t} - \psi(v)\sin\frac{2\pi r(v-t)}{T-t} - \right.$$

$$-\int\limits_{v}^{x}\sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\Biggr), \tag{108}$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (107) and (108), we obtain for the trigonometric case

$$\left(\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta\right)^2 \le \frac{C_4}{j^2}, \tag{109}$$

where $j\in\mathbb{N}$, constant C_4 is independent of j.

From (109) we finally have

$$\int\limits_{t}^{T}\int\limits_{t}^{\tau}\left(\int\limits_{u}^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le \frac{K_4}{j_s^2}, \tag{110}$$

where $j_s\in\mathbb{N}$, constant K_4 does not depend on j_s.

Combibing (93), (105) and (110), we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 \le$$

$$\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2}\le\frac{L_k}{p}, \tag{111}$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (88) and (111) we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\le\frac{G_k}{p}, \tag{112}$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use the estimate (60). Using (112) and the estimate (60) for the case $p_1=\ldots=p_k=p$ and $n=2$, we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\}\le$$

$$\le C_{2,k}\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\right)^2\le\frac{H_{2,k}}{p^2}, \tag{113}$$

where

$$C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

and $H_{2,k} = G_k^2 C_{2,k}$.

Note the well known fact.

Lemma 5. *If for the sequence of random variables ξ_p and for some $\alpha > 0$ the number series*

$$\sum_{p=1}^{\infty} \mathsf{M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

Let α and ξ_p in Lemma 5 be chosen as follows

$$\alpha = 4, \quad \xi_p = \left| J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right|.$$

Then from (113) we obtain

$$\sum_{p=1}^{\infty} \mathsf{M}\left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \right)^4 \right\} \le H_{2,k} \sum_{p=1}^{\infty} \frac{1}{p^2} < \infty. \tag{114}$$

Using Lemma 5, from (114) we obtain

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p \to \infty$$

w. p. 1, where (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \Biggr) \tag{115}$$

or (see Theorem 2)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = \, i_{g_{2s}} \ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = \, j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \tag{116}$$

where $i_1,\ldots,i_k=1,\ldots,m$ in (115) and (116). Theorem 6 is proved.

Remark 3. *From Theorem* 3 *and Lemma* 4 *we obtain*

$$\lim_{p_{q_1}\to\infty}\overline{\lim_{p_{q_2}\to\infty}}\ldots\overline{\lim_{p_{q_k}\to\infty}}\ \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\cdot\lim_{p_{q_1}\to 0}\ldots\lim_{p_{q_k}\to\infty}\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)=$$

$$=k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2\right)=0$$

for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$.

At that, $(q_1,\ldots,q_k)$ *is any permutation such that* $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$, $J[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (6) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$, $\overline{\lim}$ *means* lim sup; *another notations are the same as in Theorem* 1.

Remark 4. *Taking into account Theorem* 3 *and the estimate* (112)*, we obtain the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\}\le\frac{k!P_k(T-t)^k}{p},\tag{117}$$

where $i_1,\ldots,i_k=1,\ldots,m$ *and constant* P_k *depends only on* k.

Remark 5. *The estimates* (60) *and* (112) *imply the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{(P_k)^n\,(T-t)^{nk}}{p^n},\tag{118}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ n\in\mathbb{N}$*, and constant* P_k *depends only on* k.

Remark 6. *Consider the question on the rate of convergence w. p.* 1 *in Theorem* 6*. Using the inequality* (118)*, we obtain*

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\}\right)^{1/2n}\le\frac{Q_{n,k}}{\sqrt{p}},\tag{119}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ n\in\mathbb{N},$ *and*

$$Q_{n,k}=k!\ (n(2n-1))^{(k-1)/2}\,((2n-1)!!)^{1/2n}\ \sqrt{P_k}\ (T-t)^{k/2}.$$

According to the Lyapunov inequality, we have

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^n\right\}\right)^{1/n}\le\frac{Q_{n,k}}{\sqrt{p}}\tag{120}$$

for all $n\in\mathbb{N}$. *Following* [79] (*Lemma* 2.1), *we get*

$$\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|=\frac{p^{1/2-\varepsilon}}{p^{1/2-\varepsilon}}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\le$$

$$\le\frac{1}{p^{1/2-\varepsilon}}\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)=\frac{\eta_\varepsilon}{p^{1/2-\varepsilon}}\tag{121}$$

w. p. 1, *where*

$$\eta_\varepsilon=\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)$$

and $\varepsilon>0$ *is fixed.*

For $q>1/\varepsilon,\ q\in\mathbb{N}$ *we obtain* [79] (*see* (120))

$$\mathsf{M}\left\{|\eta_\varepsilon|^q\right\}=\mathsf{M}\left\{\left(\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)\right)^q\right\}=$$

$$=\mathsf{M}\left\{\sup_{p\in\mathbb{N}}\left(p^{(1/2-\varepsilon)q}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right)\right\}\le$$

$$\le\mathsf{M}\left\{\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right\}=$$

$$=\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\mathsf{M}\left\{\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right\}\le$$

$$\le\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\frac{(Q_{q,k})^q}{p^{q/2}}=(Q_{q,k})^q\sum_{p=1}^{\infty}\frac{1}{p^{\varepsilon q}}<\infty.\tag{122}$$

From (121) *we obtain that for all* $\varepsilon > 0$ *there exists a random variable* η_ε *such that the inequality* (121) *is fulfilled w. p. 1 for all* $p \in \mathbb{N}$. *Moreover, from the Lyapunov inequality and* (122)*, we obtain* $\mathsf{M}\left\{|\eta_\varepsilon|^q\right\} < \infty$ *for all* $q \geq 1$.

10. Conclusions

Thus, we obtain the following useful possibilities and modifications of the method based on Theorem 1 (in Sect. 15, we will consider the generalization of Theorems 1, 2 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots, \psi_k(\tau) \in L_2([t,T])$).

1. There is an explicit formula (see (4)) for calculation of expansion coefficients of the iterated Ito stochastic integral (1) with any fixed multiplicity k ($k \in \mathbb{N}$).

2. We have possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (1) [34] (also see [20]-[25]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [62]-[66] but Legendre polynomials.

4. As it turned out [7]-[55] (also see early publications [3]-[6]) it is more convenient to work with Legendre polynomials for building of approximations of the iterated Ito stochastic integrals (1). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions [7]-[55] (also see early publications [3]-[6]). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [22]-[25], [40], [45].

5. The approach to expansion of iterated stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process [62]-[66], [70], [71], [74], [75] leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$) of the iterated Ito stochastic integrals (1). Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since the partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [62] (Sect. 5.8, pp. 202–204), [65] (pp. 438-439), [66] (pp. 82-84), [71] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [63] together with the Wong–Zakai approximation [80]-[82] (see discussion in Sect. 11 of this paper for detail).

6. As we mentioned above, constructing the expansions of iterated Ito stochastic integrals from Theorem 1 we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge w. p. 1 and in the mean of degree $2n$ ($n \in \mathbb{N}$) (see Sect. 6 and 9 from this article).

7. The modification of Theorem 1 for complete orthonormal with weight $r(t_1)\ldots r(t_k) \geq 0$ systems of functions in the space $L_2([t,T]^k)$ ($k \in \mathbb{N}$) (Theorems 4, 18) as well as for some other types of

iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson measures and iterated stochastic integrals with respect to martingales) were obtained in [21], [22]-[25], [36].

8. The adaptation of Theorem 1 for iterated Stratonovich stochastic integrals of multiplicities 1 to 6 was realized in [12]-[17], [20]-[25], [27]-[30], [32], [33], [35], [37], [39]-[44].

9. Application of Theorem 1 and Theorem 12 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [22]-[25] (Chapter 7) and in [46]-[49].

11. Theorem 1 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito stochastic differential equations are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [80], [81], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich stochastic differential equations and not to iterated Ito stochastic integrals and solutions of Ito stochastic differential equations. The piecewise linear approximation as well as the regularization by convolution [80]-[82] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of stochastic differential equations is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [83], [84]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{123}$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (123) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{124}$$

From (124) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{125}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{126}$$

where $i_1,\ldots,i_k=0,1,\ldots,m,\ \ p_1,\ldots,p_k\in\mathbb{N}$,

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{127}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (125).

Let us substitute (125) into (126)

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{128}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

Consider the following iterated Stratonovich stochastic integral

$$\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{129}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuously differentialble nonrandom function on $[t,T]$, $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=\tau$; $i_1,\ldots,i_k=0,1,\ldots,m$.

To best of our knowledge [80]-[82] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [82] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (124) were not considered in [80], [81] (also see [82], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [82] for approximations of the Wiener process based on its series expansion (123) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (128) to the iterated Stratonovich stochastic integral (129) does not follow from the results of the papers [80], [81] (also see [82], Theorems 7.1, 7.2).

From the other hand, Theorems 1 from this paper and the theory built in Chapters 1 and 2 of the monographs [22]-[25] can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (129) of multiplicities 1 to 6 based on the iterated Riemann–Stieltjes integrals (126) and approximation (124) of the Wiener process. At that, the Riemann–Stieltjes integrals (126) of multiplicities 1 to 6 converge (according to Theorems 2.1–2.9 from [22]-[25]) to the appropriate Stratonovich stochastic integrals (129). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (123), (124)) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2,\ \psi_1(s),\ \psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [80]-[82]).

Let $\mathbf{b}_{\Delta}^{(i)}(t),\ t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t,\ t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)},\ i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{130}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (130) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$
= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta \mathbf{f}_{q\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{\Delta^2} \sum_{l=0}^{N-1} \Delta \mathbf{f}_{l\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} \int\limits_{l\Delta}^{(l+1)\Delta} \int\limits_{l\Delta}^{s} d\tau ds =
$$

$$
= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta \mathbf{f}_{q\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta \mathbf{f}_{l\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)}. \tag{131}
$$

Using (131) it is not difficult to show that

$$
\underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau) d\mathbf{b}_{\Delta}^{(i_2)}(s) = \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} + \frac{1}{2} \mathbf{1}_{\{i_1=i_2\}} \int\limits_0^T ds =
$$

$$
= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)}, \tag{132}
$$

where $\Delta \to 0$ if $N \to \infty$ $(N\Delta = T)$.

Obviously, (132) agrees with Theorem 7.1 (see [82], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (123) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0, T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{133}
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (125).

Let us substitute (125) into (133)

$$
\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \sum_{j_1, j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{134}
$$

where

$$
C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (128).

As we noted above, approximations of the Wiener process that are similar to (124) were not considered in [80], [81] (also see Theorems 7.1, 7.2 in [82]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [82] to the case under consideration is not obvious.

Nevertheless, in [62] (Sect. 5.8, pp. 202–204), [65] (pp. 438-439), [66] (pp. 82-84), [71] (pp. 263-264) the authors use (without rigorous proof) the Wong–Zakai approximation [80]-[82] together with the approximation of the Wiener process based on its series expansion.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [22]-[25]. More precisely, using Theorems 2.1, 2.2 [22]-[25] we obtain from (134) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{135}$$

From the other hand, by Theorem 1 from this paper (see (37)) for the case $k=2$ we obtain from (134) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{136}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from the standard relation between Stratonovich and Ito stochastic integrals and (136) we obtain (135).

12. Modification of Theorem 1 for the Case of the Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Ito Stochastic Integrals

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Define the following function on the hypercube $[t,T]^k$

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k),$$

where the function $K(t_1,\ldots,t_k)$ is defined by (2), $s\in(t,T]$ (s is fixed), and $\mathbf{1}_A$ is the indicator of the set A. So we have

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_1<\ldots<t_k<s\}}\psi_1(t_1)\ldots\psi_k(t_k)=$$

$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s\\ \\ 0, & \text{otherwise}\end{cases}, \tag{137}$$

where $k\ge 1$, $t_1,\ldots,t_k\in[t,T]$, and $s\in(t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $\bar{K}(t_1,\ldots,t_k,s)$ defined by (137) is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $\bar{K}(t_1,\ldots,t_k,s)$ $\in L_2([t,T]^k)$ is converging to $\bar{K}(t_1,\ldots,t_k,s)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|\bar{K}(t_1,\ldots,t_k,s)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{138}$$

where

$$C_{j_k\ldots j_1}(s)=\int\limits_{[t,T]^k}\bar{K}(t_1,\ldots,t_k,s)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k=$$

$$=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{139}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Note that

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{140}$$

$$=\int\limits_t^T\mathbf{1}_{\{t_k<s\}}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},$$

where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=0,1,\ldots,m$.

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{141}$$

Theorem 7 [22], [24], [25]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 4). Then*

$$J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{142}$$

where $J[\psi^{(k)}]_{s,t}$ is the iterated Ito stochastic integral (140), $s\in(t,T]$ (s is fixed),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m,$*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (139), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (141).

Proof. Let us consider the multiple stochastic integrals (12), (19). We will write $J[\Phi]_{s,t}^{(k)}$ and $J'[\Phi]_{s,t}^{(k)}$ ($s\in(t,T]$, s is fixed) if the function $\Phi(t_1,\ldots,t_k)$ in (12) and (19) is replaced by the function $\mathbf{1}_{\{t_1,\ldots,t_k<s\}}\Phi(t_1,\ldots,t_k)$.

By analogy with (20), we have

$$J'[\Phi]_{s,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}\mathbf{1}_{\{t_k<s\}}\sum_{(t_1,\ldots,t_k)}\biggl(\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\biggr)\quad \text{w. p. 1}, \tag{143}$$

where $J'[\Phi]_{s,t}^{(k)}$ is defined by (19) and

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$. At the same time permutations $(t_1,\ldots,t_k)$ when summing are performed in (143) only in the expression, which is enclosed in parentheses. Moreover, the nonrandom function $\Phi(t_1,\ldots,t_k)$ is assumed to be continuous in the corresponding closed domains of integration. The case when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domains of integration and bounded at their boundaries is also possible.

Let us write (143) as

$$J'[\Phi]^{(k)}_{s,t}=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum\limits_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,} \tag{144}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed in (144) only in the expression

$$\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}.$$

It is not difficult to notice that (143), (144) can be rewritten in the form (see (21))

$$J'[\Phi]^{(k)}_{s,t}=\sum\limits_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1,} \tag{145}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values

$$\mathbf{1}_{\{t_k<s\}}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}.$$

At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

According to Lemma 1, we have

$$J[\psi^{(k)}]_{s,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{l_k=0}^{N-1}\ldots\sum\limits_{l_1=0}^{l_2-1}\mathbf{1}_{\{\tau_{l_k}<s\}}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{l_k=0}^{N-1}\ldots\sum\limits_{l_1=0}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum\limits_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\mathbf{1}_{\{\tau_{l_k}<s\}}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum\limits_{(t_1,\ldots,t_k)}\left(\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)\quad \text{w. p. 1,} \tag{146}$$

where $K(t_1,\ldots,t_k)$ is defined by (2) and permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression

$$K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}.$$

According to Lemmas 1, 3 and (21), (145), (146), we get the following representation

$$J[\psi^{(k)}]_{s,t} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}(s) \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \left(\phi_{j_1}(t_1) \ldots \phi_{j_k}(t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \right) +$$

$$+ R_{T,t,s}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}(s) \times$$

$$\times \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0 \\ l_q \ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1}) \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} +$$

$$+ R_{T,t,s}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}(s) \left(\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} \phi_{j_1}(\tau_{l_1}) \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +$$

$$+ R_{T,t,s}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}(s) \times$$

$$\times \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right) +$$

$$+ R_{T,t,s}^{p_1,\ldots,p_k} \quad \text{w. p. 1},$$

where

$$R_{T,t,s}^{p_1,\ldots,p_k} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(\mathbf{1}_{\{t_k<s\}} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) \right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} K(t_1,\ldots,t_k) \mathbf{1}_{\{t_k<s\}} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} - \tag{147}$$

$$- \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{148}$$

w. p. 1, where permutations $(t_1,\ldots,t_k)$ when summing in (147) are performed only in the values $\mathbf{1}_{\{t_k<s\}} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time permutations $(t_1,\ldots,t_k)$ when summing in (148) are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. Moreover, the indices near upper limits of integration in the iterated stochastic integrals in (147), (148) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t,s}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 2 and (30), we have

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) \mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times$$

$$\times dt_1 \ldots dt_k, \tag{149}$$

where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral $J[\psi^{(k)}]_{s,t}$ and permutations $(t_1,\ldots,t_k)$ when summing in (149) are performed only in the values $\mathbf{1}_{\{t_k<s\}}$ and $dt_1 \ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (149) are changed correspondently.

Since $K(t_1,\ldots,t_k) \equiv 0$ if the condition $t_1 < \ldots < t_k$ is not fulfilled, then

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) \mathbf{1}_{\{t_k<s\}} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 \times$$

$$\times dt_1 \ldots dt_k, \tag{150}$$

where permutations $(t_1,\ldots,t_k)$ when summing in (150) are performed only in the values $dt_1\ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals in (150) are changed correspondently.

Then from (30), (138), and (150) we obtain

$$\mathsf{M}\left\{\left(R_{T,t,s}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)\mathbf{1}_{\{t_k<s\}}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times$$

$$\times dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\bar{K}(t_1,\ldots,t_k,s)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k\to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral $J[\psi^{(k)}]_{s,t}$. Theorem 7 is proved.

Remark 7. *Obviously from Theorem 7 for the case $s=T$ we obtain the variant of Theorem 1.*

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 7 we obtain

$$J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.$$

Consider particular cases of Theorem 7 for $k=1,\ldots,5$

$$J[\psi^{(1)}]_{s,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}(s)\zeta_{j_1}^{(i_1)},$$

$$J[\psi^{(2)}]_{s,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right),$$

$$J[\psi^{(3)}]_{s,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(s)\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$
J[\psi^{(4)}]_{s,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}(s)\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-
$$
$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

$$
J[\psi^{(5)}]_{s,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}(s)\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$
$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$
$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
$$

where $\mathbf{1}_A$ is the indicator of the set A, $C_{j_k\ldots j_1}(s)$ $(k=1,\ldots,5)$ has the form (139), $s\in(t,T]$ (s is fixed).

Remark 8. *Note that by analogy with the proof of estimate (112) we obtain the following inequality*

$$\int\limits_{[t,T]^k} \bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C^2_{j_k\ldots j_1}(s) \le \frac{G_k(s)}{p}, \tag{151}$$

where $\bar{K}(t_1,\ldots,t_k,s)$ and $C_{j_k\ldots j_1}(s)$ are defined by the equalities (137) and (139), respectively; constant $G_k(s)$ depends on k and $s-t$ ($s\in(t,T]$, s is fixed).

The following obvious modification of Theorem 3 takes place.

Theorem 8 [22], [24], [25]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 4). Then the estimate*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k} \bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}(s)\right) \tag{152}$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{s,t}$ is the stochastic integral (140), $J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (142) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$, $\bar{K}(t_1,\ldots,t_k,s)$ and $C_{j_k\ldots j_1}(s)$ are defined by the equalities (137) and (139), respectively; $s\in(t,T]$ (s is fixed); another notations are the same as in Theorem 1.11.

Remark 9. *Combining the estimates (151) and (152), we obtain*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p,\ldots,p}\right)^2\right\}\le \frac{k!P_k(s-t)^k}{p}, \tag{153}$$

where $i_1,\ldots,i_k=1,\ldots,m$, constant P_k depends only on k; another notations are the same as in (151) and (152).

Remark 10. *The analogue of the estimate (60) for the iterated Ito stochastic integral (140) has the following form*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times$$

$$\times\left(\int\limits_{[t,T]^k}\bar{K}^2(t_1,\ldots,t_k,s)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}(s)\right)^n, \tag{154}$$

where $J[\psi^{(k)}]^{p_1,\ldots,p_k}_{s,t}$ *is the expression on the right-hand side of* (142) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$, $\bar{K}(t_1,\ldots,t_k,s)$ *and* $C_{j_k\ldots j_1}(s)$ *are defined by the equalities* (137) *and* (139)*, respectively;* $s\in(t,T]$ *(s is fixed);* $i_1,\ldots,i_k=1,\ldots,m$*.*

Remark 11. *The estimates* (151) *and* (154) *imply the following inequality*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]^{p,\ldots,p}_{s,t}\right)^{2n}\right\}\le$$

$$\le(k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{(P_k)^n\,(s-t)^{nk}}{p^n},$$

where $i_1,\ldots,i_k=1,\ldots,m$, $n\in\mathbb{N}$, *and constant* P_k *depends only on* k*.*

13. Expansion of Multiple Wiener Stochastic Integral Based on Generalized Multiple Fourier Series

Let us consider the multiple stochastic integral (19)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^k\Delta\mathbf{w}^{(i_l)}_{\tau_{j_l}}\stackrel{\text{def}}{=}J'[\Phi]^{(k)}_{T,t}, \tag{155}$$

where for simplicity we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\{\tau_j\}_{j=0}^N$ is a partition of $[t,T]$, which satisfies the condition (5).

The stochastic integral with respect to the scalar standard Wiener process ($i_1=\ldots=i_k\ne 0$) and similar to (155) was considered in [85] (1951) and is called the multiple Wiener stochastic integral [85]. The case $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ [85] will be considered in Sect. 15.

Consider the following theorem on expansion of the multiple Wiener stochastic integral (155) based on generalized multiple Fourier series.

Theorem 9 [22], [24], [25]. *Suppose that* $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ *is a continuous nonrandom function on* $[t,T]^k$ *and* $\{\phi_j(x)\}_{j=0}^\infty$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$*, each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect. 4). Then the following expansions*

$$J'[\Phi]^{(k)}_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^k\zeta^{(i_l)}_{j_l}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}^{(i_1)}_{\tau_{l_1}}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}^{(i_k)}_{\tau_{l_k}}\right), \tag{156}$$

$$J'[\Phi]_{T,t}^{(k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{157}$$

converging in the mean-square sense are valid, where

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k = \bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\bigr\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),*

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{158}$$

is the Fourier coefficient, $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,\ 1,\ldots,m)$*,* $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$*, which satisfies the condition* (5); $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 2.

Proof. Using Lemma 3 and (20), (21), we get the following representation

$$J'[\Phi]_{T,t}^{(k)} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Bigl(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr) +$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\neq l_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}\quad \text{w. p. 1},$$

where

$$R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series using Lemma 2 and (30). We have

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2\times$$

$$\times dt_1\ldots dt_k=$$

$$= C_k \int\limits_{[t,T]^k} \left(\Phi(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k \to 0$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(k)}$. The expansion (156) is proved. Using (156) and Remark 2, we get the expansion (157) (see Theorem 2). Theorem 9 is proved.

Note that particular cases of the expansion (157) are determined by the equalities (36)–(42), in which the Fourier coefficient $C_{j_k\ldots j_1}$ $(k = 1,\ldots,7)$ has the form (158).

14. Reformulation of Theorems 1, 2, and 9 Using Hermite Polynomials

In [86] it was noted that Theorem 3.1 ([85], p. 162) can be applied to the case of multiple Wiener stochastic integral with respect to components of the multidimensional Wiener process. As a result, Theorems 1, 2, and 9 can be reformulated using Hermite polynomials. Consider this approach using our notations. Note that we derive the formula (163) (see below) in two different ways. One of them is not based on Theorem 3.1 [85] (see the proof of Theorem 20 below for details).

We will say that the condition $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ $(i_1,\ldots,i_k = 0,1,\ldots,m)$ *if* $m_1,\ldots,m_k$ *are multiplicities of the elements* $i_1,\ldots,i_k$*, respectively, i.e.*

$$\{i_1,\ldots,i_k\} = \{\overbrace{i_1,\ldots,i_1}^{m_1},\overbrace{i_2,\ldots,i_2}^{m_2},\ldots,\overbrace{i_r,\ldots,i_r}^{m_r}\},$$

where $r = 1,\ldots,k$*, braces mean an unordered set, and parentheses mean an ordered set. At that,* $m_1+\ldots+m_k = k$*,* $m_1,\ldots,m_k = 0,1,\ldots,k$*, and all elements with nonzero multiplicities are pairwise different.*

In this section, we consider the case $i_1,\ldots,i_k = 0,1,\ldots,m$. Let the condition $(\star\star)$ is fulfilled for the multi-index $(i_1\ldots i_k)$. Then

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)} = J'\Biggl[\underbrace{\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}}_{m_1}\underbrace{\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}}_{m_2}\ldots$$

$$\ldots\underbrace{\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}}_{m_k}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1}\overbrace{i_2\ldots i_2}^{m_2}\ldots\overbrace{i_k\ldots i_k}^{m_k})} \qquad (159)$$

w. p. 1, where $J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}$ is defined by (19) (also see (155)), $\Phi(t_1,\ldots,t_k) = \phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$ (see Sect. 4), $\{j_{g_1},\ldots,j_{g_{m_1+m_2+\ldots+m_k}}\} = \{j_1,\ldots,j_k\}$.

From (159) we have

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)} = J'\left[\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}\right]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1})}\cdot J'\left[\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}\right]_{T,t}^{(\overbrace{i_2\ldots i_2}^{m_2})}\ldots$$

$$\ldots \cdot J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}\right]_{T,t}^{(\overbrace{i_k\ldots i_k}^{m_k})} \tag{160}$$

w. p. 1, where we suppose that the condition $(\star\star)$ is fulfilled for the multi-index $(i_1\ldots i_k)$ and

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}\stackrel{\text{def}}{=}1\quad\text{for}\quad m_l=0, \tag{161}$$

braces mean an unordered set, and parentheses mean an ordered set. The detailed proof of the equality (160) will be given in Sect. 18 (see the proof of Theorem 20).

Let us consider the following multiple Wiener stochastic integral

$$J'\left[\phi_{j_{m_1+m_2+\ldots+m_{l-1}+1}}\ldots\phi_{j_{m_1+m_2+\ldots+m_l}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}\quad(m_l>0),$$

where we suppose that

(162)
$$\left\{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}},\ldots,j_{g_{m_1+m_2+\ldots+m_l}}\right\}=\left\{\underbrace{j_{h_{1,l}},\ldots,j_{h_{1,l}}}_{n_{1,l}},\underbrace{j_{h_{2,l}},\ldots,j_{h_{2,l}}}_{n_{2,l}},\ldots,\underbrace{j_{h_{d_l,l}},\ldots,j_{h_{d_l,l}}}_{n_{d_l,l}}\right\},$$

where $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$. Note that the numbers $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ depend on $(i_1,\ldots,i_k)$ and the numbers $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l depend on $\{j_1,\ldots,j_k\}$. Moreover, $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$.

Using Theorem 3.1 [85], we get w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\quad(m_l>0), \tag{163}$$

where $H_n(x)$ is the Hermite polynomial of degree n

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

or

$$H_n(x)=n!\sum_{m=0}^{[n/2]}\frac{(-1)^m x^{n-2m}}{m!(n-2m)!2^m}\quad(n\in\mathbb{N}), \tag{164}$$

and $\zeta_j^{(i)}$ $(i=0,1,\ldots,m,\ j=0,1,\ldots)$ is defined by (7).

For example,

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=x^2-1,$$
$$H_3(x)=x^3-3x,\quad H_4(x)=x^4-6x^2+3,$$
$$H_5(x)=x^5-10x^3+15x.$$

From (161) and (163) we obtain w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases},\tag{165}$$

where $\mathbf{1}_A$ denotes the indicator of the set A.

Using (160) and (165), we get w. p. 1

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right),\tag{166}$$

where notations are the same as in (162) and (163).

The equality (166) allows us to reformulate Theorems 1, 2, and 9 using the Hermite polynomials.

Theorem 10 [22], [24], [25] (reformulation of Theorems 1 and 2). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *and the condition* (162) *is also fulfilled. Furthermore, let every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 4). *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)\tag{167}$$

converging in the mean-square sense is valid, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is the iterated Itô stochastic integral* (1); $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;$ $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;$ $d_l=1,\ldots,m_l;$ $l=1,\ldots,k;$ $m_1+\ldots+m_k=k;$ *the numbers* $m_1,\ldots,m_k,$ $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l},$ $h_{1,l},\ldots,h_{d_l,l},$ d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (164); *another notations are the same as in Theorem* 1.

Theorem 11 [22], [24], [25] (reformulation of Theorems 9). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *and the condition* (162) *is also fulfilled. Furthermore, let* $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ *is a continuous nonrandom function on* $[t,T]^k$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 4*). Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)$$

converging in the mean-square sense is valid, where we denote the multiple Wiener stochastic integral (155) *as* $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$; $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;$ $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;$ $d_l=1,\ldots,m_l;$ $l=1,\ldots,k;$ $m_1+\ldots+m_k=k;$ *the numbers* $m_1,\ldots,m_k,$ $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l},$ $h_{1,l},\ldots,h_{d_l,l},$ d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (164); *another notations are the same as in Theorem* 9.

From (165) we have w. p. 1

$$J'[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{k}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=\begin{cases}H_k\left(\zeta_{j_1}^{(i_1)}\right), & \text{if}\quad i_1\ne 0\\ \\ \left(\zeta_{j_1}^{(0)}\right)^k, & \text{if}\quad i_1=0\end{cases}\qquad (k>0). \tag{168}$$

Let us show how the relation (168) can be obtained from Theorem 2. To prove (168) using Theorem 2 we choose $i_1=\ldots=i_k$ and $j_1=\ldots=j_k$ $(i_1=0,1,\ldots,m)$ in the following formula (this formula follows from a comparison of (35) and (45) or can be obtained using the recurrence relation (289))

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{169}$$

w. p. 1, where notations are the same as in Theorem 2.

The case $i_1=0$ of (168) is obvious. Simple combinatorial reasoning shows that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}=$$

$$=\frac{C_k^2\cdot C_{k-2}^2\cdot\ldots\cdot C_{k-(r-1)2}^2}{r!}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r},\tag{170}$$

where $i_1=\ldots=i_k,\ \ j_1=\ldots=j_k\ (i_1=1,\ldots,m)$, and

$$C_k^l=\frac{k!}{l!(k-l)!}$$

is the binomial coefficient.

We have

$$\frac{C_k^2\cdot C_{k-2}^2\cdot\ldots\cdot C_{k-(r-1)2}^2}{r!}=\frac{k!}{r!(k-2r)!2^r}.\tag{171}$$

Combining (169), (170), and (171), we get w. p. 1

$$J'[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{k}]_{T,t}^{(\overbrace{i_1\ldots i_1}^{k})}=\left(\zeta_{j_1}^{(i_1)}\right)^k+k!\sum_{r=1}^{[k/2]}\frac{(-1)^r}{r!(k-2r)!2^r}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r}=$$

$$=k!\sum_{r=0}^{[k/2]}\frac{(-1)^r}{r!(k-2r)!2^r}\left(\zeta_{j_1}^{(i_1)}\right)^{k-2r}=H_k\left(\zeta_{j_1}^{(i_1)}\right).$$

The relation (168) is proved using (169).

From (166) and (169) we obtain the following equality for multiple Wiener stochastic integral

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)\tag{172}$$

w. p. 1, where notations are the same as in Theorem 2 and (162), (163).

Let us make a remark about how it is possible to obtain the formula (163) without using Theorem 3.1 [85]. Consider the set of polynomials $H_n(x,y)$, $n=0,1,\ldots$ defined by [88]

(173) $$H_n(x,y)=\left(\frac{d^n}{d\alpha^n}e^{\alpha x-\alpha^2 y/2}\right)\Bigg|_{\alpha=0}\qquad (H_0(x,y)\stackrel{\sf def}{=}1).$$

It is well known that polynomials $H_n(x,y)$ are connected with the Hermite polynomials (164) by the formula [88]

(174) $$H_n(x,y)=y^{n/2}H_n\left(\frac{x}{\sqrt{y}}\right)=n!\sum_{i=0}^{[n/2]}\frac{(-1)^i x^{n-2i}y^i}{i!(n-2i)!2^i}.$$

For example,

$$H_1(x,y)=x,$$
$$H_2(x,y)=x^2-y,$$
$$H_3(x,y)=x^3-3xy,$$
$$H_4(x,y)=x^4-6x^2y+3y^2,$$
$$H_5(x,y)=x^5-10x^3y+15xy^2.$$

From (164) and (174) we get

(175) $$H_n(x,1)=H_n(x).$$

Obviously, without loss of generality, we can write

(176) $$(j_1\ldots j_k)=\bigl(\underbrace{j_1\ldots j_1}_{m_1}\ \underbrace{j_2\ldots j_2}_{m_2}\ \ldots\ \underbrace{j_r\ldots j_r}_{m_r}\bigr),$$

where $m_1+\ldots+m_r=k,\ \ m_1,\ldots,m_r=1,\ldots,k,\ \ r=1,\ldots,k,\ \ k>0$, and $j_1,\ldots,j_r$ are pairwise different.

Analyzing the proof of Theorem 1 and using (236), (259) (see the proof of Theorem 20 below), we can notice that (we suppose that the condition (176) is fulfilled)

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_g;\ q\ne g;\ q,g=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_{m_1}=0\\ l_q\ne l_g;\ q\ne g;\ q,g=1,\ldots,m_1}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\times$$

$$\times\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_{m_1+1},\ldots,l_{m_1+m_2}=0\\ l_q\ne l_g;\ q\ne g;\ q,g=m_1+1,\ldots,m_1+m_2}}^{N-1}\phi_{j_2}(\tau_{l_{m_1+1}})\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\times$$

$$\ldots$$

$$\times \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_{k-m_r+1},\ldots,l_k=0\\ l_q\ne l_g;\ q\ne g;\ q,g=k-m_r+1,\ldots,k}}^{N-1} \phi_{j_r}(\tau_{l_{k-m_r+1}})\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\sum_{l_{m_1}=0}^{N-1}\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)} -\right.$$

$$\left. - \sum_{(l_1,\ldots,l_{m_1})\in \mathrm{G}'_{1,m_1}} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\right)\times$$

$$\times \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{m_1+1}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\sum_{l_{m_1+m_2}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)} -\right.$$

$$\left. - \sum_{(l_{m_1+1},\ldots,l_{m_1+m_2})\in \mathrm{G}'_{m_1+1,m_1+m_2}} \phi_{j_2}(\tau_{l_{m_1}+1})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\right)\times$$

$$\ldots$$

$$\times \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{k-m_r+1}=0}^{N-1}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\sum_{l_k=0}^{N-1}\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)} -\right.$$

$$\left. - \sum_{(l_{k-m_r+1},\ldots,l_k)\in \mathrm{G}'_{k-m_r+1,k}} \phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{k-m_r+1}}^{(i_1)}\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}\right),$$

where the set $\mathrm{G}'_{m,n}$ is defined according to the same rule as the set G_k in (6). However, the elements of the set $\mathrm{G}'_{m,n}$ are the numbers $l_m,\ldots,l_n$ $(n>m)$, while the elements of the set G_k are the numbers $l_1,\ldots,l_k$.

We have (see the proof of Theorem 1) w. p. 1 $(i_1\ne 0)$

$$\underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\sum_{l_{m_1}=0}^{N-1}\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)} -\right.$$

$$\left. - \sum_{(l_1,\ldots,l_{m_1})\in \mathrm{G}'_{1,m_1}} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_1}(\tau_{l_{m_1}})\Delta\mathbf{w}_{\tau_{l_{m_1}}}^{(i_1)}\right) =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \left(\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1} + \sum_{r=1}^{[m_1/2]}(-1)^r\times\right.$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{m_1-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{m_1-2r}\}=\{1,2,\ldots,m_1\}}} \left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r \times
$$

$$
\left.\times\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}\right) =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \left(\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1} + \sum_{r=1}^{[m_1/2]}\frac{(-1)^r m_1!}{r!(m_1-2r)!2^r}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r \times\right.
$$

$$
\left.\times\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}\right) =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{r=0}^{[m_1/2]}\frac{(-1)^r m_1!}{r!(m_1-2r)!2^r}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)^r\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^{m_1-2r}\right) =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_1}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right),
$$

where notations are the same as in Theorems 1, 2.

Similarly we get w. p. 1

$$
\underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{m_1+1}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1+1}})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\sum_{l_{m_1+m_2}=0}^{N-1}\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}-\right.
$$

$$
\left.-\sum_{(l_{m_1+1},\ldots,l_{m_1+m_2})\in \mathrm{G}'_{m_1+1,m_1+m_2}}\phi_{j_2}(\tau_{l_{m_1+1}})\Delta\mathbf{w}_{\tau_{l_{m_1+1}}}^{(i_1)}\ldots\phi_{j_2}(\tau_{l_{m_1+m_2}})\Delta\mathbf{w}_{\tau_{l_{m_1+m_2}}}^{(i_1)}\right) =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}}\ H_{m_2}\left(\sum_{l_1=0}^{N-1}\phi_{j_2}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_2}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right),
$$

$$
\ldots
$$

$$
\underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_{k-m_r+1}=0}^{N-1}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{l_{k-m_r+1}}}^{(i_1)}\ldots\sum_{l_k=0}^{N-1}\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}-\right.
$$

$$-\sum_{(l_{k-m_r+1},\ldots,l_k)\in\mathrm{G}'_{k-m_r+1,k}}\phi_{j_r}(\tau_{l_{k-m_r+1}})\Delta\mathbf{w}_{\tau_{k-m_r+1}}^{(i_1)}\ldots\phi_{j_r}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_1)}\Biggr)=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\ H_{m_r}\left(\sum_{l_1=0}^{N-1}\phi_{j_r}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_r}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right).$$

Then

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\ H_{m_1}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_1}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)\times$$

$$\times\underset{N\to\infty}{\text{l.i.m.}}\ H_{m_2}\left(\sum_{l_1=0}^{N-1}\phi_{j_2}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_2}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right)\times\ldots$$

$$\ldots\times\underset{N\to\infty}{\text{l.i.m.}}\ H_{m_r}\left(\sum_{l_1=0}^{N-1}\phi_{j_r}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)},\ \sum_{l_1=0}^{N-1}\phi_{j_r}^2(\tau_{l_1})\left(\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\right)^2\right) \tag{177}$$

w. p. 1 for $i_1\ne 0$ and

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(0\ldots 0)}=$$

$$=\lim_{N\to\infty}\left(\sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\Delta\tau_{l_1}\right)^{m_1}\ldots\left(\sum_{l_r=0}^{N-1}\phi_{j_r}(\tau_{l_r})\Delta\tau_{l_r}\right)^{m_r}=$$

$$=\left(\int_t^T\phi_{j_1}(s)ds\right)^{m_1}\ldots\left(\int_t^T\phi_{j_r}(s)ds\right)^{m_r}=$$

$$=\left(\zeta_{j_1}^{(0)}\right)^{m_1}\ldots\left(\zeta_{j_r}^{(0)}\right)^{m_r} \tag{178}$$

for $i_1=0$, where we suppose that the condition (176) is fulfilled; also we use in (177) and (178) the same notations as in the proof of Theorem 1.

Applying (174), (175), Lemma 3, and Remark 2 to the right-hand side of (177), we finally obtain w. p. 1

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_1)}=H_{m_1}\left(\int_t^T\phi_{j_1}(s)d\mathbf{w}_s^{(i_1)},\int_t^T\phi_{j_1}^2(s)ds\right)\times$$

$$\times H_{m_2}\left(\int\limits_t^T \phi_{j_2}(s)d\mathbf{w}_s^{(i_1)},\int\limits_t^T \phi_{j_2}^2(s)ds\right)\ldots H_{m_r}\left(\int\limits_t^T \phi_{j_r}(s)d\mathbf{w}_s^{(i_1)},\int\limits_t^T \phi_{j_r}^2(s)ds\right)=$$

$$=H_{m_1}\left(\zeta_{j_1}^{(i_1)},1\right)H_{m_2}\left(\zeta_{j_2}^{(i_1)},1\right)\ldots H_{m_r}\left(\zeta_{j_r}^{(i_1)},1\right)=$$

$$=H_{m_1}\left(\zeta_{j_1}^{(i_1)}\right)H_{m_2}\left(\zeta_{j_2}^{(i_1)}\right)\ldots H_{m_r}\left(\zeta_{j_r}^{(i_1)}\right)$$

for $i_1 \neq 0$, where we suppose that the condition (176) is fulfilled. An equality similar to (163) was proved without using Theorem 3.1 [85].

Consider particular cases of the equality (172) for $k = 1,\ldots,4$ and $i_1,\ldots,i_4 = 1,\ldots,m$ (see (36)–(39)). We have w. p. 1

$$J'[\phi_{j_1}]_{T,t}^{(i_1)}=\zeta_{j_1}^{(i_1)}=H_1\left(\zeta_{j_1}^{(i_1)}\right);$$

$$J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}=$$

$$=\begin{cases}H_2\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_2)}\right), & \text{if}\quad i_1=i_2,\ j_1=j_2\\ H_1\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_2)}\right), & \text{otherwise}\end{cases};\tag{179}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_1i_1)}=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_1)}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-$$

$$-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_1)}=$$

$$=\begin{cases}H_3\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if}\quad j_1=j_2=j_3\\ H_2\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if}\quad j_1=j_2\neq j_3\\ H_1\left(\zeta_{j_1}^{(i_1)}\right)H_2\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if}\quad j_2=j_3\neq j_1\\ H_0\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_2\left(\zeta_{j_3}^{(i_1)}\right), & \text{if}\quad j_1=j_3\neq j_2\\ H_1\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if}\quad j_1\neq j_2,\ j_2\neq j_3, j_1\neq j_3\end{cases};\tag{180}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_3)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} =$$

$$= H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right),$$

where i_1, i_2, i_3 are pairwise different;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_1 i_3)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} =$$

$$= \left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}}\right)\zeta_{j_3}^{(i_3)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_3}]_{T,t}^{(i_3)} =$$

$$= \begin{cases} H_2\left(\zeta_{j_1}^{(i_1)}\right) H_0\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), & \text{if} \quad j_1 = j_2 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right), & \text{if} \quad j_1 \neq j_2 \end{cases},$$

where $i_1 = i_2 \neq i_3$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_2)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} - \mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} =$$

$$= \zeta_{j_1}^{(i_1)}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)} - \mathbf{1}_{\{j_2=j_3\}}\right) = J'[\phi_{j_1}]_{T,t}^{(i_1)} J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)} =$$

$$= \begin{cases} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_2\left(\zeta_{j_2}^{(i_2)}\right) H_0\left(\zeta_{j_3}^{(i_2)}\right), & \text{if} \quad j_2 = j_3 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_2)}\right), & \text{if} \quad j_1 \neq j_2 \end{cases},$$

where $i_1 \neq i_2 = i_3$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_2 i_1)} = \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)} =$$

$$= \zeta_{j_2}^{(i_2)}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right) = J'[\phi_{j_2}]_{T,t}^{(i_2)} J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} =$$

$$= \begin{cases} H_2\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_0\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 = j_3 \\ \\ H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_1)}\right), & \text{if} \quad j_1 \neq j_3 \end{cases},$$

where $i_1 = i_3 \neq i_2$;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1 i_1)}=\prod_{l=1}^{4}\zeta_{j_l}^{(i_1)}-$$

$$-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_1)}\zeta_{j_4}^{(i_1)}-$$

$$-\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-$$

$$-\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)}-\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{j_2=j_3\}}=$$

$$=\begin{cases}
H_4\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{I})\\
H_1\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right)H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{II})\\
H_2\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right)H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{III})\\
H_0\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_2\left(\zeta_{j_3}^{(i_1)}\right)H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{IV})\\
H_0\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right)H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{V})\\
H_1\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_2\left(\zeta_{j_3}^{(i_1)}\right)H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VI})\\
H_1\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right)H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VII})\\
H_1\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{VIII})\\
H_3\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_1\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{IX})\\
H_1\left(\zeta_{j_1}^{(i_1)}\right)H_3\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{X})\\
H_0\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_1\left(\zeta_{j_3}^{(i_1)}\right)H_3\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XI})\\
H_0\left(\zeta_{j_1}^{(i_1)}\right)H_1\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_3\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XII})\\
H_2\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_2\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XIII})\\
H_2\left(\zeta_{j_1}^{(i_1)}\right)H_2\left(\zeta_{j_2}^{(i_1)}\right)H_0\left(\zeta_{j_3}^{(i_1)}\right)H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XIV})\\
H_2\left(\zeta_{j_1}^{(i_1)}\right)H_0\left(\zeta_{j_2}^{(i_1)}\right)H_2\left(\zeta_{j_3}^{(i_1)}\right)H_0\left(\zeta_{j_4}^{(i_1)}\right), & \text{if} \quad (\mathrm{XV})
\end{cases},$$

where $H_n(x)$ is the Hermite polynomial (164) of degree n and (I)–(XV) are the following conditions

(I). $j_1 = j_2 = j_3 = j_4$,
(II). j_1, j_2, j_3, j_4 are pairwise different,
(III). $j_1 = j_2 \ne j_3, j_4;\ j_3 \ne j_4$,
(IV). $j_1 = j_3 \ne j_2, j_4;\ j_2 \ne j_4$,
(V). $j_1 = j_4 \ne j_2, j_3;\ j_2 \ne j_3$,
(VI). $j_2 = j_3 \ne j_1, j_4;\ j_1 \ne j_4$,
(VII). $j_2 = j_4 \ne j_1, j_3;\ j_1 \ne j_3$,
(VIII). $j_3 = j_4 \ne j_1, j_2;\ j_1 \ne j_2$,
(IX). $j_1 = j_2 = j_3 \ne j_4$,
(X). $j_2 = j_3 = j_4 \ne j_1$,
(XI). $j_1 = j_2 = j_4 \ne j_3$,
(XII). $j_1 = j_3 = j_4 \ne j_2$,
(XIII). $j_1 = j_2 \ne j_3 = j_4$,
(XIV). $j_1 = j_3 \ne j_2 = j_4$,
(XV). $j_1 = j_4 \ne j_2 = j_3$.

Moreover, from (160) we have w. p. 1

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_4)} = H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right),$$

where $i_1, \ldots, i_4$ are pairwise different;

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_4)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_3}^{(i_3)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right) \quad (i_1 = i_2 \ne i_3, i_4;\ i_3 \ne i_4); \tag{181}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_4)} = J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right) \quad (i_1 = i_3 \ne i_2, i_4;\ i_2 \ne i_4); \tag{182}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_1)} = J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right) \quad (i_1 = i_4 \ne i_2, i_3;\ i_2 \ne i_3); \tag{183}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_4)} = J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_4}^{(i_4)}\right) \quad (i_2 = i_3 \ne i_1, i_4;\ i_1 \ne i_4); \tag{184}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_2)} = J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_3}^{(i_3)}\right) \quad (i_2 = i_4 \ne i_1, i_3;\ i_1 \ne i_3); \tag{185}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_3)} = J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3 i_3)} H_1\left(\zeta_{j_1}^{(i_1)}\right) H_1\left(\zeta_{j_2}^{(i_2)}\right) \quad (i_3 = i_4 \ne i_1, i_2;\ i_1 \ne i_2); \tag{186}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1 i_4)} = J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_4}^{(i_4)}\right) \quad (i_1 = i_2 = i_3 \ne i_4); \tag{187}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_2)} = J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2 i_2 i_2)} H_1\left(\zeta_{j_1}^{(i_1)}\right) \quad (i_2 = i_3 = i_4 \neq i_1); \tag{188}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_1)} = J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_3}^{(i_3)}\right) \quad (i_1 = i_2 = i_4 \neq i_3); \tag{189}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_1)} = J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_1)} H_1\left(\zeta_{j_2}^{(i_2)}\right) \quad (i_1 = i_3 = i_4 \neq i_2); \tag{190}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_1 i_3 i_3)} = J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3 i_3)} \quad (i_1 = i_2 \neq i_3 = i_4); \tag{191}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_1 i_2)} = J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2 i_2)} \quad (i_1 = i_3 \neq i_2 = i_4); \tag{192}$$

$$J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_2 i_1)} = J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1 i_1)} J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_2)} \quad (i_1 = i_4 \neq i_2 = i_3). \tag{193}$$

Note that the right-hand sides of (181)–(193) contain multiple Wiener stochastic integrals of multiplicities 2 and 3. These integrals are considered in detail in (179), (180).

It should be noted that the formulas (45) (Theorem 2) and (167) (Theorem 10) are interesting from various points of view. The formulas (36)–(41) (these formulas are particular cases of (45) for $k = 1, \ldots, 6$) are convenient for numerical modeling of iterated Ito stochastic integrals of multiplicities 1 to 6. For example, in [56] and [57], approximations of iterated Ito stochastic integrals of multiplicities 1 to 6 in the Python programming language were successfully implemented using (36)–(41) and Legendre polynomials.

On the other hand, the equality (167) is interesting by a number of reasons. Firstly, this equality connects Ito's results on multiple Wiener stochastic integrals ([85], Theorem 3.1) with the theory of mean-square approximation of iterated Ito stochastic integrals presented in this paper and in the book [22]. Secondly, the equality (167) is based on the Hermite polynomials, which have the orthogonality property on $\mathbb{R}$ with a Gaussian weight. This feature opens up new possibilities in the study of iterated Ito stochastic integrals.

15. A Generalization of Theorems 1, 2, 10, and 11 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$, $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$

In this section, we will use the definition of the multiple Wiener stochastic integral from [85], [87] to generalize Theorems 1, 2, 10, and 11 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$, $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$.

Consider the following step function on the hypercube $[t,T]^k$

$$\Phi_N(t_1,\ldots,t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{194}$$

where $a_{l_1\ldots l_k}\in\mathbb{R}$ and such that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$,

$$\mathbf{1}_A(\tau)=\begin{cases}1 & \text{if } \tau\in A\\ \\ 0 & \text{otherwise}\end{cases},$$

$N\in\mathbb{N}$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (5):

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{195}$$

Let us define the multiple Wiener stochastic integral for $\Phi_N(t_1,\ldots,t_k)$ [85], [87]

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}\stackrel{\sf def}{=}\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{196}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

It is known (see [87], Lemma 9.6.4) that for any $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ there exists a sequence of step functions $\Phi_N(t_1,\ldots,t_k)$ of the form (194) such that

$$\lim_{N\to\infty}\int\limits_{[t,T]^k}\left(\Phi(t_1,\ldots,t_k)-\Phi_N(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=0. \tag{197}$$

We have

$$\Phi_N(t_1,\ldots,t_k)=\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k)=$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{198}$$

where permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$ (recall that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$).

Using (198), we get

$$\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{199}$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$= \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\neq l_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1} a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} =$$

(200) $$= J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1},$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$ and permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1 < l_2 < \ldots < l_k$. At the same time the indices near upper limits of integration in the iterated stochastic integrals in (199) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$ (see (199)). In addition, the multiple Wiener stochastic integral $J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}$ is defined by (196) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

Using (197), (200), Lemma 2 for $\Phi(t_1,\ldots,t_k)\in L_2(D_k)$, and (30) for Lebesgue integrals, we have

$$\mathsf{M}\left\{\left(J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} - J'[\Phi_M]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k =$$

$$= C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k =$$

$$= C_k\left\|\Phi_N-\Phi_M\right\|^2_{L_2([t,T]^k)} \le$$

$$\le 2C_k\left(\left\|\Phi_N-\Phi\right\|^2_{L_2([t,T]^k)}+\left\|\Phi-\Phi_M\right\|^2_{L_2([t,T]^k)}\right)^2 \to 0$$

if $N,M\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

Thus, there exists the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}.$$

We will define the multiple Wiener stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the formula [85], [87]

(201) $$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\mathsf{def}}{=} \underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)},$$

where $\Phi_N(t_1,\ldots,t_k)$ is defined by (194), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

It is easy to see that the above definition coincides with (19) if the function $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is continuous in the hypercube $[t,T]^k$.

Let us prove the following equality

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1,}\tag{202}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (201) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

The equality (202) has already been proved for the case $\Phi(t_1,\ldots,t_k)=\Phi_N(t_1,\ldots,t_k)$ (see (200)). From (200) we have

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1.}\tag{203}$$

Passing to the limit $\underset{N\to\infty}{\text{l.i.m.}}$ in the equality (203), we obtain

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1.}\tag{204}$$

Using Lemma 2 for $\Phi(t_1,\ldots,t_k)\in L_2(D_k)$, (30) for Lebesgue integrals, and (197), we get

$$\mathsf{M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k\ \to 0 \tag{205}$$

if $N\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

The relations (204) and (205) prove the equality (202). Using (202) and the isometry property of the Ito stochastic integral, we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \tag{206}$$

where $K=K(t_1,\ldots,t_k)$ is defined by (2), i.e.

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{207}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Applying (206) and the linearity property of the Ito stochastic integral, we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \tag{208}$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)\stackrel{\rm def}{=}K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l) \tag{209}$$

and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{210}$$

is the Fourier coefficient corresponding to $K(t_1,\ldots,t_k)$.

Using the Ito formula, we have

$$\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_q}^{(i_q)}\times$$

$$\times\sum_{(j_1',\ldots,j_n')}\int\limits_t^T\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(g)}\ldots\mathbf{w}_{t_n'}^{(g)}=$$

$$=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(g)}\ldots d\mathbf{w}_{t_n'}^{(g)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} \tag{211}$$

w. p. 1, where $g=0$ or $g=1$, $n,q\in\mathbb{N}$, $i_1,\ldots,i_q\neq 0,\ 1$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_d in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_d in the permutation $(i_1,\ldots,i_k)$.

The detailed proof of (211) will be given in Sect. 18 (see the proof of Theorem 20). The equality (211) means that (see (202))

$$J'[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J'[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(g\ldots g)}=$$

$$=J'[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q g\ldots g)} \tag{212}$$

w. p. 1, where $g=0$ or $g=1$, $n,q\in\{0\}\cup\mathbb{N}$, $i_1,\ldots,i_q\neq 0,\ 1$, and $J'[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\stackrel{\sf def}{=}1$ for $q=0$.

Using the equality (212), we get (160) for the case of an arbitrary complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions in $L_2([t,T])$.

Using Theorem 9.6.9 [87] (also see [85], Theorem 3.1) and (172) (also see Theorem 21 below), we get

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)=$$

$$
= \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \tag{213}
$$

w. p. 1, where notations are the same as in Theorems 2 and 10; the multiple Wiener stochastic integral $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (201).

Again applying (202), we have

$$
J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int_t^T \ldots \int_t^{t_2} \Biggl(K(t_1,\ldots,t_k) -
$$

$$
- \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \Biggr) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{214}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (201).

According to Lemma 2 for $\Phi(t_1,\ldots,t_k) \in L_2(D_k)$, (3), and (30) for Lebesgue integrals, we have

$$
\mathsf{M}\left\{ \left(J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \right)^2 \right\} \le
$$

$$
\le C_k \sum_{(t_1,\ldots,t_k)} \int_t^T \ldots \int_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =
$$

$$
= C_k \int_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k \to 0 \tag{215}
$$

if $p_1,\ldots,p_k \to \infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$.

Thus, the following theorem is proved.

Theorem 12 [22], [25] (generalization of Theorems 1, 2, and 10). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 14*) and the condition* (162) *is also fulfilled.*

Furthermore, let $\psi_l(\tau) \in L_2([t,T])$ $(l=1,\ldots,k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansions*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l=0\end{cases}\right), \tag{216}$$

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right) \tag{217}$$

converging in the mean-square sense are valid, where $[x]$ *is an integer part of a real number* x*;* $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$*;* $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$*;* $d_l=1,\ldots,m_l$*;* $l=1,\ldots,k$*;* $m_1+\ldots+m_k=k$*; the numbers* $m_1,\ldots,m_k$*,* $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$*,* $h_{1,l},\ldots,h_{d_l,l}$*,* d_l *depend on* $\{j_1,\ldots,j_k\}$*; moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$*;* $H_n(x)$ *is the Hermite polynomial* (164)*; another notations are the same as in Theorems* 1, 2, and 10.

Replacing the function $K(t_1,\ldots,t_k)$ by $\Phi(t_1,\ldots,t_k)$ we get the following theorem.

Theorem 13 [22], [25] (generalization of Theorem 11). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 14*) and the condition* (162) *is also fulfilled. Furthermore, let* $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansions*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l=0\end{cases}\right), \tag{218}$$

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)$$

converging in the mean-square sense are valid, where $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$; $m_1+\ldots+m_k=k$; *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; *the multiple Wiener stochastic integral* $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (201); $H_n(x)$ *is the Hermite polynomial* (164); *another notations are the same as in Theorem* 9, 11.

It should be noted that an analogue of the expansion (218) was obtained in [86] for the case $i_1,\ldots,i_k=1,\ldots,m$. The proof in [86] is different from the proof given in this section and Sect. 18. Note that the results of work [86], as well as the results of this article, are based on our idea [7] (2006) on the expansion of the kernel (2) (or $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$) into a generalized multiple Fourier series (see [7], Chapter 5, Theorem 5.1, pp. 235-245 or [22], Sect. 1.1.3 for details).

16. Exact Calculation of the Mean-Square Error in Theorems 1, 2, and 12

In this section, we will use the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process to generalize theorem on the exact calculation of the mean-square error in Theorems 1, 2. More precisely, we will generalize the following theorem.

Theorem 14 [22], [25], [34]. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 4*). Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \quad (219)$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right), \quad (220)$$

$$S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)}, \quad (221)$$

the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form* (4),

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)} \tag{222}$$

are independent standard Gaussian random variables for various i *or* j $(i = 1, \ldots, m)$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1, \ldots, j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1, \ldots, i_k)$ *(see* (219)); *another notations are the same as in Theorem* 1.

Let us generalize Theorem 14 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$.

Theorem 15 [22], [25]. *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t, T])$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^p\right)^2\right\} = \int\limits_{[t,T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k -$$

$$- \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \mathsf{M}\left\{J[\psi^{(k)}]_{T,t} \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{223}$$

where

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^p = \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)}, \tag{224}$$

$J'[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1 \ldots i_k)}$ *is the multiple Wiener stochastic integral defined by* (201), *the Fourier coefficient* $C_{j_k \ldots j_1}$ *has the form* (210), $K(t_1, \ldots, t_k)$ *is defined by* (207),

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j $(i = 1, \ldots, m)$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1, \ldots, j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1, \ldots, i_k)$ *(see* (223)).

Proof. First, note that the formula (224) appears due to the equality (208). Using the equality (202), we get

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1,} \tag{225}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

It is easy to see that the equality (225) can be written in the form

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1,} \tag{226}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Further proof of Theorem 15 is based on the equality (226) and is similar to the proof of Theorem 14 in [22], [34]. Theorem 15 is proved.

The equalities (213) and (226) allow us to formulate the following modification of Theorem 12.

Theorem 16. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}$$

converging in the mean-square sense is valid, where $i_1,\ldots,i_k=1,\ldots,m$; another notations are the same as in Theorems 1, 12.

Consider the following obvious generalization of Theorem 3.

Theorem 17 [22], [25]. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the estimate*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$
\leq k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right) \tag{227}
$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m$, $i_1^2+\ldots+i_k^2>0$, *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (1), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (217) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 1, 2, 12.

In addition, under the conditions of Theorem 17 we have the estimate (also see (60))

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\leq
$$

$$
\leq (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \times
$$

$$
\times\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right)^n.
$$

17. Generalization of Theorems 4, 5 to the Case of an Arbitrary Complete Orthonormal with Weight $r(x)\geq 0$ System of Functions in the Space $L_2([t,T])$ and $\psi_1(x)\sqrt{r(x)},\ \ldots,\ \psi_k(x)\sqrt{r(x)}\ \in\ L_2([t,T])$

In this section, we will use the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process to generalize Theorems 4, 5 to the case of an arbitrary complete orthonormal with weight $r(x)\geq 0$ system of functions in the space $L_2([t,T])$ and $\psi_1(x)\sqrt{r(x)},\ \ldots,$ $\psi_k(x)\sqrt{r(x)}\in L_2([t,T])$. From the results of Sect. 8, 15 we obtain the following two theorems.

Theorem 18 [22], [25]. *Suppose that* $\psi_1(x)\sqrt{r(x)},\ldots,\psi_k(x)\sqrt{r(x)}\in L_2([t,T])$, *where* $r(x)\geq 0$. *Moreover, let*

$$
\left\{\Psi_j(x)\sqrt{r(x)}\right\}_{j=0}^{\infty}
$$

is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. *Then, for the iterated Ito stochastic integral*

$$
\tilde J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{228}
$$

the following expansion

$$\tilde{J}[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\tilde{\zeta}_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (229)$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\tilde{\zeta}_j^{(i)} = \int_t^T \Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),*

$$\tilde{C}_{j_k\ldots j_1} = \int_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient, $K(t_1,\ldots,t_k)$ *is defined by* (2)*; another notations are the same as in Theorems* 1, 2, 4.

Theorem 19 [22], [25]. *Under the conditions of Theorem* 18 *the following estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t} - \tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le k!\left(\int_{[t,T]^k} K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^{k} r(t_l)\right)dt_1\ldots dt_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}^2\right)$$

is valid for the following cases:

1. $i_1,\ldots,i_k = 1,\ldots,m$ *and* $0 < T-t < \infty$,
2. $i_1,\ldots,i_k = 0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2 > 0,$ *and* $0 < T-t < 1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (228), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (229) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ *; another notations are the same as in Theorems* 4, 5, 18.

18. Proof of Theorems 12 and 13 Based on the Ito Formula and Without Explicit Use of the Multiple Wiener Stochastic Integral

Note that Theorems 12 and 13 can also be proved without explicit use of the multiple Wiener stochastic integral. To do this, we introduce the following sum of iterated Ito stochastic integrals

$$J''[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\mathsf{def}}{=} \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{230}$$

where $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$, $i_1,\ldots,i_k = 0,1,\ldots,m$, $d\mathbf{w}_\tau^{(0)} \stackrel{\mathsf{def}}{=} d\tau$; another notations are the same as in (202).

Further, using the isometry property of the Ito stochastic integral as well as the linearity property of this integral, we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J''[K]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} + J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1}, \tag{231}$$

where $K(t_1,\ldots,t_k)$ and $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ are defined by (207) and (209) correspondingly. Moreover, $J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ are defined by (230). Obviously, we can consider an analogue of (231) for $\Phi(t_1,\ldots,t_k)$ instead of $K(t_1,\ldots,t_k)$.

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in (231) and using (214), (215), (230), we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{232}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

It is easy to see that the equality (232) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \times$$

$$\times \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{233}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Further, using the Ito formula, we can prove the following equality

$$\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\qquad(234)$$

w. p. 1, where notations are the same as in Theorem 2 and (233).

The main difficulty in proving (234) using the Ito formula is related to the need to take into account various combinations of indices $i_1,\ldots,i_k=0,1,\ldots,m$. To avoid this difficulty, consider another approach, also based on the Ito formula.

First, we prove the following modification and generalization of Theorem 3.1 from [85] (1951) for the case $i_1,\ldots,i_k=0,1,\ldots,m$ using the Ito formula and without explicit use of the multiple Wiener stochastic integral.

Theorem 20 [22]. *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 14*) and the condition* (162) *is also fulfilled. Furthermore, let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^k\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\ne 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)\qquad(235)$$

w. p. 1*, where* $i_1,\ldots,i_k=0,1,\ldots,m$; $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$; $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$; $d_l=1,\ldots,m_l$; $l=1,\ldots,k$; $m_1+\ldots+m_k=k$; *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (164)*; another notations are the same as in Theorem* 10.

Proof. First, consider the case $i_1=\ldots=i_k=1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbb{N}$. By induction, we prove the following equality

$$p!\int\limits_t^T\phi_l(t_p)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_p}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{p})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_p')\ldots\int\limits_t^{t_2'}\phi_l(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_p'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} \tag{236}$$

w. p. 1, where $p\in\mathbb{N}$, $l\ne j_1,\ldots,j_q$, and

$$\sum_{(q_1,\ldots,q_n)}$$

means the sum with respect to all possible permutations $(q_1,\ldots,q_n)$.

Consider the case $p=1$. Using the Ito formula, we get w. p. 1 for $s\in[t,T]$

$$\int\limits_t^s\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$=\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+$$

$$+\int\limits_t^s\phi_l(\tau)\int\limits_t^{\tau}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}d\mathbf{w}_{\tau}^{(1)}+$$

$$+\int\limits_t^s\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\phi_l(\theta)d\mathbf{w}_{\theta}^{(1)}\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}\right)d\mathbf{w}_{\tau}^{(1)}. \tag{237}$$

Hereinafter in this section always $s\in[t,T]$. Differentiating by the Ito formula the expression in parentheses on the right-hand side of equality (237) and combining the result of differentiation with (237), we obtain w. p. 1

$$J_{(l)s,t}J_{(j_q\ldots j_1)s,t}=$$

$$=\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+$$

$$+J_{(lj_q\ldots j_1)s,t}+$$

$$+\int\limits_t^s\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_l(\theta)\phi_{j_{q-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{q-2}}(t_{q-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-2}}^{(1)}d\theta d\mathbf{w}_{\tau}^{(1)}+$$

$$+J_{(j_qlj_{q-1}\ldots j_1)s,t}+$$

$$+\int\limits_t^s \phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(\theta)\times$$

$$\times\left(\int\limits_t^{\theta}\phi_l(u)\ d\mathbf{w}_u^{(1)}\int\limits_t^{\theta}\phi_{j_{q-2}}(t_{q-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-2}}^{(1)}\right)d\mathbf{w}_{\theta}^{(1)}d\mathbf{w}_{\tau}^{(1)}, \tag{238}$$

where

$$\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} \stackrel{\sf def}{=} J_{(j_q\ldots j_1)s,t}.$$

Continuing the process of iterative application of the Ito formula, we have w. p. 1

$$J_{(l)s,t}J_{(j_q\ldots j_1)s,t}=$$

$$=J_{(lj_q\ldots j_1)s,t}+J_{(j_qlj_{q-1}\ldots j_1)s,t}+\ldots+J_{(j_q\ldots j_1l)s,t}+$$

$$+\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots$$

$$\ldots+\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_l(\tau)\phi_{j_1}(\tau)d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \tag{239}$$

Summing the equality (239) over permutations $(j_1,\ldots,j_q)$, we get

$$\sum_{(j_1,\ldots,j_q)} J_{(l)s,t}J_{(j_q\ldots j_1)s,t}=\sum_{(j_1,\ldots,j_q,l)} J_{(lj_q\ldots j_1)s,t}+S(s) \tag{240}$$

w. p. 1, where

$$S(s)=$$

$$=\sum_{(j_1,\ldots,j_q)}\left(\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots\right.$$

$$\left.\ldots+\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_l(\tau)\phi_{j_1}(\tau)d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\right). \tag{241}$$

Consider

$$\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}.$$

Applying the Ito formula, we get w. p. 1

$$\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)} =$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau +$$

$$+ \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\times$$

$$\times \left(\int\limits_t^{t_{q-1}} \phi_l(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^{t_{q-1}} \phi_{j_{q-2}}(t_{q-2})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-2}}^{(1)}\right)d\mathbf{w}_{t_{q-1}}^{(1)}.$$

By iterative application of the Ito formula (as above), we obtain w. p. 1

$$\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)} =$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau + \ldots$$

$$\ldots + \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(\tau)\phi_{j_q}(\tau)d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}. \tag{242}$$

Summing the equality (242) over permutations $(j_1,\ldots,j_q)$, we get

$$\sum_{(j_1,\ldots,j_q)} \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)} = S_1(s) \tag{243}$$

w. p. 1, where

$$S_1(s) =$$

$$= \sum_{(j_1,\ldots,j_q)} \left(\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau + \ldots\right.$$

$$\ldots+\int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(\tau)\phi_{j_q}(\tau)d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}\Biggr). \tag{244}$$

It is not difficult to see that

$$S(s)=S_1(s)\quad \text{w. p. 1.} \tag{245}$$

Moreover, due to the orthogonality of $\{\phi_j(x)\}_{j=0}^{\infty}$ and (243), (245), we have

$$S(T)=S_1(T)=0\quad \text{w. p. 1.} \tag{246}$$

Thus (see (240), (246)), the equality (236) is proved for the case $p=1$. Let us assume that the equality (236) is true for $p=2,3,\ldots,k-1$, and prove its validity for $p=k$.

From (240) for the case $q=k-1$, $j_1=\ldots=j_{k-1}=l$ we obtain

$$(J_1)_{s,t}\,(k-1)!\,(J_{k-1})_{s,t}=k!\,(J_k)_{s,t}+S_2(s) \tag{247}$$

w. p. 1, where

$$S_2(s)=S(s)\Bigr|_{j_1=\ldots=j_q=l,\ q=k-1}\quad (k\ge 2)\quad \text{and}\quad S_2(s)\stackrel{\sf def}{=}0\ \ (q=k-1,\ k=1),$$

$$\int\limits_t^s\phi_l(t_r)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_r}^{(1)}\stackrel{\sf def}{=}(J_r)_{s,t}\quad (r\in\mathbb{N})\quad \text{and}\quad (J_0)_{s,t}\stackrel{\sf def}{=}1.$$

Taking into account (241), (243)–(245) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we have

$$S_2(T)=(k-1)!\,(J_{k-2})_{T,t}\,. \tag{248}$$

Combining (247) and (248), we obtain the following recurrence relation

$$k!\,(J_k)_{T,t}=(J_1)_{T,t}\,(k-1)!\,(J_{k-1})_{T,t}-(k-1)!\,(J_{k-2})_{T,t} \tag{249}$$

w. p. 1.

Using (249) and the induction hypothesis, we get w. p. 1

$$k!\int\limits_t^T\phi_l(t_k)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$
=\int\limits_t^T \phi_l(\tau)\ d\mathbf{w}_{\tau}^{(1)}\Biggl((k-1)!\int\limits_t^T \phi_l(t_{k-1})\ldots \int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\Biggr)-
$$

$$
-(k-1)!\int\limits_t^T \phi_l(t_{k-2})\ldots \int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times
$$

$$
\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=
$$

$$
=\int\limits_t^T \phi_l(\tau)\ d\mathbf{w}_{\tau}^{(1)}\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}\int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_{k-1}')\ldots \int\limits_t^{t_2'}\phi_l(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}-
$$

$$
-(k-1)\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})}\int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_{k-2}')\ldots \int\limits_t^{t_2'}\phi_l(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \tag{250}
$$

Let $\boxed{l}$ be the symbol l which does not participate in the following sum with respect to permutations

$$
\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.
$$

Applying (240), we have w. p. 1

$$
\int\limits_t^s \phi_l(\tau)\ d\mathbf{w}_{\tau}^{(1)}\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}\int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_{k-1}')\ldots \int\limits_t^{t_2'}\phi_l(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=
$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\ d\mathbf{w}_\tau^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \left(J_{(\boxed{l} j_q\ldots j_1 \underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q \boxed{l} j_{q-1}\ldots j_1 \underbrace{l\ldots l}_{k-1})s,t} + \ldots \right.$$

$$\left. \ldots + J_{(j_q\ldots j_1 \boxed{l} \underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q\ldots j_1 l \boxed{l} \underbrace{l\ldots l}_{k-2})s,t} + \ldots + J_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k-1} \boxed{l})s,t} \right) + S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} J_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k})s,t} + S_3(s), \tag{251}$$

where

$$S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(\int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots$$

$$+ \ldots \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{\boxed{l}}(\tau)\phi_{j_1}(\tau)\times$$

$$\times \int\limits_t^{\tau} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} +$$

$$+ \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{\boxed{l}}(\tau)\phi_l(\tau)\times$$

$$\times \int\limits_t^{\tau} \phi_l(t'_{k-2})\ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\tau d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} + \ldots$$

$$
\ldots + \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_3} \phi_l(t'_2) \int\limits_t^{t'_2} \phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau d\mathbf{w}_{t'_2}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \Biggr).
$$

Using (241), (243)–(245), we get w. p. 1

$$
S_3(s) =
$$

$$
= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =
$$

$$
= (k-1) \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau \int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} +
$$

$$
+ \sum_{(j_1,\ldots,j_{q-1},\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} +
$$

$$
+ \sum_{(j_1,\ldots,j_{q-2},j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_{q-1}}(\tau)d\tau \int\limits_t^s \phi_{j_q}(t_q) \int\limits_t^{t_q} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} d\mathbf{w}_{t_q}^{(1)} + \ldots
$$

$$\ldots$$

$$\ldots + \sum_{(j_2,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_1}(\tau)d\tau \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_3} \phi_{j_2}(t_2)\times$$

$$\times \int\limits_t^{t_2} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \tag{252}$$

Applying (252) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we finally have

$$S_3(T) = (k-1) \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-2})\ldots \int\limits_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \tag{253}$$

Combining (250), (251), (253), we obtain w. p. 1

$$k! \int\limits_t^T \phi_l(t_k)\ldots \int\limits_t^{t_2} \phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(\underbrace{l,\ldots,l}_{k})} \int\limits_t^T \phi_l(t_k)\ldots \int\limits_t^{t_2} \phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_k)\ldots \int\limits_t^{t'_2} \phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}, \tag{254}$$

where $l \neq j_1,\ldots,j_q$.

The equality (236) is proved. From the other hand, (254) means that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{q+n})} = J''[\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{n})}\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots1}^{q})} \tag{255}$$

w. p. 1, where $n,q=0,1,2\ldots$; $l\neq j_1,\ldots,j_q$ and

$$J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots1}^{q})}\stackrel{\sf def}{=}1$$

for $q=0$.

Note that [88]

$$\int\limits_t^T\phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)}=$$

$$=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)},\int\limits_t^T\phi_l^2(\tau)d\tau\right)=$$

$$=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)},1\right)=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right) \tag{256}$$

w. p. 1, where $n\in\mathbb{N}$, $H_n(x,y)$ is defined by (173) (also see (174)), and $H_n(x)$ is the Hermite polynomial (164).

From (256) we have w. p. 1

$$J''[\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{n})}=n!\int\limits_t^T\phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)}=$$

$$=n!\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right)=H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right), \tag{257}$$

where $n\in\mathbb{N}$.

Combining (255) and (257), we obtain

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{q+n})}=H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right)\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots 1}^{q})} \tag{258}$$

w. p. 1, where $n,q=0,1,2\ldots;\ l\ne j_1,\ldots,j_q$.

The iterated application of the formula (258) completes the proof of Theorem 20 for the case $i_1=\ldots=i_k=1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbb{N}$.

To prove Theorem 20 for the case $i_1=\ldots=i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbb{N}$, we need to prove the following formula in addition to the previous proof

$$p!\int\limits_t^T\phi_l(t_p)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_p\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q=$$

$$=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{p})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_p')\ldots\int\limits_t^{t_2'}\phi_l(t_1')dt_1'\ldots dt_p'dt_1\ldots dt_q, \tag{259}$$

where $p\in\mathbb{N}$,

$$\sum_{(j_1,\ldots,j_d)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_d)$.

First, consider the case $p=1$. We have

$$d\left(\int\limits_t^s\phi_l(\theta)d\theta\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q\right)=$$

$$=\phi_l(s)\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q ds+$$

$$+\phi_{j_q}(s)\left(\int\limits_t^s\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{q-1}\cdot\int\limits_t^s\phi_l(\theta)d\theta\right)ds.$$

Then

$$\int\limits_t^s\phi_l(\theta)d\theta\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q=$$

$$=I_{(lj_q\ldots j_1)s,t}+$$

$$+\int\limits_t^s \phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{q-1}\cdot\int\limits_t^{\tau}\phi_l(\theta)d\theta\right)d\tau,$$

where

$$\int\limits_t^s \phi_{j_r}(t_r)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_r \stackrel{\sf def}{=} I_{(j_r\ldots j_1)s,t}. \qquad (260)$$

Continuing this process, we get

$$\int\limits_t^s \phi_l(\theta)d\theta\sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t}=\sum_{(j_1,\ldots,j_q,l)} I_{(lj_q\ldots j_1)s,t}, \qquad (261)$$

where

$$\sum_{(j_1,\ldots,j_d)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_d)$.

The equality (259) is proved for the case $p=1$. Let us assume that the equality (259) is true for $p=2,3,\ldots,k-1$, and prove its validity for $p=k$.

From (261) for $j_1=\ldots=j_q=l$, $q=k-1$ we have

$$\left(I_1\right)_{s,t}(k-1)!\left(I_{k-1}\right)_{s,t}=k!\left(I_k\right)_{s,t}, \qquad (262)$$

where $k\in\mathbb{N}$ and

$$\int\limits_t^s \phi_l(t_k)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_k \stackrel{\sf def}{=} \left(I_k\right)_{s,t}, \quad \left(I_0\right)_{s,t}\stackrel{\sf def}{=}1.$$

Using (262) and the induction hypothesis, we obtain

$$k!\left(I_k\right)_{s,t}\sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t}=\left(I_1\right)_{s,t}(k-1)!\left(I_{k-1}\right)_{s,t}\sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t}=$$

$$=I_{(l)s,t}\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t}=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t}I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t}, \qquad (263)$$

where $I_{(j_r\ldots j_1)s,t}$ is defined by (260) and $\boxed{l}$ is the symbol l which does not participate in the following sum with respect to permutations

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.$$

By analogy with (261) we have

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t} I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(I_{(\boxed{l}j_q\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\boxed{l}j_{q-1}\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + \ldots$$

$$\ldots + I_{(j_q\ldots j_1\boxed{l}\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\ldots j_1 l\boxed{l}\underbrace{l\ldots l}_{k-2})s,t} + \ldots + I_{(j_q\ldots j_1\underbrace{l\ldots l}_{k-1}\boxed{l})s,t} \Biggr) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} I_{(j_q\ldots j_1\underbrace{l\ldots l}_{k})s,t}. \tag{264}$$

Substituting $s = T$ into (263), (264) and combining (263), (264), we conlude that the equality (259) is proved for $p = k$. The equality (259) is proved.

Note that

$$n!\int\limits_t^T \phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_n = n!\frac{1}{n!}\left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n = \left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n, \tag{265}$$

where $n \in \mathbb{N}$.

After substituting (265) into (259), we have for $p = n$

$$\left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n \sum_{(j_1,\ldots,j_q)} J_{(j_q\ldots j_1)T,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{n})} J_{(j_q\ldots j_1\underbrace{l\ldots l}_{n})T,t}. \tag{266}$$

The equality (266) means that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{0\ldots 0}^{q+n})} = \left(\int\limits_t^T \phi_l(\tau)d\tau\right)^n \cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}, \tag{267}$$

where $n, q = 0, 1, 2\ldots$ and $J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(0\ldots 0)} \stackrel{\sf def}{=} 1$ for $q = 0$.

The relations (258) and (267) prove Theorem 20 for the case $i_1 = \ldots = i_k = 0, 1, \ldots, m$ and $j_1, \ldots, j_k \in \{0\} \cup \mathbb{N}$.

Remark 12. *Note that the equality (259) can be obtained in another way. Let*

$$D_q = \{(t_1,\ldots,t_q)\in [t,T]^q:\ \exists\ i\ne j\ \textit{such that}\ t_i=t_j\}$$

be the "diagonal set"of $[t,T]^q$ $(q=2,3,\ldots)$ [87]. *Since the Lebesgue meashure of the set* D_q *is equal to zero* [87], *then (see (230))*

$$J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})} = \int\limits_{[t,T]^q} \phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q. \tag{268}$$

From (268) we have

$$J''[\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p})}\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})} =$$

$$= \int\limits_{[t,T]^q} \phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q \int\limits_{[t,T]^p} \phi_{l}(t_1)\ldots\phi_{l}(t_p)dt_1\ldots dt_p =$$

$$= \int\limits_{[t,T]^{p+q}} \phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)\phi_l(t_1')\ldots\phi_l(t_p')dt_1'\ldots dt_p'dt_1\ldots dt_q =$$

$$= J''[\phi_{j_1}\ldots\phi_{j_q}\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p+q})}. \tag{269}$$

It is not difficult to see that the equality (269) is nothing but the equality (259) written in another form.

To complete the proof of Theorem 20, we need to consider the case $i_1,\ldots,i_k = 0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbb{N}$.

Obviously, the proof of Theorem 20 will be completed if we prove the following equalities

$$\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times$$

$$\times\sum_{(j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_n'}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_n'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}, \tag{270}$$

$$\sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times$$

$$\times\sum_{(j_1',\ldots,j_n')}\int\limits_t^T \phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_n'}^{(0)}=$$

$$=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_n'}(t_n')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_n'}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} \tag{271}$$

w. p. 1, where $n,q\in\mathbb{N}$, $d\mathbf{w}_\tau^{(0)}\stackrel{\text{def}}{=}d\tau$, $i_1,\ldots,i_q\ne 1$ in (270) and $i_1,\ldots,i_q\ne 0$ in (271),

$$\sum_{(j_1,\ldots,j_g)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_g)$. At the same time if j_r swapped with j_d in the permutation $(j_1,\ldots,j_g)$, then i_r swapped with i_d in the permutation $(i_1,\ldots,i_g)$.

The equalities (270) and (271) mean that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q 1\ldots 1)}=J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J''[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(1\ldots 1)}, \tag{272}$$

$$J''[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q 0\ldots 0)}=J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)}\cdot J''[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(0\ldots 0)} \tag{273}$$

w. p. 1, where $i_1,\ldots,i_q\ne 1$ in (272) and $i_1,\ldots,i_q\ne 0$ in (273).

First, we prove the equality (270). Consider the case $n=1$. Using the Ito formula, we get w. p. 1

$$\int\limits_t^s\phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)}+$$

$$+\int\limits_t^s\phi_{j_q}(\tau)\left(\int\limits_t^\tau\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{q-1}}^{(i_{q-1})}\int\limits_t^\tau\phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)}\right)d\mathbf{w}_\tau^{(i_q)}=$$

$$=\ldots=$$

$$=J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)}+J_{(j_qj_1'j_{q-1}\ldots j_1)s,t}^{(i_q1i_{q-1}\ldots i_1)}+\ldots+J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)}, \tag{274}$$

where

$$\int\limits_t^s \phi_{j_r}(t_r)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_r}^{(i_r)} \stackrel{\sf def}{=} J_{(j_r\ldots j_1)s,t}^{(i_r\ldots i_1)}, \tag{275}$$

$i_1,\ldots,i_r=0,1,\ldots,m.$

From (274) we obtain

$$\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(1)}\sum_{(j_1,\ldots,j_q)}\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\sum_{(j_1,\ldots,j_q)}\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(1)}\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\sum_{(j_1,\ldots,j_q)}\left(J_{(j_1'j_q\ldots j_1)s,t}^{(1i_q\ldots i_1)}+J_{(j_qj_1'j_{q-1}\ldots j_1)s,t}^{(i_q1i_{q-1}\ldots i_1)}+\ldots+J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)}\right)=$$

$$=\sum_{(j_1,\ldots,j_q,j_1')}J_{(j_q\ldots j_1j_1')s,t}^{(i_q\ldots i_11)} \tag{276}$$

w. p. 1, where $J_{(j_r\ldots j_1)s,t}^{(i_r\ldots i_1)}$ is defined by (275). The equality (270) is proved for the case $n=1$.

Let us assume that the equality (270) is true for $n=2,3,\ldots,k-1$, and prove its validity for $n=k$.

Applying (240), (241), (243)–(245), we obtain w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^s \phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}=$$

$$=\int\limits_t^s \phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^s \phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^s \phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\theta\int\limits_t^s \phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}. \tag{277}$$

After substituting $s=T$ in (277) and applying the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we get w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}=$$

$$=\int\limits_t^T \phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^T \phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}, \tag{278}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Using (278) and the induction hypothesis, we obtain w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_\theta^{(1)}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_\theta^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{279}$$

Further, applying the induction hypothesis, we have w. p. 1

$$\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\Biggl(\sum_{(j_1',\ldots,j_{k-2}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}+$$

$$+\sum_{(j_1',\ldots,j_{k-3}',j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-2}'\}}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-2})\int\limits_t^{t_{k-2}}\phi_{j_{k-3}'}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-3}}^{(1)}d\mathbf{w}_{t_{k-2}}^{(1)}+\ldots$$

$$\ldots+\sum_{(j_2',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_1'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_{k-1}'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\Biggr)\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\Biggl(\mathbf{1}_{\{j_k'=j_{k-1}'\}}\sum_{(j_1',\ldots,j_{k-2}')}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}+$$

$$+\mathbf{1}_{\{j_k'=j_{k-2}'\}}\sum_{(j_1',\ldots,j_{k-3}',j_{k-1}')}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-2})\int\limits_t^{t_{k-2}}\phi_{j_{k-3}'}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-3}}^{(1)}d\mathbf{w}_{t_{k-2}}^{(1)}+\ldots$$

$$\ldots+\mathbf{1}_{\{j_k'=j_1'\}}\sum_{(j_2',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_{k-1}'}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\Biggr)\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int_t^T \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \mathbf{1}_{\{j'_k=j'_{k-1}\}} \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-2})} \int_t^T \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1) \int_t^{t_1} \phi_{j'_{k-2}}(t'_{k-2})\ldots \int_t^{t'_2} \phi_{j'_1}(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\mathbf{1}_{\{j'_k=j'_{k-2}\}} \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-3},j'_{k-1})} \int_t^T \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1) \int_t^{t_1} \phi_{j'_{k-1}}(t'_{k-2})\times$$

$$\times \int_t^{t'_{k-2}} \phi_{j'_{k-3}}(t'_{k-3})\ldots \int_t^{t'_2} \phi_{j'_1}(t'_1) d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)} d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} + \ldots$$

$$\ldots$$

$$\ldots + \mathbf{1}_{\{j'_k=j'_1\}} \sum_{(j_1,\ldots,j_q,j'_2,\ldots,j'_{k-1})} \int_t^T \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int_t^{t_1} \phi_{j'_{k-2}}(t'_{k-2})\ldots \int_t^{t'_3} \phi_{j'_2}(t'_2) \int_t^{t'_2} \phi_{j'_{k-1}}(t'_1) d\mathbf{w}_{t'_1}^{(1)} d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} S_4(T). \tag{280}$$

By analogy with (242) we obtain w. p. 1

$$\int_t^T \phi_l(\tau)\phi_{j_r}(\tau)d\tau \int_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})} =$$

$$= \int_t^T \phi_l(\tau)\phi_{j_r}(\tau) \int_t^{\tau} \phi_{j_{r-1}}(t_{r-1})\ldots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})} d\tau + \ldots$$

$$\ldots + \int_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int_t^{t_2} \phi_{j_1}(t_1) \int_t^{t_1} \phi_l(\tau)\phi_{j_r}(\tau) d\tau d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})}, \tag{281}$$

where $i_1,\ldots,i_{r-1} = 0,1,\ldots,m$.

Using iteratively the Ito formula, as well as (281) and combinatorial reasoning, we obtain w. p. 1 (see Remark 13 below for details)

$$\int\limits_t^T \phi_{j_k'}(\theta)d\mathbf{w}_\theta^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\Biggl(\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}'}(t_{k-2}')\ldots$$

$$\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_\theta^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\int\limits_t^{t_{k-1}'}\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}'}(t_{k-3}')\ldots$$

$$\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)}d\mathbf{w}_\theta^{(0)}d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots$$

$$\ldots+\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_3'}\phi_{j_2'}(t_2')\int\limits_t^{t_2'}\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(0)}\times$$

$$\times d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr)=$$

$$=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')}\Biggl\{\int\limits_t^T\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-2}'}(t_{k-2}')\ldots$$

$$\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_{\theta}^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-2}'}(t_{k-2}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\int\limits_t^{t_1'}\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\mathbf{w}_{\theta}^{(0)}\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr\}+$$

$$+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-3}',j_{k-1}')}\Biggl\{\int\limits_t^T\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\times$$

$$\times\int\limits_t^{t_{k-1}'}\phi_{j_{k-3}'}(t_{k-3}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)}d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_{\theta}^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\int\limits_t^{t_{k-1}'}\phi_{j_{k-3}'}(t_{k-3}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times\int\limits_t^{t_1'}\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)}d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr\}+\ldots$$

$$\ldots+\sum_{(j_1,\ldots,j_q,j_2',\ldots,j_{k-1}')}\Biggl\{\int\limits_t^T\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots$$

$$\ldots\int\limits_t^{t_3'}\phi_{j_2'}(t_2')d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_{\theta}^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_3'}\phi_{j_2'}(t_2')\int\limits_t^{t_2'}\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(0)}\times$$

$$\times d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr\}=$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') \times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} +
$$

$$
+ \int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-2}'}(t_{k-2}') \ldots
$$

$$
\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_{k-2}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} +
$$

$$
+ \int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-3}',j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}') \times
$$

$$
\times \int\limits_t^{t_{k-1}'} \phi_{j_{k-3}'}(t_{k-3}') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_{k-3}'}^{(1)} d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} + \ldots
$$

$$
\ldots + \int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_2',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}') \ldots
$$

$$
\ldots \int\limits_t^{t_3'} \phi_{j_2'}(t_2') d\mathbf{w}_{t_2'}^{(1)} \ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') \times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} + S_4(T). \tag{282}
$$

From (279), (280), and (282) we conclude that the equality (270) is proved for $n = k$. The equality (270) is proved.

Remark 13. *It should be noted that the sums with respect to permutations*

$$
\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}
$$

in (282), containing the expressions $\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta),\ldots,\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)$*, should be understood in a special way. Let us explain this rule on the basis of the sum*

$$\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j'_{k-2}}(t'_{k-2})\ldots$$

$$\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{283}$$

More precisely, permutations $(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})$ *when summing in (283) are performed in such a way that if* j_r^* *swapped with* j_d^* *in the permutation*

$$\left(j^*_{q+k-1},\ldots,j^*_1\right)=\left(j_q,\ldots,j_1,j'_{k-1},j'_{k-2},\ldots,j'_1\right),$$

then i_r^* *swapped with* i_d^* *in the permutation*

$$\left(i^*_{q+k-1},\ldots,i^*_1\right)=\bigl(i_q,\ldots,i_1,0,\underbrace{1,\ldots,1}_{k-2}\bigr).$$

Moreover, $\bar\phi_{j_r^*}$ *swapped with* $\bar\phi_{j_d^*}$ *in the permutation*

$$\left(\bar\phi_{j^*_{q+k-1}},\ldots,\bar\phi_{j^*_1}\right)=\left(\phi_{j_q},\ldots,\phi_{j_1},\ \phi_{j'_k}\cdot\phi_{j'_{k-1}},\ \phi_{j'_{k-2}},\ldots,\phi_{j'_1}\right).$$

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})}$$

in (282) that contain the expressions $\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta),\ldots,\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)$*.*

Let us prove the equality (271). Consider the case $n=1$. By analogy with (274) and (276) we obtain

$$\int\limits_t^s\phi_{j'_1}(\theta)d\mathbf{w}_{\theta}^{(0)}\sum_{(j_1,\ldots,j_q)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\sum_{(j_1,\ldots,j_q,j'_1)}J^{(i_q\ldots i_1 0)}_{(j_q\ldots j_1 j'_1)s,t}$$

w. p. 1, where $J^{(i_r\ldots i_1)}_{(j_r\ldots j_1)s,t}$ is defined by (275). The equality (271) is proved for the case $n=1$.

Let us assume that the equality (271) is true for $n=2,3,\ldots,k-1$, and prove its validity for $n=k$.

In complete analogy with (261) we get

$$\int\limits_t^s \phi_{j'_k}(\theta)d\theta \int\limits_t^s \phi_{j'_{k-1}}(t_{k-1})\ldots \int\limits_t^{t_2} \phi_{j'_1}(t_1)dt_1\ldots dt_{k-1} =$$

$$= J^{(0\ldots 0)}_{(j'_k j'_{k-1}\ldots j'_1)s,t} + J^{(0\ldots 0)}_{(j'_{k-1} j'_k j'_{k-2}\ldots j'_1)s,t} + \ldots + J^{(0\ldots 0)}_{(j'_{k-1}\ldots j'_1 j'_k)s,t}. \tag{284}$$

Applying (284), we have

$$\sum_{(j'_1,\ldots,j'_k)} \int\limits_t^T \phi_{j'_k}(t'_k)\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}^{(0)}_{t'_1}\ldots d\mathbf{w}^{(0)}_{t'_k} =$$

$$= \sum_{(j'_1,\ldots,j'_{k-1})} \Bigl(J^{(0\ldots 0)}_{(j'_k j'_{k-1}\ldots j'_1)s,t} + J^{(0\ldots 0)}_{(j'_{k-1} j'_k j'_{k-2}\ldots j'_1)s,t} + \ldots + J^{(0\ldots 0)}_{(j'_{k-1}\ldots j'_1 j'_k)s,t} \Bigr) =$$

$$= \int\limits_t^T \phi_{j'_k}(\theta)d\theta \sum_{(j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j'_{k-1}}(t_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t_1)d\mathbf{w}^{(0)}_{t_1}\ldots d\mathbf{w}^{(0)}_{t_{k-1}}. \tag{285}$$

Using (285) and the induction hypothesis, we obtain w. p. 1

$$\sum_{(j'_1,\ldots,j'_k)} \int\limits_t^T \phi_{j'_k}(t_k)\ldots \int\limits_t^{t_2} \phi_{j'_1}(t_1)d\mathbf{w}^{(0)}_{t_1}\ldots d\mathbf{w}^{(0)}_{t_k} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_q)}_{t_q} =$$

$$= \int\limits_t^T \phi_{j'_k}(\theta)d\theta \sum_{(j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}^{(0)}_{t'_1}\ldots d\mathbf{w}^{(0)}_{t'_{k-1}} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_q)}_{t_q} =$$

$$= \int\limits_t^T \phi_{j'_k}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}^{(0)}_{t'_1}\ldots d\mathbf{w}^{(0)}_{t'_{k-1}} d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_q)}_{t_q} =$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_k'}(\theta)d\theta\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_{k-1}'}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{286}
$$

An iterative application of the Ito formula leads to the following equality

$$
\int\limits_t^T\phi_{j_k'}(\theta)d\theta\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times
$$

$$
\times\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_{k-1}'}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=
$$

$$
=J_{(j_k'j_q\ldots j_1j_{k-1}'\ldots j_1')T,t}^{(0i_q\ldots i_10\ldots 0)}+J_{(j_qj_k'j_{q-1}\ldots j_1j_{k-1}'\ldots j_1')T,t}^{(i_q0i_{q-1}\ldots i_10\ldots 0)}+\ldots J_{(j_q\ldots j_1j_k'j_{k-1}'\ldots j_1')T,t}^{(i_q\ldots i_10\ldots 0)}+
$$

$$
+J_{(j_q\ldots j_1j_{k-1}'j_k'j_{k-2}'\ldots j_1')T,t}^{(i_q\ldots i_10\ldots 0)}+\ldots+J_{(j_q\ldots j_1j_{k-1}'\ldots j_1'j_k')T,t}^{(i_q\ldots i_10\ldots 0)} \tag{287}
$$

w. p. 1.

Combining (286) and (287) we finally obtain w. p. 1

$$
\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times
$$

$$
\times\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)}=
$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}.
$$

The equality (271) is proved for $n=k$. The equality (271) is proved. Theorem 20 is proved.

To complete the proof of Theorems 12 and 13, we prove the following theorem.

Theorem 21 [22]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following representation*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{288}$$

is valid w. p. 1*, where* $i_1,\ldots,i_k=0,1,\ldots,m$*,* $[x]$ *is an integer part of a real number* x*, the sum in the second line of the formula* (288) *is the sum with respect to all possible partitions* (44)*,* $\prod_{\emptyset}\stackrel{\sf def}{=}1$*,* $\sum_{\emptyset}\stackrel{\sf def}{=}0$*; another notations are the same as in Theorems* 1, 2.

Remark 14. *It should be noted that the formulas* (236)*,* (269)*,* (272)*,* (273) *follow from* (288)*. It is only necessary to set the values of the corresponding indicators of the form* $\mathbf{1}_A$ *from the formula* (288) *equal to* 0 *or* 1.

Proof. The proof of Theorem 21 is carried out by induction using the following recurrence relation

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=J''[\phi_{j_k}]_{T,t}^{(i_k)}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})}-$$

$$-\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\ne 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})} \tag{289}$$

w. p. 1.

Let us prove the recurrence relation (289). Using iteratively the Ito formula, the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, as well as (281) and combinatorial reasoning, we obtain w. p. 1 (see Remark 15 below for details)

$$J''[\phi_{j_k}]_{T,t}^{(i_k)}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})}=$$

$$=\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(j_1,\ldots,j_{k-1})}\Biggl(\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}+$$

$$+\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}+\ldots$$

$$\ldots+\mathbf{1}_{\{i_k=i_1\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_k}(\theta)\phi_{j_1}(\theta)\times$$

$$\left.\times d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\right)=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(j_1,\ldots,j_{k-2})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\left\{\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times\right.$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}+\ldots$$

$$\left.\ldots+\int\limits_t^T\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}\right\}+$$

$$+\sum_{(j_1,\ldots,j_{k-3},j_{k-1})}\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\left\{\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_{k-3}}(t_{k-3})\ldots\right.$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}d\mathbf{w}_{\theta}^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\times$$

$$\left.\times d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\right\}+\ldots$$

$$
\ldots + \sum_{(j_2,\ldots,j_{k-1})} \mathbf{1}_{\{i_k=i_1\ne 0\}} \Biggl\{ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_1}(\theta) \int\limits_t^{\theta} \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times
$$

$$
\times d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} d\mathbf{w}_{\theta}^{(0)} + \ldots
$$

$$
\ldots + \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_k}(\theta)\phi_{j_1}(\theta) d\mathbf{w}_{\theta}^{(0)} d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} \Biggr\} =
$$

$$
= \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} +
$$

$$
+ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta) d\theta \sum_{(j_1,\ldots,j_{k-2})} \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}} \int\limits_t^T \phi_{j_{k-2}}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times
$$

$$
\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})} +
$$

$$
+ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta) d\theta \sum_{(j_1,\ldots,j_{k-3},j_{k-1})} \mathbf{1}_{\{i_k=i_{k-2}\ne 0\}} \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \int\limits_t^{t_{k-1}} \phi_{j_{k-3}}(t_{k-3}) \ldots
$$

$$
\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})} d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} + \ldots
$$

$$
\ldots + \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_1}(\theta) d\theta \sum_{(j_2,\ldots,j_{k-1})} \mathbf{1}_{\{i_k=i_1\ne 0\}} \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times
$$

$$
\times d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} =
$$

$$
= J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} + \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}} \mathbf{1}_{\{j_k=j_{k-1}\}} \cdot J''[\phi_{j_1} \ldots \phi_{j_{k-2}}]_{T,t}^{(i_1\ldots i_{k-2})} +
$$

$$
+ \mathbf{1}_{\{i_k=i_{k-2}\ne 0\}} \mathbf{1}_{\{j_k=j_{k-2}\}} \cdot J''[\phi_{j_1} \ldots \phi_{j_{k-3}} \phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-3} i_{k-1})} + \ldots
$$

$$
\ldots + \mathbf{1}_{\{i_k=i_1\ne 0\}} \mathbf{1}_{\{j_k=j_1\}} \cdot J''[\phi_{j_2} \ldots \phi_{j_{k-1}}]_{T,t}^{(i_2\ldots i_{k-1})} =
$$

$$
= J''[\phi_{j_1} \ldots \phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +
$$

$$+\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\ne 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})}. \qquad (290)$$

The equality (289) is proved. Theorem 21 is proved.

Remark 15. *It should be noted that the sums with respect to permutations*

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (290), *containing the expressions*

$$\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\ne 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta),$$

should be understood in a special way. Let us explain this rule on the basis of the sum

$$\sum_{(j_1,\ldots,j_{k-1})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}. \qquad (291)$$

More precisely, permutations $(j_1,\ldots,j_{k-1})$ *when summing in* (291) *are performed in such a way that if* j_r *swapped with* j_d *in the permutation* $(j_1,\ldots,j_{k-1})$, *then* i_r *swapped with* i_d *in the permutation* $(i_1,\ldots,i_{k-2}i_{k-1})$ (*note that* $i_{k-1}=0$). *Moreover,* $\bar{\phi}_{j_r}$ *swapped with* $\bar{\phi}_{j_d}$ *in the permutation*

$$\left(\bar{\phi}_{j_1},\ldots,\bar{\phi}_{j_{k-1}}\right)=\left(\phi_{j_1},\ldots,\phi_{j_{k-2}},\ \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\cdot\phi_{j_k}\cdot\phi_{j_{k-1}}\right),$$

where $\bar{\phi}_{j_{k-1}}(\tau)=\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\tau)\phi_{j_{k-1}}(\tau)$.

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (290) *that contain the expressions*

$$\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\ne 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta).$$

The relations (232), (235), (288) prove Theorem 12. An analogue of the formula (232) for the function $\Phi(t_1,\ldots,t_k)$ instead of $K(t_1,\ldots,t_k)$ and (235), (288) prove Theorem 13.

We also note a number of works [85], [87]-[91] in which the properties of multiple Wiener stochastic integrals were studied using measure theory, in particular, the formulas for the product of such integrals were obtained.

First of all, let us compare Theorem 21 with Proposition 5.1 from [89]. An analogue of the right-hand side of (288) for nonrandom $x_1,\ldots,x_k$ is constructed in [89] using diagrams (see the formula (5.1) in [89]). This means that the application of the formula (5.1) from [89], unlike the formula (288), is difficult when performing algebraic transformations.

Further, we note that the formula (5.1) from [89] was applied to the representation of the multiple Wiener stochastic integral somewhat differently than the formula (288). Namely, using Proposition 5.1 [89]. Let us expain this difference in more detail.

Proposition 5.1 from [89] in our degree of generality and in our notations can be written as

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=J''\Biggl[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{m_1}\underbrace{\phi_{j_2}\ldots\phi_{j_2}}_{m_2}\ldots\underbrace{\phi_{j_p}\ldots\phi_{j_p}}_{m_p}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1}\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2}\ldots\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}=$$

$$=J''[\phi_{j_1}\ldots\phi_{j_1}]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})}\cdot J''[\phi_{j_2}\ldots\phi_{j_2}]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})}\cdot\ldots\cdot J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}\tag{292}$$

w. p. 1, where

$$J''[\phi_{j_1}\ldots\phi_{j_1}]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})},\ J''[\phi_{j_2}\ldots\phi_{j_2}]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})},\ldots,\ J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}$$

are defined by the right-hand side of the formula (5.1) from [89], $m_1+\ldots+m_p=k,\ m_1,\ldots,m_p>0$, $j_q\ne j_d\ (q\ne d,\ q,d=1,\ldots,p),\ i_1,\ldots,i_k=1,\ldots,m.$

This actually means that in [89] an analogue of the formula (288) is constructed for the special case $j_1=\ldots=j_k$. Moreover, the specified analogue is based on the formula (5.1) [89] obtained using diagrams.

Comparing the formulas (288) and (292) (or (5.1) from [89]), it is easy to understand that the transition from (288) and (292) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (288) equal to 0 or 1. The reverse transition from the formula (292) to the formula (288) is not obvious. Note that the formula (288) (not the formula (292)) is convenient for the numerical integration of Ito stochastic differential equations (see [22], Chapter 5 and [56], [57] for details).

Let us turn to the comparison of Theorem 21 with another interesting work [92] (2019). As it turned out, a version of Theorem 21 was obtained in terms of Wick polynomials and for the case of vector valued random measures in [92] (see Theorem 7.2, p. 69). However, much earlier the formula (288) (Theorem 21) is obtained in our monograph [10] (2009) as part of the formula (5.30) (see [10], p. 220). Moreover, particular cases of the formula (288) were obtained even earlier in our works [7] (2006) and [9] (2007). More precisely, partiular cases $k=1,\ldots,5$ of the formula (288) were obtained in [7] (2006) as parts of the formulas on the pages 243-244 and partiular cases $k=1,\ldots,7$ of the formula (288) were obtained in [9] (2007) as parts of the formulas on the pages 208-218.

We also note that we have found an explicit expression for the Wick polynomial of degree k of the arguments $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$ (see the formula (288)), which is very convenient for the numerical simulation of iterated Ito stochastic integrals (1) [56], [57]. Note that the representation of the Wick polynomial of the arguments $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$ in terms of the product of Hermite polynomials is less convenient for the numerical simulation of iterated Ito stochastic integrals (1). For example, the expression for $J''[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_4)}$ in terms of the product of Hermite polynomials, even under the condition $i_1 = i_2 = i_3 = i_4$, already contains 15 different expressions (see Sect. 14). At the same time, all these 15 expressions are contained in one formula (288) provided that $k = 4$ and $i_1 = i_2 = i_3 = i_4$. It is very convenient, since in computer simulation using the formula (288), in addition to modeling of random variables $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$, it remains only to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (288) equal to 0 or 1.

It should be noted that in [90] (Theorem 6.1) a diagram formula was obtained for the product of two multiple Wiener stochastic integrals with respect to vector valued random measures. The formula (270) can be derived from the diagram formula [90]. Although the proof of the diagram formula [90] is much more complicated than our proof of the formula (270).

To conclude this section, we say a few words about expansions (15) and (217). The transition from the expansion (217) to the expansion (15) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (217) equal to 0 or 1. The reverse transition from the formula (15) to the formula (217) is also possible but not obvious. However, Theorems 20 and 21 provide a transition from (15) to (217) and vice versa. Note that the expansion (15) is interesting from the point of view of studying the structure of the expansion of iterated Ito stochastic integrals. On the orther hand, the expansion (217) is exceptionally convenient for applications (see [56], [57]).

19. Generalization of Theorem 7 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$\bar{K}(t_1,\ldots,t_k,s) = \mathbf{1}_{\{t_k<s\}} K(t_1,\ldots,t_k),$$

where the function $K(t_1,\ldots,t_k)$ has the form (2), $s \in (t,T]$ (s is fixed), and $\mathbf{1}_A$ is the indicator of the set A.

Further, we have (see (2))

$$\bar{K}(t_1,\ldots,t_k,s) = \mathbf{1}_{\{t_1<\ldots<t_k<s\}} \psi_1(t_1)\ldots\psi_k(t_k) =$$

$$= \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s \\ \\ 0, & \text{otherwise} \end{cases},$$

where $\bar{K}(t_1,\ldots,t_k,s) \in L_2([t,T]^k)$, $k \ge 1$, $t_1,\ldots,t_k \in [t,T]$, and $s \in (t,T]$.

Note that

$$J[\psi^{(k)}]_{s,t} = \int\limits_t^s \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \int\limits_t^T \mathbf{1}_{\{t_k<s\}} \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{293}$$

where $s \in (t, T]$ (s is fixed), $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Applying Theorem 12 to the iterated Ito stochastic integral (293), we obtain the following generalization of Theorem 7 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$.

Theorem 22. *Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$. Then, the following expansion*

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}(s) \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x,

$$C_{j_k\ldots j_1}(s) = \int\limits_{[t,T]^k} \bar{K}(t_1, \ldots, t_k, s) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k =$$

$$= \int\limits_t^s \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient, $\prod\limits_{\emptyset} \stackrel{\text{def}}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\text{def}}{=} 0$; *another notations are the same as in Theorem* 2.

Note that the estimates (152) and (154) will also be valid under the conditions of Theorem 22.

References

[1] Kulchitskiy O.Yu., Kuznetsov D.F. Approximation of multiple Ito stochastic integrals. [In Russian]. VINITI, 1679-V94 (1994), 38 pp.

[2] Kuznetsov D.F. Methods of numerical simulation of stochastic differential Ito equations solutions in problems of mechanics. Ph. D. Thesis, Saint-Petersburg State Technical University, 1996, 260 pp.

[3] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[4] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, Saint-Petersburg State Technical University, 204 pp. (ISBN 5-7422-0045-5)

[5] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
Available at: http://www.sde-kuznetsov.spb.ru/00a.pdf

[6] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[7] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[8] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[9] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[12] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[13] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[14] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[15] Kuznetsov D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing, Saarbrucken, 2012, 409 pp. Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[16] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[17] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier–Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[18] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[19] Kuznetsov D.F. On Numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[22] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2023, 996 pp.
[23] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html
[24] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html
[25] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html
[26] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR], 2023, 133 pp.
[27] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR], 2022, 221 pp.
[28] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR], 2022, 106 pp.
[29] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR], 2018, 77 pp.
[30] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor–Stratonovich expansion. [In English]. arXiv:1806.10705 [math.PR], 2018, 29 pp.
[31] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor–Ito expansion. [In English]. arXiv:1805.12527 [math.PR], 2018, 29 pp.
[32] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [In English]. arXiv:1807.02190 [math.PR], 2018, 44 pp.
[33] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR], 2022, 149 pp.
[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR], 2018, 70 pp.
[35] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.
[36] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR], 2018, 40 pp.
[37] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor–Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. [In English]. arXiv:1801.08862 [math.PR], 2018, 36 p.
[38] Kuznetsov D.F. Four new forms of the Taylor–Ito and Taylor–Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. [In English]. arXiv:2001.10192 [math.PR], 2020, 90 pp.
[39] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[41] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[42] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878

[43] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2018), 1-34.
Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[44] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier–Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR], 2018, 49 pp.

[45] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[46] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[47] Kuznetsov D.F. Development and application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2022, 57 pp.

[48] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[49] Kuznetsov D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2020), 129-162. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[50] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor–Ito and Taylor–Stratonovich expansions. [In English]. arXiv:1802.04844 [math.PR], 2018, 37 pp.

[51] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [In English]. arXiv:1801.03195 [math.PR], 2022, 159 pp.

[52] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.05654 [math.PR], 2018, 46 pp.

[53] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.07248 [math.PR], 2018, 20 pp.

[54] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [In English]. arXiv:1801.01564 [math.PR], 2018, 66 pp.

[55] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR], 2018, 29 pp.

[56] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[57] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[58] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[59] Kuznetsov D.F. Application of multiple Fourier–Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[60] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2019), 32-52.
Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[61] Kuznetsov D.F. New simple method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on expansion of the Brownian motion using Legendre polynomials and trigonometric functions. [In English]. arXiv:1807.00409 [math.PR], 2019, 23 pp.
[62] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.
[63] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.
[64] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
[65] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.
[66] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.
[67] Wiktorsson M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487.
[68] Ryden T., Wiktorsson M. On the simulation of iterated Ito integrals. Stochastic Processes and their Applications, 91, 1 (2001), 151-168.
[69] Gaines J. G., Lyons, T. J. Random generation of stochastic area integrals. SIAM J. Appl. Math., 54 (1994), 1132-1146.
[70] Gilsing H., Shardlow T. SDELab: A package for solving stochastic differential equations in MATLAB. Journal of Computational and Applied Mathematics, 2, 205 (2007), 1002-1018.
[71] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010, 868 pp.
[72] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.
[73] Rybakov K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2019), 1-31. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html
[74] Tang X., Xiao A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics, 45 (2019), 813-846.
[75] Zahri M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science, 8, 2 (2014), 186-198.
[76] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. [In Russian]. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.
[77] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.
[78] Skorohod A.V. Stochastc Processes with Independent Increments. Moscow, Nauka, 1964, 280 pp.
[79] Kloeden P.E., Neuenkirch A. The pathwise convergence of approximation schemes for stochastic differential equations. LMS Journal of Computation and Mathematics, 10 (2007), 235-253.
[80] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[81] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[82] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.
[83] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974, 696 pp.
[84] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. Ph. D. Thesis, California Institute of Technology, 2006, 225 pp.
[85] Itô K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.
[86] Rybakov K.A. Orthogonal expansion of multiple Ito stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[87] Kuo, H.-H. Introduction to Stochastic Integration. Universitext (UTX), Springer. N. Y., 2006, 289 pp.
[88] Chung K.L., Williams R.J. Introduction to stochastic integration. Progress in Probability and Stochastics. Vol. 4, Ed. Huber P., Rosenblatt M. Birkhauser, Boston, Basel, Stuttgart, 1983, 152 pp.
[89] Fox R., Taqqu, M.S. Multiple stochastic integrals with dependent integrators. Journal of Multivariate Analysis, 21 (1987), 105-127.
[90] Major P. The theory of Wiener–Itô integrals in vector valued Gaussian stationary random fields. Part I. Moscow Mathematical Journal, 20, 4 (2020), 749-812.
[91] Major P. Multiple Wiener–Itô Integrals With Applications to Limit Theorems. Second Edition. Springer. Cham, Heidelberg, New York, Dordrecht, London. 2014, 126 pp.

[92] Major, P. Wiener–Itô integral representation in vector valued Gaussian stationary random fields. arXiv:1901.04084 [math.PR]. 2019, 90 pp. DOI: https://doi.org/10.48550/arXiv.1901.04084

[93] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[94] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[95] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXACT CALCULATION OF THE MEAN-SQUARE ERROR IN THE METHOD OF EXPANSION OF ITERATED ITO STOCHASTIC INTEGRALS BASED ON GENERALIZED MULTIPLE FOURIER SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the developement of the method of expansion and mean-square approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$ (k is the multiplicity of the iterated Ito stochastic integral). We obtain the exact and approximate expressions for the mean-square approximation error of iterated Ito stochastic integrals of multiplicity k ($k \in \mathbb{N}$) from the stochastic Taylor–Ito expansion. As a result, we do not need to use redundant terms of expansions of iterated Ito stochastic integrals that complicate the numerical methods for Ito stochastic differential equations. Moreover, we proved the convergence with propability 1 for the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series for the cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series. Mean-square approximation of iterated Stratonovich stochastic integrals is also considered in the article. The results of the article can be applied to the high-order strong numerical methods for Ito stochastic differential equations as well as for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.
Keywords: Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Parseval equality, Ito stochastic differential equation, Taylor–Ito expansion, Mean-square convergence, Convergence in the mean of degree $2n$, ($n \in \mathbb{N}$), Convergence with probability 1, Expansion.

1. Introduction

In this article we develop the method of expansion and mean-square approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$ (k is the multiplicity of the iterated Ito stochastic integral), which was proposed and developed by the author of this work [1]-[51] (also see related publications [52]-[61]). Further, this method is referred to as the method of generalized multiple Fourier series.

The question of how to estimate or calculate exactly the mean-square approximation error of iterated Ito stochastic integrals for the method of generalized multiple Fourier series composes the subject of the article. From the one side the mentioned question is essentially difficult for the case of a multidimensional Wiener process, because of we need to take into account all possible combinations of components of the multidimensional Wiener process. From the other side an effective solution of the mentioned problem allows us to construct more economical numerical methods for Ito stochastic differential equations than in [62]-[64].

The results of the article (also see [1]-[51] and related publications [52]-[61]) will be useful for the implementation of high-order strong numerical methods for Ito stochastic differential equations as well as for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise. The latter methods are constucted, for example, in [65], [66].

2. Method of Generalized Multiple Fourier Series. The Case of Complete Orthonormal Systems of Continuous Functions in the Space $L_2([t,T])$ and Continuous Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)$

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continous family of σ-subfields of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider the following iterated Ito stochastic integrals

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1}$$

where $\psi_l(\tau)$ $(l=1,\ldots,k)$ are nonrandom functions at the interval $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$.

In addition, suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$ (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Sect. 5).

Let us define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ & \\ 0, & \text{otherwise}\end{cases}\quad=\quad\prod_{l=1}^k\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{2}$$

where $t_1,\ldots,t_k\in[t,T]$ for $k\ge 2$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{3}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{4}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}$$

is a norm in the space $L_2([t,T]^k)$.

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{5}$$

Theorem 1 [1] (2006) (also see [2]-[51]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{6}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m,$*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{7}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (4), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, *$\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition* (5).

Remark 1. *Further* (see Theorem 2) *we will use the following form of expansion* (6)

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right), \tag{8}$$

where

$$S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\ =\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{9}$$

where notations are the same as in Theorem 1.

Note that the version of Theorem 1 for the Haar and Rademacher–Walsh functions has been considered in [1]-[17], [27]. Some modifications of Theorem 1 for another types of iterated stochastic integrals (including iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process) as well as for complete orthonormal with weight $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in the space $L_2([t,T]^k)$ can be found in [27], [39] [46], [47] (also see [1]-[26], [28]-[38], [40]-[45], [48]-[51]). Generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Sect. 5.

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [1]-[51]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{10}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{11}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{12}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{13}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
&+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
J[\psi^{(6)}]_{T,t}=&\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
&-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
&-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
&-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
&-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
&-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
&-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
&-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
&-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
\end{aligned}
$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{15}$$

where $\mathbf{1}_A$ is the indicator of the set A.

The cases $k=7$ and $k>7$ are considered in [1]-[51] (also see Sect. 5).

3. Exact Calculation of the Mean-Square Approximation Error in The Method of Generalized Multiple Fourier Series. The Case of Complete Orthonormal Systems of Continuous Functions in the Space $L_2([t,T])$ and Continuous Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)$

Theorem 2 [29] (also see [12]-[17], [18]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{16}$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

(17)
$$J[\psi^{(k)}]_{T,t}^p=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right),$$

(18)
$$S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)},$$

the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form* (4),

(19)
$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j $(i=1,\ldots,m)$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorem* 1.

Remark 2. *Note that*

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=$$

$$=\mathsf{M}\left\{\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k=C_{j_k\ldots j_1}.$$

Therefore, from Theorem 2 *for the case of pairwise different numbers* $i_1,\ldots,i_k$ *we obtain*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2.$$

Moreover, if $i_1 = \ldots = i_k$, then from Theorem 2 we get

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\} =
$$

$$
= \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right),
$$

where

$$
\sum_{(j_1,\ldots,j_k)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

For example, for the case $k=3$ we have

$$
\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t} - J[\psi^{(3)}]_{T,t}^{p}\right)^2\right\} = \int\limits_t^T \psi_3^2(t_3)\int\limits_t^{t_3}\psi_2^2(t_2)\int\limits_t^{t_2}\psi_1^2(t_1)dt_1dt_2dt_3 -
$$

$$
- \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\left(C_{j_3j_2j_1} + C_{j_3j_1j_2} + C_{j_2j_3j_1} + C_{j_2j_1j_3} + C_{j_1j_2j_3} + C_{j_1j_3j_2}\right).
$$

Proof. Using Theorem 1 for the case $p_1 = \ldots = p_k = p$, we obtain

$$
J[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right). \tag{20}
$$

For $n > p$ we can write

$$
J[\psi^{(k)}]_{T,t}^{n} = \left(\sum_{j_1=0}^{p} + \sum_{j_1=p+1}^{n}\right)\ldots\left(\sum_{j_k=0}^{p} + \sum_{j_k=p+1}^{n}\right) C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right) =
$$

$$
= J[\psi^{(k)}]_{T,t}^{p} + \xi[\psi^{(k)}]_{T,t}^{p+1,n}. \tag{21}
$$

Let us prove that due to the special structure of random variables $S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}$ (see (11)–(15), (18)), the following relations are correct

$$
\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right\} = 0, \tag{22}
$$

$$
\mathsf{M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j_l'}^{(i_l)} - S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)}\right)\right\} = 0, \tag{23}
$$

where

$$(j_1,\ldots,j_k)\in \mathrm{K}_p,\quad (j_1',\ldots,j_k')\in \mathrm{K}_n\backslash \mathrm{K}_p$$

and

$$\mathrm{K}_n=\left\{(j_1,\ldots,j_k):\ 0\le j_1,\ldots,j_k\le n\right\},$$

$$\mathrm{K}_p=\left\{(j_1,\ldots,j_k):\ 0\le j_1,\ldots,j_k\le p\right\}.$$

For the case $i_1,\ldots,i_k=0,1,\ldots,m$ from the proof of Theorem 1 in [27] (also see [1]-[26], [28]-[51]) it follows that

$$J[\psi^{(k)}]_{T,t}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr)+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}\quad \text{w. p. 1},\tag{24}$$

where

$$R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{25}$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$, which are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals (see (25)) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

For the case $i_1,\ldots,i_k=1,\ldots,m$ and $p_1=\ldots=p_k=p$ from (24) we obtain

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\ =\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad \text{w. p. 1,} \tag{26}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in Theorem 1.

From (26) due to the moment property of the Ito stochastic integral we obtain (22). Let us prove (23). From (26) we have

$$0\le\left|\mathsf{M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j_l'}^{(i_l)}-S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)}\right)\right\}\right|=$$

$$=\left|\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.\right.$$

$$\left.\left.\times\int\limits_t^T\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}\right|\le$$

$$\le\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k}(t_k)\phi_{j_k'}(t_k)dt_k\ldots\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1=$$

$$=\sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{j_1=j_1'\}}\ldots\mathbf{1}_{\{j_k=j_k'\}}, \tag{27}$$

where where $\mathbf{1}_A$ is the indicator of the set A. Using (27), we obtain (23).

First, let us prove (27) for the cases $k=2$ and $k=3$ in detail. We have

$$\mathsf{M}\left\{\sum_{(j_1,j_2)}\sum_{(j_1',j_2')}\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\int\limits_t^T\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\right\}=$$

$$=\int\limits_t^T\phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds=$$

$$=\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}+\mathbf{1}_{\{i_1=i_2\}}\cdot\mathbf{1}_{\{j_2=j_1'\}}\mathbf{1}_{\{j_1=j_2'\}},\tag{28}$$

$$\mathsf{M}\left\{\sum_{(j_1,j_2,j_3)}\sum_{(j_1',j_2',j_3')}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\times\right.$$

$$\left.\times\int\limits_t^T\phi_{j_3'}(t_3)\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\right\}=$$

$$=\int\limits_t^T\phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_1'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_2'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds=$$

$$= \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}} + \mathbf{1}_{\{i_1=i_2\}}\cdot\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}} + \mathbf{1}_{\{i_1=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}. \tag{29}$$

From (28) and (29) we get

$$\left|\mathsf{M}\left\{\sum_{(j_1,j_2)}\sum_{(j_1',j_2')}\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\times\right.\right.$$

$$\left.\left.\times\int\limits_t^T\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\right\}\right|\le$$

$$\le \mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}} + \mathbf{1}_{\{j_2=j_1'\}}\mathbf{1}_{\{j_1=j_2'\}} =$$

$$= \sum_{(j_1',j_2')}\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}},$$

$$\left|\mathsf{M}\left\{\sum_{(j_1,j_2,j_3)}\sum_{(j_1',j_2',j_3')}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\times\right.\right.$$

$$\left.\left.\times\int\limits_t^T\phi_{j_3'}(t_3)\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\right\}\right|\le$$

$$\le \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}} + \mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+$$

$$+\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}} + \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$

$$+\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}} + \mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}} =$$

$$= \sum_{(j_1',j_2',j_3')}\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_3'\}},$$

where we used the relation

$$\int\limits_t^T \phi_i(\tau)\phi_j(\tau)d\tau = \mathbf{1}_{\{i=j\}}, \quad i,j = 0,1,2\ldots$$

Now consider the case of an arbitrary $k \in \mathbb{N}$. We have

$$\mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.$$

$$\left.\times\int\limits_t^T \phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\} =$$

$$= \mathsf{M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\times\right.$$

$$\left.\times\int\limits_t^T \phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{f}_{t_1}^{(i_1')}\ldots d\mathbf{f}_{t_k}^{(i_k')}\right\} =$$

$$= \sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\times$$

$$\times\int\limits_t^T \phi_{j_k}(t_k)\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1\ldots dt_k =$$

$$= \sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\times$$

$$\times\int\limits_t^T \phi_{j_k}(t_k)\phi_{j_k'}(t_k)dt_k\ldots\int\limits_t^{T}\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1 =$$

$$= \sum_{(j_1',\ldots,j_k')}\mathbf{1}_{\{i_k=i_k'\}}\ldots\mathbf{1}_{\{i_1=i_1'\}}\mathbf{1}_{\{j_k=j_k'\}}\ldots\mathbf{1}_{\{j_1=j_1'\}}, \tag{30}$$

where $(i_1',\ldots,i_k') = (i_1,\ldots,i_k)$. However, if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$ and if j_r' swapped with j_q' in the permutation $(j_1',\ldots,j_k')$, then i_r' swapped with i_q' in the permutation $(i_1',\ldots,i_k')$. From (30) we obtain (27). The equality (23) is proved.

Note that the formula (23) (in the light of the results of Sect. 5) can be interpreted as a consequence of the orthogonality of two random variables that are Hermite polynomials of vector random arguments.

Applying (22) and (23), we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{p}\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right\}=0.$$

Due to (17), (20), and (21) we can write

$$\xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}^{n}-J[\psi^{(k)}]_{T,t}^{p},$$

$$\underset{n\to\infty}{\text{l.i.m.}}\ \xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\stackrel{\text{def}}{=}\xi[\psi^{(k)}]_{T,t}^{p+1}.$$

We have

$$0\le\left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$=\left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1}-\xi[\psi^{(k)}]_{T,t}^{p+1,n}+\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$\le\left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1}-\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|+\left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1,n}J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$=\left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}\right|\le$$

$$\le\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^{2}\right\}}\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}}\le$$

$$\le\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^{2}\right\}}\times$$

$$\times\left(\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}-J[\psi^{(k)}]_{T,t}\right)^{2}\right\}}+\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^{2}\right\}}\right)\le$$

$$\le K\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{n}\right)^{2}\right\}}\to 0\quad\text{if}\quad n\to\infty,\tag{31}$$

where K is a constant.

From (31) it follows that

$$\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}=0$$

or

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)J[\psi^{(k)}]_{T,t}^{p}\right\}=0.$$

The last equality means that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}. \tag{32}$$

Taking into account (32), we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^{2}\right\}+$$

$$+\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}-2\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^{2}\right\}-$$

$$-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}. \tag{33}$$

Let us consider the value

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}.$$

Using (17) and (26), we get

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}. \tag{34}$$

After substituting (34) into (33), we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^{2}\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}.$$

Theorem 2 is proved.

4. Exact Calculation of the Mean-Square Approximation Error for the Cases $k=1,\ldots,5$

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}\stackrel{\rm def}{=}E_k^p,$$

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k \stackrel{\rm def}{=} I_k.$$

4.1. **The Case $k=1$.** For this case from Theorem 2 we obtain

$$E_1^p=I_1-\sum_{j_1=0}^{p}C_{j_1}^2.$$

4.2. **The Case $k=2$.** For this case from Theorem 2 we have

(I). $i_1\ne i_2$:

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2, \tag{35}$$

(II). $i_1=i_2$:

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}.$$

Example 1. Let us consider the following iterated Ito stochastic integral from the stochastic Taylor–Ito expansion [62]-[64] (also see [49] and [1]-[17])

$$I_{(00)T,t}^{(i_1i_2)}=\int\limits_t^T\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}, \tag{36}$$

where $i_1,i_2=1,\ldots,m$.

The approximation based on the expansion (11) for the integral (36) (the case of Legendre polynomials) has the following form [1]-[50]

$$I_{(00)T,t}^{(i_1i_2)p}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{p}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right). \tag{37}$$

It should be noted that the formula (37) has been derived for the first time in [54] (1997), [55] (1998) with using the another approach, which was developed in [32].

Applying (35), we obtain [1]-[50] (also see [54] (1997), [55] (1998))

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1i_2)}-I_{(00)T,t}^{(i_1i_2)p}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{p}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2). \tag{38}$$

4.3. **The Case** $k=3$**.** For the case $k=3$ from Theorem 2 we obtain

(I). $i_1\ne i_2, i_1\ne i_3, i_2\ne i_3:$

$$E_3^p=I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2, \tag{39}$$

(II). $i_1=i_2=i_3:$

$$E_3^p=I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\left(\sum_{(j_1,j_2,j_3)}C_{j_3j_2j_1}\right),$$

(III).1. $i_1=i_2\ne i_3:$

$$E_3^p=I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}, \tag{40}$$

(III).2. $i_1\ne i_2=i_3:$

$$E_3^p=I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}, \tag{41}$$

(III).3. $i_1=i_3\ne i_2:$

$$E_3^p=I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}C_{j_1j_2j_3}. \tag{42}$$

It should be noted that the formulas from the above cases (I), (III).1, (III).2, (III).3 have been derived in [1]-[17] by direct calculation.

Example 2. Let us consider the following iterated Ito stochastic integral from the stochastic Taylor–Ito expansion [62]-[64] (also see [49] and [1]-[17])

$$I_{(000)T,t}^{(i_1i_2i_3)}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}, \tag{43}$$

where $i_1,i_2,i_3=1,\ldots,m.$

The approximation based on the expansion (12) for the integral (43) (the case of Legendre polynomials and $p_1=p_2=p_3=p$) has the following form [1]-[50]

$$I_{(000)T,t}^{(i_1i_2i_3)p}=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{44}
$$

where

$$
C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1}, \tag{45}
$$

$$
\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,
$$

where $P_i(x)$ is the Legendre polynomial $(i=0,1,2,\ldots)$.

For example, using (40) we obtain

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}=
$$

$$
=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,J_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1},
$$

where $i_1=i_2\neq i_3$.

As mentioned in [30] (also see [1]-[17]), the exact values of coefficients $\bar{C}_{j_3j_2j_1}$ when $j_1,j_2,j_3=0,1,\ldots,p$ can be calculated using DERIVE (computer algebra system). In [30] (also see [1]-[17]) we can find several tables with exactly calculated Fourier–Legendre coefficients for approximations of iterated Ito stochastic integrals of multiplicities 1 to 5. In addition, in [57], [58], a database was obtained with 270,000 exactly calculated Fourier–Legendre coefficients for approximations of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6.

For the case $i_1=i_2=i_3$ we can use the following formula [62]-[64]

$$
I_{(000)T,t}^{(i_1i_1i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right)
$$

w. p. 1.

4.4. **The Case $k=4$.** For this case from Theorem 2 we obtain

(I). $i_1,\ldots,i_4$ are pairwise different:

$$
E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}^2,
$$

(II). $i_1=i_2=i_3=i_4$:

$$
E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\Biggl(\sum_{(j_1,\ldots,j_4)}C_{j_4\ldots j_1}\Biggr),
$$

(III).1. $i_1 = i_2 \ne i_3, i_4;\ i_3 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\right),$$

(III).2. $i_1 = i_3 \ne i_2, i_4;\ i_2 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\right),$$

(III).3. $i_1 = i_4 \ne i_2, i_3;\ i_2 \ne i_3:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1}\right),$$

(III).4. $i_2 = i_3 \ne i_1, i_4;\ i_1 \ne i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right),$$

(III).5. $i_2 = i_4 \ne i_1, i_3;\ i_1 \ne i_3:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right),$$

(III).6. $i_3 = i_4 \ne i_1, i_2;\ i_1 \ne i_2:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right),$$

(IV).1. $i_1 = i_2 = i_3 \ne i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1}\right),$$

(IV).2. $i_2 = i_3 = i_4 \ne i_1$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1}\right),$$

(IV).3. $i_1 = i_2 = i_4 \ne i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1}\right),$$

(IV).4. $i_1 = i_3 = i_4 \neq i_2$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1}\right),$$

(V).1. $i_1 = i_2 \neq i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right)\right),$$

(V).2. $i_1 = i_3 \neq i_2 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right)\right),$$

(V).3. $i_1 = i_4 \neq i_2 = i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right)\right).$$

4.5. **The Case $k = 5$.** For the case $k = 5$ from Theorem 2 we obtain

(I). $i_1, \ldots, i_5$ are pairwise different:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}^2,$$

(II). $i_1 = i_2 = i_3 = i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,\ldots,j_5)} C_{j_5\ldots j_1}\right),$$

(III).1. $i_1 = i_2 \neq i_3, i_4, i_5$ (i_3, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_5\ldots j_1}\right),$$

(III).2. $i_1 = i_3 \neq i_2, i_4, i_5$ (i_2, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1}\right),$$

(III).3. $i_1 = i_4 \neq i_2, i_3, i_5$ (i_2, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \right),$$

(III).4. $i_1 = i_5 \neq i_2, i_3, i_4$ (i_2, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right),$$

(III).5. $i_2 = i_3 \neq i_1, i_4, i_5$ (i_1, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right),$$

(III).6. $i_2 = i_4 \neq i_1, i_3, i_5$ (i_1, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \right),$$

(III).7. $i_2 = i_5 \neq i_1, i_3, i_4$ (i_1, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} C_{j_5\ldots j_1} \right),$$

(III).8. $i_3 = i_4 \neq i_1, i_2, i_5$ (i_1, i_2, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(III).9. $i_3 = i_5 \neq i_1, i_2, i_4$ (i_1, i_2, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right),$$

(III).10. $i_4 = i_5 \neq i_1, i_2, i_3$ (i_1, i_2, i_3 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4, i_5$ $(i_4 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\Biggr),$$

(IV).2. $i_1 = i_2 = i_4 \neq i_3, i_5$ $(i_3 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1}\Biggr),$$

(IV).3. $i_1 = i_2 = i_5 \neq i_3, i_4$ $(i_3 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).4. $i_2 = i_3 = i_4 \neq i_1, i_5$ $(i_1 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1}\Biggr),$$

(IV).5. $i_2 = i_3 = i_5 \neq i_1, i_4$ $(i_1 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).6. $i_2 = i_4 = i_5 \neq i_1, i_3$ $(i_1 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).7. $i_3 = i_4 = i_5 \neq i_1, i_2$ $(i_1 \neq i_2)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).8. $i_1 = i_3 = i_5 \neq i_2, i_4$ $(i_2 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Biggl(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\Biggr),$$

(IV).9. $i_1 = i_3 = i_4 \ne i_2, i_5$ $(i_2 \ne i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(IV).10. $i_1 = i_4 = i_5 \ne i_2, i_3$ $(i_2 \ne i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).1. $i_1 = i_2 = i_3 = i_4 \ne i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3,j_4)} C_{j_5\ldots j_1} \right),$$

(V).2. $i_1 = i_2 = i_3 = i_5 \ne i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_3,j_5)} C_{j_5\ldots j_1} \right),$$

(V).3. $i_1 = i_2 = i_4 = i_5 \ne i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).4. $i_1 = i_3 = i_4 = i_5 \ne i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(V).5. $i_2 = i_3 = i_4 = i_5 \ne i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3,j_4,j_5)} C_{j_5\ldots j_1} \right),$$

(VI).1. $i_5 \ne i_1 = i_2 \ne i_3 = i_4 \ne i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VI).2. $i_5 \neq i_1 = i_3 \neq i_2 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VI).3. $i_5 \neq i_1 = i_4 \neq i_2 = i_3 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right) \right),$$

(VI).4. $i_4 \neq i_1 = i_2 \neq i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).5. $i_4 \neq i_1 = i_5 \neq i_2 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right) \right),$$

(VI).6. $i_4 \neq i_2 = i_5 \neq i_1 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1} \right) \right),$$

(VI).7. $i_3 \neq i_2 = i_5 \neq i_1 = i_4 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \right) \right),$$

(VI).8. $i_3 \neq i_1 = i_2 \neq i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).9. $i_3 \neq i_2 = i_4 \neq i_1 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right) \right),$$

(VI).10. $i_2 \neq i_1 = i_4 \neq i_3 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).11. $i_2 \neq i_1 = i_3 \neq i_4 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).12. $i_2 \neq i_1 = i_5 \neq i_3 = i_4 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VI).13. $i_1 \neq i_2 = i_3 \neq i_4 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).14. $i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).15. $i_1 \neq i_2 = i_5 \neq i_3 = i_4 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).1. $i_1 = i_2 = i_3 \neq i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right),$$

(VII).2. $i_1 = i_2 = i_4 \neq i_3 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_5)}\left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).3. $i_1 = i_2 = i_5 \neq i_3 = i_4$:

$$E_p = I - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).4. $i_2 = i_3 = i_4 \neq i_1 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).5. $i_2 = i_3 = i_5 \neq i_1 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).6. $i_2 = i_4 = i_5 \neq i_1 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).7. $i_3 = i_4 = i_5 \neq i_1 = i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VI).8. $i_1 = i_3 = i_5 \neq i_2 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\right)\right),$$

(VII).9. $i_1 = i_3 = i_4 \neq i_2 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1}\right)\right),$$

(VII).10. $i_1 = i_4 = i_5 \neq i_2 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)}\left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1}\right)\right).$$

5. Method of Generalized Multiple Fourier Series. The Case of an Arbitrary Complete Orthonormal Systems of Functions in the Space $L_2([t,T])$ and Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$

For further consideration, let us consider the generalization of formulas (10)–(15) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (1). In order to do this, let us introduce some notations. Let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{46}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (46) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{47}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (47)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2,g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$

$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (6) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \tag{48}$$

where $[x]$ is an integer part of a real number x, $\prod\limits_{\emptyset}\stackrel{\sf def}{=}1$, $\sum\limits_{\emptyset}\stackrel{\sf def}{=}0$; another notations are the same as in Theorem 1.

another notations are the same as in Theorem **1**.

In particular, from (48) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\right.$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$\left.+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\right).$$

The last equality obviously agrees with (14).

Further, we will use the definition of the multiple Wiener stochastic integral from [68], [69] to generalize Theorem 1 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Consider the following step function on the hypercube $[t,T]^k$

$$\Phi_N(t_1,\ldots,t_k)=\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{49}$$

where $a_{l_1\ldots l_k}\in\mathbb{R}$ and such that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$,

$$\mathbf{1}_A(\tau)=\begin{cases}1 & \text{if } \tau\in A\\ \\ 0 & \text{otherwise}\end{cases},$$

$N\in\mathbb{N}$, $\{\tau_j\}_{j=0}^N$ is a partition of $[t,T]$, which satisfies the condition (5):

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{50}$$

Let us define the multiple Wiener stochastic integral for $\Phi_N(t_1,\ldots,t_k)$ [68], [69]

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}\stackrel{\text{def}}{=}\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{51}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_\tau^{(0)}=\tau$.

It is known (see [69], Lemma 9.6.4) that for any $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ there exists a sequence of step functions $\Phi_N(t_1,\ldots,t_k)$ of the form (49) such that

$$\lim_{N\to\infty}\int\limits_{[t,T]^k}\left(\Phi(t_1,\ldots,t_k)-\Phi_N(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=0. \tag{52}$$

We have

$$\Phi_N(t_1,\ldots,t_k)=\sum_{l_1,\ldots,l_k=0}^{N-1}a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k)=$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots\mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{53}$$

where permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$ (recall that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\neq q$).

Using (53), we get

$$\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{54}$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\neq l_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$= J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1,} \tag{55}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$ and permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1 < l_2 < \ldots < l_k$. At the same time the indices near upper limits of integration in the iterated stochastic integrals in (54) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$ (see (54)). In addition, the multiple Wiener stochastic integral $J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}$ is defined by (51) and

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

Since the integration of a bounded function with respect to the set of measure zero for Lebesgue integrals gives zero result, then the following formula is correct for these integrals

$$\int\limits_{[t,T]^k} |G(t_1,\ldots,t_k)|dt_1\ldots dt_k = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} |G(t_1,\ldots,t_k)|dt_1\ldots dt_k, \tag{56}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $dt_1,\ldots,dt_k$. At the same time the indexes near upper limits of integration are changed correspondently and the function $|G(t_1,\ldots,t_k)|$ is assumed to be integrable in the hypercube $[t,T]^k$.

Using (52), (55), (56), and standard moment properties of the Ito stochastic integral, we have

$$\mathsf{M}\left\{\left(J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} - J'[\Phi_M]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(\Phi_N(t_1,\ldots,t_k) - \Phi_M(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} \left(\Phi_N(t_1,\ldots,t_k) - \Phi_M(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k =$$

$$= C_k \left\|\Phi_N - \Phi_M\right\|^2_{L_2([t,T]^k)} \le$$

$$\le 2C_k \left(\left\|\Phi_N - \Phi\right\|^2_{L_2([t,T]^k)} + \left\|\Phi - \Phi_M\right\|^2_{L_2([t,T]^k)}\right)^2 \to 0$$

if $N, M \to \infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

Thus, there exists the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}.$$

We will define the multiple Wiener stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the formula

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\rm l.i.m.} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{57}$$

where $\Phi_N(t_1,\ldots,t_k)$ is defined by (49), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

Let us prove the following equality

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1}, \tag{58}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots\ d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (57) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

The equality (58) has already been proved for the case $\Phi(t_1,\ldots,t_k)=\Phi_N(t_1,\ldots,t_k)$ (see (55)). From (55) we have

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1}. \tag{59}$$

Passing to the limit $\underset{N\to\infty}{\rm l.i.m.}$ in the equality (59), we obtain

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\underset{N\to\infty}{\rm l.i.m.}\sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1}. \tag{60}$$

Using (52), (56), and standard moment properties of the Ito stochastic integral, we get

$$\mathsf{M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k\ \to 0 \tag{61}$$

if $N\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

The relations (60) and (61) prove the equality (58). From (58) we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}, \tag{62}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (1), $K=K(t_1,\ldots,t_k)$ is defined by (2), i.e.

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{63}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Applying (62) and the linearity property of the Ito stochastic integral, we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}, \tag{64}$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)\stackrel{\text{def}}{=}K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)$$

and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{65}$$

is the Fourier coefficient corresponding to $K(t_1,\ldots,t_k)$.

Again applying (58), we have

$$J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{66}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (57).

According to (3), (56), and the standard moment properties of the Ito stochastic integral, we have

$$\mathsf{M}\left\{\left(J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k\to 0 \tag{67}$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$.

Applying (64) and (67), we obtain the following expansion

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}. \tag{68}$$

In [13] (Sect. 1.14, Theorem 1.23), [51] (Theorem 5) it is shown that

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{69}$$

w. p. 1, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, $i_1,\ldots,i_k=0,1,\ldots,m$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (57), $[x]$ is an integer part of a real number x, $\prod_{\emptyset}\stackrel{\sf def}{=}1$, $\sum_{\emptyset}\stackrel{\sf def}{=}0$; another notations are the same as in Theorem 1.

Note that the right-hand side of (69) is nothing but the Hermite polynomial of random vector argument with components $\zeta_{j_1}^{(i_1)},\ldots,\zeta_{j_k}^{(i_k)}$.

The equalities (68) and (69) prove Theorem 1 for the case of an arbitrary complete orthonormal systems of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ is in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Let us find the representation of the right-hand side of (69) through the product of Hermite polynomials of scalar arguments.

We will say that the condition $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ $(i_1,\ldots,i_k=0,1,\ldots,m)$ *if* $m_1,\ldots,m_k$ *are multiplicities of the elements* $i_1,\ldots,i_k$, *respectively, i.e.*

$$\{i_1,\ldots,i_k\}=\{\overbrace{i_1,\ldots,i_1}^{m_1},\overbrace{i_2,\ldots,i_2}^{m_2},\ldots,\overbrace{i_r,\ldots,i_r}^{m_r}\},$$

where $r=1,\ldots,k$, *braces mean an unordered set, and parentheses mean an ordered set. At that,* $m_1+\ldots+m_k=k$, $m_1,\ldots,m_k=0,1,\ldots,k$, *and all elements with nonzero multiplicities are pairwise different.*

Let the condition $(\star\star)$ is fulfilled for the multi-index $(i_1\ldots i_k)$. Then

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}=J'\Biggl[\underbrace{\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}}_{m_1}\underbrace{\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}}_{m_2}\ldots$$

$$\ldots\underbrace{\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}}_{m_k}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1}\overbrace{i_2\ldots i_2}^{m_2}\ldots\overbrace{i_k\ldots i_k}^{m_k})} \tag{70}$$

w. p. 1, where $J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}$ is defined by (57), $\Phi(t_1,\ldots,t_k)=\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, $\{j_{g_1},\ldots,j_{g_{m_1+m_2+\ldots+m_k}}\}$ $=\{j_1,\ldots,j_k\}$, braces mean an unordered set, and parentheses mean an ordered set.

From (70) we have

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}=$$

$$=J'\left[\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}\right]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1})}\cdot J'\left[\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}\right]_{T,t}^{(\overbrace{i_2\ldots i_2}^{m_2})}\cdot\ldots$$

$$\ldots\cdot J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}\right]_{T,t}^{(\overbrace{i_k\ldots i_k}^{m_k})} \tag{71}$$

w. p. 1, where

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})} \stackrel{\sf def}{=} 1 \quad \hbox{for} \quad m_l=0. \tag{72}$$

The detailed proof of the equality (71) is given in [13] (Sect. 1.14), [51], Sect. 2.2).

Let us consider the following multiple Wiener stochastic integral

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})} \qquad (m_l>0),$$

where we suppose that

$$\left\{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}},\ldots,j_{g_{m_1+m_2+\ldots+m_l}}\right\}=$$
$$=\bigl\{\underbrace{j_{h_{1,l}},\ldots,j_{h_{1,l}}}_{n_{1,l}},\underbrace{j_{h_{2,l}},\ldots,j_{h_{2,l}}}_{n_{2,l}},\ldots,\underbrace{j_{h_{d_l,l}},\ldots,j_{h_{d_l,l}}}_{n_{d_l,l}}\bigr\}, \tag{73}$$

where $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;\ \ n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;\ \ d_l=1,\ldots,m_l;\ \ l=1,\ldots,k.$ Note that the numbers $m_1,\ldots,m_k,\ g_1,\ldots,g_k$ depend on $(i_1,\ldots,i_k)$ and the numbers $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l depend on $\{j_1,\ldots,j_k\}$. Moreover, $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$.

Using Theorem 9.6.9 [69] (also see [68], Theorem 3.1), we get w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \hbox{if} \quad i_l\ne 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \hbox{if} \quad i_l=0\end{cases} \qquad (m_l>0), \tag{74}$$

where $H_n(x)$ is the Hermite polynomial of degree n

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

or

$$H_n(x)=n!\sum_{m=0}^{[n/2]}\frac{(-1)^m x^{n-2m}}{m!(n-2m)!2^m} \quad (n\in\mathbb{N}), \tag{75}$$

and $\zeta_j^{(i)}$ $(i=0,1,\ldots,m,\ j=0,1,\ldots)$ is defined by (7).

Note that the equality (74) is proved in [13], [51] using the Ito formula (see the detailed proof in [13] (Sect. 1.14) or [51], Sect. 2.2).

From (72) and (74) we obtain w. p. 1

$$J'\left[\phi_{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_l}}}\right]_{T,t}^{(\overbrace{i_l\ldots i_l}^{m_l})}=$$

$$=\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}, \tag{76}$$

where $\mathbf{1}_A$ denotes the indicator of the set A.

Using (69), (71), and (76), we get w. p. 1

$$J'\left[\phi_{j_1}\ldots\phi_{j_k}\right]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)= \tag{77}$$

$$=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{78}$$

w. p. 1, where the multiple Wiener stochastic integral $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (57); another notations are the same as in (69).

Thus, the following theorem is proved.

Theorem 3 [13] (Sect. 1.11), [27] (Sect. 15). *Suppose that the condition* $(\star\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *and the condition* (73) *is also fulfilled. Furthermore, let* $\psi_l(\tau)\in L_2([t,T])$ $(l=1,\ldots,k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansions*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right), \tag{79}$$

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{80}$$

converging in the mean-square sense are valid, where $[x]$ *is an integer part of a real number* x, $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l$, $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l$, $d_l=1,\ldots,m_l$, $l=1,\ldots,k$; $m_1+\ldots+m_k=k$, *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$; $H_n(x)$ *is the Hermite polynomial* (75); *another notations are the same as in* (69) *and in Theorem* 1.

It should be noted that an analogue of the expansion (79) was considered in [70]. However, the proof of an analogue of the expansion (79) from [70] is different from the proof given in this section (see [51], Sect. 4 for details).

6. Exact Calculation of the Mean-Square Approximation Error in The Method of Generalized Multiple Fourier Series. The Case of an Arbitrary Complete Orthonormal Systems of Functions in the Space $L_2([t,T])$ and Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$

In this section, we generalize Theorem 2 to the case of an arbitrary complete orthonormal systems of functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 4 [27]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{81}$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{82}$$

$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ *is the multiple Wiener stochastic integral defined by* (57), *the Fourier coefficient* $C_{j_k\ldots j_1}$ *has the form* (65), $K(t_1,\ldots,t_k)$ *is defined by* (63),

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j $(i=1,\ldots,m)$,

$$\sum\limits_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$ *(see* (81)*).*

Proof. First, note that the formula (82) appears due to (68). Using the equality (58), we get

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\sum\limits_{(t_1,\ldots,t_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad\text{w. p. 1,}\tag{83}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

It is easy to see that the equality (83) can be written in the form

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\sum\limits_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad\text{w. p. 1,}\tag{84}$$

where

$$\sum\limits_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation.

Further proof of Theorem 4 is based on the equality (84) and is similar to the proof of Theorem 2. Theorem 4 is proved.

7. Estimate for the Mean-Square Approximation Error in the Method of Generalized Multiple Fourier Series

In this section, we prove the useful estimate for the mean-square error of approximation based on Theorem 3.

Theorem 5. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the estimate*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

(85) $$\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right)$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $J[\psi^{(k)}]_{T,t}$ *is the iterated Ito stochastic integral* (1), $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (80) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 3.

Proof. Using (64), (67), (68), Theorem 3, orthonormality of the system $\{\phi_j(x)\}_{j=0}^{\infty}$, and the elementary inequality

(86) $$(a_1+a_2+\ldots+a_n)^2\le n\left(a_1^2+a_2^2+\ldots+a_n^2\right),$$

we obtain for the case $i_1,\ldots,i_k=1,\ldots,m$ $(0<T-t<\infty)$ the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=k!\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k$$

$$=k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right).$$

Similarly using standard moment properties of stochastic integrals, we get

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$=C_k\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right),$$

where $i_1,\ldots,i_k=0,1,\ldots,m$ $(i_1^2+\ldots+i_k^2>0)$, and C_k is a constant.

It is not difficult to see that C_k depends on k (k is the multiplicity of the iterated Ito stochastic integral) and $T-t$ ($T-t$ is the length of integration interval for the iterated Ito stochastic integral). Moreover, C_k has the following form

$$C_k = k! \cdot \max\biggl\{(T-t)^{\alpha_1},\ (T-t)^{\alpha_2},\ \ldots,\ (T-t)^{\alpha_{k!}}\biggr\},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{k!} = 0,\ 1, \ldots,\ k-1$.

Then for the case $i_1, \ldots, i_k = 0, 1, \ldots, m$, $i_1^2 + \ldots + i_k^2 > 0$ $(0 < T-t < 1)$ we obtain (85). Theorem 5 is proved.

Example 3. Let us consider the estimate (85) for the iterated Ito stochastic integral $I_{(000)T,t}^{(i_1 i_2 i_3)}$ defined by (43)

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)p}\right)^2\right\} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2\right) \quad (i_1, i_2, i_3 = 1, \ldots, m),$$

where $C_{j_3 j_2 j_1}$ has the form (45).

8. Proof of Convergence With Probability 1 in the Method of Generalized Multiple Fourier Series. The Cases of Complete Orthonormal Systems of Legendre Polynomials and Trigonometric Functions in the Space $L_2([t,T])$

Remind that in a lot of author's publications [1]-[51] the convergence in Theorem 1 has been considered in different probability meanings. For example, the mean-square convergence [1] (2006) (also see [2]-[51]) and convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [3] (2007) (also see [4]-[17], [27]) have been proved. For some specific iterated Ito stochastic integrals of mutiplicities 1 and 2 the convergence with probability 1 also has been proved [3] (2007) (also see [4]-[17], [30]). However, these examples are narrow particular cases of the iterated Ito stochastic integrals (1).

In this section, we formulate and prove the theorem on convergence with probability 1 (w. p. 1) for expansions of iterated Ito stochastic integrals of multiplicity k ($k \in \mathbb{N}$) from Theorems 1, 3.

Let us remind the well-known fact from the mathematical analysis which is connected to existence of iterated limits.

Proposition 1. *Let* $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$\lim_{n,m\to\infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \text{for all} \quad m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \text{for all} \quad n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Theorem 6 [13]-[16], [27], [29], [30]. *Let $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are continuously differentiable nonrandom functions at the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to\ J[\psi^{(k)}]_{T,t} \quad \textit{if} \quad p \to \infty$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is the expression on the right-hand sides of (6) and (48) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1 = \ldots = p_k = p$, i.e.

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr)$$

or

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $i_1, \ldots, i_k = 1, \ldots, m$.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2, \tag{87}$$

where

$$K(t_1,\ldots,t_k) = \begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases} = \prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{88}$$

where $t_1, \ldots, t_k \in [t, T]$ for $k \geq 2$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$, $\mathbf{1}_A$ denotes the indicator of the set A,

(89) $$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient.

Using (88), we obtain

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k.$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

If $p_1=\ldots=p_k=p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\sf def}{=}\sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

and so on.

Lemma 1. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}=$$

(90) $$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C^2_{j_k\ldots j_1}=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty}C^2_{j_k\ldots j_1}$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Proof. Let us consider the value

(91) $$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C^2_{j_k\ldots j_1}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\}=\{1,\ldots,k\}$.

Obviously, (91) is the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le\sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2<\infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{92}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{93}$$

and equality of limits (92) and (93).

Taking into account the above reasoning, we have

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{94}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists (see the Parseval equality (87)), then from Proposition 1 we have

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 = \lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 = \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2. \tag{95}$$

Using (94) and Proposition 1, we get

$$\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 = \lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 = \sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2. \tag{96}$$

Combining (96) and (95), we obtain

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2.$$

Repeating the above reasoning, we complete the proof of Lemma 1.
Further, let us show that for $s = 1,\ldots,k$

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2. \tag{97}$$

Using the arguments which we used when proving Lemma 1, we have

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_s=0}^{p}\sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty} C_{j_k\ldots j_1}^2 \tag{98}$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\} = \{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we have

$$\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}C_{j_k\ldots j_1}^2=\ldots=$$

$$=\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty}\sum_{j_s=0}^{p}C_{j_k\ldots j_1}^2. \tag{99}$$

Using (98), (99), and Lemma 1, we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-$$

$$-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.$$

The equality (97) is proved.

Applying the Parseval equality and Lemma 1, we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}+$$

$$+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=\ldots=$$

$$=\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+$$

$$+\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\ldots+\sum_{j_1=0}^{p}\ldots\sum_{j_{k-1}=0}^{p}\sum_{j_k=p+1}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=0}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+$$

$$+\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\ldots+\sum_{j_1=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\sum_{j_k=p+1}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{s=1}^{k}\left(\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\right). \tag{100}$$

Note that deriving (100), we used the following

$$\sum_{j_1=0}^{p}\ldots\sum_{j_{s-1}=0}^{p}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\lim_{m_{s-1}\to\infty}\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$=\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le\ldots\le$$

$$\le\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2,$$

where $m_1,\ldots,m_{s-1}>p$.

Denote

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,$$

where $s=1,\ldots,k-1$.

Let us remind the Dini Theorem, which we will use further.

Theorem (Dini). *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a,b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a,b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a,b]$.*

For $s<k$ due to the Parseval equality and Dini Theorem as well as (97) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$\stackrel{(97)}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=$$

$$\stackrel{\text{(Dini Th.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\int\limits_t^{t_k}\psi_{k-1}^2(t_{k-1})\left(C_{j_{k-2}\ldots j_1}(t_{k-1})\right)^2\times$$

$$\times dt_{k-1}dt_k\le$$

$$\le C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=$$

$$\stackrel{\text{(Dini Th.)}}{=}C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=$$

$$\stackrel{\text{(Parseval Eq.)}}{=} C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_{k-2}^2(\theta)\left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le K\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le$$

$$\le \ldots \le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\int\limits_t^T\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$\stackrel{\text{(Dini Th.)}}{=} C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \tag{101}$$

where constants C, K depend on $T-t$ and constant C_k depends on k and $T-t$.

Let us explane more precisely how we obtain (101). For any function $g(s)\in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty}\left(\int\limits_t^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_t^{T}\mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s)ds=\int\limits_t^{\tau}g^2(s)ds. \tag{102}$$

The equality (102) has been applied repeatedly when we obtaining (101). Using the integration order replacement in Riemann integrals, we have

$$C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s=$$

$$=\int\limits_t^{\tau}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{\tau}\phi_{j_2}(t_2)\psi_2(t_2)\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1 \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\tilde{C}_{j_s\ldots j_1}(\tau).$$

For $l=1,\ldots,s$ we will use the following notation

$$\tilde{C}_{j_s\ldots j_l}(\tau,\theta)=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.$$

Applying the Parseval equality and Dini Theorem, from (101) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=$$

$$=C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_1}(\tau)\right)^2d\tau=$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \qquad (103)$$

$$\stackrel{\text{(Dini Th.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \qquad (104)$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t_1}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le C_k'\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2d\tau\le$$

$$\le\ldots\le$$

$$\leq C_k^{''} \sum_{j_s=p+1}^{\infty} \int_t^T \int_t^{\tau} \psi_{s-1}^2(t_{s-1}) \left(\tilde{C}_{j_s}(\tau, t_{s-1})\right)^2 dt_{s-1} d\tau \leq$$

$$\leq \tilde{C}_k \sum_{j_s=p+1}^{\infty} \int_t^T \int_t^{\tau} \left(\int_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta \right)^2 du d\tau, \tag{105}$$

where constants $C_k^{'}$, $C_k^{''}$, $\tilde{C}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (105). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty} \left(\int_{\theta}^{\tau} \phi_j(s) g(s) ds \right)^2 = \sum_{j=0}^{\infty} \left(\int_t^T \mathbf{1}_{\{\theta < s < \tau\}} \phi_j(s) g(s) ds \right)^2 =$$

$$= \int_t^T \left(\mathbf{1}_{\{\theta < s < \tau\}} \right)^2 g^2(s) ds = \int_{\theta}^{\tau} g^2(s) ds. \tag{106}$$

The equality (106) has been applied repeatedly when we obtaining (105). Let us explane more precisely the passing from (103) to (104) (the same steps were used when we were deriving (105)).

We have

$$\int_t^T \int_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau - \sum_{j_2=0}^{n} \int_t^T \int_t^{\tau} \psi_1^2(t_1) \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau =$$

$$= \int_t^T \int_t^{\tau} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau =$$

$$= \lim_{N \to \infty} \sum_{j=0}^{N-1} \int_t^{\tau_j} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2 dt_1 \Delta\tau_j, \tag{107}$$

where $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition (5).

Since the non-decreasing functional sequence $u_n(\tau_j, t_1)$ and its limit function $u(\tau_j, t_1)$ are continuous on the interval $[t, \tau_j] \subseteq [t,T]$ with respect to t_1, where

$$u_n(\tau_j, t_1) = \sum_{j_2=0}^{n} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2,$$

$$u(\tau_j, t_1) = \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2 = \int_{t_1}^{\tau_j} \psi_2^2(t_2) \left(\tilde{C}_{j_s \ldots j_3}(\tau_j, t_2)\right)^2 dt_2,$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j, t_1)$ to $u(\tau_j, t_1)$ at the interval $[t, \tau_j] \subseteq [t, T]$ with respect to t_1. As a result, we obtain

$$\sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2 < \varepsilon, \quad t_1 \in [t, \tau_j] \tag{108}$$

for $n > N(\varepsilon)$ ($N(\varepsilon)$ exists for any $\varepsilon > 0$ and it does not depend on t_1).

From (107) and (108) we get

$$\lim_{N\to\infty} \sum_{j=0}^{N-1} \int\limits_t^{\tau_j} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau_j, t_1)\right)^2 dt_1 \Delta\tau_j \le \varepsilon \lim_{N\to\infty} \sum_{j=0}^{N-1} \int\limits_t^{\tau_j} \psi_1^2(t_1) dt_1 \Delta\tau_j =$$

$$= \varepsilon \int\limits_t^{T} \int\limits_t^{\tau} \psi_1^2(t_1) dt_1 d\tau. \tag{109}$$

From (109) we have

$$\lim_{n\to\infty} \int\limits_t^{T} \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s \ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = 0.$$

This fact completes the proof of passing from (103) to (104).

Let us estimate the integral

$$\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta \tag{110}$$

from (105) for the case when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

Note that the estimates for the integral

$$\int\limits_t^{\tau} \phi_j(\theta)\psi(\theta) d\theta, \quad j \ge p + 1 \tag{111}$$

have been obtained in [22], [31], where $\psi(\theta)$ is a continuously differentiable function on the interval $[t, T]$. The same estimates can also be found in early publications [10]-[12], [17] or in [13]-[16] (2020-2023).

Let us estimate the integral (110) using the approach from [22], [31]. First consider the case of Legendre polynomials. Then $\phi_j(s)$ looks as follows

$$\phi_j(\theta) = \sqrt{\frac{2j+1}{T-t}} P_j \left(\left(\theta - \frac{T+t}{2}\right) \frac{2}{T-t} \right), \quad j \ge 0, \tag{112}$$

where $P_j(x)$ is the Legendre polynomial.

Further, we have

$$\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta=\frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy=$$

$$=\frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x))-P_{j-1}(z(x)))\psi(x)-(P_{j+1}(z(v))-P_{j-1}(z(v)))\psi(v)-$$

$$-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y)-P_{j-1}(y))\psi'(u(y))dy\Biggr), \tag{113}$$

where $x,v\in(t,T)$, $j\ge p+1$, $u(y)$ and $z(x)$ are defined by the following relations

$$u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t},$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (113) we used the following well-known property of the Legendre polynomials

$$\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),\quad j=1,2,\ldots$$

From (113) and the well-known estimate for the Legendre polynomials

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{114}$$

where constant K does not depend on y and j, it follows that

$$\left|\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+C_1\right), \tag{115}$$

where constants C,C_1 do not depend on j $(j>0)$ and $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$.

From (115) we obtain

$$\left(\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta\right)^2<\frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}}+\frac{1}{(1-(z(v))^2)^{1/2}}+C_3\right), \tag{116}$$

where constants C_2,C_3 do not depend on j $(j>0)$.

Let us apply (116) for estimation of the right-hand side of (105). We have

$$\int\limits_t^T\int\limits_t^{\tau}\left(\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/2}}+\int\limits_{-1}^{1}\int\limits_{-1}^{x}\frac{dy}{(1-y^2)^{1/2}}dx+K_2\right)\le$$

$$\le \frac{K_3}{j_s^2}, \tag{117}$$

where constants K_1, K_2, K_3 are independent of j_s ($j_s > 0$).

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r\end{cases} \tag{118}$$

where $r=1,2,\ldots$

Using the system of functions (118), we have

$$\int\limits_v^x\phi_{2r-1}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x\sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=-\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\cos\frac{2\pi r(x-t)}{T-t}-\psi(v)\cos\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x\cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{119}$$

$$\int\limits_v^x\phi_{2r}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x\cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\sin\frac{2\pi r(x-t)}{T-t}-\psi(v)\sin\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x\sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{120}$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (119) and (120), we obtain for the trigonometric case

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 \le \frac{C_4}{j^2}, \tag{121}$$

where constant C_4 is independent of j $(j>0)$.

From (121) we finally have

$$\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le \frac{K_4}{j_s^2}, \tag{122}$$

where constant K_4 does not depend on j_s $(j_s>0)$.

Combibing (105), (117), and (122), we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 \le$$

$$\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2}\le\frac{L_k}{p}, \tag{123}$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (100) and (123) we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\le\frac{G_k}{p}, \tag{124}$$

where constant G_k depends on k and $T-t$.

For the further consideration we consider the following theorem.

Theorem 7 [3] (2007) (also see [4]-[17]). *Under the conditions of Theorems* 1, 3 *the following estimate is correct*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \times$$

$$\times\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n \tag{125}$$

for $n \in \mathbb{N}$, *where* $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (48) *before passing to the limit, i.e.*

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $i_1,\ldots,i_k = 1,\ldots,m$ *and the remainder notations are the same as in Theorems* 1, 3.

Using (124) and Theorem 7 for the case $p_1 = \ldots = p_k = p$ and $n = 2$ (see (125)), we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le$$

$$\le C_{2,k}\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2\right)^2 \le \frac{H_{2,k}}{p^2}, \tag{126}$$

where $H_{2,k} = G_k^2 C_{2,k}$ and

$$C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

Let us consider the the well-known fact.

Proposition 2. *If for the sequence of random variables* ξ_p *and for some real* $\alpha > 0$ *the number series*

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p *converges to zero w. p. 1.*

Let us put

$$\xi_p = \left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|$$

and $\alpha = 4$.

Then from (126) we obtain

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \le H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty. \tag{127}$$

From (127) we get

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to\ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p\to\infty$$

w. p. 1, where (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr) \tag{128}$$

or (see (48))

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{129}$$

where $i_1,\ldots,i_k=1,\ldots,m$ in (128) and (129). Theorem 6 is proved.

9. Mean-Square Approximation of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6

This section is devoted to the mean-square approximation of iterated Stratonovich stochastic integrals. We consider the adaptation of Theorems 1, 3 for iterated Stratonovich stochastic integrals of multiplicities 1 to 6. Also we consider the question on the exact calculation of the mean-square approximation errors for the following iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)},\quad I_{(1)T,t}^{*(i_1)},\quad I_{(00)T,t}^{*(i_1i_2)},\quad I_{(000)T,t}^{*(i_1i_2i_3)},\quad i_1,i_2,i_3=1,\ldots,m,$$

where

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)} \tag{130}$$

is the iterated Stratonovich stochastic integral; $i_1,\ldots,i_k=1,\ldots,m;\ \ l_1,\ldots,l_k=0,1,\ldots$

Let us first formulate some old results.

Theorem 8 [13], [22] (also see [6]-[12], [14]-[21], [23], [26], [28], [30], [31], [35], [37], [40]-[45], [57], [58]). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or* *trigonometric functions in the space* $L_2([t,T])$*. Moreover,* $\psi_1(\tau)$, $\psi_2(\tau)$ *are continuously differentiable functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \tag{131}$$

that converges in the mean-square sense is valid, where

$$C_{j_2j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 \tag{132}$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

Theorem 9 [13], [22] (also see [6]-[12], [14]-[21], [23], [26], [28], [30], [31], [37], [40]-[44], [57], [58]). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{133}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Theorem 10 [13], [22] (also see [7]-[12], [14]-[21], [23], [26], [28], [30], [31], [37], [40]-[44], [57], [58]). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1, i_2, i_3, i_4 = 0, 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \tag{134}$$

that converges in the mean-square sense is valid, where

$$C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [13] (Sect. 2.10–2.16), [31] (Sect. 13–19), [36] (Sect. 5–11), [37] (Sect. 7–13). Let us formulate four theorems that were obtained using this approach.

Theorem 11 [13], [31], [36], [37]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{135}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{136}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (135) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (136)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*); another notations are the same as in Theorem* 1*.*

Theorem 12 [13], [31], [36], [37]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{137}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{138}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{139}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (137), (138) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (139)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$C_{j_4 j_3 j_2 j_1}=\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 11.

Theorem 13 [13], [31], [36], [37]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\ \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{140}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{141}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{142}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (140), (141) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (142)*, constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorem 11, 12.

Theorem 14 [13], [31], [36], [37]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{143}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1, \ldots, i_6 = 0, 1, \ldots, m$,

$$C_{j_6 \ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_6;$$

another notations are the same as in Theorems 11–13.

Consider the question on the exact calculation of the mean-square approximation errors for the following iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)}, \quad I_{(1)T,t}^{*(i_1)}, \quad I_{(00)T,t}^{*(i_1 i_2)}, \quad I_{(000)T,t}^{*(i_1 i_2 i_3)}, \quad i_1, i_2, i_3 = 1, \ldots, m. \tag{144}$$

We assume that the stochastic integrals (144) are approximated using Theorems 1, 8, 9 and the Legendre polynomial system. Since $I_{(0)T,t}^{(i_1)} = I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{(i_1)} = I_{(1)T,t}^{*(i_1)}$ w. p. 1, we can use the following formulas (see (10) for the case of Legendre polynomilas)

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t} \zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right)$$

to approximate the stochastic integrals $I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{*(i_1)}$. In this case, we will have zero mean-square approximation errors.

To approximate the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1 i_2)}$ we can use the formula (see (37))

$$I_{(00)T,t}^{*(i_1 i_2)p} = \frac{T-t}{2} \left(\zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{i=1}^{p} \frac{1}{\sqrt{4i^2-1}} \left(\zeta_{i-1}^{(i_1)} \zeta_i^{(i_2)} - \zeta_i^{(i_1)} \zeta_{i-1}^{(i_2)} \right) \right). \tag{145}$$

The mean-square approximation error for (145) will be determined by the formula (38) ($i_1 \neq i_2$). For the case $i_1 = i_2$ we can use the well-known equality [62]

$$I_{(00)T,t}^{*(i_1 i_1)} = \frac{T-t}{2} \left(\zeta_0^{(i_1)} \right)^2 \quad \text{w. p. 1.}$$

Consider now the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1 i_2 i_3)}$ of multiplicity 3 ($i_1, i_2, i_3 = 1, \ldots, m$). For the case of pairwise different i_1, i_2, i_3 we can use the formula (39). In the case $i_1 = i_2 = i_3$, to approximate the stochastic integral $I_{(000)T,t}^{*(i_1 i_1 i_1)}$, we use the well-known equality [62]

$$I_{(000)T,t}^{*(i_1 i_1 i_1)} = \frac{(T-t)^{3/2}}{6} \left(\zeta_0^{(i_1)} \right)^3 \quad \text{w. p. 1.}$$

Thus, it remains to consider the following three cases

$$i_1 = i_2 \neq i_3, \tag{146}$$

$$i_1 \neq i_2 = i_3, \tag{147}$$

$$i_1 = i_3 \neq i_2. \tag{148}$$

Taking into account the standard relations between Ito and Stratonovich stochastic integrals [62] and Theorem 1 (the case $k=3$) together with Theorem 9, we obtain

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)p}\right)^2\right\}=
$$

$$
={\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}+\frac{1}{2}{\bf 1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{\tau}dsd{\bf f}_{\tau}^{(i_3)}+\frac{1}{2}{\bf 1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{\tau}d{\bf f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1 i_2 i_3)p}\right)^2\right\}=
$$

$$
={\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)p}+I_{(000)T,t}^{(i_1 i_2 i_3)p}+{\bf 1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd{\bf f}_{\tau}^{(i_3)}\ +\right.\right.
$$

$$
\left.\left.+{\bf 1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d{\bf f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1 i_2 i_3)p}\right)^2\right\}, \tag{149}
$$

where $I_{(000)T,t}^{(i_1 i_2 i_3)}$ and $I_{(000)T,t}^{(i_1 i_2 i_3)p}$ are defined by the relations (43), (44). Moreover, $I_{(000)T,t}^{*(i_1 i_2 i_3)p}$ has the form (see Theorem 9)

$$
I_{(000)T,t}^{*(i_1 i_2 i_3)p}=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{150}
$$

Substituting (44) and (150) into (149) yields

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)p}\right)^2\right\}=
$$

$$
={\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)p}+{\bf 1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd{\bf f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)+\right.\right.
$$

$$
\left.\left.+{\bf 1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d{\bf f}_{s}^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{p}C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)-{\bf 1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{p}C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \tag{151}
$$

Consider the case (146). From (151) we obtain

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)p}\right)^2\right\}=
$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p} + \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{152}$$

According to the formulas (17), (26), the quantity

$$I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p}$$

includes only iterated Ito stochastic integrals of multiplicity 3. At the same time, the quantity

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}$$

contains only iterated Ito stochastic integrals of multiplicity 1. This means that from (152) we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)} - I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(\tau-t)d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{153}$$

The relation (40) implies that

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\} = \frac{(T-t)^3}{6} -$$

$$- \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}, \tag{154}$$

where $i_1 = i_2 \neq i_3$.

We have

$$\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(\tau-t)d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} = \frac{1}{4}\int\limits_t^T(\tau-t)^2 d\tau -$$

$$- \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\int\limits_t^T(\tau-t)\phi_{j_3}(\tau)d\tau + \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}\right)^2, \tag{155}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

Using the orthogonality property of Legendre polynomials, we obtain

$$\int\limits_t^T (\tau-t)\phi_{j_3}(\tau)d\tau = \frac{(T-t)^{3/2}}{2}\begin{cases} 1, & j_3=0\\ 1/\sqrt{3}, & j_3=1.\\ 0, & j_3\ge 2\end{cases} \tag{156}$$

Combining (153)–(156), we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\}=\frac{(T-t)^3}{4}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}-$$

$$-\frac{(T-t)^{3/2}}{2}\sum_{j_1=0}^{p}\left(C_{0j_1j_1}+\frac{1}{\sqrt{3}}C_{1j_1j_1}\right)+\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}\right)^2, \tag{157}$$

where $i_1=i_2\ne i_3$.

Consider the case (147). From (151) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}+\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}+\frac{1}{2}\int\limits_t^T(T-s)d\mathbf{f}_s^{(i_1)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(T-s)d\mathbf{f}_s^{(i_1)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}+$$

$$+\frac{1}{4}\int\limits_t^T(T-s)^2ds-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\int\limits_t^T(T-s)\phi_{j_1}(s)ds+$$

$$+\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2, \tag{158}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

The relation (41) implies that

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}=\frac{(T-t)^3}{6}-$$

$$-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}, \tag{159}$$

where $i_1\ne i_2=i_3$.

Moreover,

$$\int\limits_t^T(T-s)\phi_{j_1}(s)ds=\frac{(T-t)^{3/2}}{2}\begin{cases}1, & j_1=0\\ -1/\sqrt{3}, & j_1=1.\\ 0, & j_1\ge 2\end{cases} \tag{160}$$

Combining (158)–(160), we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\}=\frac{(T-t)^3}{4}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}-$$

$$-\frac{(T-t)^{3/2}}{2}\sum_{j_3=0}^{p}\left(C_{j_3j_30}-\frac{1}{\sqrt{3}}C_{j_3j_31}\right)+\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}\right)^2, \tag{161}$$

where $i_1\ne i_2=i_3$.

Consider the case (148). From (151) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}-\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\}+\mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}=$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\} + \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}\right)^2. \tag{162}$$

The relation (42) implies that

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)} - I_{(000)T,t}^{(i_1i_2i_3)p}\right)^2\right\} = \frac{(T-t)^3}{6} -$$

$$- \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3}, \tag{163}$$

where $i_1 = i_3 \neq i_2$.

Combining (162) and (163), we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)} - I_{(000)T,t}^{*(i_1i_2i_3)p}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3} +$$

$$+ \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}\right)^2, \tag{164}$$

where $i_1 = i_3 \neq i_2$.

Thus, the exact calculaton of the mean-square approximation error for the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1i_2i_3)}$ $(i_1, i_2, i_3 = 1, \ldots, m)$ is given by the formulas (39), (157), (161), and (164).

References

[1] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-227
Available at: http://www.sde-kuznetsov.spb.ru/06.pdf

[2] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf

[3] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, xxxii+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf

[4] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, xxxiv+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf

[5] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: Saint-Petersburg, 2010, xxx+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf

[6] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. [In Russian]. Differential Equations and Control Processes, 3 (2010), A.1-A.257.
Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[7] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf

[8] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233
Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf

[9] Kuznetsov D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing: Saarbrucken, 2012 , 409 pp. Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[10] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House: Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf

[11] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Differential Equations and Control Processes, 1 (2017), A.1–A.385. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[12] Kuznetsov D.F. Stochastic differential equations: theory and practice of numerical solution. With programs on MATLAB, 5th Edition. [In Russian]. Differential Equations and Control Processes, 2 (2017), A.1-A.1000. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[13] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [In English]. arXiv:2003.14184v45 [math.PR], 2023, 996 pp.

[14] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Differential Equations and Control Processes, 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[15] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. [In English]. Differential Equations and Control Processes, 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[16] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs (Third Edition). [In English]. Differential Equations and Control Processes, 1 (2023), A.1-A.947. Available at:
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[17] Kuznetsov D.F. Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Edition. [In Russian]. Differential Equations and Control Processes, 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[18] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[19] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[20] Kuznetsov D.F. On Numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[21] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[22] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals, based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[23] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

[24] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Differential Equations and Control Processes. 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[25] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020). DOI: http://10.1137/S0040585X97T989878

[26] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Differential Equations and Control Processes, 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[27] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.

[28] Kuznetsov D.F. Developement and Application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [In English]. arXiv:1712.08991 [math.PR]. 2017, 57 pp.

[29] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2023, 70 pp.

[30] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[31] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2023, 222 pp.

[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2023, 80 pp.

[33] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [In English]. arXiv:1801.01564 [math.PR]. 2018, 66 pp.

[34] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and repeated Fourier series. [In English]. arXiv:1801.05654 [math.PR]. 2018, 46 pp.

[35] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.07248 [math.PR]. 2018, 20 pp.

[36] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.

[37] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [In English]. arXiv:1801.03195 [math.PR]. 2023, 158 pp.

[38] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[39] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[40] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.

[41] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. [In English]. arXiv:1806.10705 [math.PR]. 2018, 29 pp.

[42] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Ito expansion. [In English]. arXiv:1805.12527 [math.PR]. 2018, 29 pp.

[43] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. [In English]. arXiv:1802.04844 [math.PR]. 2018, 37 pp.

[44] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR], 2018, 44 pp.

[45] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR]. 2023, 49 pp.
[46] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[47] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.
[48] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor-Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. [In English]. arXiv:1801.08862 [math.PR], 2018, 36 pp.
[49] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. [In English]. arXiv:2001.10192 [math.PR], 2020, 90 pp.
[50] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Differential Equations and Control Processes, 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html
[51] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR], 2023, 58 pp.
[52] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. [In Russian]. Differential Equations and Control Processes, 4, (2019), 32-52. Available at: https://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html
[53] Kuznetsov D.F. New simple method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on expansion of the Brownian motion using Legendre polynomials and trigonometric functions. [In English]. arXiv:1807.00409 [math.PR], 2019, 23 pp.
[54] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Differential Equations and Control Processes, 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html
[55] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Differential Equations and Control Processes, 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html
[56] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: 10.1615/JAutomatInfScien.v32.i12.80 Available at: http://www.sde-kuznetsov.spb.ru/00a.pdf
[57] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html
[58] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.
[59] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.
[60] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier-Legendre series. Journal of Physics: Conference Series. Vol. 1925, 2021, article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010
[61] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf
[62] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.
[63] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.
[64] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.
[65] Jentzen A., Röckner M. A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.
[66] Becker S., Jentzen A., Kloeden P.E. An exponential Wagner-Platen type scheme for SPDEs. SIAM J. Numer. Anal. 54, 4 (2016), 2389-2426.

[67] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982. 612 pp.

[68] Itô K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.

[69] Kuo, H.-H. Introduction to Stochastic Integration. Universitext (UTX), Springer. N. Y., 2006, 289 pp.

[70] Rybakov K.A. Orthogonal expansion of multiple Ito stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STOCHASTIC INTEGRALS WITH RESPECT TO MARTINGALE POISSON MEASURES AND WITH RESPECT TO MARTINGALES BASED ON GENERALIZED MULTIPLE FOURIER SERIES

DMITRIY F. KUZNETSOV

Abstract. We consider some versions and generalizations of the approach to the expansion of iterated Ito stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on generalized multiple Fourier series. Expansions of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales were obtained. For the iterated stochastic integrals with respect to martingales, we have proved a theorem which gives a generalization of the expansion for iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series. Also we consider a modification of the mentioned expansion of iterated Ito stochastic integrals for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \geq 0$ systems of functions in the space $L_2([t,T]^k)$. Mean-square convergence of the considered expansions is proved. An example of the expansion of iterated (double) stochastic integrals with respect to martingales using the system of Bessel functions is considered.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
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1. Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a non-decreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau,\quad \mathbf{x}_0=\mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The non-random functions $\mathbf{a}:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $B:\mathbb{R}^n\times[0,T]\to\mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\}<\infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ are independent when $t>0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[17]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau,\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in (3), we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[7]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k,\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [8]-[17].

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[5], [10]-[61].

The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case can be investigated using the Ito formula [2]-[4].

Note that even for the mentioned coincidence $(i_1 = \ldots = i_k \neq 0)$ but for different functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ the mentioned difficulties persist. As a result, relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, can not be represented effectively in a finite form (with respect to the mean-square criterion of approximation) using the system of standard Gaussian random variables.

Usually, approaches to the expansion of iterated stochastic integrals (2) and (3) are based on the expansion of the Wiener process.

For example, in [3] (also see [2], [4]) Milstein G.N. proposed to expand (2) or (3) (the case $k = 2$ and $i_1 \neq i_2$; $i_1, i_2 = 1, \ldots, m$) into the iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as a trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of (2) or (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_\tau$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. The above procedure leads to iterated application of the operation of limit transition and does not lead to a general expansion of (2) or (3) which is valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals were presented in [2] $(k = 1, 2, 3)$ and in [3], [4] $(k = 1, 2)$ for the simplest case $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$. Moreover, generally speaking, the convergence of approximations to the appropriate stochastic integrals (3) is not proved rigorously for $k = 3$ in [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82–84), [62] (pp. 438–439), [63] (pp. 263–264) (see [15]-[18] (Sect. 6.2), [43], [45]-[48] for details).

Note that in [60], [61] a method for the expansion of double [60], [61] and triple [60] Ito stochastic integrals (2) $(k = 2, 3;\ \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1;\ i_1, i_2, i_3 = 0, 1, \ldots, m)$ based on the expansion of the Wiener process using Haar functions [61] and trigonometric functions [60], [61] has been considered. The restrictions of this method [60], [61] are also connected with the iterated application of the operation of limit transition at least starting from the second or third multiplicity of iterated stochastic integrals.

A more effective and general approach to the expansion of iterated Ito stochastic integrals (2) of arbitrary multiplicity k $(k \in \mathbb{N})$ based on generalized multiple Fourier series (converging in the sense of norm in Hilbert space $L_2([t, T]^k)$) was proposed and developed by the author of this paper in [10] (2006) (also see [11]-[36], [41]-[50], [52]-[55]). Hereinafter, this method is referred to as the method of generalized multiple Fourier series. As it turned out, the method of generalized multiple Fourier series can be adapted for the iterated Stratonovich stochastic integrals (3) at least for the multiplicities 1 to 6 [11]-[18], [23]-[25], [31], [37], [38], [42], [47]-[49], [51], [54], [57], [58]. Expansions of these iterated Stratonovich stochastic integrals turned out simpler than the appropriate expansions for the iterated Ito stochastic integrals (2).

The problem of iterated application of the operation of limit transition (see above) not appears in the method of generalized multiple Fourier series [10]-[36], [41]-[50], [52]-[55]. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous non-random function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated non-random function is expanded in the hypercube into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of the generalized multiple Fourier series for the mentioned non-random function of k variables which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2). Recall that this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following new and useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (6)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square approximation error of the iterated Ito stochastic integral (2) of arbitrary multiplicity k [12]-[18], [26], [44].

3. Since the used multiple Fourier series is a generalized in the sense that it is constructed using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only the trigonometric functions as in [2]-[4] but the Legendre polynomials.

4. As it turned out [10]-[36], [41]-[50], [52]-[55] it is more convenient to work with the Legendre polynomials for constructing the approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [15]-[18], [30], [41].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process as well as the approach from [60], [61] lead to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 1, 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$; $i_1,i_2,i_3=1,\ldots,m$) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series.

However, in [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82–84), [62] (pp. 438–439), [63] (pp. 263–264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [64]-[66] (see [15]-[18] (Sect. 6.2), [43], [45]-[48] for details).

The method of generalized multiple Fourier series allows some generalizations and modifications in several directions.

Recently, the method of generalized multiple Fourier series (see Theorems 1, 2 below) was applied to the expansion and mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [15]-[18] (Chapter 7), [32]-[35]. These results can be directly applied to the construction of high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations with non-linear multiplicative trace class noise [15]-[18] (Chapter 7), [32]-[35].

In this article, we demonstrate that the method of generalized multiple Fourier series is essentially general and allows some transformations for other types of iterated stochastic integrals. We will consider versions of the method of generalized multiple Fourier series for iterated stochastic integrals with respect to martingale Poisson measures and for iterated stochastic integrals with respect to martingales. The mentioned results are sufficiently natural according to general properties of martingales.

In Sect. 2, we formulate Theorem 1 on expansion of the iterated Ito stochastic integrals (2) of arbitrary multiplicity k based on generalized multiple Fourier series (method of generalized multiple Fourier series) [10]-[36], [41]-[50], [52]-[55]. Sect. 3 is devoted to a generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. In Sect. 4, we define the stochastic integral with respect to the martingale Poisson measure and consider some properties of this integral. Sect. 5 is devoted to a version of Theorem 1 for the iterated stochastic integrals with respect to martingale Poisson measures. In Sect. 6, we consider a generalization of Theorem 1 for the case of iterated stochastic integrals with respect to martingales. Sect. 7 is devoted to versions of Theorems 1, 2 for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k) \geq 0$ systems of functions in the space $L_2([t,T]^k)$. In Sect. 8, we consider one modification of theorems from Sect. 6 and 7. Sect. 9 is devoted to an example of the application of results from Sect. 8.

We will say that the function $f(x) : [t,T] \to \mathbb{R}^1$ satisfies the condition $(\star)$, if it is continuous at the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity as well as it is right-continuous at the interval $[t,T]$.

Let us suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$.

It is clear that complete orthonormal systems $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions in the space $L_2([t,T])$ satisfy the condition $(\star)$.

Let us consider some examples of systems satisfying the condition $(\star)$.

Example 1. The system of Legendre polynomials

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \quad x \in [t,T], \tag{4}$$

where $P_j(y)$, $y \in [-1,1]$ is the Legendre polynomial

$$P_j(y) = \frac{1}{2^j j!}\frac{d^j}{dy^j}\left(y^2-1\right)^j.$$

Example 2. The system of trigonometric functions

$$\phi_j(x) = \frac{1}{\sqrt{T-t}}\begin{cases} 1, & j = 0 \\ \sqrt{2}\sin\left(2\pi r(x-t)/(T-t)\right), & j = 2r-1, \\ \sqrt{2}\cos\left(2\pi r(x-t)/(T-t)\right), & j = 2r \end{cases}$$

where $x \in [t,T]$, $r = 1, 2, \ldots$

Example 3. The system of Haar functions

$$\phi_0(x) = \frac{1}{\sqrt{T-t}}, \quad \phi_{nj}(x) = \frac{1}{\sqrt{T-t}}\varphi_{nj}\left(\frac{x-t}{T-t}\right), \quad x \in [t,T],$$

where $n = 0, 1, \ldots,\ \ j = 1, 2, \ldots, 2^n$, and the functions $\varphi_{nj}(x)$ have the following form

$$\varphi_{nj}(x) = \begin{cases} 2^{n/2}, & x \in [(j-1)/2^n,\ (j-1)/2^n + 1/2^{n+1}) \\ \\ -2^{n/2}, & x \in [(j-1)/2^n + 1/2^{n+1},\ j/2^n) \\ \\ 0, & \text{otherwise} \end{cases},$$

where $n = 0, 1, \ldots,\ \ j = 1, 2, \ldots, 2^n$ (we choose the values of Haar functions in the points of discontinuity in order they will be right-continuous).

Example 4. The system of Rademacher–Walsh functions

$$\phi_0(x) = \frac{1}{\sqrt{T-t}},$$

$$\phi_{m_1\ldots m_k}(x) = \frac{1}{\sqrt{T-t}}\varphi_{m_1}\left(\frac{x-t}{T-t}\right)\ldots\varphi_{m_k}\left(\frac{x-t}{T-t}\right), \quad x \in [t, T],$$

where $0 < m_1 < \ldots < m_k,\ \ m_1, \ldots, m_k = 1, 2, \ldots,\ \ k = 1, 2, \ldots,$

$$\varphi_m(x) = (-1)^{[2^m x]},$$

$x \in [0, 1],\ \ m = 1, 2, \ldots,\ \ [y]$ is an integer part of a real number y.

2. Method of Expansion of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series Converging in the Mean

Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a non-random function from the space $L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1 < \ldots < t_k \\ \\ 0 & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2 \tag{5}$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. The function $K(t_1, \ldots, t_k)$ belongs to the space $L_2([t, T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{6}$$

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{7}$$

Theorem 1 [10] (2006), [11]-[36], [41]-[50], [52]-[55]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function of which for finite j satisfies the condition $(\star)$ (see Sect. 1). Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{8}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)} \tag{9}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (6), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval $[t,T]$ which satisfies the condition* (7).

Let us consider transformed particular cases of Theorem 1 for $k=1,\ldots,5$ [10]-[36], [41]-[50], [52]-[55] (the cases $k=6$ and 7 can be found in [11]-[17], [43])

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{10}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{11}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{12}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{13}$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{14}$$

where $\mathbf{1}_A$ is the indicator of the set A.

The convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) is proved for approximations from Theorem 1 in [11]-[25], [43]. In [15]-[17], [43]-[45], the convergence with probability 1 (further w. p. 1) is proved for expansions of iterated Ito stochastic integrals of arbitrary multiplicity k ($k\in\mathbb{N}$) from Theorem 1 for the cases of Legendre polynomials and trigonometric functions.

As follows from Theorem 1, the expansion (8) is valid for discontinuous complete orthonormal systems of functions in $L_2([t,T])$ satisfying the condition $(\star)$. For example, Theorem 1 is valid for the system of Haar functions as well as for the system of Rademacher–Walsh functions [10]-[25], [43].

3. Generalization of Theorem 1 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$

Consider a generalization of formulas (10)–(14) for the case of arbitrary multiplicity k of the iterated Ito stochastic integrals (2). In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{15}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (15) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{16}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (16)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$

$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$

$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [15] (Sect. 1.11), [36], [43] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(17)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x, $\prod_{\emptyset}\stackrel{\mathsf{def}}{=}1$, $\sum_{\emptyset}\stackrel{\mathsf{def}}{=}0$; *another notations are the same as in Theorem* 1.

In particular from (17) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (14).

It should be noted that an analogue of Theorem 2 (the case $i_1,\ldots,i_k=1,\ldots,m$) was considered in [67] using the Hermite polynomials and Wick product. Note that we use another notations [15]

(Sect. 1.11), [36], [43] (Sect. 15) in comparison with [67]. Moreover, the proof from [67] is different from the proof given in [15] (Sect. 1.11), [36], [43] (Sect. 15). See Sect. 4 in [36] for details.

Below we demonstrate that an approach to the expansion of iterated Ito stochastic integrals considered in Theorems 1, 2 is essentially general and allows some transformations for other types of iterated stochastic integrals.

Note that Theorems 1, 2 allow to calculate exactly the mean-square approximation error of the iterated Ito stochastic integrals (2) of arbitrary multiplicity k (see [13]–[18], [44]). In these papers we consider approxinations of iterated Ito stochastic integrals as the expression on the right-hand side of (17) before passing to the limit with respect to $p_1, \ldots, p_k$.

4. Stochastic Integral with Respect to Martingale Poisson Measure

Let us consider the Poisson random measure in the space $[0,T]\times\mathbf{Y}$ ($\mathbb{R}^n \stackrel{\rm def}{=} \mathbf{Y}$). We will denote the values of this measure at the set $\Delta\times A$ ($\Delta\subseteq[0,T]$, $A\subset\mathbf{Y}$) as $\nu(\Delta,A)$. Let us assume that

$$\mathsf{M}\left\{\nu(\Delta,A)\right\}=|\Delta|\Pi(A),$$

where $|\Delta|$ is the Lebesgue measure of Δ, $\Pi(A)$ is a measure on σ-algebra $\mathcal{B}$ of Borel sets of $\mathbf{Y}$, and $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ consisting of sets $A\subset\mathcal{B}$ which satisfy the condition $\Pi(A)<\infty$.

Let us consider the martingale Poisson measure

$$\tilde\nu(\Delta,A)=\nu(\Delta,A)-|\Delta|\Pi(A).$$

Let $(\Omega,{\rm F},\mathsf{P})$ be a fixed probability space, let $\{{\rm F}_t,\ t\in[0,T]\}$ be a non-decreasing family of σ-algebras ${\rm F}_t\subset{\rm F}$.

Assume that:

1. The random variables $\nu([0,t),A)$ are ${\rm F}_t$-measurable for all $A\subseteq\mathcal{B}_0$.
2. The random variables $\nu([t,t+h),A)$, $A\subseteq\mathcal{B}_0$, $h>0$ do not depend on σ-algebra ${\rm F}_t$.

Let us define the class $H_l(\Pi,[0,T])$ of random functions $\varphi:[0,T]\times\mathbf{Y}\times\Omega\to\mathbb{R}^1$, that are ${\rm F}_t$-measurable for all $t\in[0,T]$, $\mathbf{y}\in\mathbf{Y}$ and satisfy the following condition

$$\int\limits_0^T\int\limits_{\mathbf{Y}}\mathsf{M}\left\{|\varphi(t,\mathbf{y})|^l\right\}\Pi(d\mathbf{y})dt<\infty.$$

Let us consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$ which satisfies the condition (7).

For $\varphi(t,\mathbf{y})\in H_2(\Pi,[0,T])$ let us define the stochastic integral with respect to the martingale Poisson measure as the following mean-square limit [1]

$$\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\tilde\nu(dt,d\mathbf{y})\stackrel{\rm def}{=}\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi^{(N)}(t,\mathbf{y})\tilde\nu(dt,d\mathbf{y}),\tag{18}$$

where $\varphi^{(N)}(t,\mathbf{y})$ is any sequense of step functions from the class $H_2(\Pi,[0,T])$ such that

$$\lim_{N\to\infty}\int\limits_0^T\int\limits_{\mathbf{Y}}\mathsf{M}\left\{\left|\varphi(t,\mathbf{y})-\varphi^{(N)}(t,\mathbf{y})\right|^2\right\}\Pi(d\mathbf{y})dt=0.$$

It is well known [1] that the stochastic integral (18) exists, it does not depend on selection of the sequence $\varphi^{(N)}(t,\mathbf{y})$ and it satisfies w. p. 1 the following properties

$$\mathsf{M}\left\{\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\Bigg|\mathrm{F}_0\right\}=0,$$

$$\int\limits_0^T\int\limits_{\mathbf{Y}}(\alpha\varphi_1(t,\mathbf{y})+\beta\varphi_2(t,\mathbf{y}))\tilde{\nu}(dt,d\mathbf{y})=\alpha\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi_1(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})+\beta\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi_2(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}),$$

$$\mathsf{M}\left\{\left|\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\right|^2\Bigg|\mathrm{F}_0\right\}=\int\limits_0^T\int\limits_{\mathbf{Y}}\mathsf{M}\left\{|\varphi(t,\mathbf{y})|^2\Big|\mathrm{F}_0\right\}\Pi(d\mathbf{y})dt,$$

where α, β are some real constants and $\varphi_1(t,\mathbf{y})$, $\varphi_2(t,\mathbf{y})$, $\varphi(t,\mathbf{y})$ from the class $H_2(\Pi,[0,T])$.

The stochastic integral

$$\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\nu(dt,d\mathbf{y})$$

with respect to the Poisson random measure will be defined as follows [1]

$$\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\nu(dt,d\mathbf{y})=\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})+\int\limits_0^T\int\limits_{\mathbf{Y}}\varphi(t,\mathbf{y})\Pi(d\mathbf{y})dt,$$

where we suppose that the right-hand side of the last equality exists.

According to the Ito formula for Ito processes with jump component, we obtain w. p. 1 [1]

$$(z_t)^n=\int\limits_0^t\int\limits_{\mathbf{Y}}\Big((z_{\tau-}+\gamma(\tau,\mathbf{y}))^n-(z_{\tau-})^n\Big)\nu(d\tau,d\mathbf{y}), \tag{19}$$

where $n\in\mathbb{N}$,

$$z_t=\int\limits_0^t\int\limits_{\mathbf{Y}}\gamma(\tau,\mathbf{y})\nu(d\tau,d\mathbf{y}).$$

We suppose that the function $\gamma(\tau,\mathbf{y})$ satisfies the conditions of existence of the right-hand side of (19) [1].

Let us consider [1] the useful estimate for moments of the stochastic integral with respect to the Poisson random measure

$$a_n(T)\le\max_{j\in\{n,\ 1\}}\left\{\left(\int\limits_0^T\int\limits_{\mathbf{Y}}\left(\left((b_n(\tau,\mathbf{y}))^{1/n}+1\right)^n-1\right)\Pi(d\mathbf{y})d\tau\right)^j\right\}, \tag{20}$$

where

$$a_n(t) = \sup_{0\le\tau\le t} \mathsf{M}\biggl\{|z_\tau|^n\biggr\},\quad b_n(\tau,\mathbf{y}) = \mathsf{M}\biggl\{|\gamma(\tau,\mathbf{y})|^n\biggr\}.$$

We suppose that the right-hand side of (20) exists. Since

$$\tilde{\nu}(dt, d\mathbf{y}) = \nu(dt, d\mathbf{y}) - \Pi(d\mathbf{y})dt,$$

then according to the Minkowski inequality, we obtain

$$\left(\mathsf{M}\biggl\{|\tilde{z}_t|^{2n}\biggr\}\right)^{1/2n} \le \left(\mathsf{M}\biggl\{|z_t|^{2n}\biggr\}\right)^{1/2n} + \left(\mathsf{M}\biggl\{|\hat{z}_t|^{2n}\biggr\}\right)^{1/2n}, \tag{21}$$

where

$$\hat{z}_t \stackrel{\text{def}}{=} \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})d\tau$$

and

$$\tilde{z}_t = \int\limits_0^t \int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\tilde{\nu}(d\tau, d\mathbf{y}).$$

The value $\mathsf{M}\biggl\{|\hat{z}_\tau|^{2n}\biggr\}$ can be estimated using the well known inequality [1]

$$\mathsf{M}\left\{|\hat{z}_t|^{2n}\right\} \le t^{2n-1}\int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \varphi(\tau,\mathbf{y})\Pi(d\mathbf{y})\right|^{2n}\right\}d\tau, \tag{22}$$

where we suppose that

$$\int\limits_0^t \mathsf{M}\left\{\left|\int\limits_{\mathbf{Y}} \gamma(\tau,\mathbf{y})\Pi(d\mathbf{y})\right|^{2n}\right\}d\tau < \infty.$$

5. Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures Based on Generalized Multiple Fourier Series

Let us consider the following iterated stochastic integrals

$$P[\chi^{(k)}]_{T,t} = \int\limits_t^T \int\limits_{\mathbf{X}} \chi_k(t_k,\mathbf{y}_k)\ldots\int\limits_t^{t_2}\int\limits_{\mathbf{X}} \chi_1(t_1,\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)\ldots\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y}_k), \tag{23}$$

where $i_1,\ldots,i_k = 0,1,\ldots,m$, $\mathbb{R}^n \stackrel{\text{def}}{=} \mathbf{X}$,

$$\chi_l(\tau,\mathbf{y}) = \psi_l(\tau)\varphi_l(\mathbf{y})\quad (l = 1,\ldots,k),$$

every function $\psi_l(\tau):[t,T]\to\mathbb{R}^1$ $(l=1,\ldots,k)$ and every function $\varphi_l(\mathbf{y}):\mathbf{X}\to\mathbb{R}^1$ $(l=1,\ldots,k)$ is such that

$$\chi_l(s,\mathbf{y})\in H_2(\Pi,[t,T])\quad (l=1,\ldots,k),$$

where definition of the class $H_2(\Pi,[t,T])$ is given above,

$$\nu^{(i)}(dt,d\mathbf{y})\quad (i=1,\ldots,m)$$

are independent Poisson random measures for various i which are defined on $[0,T]\times\mathbf{X}$,

$$\tilde{\nu}^{(i)}(dt,d\mathbf{y})=\nu^{(i)}(dt,d\mathbf{y})-\Pi(d\mathbf{y})dt\quad (i=1,\ldots,m)$$

are independent martingale Poisson measures for various i,

$$\tilde{\nu}^{(0)}(dt,d\mathbf{y})\stackrel{\rm def}{=}\Pi(d\mathbf{y})dt.$$

Let us formulate an analoque of Theorem 1 for the iterated stochastic integrals (23).

Theorem 3 [12]-[18]. *Suppose that the following conditions are fulfilled*:
1. *Every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous non-random function at the interval* $[t,T]$.
2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1).
3. *For* $l=1,\ldots,k$ *and* $q=2^{k+1}$ *the following condition is fulfilled*

$$\int\limits_{\mathbf{X}}|\varphi_l(\mathbf{y})|^q\,\Pi(d\mathbf{y})<\infty.$$

Then, for the iterated stochastic integral with respect to martingale Poisson measures $P[\chi^{(k)}]_{T,t}$ *defined by* (23) *the following expansion*

$$P[\chi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{g=1}^{k}\pi_{j_g}^{(g,i_g)}-$$

$$-\underset{N\to\infty}{\rm l.i.m.}\sum_{(l_1,\ldots,l_k)\in{\rm G}_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\int\limits_{\mathbf{X}}\varphi_g(\mathbf{y})\tilde{\nu}^{(i_g)}([\tau_{l_g},\tau_{l_g+1}),d\mathbf{y})\Biggr)\tag{24}$$

converging in the mean-square sense is valid, where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *which satisfies the condition* (7),

$${\rm G}_k={\rm H}_k\backslash{\rm L}_k,\quad {\rm H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$${\rm L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$, *random variables*

$$\pi_j^{(g,i_g)}=\int\limits_t^T\phi_j(\tau)\int\limits_{\mathbf{X}}\varphi_g(\mathbf{y})\tilde{\nu}^{(i_g)}(d\tau,d\mathbf{y})$$

are independent for various $i_g \neq 0$ *and uncorrelated for various* j,

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k \\ \\ 0 & \text{otherwise} \end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\geq 2$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Proof. The scheme of the proof of Theorem 3 is the same as the scheme of the proof of Theorem 1 (see [10]-[25], [43] for details). Some differences will take place in the proof of the following lemmas (Lemmas 1, 2) and in the final part of the proof of Theorem 3.

Lemma 1 [11]-[25]. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous function at the interval* $[t,T]$ *and every function* $\varphi_l(\mathbf{y})$ $(l=1,\ldots,k)$ *is such that*

$$\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^2\,\Pi(d\mathbf{y})<\infty.$$

Then, the following equality

$$P[\bar{\chi}^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\int\limits_{\mathbf{X}}\chi_l(\tau_{j_l},\mathbf{y})\bar{\nu}^{(i_l)}([\tau_{j_l},\tau_{j_l+1}),d\mathbf{y}) \tag{25}$$

is valid w. p. 1, where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *which satisfies the condition* (7),

$$\bar{\nu}^{(i)}([\tau,s),d\mathbf{y}) = \begin{cases} \tilde{\nu}^{(i)}([\tau,s),d\mathbf{y}) \\ \\ \nu^{(i)}([\tau,s),d\mathbf{y}) \end{cases}\qquad (i=0,1,\ldots,m),$$

the integral $P[\bar{\chi}^{(k)}]_{T,t}$ *differs from the integral* $P[\chi^{(k)}]_{T,t}$ *(see* (23)*) by the fact that in* $P[\bar{\chi}^{(k)}]_{T,t}$ *we use* $\bar{\nu}^{(i_l)}(dt_l,d\mathbf{y}_l)$ *instead of* $\tilde{\nu}^{(i_l)}(dt_l,d\mathbf{y}_l)$ $(l=1,\ldots,k)$.

Proof. Using the moment properties of stochastic integrals with respect to Poisson random measures (see above) and conditions of Lemma 1, it is easy to notice that the integral sum of the integral $P[\bar{\chi}^{(k)}]_{T,t}$ under the conditions of Lemma 1 can be represented as a sum of the expression from the right-hand side of (25) before passing to the limit $\underset{N\to\infty}{\text{l.i.m.}}$ and the value which converges to zero in the mean-square sense if $N\to\infty$.

Note that in the case when the functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1) we can suppose that among the points $\tau_j,\ j=0,1,\ldots,N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l=1,\ldots,k)$. Further, we can apply the argumentation as in Sect. 4 from [43] (also see [10]-[18]).

Let us consider the following multiple and iterated stochastic integrals

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l(\mathbf{y})\tilde{\nu}^{(i_l)}([\tau_{j_l},\tau_{j_l+1}),d\mathbf{y}) \stackrel{\text{def}}{=} P[\Phi]_{T,t}^{(k)},$$

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}) \ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k,d\mathbf{y}) \stackrel{\text{def}}{=} \hat{P}[\Phi]_{T,t}^{(k)},$$

where the sense of notations of the formula (25) is saved and $\Phi(t_1,\ldots,t_k) : [t,T]^k \to \mathbb{R}^1$ is a bounded non-random function.

Note that if the functions $\varphi_l(\mathbf{y})$ $(l=1,\ldots,k)$ satisfy the conditions of Lemma 1 and the function $\Phi(t_1,\ldots,t_k)$ is continuous in the domain of integration, then for the integral $\hat{P}[\Phi]_{T,t}^{(k)}$ the equality similar to (25) is valid w. p. 1.

Lemma 2 [11]-[25]. *Assume that the following conditions are fulfilled*:

$$g_l(\tau,\mathbf{y}) = h_l(\tau)\varphi_l(\mathbf{y}) \quad (l=1,\ldots,k),$$

where the functions $h_l(\tau) : [t,T] \to \mathbb{R}^1$ $(l=1,\ldots,k)$ *satisfy the condition* $(\star)$ (*see Sect.* 1) *and the functions* $\varphi_l(\mathbf{y}) : \mathbf{X} \to \mathbb{R}^1$ $(l=1,\ldots,k)$ *satisfy the condition*

$$\int\limits_{\mathbf{X}} |\varphi_l(\mathbf{y})|^p\, \Pi(d\mathbf{y}) < \infty \quad \text{for} \quad p = 2^{k+1}.$$

Then

$$\prod_{l=1}^{k} \int\limits_t^T \int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}) = P[\Phi]_{T,t}^{(k)} \quad \text{w. p. } 1,$$

where $i_l = 0,1,\ldots,m$ $(l=1,\ldots,k)$ *and*

$$\Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} h_l(t_l).$$

Proof. Let us introduce the following notations

$$J[\bar{g}_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} \int\limits_{\mathbf{X}} g_l(\tau_j,\mathbf{y})\bar{\nu}^{(i_l)}([\tau_j,\tau_{j+1}),d\mathbf{y}),$$

$$J[\bar{g}_l]_{T,t} \stackrel{\text{def}}{=} \int\limits_t^T \int\limits_{\mathbf{X}} g_l(s,\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}),$$

where $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (7).

It is easy to see that

$$\prod_{l=1}^{k} J[\bar{g}_l]_N - \prod_{l=1}^{k} J[\bar{g}_l]_{T,t} =$$

$$
=\sum_{l=1}^{k}\left(\prod_{q=1}^{l-1}J[\bar{g}_q]_{T,t}\right)\left(J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}\right)\left(\prod_{q=l+1}^{k}J[\bar{g}_q]_N\right).
$$

Using the Minkowski inequality and the inequality of Cauchy–Bunyakovsky together with estimates of moments of stochastic integrals with respect to Poisson random measures (see Sect. 4) and conditions of Lemma 2, we obtain

$$
\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k}J[\bar{g}_l]_N-\prod_{l=1}^{k}J[\bar{g}_l]_{T,t}\right|^2\right\}\right)^{1/2}\le C_k\sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{26}
$$

where $C_k<\infty$.

We have

$$
J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}=\sum_{q=0}^{N-1}J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q},
$$

where

$$
J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q}=\int\limits_{\tau_q}^{\tau_{q+1}}\int\limits_{\mathbf{X}}\left(g_l(\tau_q,\mathbf{y})-g_l(s,\mathbf{y})\right)\bar{\nu}^{(i_l)}(ds,d\mathbf{y}).
$$

Let as introduce the notation

$$
h_l^{(N)}(s)=h_l(\tau_q),\quad s\in[\tau_q,\tau_{q+1}),\quad q=0,1,\ldots,N-1.
$$

Then

$$
J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}=\sum_{q=0}^{N-1}J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q}=
$$

$$
=\int\limits_{t}^{T}\left(h_l^{(N)}(s)-h_l(s)\right)\int\limits_{\mathbf{X}}\phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y}).
$$

Applying the estimate (20) for $n=4$ and the estimates (21), (22) for $n=2$ to the value

$$
\mathsf{M}\left\{\left|\int\limits_{t}^{T}\left(h_l^{(N)}(s)-h_l(s)\right)\int\limits_{\mathbf{X}}\phi_l(\mathbf{y})\bar{\nu}^{(i_l)}(ds,d\mathbf{y})\right|^4\right\},
$$

taking into account (26) together with the conditions of Lemma 2 and the following estimate

$$
|h_l(\tau_q)-h_l(s)|<\varepsilon,\quad s\in[\tau_q,\tau_{q+1}],\quad q=0,1,\ldots,N-1, \tag{27}
$$

where ε is an arbitrary small positive real number, we obtain that the right-hand side of (26) converges to zero when $N\to\infty$. Considering this fact, we come to the statement of Lemma 2.

It should be noted that (27) is valid if the functions $h_l(s)$ are continuous at the interval $[t,T]$, i.e. these functions are uniformly continuous at this interval. So, $|h_l(\tau_q)-h_l(s)|<\varepsilon$ if $s\in[\tau_q,\tau_{q+1}]$, where $|\tau_{q+1} - \tau_q| < \delta(\varepsilon)$, $q = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on points of the interval $[t, T]$).

In the case when the functions $h_l(s)$ ($l = 1, \ldots, k$) satisfy the condition ($\star$) (see Sect. 1) we can suppose that among the points τ_q, $q = 0, 1, \ldots, N$ there are all points of jumps of the functions $h_l(s)$ ($l = 1, \ldots, k$). Further, we can apply the argumentation as in Sect. 4 from [43] (also see [10]-[18]).

Obviously, if $i_l = 0$ for some $l = 1, \ldots, k$, then we also come to the statement of Lemma 2. Lemma 2 is proved.

Proving Theorem 3 according to the scheme used for the proof of Theorem 1 in [43] or Theorem 1.1 in [15]-[18] (also see [10] (Theorem 5.1, P. 236-237), [12] (Theorem 1, P. A.22-A.23), [13] (Theorem 5.1, P. A.250), [14] (Theorem 5.1, P. A.252-A.253)) and using Lemmas 1, 2 together with estimates for moments of stochastic integrals with respect to Poisson random measures (see Sect. 4), we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le$$

$$\le C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y}) \Pi(d\mathbf{y}) \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 \times$$

$$\times dt_1 \ldots dt_k = \tag{28}$$

$$= C_k \prod_{l=1}^{k} \int\limits_{\mathbf{X}} \varphi_l^2(\mathbf{y}) \Pi(d\mathbf{y}) \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 \times$$

$$\times dt_1 \ldots dt_k \le$$

$$\le \bar{C}_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 dt_1 \ldots dt_k \to 0$$

if $p_1, \ldots, p_k \to \infty$, where constant $\bar{C}_k$ depends only on k (multiplicity of the iterated stochastic integral with respect to martingale Poisson measures). At that permutations $(t_1, \ldots, t_k)$ when summing

$$\sum_{(t_1,\ldots,t_k)}$$

in (28) are performed only in the values $dt_1 \ldots dt_k$ and indexes near upper limits of integration are changed correspondently. Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right) \times$$

$$\times \int\limits_{\mathbf{X}} \varphi_1(\mathbf{y}) \tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}) \ldots \int\limits_{\mathbf{X}} \varphi_k(\mathbf{y}) \tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}), \tag{29}$$

where permutations $(t_1, \ldots, t_k)$ when summing

$$\sum_{(t_1,\ldots,t_k)}$$

in (29) are performed only in the values

$$\varphi_1(\mathbf{y})\tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y})\ldots\varphi_k(\mathbf{y})\tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}).$$

At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover, $\varphi_r(\mathbf{y})$ swapped with $\varphi_q(\mathbf{y})$ in the permutation $(\varphi_1(\mathbf{y}),\ldots,\varphi_k(\mathbf{y}))$. Theorem 3 is proved.

Let us consider an example of Theorem 3 usage. Suppose that $i_1 \neq i_2,\ i_1, i_2 = 1,\ldots,m$. According to Theorem 3, we obtain

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_2(\mathbf{y}_2)\int\limits_t^{t_2}\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}_1)\tilde{\nu}^{(i_2)}(dt_2, d\mathbf{y}_2) =$$

$$= \frac{T-t}{2}\left(\pi_0^{(1,i_1)}\pi_0^{(2,i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\pi_{i-1}^{(1,i_1)}\pi_i^{(2,i_2)} - \pi_i^{(1,i_1)}\pi_{i-1}^{(2,i_2)}\right)\right),$$

$$\int\limits_t^T\int\limits_{\mathbf{X}}\varphi_1(\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}_1) = \sqrt{T-t}\,\pi_0^{(1,i_1)},$$

where

$$\pi_j^{(l,i_l)} = \int\limits_t^T\phi_j(\tau)\int\limits_{\mathbf{X}}\varphi_l(\mathbf{y})\tilde{\nu}^{(i_l)}(d\tau, d\mathbf{y}) \quad (l = 1,\ 2)$$

and $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

6. Expansion of Iterated Stochastic Integrals with Respect to Martingales

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a fixed probability space, let $\{\mathrm{F}_t,\ t \in [0,T]\}$ be a non-decreasing family of σ-algebras $\mathrm{F}_t \subset \mathrm{F}$, and let $\mathrm{M}_2(\rho, [0,T])$ be a class of F_t-measurable for each $t \in [0,T]$ martingales M_t satisfying the conditions

$$\mathsf{M}\left\{(M_s - M_t)^2\right\} = \int\limits_t^s\rho(\tau)d\tau, \tag{30}$$

$$\mathsf{M}\left\{|M_s - M_t|^p\right\} \leq C_p|s-t|, \quad p = 3, 4, \ldots,$$

where $0 \leq t < s \leq T$, $\rho(\tau)$ is a non-negative and continuously differentiable non-random function at the interval $[0,T]$, $C_p < \infty$ is a constant.

Let us define the class $H_2(\rho, [0,T])$ of stochastic processes $\xi_t,\ t \in [0,T]$ which are F_t-measurable for all $t \in [0,T]$ and satisfy the condition

$$\int\limits_0^T \mathsf{M}\left\{|\xi_t|^2\right\}\rho(t)dt<\infty.$$

For any partition $\{\tau_j^{(N)}\}_{j=0}^N$ of the interval $[0,T]$ such that

$$0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\quad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \text{ if }\ N\to\infty \tag{31}$$

we will define the sequense of step functions $\xi^{(N)}(t,\omega)$ by the following relation

$$\xi^{(N)}(t,\omega)=\xi_j(\omega)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),$$

where $j=0,1,\ldots,N-1,\ \ N=1,2,\ldots.$

Let us define the stochastic integral with respect to martingale from the process $\xi(t,\omega)\in H_2(\rho,[0,T])$ as the following mean-square limit [1]

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(M\left(\tau_{j+1}^{(N)},\omega\right)-M\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\xi_\tau dM_\tau, \tag{32}$$

where $\xi^{(N)}(t,\omega)$ is any step function from the class $H_2(\rho,[0,T])$ which converges to the function $\xi(t,\omega)$ in the following sense

$$\lim_{N\to\infty}\int\limits_0^T\mathsf{M}\left\{\left|\xi^{(N)}(t,\omega)-\xi(t,\omega)\right|^2\right\}\rho(t)dt=0.$$

It is well known [1] that the stochastic integral

$$\int\limits_0^T\xi_t dM_t$$

exists and it does not depend on the selection of sequence $\xi^{(N)}(t,\omega)$ and it satisfies w. p. 1 the following properties

$$\mathsf{M}\left\{\int\limits_0^T\xi_t dM_t\bigg|\mathrm{F}_0\right\}=0,$$

$$\mathsf{M}\left\{\left|\int\limits_0^T\xi_t dM_t\right|^2\bigg|\mathrm{F}_0\right\}=\mathsf{M}\left\{\int\limits_0^T\xi_t^2\rho(t)dt\bigg|\mathrm{F}_0\right\},$$

$$\int\limits_0^T(\alpha\xi_t+\beta\psi_t)dM_t=\alpha\int\limits_0^T\xi_t dM_t+\beta\int\limits_0^T\psi_t dM_t,$$

where $\xi_t,\ \phi_t \in H_2(\rho,[0,T]),\ \ \alpha,\ \beta \in \mathbb{R}^1$.

Let $Q_4(\rho,[0,T])$ be the class of martingales $M_t,\ t\in[0,T]$ which satisfy the following conditions:

1. $M_t,\ t\in[0,T]$ belongs to the class $\text{M}_2(\rho,[0,T])$.

2. For some $\alpha>0$ the following estimate is correct

$$\mathsf{M}\left\{\left|\int\limits_t^{\tau} g(s)dM_s\right|^4\right\} \le K_4\int\limits_t^{\tau}|g(s)|^{\alpha}ds, \tag{33}$$

where $0\le t<\tau\le T,\ \ g(s)$ is a bounded non-random function at the interval $[0,T],\ \ K_4<\infty$ is a constant.

Let $G_n(\rho,[0,T])$ be the class of martingales $M_t,\ t\in[0,T]$ which satisfy the following conditions:

1. $M_t,\ t\in[0,T]$ belongs to the class $\text{M}_2(\rho,[0,T])$.

2. The following estimate is correct

$$\mathsf{M}\left\{\left|\int\limits_t^{\tau} g(s)dM_s\right|^n\right\} <\infty,$$

where $0\le t<\tau\le T,\ \ n\in\mathbb{N},\ \ g(s)$ is the same function as in the definition of $Q_4(\rho,[0,T])$.

Let us remind that if $(\xi_t)^n\in H_2(\rho,[0,T])$ with $\rho(t)\equiv 1$, then the following estimate is correct [1]

$$\mathsf{M}\left\{\left|\int\limits_t^{\tau}\xi_s ds\right|^{2n}\right\}\le(\tau-t)^{2n-1}\int\limits_t^{\tau}\mathsf{M}\left\{|\xi_s|^{2n}\right\}ds,\quad 0\le t<\tau\le T. \tag{34}$$

Let us consider the iterated stochastic integral with respect to martingales

$$J[\psi^{(k)}]_{T,t}^M=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}\quad(i_1,\ldots,i_k=0,1,\ldots,m), \tag{35}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function at the interval $[t,T]$, $M^{(r,i)}$ $(r=1,\ldots,k)$ are independent martingales for various $i=1,\ldots,m$, $M_{\tau}^{(r,0)}\stackrel{\text{def}}{=}\tau$.

Let us formulate the following theorem.

Theorem 4 [12]-[18]. *Suppose that the following conditions are fulfilled*:

1. *Every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous non-random function at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1*).*

3. $M_{\tau}^{(l,i_l)}\in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ *with* $n=2^{k+1}$, $i_l=1,\ldots,m,\ \ l=1,\ldots,k$.

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^M$ *with respect to martingales defined by* (35) *the following expansion*

$$J[\psi^{(k)}]_{T,t}^M=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k\xi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\text{G}_k}\phi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)$$

converging in the mean-square sense is valid, where $i_1,\ldots,i_k=0,1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *which satisfies the condition* (7), $\Delta M_{\tau_j}^{(r,i)}=M_{\tau_{j+1}}^{(r,i)}-M_{\tau_j}^{(r,i)}$ $(i=0,1,\ldots,m,\ \ r=1,\ldots,k)$,

$$
\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},
$$

$$
\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},
$$

l.i.m. *is a limit in the mean-square sense,*

$$
\xi_j^{(l,i_l)}=\int\limits_t^T\phi_j(s)dM_s^{(l,i_l)}
$$

are independent for various $i_l=1,\ldots,m$, $l=1,\ldots,k$ *and uncorrelated for various* j *(if* $\rho(\tau)$ *is a constant,* $i_l\ne 0$*) random variables,*

$$
C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k
$$

is the Fourier coefficient,

$$
K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & for\ \ t_1<\ldots<t_k\\ \\ 0 & otherwise\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2
$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Remark 1. *Note that from Theorem* 4 *for the case* $\rho(\tau)\equiv 1$ *we obtain the variant of Theorem* 1.

Proof. The scheme of the proof of Theorem 4 is the same with the scheme of the proof of Theorem 1 in [43] or Theorem 1.1 in [15]-[18] (also see [10]-[25], [43]). Some differences will take place in the proof of the following lemmas (Lemmas 3, 4) and in the final part of the proof of Theorem 4.

Lemma 3. *Suppose that* $M_\tau^{(r,i)}\in\mathrm{M}_2(\rho,[t,T])$, $M_\tau^{(r,0)}=\tau$ $(i=0,1,\ldots,m,\ \ r=1,\ldots,k)$, *and every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous non-random function at the interval* $[t,T]$. *Then*

$$
J[\psi^{(k)}]_{T,t}^{M}=\underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta M_{\tau_{j_l}}^{(l,i_l)}\quad w.\ p.\ 1, \tag{36}
$$

where $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[0,T]$ *satisfying the condition* (7).

Proof. According to properties of the stochastic integral with respect to martingale, we have [1]

$$
\mathsf{M}\left\{\left(\int\limits_t^\tau\xi_s dM_s^{(l,i_l)}\right)^2\right\}=\int\limits_t^\tau\mathsf{M}\left\{|\xi_s|^2\right\}\rho(s)ds, \tag{37}
$$

$$\mathsf{M}\left\{\left(\int\limits_t^\tau \xi_s ds\right)^2\right\} \le (\tau - t)\int\limits_t^\tau \mathsf{M}\left\{|\xi_s|^2\right\} ds, \tag{38}$$

where $\xi_s \in H_2(\rho, [0,T])$, $0 \le t < \tau \le T$, $i_l = 1,\ldots,m$, $l = 1,\ldots,k$. Then the integral sum of the integral $J[\psi^{(k)}]^M_{T,t}$ under the conditions of Lemma 3 can be represented as a sum of the expression from the right-hand side of (36) before passing to the limit and the value which converges to zero in the mean-square sense if $N \to \infty$. More detailed proof of the analogous lemma for the case $\rho(\tau) \equiv 1$ can be found in [10]-[25], [43].

In the case when the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1) we can suppose that among the points τ_j, $j = 0,1,\ldots,N$ there are all points of jumps of the functions $\psi_l(\tau)$ $(l = 1,\ldots,k)$. Then can apply the argumentation as in Sect. 4 from [43] (also see [10]-[18]).

Let us define the folloing multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^k \Delta M^{(l,i_l)}_{\tau_{j_l}} \stackrel{\rm def}{=} I[\Phi]^{(k)}_{T,t}, \tag{39}$$

where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[0,T]$ satisfying the condition (7) and $\Phi(t_1,\ldots,t_k)$: $[t,T]^k \to \mathbb{R}^1$ is a bounded non-random function.

Lemma 4. *Suppose that $M^{(l,i_l)}_s \in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ with $n = 2^{k+1}$, $k \in \mathbb{N}$ $(i_l = 0,1,\ldots,m$, $l = 1,\ldots,k)$ and the functions $g_1(s),\ldots,g_k(s)$ satisfy the condition $(\star)$ (see Sect. 1). Then*

$$\prod_{l=1}^k \int\limits_t^T g_l(s) dM^{(l,i_l)}_s = I[\Phi]^{(k)}_{T,t} \quad \text{w. p. } 1,$$

where

$$\Phi(t_1,\ldots,t_k) = \prod_{l=1}^k g_l(t_l).$$

Proof. Let us denote

$$J[g_l]_N \stackrel{\rm def}{=} \sum_{j=0}^{N-1} g_l(\tau_j)\Delta M^{(l,i_l)}_{\tau_j}, \quad J[g_l]_{T,t} \stackrel{\rm def}{=} \int\limits_t^T g_l(s) dM^{(l,i_l)}_s,$$

where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition (7).

Note that

$$\prod_{l=1}^k J[g_l]_N - \prod_{l=1}^k J[g_l]_{T,t} =$$

$$= \sum_{l=1}^k \left(\prod_{q=1}^{l-1} J[g_q]_{T,t}\right)\left(J[g_l]_N - J[g_l]_{T,t}\right)\left(\prod_{q=l+1}^k J[g_q]_N\right).$$

Using the Minkowski inequality and the inequality of Cauchy-Bunyakovsky as well as the conditions of Lemma 4, we obtain

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k}J[g_l]_N-\prod_{l=1}^{k}J[g_l]_{T,t}\right|^2\right\}\right)^{1/2}\le C_k\sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|J[g_l]_N-J[g_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{40}$$

where $C_k<\infty$ is a constant.

We have

$$J[g_l]_N-J[g_l]_{T,t}=\sum_{q=0}^{N-1}J[\Delta g_l]_{\tau_{q+1},\tau_q},$$

$$J[\Delta g_l]_{\tau_{q+1},\tau_q}=\int\limits_{\tau_q}^{\tau_{q+1}}\left(g_l(\tau_q)-g_l(s)\right)dM_s^{(l,i_l)}.$$

Let as introduce the notation

$$g_l^{(N)}(s)=g_l(\tau_q),\quad s\in[\tau_q,\tau_{q+1}),\quad q=0,1,\ldots,N-1.$$

Then

$$J[\bar{g}_l]_N-J[\bar{g}_l]_{T,t}=\sum_{q=0}^{N-1}J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q}=$$

$$=\int\limits_t^T\left(g_l^{(N)}(s)-g_l(s)\right)dM_s^{(l,i_l)}.$$

Applying the estimate (33), we obtain

$$\mathsf{M}\left\{\left|\int\limits_t^T\left(g_l^{(N)}(s)-g_l(s)\right)dM_s^{(l,i_l)}\right|^4\right\}\le K_4\int\limits_t^T\left|g_l^{(N)}(s)-g_l(s)\right|^\alpha ds=$$

$$=K_4\sum_{q=0}^{N-1}\int\limits_{\tau_q}^{\tau_{q+1}}\left|g_l(\tau_q)-g_l(s)\right|^\alpha ds<K_4\varepsilon^\alpha\sum_{q=0}^{N-1}(\tau_{q+1}-\tau_q)=$$

$$=K_4\varepsilon^\alpha(T-t). \tag{41}$$

Note that deriving (41) we used the estimate

$$|g_l(\tau_q)-g_l(s)|<\varepsilon,\quad s\in[\tau_q,\tau_{q+1}],\quad q=0,1,\ldots,N-1, \tag{42}$$

where ε is an arbitrary small positive real number.

Note that (42) is valid if the functions $g_l(s)$ are continuous at the interval $[t,T]$, i.e. these functions are uniformly continuous at this interval. So, $|g_l(\tau_q)-g_l(s)|<\varepsilon$ if $s\in[\tau_q,\tau_{q+1}]$, where $|\tau_{q+1}-\tau_q|<\delta(\varepsilon)$, $q=0,1,\ldots,N-1$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on points of the interval $[t,T]$).

Thus, taking into account (41), we obtain that the right-hand side of (40) converges to zero when $N\to\infty$. Considering this fact, we come to the statement of Lemma 4.

In the case when the functions $g_l(s)$ $(l=1,\ldots,k)$ satisfy the condition $(\star)$ (see Sect. 1) we can suppose that among the points τ_q, $q=0,1,\ldots,N$ there are all points of jumps of the functions $g_l(s)$ $(l=1,\ldots,k)$. Further, we can apply the argumentation as in Sect. 4 from [43] (also see [10]-[18]).

Obviously, if $i_l=0$ for some $l=1,\ldots,k$, then we also come to the statement of Lemma 4 with using (38). Lemma 4 is proved.

Proving Theorem 4 according to the scheme used for the proof of Theorem 1 in [43] or Theorem 1.1 in [15]-[18] (also see [10] (Theorem 5.1, P. 236-237), [12] (Theorem 1, P. A.22-A.23), [13] (Theorem 5.1, P. A.250), [14] (Theorem 5.1, P. A.252-A.253)) and using Lemmas 3, 4 together with the estimates (37), (38) for moments of stochastic integrals with respect to martingales, we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2\times$$

$$\times\tilde\rho_1(t_1)dt_1\ldots\tilde\rho_k(t_k)dt_k\le \tag{43}$$

$$\le \bar C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=$$

$$=\bar C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k\to 0$$

when $p_1,\ldots,p_k\to\infty$, where constant $\bar C_k$ depends only on k (multiplicity of the iterated stochastic integral with respect to martingales) and $\tilde\rho_l(s)\equiv\rho(s)$ or $\tilde\rho_l(s)\equiv 1$ $(l=1,\ldots,k)$. At that permutations $(t_1,\ldots,t_k)$ when summing

$$\sum_{(t_1,\ldots,t_k)}$$

in (43) are performed only in the values $dt_1\ldots dt_k$ and indexes near upper limits of integration are changed correspondently. Moreover, $R_{T,t}^{p_1,\ldots,p_k}$ has the following form

$$R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)\times$$

$$\times dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}, \tag{44}$$

where permutations $(t_1,\ldots,t_k)$ when summing

$$\sum_{(t_1,\ldots,t_k)}$$

in (44) are performed only in the values

$$dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)}.$$

At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Moreover, r swapped with q in the permutation $(1,\ldots,k)$. Theorem 4 is proved.

7. Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Complete Orthonormal With Weight $r(t_1)\ldots r(t_k)\ge 0$ Systems of Functions in the Space $L_2([t,T]^k)$

In this section, we consider modifications of Theorems 1, 2 for the case of complete orthonormal with weight $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in the space $L_2([t,T]^k)$, $k\in\mathbb{N}$.

Let $\{\Psi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal with weight $r(x)\ge 0$ system of functions in the space $L_2([t,T])$. It is well known that the Fourier series with respect to the system

$$\{\Psi_j(x)\}_{j=0}^{\infty}$$

of the function $f(x)$ $\left(f(x)\sqrt{r(x)}\in L_2([t,T])\right)$ converges to the function $f(x)$ in the mean-square sense with weight $r(x)$, i.e.

$$\lim_{p\to\infty}\int\limits_t^T\left(f(x)-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\right)^2 r(x)dx=0, \tag{45}$$

where

$$\tilde C_j=\int\limits_t^T f(x)\Psi_j(x)r(x)dx \tag{46}$$

is the Fourier coefficient.

Obviously, the relation (45) can be obtained if we will expand the function $f(x)\sqrt{r(x)}\in L_2([t,T])$ into a usual Fourier series with respect to the complete orthonormal with weight 1 system of functions

$$\left\{\Psi_j(x)\sqrt{r(x)}\right\}_{j=0}^{\infty}$$

in the space $L_2([t,T])$. Then

$$\lim_{p\to\infty}\int\limits_t^T\left(f(x)\sqrt{r(x)}-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\sqrt{r(x)}\right)^2 dx=$$

$$=\lim_{p\to\infty}\int\limits_t^T\left(f(x)-\sum_{j=0}^{p}\tilde C_j\Psi_j(x)\right)^2 r(x)dx=0, \tag{47}$$

where $\tilde{C}_j$ has the form (46).

Let us consider an obvious generalization of this approach to the case of several variables. Let us expand the function $K(t_1,\ldots,t_k)$ such that

$$K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}\in L_2([t,T]^k)$$

using the complete orthonormal system of functions

$$\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)},\quad j_l=0,1,2,\ldots,\quad l=1,\ldots,k$$

in the space $L_2([t,T]^k)$ into the generalized multiple Fourier series.

It is well known that the mentioned generalized multiple Fourier series converges in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)^2\times$$

$$\times dt_1\ldots dt_k=$$

$$=\lim_{p_1,\ldots,p_k\to\infty}\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\times$$

$$\times\left(\prod_{l=1}^{k}r(t_l)\right)dt_1\ldots dt_k=0, \tag{48}$$

where

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k.$$

Let us consider the following iterated Ito stochastic integrals

$$\tilde{J}[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{49}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$, $\mathbf{w}_\tau^{(0)}=\tau$, and $i_1,\ldots,i_k=0,1,\ldots,m$.

So, we obtain the following modification of Theorem 1.

Theorem 5 [14]-[17], [53]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\Psi_j(x)\sqrt{r(x)}$ of which for finite j satisfies the condition $(\star)$ (see Sect. 1). Then*

$$\tilde{J}[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{r(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{50}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$ *which satisfies the condition* (7),

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k \tag{51}$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ & \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Proof. According to Lemmas 1–3 in [43] or Lemmas 1.1–1.3 in [15]-[18] (also see [10]-[14]), we get the following representation w. p. 1

$$\tilde{J}[\psi^{(k)}]_{T,t}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\prod_{l=1}^{k}\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times$$

$$\times \underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times$$

$$\times\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\times$$

$$\times\left(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\sqrt{r(\tau_{l_1})}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\sqrt{(\tau_{l_k})}\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+\tilde{R}_{T,t}^{p_1,\ldots,p_k},$$

where

$$\tilde{R}_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int_t^T\ldots\int_t^{t_2}\left(K(t_1,\ldots,t_k)\prod_{l=1}^{k}\sqrt{r(t_l)}-\right.$$

$$\left.-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped i_q in the permutation $(i_1,\ldots,i_k)$.

Let us evaluate the remainder $\tilde{R}_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to Lemma 2 in [43] or Lemma 1.2 in [15] (also see [16]-[18]), we have

$$\mathsf{M}\left\{\left(\tilde{R}_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Biggl(K(t_1,\ldots,t_k)\prod_{l=1}^k\sqrt{r(t_l)}-$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^k\left(\Psi_{j_l}(t_l)\sqrt{r(t_l)}\right)\Biggr)^2 dt_1\ldots dt_k= \tag{52}$$

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^k\Psi_{j_l}(t_l)\right)^2\times$$

$$\times\left(\prod_{l=1}^k r(t_l)\right)dt_1\ldots dt_k\to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral (49). Theorem 5 is proved.

Let us formulate the following theorem (the version of Theorem 3 in [44]).

Theorem 6 [15]-[18]. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous non-random function on* $[t,T]$ *and* $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ $(r(x)\ge 0)$ *is a complete orthonormal system of functions in the space* $L_2([t,T])$, *each function* $\Psi_j(x)\sqrt{r(x)}$ *of which for finite* j *satisfies the condition* $(\star)$ *(see Sect.* 1*). Then the estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t}-\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^k r(t_l)\right)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}^2\right) \tag{53}$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,$ $i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (49), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (50) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 5.

Consider the following generalizations of Theorems 5, 6.

Theorem 7 [15] (Sect. 1.13), [43] (Sect. 17). *Let* $\psi_1(x)\sqrt{r(x)},\ldots,\psi_k(x)\sqrt{r(x)}\in L_2([t,T])$, *where* $r(x)\ge 0$. *Furthermore, let* $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then, for the iterated Ito stochastic integral*

$$\tilde{J}[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\sqrt{r(t_k)}\ldots\int\limits_t^{t_2}\psi_1(t_1)\sqrt{r(t_1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{54}$$

the following expansion

$$\tilde{J}[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\tilde{\zeta}_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\tilde{\zeta}_{j_{q_l}}^{(i_{q_l})}\right) \tag{55}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\Psi_j(s)\sqrt{r(s)}d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$),

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient, $K(t_1,\ldots,t_k)$ *is defined by* (5); *another notations are the same as in Theorems* 1, 2, 5.

Theorem 8 [15] (Sect. 1.13), [43] (Sect. 17). *Let* $\psi_1(x)\sqrt{r(x)},\ldots,\psi_k(x)\sqrt{r(x)}\in L_2([t,T])$, *where* $r(x)\ge 0$. *Furthermore, let* $\{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following estimate*

$$\mathsf{M}\left\{\left(\tilde{J}[\psi^{(k)}]_{T,t}-\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)\left(\prod_{l=1}^{k}r(t_l)\right)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}^2\right)$$

is valid for the following cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ *and* $0<T-t<\infty$,
2. $i_1,\ldots,i_k=0,1,\ldots,m,\ \ i_1^2+\ldots+i_k^2>0,$ *and* $0<T-t<1$,

where $\tilde{J}[\psi^{(k)}]_{T,t}$ *is the stochastic integral* (54), $\tilde{J}[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is the expression on the right-hand side of* (55) *before passing to the limit* $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$; *another notations are the same as in Theorem* 2, 7.

8. One Modification of Theorems 4 and 5

Let us compare (52) and (43). If we suppose that $r(x)\ge 0$ and

$$\frac{\rho(x)}{r(x)}\le C<\infty,$$

where $\rho(x)$ as in (30), then

$$\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\times$$

$$\times\rho(t_1)dt_1\ldots\rho(t_k)dt_k=$$

$$=\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\times$$

$$\times\frac{\rho(t_1)}{r(t_1)}r(t_1)dt_1\ldots\frac{\rho(t_k)}{r(t_k)}r(t_k)dt_k\le$$

$$\le C_k'\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\prod_{l=1}^{k}\Psi_{j_l}(t_l)\right)^2\times$$

$$\times r(t_1)dt_1\ldots r(t_k)dt_k,$$

where C_k' is a constant, $\{\Psi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal with weight $r(x)\ge 0$ system of functions in the space $L_2([t,T])$, and the Fourier coefficient $\tilde{C}_{j_k\ldots j_1}$ has the form (51).

So, we obtain the following modification of Theorems 4 and 5.

Theorem 9 [15], [53]. *Suppose that the following conditions are fulfilled*:

1. *Every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous non-random function at the interval* $[t,T]$.
2. $M_\tau^{(l,i_l)}\in Q_4(\rho,[t,T])$, $G_n(\rho,[t,T])$ *with* $n=2^{k+1}$, $i_l=1,\ldots,m$, $l=1,\ldots,k$ $(k\in\mathbb{N})$.
3. $\{\Psi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal with weight* $r(\tau)\ge 0$ *system of functions in the space* $L_2([t,T])$, *each function of which for finite* j *satisfies the condition* $(\star)$ (*see Sect.* 1). *Moreover,*

$$\frac{\rho(x)}{r(x)}\le C<\infty.$$

Then, for the iterated stochastic integral $J[\psi^{(k)}]_{T,t}^M$ *with respect to martingales defined by* (35) *the following expansion*

$$J[\psi^{(k)}]_{T,t}^{M}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}\tilde{C}_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\Psi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\Psi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr)$$

converging in the mean-square sense is valid, where $i_1,\ldots,i_k=1,\ldots,m$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$ *which satisfies the condition* (31), $\Delta M_{\tau_j}^{(r,i)}=M_{\tau_{j+1}}^{(r,i)}-M_{\tau_j}^{(r,i)}$ $(i=1,\ldots,m,\ r=1,\ldots,k)$,

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,*

$$\xi_j^{(l,i_l)}=\int\limits_t^T\Psi_j(s)dM_s^{(l,i_l)}$$

are independent for various $i_l=1,\ldots,m$ $(l=1,\ldots,k)$ *and uncorrelated for various* j *(if* $i_l\neq 0$, $\rho(x)\equiv r(x)$*) random variables,*

$$\tilde{C}_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\Bigl(\Psi_{j_l}(t_l)r(t_l)\Bigr)dt_1\ldots dt_k$$

is the Fourier coefficient,

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ & \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\geq 2$$

and $K(t_1)\equiv\psi_1(t_1)$ *for* $t_1\in[t,T]$.

Remark 2. *Note that if* $\rho(\tau),r(\tau)\equiv 1$ *in Theorem* 9*, then we obtain the variant of Theorem* 1.

9. Example on Application of Theorem 9 for the System of Bessel Functions

Let us consider the following boundary-value problem

$$\bigl(p(x)\Phi'(x)\bigr)'+q(x)\Phi(x)=-\lambda r(x)\Phi(x),$$

$$\alpha\Phi(a)+\beta\Phi'(a)=0,\quad \gamma\Phi(b)+\delta\Phi'(b)=0,$$

where the functions $p(x)$, $q(x)$, $r(x)$ satisfy the well known conditions and α, β, γ, δ, λ are real numbers.

It is well known (Steklov V.A.) that the eigenfunctions $\Phi_0(x)$, $\Phi_1(x)$, $\ldots$ of this boundary-value problem form a complete orthonormal with weight $r(x)$ system of functions in the space $L_2([a,b])$. It means that the Fourier series of the function $\sqrt{r(x)}f(x) \in L_2([a,b])$ with respect to the system of functions

$$\sqrt{r(x)}\Phi_0(x), \quad \sqrt{r(x)}\Phi_1(x), \quad \ldots$$

converges in the mean-square sense to the function $\sqrt{r(x)}f(x)$ at the interval $[a,b]$. Moreover, the Fourier coefficients are defined by the formula

$$C_j = \int\limits_a^b r(x)f(x)\Phi_j(x)dx. \tag{56}$$

It is known that when solving the problem on oscillations of a circular membrane (general case), a boundary-value problem arises for the following Euler–Bessel equation

$$r^2R''(r) + rR'(r) + \left(\lambda^2r^2 - n^2\right)R(r) = 0 \quad (\lambda \in \mathbb{R}, \quad n \in \mathbb{N}). \tag{57}$$

The eigenfunctions of this problem, taking into account specific boundary conditions, are the following functions

$$J_n\left(\mu_j\frac{r}{L}\right), \tag{58}$$

where $\tau \in [0,L]$ and μ_j $(j = 0,1,2,\ldots)$ are positive roots of the Bessel function $J_n(\mu)$ $(n = 0,1,2,\ldots)$ numbered in ascending order.

The problem on radial oscillations of a circular membrane leads to the boundary-value problem for the equation (57) for $n = 0$, the eigenfunctions of which are the functions (58) when $n = 0$.

Let us analyze the system of functions

$$\Psi_j(\tau) = \frac{\sqrt{2}}{TJ_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T}\tau\right), \quad j = 0,1,2,\ldots, \tag{59}$$

where

$$J_n(x) = \sum_{m=0}^{\infty}(-1)^m\left(\frac{x}{2}\right)^{n+2m}\frac{1}{\Gamma(m+1)\Gamma(m+n+1)}$$

is the Bessel function of the first kind and

$$\Gamma(z) = \int\limits_0^{\infty} e^{-x}x^{z-1}dx$$

is the gamma-function, μ_j are positive roots of the function $J_n(x)$ numbered in ascending order, and n is a natural number or zero.

Due to the well known properties of the Bessel functions, the system $\{\Psi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions with weight τ in the space $L_2([0,T])$.

Let us use the system of functions (59) in Theorem 9.

Consider the following iterated stochastic integral with respect to martingales

$$\int\limits_0^T\int\limits_0^s dM_\tau^{(1)}dM_s^{(2)},$$

where

$$M_s^{(i)}=\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(i)}\quad (i=1,2),$$

$\mathbf{f}_\tau^{(i)}$ $(i=1,2)$ are independent standard Wiener processes, $M_s^{(i)}$ $(i=1,2)$ are martingales (here $\rho(\tau)\equiv\tau$), $0\le s\le T$. In addition, $M_s^{(i)}$ has a Gaussian distribution.

It is obvious that the conditions of Theorem 9 are fulfilled for $k=2$. Using Theorem 9, we obtain

$$\int\limits_0^T\int\limits_0^s dM_\tau^{(1)}dM_s^{(2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\tilde{C}_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)}=\int\limits_0^T\Psi_j(\tau)dM_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,2,\ \ j=0,1,2,\ldots)$,

$$\mathsf{M}\left\{\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)}\right\}=0,$$

$$\tilde{C}_{j_2j_1}=\int\limits_0^T s\Psi_{j_2}(s)\int\limits_0^s\tau\Psi_{j_1}(\tau)d\tau ds.$$

It is obvious that we can get this result using the another approach: we can use Theorems 1, 2 for the iterated Ito stochastic integral

$$\int\limits_0^T\sqrt{s}\int\limits_0^s\sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)},$$

and as a system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in Theorems 1, 2 we can take

$$\phi_j(s)=\frac{\sqrt{2s}}{TJ_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T}s\right),\quad j=0,1,2,\ldots.$$

As a result, we obtain

$$\int\limits_0^T \sqrt{s}\int\limits_0^s \sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where

$$\zeta_j^{(i)} = \int\limits_0^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,2,\ \ j=0,1,2,\ldots)$,

$$\mathsf{M}\left\{\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)}\right\}=0,\quad C_{j_2j_1}=\int\limits_0^T\sqrt{s}\phi_{j_2}(s)\int\limits_0^s\sqrt{\tau}\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient. Obviously that $C_{j_2j_1}=\tilde{C}_{j_2j_1}$.

Easy calculation demonstrates that

$$\tilde{\phi}_j(s)=\frac{\sqrt{2(s-t)}}{(T-t)J_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T-t}(s-t)\right),\quad j=0,1,2,\ldots$$

is a complete orthonormal system of functions in the space $L_2([t,T])$.

Then, using Theorems 1, 2, we obtain

$$\int\limits_t^T \sqrt{s-t}\int\limits_t^s \sqrt{\tau-t}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)},$$

where

$$\tilde{\zeta}_j^{(i)} = \int\limits_t^T \tilde{\phi}_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j $(i=1,2,\ \ j=0,1,2,\ldots)$,

$$\mathsf{M}\left\{\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)}\right\}=0,\quad C_{j_2j_1}=\int\limits_t^T\sqrt{s-t}\tilde{\phi}_{j_2}(s)\int\limits_t^s\sqrt{\tau-t}\tilde{\phi}_{j_1}(\tau)d\tau ds.$$

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.

[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.

[5] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Berlin, Springer, 1994, 292 pp.

[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.

[7] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor Expansions. Math. Nachr. 151 (1991), 33-50.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: https://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[12] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[15] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2023, 992 pp.

[16] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[17] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[18] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[23] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html
[24] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)
[25] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)
[26] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096
[27] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056
[28] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.
[29] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[30] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116
[31] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[32] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[33] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html
[34] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.
[35] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html
[36] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR]. 2023, 56 pp.
[37] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html
[38] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)
[39] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[40] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[41] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[42] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[43] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 143 pp.

[44] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 70 pp.

[45] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[46] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [In English]. arXiv:1801.03195 [math.PR]. 2023, 159 pp.

[47] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2023, 221 pp.

[48] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [In English]. arXiv:1801.01564 [math.PR]. 2018, 66 pp.

[49] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.

[50] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [In English]. arXiv:1712.08991 [math.PR]. 2022, 57 pp.

[51] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[52] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[53] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[54] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 149 pp.

[55] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[56] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[57] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[58] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[59] Allen E. Approximation of triple stochastic integrals through region subdivision. Communicat. in Appl. Anal. Special Tribute Issue to Prof. V. Lakshmikantham. 17 (2013), 355-366.

[60] Averina T.A., Prigarin S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[61] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].
[62] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.
[63] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.
[64] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[65] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[66] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[67] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[68] Skhorohod A.V. Stochastc Processes with Independent Augments. [In Russian]. Moscow, Nauka Publ., 1964. 280 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

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The Proof of Convergence with Probability 1 in the Method of Expansion of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to the formulation and proof of the theorem on convergence with probability 1 of expansion of iterated Itô stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the sense of norm in Hilbert space. The cases of multiple Fourier–Legendre series and multiple trigonomertic Fourier series are considered in detail. The proof of the mentioned theorem is based on the general properties of multiple Fourier series as well as on the estimate for the fourth moment of approximation error in the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series.

Key words: Iterated Itô stochastic integral, generalized multiple Fourier series, multiple Fourier–Legendre series, multiple trigonometric Fourier series, Parseval equality, Legendre polynomials, convergence with probability 1, mean-square convergence, convergence in the mean of arbitrary degree, expansion, approximation.

1 Introduction

The beginning of an intensive study of the problem of mean-square approximation of iterated Itô and Stratonovich stochastic integrals in the context of the numerical solution of Itô stochastic differential equations dates back to the 1980s–1990s. To date, there are many publications on the mentioned problem [1]-[36] (also see bibliographic references in these works). There are various approaches to solving the problem of the mean-square approximation of iterated stochastic integrals. Among them, we note the approach based on the Karhunen–Loeve expansion of the Brownian bridge process [1]-[4], [13], [18], [21], approach based on the expansion of the Wiener process using various basis systems of functions [6], [10], [30], [31], approach based on the conditional joint characteristic function of a stochastic integral of multiplicity 2 [11], [12] as well as an approach based on multiple integral sums [1], [19].

The use of multiple and iterated generalized Fourier series by various complete orthonormal systems of functions in the space $L_2([t,T])$ for the expansion of iterated Itô and Stratonovich stochastic integrals was reflected in a number of author's works [7]-[9], [14]-[17], [20], [22]-[29], [35]. The mentioned results based on generalized multiple and iterated Fourier series are systematized in the monograph [36] (2022).

The idea of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series is as follows: the iterated Itô stochastic integral of multiplicity k ($k \in \mathbb{N}$) is represented as a multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is an interval of integration of the iterated Itô stochastic integral. Then, the indicated nonrandom function is expanded into the generalized multiple Fourier series converging in the sense of norm in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come [14] (2006) to the mean-square converging expansion of the iterated Itô stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of multiplicity k of the iterated Itô stochastic integral.

In a lot of author's publications the convergence of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series has been considered in different probabilistic meanings. For example, the mean-square convergence [14]-[17], [20], [22]-[29], [35], [36] and convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [15]-[17], [20], [22], [23], [36] have been proved. On the examples of specific iterated Itô stochastic integrals of mutiplicities 1 and 2 the convergence with probability 1 also has been considered [15]-[17], [20], [22], [23]. This article is devoted to the development of the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series. Namely, we formulate and prove the theorem on convergence with probability 1 of the mentioned method for an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Itô stochastic integrals. Moreover, the cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series are considered in detail.

2 Method of Expansion of Iterated Itô Stochastic Integrals of Multiplicity k ($k \in \mathbb{N}$) Based on Generalized Multiple Fourier Series

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a non-decreasing right-continuous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{w}_t^{(i)}$ ($i = 1, \ldots, m$) of this process are independent.

Let us consider an efficient method [14]-[17], [20], [22]-[29], [35], [36] of the expansion and mean-square approximation of iterated Itô stochastic integrals of the form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1}$$

where $0 \le t < T < \infty$, $\psi_l(\tau)$ ($l = 1, \ldots, k$) are nonrandom functions from the space $L_2([t,T])$, $\mathbf{w}_\tau^{(i)}$ ($i = 1, \ldots, m$) are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$ and define the following function on the hypercube $[t,T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ & \\ 0, & \text{otherwise} \end{cases}, \tag{2}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ converges to $K(t_1,\ldots,t_k)$ on the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \qquad (3)$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \qquad (4)$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\qquad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\quad\text{if}\quad N\to\infty, \qquad (5)$$

where $\Delta\tau_j=\tau_{j+1}-\tau_j$.

Theorem 1 [14] (2006), [15]-[17], [20], [22]-[29], [35], [36]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k},$$

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le k!\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right), \qquad (6)$$

where

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr) \tag{7}$$

and

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i *or* j (*if* $i\neq 0$), $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (4), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is the discretization* (5), *the estimate* (6) *is valid for* $T-t\in(0,\infty)$ *and* $i_1,\ldots,i_k=1,\ldots,m$ *or* $T-t\in(0,1)$ *and* $i_1,\ldots,i_k=0,1,\ldots,m.$

Note that in [14]-[17], [20], [22], [23], [36] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Some modifications of Theorem 1 for another types of iterated stochastic integrals as well as for complete orthonormal with weight $r(t_1)\ldots r(t_k)\geq 0$ systems of functions in the space $L_2([t,T]^k)$ can be found in [14]-[17], [20], [22], [23], [36]. Application of Theorem 1 and Theorem 4 (see below) to the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process is presented in [29], [36] (Chapter 7), [38], [39].

Obtain transformed particular cases of Theorem 1 for $k=1,\ldots,5$ [14]-[17], [20], [22]-[29], [35], [36]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{10}$$

$$
J[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (11)
$$

$$
J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (12)
$$

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$\left.+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \qquad (13)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the generalization of the formulas (9)–(13) for the case of an arbitrary k ($k\in\mathbb{N}$).

Theorem 2 [16] (2009), [17], [20], [22], [23], [29], [36]. *Under the conditions of Theorem* 1 *the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right) \qquad (14)$$

converging in the mean-square sense is valid, where $[\cdot]$ *is an integer part of a real number,*

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}$$

means the sum with respect to all possible permutations of the set

$$(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}),$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, *braces mean an unordered set, and parentheses mean an ordered set*; *another notations are the same as in Theorem* 1.

For further consideration, we need the following statement.

Theorem 3 [15] (2007), [16], [17], [20], [22], [23], [36]. *Under the conditions of Theorem* 1 *the following estimate*

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le
$$

$$
\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \times
$$

$$
\times\ \left(\ \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n \tag{15}
$$

is valid, where $n\in\mathbb{N}$; *another notations are the same as in Theorem* 1.

Since according to the Parseval's equality

$$
\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\ \to\ 0
$$

if $p_1,\ldots,p_k\to\infty$, then the inequality (15) means that the expansions of iterated Itô stochastic integrals in Theorem 1 converge in the mean of degree $2n$ ($n\in\mathbb{N}$).

Let us consider the generalization of Theorems 1–3 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 4 [36] (Sect. 1.11), [37] (Sect. 15). *Suppose that* $\psi_1(\tau),\ \ldots,$ $\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then*

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}, \tag{16}
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2}\right\}\le
$$

$$
\le k!\left(\ \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),
$$

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times$$

$$\times\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n,$$

where $n\in\mathbb{N}$,

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\times\left.\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right),\quad(17)$$

where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorems* 1–3*.*

It should be noted that an analogue of the expansion (16) under the conditions of Theorem 4 was considered in [40]. Note that we use another notations [36] (Sect. 1.11), [37] (Sect. 15) in comparison with [40]. Moreover, the proof of an analogue of (16) from [40] is somewhat different from the proof given in [36] (Sect. 1.11), [37] (Sect. 15).

Also note the following theorem.

Theorem 5 [36] (Sect. 1.12), [41] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2}\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\},$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the value* $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ *is defined by* (17) ($p_1=\ldots=p_k=p$); *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Let us consider the following iterated Itô stochastic integrals from the Taylor–Itô expansion [3]

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{18}$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, $\lambda_l=1$ if $i_l=1,\ldots,m$ and $\lambda_l=0$ if $i_l=0$ $(l=1,\ldots,k)$. Remind that $\mathbf{w}_\tau^{(i)}$, $i=1,\ldots,m$ are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)}=\tau$.

For example, using Theorems 1, 4 (see (9)-(11)) and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ we obtain the following approximations of the iterated Itô stochastic integrals (18) [14]-[17], [20], [22]-[29], [35], [36] (also see early publications [8], [9])

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},\tag{19}$$

$$J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),\tag{20}$$

$$J_{(10)T,t}^{(i_10)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),\tag{21}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),\tag{22}$$

$$J_{(11)T,t}^{(i_1 i_1)} = \frac{1}{2}(T-t)\left(\left(\zeta_0^{(i_1)}\right)^2 - 1\right),$$

$$J_{(111)T,t}^{(i_1 i_2 i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (23)$$

$$J_{(111)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right),$$

where

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

where the Gaussian random variable $\zeta_j^{(i)}$ (if $i \ne 0$) is defined by (8) and $P_j(x)$ ($j = 0, 1, 2, \ldots$) is the Legendre polynomial [42].

Note that formula (22) has been obtained for the first time in [8] (1997). For pairwise different $i_1, i_2, i_3 = 1, \ldots, m$ we have [8], [9], [14]-[17], [20], [22]-[29], [35]

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1 i_2)} - J_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \qquad (24)$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)} - J_{(111)T,t}^{(i_1 i_2 i_3)p}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2. \qquad (25)$$

The problem of the exact calculation of the mean-square error of approximation in Theorems 1, 4 is solved completely for an arbitrary k ($k \in \mathbb{N}$) and any possible combinations of the numbers $i_1, \ldots, i_k = 1, \ldots, m$ in Theorem 5 (also see [23], [36], [41]).

3 Convergence With Probability 1 of Expansions of Iterated Itô Stochastic Integrals of Multiplicity k $(k \in \mathbb{N})$ in Theorems 1, 2

Let us address now to the convergence with probability 1 (w. p. 1) in Theorem 1. As we mentioned above this question has been studied for simplest iterated Itô stochastic integrals of multiplicities 1 and 2 in [15]-[17], [20], [22], [23], [36].

In this section, we formulate and prove the general result on convergence w. p. 1 of expansions of iterated Itô stochastic integrals in Theorems 1, 2 for the case of multiplicity k $(k \in \mathbb{N})$ for these integrals.

Theorem 6. *Let $\psi_l(\tau)$ $(l = 1, \ldots, k)$ are continuously differentiable nonrandom functions on the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then $J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \to J[\psi^{(k)}]_{T,t}$ if $p \to \infty$ w. p. 1, where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is defined as the right-hand side of (14) before passing to the limit for the case $p_1 = \ldots = p_k = p$, i.e. (see Theorem 2)*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times\right.$$

$$\left. \times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}\right),$$

where $i_1, \ldots, i_k = 1, \ldots, m$, another notations are the same as in Theorems 1, 2.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1, \ldots, t_k)dt_1 \ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2, \qquad (26)$$

where

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases},$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k$$

is the Fourier coefficient.

Taking into account the definitions of $K(t_1,\ldots,t_k)$ and $C_{j_k\ldots j_1}$, we obtain

$$C_{j_k\ldots j_1}=\int\limits_{t}^{T}\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_{t}^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k. \tag{27}$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathrm{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

If $p_1=\ldots=p_k=p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}\stackrel{\mathrm{def}}{=}\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathrm{def}}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\stackrel{\mathrm{def}}{=}\sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}$$

and so on.

Using the Parseval equality and Lemma 2 (see Appendix) we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$=\sum_{j_1,\ldots,j_k=0}^{\infty}C^2_{j_k\ldots j_1}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C^2_{j_k\ldots j_1}=$$

$$= \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} \sum_{j_3=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} +$$

$$+ \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 =$$

$$= \ldots =$$

$$= \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \sum_{j_4=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{p} \ldots \sum_{j_{k-1}=0}^{p} \sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \sum_{j_1=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_2=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+ \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=p+1}^{\infty} \sum_{j_4=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{\infty} \ldots \sum_{j_{k-1}=0}^{\infty} \sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{s=1}^{k} \left(\sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \right). \qquad (28)$$

Note that deriving (28) we use the following

$$\sum_{j_1=0}^{p} \ldots \sum_{j_{s-1}=0}^{p} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \sum_{j_1=0}^{m_1} \ldots \sum_{j_{s-1}=0}^{m_{s-1}} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \lim_{m_{s-1}\to\infty} \sum_{j_1=0}^{m_1} \ldots \sum_{j_{s-1}=0}^{m_{s-1}} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$
=\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le\ldots\le
$$

$$
\le\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2,
$$

where $m_1,\ldots,m_{s-1}>p$.

Denote

$$
C_{j_s\ldots j_1}(\tau)=\int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,
$$

where $s=1,\ldots,k-1$.

For $s<k$ due to Lemma 3, Dini Theorem (see Appendix) and Parseval equality we obtain

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2dt_k=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\int\limits_t^{t_k}\psi_{k-1}^2(\tau)\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau dt_k\le
$$

$$
\le M\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=
$$

$$
=M\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2d\tau=
$$

$$= M \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-3}=0}^{\infty} \int_t^T \int_t^{\tau} \psi_{k-2}^2(\theta) \left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le M' \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_{k-3}=0}^{\infty} \int_t^T \left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le \ldots \le$$

$$\le M_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \int_t^T \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$= M_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int_t^T \sum_{j_1=0}^{\infty} \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau, \qquad (29)$$

where constants M, M' depend on $T-t$ and constant M_k depends on $T-t$ and k.

Let us explane more precisely how we obtain (29). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty} \left(\int_t^{\tau} \phi_j(s) g(s) ds\right)^2 = \sum_{j=0}^{\infty} \left(\int_t^{T} \mathbf{1}_{\{s<\tau\}} \phi_j(s) g(s) ds\right)^2 =$$

$$= \int_t^T \left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s) ds = \int_t^{\tau} g^2(s) ds. \qquad (30)$$

Equality (30) has been applied repeatedly when we obtaining (29).

Using the replacement of integration order for Riemann integrals, we have

$$C_{j_s\ldots j_1}(\tau) = \int_t^{\tau} \phi_{j_s}(t_s)\psi_s(t_s) \ldots \int_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1) dt_1 \ldots dt_s =$$

$$= \int_t^{\tau} \phi_{j_1}(t_1)\psi_1(t_1) \int_{t_1}^{\tau} \phi_{j_2}(t_2)\psi_2(t_2) \ldots \int_{t_{s-1}}^{\tau} \phi_{j_s}(t_s)\psi_s(t_s) dt_s \ldots dt_2 dt_1.$$

For $l = 1, \ldots, s$ we will use the following notation

$$\tilde{C}_{j_s\ldots j_l}(\tau,\theta)=$$

$$=\int\limits_{\theta}^{\tau}\phi_{j_l}(t_l)\psi_l(t_l)\int\limits_{t_l}^{\tau}\phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots\int\limits_{t_{s-1}}^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.$$

Using the Parseval equality and Dini Theorem (see Appendix), from (29) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C^2_{j_k\ldots j_1}\le$$

$$\le M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\sum_{j_1=0}^{\infty}\left(C_{j_s\ldots j_1}(\tau)\right)^2d\tau=$$

$$=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_2=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \qquad (31)$$

$$=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau= \qquad (32)$$

$$=M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t_1}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le M_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_1^2(t_1)\int\limits_{t}^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2dt_1d\tau\le$$

$$\le M_k^{'}\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2dt_2d\tau\le\ldots\le$$

$$\le M_k^{''}\sum_{j_s=p+1}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_{s-1}^2(t_{s-1})\left(\tilde{C}_{j_s}(\tau,t_{s-1})\right)^2dt_{s-1}d\tau\le$$

$$\le\tilde{M}_k\sum_{j_s=p+1}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\left(\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2dud\tau, \qquad (33)$$

where constants $M_k^{'}$, $M_k^{''}$, and $\tilde{M}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (33). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty}\left(\int\limits_{\theta}^{\tau}\phi_j(s)g(s)ds\right)^2=\sum_{j=0}^{\infty}\left(\int\limits_{t}^{T}\mathbf{1}_{\{\theta<s<\tau\}}\phi_j(s)g(s)ds\right)^2=$$

$$=\int\limits_{t}^{T}\left(\mathbf{1}_{\{\theta<s<\tau\}}\right)^2 g^2(s)ds=\int\limits_{\theta}^{\tau}g^2(s)ds. \qquad (34)$$

Equality (34) has been applied repeatedly when we obtain (33).

Let us explane more precisely the passing from (31) to (32) (the same steps have been used when we derived (33)).

We have

$$\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau-\sum_{j_2=0}^{n}\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau=$$

$$=\int\limits_{t}^{T}\int\limits_{t}^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau=$$

$$=\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_{t}^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2 dt_1\Delta\tau_j, \qquad (35)$$

where $\{\tau_j\}_{j=0}^{N}$ is the partition of the interval $[t,T]$, which satisfies the condition (5).

Since the non-decreasing functional sequence $u_n(\tau_j,t_1)$ and its limit function $u(\tau_j,t_1)$ are continuous on the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1, where

$$u_n(\tau_j,t_1)=\sum_{j_2=0}^{n}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2,$$

$$u(\tau_j,t_1)=\sum_{j_2=0}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2=\int\limits_{t_1}^{\tau_j}\psi_2^2(t_2)\left(\tilde{C}_{j_s\ldots j_3}(\tau_j,t_2)\right)^2 dt_2,$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j,t_1)$ to $u(\tau_j,t_1)$ at the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1. As a result, we obtain

$$\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2<\varepsilon,\quad t_1\in[t,\tau_j] \tag{36}$$

for $n>N(\varepsilon)$ ($N(\varepsilon)$ exists for any $\varepsilon>0$ and it does not depend on t_1).

From (35) and (36) we obtain

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau_j,t_1)\right)^2dt_1\Delta\tau_j\le$$

$$\le\varepsilon\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)dt_1\Delta\tau_j=$$

$$=\varepsilon\int\limits_t^{T}\int\limits_t^{\tau}\psi_1^2(t_1)dt_1d\tau. \tag{37}$$

From (37) we get

$$\lim_{n\to\infty}\int\limits_t^{T}\int\limits_t^{\tau}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2dt_1d\tau=0.$$

This fact completes the proof of passing from (31) to (32).

Let us estimate the integral

$$\int\limits_u^{\tau}\phi_{j_s}(\theta)\psi_s(\theta)d\theta \tag{38}$$

from (33) for the cases when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Note that the estimates for the integral

$$\int\limits_t^{\tau}\phi_j(\theta)\psi(\theta)d\theta,\quad j\ge p+1 \tag{39}$$

have been obtained in [20], [22], [23], [36]. Here $\psi(\theta)$ is a continuously differentiable function on the interval $[t, T]$,

Let us estimate the integral (38) using the approach from [20], [22], [23], [36].

First consider the case of Legendre polynomials. Then $\phi_j(\theta)$ looks as follows

$$\phi_j(\theta)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(\theta-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j\ge 0,$$

where $P_j(x)$ $(j=0,1,2\ldots)$ is the Legendre polynomial.

Further, we have

$$\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta=\frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy=$$

$$=\frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x))-P_{j-1}(z(x)))\psi(x)-$$

$$-(P_{j+1}(z(v))-P_{j-1}(z(v)))\psi(v)-$$

$$-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y)-P_{j-1}(y))\psi'(u(y))dy\Biggr),\tag{40}$$

where $x,v\in(t,T)$, $j\ge p+1$, and $u(y)$, $z(x)$ are defined by the following relations

$$u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t},$$

ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (40) we used the following well-known property of the Legendre polynomials [42]

$$\frac{dP_{j+1}}{dx}(x)-\frac{dP_{j-1}}{dx}(x)=(2j+1)P_j(x),\quad j=1,2,\ldots$$

From (40) and the well-known estimate for the Legendre polynomials [46]

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N},$$

where constant K does not depend on y and j, it follows that

$$\left|\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + \frac{1}{(1-(z(v))^2)^{1/4}} + C_1\right), \qquad (41)$$

where $z(x), z(v) \in (-1,1)$, $x, v \in (t,T)$ and constants C, C_1 does not depend on j.

From (41) we obtain

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 < \frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + \frac{1}{(1-(z(v))^2)^{1/2}} + C_3\right), \qquad (42)$$

where constants C_2, C_3 does not depend on j.

Let us apply (42) for the estimate of the right-hand side of (33). We have

$$\int\limits_t^T \int\limits_t^\tau \left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^1 \frac{dy}{\left(1-y^2\right)^{1/2}} + \int\limits_{-1}^1 \int\limits_{-1}^x \frac{dy}{\left(1-y^2\right)^{1/2}}dx + K_2\right) \le$$

$$\le \frac{K_3}{j_s^2}, \qquad (43)$$

where constants K_1, K_2, K_3 are independent of j_s.

Now consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta) = \frac{1}{\sqrt{T-t}}\begin{cases} 1, & j=0 \\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1, \\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r \end{cases} \qquad (44)$$

where $r = 1, 2, \ldots$

Using the system of functions (44), we have

$$\int\limits_{v}^{x} \phi_{2r-1}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}}\int\limits_{v}^{x} \sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$= -\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(\psi(x)\cos\frac{2\pi r(x-t)}{T-t} - \psi(v)\cos\frac{2\pi r(v-t)}{T-t} -$$

$$- \int\limits_{v}^{x} \cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\Biggr), \qquad (45)$$

$$\int\limits_{v}^{x} \phi_{2r}(\theta)\psi(\theta)d\theta = \sqrt{\frac{2}{T-t}}\int\limits_{v}^{x} \cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta =$$

$$= \sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\Biggl(\psi(x)\sin\frac{2\pi r(x-t)}{T-t} - \psi(v)\sin\frac{2\pi r(v-t)}{T-t} -$$

$$- \int\limits_{v}^{x} \sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\Biggr), \qquad (46)$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (45) and (46), we obtain for the trigonometric case

$$\left(\int\limits_{v}^{x} \phi_j(\theta)\psi(\theta)d\theta\right)^2 \le \frac{C_4}{j^2}, \qquad (47)$$

where constant C_4 is independent of j.

From (47) we finally have

$$\int\limits_{t}^{T}\int\limits_{t}^{\tau}\left(\int\limits_{u}^{\tau} \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau \le \frac{K_4}{j_s^2}, \qquad (48)$$

where constant K_4 is independent of j_s.

Combining (33), (43) and (48), we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \le$$

$$\leq L_k \sum_{j_s=p+1}^{\infty} \frac{1}{j_s^2} \leq L_k \int\limits_p^{\infty} \frac{dx}{x^2} = \frac{L_k}{p}, \tag{49}$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (28) and (49) we obtain

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2 \leq \frac{G_k}{p}, \tag{50}$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use estimate (15). Using (50) and the estimate (15) for the case $p_1=\ldots=p_k=p$ and $n=2$, we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \leq$$

$$\leq C_{2,k}\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}^2\right)^2 \leq$$

$$\leq \frac{H_{2,k}}{p^2}, \tag{51}$$

where

$$C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

and $H_{2,k} = G_k^2 C_{2,k}$.

Let us consider Lemma 1 (see Appendix) with

$$\xi_p = \left|J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right| \quad \text{and} \quad \alpha = 4.$$

Then from (51) we get

$$\sum_{p=1}^{\infty} \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\} \leq H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty. \tag{52}$$

Using Lemma 1 (see Appendix) and the estimate (52), we obtain

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p\to\infty \quad \text{w. p. 1},$$

where (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \ \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr) \tag{53}$$

or (see Theorem 2)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{54}$$

where $i_1,\ldots,i_k=1,\ldots,m$ in (53) and (54). The proof of Theorem 6 is completed.

4 Appendix

Lemma 1 [43]. *If for the sequence of random variables ξ_p and for some $\alpha>0$ the number series*

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

Lemma 2. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty} C_{j_k\ldots j_1}^2 = \sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2 \tag{55}$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$, *where* $C_{j_k\ldots j_1}$ *is defined by* (27).

Proof. Let us remind the well-known fact from the mathematical analysis, which is connected to existence of iterated limits.

Proposition 1 [44]. *Let* $\bigl\{x_{n,m}\bigr\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$\lim_{n,m\to\infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \text{for any} \ \ m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \text{for any} \ \ n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \tag{56}$$

for any permutation $(q_l,\ldots,q_k)$, where $l=1,2,\ldots,k$, $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$.

Obviously, (56) is the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \le \sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 \le$$

$$\le \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}^2 < \infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C_{j_k\ldots j_1}^2 = \sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty} C_{j_k\ldots j_1}^2$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{57}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{58}$$

and equality of the limits (57) and (58).

Consequently,

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{59}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists (see the Parseval equality (26)), then from Proposition 1 we have

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{60}$$

Using (59) and Proposition 1, we have

$$\sum_{j_{q_2}=0}^{\infty}\ \sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} =$$

$$= \lim_{q,p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} = \sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{61}$$

Combining (61) and (60), we obtain

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2}=0}^{\infty}\ \sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

Repeating the above steps, we complete the proof of Lemma 2.

Lemma 3. *The following equality takes place*

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1}, \tag{62}$$

where $s = 1,\ldots,k$ *and* $C_{j_k\ldots j_1}$ *is defined by* (27).

Proof. Applying the arguments that we used in the proof of Lemma 2, we obtain

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}\ldots\sum_{j_{s-1}=0}^{n}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{n}\ldots\sum_{j_k=0}^{n} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=0}^{p}\ \sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} \tag{63}$$

for any permutation $(q_1,\ldots,q_{k-1})$ such that $\{q_1,\ldots,q_{k-1}\} = \{1,\ldots,s-1,s+1,\ldots,k\}$, where p is a fixed natural number.

Obviously, we have

$$\sum_{j_s=0}^{p}\sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_s=0}^{p}\ldots\sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} = \ldots =$$

$$= \sum_{j_{q_1}=0}^{\infty} \dots \sum_{j_{q_{k-1}}=0}^{\infty} \sum_{j_s=0}^{p} C_{j_k \dots j_1}^2. \tag{64}$$

Using (63), (64) and Lemma 2, we obtain

$$\sum_{j_1=0}^{\infty} \dots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2 =$$

$$= \sum_{j_1=0}^{\infty} \dots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2 - \sum_{j_1=0}^{\infty} \dots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=0}^{p} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2 =$$

$$= \sum_{j_s=0}^{\infty} \sum_{j_{s-1}=0}^{\infty} \dots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2 - \sum_{j_s=0}^{p} \sum_{j_{s-1}=0}^{\infty} \dots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2 =$$

$$= \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \dots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \dots \sum_{j_k=0}^{\infty} C_{j_k \dots j_1}^2.$$

The equality (4) is proved.

Theorem (Dini) [45]. *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a, b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a, b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a, b]$.*

Bibliography

[1] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[2] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992. 632 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[5] Gaines J. G., Lyons, T. J. Random generation of stochastic area integrals. SIAM Journal of Applied Mathematics, 54 (1994), 1132-1146.

[6] Averina T.A., Prigarin S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[7] Kuznetsov D.F. Methods of numerical simulation of stochastic differential Ito equations solutions in problems of mechanics. Ph. D., St.-Petersburg, 1996. 260 p.

[8] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. Differential Equations and Control Processes, 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[9] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. Differential Equations and Control Processes, 1 (1998), 66-367. Available at:
http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html

[10] Prigarin S.M., Belov S.M. One application of series expansions of Wiener process. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[11] Wiktorsson M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487,

[12] Ryden T., Wiktorsson M. On the simulation of iterated Ito integrals. Stochastic Processes and their Applications, 91, 1 (2001), 151-168.

[13] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.

[14] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. Polytechnical University Publ., St.-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. Polytechnical University Publ., St.-Petersburg, 2007, 770 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-229

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. Polytechnical University Publ., St.-Petersburg, 2009, 768 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-230

[17] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. Differential Equations and Control Processes, 3 (2010), A.1-A.257. Available at:
http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[18] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[19] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[20] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. Polytechnical University Publ., St.-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234

[21] Zahri M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science, 8, 2 (2014), 186-198.

[22] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. Differential Equations and Control Processes, 1 (2017), A.1–A.385. Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[23] Kuznetsov D.F. Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Edition. Differential Equations and Control Processes, 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[24] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[25] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[26] Kuznetsov D.F. On Numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[27] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[28] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals, based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. Available at:
http://matem.anrb.ru/en/article?art_id=604.

[29] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Differential Equations and Control Processes, 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[30] Rybakov K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. Differential Equations and Control Processes, 4 (2019), 1-31. Available at:
https://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html

[31] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion

using Legendre polynomials and trigonometric functions. Differential Equations and Control Processes, 4 (2019), 32-52. Available at: https://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[32] Tang X., Xiao A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics, 45 (2019), 813-846.

[33] Chugai K.N., Kosachev I.M., Rybakov K.A. Approximate filtering methods in continuous-time stochastic systems. Advances in Theory and Practice of Computational Mechanics, ed. by L.C. Jain, M.N. Favorskaya, I.S. Nikitin, and D.L. Reviznikov. Springer Publ., 2020. 351-371. DOI: https://doi.org/10.1007/978-981-15-2600-8_24

[34] Rybakov K.A. Modeling and analysis of output processes of linear continuous stochastic systems based on orthogonal expansions of random functions. Journal of Computer and Systems Sciences International, 59, 3 (2020), 322-337. DOI: https://doi.org/10.1134/S1064230720030156

[35] Kuznetsov D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: https://doi.org/10.1134/S0965542520030100

[36] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 869 pp. [in English].

[37] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [in English].

[38] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[39] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods

for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[40] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differential Equations and Control Processes, 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[41] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals, based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 67 pp. [in English].

[42] Suetin P.K. Classical orthogonal polynomials. 3rd Edition. Moscow, Fizmatlit, 2005. 480 pp.

[43] Shiryaev A.N. Probability. Springer-Verlag, New York, 1996. 624 pp.

[44] Il'in V.A., Poznyak E.G. Foundations of mathematical analysis. Part I. Moscow, Nauka, 1967. 572 pp.

[45] Il'in V.A., Poznyak E.G. Foundations of mathematical analysis. Part II. Moscow, Nauka, 1973. 448 pp.

[46] Hobson E.W. The theory of spherical and ellipsoidal harmonics. Cambridge, Cambridge University Press, 1931. 502 p.

Chapter 2.

Expansions of Iterated Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series

EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS BASED ON GENERALIZED MULTIPLE FOURIER SERIES: MULTIPLICITIES 1 TO 6 AND BEYOND

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 6 on the basis of the method of generalized multiple Fourier series that converge in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k=1,\ldots,6$. The rate of mean-square convergence of the mentioned expansions for the case of multiple Fourier–Legendre series and for the case of multiple trigonometric Fourier series is found. Sufficient conditions are given for the expansion of iterated Stratonovich stochastic integrals of arbitrary miltiplicity k ($k\in\mathbb{N}$). The considered expansions contain only one operation of the limit transition in contrast to its existing analogues. This property is very important for the mean-square approximation of iterated stochastic integrals. The results of the article can be applied to the numerical integration of Ito stochastic differential equations with multidimensional non-commutative noises.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.

Keywords: Iterated Stratonovich stochastic integral, Iterated Ito stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Wong–Zakai type theorem, Expansion, Mean-square convergence.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[5]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ in [2]-[5]. At the same time $\psi_l(\tau)\equiv (t-\tau)^{q_l}$ $(l=1,\ldots,k;\ q_1,\ldots,q_k=0,1,2,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ in [6]-[29].

The construction of effective expansions (converging in the mean-square sense) for collections of iterated Stratonovich stochastic integrals (3) composes the subject of this article.

The problem of effective jointly numerical modeling (in the sense of the mean-square convergence criterion) of the iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[55]. The only exception is connected with a narrow particular case when $i_1=\ldots=i_k\neq 0$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv\psi(\tau)$. This case allows the investigation with using of the Ito formula [2]-[5].

Seems that iterated stochastic integrals may be approximated by multiple integral sums of different types [3], [5], [52]. However, this approach implies partitioning of the interval of integration $[t,T]$ of iterated stochastic integrals (the length $T-t$ of this interval is a rather small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant calculating costs [10].

In [3] (also see [2], [4], [5], [53], [54]) Milstein G.N. proposed to expand (2), (3) (the case $k=2$ and $\psi_1(\tau),\psi_2(\tau)\equiv 1$) in iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as the trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals (2), (3) were presented in [2], [4], [53], [54] $(k=1,2,3)$ and in [3], [5] $(k=1,2)$ for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1;\ i_1,i_2,i_3=0,1,\ldots,m$. Moreover, the authors of the works [2] (Sect. 5.8, pp. 202-204), [4] (pp. 82-84), [53] (pp. 438-439), [54] (pp. 263-264) use the Wong–Zakai approximation [56]-[59] (without rigorous proof) within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process. See discussion in Sect. 6 of this paper for details.

Note that in [55] the method (similar to the Milstein approach) of expansion of the double Ito stochastic integrals (2) $(k=2;\ \psi_1(\tau),\psi_2(\tau)\equiv 1;\ i_1,i_2=1,\ldots,m)$ based on the series expansion of the Wiener process [60] using Haar basis functions and trigonometric basis functions has been considered.

It is necessary to note that the approach based on the Karhunen–Loeve expansion [3] excelled in several times (or even in several orders) the methods of integral sums [3], [5], [52] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [6], [7] (also see [14]-[19], [22], [24], [26]-[29]) where $J^*[\psi^{(k)}]_{T,t}$ (see (3)) was represented as a multiple stochastic integral from the certain discontinuous nonrandom function of k variables, and the function was then expressed as the generalized iterated Fourier series by complete systems of continuously differentiable functions that are orthonormal in the space $L_2([t,T])$. As a result, the general iterated series expansion in terms of products of standard Gaussian random variables was obtained in [6], [7] (also see [14]-[19], [22], [24], [26]-[29]) for (3) with arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series. It was shown [6], [7] (also see [14]-[19], [22], [24], [26]-[29]) that the method of generalized iterated Fourier series leads to the approach based on the Karhunen–Loeve expansion [3] in the case of trigonometric system of functions and to a substantially simpler expansion of (3) in the case of Legendre polynomials system.

2. Method of Generalized Multiple Fourier Series

In the previous section we paid attention on the fact that the approach based on the Karhunen–Loeve expansion [3] and the method of generalized iterated Fourier series [6], [7] (also see [14]-[19], [22], [24], [26]-[29]) leads to iterated application of the operation of limit transition. This means that these methods may not converge in the mean-square sense to the appropriate stochastic integrals (3) for some methods of series summation. As we noted above, where is no rigorous proof how to overcome the mentioned problem (iterated application of the operation of limit transition) in the papers [2] (Sect. 5.8, pp. 202-204), [4] (pp. 82-84), [53] (pp. 438-439), [54] (pp. 263-264). Nevetheless, this problem not appears in the method, which is proposed for (2) in Theorem 1 (see below).

Let us consider the efficient approach to expansion of the iterated Ito stochastic integrals (2) [10]-[22], [24]-[44] (the so-called method of generalized multiple Fourier series).

The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then the indicated nonrandom function is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series that converges in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorem 1 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2).

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Sect. 13 (see Theorem 18)). Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & \text{for}\ \ t_1<\ldots<t_k\\ & \\ 0, & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{4}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

Theorem 1 [10] (2006), [11]-[22], [24]-[44], [51]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (5), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (6).

It was shown [12]-[19], [22], [24], [26]-[29], [36] that Theorem 1 is valid for convergence in the mean of degree $2n$ $(n\in\mathbb{N})$ and for convergence with probability 1 [26]-[29], [25]. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ also can be applied in Theorem 1 [12]-[19], [22], [24], [26]-[29], [36]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [24], [26]-[29], [37]. The generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ is given in [26] (Sect. 1.11), [36] (Sect. 15), [51] (see Theorem 18 from this paper).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [10]-[22], [24]-[44]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{10}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{11}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{12}$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{13}$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{14}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (5)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) (see [20], [22], [24], [26]-[29], [35]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [2]-[5] but Legendre polynomials.

4. As it turned out (see [6]-[22], [24]-[44]), it is more convenient to work with Legendre polynomials for construction the approximations of iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions (see [6]-[22], [24]-[44]). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [26]-[29], [40], [41].

5. As we noted above, the approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see similar approach [55]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$; $i_1,i_2,i_3=0,1,\ldots,m$) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since the partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the

condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202-204), [4] (pp. 82-84), [53] (pp. 438-439), [54] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [56]-[59] (see discussion in Sect. 6 of this paper for details).

Note that the correctness of formulas (9)–(14) can be verified by the fact that if $i_1 = \ldots = i_6 = i = 1, \ldots, m$ and $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv \psi(\tau)$, then we can derive from (9)–(14) the well known equalities [11]-[19], [22], [24], [26]-[29]

$$J[\psi^{(1)}]_{T,t} = \frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t} = \frac{1}{2!}\left(\delta^2_{T,t} - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t} = \frac{1}{3!}\left(\delta^3_{T,t} - 3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t} = \frac{1}{4!}\left(\delta^4_{T,t} - 6\delta^2_{T,t}\Delta_{T,t} + 3\Delta^2_{T,t}\right),$$

$$J[\psi^{(5)}]_{T,t} = \frac{1}{5!}\left(\delta^5_{T,t} - 10\delta^3_{T,t}\Delta_{T,t} + 15\delta_{T,t}\Delta^2_{T,t}\right),$$

$$J[\psi^{(6)}]_{T,t} = \frac{1}{6!}\left(\delta^6_{T,t} - 15\delta^4_{T,t}\Delta_{T,t} + 45\delta^2_{T,t}\Delta^2_{T,t} - 15\Delta^3_{T,t}\right)$$

w. p. 1, where

$$\delta_{T,t} = \int\limits_t^T \psi(\tau)d\mathbf{f}_\tau^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(\tau)d\tau.$$

The above relations can be independently obtained using the Ito formula and Hermite polynomials.

The results of Sect. 3–5 adapt Theorem 1 for the iterated Stratonovich stochastic integrals (3) of multiplicities 2–4. The case of multiplicity 1 follows from (9).

3. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2

Theorem 2 [17]-[19], [22], [24], [26]-[29]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$ is twice continuously differentiable nonrandom function on $[t,T]$. Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m),$$

where notations are the same as in Theorem 1.

Proof. In accordance to the standard relations between Ito and Stratonovich stochastic integrals [2] we have w. p. 1

(15)
$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1.$$

From the other hand, according to (10), we obtain

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

(16)
$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1}.$$

From (15) and (16) it folows that Theorem 2 will be proved if

(17)
$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}.$$

Note that in this section we present two different proofs of the existence of a limit on the right-hand side of (17) for the polynomial and trigonometric cases.

Let us prove (17). Consider the function

(18)
$$K^*(t_1,t_2) = K(t_1,t_2) + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_1),$$

where $t_1, t_2 \in [t, T]$ and $K(t_1, t_2)$ has the form (4) for $k = 2$.

Let us expand the function $K^*(t_1, t_2)$ defined by (18) using the variable t_1, when t_2 is fixed, into the generalized Fourier series at the interval (t, T)

(19)
$$K^*(t_1,t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2)\phi_{j_1}(t_1) \quad (t_1 \ne t, T),$$

where

$$C_{j_1}(t_2) = \int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1 = \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The equality (19) is fulfilled pointwise in each point of the interval (t, T) with respect to the variable t_1, when $t_2 \in [t, T]$ is fixed, due to the piecewise smoothness of the function $K^*(t_1, t_2)$ with respect to the variable $t_1 \in [t, T]$ (t_2 is fixed).

Note that due to the well-known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (19) converges when $t_1 = t$, $t_1 = T$.

Obtaining (19) we also used the fact that the right-hand side of (19) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

The function $C_{j_1}(t_2)$ is a continuously differentiable one at the interval $[t,T]$. Let us expand it into the generalized Fourier series at the interval (t,T)

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\neq t,T), \tag{20}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and the equality (20) is fulfilled pointwise at any point of the interval (t,T) (the right-hand side of (20) converges when $t_2=t$, $t_2=T$).

Let us substitute (20) into (19)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2. \tag{21}$$

Note that the series on the right-hand side of (21) converges at the boundary of the square $[t,T]^2$.

It is easy to see that substituting $t_1=t_2$ into (21), we obtain

$$\frac{1}{2}\psi_1(t_1)\psi_2(t_1)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1). \tag{22}$$

From (22) we formally have

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\int\limits_t^T\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\int\limits_t^T C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}=$$

$$=\lim_{p_1\to\infty}\lim_{p_2\to\infty}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}=\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{23}$$

Let us explain the second step in (23) (the fourth step in (23) follows from the orthonormality of the functions $\phi_j(s)$ at the interval $[t, T]$).

We have

$$\left|\int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1}\int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1\right| \le$$

$$\le \int\limits_t^T |\psi_2(t_1)G_{p_1}(t_1)|\,dt_1 \le C\int\limits_t^T |G_{p_1}(t_1)|\,dt_1, \tag{24}$$

where $C < \infty$ and

$$\sum_{j=p+1}^{\infty}\int\limits_t^{\tau}\psi_1(s)\phi_j(s)ds\phi_j(\tau) \stackrel{\mathsf{def}}{=} G_p(\tau).$$

Let us consider the case of Legendre polynomials. Then

$$|G_{p_1}(t_1)| = \frac{1}{2}\left|\sum_{j_1=p_1+1}^{\infty}(2j_1+1)\int\limits_{-1}^{z(t_1)}\psi_1(u(y))P_{j_1}(y)dyP_{j_1}(z(t_1))\right|, \tag{25}$$

where

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}, \tag{26}$$

and $P_j(s)$ $(j = 0, 1, \ldots)$ is the Legendre polynomial.

From (25) and the well-known formula

$$\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j = 1, 2, \ldots \tag{27}$$

it follows that

$$|G_{p_1}(t_1)| = \frac{1}{2}\left|\sum_{j_1=p_1+1}^{\infty}\left\{(P_{j_1+1}(z(t_1)) - P_{j_1-1}(z(t_1)))\,\psi_1(t_1) - \right.\right.$$

$$\left.\left. -\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}(P_{j_1+1}(y) - P_{j_1-1}(y))\,\psi_1'(u(y))dy\right\}P_{j_1}(z(t_1))\right| \le$$

$$\le C_0\left|\sum_{j_1=p_1+1}^{\infty}(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1)) - P_{j_1-1}(z(t_1))P_{j_1}(z(t_1)))\right| +$$

$$+\frac{T-t}{4}\left|\sum_{j_1=p_1+1}^{\infty}\left\{\psi_1'(t_1)\left(\frac{1}{2j_1+3}(P_{j_1+2}(z(t_1)) - P_{j_1}(z(t_1))) - \right.\right.\right.$$

$$-\frac{1}{2j_1-1}(P_{j_1}(z(t_1))-P_{j_1-2}(z(t_1)))\Biggr)-$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(t_1)}\Biggl(\frac{1}{2j_1+3}(P_{j_1+2}(y)-P_{j_1}(y))-$$

$$-\frac{1}{2j_1-1}(P_{j_1}(y)-P_{j_1-2}(y))\Biggr)\psi_1''(u(y))dy\Biggr\}P_{j_1}(z(t_1))\Biggr|, \tag{28}$$

where C_0 is a constant, ψ_1' and ψ_1'' are derivatives of the function $\psi_1(s)$ with respect to the variable $u(y)$.

Using (28) and the well-known estimate for Legendre polynomials

$$|P_n(y)|<\frac{K}{\sqrt{n+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad n\in\mathbb{N}, \tag{29}$$

where constant K does not depend on y and n, we have

$$|G_{p_1}(t_1)|<$$

$$<C_0\left|\lim_{n\to\infty}\sum_{j_1=p_1+1}^{n}(P_{j_1+1}(z(t_1))P_{j_1}(z(t_1))-P_{j_1-1}(z(t_1))P_{j_1}(z(t_1)))\right|+$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}}+\int\limits_{-1}^{z(t_1)}\frac{dy}{(1-y^2)^{1/4}}\frac{1}{(1-(z(t_1))^2)^{1/4}}\right)<$$

$$<C_0\left|\lim_{n\to\infty}(P_{n+1}(z(t_1))P_n(z(t_1))-P_{p_1}(z(t_1))P_{p_1+1}(z(t_1)))\right|+$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}}+C_2\frac{1}{(1-(z(t_1))^2)^{1/4}}\right)<$$

$$<C_3\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{p_1}\right)\frac{1}{(1-(z(t_1))^2)^{1/2}}+$$

$$+C_1\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}}+C_2\frac{1}{(1-(z(t_1))^2)^{1/4}}\right)\le$$

$$\le C_4\left(\left(\frac{1}{p_1}+\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)\frac{1}{(1-(z(t_1))^2)^{1/2}}+\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\frac{1}{(1-(z(t_1))^2)^{1/4}}\right)\le$$

$$\le\frac{K}{p_1}\left(\frac{1}{(1-(z(t_1))^2)^{1/2}}+\frac{1}{(1-(z(t_1))^2)^{1/4}}\right), \tag{30}$$

where $C_0, C_1, \ldots, C_4, K$ are constants, $t_1 \in (t, T)$.

Note that in (30) we used the following inequality

$$\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \le \int\limits_{p_1}^{\infty} \frac{dx}{x^2} = \frac{1}{p_1}. \tag{31}$$

From (24) and (30) we get

$$\left| \int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{p_1} \int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 \right| <$$

$$< \frac{K}{p_1} \left(\int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/4}} \right) \to 0$$

if $p_1 \to \infty$. So we obtain

$$\frac{1}{2} \int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 =$$

$$= \sum_{j_1=0}^{\infty} \int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty} \int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 =$$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \int\limits_t^T C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{32}$$

In (32) we used the fact that the Fourier–Legendre series

$$\sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_2}(t_1)$$

of the smooth function $C_{j_1}(t_1)$ converges uniformly to this function at the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon > 0$, converges to this function at any point $t_1 \in (t, T)$, and converges to $C_{j_1}(t+0)$ and $C_{j_1}(T-0)$ when $t_1 = t$, $t_1 = T$ [61].

More precisely, we have

$$\int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = \int\limits_{t+\varepsilon}^{T-\varepsilon} \sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 + A_\varepsilon + B_\varepsilon =$$

$$= \sum_{j_2=0}^{\infty} C_{j_2j_1} \int\limits_{t+\varepsilon}^{T-\varepsilon} \phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 + A_\varepsilon + B_\varepsilon =$$

$$= \sum_{j_2=0}^{\infty} C_{j_2 j_1} \left(\int\limits_t^T - \int\limits_t^{t+\varepsilon} - \int\limits_{T-\varepsilon}^T \right) \phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 + A_\varepsilon + B_\varepsilon =$$

$$= \sum_{j_2=0}^{\infty} C_{j_2 j_1} \Bigl(\mathbf{1}_{\{j_1=j_2\}} - \varepsilon\bigl(\phi_{j_2}(\lambda)\phi_{j_1}(\lambda) + \phi_{j_2}(\theta)\phi_{j_1}(\theta)\bigr) \Bigr) + A_\varepsilon + B_\varepsilon =$$

$$= C_{j_1 j_1} - \varepsilon \left(\sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(\lambda)\phi_{j_1}(\lambda) + \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(\theta)\phi_{j_1}(\theta) \right) + A_\varepsilon + B_\varepsilon, \qquad (33)$$

where $\theta \in [t, t+\varepsilon]$, $\lambda \in [T-\varepsilon, T]$, and

$$A_\varepsilon = \int\limits_t^{t+\varepsilon} \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1, \quad B_\varepsilon = \int\limits_{T-\varepsilon}^{T} \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1.$$

In obtaining (33) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j = 0, 1, 2 \ldots$

Further, we have $|A_\varepsilon| + |B_\varepsilon| \le \varepsilon C$, where $C < \infty$ is a constant. Performing the passage to the limit $\lim\limits_{\varepsilon \to +0}$ in the equality (33), we get

$$\int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = C_{j_1 j_1}.$$

Then (see (32))

$$\sum_{j_1=0}^{\infty} \int\limits_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_1)\phi_{j_1}(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}$$

and the relation (17) is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. The complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of trigonometric functions in the space $L_2([t,T])$ has the following form

$$\phi_j(\theta) = \frac{1}{\sqrt{T-t}} \begin{cases} 1, & j = 0 \\ \sqrt{2}\sin\left(2\pi r(\theta - t)/(T-t)\right), & j = 2r-1, \\ \sqrt{2}\cos\left(2\pi r(\theta - t)/(T-t)\right), & j = 2r \end{cases} \qquad (34)$$

where $r = 1, 2, \ldots$

We have

$$S_{2p_1} \stackrel{\mathsf{def}}{=} \left| \int\limits_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 - \sum_{j_1=0}^{2p_1} \int\limits_t^T C_{j_1}(t_1)\phi_{j_1}(t_1)dt_1 \right| =$$

$$=\left|\int\limits_t^T\sum_{j_1=2p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|=$$

$$=\frac{2}{T-t}\left|\int\limits_t^T\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^{t_1}\psi_1(s)\sin\frac{2\pi j_1(s-t)}{T-t}ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.$$

$$\left.\left.+\int\limits_t^{t_1}\psi_1(s)\cos\frac{2\pi j_1(s-t)}{T-t}ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\right)dt_1\right|=$$

$$=\frac{1}{\pi}\left|\int\limits_t^T\left(\psi_1(t)\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}+\right.\right.$$

$$+\frac{T-t}{2\pi}\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\left(\psi_1'(t_1)-\psi_1'(t)\cos\frac{2\pi j_1(t_1-t)}{T-t}-\right.$$

$$-\int\limits_t^{t_1}\sin\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \sin\frac{2\pi j_1(t_1-t)}{T-t}-$$

$$\left.\left.\left.-\int\limits_t^{t_1}\cos\frac{2\pi j_1(s-t)}{T-t}\psi_1''(s)ds\ \cos\frac{2\pi j_1(t_1-t)}{T-t}\right)\right)dt_1\right|\le$$

$$\le C_1\left|\int\limits_t^T\psi_2(t_1)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right|+\frac{C_2}{p_1}=$$

$$=C_1\left|\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1}\int\limits_t^T\psi_2(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1\right|+\frac{C_2}{p_1},\tag{35}$$

where constants C_1, C_2 do not depend on p_1.

Here we used the fact that the functional series

$$\sum_{j_1=1}^{\infty}\frac{1}{j_1}\sin\frac{2\pi j_1(t_1-t)}{T-t}\tag{36}$$

converges uniformly at the interval $[t+\varepsilon, T-\varepsilon]$ for any $\varepsilon>0$ due to Drichlet–Abel Theorem, and converges to zero at the points t and T. Moreover, the series (36) (with accuracy to a linear transformation) is the trigonometric Fourier series of the smooth function $K(t_1)=t_1-t$, $t_1\in[t,T]$. So the series (36) converges to the smooth function at any point $t_1\in(t,T)$.

From (35) we obtain

$$S_{2p_1}=\left|\int\limits_t^T\sum_{j_1=2p_1+1}^{\infty}\psi_2(t_1)\phi_{j_1}(t_1)\int\limits_t^{t_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta dt_1\right|\le$$

$$
\leq C_3 \left| \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(\psi_2(T) - \psi_2(t) - \int\limits_t^T \cos\frac{2\pi j_1(s-t)}{T-t} \psi_2'(s) ds \right) \right| + \frac{C_2}{p_1} \leq \frac{C_4}{p_1}, \tag{37}
$$

where constants C_2, C_3, C_4 do not depend on p_1.

Further,

$$
S_{2p_1-1} = \left| \int\limits_t^T \sum_{j_1=2p_1}^{\infty} \psi_2(t_1) \phi_{j_1}(t_1) \int\limits_t^{t_1} \psi_1(\theta) \phi_{j_1}(\theta) d\theta dt_1 \right| =
$$

$$
= \left| S_{2p_1} + \int\limits_t^T \psi_2(t_1) \phi_{2p_1}(t_1) \int\limits_t^{t_1} \psi_1(\theta) \phi_{2p_1}(\theta) d\theta dt_1 \right| \leq
$$

$$
\leq S_{2p_1} + \frac{2}{T-t} \left| \int\limits_t^T \psi_2(t_1) \cos\frac{2\pi p_1(t_1-t)}{T-t} \int\limits_t^{t_1} \psi_1(\theta) \cos\frac{2\pi p_1(\theta-t)}{T-t} d\theta dt_1 \right|. \tag{38}
$$

Moreover,

$$
\int\limits_t^T \psi_2(t_1) \cos\frac{2\pi p_1(t_1-t)}{T-t} \int\limits_t^{t_1} \psi_1(\theta) \cos\frac{2\pi p_1(\theta-t)}{T-t} d\theta dt_1 =
$$

$$
= \frac{T-t}{2\pi p_1} \int\limits_t^T \psi_2(t_1) \cos\frac{2\pi p_1(t_1-t)}{T-t} \left(\psi_1(t_1) \sin\frac{2\pi p_1(t_1-t)}{T-t} - \right.
$$

$$
\left. - \int\limits_t^T \psi_1'(\theta) \sin\frac{2\pi p_1(\theta-t)}{T-t} d\theta \right) dt_1. \tag{39}
$$

The relations (37)–(39) imply that

$$
S_{2p_1-1} \leq \frac{C_5}{p_1}, \tag{40}
$$

where constant C_5 is independent of p_1.

From (37) and (40) we get

$$
S_{p_1} = \left| \int\limits_t^T \sum_{j_1=p_1+1}^{\infty} \psi_2(t_1) \phi_{j_1}(t_1) \int\limits_t^{t_1} \psi_1(\theta) \phi_{j_1}(\theta) d\theta dt_1 \right| \leq \frac{K}{p_1} \to 0 \tag{41}
$$

if $p_1 \to \infty$, where constant K does not depend on p_1 ($p_1 \in \mathbb{N}$). Further steps are similar to the proof of (17) for the case of Legendre polynomials. Theorem 2 is proved.

Note that the estimate (41) will be used further.

Lemma 1. *Under the conditions of Theorem* 2 *the following limit*

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}C_{j_1j_1}$$

exists, where $C_{j_1j_1}$ *is defined by* (5) *for* $k=2$ *and* $j_1=j_2$, *i.e.*

$$C_{j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$

Lemma 1 has already been proved in this section. Further, in this section, another proof of Lemma 1 is given. This will allow us to obtain useful estimates that will be used later.

Consider another proof of Lemma 1. We will prove that

$$\sum_{j_1=0}^{n}C_{j_1j_1}$$

is the Cauchy sequence for the cases of Legendre polynomials and trigonometric functions.

Consider the case of Legendre polynomials. Below in this section we write $\lim\limits_{n,m\to\infty}$ instead of $\lim\limits_{\substack{n,m\to\infty\\ n>m}}$. Fix $n>m$ ($n,m\in\mathbb{N}$). We have

$$\sum_{j_1=m+1}^{n}C_{j_1j_1}=\sum_{j_1=m+1}^{n}\int\limits_t^T\psi_2(s)\phi_{j_1}(s)\int\limits_t^{s}\psi_1(\tau)\phi_{j_1}(\tau)d\tau ds=$$

$$=\frac{T-t}{4}\sum_{j_1=m+1}^{n}(2j_1+1)\int\limits_{-1}^{1}\psi_2(u(x))P_{j_1}(x)\int\limits_{-1}^{x}\psi_1(u(y))P_{j_1}(y)dydx=$$

$$=\frac{T-t}{4}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_1(u(x))\psi_2(u(x))\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-$$

$$-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_2(u(x))P_{j_1}(x)\int\limits_{-1}^{x}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(u(y))dydx=$$

$$=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(u(x))\psi_2(u(x))\sum_{j_1=m+1}^{n}\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-$$

$$-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(u(y))\int\limits_{y}^{1}P_{j_1}(x)\psi_2(u(x))dxdy=$$

$$=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(u(x))\psi_2(u(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx+$$

$$+\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\frac{1}{2j_1+1}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(u(y))\times$$

$$\times\Biggl(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_2(u(y))+$$

$$+\frac{T-t}{2}\int\limits_{y}^{1}\left(P_{j_1+1}(x)-P_{j_1-1}(x)\right)\psi_2'(u(x))dx\Biggr)dy, \tag{42}$$

where ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $u(y)$ (see (26)).

Applying the estimate (29) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we finally obtain

$$\left|\sum_{j_1=m+1}^{n}C_{j_1j_1}\right|\le C_1\left(\frac{1}{n}+\frac{1}{m}\right)\int\limits_{-1}^{1}\frac{dx}{\left(1-x^2\right)^{1/2}}+$$

$$+C_2\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/2}}+\int\limits_{-1}^{1}\frac{1}{\left(1-y^2\right)^{1/4}}\int\limits_{y}^{1}\frac{dx}{\left(1-x^2\right)^{1/4}}dy\right)\le$$

$$\le C_3\left(\frac{1}{n}+\frac{1}{m}+\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\right)\to 0 \tag{43}$$

if $n,m\to\infty$ $(n>m)$, where constants C_1,C_2,C_3 do not depend on n and m. The relation (43) completes the proof of Lemma 1 for the polynomial case.

Consider the trigonometric case. Fix $n>m$ $(n,m\in\mathbb{N})$. Denote

$$S_{n,m}\stackrel{\sf def}{=}\sum_{j_1=m+1}^{n}C_{j_1j_1}=\sum_{j_1=m+1}^{n}\int\limits_{t}^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$

By analogy with (42) we obtain

$$S_{2n,2m}=\sum_{j_1=2m+1}^{2n}\int\limits_{t}^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\frac{2}{T-t}\sum_{j_1=m+1}^{n}\left(\int\limits_{t}^{T}\psi_2(t_2){\rm sin}\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2+\right.$$

$$\left.+\int\limits_{t}^{T}\psi_2(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_{t}^{t_2}\psi_1(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2\right)=$$

$$=\frac{T-t}{2\pi^2}\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\psi_1(t)\left(\psi_2(t)-\psi_2(T)+\int\limits_{t}^{T}\psi_2'(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}dt_2\right)-\right.$$

$$-\int\limits_{t}^{T}\psi_1'(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}\left(\psi_2(T)-\psi_2(t_1){\rm cos}\frac{2\pi j_1(t_1-t)}{T-t}-\int\limits_{t_1}^{T}\psi_2'(t_2){\rm cos}\frac{2\pi j_1(t_2-t)}{T-t}dt_2\right)dt_1+$$

$$+\int\limits_t^T \psi_1'(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}\left(\psi_2(t_1){\rm sin}\frac{2\pi j_1(t_1-t)}{T-t}+\int\limits_{t_1}^T \psi_2'(t_2){\rm sin}\frac{2\pi j_1(t_2-t)}{T-t}dt_2\right)dt_1\Biggr), \qquad (44)$$

where $\psi_1'(\tau)$, $\psi_2'(\tau)$ are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable τ.

From (44) we get

$$\left|S_{2n,2m}\right| \le C\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\to 0 \qquad (45)$$

if $n,m\to\infty$ $(n>m)$, where constant C does not depend on n and m.

Further,

$$S_{2n-1,2m}=S_{2n,2m}-\frac{2}{T-t}\int\limits_t^T \psi_2(t_2){\rm cos}\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2} \psi_1(t_1){\rm cos}\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2, \qquad (46)$$

$$S_{2n,2m-1}=S_{2n,2m}+\frac{2}{T-t}\int\limits_t^T \psi_2(t_2){\rm cos}\frac{2\pi m(t_2-t)}{T-t}\int\limits_t^{t_2} \psi_1(t_1){\rm cos}\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2, \qquad (47)$$

$$S_{2n-1,2m-1}=S_{2n,2m-1}-\frac{2}{T-t}\int\limits_t^T \psi_2(t_2){\rm cos}\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2} \psi_1(t_1){\rm cos}\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2=$$

$$=S_{2n,2m}+\frac{2}{T-t}\int\limits_t^T \psi_2(t_2){\rm cos}\frac{2\pi m(t_2-t)}{T-t}\int\limits_t^{t_2} \psi_1(t_1){\rm cos}\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2-$$

$$-\frac{2}{T-t}\int\limits_t^T \psi_2(t_2){\rm cos}\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2} \psi_1(t_1){\rm cos}\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2. \qquad (48)$$

Integrating by parts in (46)–(48), we obtain

$$\left|S_{2n-1,2m}\right|\le\left|S_{2n,2m}\right|+\frac{C_1}{n},\quad \left|S_{2n,2m-1}\right|\le\left|S_{2n,2m}\right|+\frac{C_1}{m}, \qquad (49)$$

$$\left|S_{2n-1,2m-1}\right|\le\left|S_{2n,2m}\right|+C_1\left(\frac{1}{m}+\frac{1}{n}\right), \qquad (50)$$

where constant C_1 does not depend on n and m.

The relations (45), (49), (50) imply that

$$\lim_{n,m\to\infty}\left|S_{2n,2m}\right|=\lim_{n,m\to\infty}\left|S_{2n-1,2m}\right|=\lim_{n,m\to\infty}\left|S_{2n,2m-1}\right|=\lim_{n,m\to\infty}\left|S_{2n-1,2m-1}\right|=0. \qquad (51)$$

From (51) we get

$$\lim_{n,m\to\infty}|S_{n,m}|=0. \tag{52}$$

The relation (52) completes the proof.

To conclude this section, we note that in [26], Sect. 2.1.4 (also see [64]) the following formula

$$\sum_{j=0}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(\tau)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau \tag{53}$$

is proved, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

Let us consider the proof of (53) from [26], Sect. 2.1.4. First consider the case $\psi_1(\tau)\equiv\psi_2(\tau)$ or

$$\psi_1(\tau)=\psi_2(\tau)\int\limits_t^{\tau}g(\theta)d\theta, \tag{54}$$

where $\tau\in[t,T]$ and $\psi_1(\tau),\psi_2(\tau),g(\tau)\in L_2([t,T])$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$.

Using Fubini's Theorem, Lebesgue's Dominated Convergence Theorem and the Parseval equality, we have (see (54))

$$\sum_{j=0}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_2(t_1)\phi_j(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\int\limits_{\tau}^T\psi_2(t_1)\phi_j(t_1)\int\limits_{t_1}^T\psi_2(t_2)\phi_j(t_2)dt_2dt_1d\tau=$$

$$=\frac{1}{2}\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\left(\int\limits_{\tau}^T\psi_2(t_1)\phi_j(t_1)dt_1\right)^2d\tau= \tag{55}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\sum_{j=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<t_1\}}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2d\tau= \tag{56}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\int\limits_t^T\mathbf{1}_{\{\tau<t_1\}}\psi_2^2(t_1)dt_1d\tau=\frac{1}{2}\int\limits_t^T g(\tau)\int\limits_{\tau}^T\psi_2^2(t_1)dt_1d\tau=$$

$$= \frac{1}{2}\int\limits_t^T \psi_2^2(t_1)\int\limits_t^{t_1} g(\tau)d\tau dt_1 = \tag{57}$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1, \tag{58}$$

where the transition from (55) to (56) is based on Lebesgue's Dominated Convergence Theorem. The integrable majorant exists due to Parseval's equality

$$|g(\tau)|\sum_{j=0}^{q}\left(\int\limits_{\tau}^{T}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 \le |g(\tau)|\sum_{j=0}^{\infty}\left(\int\limits_{t}^{T}\mathbf{1}_{\{\tau<t_1\}}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2 \le C\,|g(\tau)|\,,\quad C<\infty.$$

From the other hand, using Fubini's Theorem and the generalized Parseval equality as well as (57), we get

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_2(t_1)\phi_j(t_1)dt_1dt_2 =$$

$$= \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}g(\tau)d\tau\int\limits_t^{t_2}\psi_2(t_1)\phi_j(t_1)dt_1dt_2 =$$

$$= \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}g(\tau)d\tau dt_2dt_1 =$$

$$= \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)dt_1\int\limits_t^{T}\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}g(\tau)d\tau dt_2 -$$

$$- \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)\int\limits_t^{t_1}\psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}g(\tau)d\tau dt_2dt_1 =$$

$$= \int\limits_t^T \psi_2(t_1)\cdot\psi_2(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1 - \frac{1}{2}\int\limits_t^T \psi_2^2(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_2^2(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1 = \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1. \tag{59}$$

In addition, for the case $\psi_1(\tau)\equiv\psi_2(\tau)$, using the Parseval equality, we obtain

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2 =$$

$$
= \frac{1}{2} \sum_{j=0}^{\infty} \left(\int_t^T \psi_1(t_1)\phi_j(t_1)dt_1 \right)^2 = \frac{1}{2} \int_t^T \psi_1^2(t_1)dt_1. \tag{60}
$$

Suppose that $\psi_2(\tau) = (\tau - t)^l, g(\tau) = k(\tau - t)^{k-1}$, where $l = 0, 1, 2, \ldots,\ k = 1, 2, \ldots$ From (54) we have

$$
\psi_1(\tau) = \psi_2(\tau) \int_t^\tau g(\theta)d\theta = k(\tau - t)^l \int_t^\tau (\theta - t)^{k-1} d\theta = (\tau - t)^{l+k}.
$$

Taking into account (58)–(60), we obtain

$$
\sum_{j=0}^{\infty} \int_t^T (t_2 - t)^l \phi_j(t_2) \int_t^{t_2} (t_1 - t)^{l+k} \phi_j(t_1) dt_1 dt_2 =
$$

$$
= \sum_{j=0}^{\infty} \int_t^T (t_2 - t)^{l+k} \phi_j(t_2) \int_t^{t_2} (t_1 - t)^l \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_t^T (\tau - t)^{2l+k} d\tau, \tag{61}
$$

where $k, l = 0, 1, 2, \ldots$

The equality similar to (61) was obtained in [64] using other arguments. In addition, the formula similar to (61) was used in [64] to generalize the equality (53) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. Consider this approach [64] in more detail.

Let us rewrite the equality (61) in the following form

$$
\sum_{j=0}^{\infty} \int_t^T (t_2 - t)^l \phi_j(t_2) \int_t^{t_2} (t_1 - t)^m \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_t^T (\tau - t)^l (\tau - t)^m d\tau, \tag{62}
$$

where $l, m = 0, 1, 2, \ldots$

Since the equality (62) is valid for monomials with respect to $\tau - t$ ($\tau \in [t,T]$), it will obviously also be valid for Legendre polynomials that form a complete orthonormal system of functions in the space $L_2([t,T])$ and finite linear combinations of Legendre polynomials.

Let $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$ and $\psi_1^{(p)}(\tau), \psi_2^{(q)}(\tau)$ be approximations of the functions $\psi_1(\tau), \psi_2(\tau)$, respectively, which are partial sums of the corresponding Fourier–Legendre series. Then we have (see (62))

$$
\sum_{j=0}^{\infty} \int_t^T \psi_2^{(q)}(t_2) \phi_j(t_2) \int_t^{t_2} \psi_1^{(p)}(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_t^T \psi_1^{(p)}(\tau) \psi_2^{(q)}(\tau) d\tau, \tag{63}
$$

where $p, q \in \mathbb{N}$, the series converges absolutly and its sum does not depend on a basis system $\{\phi_j(x)\}_{j=0}^{\infty}$.

Let us fix q in (63). The right-hand side of (63) for a fixed q defines (as a scalar product in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\psi_2^{(q)}$. The left-hand side of the equality (63) has the same properties. Let us implement the passage to the limit $\lim\limits_{p \to \infty}$ in (63)

(64) $$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2^{(q)}(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(\tau)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2^{(q)}(\tau)d\tau,$$

where $q\in\mathbb{N}$. The equality (64) defines a linear bounded functional in $L_2([t,T])$ given by the function ψ_1. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (64)

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(\tau)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. The proof is completed.

4. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3

Theorem 3 [18], [19], [22], [24], [26]-[29]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(s)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(s)$, $\psi_3(s)$ are twice continuously differentiable nonrandom functions on $[t,T]$. Then*

(65) $$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m),$$

where notations are the same as in Theorem 1.

Proof. Let us consider the case of Legendre polynomials. From (11) for the case $p_1=p_2=p_3=p$ and the standard relation between Ito and Stratonovich stochastic integrals (2), (3) of third multiplicity it follows that Theorem 3 will be proved if w. p. 1

(66) $$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T\psi_3(s)\int\limits_t^s\psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_s^{(i_3)},$$

(67) $$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\psi_3(s)\psi_2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds,$$

(68) $$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0.$$

Let us prove (66). Using Theorem 1 when $k=1$ (also see (9)), we can write

$$\frac{1}{2}\int\limits_t^T\psi_3(s)\int\limits_t^s\psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3}=\int\limits_t^T\phi_{j_3}(s)\psi_3(s)\int\limits_t^s\psi_2(s_1)\psi_1(s_1)ds_1ds.$$

We have

$$E_p\stackrel{\rm def}{=}{\sf M}\left\{\left(\sum_{j_1=0}^p\sum_{j_3=0}^p C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\sum_{j_3=0}^p\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$={\sf M}\left\{\left(\sum_{j_3=0}^p\left(\sum_{j_1=0}^p C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\}=\sum_{j_3=0}^p\left(\sum_{j_1=0}^p C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=$$

$$=\sum_{j_3=0}^p\left(\sum_{j_1=0}^p\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds-\right.$$

$$\left.-\frac{1}{2}\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_1(s_1)\psi_2(s_1)ds_1ds\right)^2=$$

$$=\sum_{j_3=0}^p\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\left(\sum_{j_1=0}^p\psi_2(s_1)\phi_{j_1}(s_1)\times\right.\right.$$

$$\left.\left.\times\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2-\frac{1}{2}\psi_1(s_1)\psi_2(s_1)\right)ds_1ds\right)^2. \qquad (69)$$

Let us substitute $t_1=t_2=s_1$ into (19). Then for all $s_1\in(t,T)$

$$\sum_{j_1=0}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2=\frac{1}{2}\psi_1(s_1)\psi_2(s_1). \qquad (70)$$

From (69) and (70) it follows that

$$E_p=\sum_{j_3=0}^p\left(\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\sum_{j_1=p+1}^{\infty}\psi_2(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds\right)^2. \qquad (71)$$

From (71) and (30) we obtain

$$E_p<C_1\sum_{j_3=0}^p\left(\int\limits_t^T|\phi_{j_3}(s)|\frac{1}{p}\left(\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/2}}+\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/4}}\right)ds\right)^2\le$$

$$
\leq \frac{C_2}{p^2} \sum_{j_3=0}^{p} \left(\int\limits_t^T |\phi_{j_3}(s)| ds \right)^2 \leq \frac{C_2(T-t)}{p^2} \sum_{j_3=0}^{p} \int\limits_t^T \phi_{j_3}^2(s) ds = \frac{C_3 p}{p^2} \ \ \to \ 0
$$

if $p \to \infty$, where constants C_1, C_2, C_3 do not depend on p. The equality (66) is proved.

Let us prove (67). Using the Ito formula, we have

$$
\frac{1}{2} \int\limits_t^T \psi_3(s)\psi_2(s) \int\limits_t^s \psi_1(s_1) d\mathbf{f}_{s_1}^{(i_1)} ds = \frac{1}{2} \int\limits_t^T \psi_1(s_1) \int\limits_{s_1}^T \psi_3(s)\psi_2(s) ds d\mathbf{f}_{s_1}^{(i_1)} \quad \text{w. p. 1.}
$$

Using Theorem 1 for $k = 1$ (also see (9)), we obtain

$$
\frac{1}{2} \int\limits_t^T \psi_1(s) \int\limits_s^T \psi_3(s_1)\psi_2(s_1) ds_1 d\mathbf{f}_s^{(i_1)} = \frac{1}{2} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} C_{j_1}^{*} \zeta_{j_1}^{(i_1)},
$$

where

$$
C_{j_1}^{*} = \int\limits_t^T \psi_1(s)\phi_{j_1}(s) \int\limits_s^T \psi_3(s_1)\psi_2(s_1) ds_1 ds. \tag{72}
$$

We have

$$
E_p' \stackrel{\text{def}}{=} \mathsf{M}\left\{ \left(\sum_{j_1=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} - \frac{1}{2} \sum_{j_1=0}^{p} C_{j_1}^{*} \zeta_{j_1}^{(i_1)} \right)^2 \right\} =
$$

$$
= \mathsf{M}\left\{ \left(\sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^{*} \right) \zeta_{j_1}^{(i_1)} \right)^2 \right\} =
$$

$$
= \sum_{j_1=0}^{p} \left(\sum_{j_3=0}^{p} C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^{*} \right)^2, \tag{73}
$$

$$
C_{j_3 j_3 j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s) \int\limits_t^s \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 ds =
$$

$$
= \int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2) \int\limits_{s_2}^T \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s) ds ds_1 ds_2. \tag{74}
$$

From (72)–(74) we obtain

$$E'_p = \sum_{j_1=0}^{p} \left(\int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2) \int\limits_{s_2}^T \left(\sum_{j_3=0}^{p} \psi_2(s_1)\phi_{j_3}(s_1) \times \right.\right.$$

$$\left.\left. \times \int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds - \frac{1}{2}\psi_3(s_1)\psi_2(s_1) \right) ds_1 ds_2 \right)^2 . \tag{75}$$

Let us prove the following equality for all $s_1 \in (t, T)$

$$\sum_{j_3=0}^{\infty} \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds = \frac{1}{2}\psi_2(s_1)\psi_3(s_1). \tag{76}$$

Denote

$$K_1^*(t_1, t_2) = K_1(t_1, t_2) + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_2(t_1)\psi_3(t_1), \tag{77}$$

where

$$K_1(t_1, t_2) = \psi_2(t_1)\psi_3(t_2)\mathbf{1}_{\{t_1<t_2\}}, \quad t_1, t_2 \in [t, T].$$

Let us expand the function $K_1^*(t_1, t_2)$ using the variable t_2, when t_1 is fixed, into the Fourier–Legendre series at the interval (t, T)

$$K_1^*(t_1, t_2) = \sum_{j_3=0}^{\infty} \psi_2(t_1) \int\limits_{t_1}^T \psi_3(t_2)\phi_{j_3}(t_2)dt_2 \cdot \phi_{j_3}(t_2) \quad (t_2 \neq t, T). \tag{78}$$

The equality (78) is fulfilled pointwise in each point of the interval (t, T) with respect to the variable t_2, when $t_1 \in [t, T]$ is fixed, due to piecewise smoothness of the function $K_1^*(t_1, t_2)$ with respect to the variable $t_2 \in [t, T]$ (t_1 is fixed).

Obtaining (78) we also used the fact that the right-hand side of (78) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K_1(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K_1(t_1, t_1 - 0) + K_1(t_1, t_1 + 0)\right) = \frac{1}{2}\psi_2(t_1)\psi_3(t_1) = K_1^*(t_1, t_1).$$

Let us substitute $t_1 = t_2$ into (78). Then we have (76).

From (75) and (76) we obtain

$$E'_p = \sum_{j_1=0}^{p} \left(\int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2) \int\limits_{s_2}^T \sum_{j_3=p+1}^{\infty} \psi_2(s_1)\phi_{j_3}(s_1) \int\limits_{s_1}^T \psi_3(s)\phi_{j_3}(s)ds ds_1 ds_2 \right)^2 . \tag{79}$$

Analogously to (30) we obtain for the twice continuously differentiable function $\psi_3(s)$ the following estimate

$$\left|\sum_{j_3=p+1}^{\infty}\phi_{j_3}(s_1)\int\limits_{s_1}^{T}\psi_3(s)\phi_{j_3}(s)ds\right|<$$

$$<\frac{C}{p}\left(\frac{1}{(1-(z(s_1))^2)^{1/2}}+\frac{1}{(1-(z(s_1))^2)^{1/4}}\right), \tag{80}$$

where constant C does not depend on p, $s_1\in(t,T)$, and $z(s_1)$ is defined by (26). Further consideration is similar to the proof of (66). The relation (67) is proved.

Let us prove (68). We have

$$E_p''\stackrel{\sf def}{=}{\sf M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\}=\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_3j_1}\right)^2, \tag{81}$$

$$C_{j_1j_3j_1}=\int\limits_t^T\psi_3(s)\phi_{j_1}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds=$$

$$=\int\limits_t^T\psi_2(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2\int\limits_{s_1}^T\psi_3(s)\phi_{j_1}(s)dsds_1. \tag{82}$$

Let us substitute (82) into (81)

$$E_p''=\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_2(s_1)\phi_{j_3}(s_1)\sum_{j_1=0}^{p}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T\psi_3(s)\phi_{j_1}(s)dsds_1\right)^2. \tag{83}$$

The generalized Parseval equality gives

$$\sum_{j_1=0}^{\infty}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T\psi_3(s)\phi_{j_1}(s)ds=$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\mathbf{1}_{\{\theta<s_1\}}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_t^T\mathbf{1}_{\{s>s_1\}}\psi_3(s)\phi_{j_1}(s)ds=$$

$$=\int\limits_t^T\mathbf{1}_{\{\tau<s_1\}}\psi_1(\tau)\mathbf{1}_{\{\tau>s_1\}}\psi_3(\tau)d\tau=0. \tag{84}$$

Using (83) and (84), we get

$$E_p'' = \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_2(s_1)\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^T\psi_3(s)\phi_{j_1}(s)dsds_1\right)^2. \tag{85}$$

We have

$$\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds = \frac{\sqrt{T-t}\sqrt{2j_1+1}}{2}\int\limits_{-1}^{z(x)}P_{j_1}(y)\psi(u(y))dy =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\Biggl((P_{j_1+1}(z(x)) - P_{j_1-1}(z(x)))\psi_1(x) -$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(x)}((P_{j_1+1}(y) - P_{j_1-1}(y))\psi_1{}'(u(y))dy\Biggr), \tag{86}$$

where $x \in (t,T)$, $j_1 \ge p+1$, $z(x)$ and $u(y)$ are defined by (26), $\psi_1{}'$ is a derivative of the function $\psi_1(s)$ with respect to the variable $u(y)$.

Note that in (86) we used the following well-known property of Legendre polynomials

$$P_{j+1}(-1) = -P_j(-1), \quad j = 0, 1, 2, \ldots$$

and (27).

From (29) and (86) we get

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds\right| < \frac{C}{j_1}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + C_1\right), \quad x \in (t,T), \tag{87}$$

where constants C, C_1 do not depend on j_1.

Similarly to (87) and due to

$$P_j(1) = 1, \quad j = 0, 1, 2, \ldots$$

we obtain for the integral (like the integral, which is on the left-hand side of (87), but with integration limits x and T) the estimate (87).

From the formula (87) and its analogue for the integral with integration limits x and T we have

$$\left|\int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds\int\limits_x^T\psi_3(s)\phi_{j_1}(s)ds\right| < \frac{K}{j_1^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + K_1\right), \tag{88}$$

where $x \in (t,T)$ and constants K, K_1 do not depend on j_1.

The estimate (29) can be rewritten for the function $\phi_j(s)$ in the following form

$$|\phi_j(s)| < \sqrt{\frac{2j+1}{j+1}}\frac{K}{\sqrt{T-t}}\frac{1}{(1-z^2(s))^{1/4}} < \frac{K_1}{\sqrt{T-t}}\frac{1}{(1-z^2(s))^{1/4}}, \tag{89}$$

where $K_1 = K\sqrt{2}$, $s \in (t, T)$, $j \in \mathbb{N}$.

Let us estimate the right-hand side of (85) using (88)

$$E_p'' \le L \sum_{j_3=0}^{p} \left(\int_t^T |\phi_{j_3}(s_1)| \sum_{j_1=p+1}^{\infty} \left| \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \int_{s_1}^{T} \psi_3(s)\phi_{j_1}(s)ds \right| ds_1 \right)^2 <$$

$$< L_1 \sum_{j_3=0}^{p} \left(\int_t^T |\phi_{j_3}(s_1)| \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \left(\frac{1}{(1-(z(s_1))^2)^{1/2}} + K_1 \right) ds_1 \right)^2 <$$

$$< \frac{L_2}{p^2} \sum_{j_3=0}^{p} \left(\int_t^T \frac{ds_1}{(1-(z(s_1))^2)^{3/4}} + K_1 \int_t^T \frac{ds_1}{(1-(z(s_1))^2)^{1/4}} \right)^2 =$$

$$= \frac{L_2(T-t)^2}{4p^2} \sum_{j_3=0}^{p} \left(\int_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} + K_1 \int_{-1}^{1} \frac{dy}{(1-y^2)^{1/4}} \right)^2 \le$$

$$\le \frac{L_3 p}{p^2} = \frac{L_3}{p} \to 0 \qquad (90)$$

if $p \to \infty$, where constants L, L_1, L_2, L_3 do not depend on p and we used (31), (89) in (90). The relation (68) is proved. Theorem 3 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. Analogously to (41) we obtain

$$\left| \int_{s_2}^{T} \sum_{j_3=p+1}^{\infty} \psi_2(s_1)\phi_{j_3}(s_1) \int_{s_1}^{T} \psi_3(s)\phi_{j_3}(s)ds ds_1 \right| \le \frac{K_1}{p}, \qquad (91)$$

where $s_2 \in (t, T)$ and constant K_1 does not depend on p.

Using (41) for $T = s$ and (71), we obtain

$$E_p \le K \sum_{j_3=0}^{p} \left(\int_t^T \left| \int_t^s \sum_{j_1=p+1}^{\infty} \psi_2(s_1)\phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2 ds_1 \right| ds \right)^2 \le$$

$$\le K \sum_{j_3=0}^{p} \left((T-t)\frac{K_1}{p} \right)^2 \le \frac{K_2}{p^2} \sum_{j_3=0}^{p} (T-t)^2 \le \frac{L}{p} \to 0 \qquad (92)$$

if $p \to \infty$, where constants K, K_1, K_2, L do not depend on p.

Analogously, using (91) and (79), we obtain that $E_p' \to 0$ if $p \to \infty$.

Integrating by parts, we have

$$\int_t^s \phi_{2r-1}(\theta)\psi(\theta)d\theta = \frac{\sqrt{2}}{\sqrt{T-t}} \int_t^s \psi(\theta) \sin\frac{2\pi r(\theta - t)}{T-t} d\theta =$$

$$
= \sqrt{\frac{T-t}{2}} \frac{1}{\pi r} \left(-\psi(s) \cos\frac{2\pi r(s-t)}{T-t} + \psi(t) + \right.
$$

$$
\left. + \int\limits_t^s \psi'(\theta) \cos\frac{2\pi r(\theta-t)}{T-t} d\theta \right),
$$

$$
\int\limits_t^s \phi_{2r}(\theta)\psi(\theta)d\theta = \frac{\sqrt{2}}{\sqrt{T-t}} \int\limits_t^s \psi(\theta) \cos\frac{2\pi r(\theta-t)}{T-t} d\theta =
$$

$$
= \sqrt{\frac{T-t}{2}} \frac{1}{\pi r} \left(\psi(s) \sin\frac{2\pi r(s-t)}{T-t} - \int\limits_t^s \psi'(\theta) \sin\frac{2\pi r(\theta-t)}{T-t} d\theta \right),
$$

where $\phi_{2r-1}(\theta)$, $\phi_{2r}(\theta)$ are defined by (34) $(r = 1, 2, \ldots)$ and $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ (we suppose that $\psi(\theta)$ is a continuously differentiable nonrandom function on $[t, T]$).

Then

$$
\left| \int\limits_t^s \phi_{2r-1}(\theta)\psi(\theta)d\theta \right| \le \frac{C}{r} = \frac{2C}{2r} < \frac{2C}{2r-1}, \tag{93}
$$

$$
\left| \int\limits_t^s \phi_{2r}(\theta)\psi(\theta)d\theta \right| \le \frac{C}{r} = \frac{2C}{2r}, \tag{94}
$$

where constant C does not depend on r $(r = 1, 2, \ldots)$.

From (93), (94) we get

$$
\left| \int\limits_t^s \phi_{j_1}(\theta)\psi(\theta)d\theta \right| \le \frac{K}{j_1}, \tag{95}
$$

where constant K is independent of j_1 $(j_1 = 1, 2, \ldots)$.

Analogously, we obtain

$$
\left| \int\limits_s^T \phi_{j_1}(\theta)\psi(\theta)d\theta \right| \le \frac{K}{j_1}, \tag{96}
$$

where constant K does not depend on j_1 $(j_1 = 1, 2, \ldots)$.

From (95) and (96) we have

$$
\left| \int\limits_t^x \psi_1(s)\phi_{j_1}(s)ds \int\limits_x^T \psi_3(s)\phi_{j_1}(s)ds \right| < \frac{C_1}{j_1^2} \quad (j_1 \ne 0), \tag{97}
$$

where constant C_1 does not depend on j_1.

Using (85) and (97), we obtain

$$E_p'' \le L\sum_{j_3=0}^{p}\left(\int\limits_t^T |\phi_{j_3}(s_1)|\sum_{j_1=p+1}^{\infty}\left|\int\limits_t^{s_1}\psi_1(\theta)\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^{T}\psi_3(s)\phi_{j_1}(s)ds\right|ds_1\right)^2 \le$$

$$\le L_1\sum_{j_3=0}^{p}\left((T-t)\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\right)^2 \le \frac{L_1}{p^2}\sum_{j_3=0}^{p}(T-t)^2 \le$$

$$\le \frac{L_2}{p} \to 0 \qquad (98)$$

if $p\to\infty$, where constants L, L_1, L_2 do not depend on p. Theorem 3 is proved for the trigonometric case. Theorem 3 is proved.

5. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4

In this section, we will develop the approach to expansion of iterated Stratonovich stochatic integrals based on Theorem 1 for the stochastic integrals of multiplicity 4.

Theorem 4 [17]-[19], [22], [24], [26]-[29]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then*

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \qquad (99)$$

where $C_{j_4j_3j_2j_1}$ is defined by (5) for $k=4$ and $\psi_1(s),\ldots,\psi_4(s)\equiv 1$; another notations are the same as in Theorem 1.

Proof. From (12) it follows that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{T,t}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}A_1^{(i_3i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}A_2^{(i_2i_4)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}A_3^{(i_2i_3)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}A_4^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}A_5^{(i_1i_3)}+\mathbf{1}_{\{i_3=i_4\ne 0\}}A_6^{(i_1i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}B_1-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}B_2-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}B_3, \qquad (100)$$

where $J[\psi^{(4)}]_{T,t}$ is defined by (2) for $\psi_1(s),\ldots,\psi_4(s)\equiv 1$ and $i_1,\ldots,i_4=0,1,\ldots,m$,

$$A_1^{(i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$A_2^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_3} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$A_3^{(i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_4} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$A_4^{(i_1 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)},$$

$$A_5^{(i_1 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_4 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)},$$

$$A_6^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2,j_1=0}^{p} C_{j_3 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$B_1 = \lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} C_{j_4 j_4 j_1 j_1}, \quad B_2 = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_3 j_4 j_3 j_4},$$

$$B_3 = \lim_{p\to\infty} \sum_{j_4,j_3=0}^{p} C_{j_4 j_3 j_3 j_4}.$$

Using the integration order replacement in Riemann integrals, Theorem 1 for $k = 2$ (see (10)) and (17), Parseval's equality and integration order replacement technique for Ito stochastic integrals [17] (also see [26]-[29], Chapter 3) or Ito's formula, we obtain

$$A_1^{(i_3 i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \sum_{j_1=0}^{p} \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \left((s_1 - t) - \sum_{j_1=p+1}^{\infty} \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 \right) ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3=0}^{p} \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1)(s_1 - t) ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \Delta_1^{(i_3 i_4)} =$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^s (s_1-t)d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_3}(s_1)(s_1-t)ds_1ds-\Delta_1^{(i_3i_4)}=$$

(101) $$=\frac{1}{2}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T(s_1-t)ds_1-\Delta_1^{(i_3i_4)}\quad \text{w. p. 1},$$

where

$$\Delta_1^{(i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^p\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

(102) $$a_{j_4j_3}^p=\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2ds_1ds.$$

Let us consider $A_2^{(i_2i_4)}$

$$A_2^{(i_2i_4)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_3}(s_3)ds_3\int\limits_{s_2}^s\phi_{j_3}(s_1)ds_1ds_2ds\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\left(\int\limits_t^s\phi_{j_3}(s_3)ds_3\right)^2\int\limits_t^s\phi_{j_2}(s_2)ds_2ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_2)\left(\int\limits_t^{s_2}\phi_{j_3}(s_3)ds_3\right)^2ds_2ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_2)\left(\int\limits_{s_2}^s\phi_{j_3}(s_1)ds_1\right)^2ds_2ds\right)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)(s-t)\int\limits_t^s\phi_{j_2}(s_2)ds_2ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_2}(s_2)(s_2-t)ds_2ds-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_2)(s-t+t-s_2)ds_2ds\Biggr)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\Delta_2^{(i_2i_4)}+\Delta_1^{(i_2i_4)}+\Delta_3^{(i_2i_4)}=-\Delta_2^{(i_2i_4)}+\Delta_1^{(i_2i_4)}+\Delta_3^{(i_2i_4)}\quad \text{w. p. 1}, \tag{103}$$

where

$$\Delta_2^{(i_2i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p} b_{j_4j_2}^{p}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p} c_{j_4j_2}^{p}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$b_{j_4j_2}^{p}=\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\sum_{j_3=p+1}^{\infty}\left(\int\limits_t^s \phi_{j_3}(s_1)ds_1\right)^2\int\limits_t^s \phi_{j_2}(s_1)ds_1ds, \tag{104}$$

$$c_{j_4j_2}^{p}=\frac{1}{2}\int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_3)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_3}^s \phi_{j_3}(s_1)ds_1\right)^2 ds_3ds. \tag{105}$$

Let us consider $A_5^{(i_1i_3)}$

$$A_5^{(i_1i_3)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_4}(s_2)\int\limits_{s_2}^T \phi_{j_3}(s_1)\int\limits_{s_1}^T \phi_{j_4}(s)dsds_1ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_1)\int\limits_{s_1}^T \phi_{j_4}(s)ds\int\limits_{s_3}^{s_1} \phi_{j_4}(s_2)ds_2ds_1ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}\Biggl(\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\left(\int\limits_{s_3}^T \phi_{j_4}(s)ds\right)^2\int\limits_{s_3}^T \phi_{j_3}(s_1)ds_1ds_3-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_1)\left(\int\limits_{s_3}^{s_1} \phi_{j_4}(s_2)ds_2\right)^2 ds_1ds_3-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(s_3)\int\limits_{s_3}^T \phi_{j_3}(s_1)\left(\int\limits_{s_1}^T \phi_{j_4}(s)ds\right)^2 ds_1ds_3\Biggr)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}\Biggl(\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)(T-s_3)\int\limits_{s_3}^T\phi_{j_3}(s_1)ds_1ds_3-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_3}(s_1)(s_1-s_3)ds_1ds_3-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_3}(s_1)(T-s_1)ds_1ds_3\Biggr)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}=-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}\quad \text{w. p. 1,} \tag{106}
$$

where

$$
\Delta_4^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}d_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_5^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}e_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
\Delta_6^{(i_1i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}f_{j_3j_1}^p\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},
$$

$$
d_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^T\phi_{j_4}(s)ds\right)^2\int\limits_{s_3}^T\phi_{j_3}(s)dsds_3, \tag{107}
$$

$$
e_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_3}(s)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^s\phi_{j_4}(s_1)ds_1\right)^2dsds_3, \tag{108}
$$

$$
f_{j_3j_1}^p=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_4}(s_1)ds_1\right)^2ds_2ds_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_4}(s_1)ds_1\right)^2\int\limits_t^{s_2}\phi_{j_1}(s_3)ds_3ds_2. \tag{109}
$$

Moreover,

$$
A_3^{(i_2i_3)}+A_5^{(i_2i_3)}=
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \left(C_{j_4j_3j_2j_4}+C_{j_4j_3j_4j_2}\right)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 ds_1 ds \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 \int\limits_{s_1}^{T} \phi_{j_4}(s) ds ds_1 \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} \left(\int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \int\limits_t^{T} \phi_{j_4}(s_3) ds_3 \int\limits_{s_1}^{T} \phi_{j_4}(s) ds ds_2 ds_1 - \right.
$$

$$
\left. - \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left(\int\limits_{s_1}^{T} \phi_{j_4}(s) ds \right)^2 ds_2 ds_1 \right) \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_2=0}^{p} \int\limits_t^T \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_2}(s_2) \left((T-s_1) - \sum_{j_4=0}^{p} \left(\int\limits_{s_1}^{T} \phi_{j_4}(s_3) ds_3 \right)^2 \right) ds_2 ds_1 \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} =
$$

$$
= 2\Delta_6^{(i_2i_3)} \quad \text{w. p. 1.} \tag{110}
$$

Then

$$
A_3^{(i_2i_3)} = 2\Delta_6^{(i_2i_3)} - A_5^{(i_2i_3)} = \Delta_4^{(i_2i_3)} - \Delta_5^{(i_2i_3)} + \Delta_6^{(i_2i_3)} \quad \text{w. p. 1.} \tag{111}
$$

Let us consider $A_4^{(i_1i_4)}$

$$
A_4^{(i_1i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_1=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3) \int\limits_{s_3}^{s} \phi_{j_3}(s_2) \int\limits_{s_2}^{s} \phi_{j_3}(s_1) ds_1 ds_2 ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3) \sum_{j_3=0}^{p} \left(\int\limits_{s_3}^{s} \phi_{j_3}(s_2) ds_2 \right)^2 ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} =
$$

$$
= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_1=0}^{p} \frac{1}{2} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_1}(s_3)(s-s_3) ds_3 ds \zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \Delta_3^{(i_1i_4)} =
$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^s(s-s_3)d\mathbf{w}_{s_3}^{(i_1)}d\mathbf{w}_s^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\ne 0\}}\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)(s-s_3)ds_3ds-\Delta_3^{(i_1i_4)}=$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\sum_{j_4=0}^{\infty}\int\limits_t^T(s-t)\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_3)ds_3ds-\right.$$

$$\left.-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s(s_3-t)\phi_{j_4}(s_3)ds_3ds\right)-\Delta_3^{(i_1i_4)}=$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-\Delta_3^{(i_1i_4)}\quad \text{w. p. 1.}\tag{112}$$

Let us consider $A_6^{(i_1i_2)}$

$$A_6^{(i_1i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2,j_1=0}^{p}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\int\limits_{s_2}^T\phi_{j_3}(s_1)\int\limits_{s_1}^T\phi_{j_3}(s)dsds_1ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)\sum_{j_3=0}^{p}\left(\int\limits_{s_2}^T\phi_{j_3}(s)ds\right)^2ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_3)\int\limits_{s_3}^T\phi_{j_2}(s_2)(T-s_2)ds_2ds_3\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_1}(s_3)ds_3ds_2\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\Delta_6^{(i_1i_2)}=$$

$$=\frac{1}{2}\int\limits_t^T(T-s_2)\int\limits_t^{s_2}d\mathbf{w}_{s_3}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_2=0}^{\infty}\int\limits_t^T\phi_{j_2}(s_2)(T-s_2)\int\limits_t^{s_2}\phi_{j_2}(s_3)ds_3ds_2-\Delta_6^{(i_1i_2)}=$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T(T-s_2)ds_2-\Delta_6^{(i_1i_2)}\quad \text{w. p. 1.} \tag{113}$$

Let us consider B_1, B_2, B_3

$$B_1=\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2ds_1ds=$$

$$=\lim_{p\to\infty}\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^{s}\phi_{j_4}(s_1)(s_1-t)ds_1ds-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^p=$$

$$=\frac{1}{4}\int\limits_t^T(s_1-t)ds_1-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^p, \tag{114}$$

$$B_2=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\int\limits_t^{s_2}\phi_{j_4}(s_3)ds_3\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1ds_2ds=$$

$$=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\left(\int\limits_t^{s}\phi_{j_4}(s_3)ds_3\right)^2\int\limits_t^{s}\phi_{j_3}(s_2)ds_2ds-\right.$$

$$-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\left(\int\limits_t^{s_2}\phi_{j_4}(s_3)ds_3\right)^2ds_2ds-$$

$$\left.-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^{s}\phi_{j_3}(s_2)\left(\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1\right)^2ds_2ds\right)=$$

$$=\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(s-t)\int\limits_t^{s}\phi_{j_3}(s_2)ds_2ds-\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p-$$

$$-\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^{s}(s_2-t)\phi_{j_3}(s_2)ds_2ds+\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p-$$

$$
-\sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_3}(s_2)(s-t+t-s_2)ds_2ds+\lim_{p\to\infty}\sum_{j_3=0}^{p}c^p_{j_3j_3}=
$$

$$
=\lim_{p\to\infty}\sum_{j_3=0}^{p}a^p_{j_3j_3}+\lim_{p\to\infty}\sum_{j_3=0}^{p}c^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p}b^p_{j_3j_3}. \qquad (115)
$$

Moreover,

$$
B_2+B_3=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(C_{j_3j_4j_3j_4}+C_{j_3j_4j_4j_3}\right)=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2ds_1ds=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3ds_2\int\limits_{s_1}^T\phi_{j_3}(s)dsds_1=
$$

$$
=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}\left(\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\int\limits_t^T\phi_{j_3}(s_2)ds_2\int\limits_{s_1}^T\phi_{j_3}(s)dsds_3ds_1-\right.
$$

$$
\left.-\int\limits_t^T\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)\left(\int\limits_{s_1}^T\phi_{j_3}(s)ds\right)^2ds_3ds_1\right)=
$$

$$
=\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1-
$$

$$
-\sum_{j_4=0}^{\infty}\int\limits_t^T\phi_{j_4}(s_1)(T-s_1)\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3ds_1+2\lim_{p\to\infty}\sum_{j_4=0}^{p}f^p_{j_4j_4}=2\lim_{p\to\infty}\sum_{j_4=0}^{p}f^p_{j_4j_4}. \qquad (116)
$$

Therefore,

$$
B_3=2\lim_{p\to\infty}\sum_{j_3=0}^{p}f^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p}a^p_{j_3j_3}-\lim_{p\to\infty}\sum_{j_3=0}^{p}c^p_{j_3j_3}+\lim_{p\to\infty}\sum_{j_3=0}^{p}b^p_{j_3j_3}. \qquad (117)
$$

After substituting the relations (101)–(117) into (100), we obtain

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}=J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1}ds_2d\mathbf{w}^{(i_3)}_{s_1}d\mathbf{w}^{(i_4)}_s+
$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1+R=J^*[\psi^{(4)}]_{T,t}+R\quad \text{w. p. 1}, \tag{118}$$

where

$$R=-\mathbf{1}_{\{i_1=i_2\neq 0\}}\Delta_1^{(i_3i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\left(-\Delta_2^{(i_2i_4)}+\Delta_1^{(i_2i_4)}+\Delta_3^{(i_2i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\left(\Delta_4^{(i_2i_3)}-\Delta_5^{(i_2i_3)}+\Delta_6^{(i_2i_3)}\right)-\mathbf{1}_{\{i_2=i_3\neq 0\}}\Delta_3^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_2=i_4\neq 0\}}\left(-\Delta_4^{(i_1i_3)}+\Delta_5^{(i_1i_3)}+\Delta_6^{(i_1i_3)}\right)-\mathbf{1}_{\{i_3=i_4\neq 0\}}\Delta_6^{(i_1i_2)}-$$

$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\left(\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p+\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p-\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p\right)-$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\left(2\lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^p-\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p-\right.$$

$$\left.-\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p+\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p\right)+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p. \tag{119}$$

From (118) and (119) it follows that Theorem 4 will be proved if

(120)

$$\Delta_k^{(ij)}=0\ \text{w. p. 1},\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^p=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^p=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^p=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^p=0,$$

where $k=1,2,\ldots,6,\quad i,j=0,1,\ldots,m$.

Let us consider the case of Legendre polynomials. Let us prove that $\Delta_1^{(i_3i_4)}=0$ w. p. 1. We have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^p\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(2a_{j_3j_3}^pa_{j_3'j_3'}^p+\left(a_{j_3j_3'}^p\right)^2+2a_{j_3j_3'}^pa_{j_3'j_3}^p+\left(a_{j_3'j_3}^p\right)^2\right)+3\sum_{j_3'=0}^{p}\left(a_{j_3'j_3'}^p\right)^2=$$

$$=\left(\sum_{j_3=0}^{p}a_{j_3j_3}^p\right)^2+\sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(a_{j_3j_3'}^p+a_{j_3'j_3}^p\right)^2+2\sum_{j_3'=0}^{p}\left(a_{j_3'j_3'}^p\right)^2\quad (i_3=i_4\neq 0), \tag{121}$$

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\sum_{j_3,j_4=0}^{p}\left(a_{j_4j_3}^{p}\right)^2\quad (i_3\ne i_4,\ i_3\ne 0,\ i_4\ne 0), \tag{122}
$$

$$
\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=\begin{cases}(T-t)\sum\limits_{j_4=0}^{p}\left(a_{j_4,0}^{p}\right)^2 & \text{if}\quad i_3=0,\ i_4\ne 0\\ (T-t)\sum\limits_{j_3=0}^{p}\left(a_{0,j_3}^{p}\right)^2 & \text{if}\quad i_4=0,\ i_3\ne 0\\ (T-t)^2\left(a_{00}^{p}\right)^2 & \text{if}\quad i_3=i_4=0\end{cases}. \tag{123}
$$

Consider the case $i_3=i_4\ne 0$

$$
a_{j_4j_3}^{p}=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times
$$

$$
\times\int\limits_{-1}^{1}P_{j_4}(y)\int\limits_{-1}^{y}P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}(2j_1+1)\left(\int\limits_{-1}^{y_1}P_{j_1}(y_2)dy_2\right)^2dy_1dy=
$$

$$
=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times
$$

$$
\times\int\limits_{-1}^{1}P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2\int\limits_{y_1}^{1}P_{j_4}(y)dydy_1=
$$

$$
=\frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times
$$

$$
\times\int\limits_{-1}^{1}P_{j_3}(y_1)\left(P_{j_4-1}(y_1)-P_{j_4+1}(y_1)\right)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2dy_1
$$

if $j_4\ne 0$ and

$$
a_{j_4j_3}^{p}=\frac{(T-t)^2\sqrt{2j_3+1}}{32}\int\limits_{-1}^{1}P_{j_3}(y_1)(1-y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2dy_1
$$

if $j_4=0$.

From (29) and the estimate $|\ P_j(y)\ |\le 1,\ y\in[-1,1]$ we obtain

$$|P_j(y)| = \sqrt{|P_j(y)|}\cdot\sqrt{|P_j(y)|} \le \frac{C}{j^{1/4}(1-y^2)^{1/8}},\quad y\in(-1,1),\quad j\in\mathbb{N}. \tag{124}$$

Using (29) and (124), we get

$$|a^p_{j_4j_3}| \le \frac{C_0}{(j_4)^{3/4}}\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{7/8}} \le \frac{C_1}{p\,(j_4)^{3/4}}\quad (j_3\ne 0, j_4\ge 2), \tag{125}$$

$$|a^p_{0j_3}| + |a^p_{1j_3}| \le C_0\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}} \le \frac{C_1}{p}\quad (j_3\ne 0), \tag{126}$$

$$|a^p_{j_40}| + |a^p_{00}| \le C_0\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/2}} \le \frac{C_1}{p}\quad (j_4\ge 1), \tag{127}$$

where constants C_0, C_1 do not depend on p.

Taking into account (121), (125)–(127), we have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a^p_{j_4j_3}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\right)^2\right\} = \left(a^p_{00} + \sum_{j_3=1}^{p} a^p_{j_3j_3}\right)^2 + \sum_{j_3'=1}^{p}\left(a^p_{0j_3'} + a^p_{j_3'0}\right)^2 +$$

$$+\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\left(a^p_{j_3j_3'} + a^p_{j_3'j_3}\right)^2 + 2\left(\sum_{j_3'=1}^{p}\left(a^p_{j_3'j_3'}\right)^2 + (a_{00})^2\right) \le$$

$$\le K_0\left(\frac{1}{p} + \frac{1}{p}\sum_{j_3=1}^{p}\frac{1}{(j_3)^{3/4}}\right)^2 + \frac{K_1}{p} + K_2\sum_{j_3'=1}^{p}\sum_{j_3=1}^{j_3'-1}\frac{1}{p^2}\left(\frac{1}{(j_3')^{3/4}} + \frac{1}{(j_3)^{3/4}}\right)^2 \le$$

$$\le K_0\left(\frac{1}{p} + \frac{1}{p}\int\limits_{0}^{p}\frac{dx}{x^{3/4}}\right)^2 + \frac{K_1}{p} + \frac{K_3}{p}\sum_{j_3=1}^{p}\frac{1}{(j_3)^{3/2}} \le$$

$$\le K_0\left(\frac{1}{p} + \frac{4}{p^{3/4}}\right)^2 + \frac{K_1}{p} + \frac{K_3}{p}\left(1 + \int\limits_{1}^{p}\frac{dx}{x^{3/2}}\right) \le$$

$$\le \frac{K_4}{p} + \frac{K_3}{p}\left(3 - \frac{2}{\sqrt{p}}\right) \le \frac{K_5}{p}\ \to 0$$

if $p \to \infty$ ($i_3 = i_4 \ne 0$).

The same result for the cases (122), (123) also follows from the estimates (125)–(127). Therefore,

$$\Delta_1^{(i_3 i_4)} = 0 \quad \text{w. p. 1.} \tag{128}$$

It is not difficult to see that the formulas

$$\Delta_2^{(i_2 i_4)} = 0, \quad \Delta_4^{(i_1 i_3)} = 0, \quad \Delta_6^{(i_1 i_3)} = 0 \quad \text{w. p. 1} \tag{129}$$

can be proved similarly to the proof of the relation (128).

Moreover, from the estimates (125)–(127) we obtain

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} a_{j_3 j_3}^{p} = 0. \tag{130}$$

The relations

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} b_{j_3 j_3}^{p} = 0 \quad \text{and} \quad \lim_{p\to\infty} \sum_{j_3=0}^{p} f_{j_3 j_3}^{p} = 0 \tag{131}$$

can also be proved analogously to (130).

Let us consider $\Delta_3^{(i_2 i_4)}$

$$\Delta_3^{(i_2 i_4)} = \Delta_4^{(i_2 i_4)} + \Delta_6^{(i_2 i_4)} - \Delta_7^{(i_2 i_4)} = -\Delta_7^{(i_2 i_4)} \quad \text{w. p. 1,} \tag{132}$$

where

$$\Delta_7^{(i_2 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_2, j_4=0}^{p} g_{j_4 j_2}^{p} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},$$

$$g_{j_4 j_2}^{p} = \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_1) \sum_{j_1=p+1}^{\infty} \left(\int_{s_1}^T \phi_{j_1}(s_2) ds_2 \int_s^T \phi_{j_1}(s_2) ds_2 \right) ds_1 ds =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 \int_t^s \phi_{j_2}(s_1) \int_{s_1}^T \phi_{j_1}(s_2) ds_2 ds_1 ds. \tag{133}$$

The last step in (133) follows from the estimate

$$\left| g_{j_4 j_2}^{p} \right| \le K \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int_{-1}^{1} \frac{1}{(1-y^2)^{1/2}} \int_{-1}^{y} \frac{1}{(1-x^2)^{1/2}} dx dy \ \le \frac{K_1}{p}.$$

Note that

$$g_{j_4j_4}^p=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\right)^2, \tag{134}$$

$$g_{j_4j_2}^p+g_{j_2j_4}^p=\sum_{j_1=p+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds, \tag{135}$$

and

$$g_{j_4j_2}^p=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_2+1)}}{16}\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)\times$$

$$\times\int\limits_{-1}^{y_1}P_{j_2}(y)\left(P_{j_1-1}(y)-P_{j_1+1}(y)\right)dydy_1,\quad j_4,j_2\le p.$$

Due to the orthogonality of the Legendre polynomials we obtain

$$g_{j_4j_2}^p+g_{j_2j_4}^p=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_2+1)}}{16}\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)dy_1\times$$

$$\times\int\limits_{-1}^{1}P_{j_2}(y)\left(P_{j_1-1}(y)-P_{j_1+1}(y)\right)dy=$$

$$=\frac{(T-t)^2(2p+1)}{16}\frac{1}{2p+3}\left(\int\limits_{-1}^{1}P_p^2(y_1)dy_1\right)^2\begin{cases}1 & \text{if } j_2=j_4=p\\ 0 & \text{otherwise}\end{cases}=$$

$$=\frac{(T-t)^2}{4(2p+3)(2p+1)}\begin{cases}1 & \text{if } j_2=j_4=p\\ 0 & \text{otherwise}\end{cases}, \tag{136}$$

$$g_{j_4j_4}^p=\frac{(T-t)^2(2j_4+1)}{16}\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\cdot\frac{1}{2}\left(\int\limits_{-1}^{1}P_{j_4}(y_1)\left(P_{j_1-1}(y_1)-P_{j_1+1}(y_1)\right)dy_1\right)^2=$$

$$
= \frac{(T-t)^2(2p+1)}{32}\frac{1}{2p+3}\left(\int\limits_{-1}^{1}P_p^2(y_1)dy_1\right)^2 \begin{cases} 1 & \text{if } j_4 = p\\ \\ 0 & \text{otherwise}\end{cases} =
$$

(137)
$$
= \frac{(T-t)^2}{8(2p+3)(2p+1)} \begin{cases} 1 & \text{if } j_4 = p\\ \\ 0 & \text{otherwise}\end{cases}.
$$

From (121), (136), (5) it follows that

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p} g_{j_4j_2}^p \zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\right)^2\right\} = \left(\sum_{j_3=0}^{p} g_{j_3j_3}^p\right)^2 + \sum_{j_3'=0}^{p}\sum_{j_3=0}^{j_3'-1}\left(g_{j_3j_3'}^p + g_{j_3'j_3}^p\right)^2 + 2\sum_{j_3'=0}^{p}\left(g_{j_3'j_3'}^p\right)^2 =
$$

$$
= \left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2 + \ 0 \ + 2\left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2 \to 0
$$

if $p\to\infty \quad (i_2 = i_4 \neq 0)$.

Let us consider the case $i_2 \neq i_4,\ i_2 \neq 0,\ i_4 \neq 0$. It is not difficult to see that

$$
g_{j_4j_2}^p = \int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_2}(s_1)F_p(s,s_1)ds_1ds = \int\limits_{[t,T]^2} K_p(s,s_1)\phi_{j_4}(s)\phi_{j_2}(s_1)ds_1ds
$$

is a coefficient of the double Fourier–Legendre series of the function

(138)
$$
K_p(s,s_1) = \mathbf{1}_{\{s_1<s\}}F_p(s,s_1),
$$

where

$$
\sum_{j_1=p+1}^{\infty}\int\limits_{s_1}^{T}\phi_{j_1}(s_2)ds_2\int\limits_{s}^{T}\phi_{j_1}(s_2)ds_2 \stackrel{\mathsf{def}}{=} F_p(s,s_1).
$$

The Parseval equality in this case has the form

(139)
$$
\lim_{p_1\to\infty}\sum_{j_4,j_2=0}^{p_1}\left(g_{j_4j_2}^p\right)^2 = \int\limits_{[t,T]^2}\left(K_p(s,s_1)\right)^2 ds_1ds = \int\limits_t^T\int\limits_t^s\left(F_p(s,s_1)\right)^2 ds_1ds.
$$

From (29) we obtain

$$\left|\int\limits_{s_1}^{T}\phi_{j_1}(\theta)d\theta\right| = \frac{1}{2}\sqrt{2j_1+1}\sqrt{T-t}\left|\int\limits_{z(s_1)}^{1}P_{j_1}(y)dy\right| =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\left|P_{j_1-1}(z(s_1)) - P_{j_1+1}(z(s_1))\right| \le \frac{K}{j_1}\frac{1}{\left(1-z^2(s_1)\right)^{1/4}}, \tag{140}$$

where $z(s_1)$ is defined by (26) and $s_1 \in (t,T)$.

Using (140), we have

$$\left(F_p(s,s_1)\right)^2 \le \frac{C^2}{p^2}\frac{1}{\left(1-z^2(s)\right)^{1/2}}\frac{1}{\left(1-z^2(s_1)\right)^{1/2}}, \quad s,\ s_1 \in (t,T). \tag{141}$$

From (141) it follows that $|F_p(s,s_1)| \le M_\varepsilon/p$ in the domain

$$D_\varepsilon = \{(s,s_1):\ s\in[t+\varepsilon, T-\varepsilon],\ s_1\in[t+\varepsilon,s]\} \quad \forall\ \varepsilon>0,$$

where constant M_ε does not depend on s, s_1. Then we have the uniform convergence

$$\sum_{j_1=0}^{p}\int\limits_{s}^{T}\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^{T}\phi_{j_1}(\theta)d\theta \to \sum_{j_1=0}^{\infty}\int\limits_{s}^{T}\phi_{j_1}(\theta)d\theta\int\limits_{s_1}^{T}\phi_{j_1}(\theta)d\theta \tag{142}$$

at the set D_ε if $p\to\infty$.

Due to continuity of the function on the left-hand side of (142) we obtain continuity of the limit function on the right-hand side of (142) at the set D_ε.

Using this fact and (141), we obtain

$$\int\limits_{t}^{T}\int\limits_{t}^{s}\left(F_p(s,s_1)\right)^2 ds_1 ds = \lim_{\varepsilon\to+0}\int\limits_{t+\varepsilon}^{T-\varepsilon}\int\limits_{t+\varepsilon}^{s}\left(F_p(s,s_1)\right)^2 ds_1 ds \le$$

$$\le \frac{C^2}{p^2}\lim_{\varepsilon\to+0}\int\limits_{t+\varepsilon}^{T-\varepsilon}\int\limits_{t+\varepsilon}^{s}\frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}}\frac{ds}{\left(1-z^2(s)\right)^{1/2}} =$$

$$= \frac{C^2}{p^2}\int\limits_{t}^{T}\int\limits_{t}^{s}\frac{ds_1}{\left(1-z^2(s_1)\right)^{1/2}}\frac{ds}{\left(1-z^2(s)\right)^{1/2}} =$$

$$= \frac{K}{p^2}\int\limits_{-1}^{1}\int\limits_{-1}^{y}\frac{dy_1}{\left(1-y_1^2\right)^{1/2}}\frac{dy}{\left(1-y^2\right)^{1/2}} < \frac{K_1}{p^2}, \tag{143}$$

where constant K_1 does not depend on p.

From (143) and (139) we obtain

$$0 \le \sum_{j_2,j_4=0}^{p} \left(g^p_{j_4 j_2}\right)^2 \le \lim_{p_1\to\infty} \sum_{j_2,j_4=0}^{p_1} \left(g^p_{j_4 j_2}\right)^2 = \sum_{j_2,j_4=0}^{\infty} \left(g^p_{j_4 j_2}\right)^2 \le \frac{K_1}{p^2} \to 0 \tag{144}$$

if $p\to\infty$. The case $i_2 \ne i_4$, $i_2 \ne 0$, $i_4 \ne 0$ is proved.

The same result for the cases

1) $i_2 = 0$, $i_4 \ne 0$,
2) $i_4 = 0$, $i_2 \ne 0$,
3) $i_2 = 0$, $i_4 = 0$

can also be obtained. Then $\Delta_7^{(i_2 i_4)} = 0$ and $\Delta_3^{(i_2 i_4)} = 0$ w. p. 1.

Let us consider $\Delta_5^{(i_1 i_3)}$

$$\Delta_5^{(i_1 i_3)} = \Delta_4^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)} - \Delta_8^{(i_1 i_3)} \quad \text{w. p. 1},$$

where

$$\Delta_8^{(i_1 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_3,j_1=0}^{p} h^p_{j_3 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_3)}_{j_3},$$

$$h^p_{j_3 j_1} = \int\limits_t^T \phi_{j_1}(s_3) \int\limits_{s_3}^T \phi_{j_3}(s) F_p(s_3,s) ds ds_3.$$

Analogously, we obtain that $\Delta_8^{(i_1 i_3)} = 0$ w. p. 1. Here we consider the function

$$K_p(s,s_3) = \mathbf{1}_{\{s_3<s\}} F_p(s_3,s)$$

and the relation

$$h^p_{j_3 j_1} = \int\limits_{[t,T]^2} K_p(s,s_3)\phi_{j_1}(s_3)\phi_{j_3}(s) ds ds_3 \quad (i_1 \ne i_3,\ i_1 \ne 0,\ i_3 \ne 0)$$

for the case $i_1 \ne i_3$, $i_1 \ne 0$, $i_3 \ne 0$.

For the case $i_1 = i_3 \ne 0$ we use (see (134), (135))

$$h^p_{j_1 j_1} = \sum_{j_4=p+1}^{\infty} \frac{1}{2} \left(\int\limits_t^T \phi_{j_1}(s) \int\limits_s^T \phi_{j_4}(s_1) ds_1 ds \right)^2,$$

$$h^p_{j_3 j_1} + h^p_{j_1 j_3} = \sum_{j_4=p+1}^{\infty} \int\limits_t^T \phi_{j_1}(s) \int\limits_s^T \phi_{j_4}(s_2) ds_2 ds \int\limits_t^T \phi_{j_3}(s) \int\limits_s^T \phi_{j_4}(s_2) ds_2 ds.$$

Let us prove that

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3} = 0. \tag{145}$$

We have

$$c^p_{j_3j_3} = f^p_{j_3j_3} + d^p_{j_3j_3} - g^p_{j_3j_3}. \tag{146}$$

Moreover,

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3j_3} = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} d^p_{j_3j_3} = 0, \tag{147}$$

where the first equality in (147) has been proved earlier. Analogously, we can prove the second equality in (147).

From (5) we obtain

$$0 \le \lim_{p\to\infty}\sum_{j_3=0}^{p} g^p_{j_3j_3} \le \lim_{p\to\infty}\frac{(T-t)^2}{8(2p+3)(2p+1)} = 0.$$

So the equality (145) is proved. The relations (120) are proved for the polynomial case. Theorem 4 is proved for the case of Legendre polynomials.

Let us consider the trigonometric case. According to (102), we have

$$a^p_{j_4j_3} = \frac{1}{2}\int\limits_t^T \phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2\int\limits_{s_1}^T\phi_{j_4}(s)dsds_1. \tag{148}$$

Moreover (see (95), (96)),

$$\left|\int\limits_t^{s_1}\phi_j(s_2)ds_2\right| \le \frac{K}{j}, \quad \left|\int\limits_{s_1}^{T}\phi_j(s_2)ds_2\right| \le \frac{K}{j}, \tag{149}$$

where constant K does not depend on j $(j = 1, 2, \ldots)$.

Note that

$$\int\limits_{s_1}^T \phi_0(s)ds = \frac{T-s_1}{\sqrt{T-t}}.$$

Using (148) and (149), we obtain

$$\left|a^p_{j_4j_3}\right| \le \frac{C_1}{j_4}\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2} \le \frac{C_1}{pj_4} \quad (j_4 \ne 0), \qquad \left|a^p_{0j_3}\right| \le \frac{C_1}{p}, \tag{150}$$

where constant C_1 does not depend on p.

Taking into account (121)–(123) and (150), we obtain that $\Delta_1^{(i_3i_4)}=0$ w. p. 1. Analogously, we get $\Delta_2^{(i_2i_4)}=0$, $\Delta_4^{(i_1i_3)}=0$, $\Delta_6^{(i_1i_3)}=0$ w. p. 1 and

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^{p}=0,\quad \lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^{p}=0.$$

Let us consider $\Delta_3^{(i_2i_4)}$ for the case $i_2=i_4\ne 0$. For the values $g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}$ and $g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}$ $(m\in\mathbb{N})$ we have (see (135))

$$g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}=\sum_{j_1=2m+1}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds=$$

$$=\sum_{r=m+1}^{\infty}\Biggl(\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r-1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2r-1}(s_2)ds_2ds+$$

$$+\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2r}(s_2)ds_2ds\Biggr),\tag{151}$$

$$g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}=\sum_{j_1=2m}^{\infty}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{j_1}(s_2)ds_2ds=$$

$$=g_{j_4j_2}^{2m}+g_{j_2j_4}^{2m}+\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2m}(s_2)ds_2ds\int\limits_t^T\phi_{j_2}(s)\int\limits_s^T\phi_{2m}(s_2)ds_2ds,\tag{152}$$

where

$$\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r-1}(s_2)ds_2ds=\sqrt{\frac{2}{T-t}}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\sin\frac{2\pi r(s_2-t)}{T-t}ds_2ds=$$

$$=\frac{\sqrt{2}\sqrt{T-t}}{2\pi r}\int\limits_t^T\phi_{j_4}(s)\left(\cos\frac{2\pi r(s-t)}{T-t}-1\right)ds,$$

$$\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\phi_{2r}(s_2)ds_2ds=\sqrt{\frac{2}{T-t}}\int\limits_t^T\phi_{j_4}(s)\int\limits_s^T\cos\frac{2\pi r(s_2-t)}{T-t}ds_2ds=$$

$$=\frac{\sqrt{2}\sqrt{T-t}}{2\pi r}\int\limits_t^T\phi_{j_4}(s)\left(-\sin\frac{2\pi r(s-t)}{T-t}\right)ds,$$

where $2r-1,\ 2r \ge p+1$, and $j_2, j_4 = 0, 1, \ldots, p$.

Due to orthogonality of the trigonometric functions we have

$$\int\limits_t^T \phi_{j_4}(s) \int\limits_s^T \phi_{2r-1}(s_2) ds_2 ds = \frac{\sqrt{2}(T-t)}{2\pi r} \cdot \begin{cases} -1 & \text{if } \ j_4 = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{153}$$

$$\int\limits_t^T \phi_{j_4}(s) \int\limits_s^T \phi_{2r}(s_2) ds_2 ds = 0, \tag{154}$$

where $2r-1,\ 2r \ge p+1$, and $j_4 = 0, 1, \ldots, p$.

Using (151), (153), and (154), we obtain

$$g_{j_4 j_2}^{2m} + g_{j_2 j_4}^{2m} = \sum_{j_1=m+1}^{\infty} \frac{(T-t)^2}{2\pi^2 j_1^2} \cdot \begin{cases} 1 & \text{if } \ j_2 = j_4 = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$g_{j_4 j_4}^{2m} = \frac{1}{2}\left(g_{j_4 j_2}^{2m} + g_{j_2 j_4}^{2m}\right)\Biggr|_{j_2=j_4} = \sum_{j_1=m+1}^{\infty} \frac{(T-t)^2}{4\pi^2 j_1^2} \cdot \begin{cases} 1 & \text{if } \ j_4 = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore (see (31)),

$$\begin{cases} \left|g_{j_4 j_2}^{2m} + g_{j_2 j_4}^{2m}\right| \le K_1/(2m) & \text{if } \ j_2 = j_4 = 0 \\ g_{j_4 j_2}^{2m} + g_{j_2 j_4}^{2m} = 0 & \text{otherwise} \end{cases}, \tag{155}$$

$$\begin{cases} \left|g_{j_4 j_4}^{2m}\right| \le K_1/(2m) & \text{if } \ j_4 = 0 \\ g_{j_4 j_4}^{2m} = 0 & \text{otherwise} \end{cases}, \tag{156}$$

where constant K_1 does not depend on $p = 2m$.

For $p = 2m-1$ from (152) and (154) we have

$$g_{j_4 j_2}^{2m-1} + g_{j_2 j_4}^{2m-1} = \sum_{j_1=m+1}^{\infty} \frac{(T-t)^2}{2\pi^2 j_1^2} \cdot \begin{cases} 1 \ \text{ or } \ 0 & \text{if } j_2 = j_4 = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{157}$$

The relation (157) implies that

(158) $$g_{j_4j_4}^{2m-1}=\frac{1}{2}\left(g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}\right)\Biggl|_{j_2=j_4}=\sum_{j_1=m+1}^{\infty}\frac{(T-t)^2}{4\pi^2j_1^2}\cdot\begin{cases}1\ \text{ or }\ 0 & \text{if } j_4=0\\ \\ 0 & \text{otherwise}\end{cases}.$$

Using (157) and (158), we obtain

(159) $$\begin{cases}\left|g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}\right|\le\ K_2/(2m-1) & \text{if }\ j_2=j_4=0\\ \\ g_{j_4j_2}^{2m-1}+g_{j_2j_4}^{2m-1}=0 & \text{otherwise}\end{cases},$$

(160) $$\begin{cases}\left|g_{j_4j_4}^{2m-1}\right|\le\ K_2/(2m-1) & \text{if }\ j_4=0\\ \\ g_{j_4j_4}^{2m-1}=0 & \text{otherwise}\end{cases},$$

where constant K_2 does not depend on $p=2m-1$.

The relations (155), (156), (159), and (160) imply the following formulas

(161) $$\begin{cases}\left|g_{j_4j_2}^{p}+g_{j_2j_4}^{p}\right|\le\ K_3/p & \text{if }\ j_2=j_4=0\\ \\ g_{j_4j_2}^{p}+g_{j_2j_4}^{p}=0 & \text{otherwise}\end{cases},\qquad \begin{cases}\left|g_{j_4j_4}^{p}\right|\le\ K_3/p & \text{if }\ j_4=0\\ \\ g_{j_4j_4}^{p}=0 & \text{otherwise}\end{cases},$$

where constant K_3 does not depend on p ($p\in\mathbb{N}$). Moreover, $g_{j_4j_4}^{p}\ge 0$ (see (134)).

From (121) and (161) it follows that $\Delta_7^{(i_2i_4)}=0$ and $\Delta_3^{(i_2i_4)}=0$ w. p. 1 for $i_2=i_4\ne 0$. Analogously to the polynomial case, we obtain $\Delta_7^{(i_2i_4)}=0$ and $\Delta_3^{(i_2i_4)}=0$ w. p. 1 for $i_2\ne i_4$, $i_2\ne 0$, $i_4\ne 0$. The similar arguments prove that $\Delta_5^{(i_1i_3)}=0$ w. p. 1.

Taking into account (146), (161) and the relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}f_{j_3j_3}^{p}=\lim_{p\to\infty}\sum_{j_3=0}^{p}d_{j_3j_3}^{p}=0,$$

which follow from the estimates

(162) $$|f_{jj}^{p}|\le\frac{C_1}{pj},\quad |d_{jj}^{p}|\le\frac{C_1}{pj}\quad (j\ne 0),\quad |f_{00}^{p}|\le\frac{C_1}{p},\quad |d_{00}^{p}|\le\frac{C_1}{p},$$

we obtain

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^{p}=-\lim_{p\to\infty}\sum_{j_3=0}^{p}g_{j_3j_3}^{p},$$

$$0 \le \lim_{p\to\infty} \sum_{j_3=0}^{p} g_{j_3 j_3}^{p} \le \lim_{p\to\infty} \frac{K_3}{p} = 0.$$

Note that the estimates (162) can be obtained by analogy with (150); constant C_1 in (162) has the same meaning as constant C_1 in (150).

Finally, we have

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} c_{j_3 j_3}^{p} = 0.$$

The relations (120) are proved for the trigonometric case. Theorem 4 is proved for the trigonometric case. Theorem 4 is proved.

Remark 1. *It should be noted that the proof of Theorem 4 can be somewhat simplified. More precisely, instead of (121)–(123), we can use only one and rather simple estimate.*

We have

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \left(\zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_3=i_4\ne 0\}} \mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \left(\zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}} \mathbf{1}_{\{j_3=j_4\}}\right) + \mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_4=0}^{p} a_{j_4 j_4}^{p}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \left(\zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}} \mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} +$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}} \left(\sum_{j_4=0}^{p} a_{j_4 j_4}^{p}\right)^2. \tag{163}$$

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_k=0 \\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J'[\Phi]_{T,t}^{(i_1\ldots i_k)}, \tag{164}$$

where for simplicity we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k \to \mathbb{R}$ *is a continuous nonrandom function on* $[t,T]^k$. *Moreover,* $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$, *which satisfies the condition* (6), $i_1,\ldots,i_k = 0,1,\ldots,m$.

The stochastic integral with respect to the scalar standard Wiener process ($i_1 = \ldots = i_k \neq 0$) *and similar to* (164) *was considered in* [57] (1951) *and is called the multiple Wiener stochastic integral* [57].

The expression

$$\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)$$

can be interpreted as the multiple Wiener stochastic integral (164) *of multiplicity* 2 *with nonrandom integrand function*

$$\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\phi_{j_3}(t_3)\phi_{j_4}(t_4).$$

Note that the following estimate is true [57] (*also see* [26], *Sect.* 2.3)

$$\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \leq C_k \int\limits_{[t,T]^k} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k, \tag{165}$$

where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (164) *and* C_k *is a constant.*

Then

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} \leq$$

$$\leq C_2 \int\limits_{[t,T]^2}\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\phi_{j_3}(t_3)\phi_{j_4}(t_4)\right)^2 dt_3dt_4 = C_2\sum_{j_3,j_4=0}^{p}\left(a_{j_4 j_3}^{p}\right)^2. \tag{166}$$

From (163) *and* (166) *we obtain*

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \leq C_2\sum_{j_3,j_4=0}^{p}\left(a_{j_4 j_3}^{p}\right)^2 + \mathbf{1}_{\{i_3=i_4\neq 0\}}\left(\sum_{j_4=0}^{p} a_{j_4 j_4}^{p}\right)^2. \tag{167}$$

Obviously, the estimate (167) *can be used in the proof of Theorem* 4 *instead of* (121)–(123). *The estimate* (167) *can be refined. We have* [26] (*see the relation* (1.87), *Sect.* 1.2)

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\} =$$

$$= \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^{p}\right)^2 + \mathbf{1}_{\{i_3=i_4\ne 0\}} \sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} a_{j_3 j_4}^{p} \le$$

$$\le \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^{p}\right)^2 + \mathbf{1}_{\{i_3=i_4\ne 0\}} \frac{1}{2} \sum_{j_3,j_4=0}^{p} \left(\left(a_{j_4 j_3}^{p}\right)^2 + \left(a_{j_3 j_4}^{p}\right)^2\right) =$$

$$= \left(1 + \mathbf{1}_{\{i_3=i_4\ne 0\}}\right) \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^{p}\right)^2. \tag{168}$$

Combining (163) *and* (168), *we finally have*

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{p} a_{j_4 j_3}^{p} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \left(1 + \mathbf{1}_{\{i_3=i_4\ne 0\}}\right) \sum_{j_3,j_4=0}^{p} \left(a_{j_4 j_3}^{p}\right)^2 +$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}} \left(\sum_{j_4=0}^{p} a_{j_4 j_4}^{p}\right)^2. \tag{169}$$

6. Theorems 1–4 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s \in [0, T]$. Let $\mathbf{f}_s^{(i)p}$, $p \in \mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i = 1, \ldots, m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [56], [58], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [56]-[59] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s \in [0, T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [60], [62]

$$\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^\tau \phi_j(s) ds\ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}, \tag{170}$$

where $\tau \in [t,T]$, $t \ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (170) converges for any $\tau \in [t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p} = \sum_{j=0}^{p} \int\limits_t^{\tau} \phi_j(s)ds\ \zeta_j^{(i)}. \tag{171}$$

From (171) we obtain

$$d\mathbf{f}_\tau^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau)\zeta_j^{(i)} d\tau. \tag{172}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{173}$$

where $p_1,\ldots,p_k \in \mathbb{N}$, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$d\mathbf{w}_\tau^{(i)p} = \begin{cases} d\mathbf{f}_\tau^{(i)p} & \text{for} \quad i = 1,\ldots,m \\ \\ d\tau & \text{for} \quad i = 0 \end{cases}, \tag{174}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (172).

Let us substitute (174) into (173)

$$\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{175}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_s^{(0)} = s$,

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [56]-[59] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [59] (see Definition 7.1, pp. 480-481). Moreover, approximations of the Wiener process that are similar to (171) were not considered in [56], [58] (also see [59], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [59] for approximations of the Wiener process based on its series expansion (170) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (175) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [56], [58] (also see [59], Theorems 7.1, 7.2).

From the other hand, Theorems 1–4 from this paper can be considered as the proof of the Wong–Zakai approximation based on the iterated Riemann–Stieltjes integrals (173) of multiplicities 1 to 4 and the approximation (171) of the Wiener process. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 1–4) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (170), (171), and Theorems 2–4) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2,\ \psi_1(s),\ \psi_2(s)\equiv 1;$ $i_1,i_2=1,\ldots,m.$

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [56]-[59]).

Let $\mathbf{b}_{\Delta}^{(i)}(t),\ t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t,\ t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)},\ i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{176}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (176) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$
= \sum_{l=0}^{N-1}\sum_{q=0}^{l-1} \Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{\Delta^2}\sum_{l=0}^{N-1} \Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)} \int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s} d\tau ds =
$$

$$
= \sum_{l=0}^{N-1}\sum_{q=0}^{l-1} \Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{2}\sum_{l=0}^{N-1} \Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{177}
$$

Using (177) it is not difficult to show that

$$
\underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s) = \int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds =
$$

$$
= \int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{178}
$$

where $\Delta \to 0$ if $N \to \infty$ $(N\Delta = T)$.

Obviously, (178) agrees with Theorem 7.1 (see [59], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (170) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}, \quad i_1, i_2 = 1,\ldots,m, \tag{179}
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (172).

Let us substitute (172) into (179)

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p} = \sum_{j_1,j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{180}
$$

where

$$
C_{j_2j_1} = \int\limits_0^T \phi_{j_2}(s)\int\limits_0^s \phi_{j_1}(\tau)d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (175).

As we noted above, approximations of the Wiener process that are similar to (171) were not considered in [56], [58] (also see Theorems 7.1, 7.2 in [59]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [59] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [26]-[29]. More precisely, using Theorem 2, we obtain from (180) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{181}$$

From the other hand, by Theorem 1 (see (10)) for the case $k=2$ we obtain from (180) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{182}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from (15) and (182) we obtain (181).

7. Modification of Theorem 1 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Ito Stochastic Integrals

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Define the following function on the hypercube $[t,T]^k$

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_k<s\}}K(t_1,\ldots,t_k),$$

where the function $K(t_1,\ldots,t_k)$ is defined by (4), $s\in(t,T]$ (s is fixed), and $\mathbf{1}_A$ is the indicator of the set A. So we have

$$\bar{K}(t_1,\ldots,t_k,s)=\mathbf{1}_{\{t_1<\ldots<t_k<s\}}\psi_1(t_1)\ldots\psi_k(t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k<s\\ \\ 0, & \text{otherwise}\end{cases}, \tag{183}$$

where $k\ge 1$, $t_1,\ldots,t_k\in[t,T]$, and $s\in(t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $\bar{K}(t_1,\ldots,t_k,s)$ defined by (183) is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $\bar{K}(t_1,\ldots,t_k,s)\in L_2([t,T]^k)$ is converging to $\bar{K}(t_1,\ldots,t_k,s)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|\bar{K}(t_1,\ldots,t_k,s)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}(s)=\int\limits_{[t,T]^k}\bar{K}(t_1,\ldots,t_k,s)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k=$$

$$=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{184}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Note that

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{185}$$

$$=\int\limits_t^T\mathbf{1}_{\{t_k<s\}}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1},$$

where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=0,1,\ldots,m$.

Consider the partition $\{\tau_j\}_{j=0}^N$ of $[t,T]$, which satisfies the condition (6).

We will say that the function $f(x):[t,T]\to\mathbb{R}$ satisfies the condition $(\star)$ if it is continuous on the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity as well as it is right-continuous on the interval $[t,T]$.

Theorem 5 [26]-[29], [36]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j satisfies the condition $(\star)$. Then*

$$J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{186}$$

where $J[\psi^{(k)}]_{s,t}$ *is the iterated Ito stochastic integral* (185), $s\in(t,T]$ (s is fixed),

$$
\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},
$$

$$
\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},
$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$), $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (184), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of* $[t,T]$, *which satisfies the condition* (6).

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 5 we obtain

$$
J[\psi^{(k)}]_{s,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}(s)\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.
$$

Consider particular cases of Theorem 5 for $k=1,\ldots,5$ [27]–[29], [36]

$$
J[\psi^{(1)}]_{s,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}(s)\zeta_{j_1}^{(i_1)},\tag{187}
$$

$$
J[\psi^{(2)}]_{s,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(s)\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),\tag{188}
$$

$$
\begin{aligned}
J[\psi^{(3)}]_{s,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(s)\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),
\end{aligned}\tag{189}
$$

$$
\begin{aligned}
J[\psi^{(4)}]_{s,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}(s)\Biggl(\prod_{l=1}^4\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
\end{aligned}
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{190}
$$

$$
\begin{gathered}
J[\psi^{(5)}]_{s,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}(s)\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{gathered}
$$

where $\mathbf{1}_A$ is the indicator of the set A, $C_{j_k\ldots j_1}(s)$ $(k=1,\ldots,5)$ has the form (184), $s\in(t,T]$ (s is fixed).

Note that in [26] (see Sect. 1.15) Theorem 5 is generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

8. Modification of Theorem 2 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 2 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 6 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$
J^*[\psi^{(2)}]_{s,t}=\int\limits_t^{*s}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)
$$

the following expansion

$$J^*[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{191}$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ *(*s *is fixed),*

$$C_{j_2j_1}(s) = \int\limits_t^s \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2, \tag{192}$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j*.*

The condition of continuity of the functions $\psi_1(\tau),\psi_2(\tau)$ is related to the definition [2] of the Stratonovich stochastic integral that we use.

Proof. The case $s=T$ follows from (53). Below we consider the case $s\in(t,T)$. In accordance to the standard relations between Stratonovich and Ito stochastic integrals we have w. p. 1

$$J^*[\psi^{(2)}]_{s,t} = J[\psi^{(2)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1, \tag{193}$$

where $s\in(t,T]$ (s is fixed), $\mathbf{1}_A$ is the indicator of the set A.

From the other side according to (188) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ (see [26], Sect. 1.15), we have

$$J[\psi^{(2)}]_{s,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1j_1}(s). \tag{194}$$

From (193) and (194) it follows that Theorem 6 will be proved if

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}(s), \tag{195}$$

where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

Let us rewrite (53) in the form

$$\frac{1}{2}\int\limits_t^T \bar{\psi}_1(\tau)\bar{\psi}_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_t^T \bar{\psi}_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\bar{\psi}_1(t_1)\phi_j(t_1)dt_1dt_2, \tag{196}$$

where $\bar{\psi}_1(\tau), \bar{\psi}_2(\tau) \in L_2([t,T])$.

Suppose that

$$\bar{\psi}_1(\tau) = \psi_1(\tau)\mathbf{1}_{\{\tau<s\}}, \quad \bar{\psi}_2(\tau) = \psi_2(\tau)\mathbf{1}_{\{\tau<s\}}, \tag{197}$$

where $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, $s \in (t,T)$ (s is fixed).

Combining (196) and (197), we get

$$\frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)\mathbf{1}_{\{\tau<s\}}d\tau =$$

$$= \sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\mathbf{1}_{\{t_2<s\}}\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\mathbf{1}_{\{t_1<s\}}\phi_j(t_1)dt_1dt_2,$$

i.e.

$$\frac{1}{2}\int\limits_t^s \psi_1(\tau)\psi_2(\tau)d\tau = \sum_{j=0}^{\infty}\int\limits_t^s \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2.$$

The equality (195) is proved. Theorem 6 is proved.

Let us reformulate Theorem 6 in terms on the convergence of solution of system of ordinary differential equations (ODEs) to the solution of system of Stratnovich SDEs (the so-called Wong–Zakai type theorem).

By analogy with (175) for $k=2$, $i_1, i_2 = 1,\ldots,m$, and $s \in (t,T]$ (s is fixed) we obtain

$$\int\limits_t^s \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{198}$$

where $p_1, p_2 \in \mathbb{N}$ and $d\mathbf{f}_{\tau}^{(i)p}$ is defined by (172); another notations are the same as in Theorem 6.

The iterated Riemann–Stiltjes integrals

$$Y_{s,t}^{(i_1i_2)p_1p_2} = \int\limits_t^s \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}d\mathbf{f}_{t_2}^{(i_2)p_2},$$

$$X_{s,t}^{(i_1)p_1} = \int\limits_t^s \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p_1}$$

are the solution of the following system of ODEs

$$\begin{cases} dY_{s,t}^{(i_1i_2)p_1p_2} = \psi_2(s)X_{s,t}^{(i_1)p_1}d\mathbf{f}_s^{(i_2)p_2}, & Y_{t,t}^{(i_1i_2)p_1p_2} = 0 \\ \\ dX_{s,t}^{(i_1)p_1} = \psi_1(s)d\mathbf{f}_s^{(i_1)p_1}, & X_{t,t}^{(i_1)p_1} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$
Y_{s,t}^{(i_1 i_2)} = \int\limits_t^{*s} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)},
$$

$$
X_{s,t}^{(i_1)} = \int\limits_t^{*s} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}
$$

are the solution of the following system of Stratonovich SDEs

$$
\begin{cases}
dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\
\\
dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0
\end{cases},
$$

where $* d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorem 6 and (187) we obtain the following theorem.

Theorem 7 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau)$ are continuous functions on $[t,T]$. Then for any fixed s $(s \in (t,T])$*

$$
\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1 i_2)p_1 p_2} = Y_{s,t}^{(i_1 i_2)}, \quad \underset{p_1\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)p_1} = X_{s,t}^{(i_1)}.
$$

9. Modification of Theorem 3 for the Case of Integration Interval $[t,s]$ $(s \in (t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 3 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 8 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{s,t} = \int\limits_t^{*s} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)
$$

the following expansion

$$
J^*[\psi^{(3)}]_{s,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}
$$

that converges in the mean-square sense is valid, where $s \in (t,T]$ (s is fixed),

$$C_{j_3j_2j_1}(s)=\int\limits_t^s\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. The case $s=T$ is considered in Theorem 3. Below we consider the case $s\in(t,T)$. First, let us consider the case of Legendre polynomials. From (189) for the case $p_1=p_2=p_3=p$ and the standard relation between Ito and Stratonovich stochastic integrals (2), (3) of third multiplicity we conclude that Theorem 8 will be proved if w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}(s)\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^s\psi_3(\tau)\int\limits_t^{\tau}\psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_\tau^{(i_3)},\tag{199}$$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^s\psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau,\tag{200}$$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}(s)\zeta_{j_3}^{(i_2)}=0.\tag{201}$$

The proof of the formulas (199), (201) is absolutely similar to the proof of the formulas (66), (68). It is only necessary to replace the interval of integration $[t,T]$ by $[t,s]$ in the proof of the formulas (66), (68) and use Theorem 5 instead of Theorem 1. Also in the case (201) it is necessary to use the estimate (87).

Let us prove (200). Using Theorem 5 for $k=2$ (see (188) for $i_1=1,\ldots,m,\ i_2=0$), we obtain w. p. 1 (also see (367), (368))

$$\frac{1}{2}\int\limits_t^s\psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}d\tau=\frac{1}{2}\ \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}C_{j_1}^{*}(s)\zeta_{j_1}^{(i_1)},$$

where

$$C_{j_1}^{*}(s)=\int\limits_t^s\psi_3(\tau)\psi_2(\tau)\int\limits_t^{\tau}\psi_1(s_1)\phi_{j_1}(s_1)ds_1d\tau=$$

$$=\int\limits_t^s\psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^s\psi_3(\tau)\psi_2(\tau)d\tau ds_1.\tag{202}$$

We have

$$E'_p(s) \stackrel{\sf def}{=} {\sf M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^{*}(s)\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$={\sf M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)-\frac{1}{2}C_{j_1}^{*}(s)\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

(203) $$=\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1}(s)-\frac{1}{2}C_{j_1}^{*}(s)\right)^2,$$

$$C_{j_3j_3j_1}(s)=\int\limits_t^s \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_t^{\tau}\psi_1(s_1)\phi_{j_1}(s_1)ds_1d\tau d\theta=$$

(204) $$=\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\int\limits_{s_1}^s\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1.$$

From (202)–(204) we obtain

$$E'_p(s)=\sum_{j_1=0}^{p}\left(\int\limits_t^s\psi_1(s_1)\phi_{j_1}(s_1)\left(\sum_{j_3=0}^{p}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau-\right.\right.$$

(205) $$\left.\left.-\frac{1}{2}\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau\right)ds_1\right)^2.$$

Let us show that

(206) $$\sum_{j_3=0}^{\infty}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau=\frac{1}{2}\int\limits_{s_1}^{s}\psi_3(\tau)\psi_2(\tau)d\tau.$$

Using (196) and Fubini's Theorem, we have

(207) $$\frac{1}{2}\int\limits_t^T\bar\psi_1(\tau)\bar\psi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_t^T\bar\psi_1(t_1)\phi_j(t_1)\int\limits_{t_1}^{T}\bar\psi_2(t_2)\phi_j(t_2)dt_2dt_1,$$

where $\bar\psi_1(\tau),\bar\psi_2(\tau)\in L_2([t,T])$.

Suppose that

(208) $$\bar\psi_1(\tau)=\psi_2(\tau)\mathbf{1}_{\{s_1<\tau<s\}},\quad \bar\psi_2(\tau)=\psi_3(\tau)\mathbf{1}_{\{\tau<s\}}.$$

Using (207) and (208), we get (206). Combining (205) and (206), we obtain

$$E'_p(s) = \sum_{j_1=0}^{p}\left(\int\limits_t^s \psi_1(s_1)\phi_{j_1}(s_1)\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau ds_1\right)^2 \le$$

$$\le K\sum_{j_1=0}^{p}\left(\int\limits_t^s |\phi_{j_1}(s_1)|\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| ds_1\right)^2, \qquad (209)$$

where constant K does not depend on p.

Let us estimate the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right|.$$

Note that, by virtue of the additivity property of the integral, we obtain

$$\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau =$$

$$= \int\limits_t^s \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta -$$

$$- \int\limits_t^{s_1} \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta -$$

$$- \int\limits_{s_1}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta\int\limits_t^{s_1} \psi_2(\tau)\phi_{j_3}(\tau)d\tau.$$

Further, we have

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^s \psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^s \psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right| \le$$

$$\le \left|\sum_{j_3=p+1}^{\infty}\int\limits_t^s \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta\right| +$$

$$+ \left|\sum_{j_3=p+1}^{\infty}\int\limits_t^{s_1} \psi_3(\theta)\phi_{j_3}(\theta)\int\limits_t^{\theta} \psi_2(\tau)\phi_{j_3}(\tau)d\tau d\theta\right| +$$

$$+\sum_{j_3=p+1}^{\infty}\left|\int\limits_{s_1}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta\int\limits_{t}^{s_1}\psi_2(\tau)\phi_{j_3}(\tau)d\tau\right|. \tag{210}$$

Applying the estimate (312), we can write

$$\left|\sum_{j_1=p+1}^{\infty}C_{j_1j_1}(s)\right|\le\frac{C}{p}\left(1+\frac{1}{\left(1-(z(s))^2\right)^{1/4}}\right), \tag{211}$$

where $s\in(t,T)$, constant C does not depend on p, $z(s)$ has the form (26), and $C_{j_1j_1}(s)$ is defined by (192) for the case $j_1=j_2$.

Let us estimate the integral

$$\int\limits_{u}^{\tau}\phi_j(\theta)\psi(\theta)d\theta\quad(j\ne 0), \tag{212}$$

where $\psi(\theta)$ is a continuously differentiable function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

We have

$$\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta=\frac{\sqrt{T-t}\sqrt{2j+1}}{2}\int\limits_{z(v)}^{z(x)}P_j(y)\psi(u(y))dy=$$

$$=\frac{\sqrt{T-t}}{2\sqrt{2j+1}}\Biggl((P_{j+1}(z(x))-P_{j-1}(z(x)))\psi(x)-(P_{j+1}(z(v))-P_{j-1}(z(v)))\psi(v)-$$

$$-\frac{T-t}{2}\int\limits_{z(v)}^{z(x)}((P_{j+1}(y)-P_{j-1}(y))\psi'(u(y))dy\Biggr), \tag{213}$$

where $x,v\in(t,T)$, $u(y)$ and $z(x)$ are defined by (26), ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (213) we used (27). From (213) and (29) it follows that

$$\left|\int\limits_{v}^{x}\phi_j(\theta)\psi(\theta)d\theta\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+C_1\right), \tag{214}$$

where $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$ and constants C, C_1 do not depend on j.

Applying the estimates (87), (211) , and (214) to the right-hand side of (210) gives

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{s_1}^{s}\psi_2(\tau)\phi_{j_3}(\tau)\int\limits_{\tau}^{s}\psi_3(\theta)\phi_{j_3}(\theta)d\theta d\tau\right|\le\frac{L}{p}\left(1+\frac{1}{\left(1-(z(s_1))^2\right)^{1/4}}\right)\times$$

$$\times\left(1+\frac{1}{(1-(z(s))^2)^{1/4}}+\frac{1}{(1-(z(s_1))^2)^{1/4}}\right), \tag{215}$$

where $s, s_1 \in (t, T)$ and constant L is independent of p.

Combining the estimates (89), (209), and (215), we finally obtain

$$E_p'(s) \le \frac{L(s)p}{p^2} = \frac{L(s)}{p}$$

if $p \to \infty$, where constant $L(s)$ (s is fixed, $s \in (t, T)$) does not depend on p. The relation (200) is proved for the polynomial case. Theorem 8 is proved for the case of Legendre polynomials.

For the trigonometric case, by analogy with the proof of Lemma 1 (Sect. 3), we obtain the following analog of (211)

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le \frac{C}{p}, \tag{216}$$

where $s \in [t, T]$, constant C does not depend on p, and $C_{j_1 j_1}(s)$ is defined by (192) for the case $j_1 = j_2$.

Note the following obvious estimates for the trigonometric case

$$\left|\int\limits_{s_1}^{s} \psi_3(\theta)\phi_j(\theta)d\theta\right| \le \frac{C}{j}, \quad \left|\int\limits_{t}^{s_1} \psi_2(\tau)\phi_j(\tau)d\tau\right| \le \frac{C}{j} \quad (j \ne 0), \tag{217}$$

where $s, s_1 \in [t, T]$, constant C does not depend on p.

Applying (209), (210), (216), and (217), we obtain the assertion of Theorem 8 for the trigonometric case. Theorem 8 is proved.

Let us reformulate Theorem 8 in terms on the convergence of solution of system of ODEs to the solution of system of Stratonovich SDEs (the so-called Wong–Zakai type theorem).

By analogy with (175) for the case $k = 3$, $p_1 = p_2 = p_3 = p$, $i_1, i_2, i_3 = 1, \ldots, m$, and $s \in (t, T]$ (s is fixed) we obtain

$$\int\limits_t^s \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{218}$$

where $p \in \mathbb{N}$ and $d\mathbf{f}_\tau^{(i)p}$ is defined by (172); another notations are the same as in Theorem 8.

The iterated Riemann–Stiltjes integrals

$$Z_{s,t}^{(i_1 i_2 i_3)p} = \int\limits_t^s \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p} d\mathbf{f}_{t_3}^{(i_3)p},$$

$$Y_{s,t}^{(i_1 i_2)p} = \int\limits_t^s \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p} d\mathbf{f}_{t_2}^{(i_2)p},$$

$$
X_{s,t}^{(i_1)p} = \int\limits_t^s \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)p}
$$

are the solution of the following system of ODEs

$$
\begin{cases}
dZ_{s,t}^{(i_1 i_2 i_3)p} = \psi_3(s) Y_{s,t}^{(i_1 i_2)p} d\mathbf{f}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\
\\
dY_{s,t}^{(i_1 i_2)p} = \psi_2(s) X_{s,t}^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\
\\
dX_{s,t}^{(i_1)p} = \psi_1(s) d\mathbf{f}_s^{(i_1)p}, & X_{t,t}^{(i_1)p_1} = 0
\end{cases}.
$$

From the other hand, the iterated Stratonovich stochastic integrals

$$
Z_{s,t}^{(i_1 i_2 i_3)} = {\int\limits_t^{*}}^s \psi_3(t_3) {\int\limits_t^{*}}^{t_3} \psi_2(t_2) {\int\limits_t^{*}}^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)},
$$

$$
Y_{s,t}^{(i_1 i_2)} = {\int\limits_t^{*}}^s \psi_2(t_2) {\int\limits_t^{*}}^{t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)},
$$

$$
X_{s,t}^{(i_1)} = {\int\limits_t^{*}}^s \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)}
$$

are the solution of the following system of Stratonovich SDEs

$$
\begin{cases}
dZ_{s,t}^{(i_1 i_2 i_3)} = \psi_3(s) Y_{s,t}^{(i_1 i_2)} * d\mathbf{f}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\
\\
dY_{s,t}^{(i_1 i_2)} = \psi_2(s) X_{s,t}^{(i_1)} * d\mathbf{f}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\
\\
dX_{s,t}^{(i_1)} = \psi_1(s) * d\mathbf{f}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0
\end{cases},
$$

where $* d\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 7 and 8 we obtain the following theorem.

Theorem 9 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(\tau)$*,* $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$*. Then for any fixed* s $(s \in (t,T])$

$$\underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1i_2i_3)p}=Z_{s,t}^{(i_1i_2i_3)},\quad \underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1i_2)p}=Y_{s,t}^{(i_1i_2)},$$

$$\underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)p}=X_{s,t}^{(i_1)}.$$

10. Modification of Theorem 4 for the Case of Integration Interval $[t,s]$ $(s\in(t,T])$ of Iterated Stratonovich Stochastic Integrals of Multiplicity 4 and Wong–Zakai Type Theorem

Let us prove the following theorem.

Theorem 10 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{s,t}=\int\limits_t^{*s}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(4)}]_{s,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $s\in(t,T]$ (s is fixed),

$$C_{j_4j_3j_2j_1}(s)=\int\limits_t^{s}\phi_{j_4}(s_4)\int\limits_t^{s_4}\phi_{j_3}(s_3)\int\limits_t^{s_3}\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_1}(s_1)ds_1ds_2ds_3ds_4$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$.

Proof. The case $s=T$ is considered in Theorem 4. Below we consider the case $s\in(t,T)$. The relation (190) (in the case when $p_1=\ldots=p_4=p\to\infty$) implies that

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=J[\psi^{(4)}]_{s,t}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}A_1^{(i_3i_4)}(s)+\mathbf{1}_{\{i_1=i_3\ne 0\}}A_2^{(i_2i_4)}(s)+\mathbf{1}_{\{i_1=i_4\ne 0\}}A_3^{(i_2i_3)}(s)+\mathbf{1}_{\{i_2=i_3\ne 0\}}A_4^{(i_1i_4)}(s)+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}A_5^{(i_1i_3)}(s)+\mathbf{1}_{\{i_3=i_4\ne 0\}}A_6^{(i_1i_2)}(s)-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}B_1(s)-$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}B_2(s)-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}B_3(s), \tag{219}$$

where $J[\psi^{(4)}]_{s,t}$ has the form (185) for $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$ and $i_1,\ldots,i_4=0,1,\ldots,m$,

$$A_1^{(i_3i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}C_{j_4j_3j_1j_1}(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$A_2^{(i_2i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}C_{j_4j_3j_2j_3}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$A_3^{(i_2i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}C_{j_4j_3j_2j_4}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$A_4^{(i_1i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}C_{j_4j_3j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)},$$

$$A_5^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}C_{j_4j_3j_4j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$A_6^{(i_1i_2)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$B_1(s)=\lim_{p\to\infty}\sum_{j_1,j_4=0}^{p}C_{j_4j_4j_1j_1}(s),\quad B_2(s)=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}C_{j_3j_4j_3j_4}(s),$$

$$B_3(s)=\lim_{p\to\infty}\sum_{j_4,j_3=0}^{p}C_{j_4j_3j_3j_4}(s).$$

Using the integration order replacement in Riemann integrals, Theorem 5 for $k=2$ (see (188)) and (195), Parseval's equality and the integration order replacement technique for Ito stochastic integrals (see [26]-[29], Chapter 3) or Ito's formula, we obtain (see the derivation of the formula (101))

$$A_1^{(i_3i_4)}(s)=\frac{1}{2}\int\limits_t^s\int\limits_t^\tau\int\limits_t^{s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_\tau^{(i_4)}+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^s(s_1-t)ds_1-\Delta_1^{(i_3i_4)}(s)\quad \text{w. p. 1}, \tag{220}$$

where

$$\Delta_1^{(i_3i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_4=0}^{p}a_{j_4j_3}^{p}(s)\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$a^p_{j_4j_3}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_3}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2\right)^2ds_1d\tau.$$

Let us consider $A_2^{(i_2i_4)}(s)$ (see the derivation of the formula (103))

(221) $$A_2^{(i_2i_4)}(s)=-\Delta_2^{(i_2i_4)}(s)+\Delta_1^{(i_2i_4)}(s)+\Delta_3^{(i_2i_4)}(s)\ \text{w. p. 1},$$

where

$$\Delta_2^{(i_2i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p}b^p_{j_4j_2}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2i_4)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2=0}^{p}c^p_{j_4j_2}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$b^p_{j_4j_2}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_4}(\tau)\sum_{j_3=p+1}^{\infty}\left(\int\limits_t^{\tau}\phi_{j_3}(s_1)ds_1\right)^2\int\limits_t^{\tau}\phi_{j_2}(s_1)ds_1d\tau,$$

$$c^p_{j_4j_2}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_2}(s_3)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_3}^{\tau}\phi_{j_3}(s_1)ds_1\right)^2ds_3d\tau.$$

Let us consider $A_5^{(i_1i_3)}(s)$ (see the derivation of the formula (106))

(222) $$A_5^{(i_1i_3)}(s)=-\Delta_4^{(i_1i_3)}(s)+\Delta_5^{(i_1i_3)}(s)+\Delta_6^{(i_1i_3)}(s)\quad\text{w. p. 1},$$

where

$$\Delta_4^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}d^p_{j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$\Delta_5^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}e^p_{j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$\Delta_6^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}f^p_{j_3j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$d^p_{j_3j_1}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_1}(s_3)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^{s}\phi_{j_4}(\tau)d\tau\right)^2\int\limits_{s_3}^{s}\phi_{j_3}(\tau)d\tau ds_3,$$

$$e^p_{j_3 j_1}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3)\int\limits_{s_3}^s \phi_{j_3}(\tau)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_3}^{\tau}\phi_{j_4}(s_1)ds_1\right)^2 d\tau ds_3,$$

$$f^p_{j_3 j_1}(s) = \frac{1}{2}\int\limits_t^s \phi_{j_1}(s_3)\int\limits_{s_3}^s \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1\right)^2 ds_2 ds_3 =$$

$$= \frac{1}{2}\int\limits_t^s \phi_{j_3}(s_2)\sum_{j_4=p+1}^{\infty}\left(\int\limits_{s_2}^{s}\phi_{j_4}(s_1)ds_1\right)^2 \int\limits_t^{s_2}\phi_{j_1}(s_3)ds_3 ds_2.$$

Moreover (see the derivation of the formula (111)),

$$A_3^{(i_2 i_3)}(s) = 2\Delta_6^{(i_2 i_3)}(s) - A_5^{(i_2 i_3)}(s) = \Delta_4^{(i_2 i_3)}(s) - \Delta_5^{(i_2 i_3)}(s) + \Delta_6^{(i_2 i_3)}(s) \quad \text{w. p. 1.} \tag{223}$$

Let us consider $A_4^{(i_1 i_4)}(s)$ (see the derivation of the formula (112))

$$A_4^{(i_1 i_4)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_\tau^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} - \Delta_3^{(i_1 i_4)}(s) \quad \text{w. p. 1.} \tag{224}$$

Let us consider $A_6^{(i_1 i_2)}(s)$ (see the derivation of the formula (113))

$$A_6^{(i_1 i_2)}(s) = \frac{1}{2}\int\limits_t^s \int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_\tau^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1 +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^s (s-s_2)ds_2 - \Delta_6^{(i_1 i_2)}(s) \quad \text{w. p. 1.} \tag{225}$$

Further, we have w. p. 1 (see the derivation of the formula (110))

$$A_3^{(i_2 i_3)}(s) + A_5^{(i_2 i_3)}(s) =$$

$$= \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_2=0}^{p}\int\limits_t^s \phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1 \zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{226}$$

Using (226) and the generalized Parseval equality, we obtain w. p. 1

$$A_3^{(i_2 i_3)}(s) + A_5^{(i_2 i_3)}(s) =$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\sum_{j_4=0}^{p}\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=$$

$$=-\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2\sum_{j_4=p+1}^{\infty}\int\limits_t^{s_1}\phi_{j_4}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_4}(\tau)d\tau ds_1\times$$

$$\times\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=$$

$$=\Delta_6^{(i_2i_3)}(s)+\Delta_2^{(i_2i_3)}(s)-\Delta_9^{(i_2i_3)}(s), \tag{227}$$

where

$$\Delta_9^{(i_2i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_2=0}^{p}q_{j_2j_3}^{p}(s)\zeta_{j_2}^{(i_3)}\zeta_{j_3}^{(i_3)},$$

$$q_{j_2j_3}^{p}(s)=\frac{1}{2}\int\limits_t^s\phi_{j_3}(s_1)\int\limits_t^{s_1}\phi_{j_2}(s_2)ds_2ds_1\sum_{j_4=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_4}(\tau)d\tau\right)^2.$$

From (222) and (227) we get

$$A_3^{(i_2i_3)}(s)=\Delta_2^{(i_2i_3)}(s)+\Delta_4^{(i_2i_3)}(s)-\Delta_5^{(i_2i_3)}(s)-\Delta_9^{(i_2i_3)}(s)\quad \text{w. p. 1.} \tag{228}$$

Let us consider $B_1(s), B_2(s), B_3(s)$ (see the derivation of the formulas (114), (115))

$$B_1(s)=\frac{1}{4}\int\limits_t^s(s_1-t)ds_1-\lim_{p\to\infty}\sum_{j_4=0}^{p}a_{j_4j_4}^{p}(s), \tag{229}$$

$$B_2(s)=\lim_{p\to\infty}\sum_{j_3=0}^{p}a_{j_3j_3}^{p}(s)+\lim_{p\to\infty}\sum_{j_3=0}^{p}c_{j_3j_3}^{p}(s)-\lim_{p\to\infty}\sum_{j_3=0}^{p}b_{j_3j_3}^{p}(s). \tag{230}$$

Moreover (see the derivation of the formula (116)),

$$B_2(s)+B_3(s)=$$

$$=\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^s\phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)ds_2\sum_{j_3=0}^{p}\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_3}(\tau)d\tau ds_1. \tag{231}$$

Using (231) and the generalized Parseval equality, we obtain

$$B_2(s)+B_3(s)=$$

$$= -\lim_{p\to\infty}\sum_{j_4=0}^{p}\int\limits_t^s \phi_{j_4}(s_1)\int\limits_t^{s_1}\phi_{j_4}(s_2)ds_2\sum_{j_3=p+1}^{\infty}\int\limits_t^{s_1}\phi_{j_3}(s_3)ds_3\int\limits_{s_1}^{s}\phi_{j_3}(\tau)d\tau ds_1 =$$

$$= \lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}(s) + \lim_{p\to\infty}\sum_{j_4=0}^{p} b^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} q^p_{j_4j_4}(s). \tag{232}$$

Combining (230) and (232), we have

$$B_3(s) = 2\lim_{p\to\infty}\sum_{j_4=0}^{p} b^p_{j_4j_4}(s) + \lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} c^p_{j_4j_4}(s) -$$

$$- \lim_{p\to\infty}\sum_{j_4=0}^{p} a^p_{j_4j_4}(s) - \lim_{p\to\infty}\sum_{j_4=0}^{p} q^p_{j_4j_4}(s). \tag{233}$$

After substituting the relations (220)–(225), (228)–(230), (233) into (219), we obtain

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}(s)\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4} =$$

$$= J[\psi^{(4)}]_{s,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^s\int\limits_t^{\tau}\int\limits_t^{s_1} ds_2 d\mathbf{w}^{(i_3)}_{s_1}d\mathbf{w}^{(i_4)}_{\tau} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^s\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}^{(i_1)}_{\tau}ds_1 d\mathbf{w}^{(i_4)}_{s_2} + \frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^s\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}^{(i_1)}_{\tau}d\mathbf{w}^{(i_2)}_{s_2}ds_1 +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2 ds_1 + R(s) = J^*[\psi^{(4)}]_{s,t} +$$

$$+R(s) \quad \text{w. p. 1}, \tag{234}$$

where

$$R(s) = -\mathbf{1}_{\{i_1=i_2\ne 0\}}\Delta_1^{(i_3i_4)}(s) + \mathbf{1}_{\{i_1=i_3\ne 0\}}\left(-\Delta_2^{(i_2i_4)}(s) + \Delta_1^{(i_2i_4)}(s) + \Delta_3^{(i_2i_4)}(s)\right) +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\left(\Delta_2^{(i_2i_3)}(s) + \Delta_4^{(i_2i_3)}(s) - \Delta_5^{(i_2i_3)}(s) - \Delta_9^{(i_2i_3)}(s)\right) - \mathbf{1}_{\{i_2=i_3\ne 0\}}\Delta_3^{(i_1i_4)}(s) +$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(-\Delta_4^{(i_1i_3)}(s) + \Delta_5^{(i_1i_3)}(s) + \Delta_6^{(i_1i_3)}(s)\right) - \mathbf{1}_{\{i_3=i_4\ne 0\}}\Delta_6^{(i_1i_2)}(s) -$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}(s) + \lim_{p\to\infty}\sum_{j_3=0}^{p} c^p_{j_3j_3}(s) - \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3j_3}(s)\right) -$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\Biggl(2\lim_{p\to\infty}\sum_{j_4=0}^{p} b^p_{j_4j_4}(s)+\lim_{p\to\infty}\sum_{j_4=0}^{p} f^p_{j_4j_4}(s)-\lim_{p\to\infty}\sum_{j_4=0}^{p} c^p_{j_4j_4}(s)-$$

$$-\lim_{p\to\infty}\sum_{j_4=0}^{p} a^p_{j_4j_4}(s)-\lim_{p\to\infty}\sum_{j_4=0}^{p} q^p_{j_4j_4}(s)\Biggr)+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3j_3}(s). \tag{235}$$

Let us prove that

$$R(s)=0\quad \text{w. p. 1.} \tag{236}$$

Consider the case of Legendre polynomials. Let us prove that

$$\Delta_1^{(i_3i_4)}(s)=0\quad \text{w. p. 1.} \tag{237}$$

We have

$$a^p_{j_4j_3}(s)=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times\int\limits_{-1}^{z(s)} P_{j_4}(y)\int\limits_{-1}^{y} P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}(2j_1+1)\left(\int\limits_{-1}^{y_1} P_{j_1}(y_2)dy_2\right)^2 dy_1dy=$$

$$=\frac{(T-t)^2\sqrt{(2j_4+1)(2j_3+1)}}{32}\times$$

$$\times\int\limits_{-1}^{z(s)} P_{j_3}(y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2\int\limits_{y_1}^{z(s)} P_{j_4}(y)dydy_1=$$

$$=\frac{(T-t)^2\sqrt{2j_3+1}}{32\sqrt{2j_4+1}}\times$$

$$\times\int\limits_{-1}^{z(s)} P_{j_3}(y_1)\left(\left(P_{j_4+1}(z(s))-P_{j_4-1}(z(s))\right)-\left(P_{j_4+1}(y_1)-P_{j_4-1}(y_1)\right)\right)\times$$

$$\times\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4\neq 0$ and

$$a^p_{j_4 j_3}(s) = \frac{(T-t)^2\sqrt{2j_3+1}}{32}\times$$

$$\times \int\limits_{-1}^{z(s)} P_{j_3}(y_1)(z(s)-y_1)\sum_{j_1=p+1}^{\infty}\frac{1}{2j_1+1}\left(P_{j_1+1}(y_1)-P_{j_1-1}(y_1)\right)^2 dy_1$$

if $j_4 = 0$, where $z(s)$ is defined by (26).

We can assume that $s \in (t, T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 4. Now the further proof of the equality (237) is completely analogous to the proof of the equality (128).

It is not difficult to see that the formulas

$$\Delta_2^{(i_2 i_4)}(s) = 0, \quad \Delta_4^{(i_1 i_3)}(s) = 0, \quad \Delta_6^{(i_1 i_3)}(s) = 0, \quad \Delta_9^{(i_2 i_3)}(s) = 0 \quad \text{w. p. 1} \tag{238}$$

can be proved similarly with the proof of (237).

Moreover, the relations

$$\lim_{p\to\infty}\sum_{j_3=0}^{p} a^p_{j_3 j_3}(s) = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} b^p_{j_3 j_3}(s) = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} f^p_{j_3 j_3}(s) = 0, \quad \lim_{p\to\infty}\sum_{j_3=0}^{p} q^p_{j_3 j_3}(s) = 0 \tag{239}$$

can also be proved analogously with (130), (131).

Let us consider $\Delta_3^{(i_2 i_4)}(s)$ and prove that

$$\Delta_3^{(i_2 i_4)}(s) = 0 \quad \text{w. p. 1.} \tag{240}$$

We have

$$\Delta_3^{(i_2 i_4)}(s) = \Delta_4^{(i_2 i_4)}(s) + \Delta_6^{(i_2 i_4)}(s) - \Delta_7^{(i_2 i_4)}(s) =$$

$$= -\Delta_7^{(i_2 i_4)}(s) \quad \text{w. p. 1,} \tag{241}$$

where

$$\Delta_7^{(i_2 i_4)}(s) = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_2, j_4=0}^{p} g^p_{j_4 j_2}(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$g^p_{j_4 j_2}(s) = \int\limits_t^s \phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_2}(s_1)\sum_{j_1=p+1}^{\infty}\left(\int\limits_{s_1}^{s}\phi_{j_1}(s_2)ds_2\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2\right)ds_1 d\tau.$$

Note that (see (134))

$$g^p_{j_4j_4}(s)=\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_\tau^s\phi_{j_1}(s_2)ds_2d\tau\right)^2. \tag{242}$$

The proof of (240) for the case $i_2=i_4\ne 0$ differs from the proof of the equality

$$\Delta_3^{(i_2i_4)}=0\quad \text{w. p. 1}$$

for the case $i_2=i_4\ne 0$ (see the proof of Theorem 4). In our case we will use Parseval's equality instead of the orthogonality property of the Legendre polynomials.

Using the Parseval equality, we obtain

$$\sum_{j_4=0}^{p}g^p_{j_4j_4}(s)=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_\tau^s\phi_{j_1}(s_2)ds_2d\tau\right)^2=$$

$$=\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\frac{1}{2}\left(\int\limits_t^s\phi_{j_4}(\tau)\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2-\int\limits_t^\tau\phi_{j_1}(s_2)ds_2\right)d\tau\right)^2\le$$

$$\le\sum_{j_4=0}^{p}\left(\int\limits_t^s\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+\sum_{j_4=0}^{p}\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2=$$

$$=\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+$$

$$+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{p}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2\le$$

$$\le\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)d\tau\right)^2\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+$$

$$+\sum_{j_1=p+1}^{\infty}\sum_{j_4=0}^{\infty}\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^\tau\phi_{j_1}(s_2)ds_2d\tau\right)^2=$$

$$=\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2d\tau\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_2)ds_2\right)^2+\sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}\right)^2\left(\int\limits_t^\tau\phi_{j_1}(s_2)ds_2\right)^2d\tau=$$

$$= (s-t)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_2)ds_2\right)^2 + \sum_{j_1=p+1}^{\infty}\int\limits_t^s\left(\int\limits_t^{\tau} \phi_{j_1}(s_2)ds_2\right)^2 d\tau. \tag{243}$$

We can assume that $s \in (t, T)$ $(z(s) \neq \pm 1)$ since the case $s = T$ has already been considered in Theorem 4. Then from (243) and (87) we obtain

$$0 \le \sum_{j_4=0}^{p} g^p_{j_4 j_4}(s) \le \frac{C(s)}{p}, \tag{244}$$

where constant $C(s)$ (s is fixed) is independent of p.

Combining (29) and (124) with (213), we obtain

$$\left|\int\limits_{s_1}^{s} \phi_j(\theta)d\theta\right| < \frac{K}{j^{1/2+m/4}}\left(\frac{1}{(1-z^2(s))^{m/8}} + \frac{1}{(1-z^2(s_1))^{m/8}}\right), \tag{245}$$

where $s, s_1 \in (t, T)$, $m = 1$ or $m = 2$, $z(s)$ is defined by (26), constant K does not depend on j.

Using the Parseval equality, we get

$$\lim_{p_1\to\infty}\sum_{j_4,j_2=0}^{p_1}\left(g^p_{j_4 j_2}(s)\right)^2 = \int\limits_{[t,T]^2}\left(K_p(\tau, s_1, s)\right)^2 ds_1 d\tau = \int\limits_t^s\int\limits_t^{\tau}\left(F_p(\tau, s_1, s)\right)^2 ds_1 d\tau, \tag{246}$$

where

$$g^p_{j_4 j_2}(s) = \int\limits_t^T \mathbf{1}_{\{\tau<s\}}\phi_{j_4}(\tau)\int\limits_t^{\tau}\phi_{j_2}(s_1)F_p(\tau, s_1, s)ds_1 d\tau = \int\limits_{[t,T]^2} K_p(\tau, s_1, s)\phi_{j_4}(\tau)\phi_{j_2}(s_1)ds_1 d\tau$$

is a coefficient of the double Fourier–Legendre series of the function

$$K_p(\tau, s_1, s) = \mathbf{1}_{\{\tau<s\}}\mathbf{1}_{\{s_1<\tau<s\}}F_p(\tau, s_1, s),$$

where

$$\sum_{j_1=p+1}^{\infty}\int\limits_{s_1}^{s}\phi_{j_1}(s_2)ds_2\int\limits_{\tau}^{s}\phi_{j_1}(s_2)ds_2 \stackrel{\sf def}{=} F_p(\tau, s_1, s). \tag{247}$$

From (245) for $m = 1$ and $m = 2$ we have

$$|F_p(\tau, s_1, s)| <$$

$$< \sum_{j_1=p+1}^{\infty}\frac{K_1}{(j_1)^{7/4}}\left(\frac{1}{(1-z^2(s))^{1/8}} + \frac{1}{(1-z^2(s_1))^{1/8}}\right)\times$$

$$\times\left(\frac{1}{(1-z^2(s))^{1/4}}+\frac{1}{(1-z^2(\tau))^{1/4}}\right)\le$$

$$\le\frac{K_2}{p^{3/4}}\left(\frac{1}{(1-z^2(s))^{1/8}}+\frac{1}{(1-z^2(s_1))^{1/8}}\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+\frac{1}{(1-z^2(\tau))^{1/4}}\right), \tag{248}$$

where $s, s_1, \tau\in(t,T)$, constant K_2 is independent of p and we used the estimate (409) in (248).

The relations (246) and (248) imply the estimate

$$\sum_{j_2,j_4=0}^{p}\left(g_{j_4j_2}^p(s)\right)^2\le\frac{C_1(s)}{p^{3/2}} \tag{249}$$

for the case $s\in(t,T)$ or $z(s)\in(-1,1)$ (the case $s=T$ has already been considered in Theorem 4), where constant $C_1(s)$ (s is fixed) does not depend on p.

Then from analogue of (169) for $s\in(t,T)$ (s is fixed), (244), and (249) we have

$$\mathsf{M}\left\{\left(\sum_{j_2,j_4=0}^{p}g_{j_4j_2}^p(s)\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\left(1+\mathbf{1}_{\{i_2=i_4\ne 0\}}\right)\sum_{j_2,j_4=0}^{p}\left(g_{j_4j_2}^p(s)\right)^2+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\sum_{j_4=0}^{p}g_{j_4j_4}^p(s)\right)^2\le\frac{C_2(s)}{p^{3/2}}\ \to 0$$

if $p\to\infty$, where constant $C_2(s)$ (s is fixed) does not depend on p. The equality (240) is proved.

Let us consider $\Delta_5^{(i_1i_3)}(s)$

$$\Delta_5^{(i_1i_3)}(s)=\Delta_4^{(i_1i_3)}(s)+\Delta_6^{(i_1i_3)}(s)-\Delta_8^{(i_1i_3)}(s)\quad \text{w. p. 1},$$

where

$$\Delta_8^{(i_1i_3)}(s)=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3,j_1=0}^{p}h_{j_3j_1}^p(s)\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$h_{j_3j_1}^p(s)=\int\limits_t^s\phi_{j_1}(s_3)\int\limits_{s_3}^s\phi_{j_3}(\tau)F_p(s_3,\tau,s)d\tau ds_3,$$

where $F_p(s_3,\tau,s)$ is defined by (247).

Analogously to (240), we obtain that $\Delta_8^{(i_1i_3)}(s)=0$ w. p. 1. In this case we consider the function

$$K_p(s_3,\tau,s)=\mathbf{1}_{\{s_3<s\}}\mathbf{1}_{\{s_3<\tau<s\}}F_p(s_3,\tau,s)$$

and the relation

$$h_{j_3j_1}^p(s)=\int\limits_{[t,T]^2}K_p(s_3,\tau,s)\phi_{j_1}(s_3)\phi_{j_3}(\tau)d\tau ds_3.$$

For the case $i_1 = i_3 \neq 0$ we use (see (242))

$$h^p_{j_1 j_1}(s) = \sum_{j_4=p+1}^{\infty} \frac{1}{2} \left(\int\limits_t^s \phi_{j_1}(\tau) \int\limits_\tau^s \phi_{j_4}(s_1) ds_1 d\tau \right)^2 .$$

Let us prove that

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} c^p_{j_3 j_3}(s) = 0. \tag{250}$$

We have

$$c^p_{j_3 j_3}(s) = f^p_{j_3 j_3}(s) + d^p_{j_3 j_3}(s) - g^p_{j_3 j_3}(s). \tag{251}$$

Moreover,

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} f^p_{j_3 j_3}(s) = 0, \quad \lim_{p\to\infty} \sum_{j_3=0}^{p} d^p_{j_3 j_3}(s) = 0, \tag{252}$$

where the first equality in (252) has been proved earlier. Analogously, we can prove the second equality in (252).

From (244) we obtain

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} g^p_{j_3 j_3}(s) = 0.$$

So, (250) is proved. The relation (236) is proved for the polynomial case. Theorem 10 is proved for the case of Legendre polynomials.

It is easy to see that the trigonometric case is considered by analogy with the case of Legendre polynomials using the estimate

$$\left| \int\limits_\tau^s \phi_j(\theta) d\theta \right| \le \frac{C}{j} \quad (j \neq 0),$$

where constant C is independent of p, $t \le \tau < s \le T$, and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t, T])$. Theorem 10 is proved.

Let us reformulate Theorem 10 in terms on the convergence of solution of system of ODEs to the solution of system of Stratonovich SDEs.

By analogy with (175) for the case $k = 4$, $p_1 = \ldots = p_4 = p$, $i_1, \ldots, i_4 = 0, 1, \ldots, m$, and $s \in (t, T]$ (s is fixed) we obtain

$$\int\limits_t^s \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}^{(i_1)p}_{t_1} d\mathbf{w}^{(i_2)p}_{t_2} d\mathbf{w}^{(i_3)p}_{t_3} d\mathbf{w}^{(i_4)p}_{t_4} = \sum_{j_1, j_2, j_3, j_4 = 0}^{p} C_{j_4 j_3 j_2 j_1}(s) \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4},$$

where $p \in \mathbb{N}$ and $d\mathbf{w}^{(i)p}_\tau$ is defined by (174); another notations are the same as in Theorem 10.

The iterated Riemann–Stiltjes integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)p} = \int_t^s \int_t^{t_4} \int_t^{t_3} \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p} d\mathbf{w}_{t_4}^{(i_4)p}, \tag{253}$$

$$Z_{s,t}^{(i_1 i_2 i_3)p} = \int_t^s \int_t^{t_3} \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p} d\mathbf{w}_{t_3}^{(i_3)p}, \tag{254}$$

$$Y_{s,t}^{(i_1 i_2)p} = \int_t^s \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} d\mathbf{w}_{t_2}^{(i_2)p}, \tag{255}$$

$$X_{s,t}^{(i_1)p} = \int_t^s d\mathbf{w}_{t_1}^{(i_1)p} \tag{256}$$

are the solution of the following system of ODEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)p} = Z_{s,t}^{(i_1 i_2 i_3)p} d\mathbf{w}_s^{(i_4)p}, & V_{t,t}^{(i_1 i_2 i_3 i_4)p} = 0 \\ \\ dZ_{s,t}^{(i_1 i_2 i_3)p} = Y_{s,t}^{(i_1 i_2)p} d\mathbf{w}_s^{(i_3)p}, & Z_{t,t}^{(i_1 i_2 i_3)p} = 0 \\ \\ dY_{s,t}^{(i_1 i_2)p} = X_{s,t}^{(i_1)p} d\mathbf{w}_s^{(i_2)p}, & Y_{t,t}^{(i_1 i_2)p} = 0 \\ \\ dX_{s,t}^{(i_1)p} = 1 \cdot d\mathbf{w}_s^{(i_1)p}, & X_{t,t}^{(i_1)p} = 0 \end{cases}.$$

From the other hand, the iterated Stratonovich stochastic integrals

$$V_{s,t}^{(i_1 i_2 i_3 i_4)} = \int_t^{*s} \int_t^{*t_4} \int_t^{*t_3} \int_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)}, \tag{257}$$

$$Z_{s,t}^{(i_1 i_2 i_3)} = \int_t^{*s} \int_t^{*t_3} \int_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}, \tag{258}$$

$$Y_{s,t}^{(i_1 i_2)} = \int_t^{*s} \int_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \tag{259}$$

$$X_{s,t}^{(i_1)} = \int_t^{*s} d\mathbf{w}_{t_1}^{(i_1)} \tag{260}$$

are the solution of the following system of Stratonovich SDEs

$$\begin{cases} dV_{s,t}^{(i_1 i_2 i_3 i_4)} = Z_{s,t}^{(i_1 i_2 i_3)} * d\mathbf{w}_s^{(i_4)}, & V_{t,t}^{(i_1 i_2 i_3 i_4)} = 0 \\ \\ dZ_{s,t}^{(i_1 i_2 i_3)} = Y_{s,t}^{(i_1 i_2)} * d\mathbf{w}_s^{(i_3)}, & Z_{t,t}^{(i_1 i_2 i_3)} = 0 \\ \\ dY_{s,t}^{(i_1 i_2)} = X_{s,t}^{(i_1)} * d\mathbf{w}_s^{(i_2)}, & Y_{t,t}^{(i_1 i_2)} = 0 \\ \\ dX_{s,t}^{(i_1)} = 1 * d\mathbf{w}_s^{(i_1)}, & X_{t,t}^{(i_1)} = 0 \end{cases},$$

where $* d\mathbf{w}_s^{(i)}$, $i = 0, 1, \ldots, m$ is the Stratonovich differential.

Then from Theorems 7, 9, and 10 we obtain the following theorem.

Theorem 11 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then for any fixed* s $(s \in (t, T])$

$$\underset{p\to\infty}{\text{l.i.m.}}\ V_{s,t}^{(i_1 i_2 i_3 i_4)p} = V_{s,t}^{(i_1 i_2 i_3 i_4)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ Z_{s,t}^{(i_1 i_2 i_3)p} = Z_{s,t}^{(i_1 i_2 i_3)},$$

$$\underset{p\to\infty}{\text{l.i.m.}}\ Y_{s,t}^{(i_1 i_2)p} = Y_{s,t}^{(i_1 i_2)}, \quad \underset{p\to\infty}{\text{l.i.m.}}\ X_{s,t}^{(i_1)p} = X_{s,t}^{(i_1)}.$$

11. Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4 in Theorems 2–4

Let us prove the following Theorem.

Theorem 12 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Moreover,* $\psi_1(\tau), \psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C}{p} \tag{261}$$

is valid, where constant C *is independent of* p,

$$C_{j_2 j_1} = \int\limits_t^T \psi_2(s_2)\phi_{j_2}(s_2) \int\limits_t^{s_2} \psi_1(s_1)\phi_{j_1}(s_1) ds_1 ds_2,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Proof. From (15) we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \mathsf{M}\Biggl\{\Biggl(J[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right) +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{p} C_{j_1 j_1}\Biggr)^2\Biggr\} =$$

$$= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\} +$$

$$+\left(\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{p} C_{j_1 j_1}\right)^2 =$$

$$= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{p,p}\right)^2\right\} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1 j_1}\right)^2, \tag{262}$$

where (see (10))

$$J[\psi^{(2)}]_{T,t}^{p,p} = \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right).$$

In [26] (Sect. 1.7.2, Remark 1.7) it is shown that

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\}\le\frac{k!P_k(T-t)^k}{p},\tag{263}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2), $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is the expression on the right-hand side of (7) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=p$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$, $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, constant P_k depends only on k, $i_1,\ldots,i_k=1,\ldots,m$.

From (263) we get

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^{p,p}\right)^2\right\}\le\frac{C_1}{p},\tag{264}$$

where constant C_1 is independent of p.

Using (53), we obtain

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p}C_{j_1j_1}=\sum_{j_1=p+1}^{\infty}C_{j_1j_1}.\tag{265}$$

The estimate (43) implies that (polynomial case)

$$\left|\sum_{j_1=p+1}^{\infty}C_{j_1j_1}\right|\le C_2\left(\frac{1}{p}+\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\right),\tag{266}$$

where constant C_2 does not depend on p.

From (31) and (266) we have

$$S_p\stackrel{\mathsf{def}}{=}\left|\sum_{j_1=p+1}^{\infty}C_{j_1j_1}\right|\le\frac{C_3}{p},\tag{267}$$

where constant C_3 is independent of p.

Applying the ideas that we used to obtain the relations (45), (49), (50), we can prove the following estimates for the trigonometric case

$$S_{2p}=\left|\sum_{j_1=2p+1}^{\infty}C_{j_1j_1}\right|\le\frac{K_1}{p},\tag{268}$$

$$S_{2p-1}=\left|\sum_{j_1=2p}^{\infty}C_{j_1j_1}\right|\le S_{2p}+\frac{K_2}{p},\tag{269}$$

where constants K_1,K_2 do not depend on p.

From (268) and (269) we get the estimate (267) for the trigonometric case. Combining (262), (264), (265), and (267), we obtain (261). Theorem 12 is proved.

Let us consider an analogue of Theorem 12 for iterated Stratonovich stochastic integrals of multiplicity 3.

Theorem 13 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable nonrandom function on* $[t,T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p}\qquad (270)$$

is valid, where constant C *is independent of* p,

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_2}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds,$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. Using standard relations between Stratonovich and Ito stochastic integrals, we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}+\right.\right.$$

$$\left.\left.+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3\;-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}+\right.\right.$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\Biggr)^2\Biggr\}, \tag{271}
$$

where (see (11))

$$
J[\psi^{(3)}]_{T,t}^{p,p,p}=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr).
$$

From (271) and the elementary inequality $(a+b+c+d)^2\le 4\left(a^2+b^2+c^2+d^2\right)$ we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le 4\left(\mathsf{M}\left\{\left(J[\psi^{(3)}]_{s,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}\right)^2\right\}+\mathbf{1}_{\{i_1=i_2\}}E_p^{(1)}+\mathbf{1}_{\{i_2=i_3\}}E_p^{(2)}+\mathbf{1}_{\{i_1=i_3\}}E_p^{(3)}\right), \tag{272}
$$

where

$$
E_p^{(1)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},
$$

$$
E_p^{(2)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3-\sum_{j_1,j_3=0}^{p}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\},
$$

$$
E_p^{(3)}=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\}.
$$

From (263) we have

(273)
$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}-J[\psi^{(3)}]_{T,t}^{p,p,p}\right)^2\right\}\le\frac{C_1}{p},$$

where constant C_1 is independent of p.

Moreover, from (90) and (98) we have the following estimate

(274)
$$E_p^{(3)}\le\frac{C_2}{p}$$

for the polynomial and trigonometric cases, where constant C_2 does not depend on p.

Using Theorem 1 for $k=1$ (also see (9)), we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi_3(s)\int\limits_t^s\psi_2(s_1)\psi_1(s_1)ds_1d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde C_{j_3}=\int\limits_t^T\phi_{j_3}(s)\psi_3(s)\int\limits_t^s\psi_2(s_1)\psi_1(s_1)ds_1ds.$$

Applying the Ito formula, we have

$$\int\limits_t^T\psi_3(s)\psi_2(s)\int\limits_t^s\psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds=\int\limits_t^T\psi_1(s_1)\int\limits_{s_1}^T\psi_3(s)\psi_2(s)dsd\mathbf{f}_{s_1}^{(i_1)}\quad\text{w. p. 1.}$$

Using Theorem 1 for $k=1$, we have w. p. 1

$$\frac{1}{2}\int\limits_t^T\psi_1(s)\int\limits_s^T\psi_3(s_1)\psi_2(s_1)ds_1d\mathbf{f}_s^{(i_1)}=\frac{1}{2}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}C^*_{j_1}\zeta_{j_1}^{(i_1)},$$

where

$$C^*_{j_1}=\int\limits_t^T\psi_1(s)\phi_{j_1}(s)\int\limits_s^T\psi_3(s_1)\psi_2(s_1)ds_1ds.$$

Further, we get

(275)
$$E_p^{(1)}\le 2G_p^{(1)}+2G_p^{(2)},$$

(276)
$$E_p^{(2)}\le 2H_p^{(1)}+2H_p^{(2)},$$

where

$$G_p^{(1)}=\mathsf{M}\left\{\frac{1}{4}\left(\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}-\sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$G_p^{(2)} = \mathsf{M}\left\{\left(\frac{1}{2}\sum_{j_3=0}^{p}\tilde C_{j_3}\zeta_{j_3}^{(i_3)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$H_p^{(1)} = \mathsf{M}\left\{\frac{1}{4}\left(\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3 - \sum_{j_1=0}^{p} C_{j_1}^{*}\zeta_{j_1}^{(i_1)}\right)^2\right\},$$

$$H_p^{(2)} = \mathsf{M}\left\{\left(\frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^{*}\zeta_{j_1}^{(i_1)} - \sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}.$$

From (263) we have

$$G_p^{(1)} \le \frac{C_2}{p}, \quad H_p^{(1)} \le \frac{C_2}{p}, \tag{277}$$

where constant C_2 is independent of p.

The estimates

$$G_p^{(2)} \le \frac{C_3}{p}, \quad H_p^{(2)} \le \frac{C_3}{p} \tag{278}$$

are proved in Sect. 4 (see the proof of Theorem 3) for the polynomial and trigonometric cases; constant C_3 does not depend on p. Combining the estimates (272)–(278), we obtain the inequality (270). Theorem 13 is proved.

Consider an analogue of Theorem 13 for iterated Stratonovich stochastic integrals of fourth multiplicity.

Theorem 14 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p} \tag{279}$$

is valid, where constant C is independent of p,

$$C_{j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(s_4)\int\limits_t^{s_4}\phi_{j_3}(s_3)\int\limits_t^{s_3}\phi_{j_2}(s_2)\int\limits_t^{s_2}\phi_{j_1}(s_1)ds_1ds_2ds_3ds_4,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. First, let us prove that Theorem 3 is valid for the case $i_1, i_2, i_3 = 0, 1, \ldots, m$. The case $i_1, i_2, i_3 = 1, \ldots, m$ has been proved in Theorem 3. From (11) and the standard relation between Stratonovich and Ito stochastic integrals (2), (3) of third multiplicity we have that Theorem 3 is valid for the following cases

$$i_1 = i_2 = 0, \quad i_3 = 1, \ldots, m,$$

$$i_1 = i_3 = 0, \quad i_2 = 1, \ldots, m,$$

$$i_2 = i_3 = 0, \quad i_1 = 1, \ldots, m.$$

Thus, it remains to consider the following three cases

$$i_1,\ i_2 = 1, \ldots, m, \quad i_3 = 0, \tag{280}$$

$$i_2,\ i_3 = 1, \ldots, m, \quad i_1 = 0, \tag{281}$$

$$i_1,\ i_3 = 1, \ldots, m, \quad i_2 = 0. \tag{282}$$

The relation (11) and the standard relation between Stratonovich and Ito stochastic integrals (2), (3) of third multiplicity imply that for the case (280) we need to prove the following equality

$$\sum_{j_1=0}^{\infty} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1) dt_1 dt_2 dt_3 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_1(t_1)\psi_2(t_1) dt_1 dt_3. \tag{283}$$

Using the relation (17), we get

$$\sum_{j_1=0}^{\infty} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1) dt_1 dt_2 dt_3 =$$

$$= \sum_{j_1=0}^{\infty} \int\limits_t^T \phi_{j_1}(t_1)\psi_1(t_1) \int\limits_{t_1}^T \phi_{j_1}(t_2)\psi_2(t_2) \int\limits_{t_2}^T \psi_3(t_3) dt_3 dt_2 dt_1 =$$

$$= \sum_{j_1=0}^{\infty} \int\limits_t^T \phi_{j_1}(t_1)\psi_1(t_1) \int\limits_{t_1}^T \phi_{j_1}(t_2)\tilde{\psi}_2(t_2) dt_2 dt_1 =$$

$$= \sum_{j_1=0}^{\infty} \int\limits_t^T \phi_{j_1}(t_2)\tilde{\psi}_2(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_1(t_2)\tilde{\psi}_2(t_2)dt_2, \tag{284}$$

where

$$\tilde{\psi}_2(t_2) = \psi_2(t_2) \int\limits_{t_2}^T \psi_3(t_3)dt_3. \tag{285}$$

From (284) and (285) we obtain

$$\sum_{j_1=0}^{\infty} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_1(t_2)\psi_2(t_2) \int\limits_{t_2}^T \psi_3(t_3)dt_3dt_2 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_1(t_2)\psi_2(t_2)dt_2dt_3. \tag{286}$$

The relation (283) is proved.

From (11) and the standard relation between Stratonovich and Ito stochastic integrals (2), (3) of third multiplicity it follows that for the case (281) we need to prove the following equality

$$\sum_{j_2=0}^{\infty} \int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_3(t_3)\psi_2(t_3) \int\limits_t^{t_3} \psi_1(t_1)dt_1dt_3. \tag{287}$$

Using the relation (17), we have

$$\sum_{j_2=0}^{\infty} \int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2)\psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \sum_{j_2=0}^{\infty} \int\limits_t^T \phi_{j_2}(t_3)\psi_3(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2)\bar{\psi}_2(t_2)dt_2dt_3 =$$

$$=\frac{1}{2}\int\limits_t^T\psi_3(t_3)\bar\psi_2(t_3)dt_3,$$

where

$$\bar\psi_2(t_2)=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1. \qquad (288)$$

The relation (287) is proved.

The relation (11) and the standard relation between Stratonovich and Ito stochastic integrals (2), (3) of third multiplicity imply that for the case (282) we need to prove the following equality

$$\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3=0. \qquad (289)$$

We have

$$\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3=$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{t_3}\psi_2(t_2)dt_2dt_1dt_3=$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\left(\int\limits_{t_1}^{T}\psi_2(t_2)dt_2-\int\limits_{t_3}^{T}\psi_2(t_2)dt_2\right)dt_1dt_3=$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)dt_2dt_1dt_3-$$

$$-\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)\int\limits_{t_3}^{T}\psi_2(t_2)dt_2dt_1dt_3=$$

$$=\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde\psi_1(t_1)dt_1dt_3-$$

$$-\sum_{j_1=0}^{\infty}\int\limits_t^T\phi_{j_1}(t_3)\tilde\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3=$$

$$=\frac{1}{2}\int\limits_t^T\psi_3(t_1)\tilde\psi_1(t_1)dt_1-\frac{1}{2}\int\limits_t^T\tilde\psi_3(t_1)\psi_1(t_1)dt_1=$$

$$
=\frac{1}{2}\int\limits_t^T \psi_3(t_1)\psi_1(t_1)\int\limits_{t_1}^T \psi_2(t_2)dt_2dt_1-\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_3(t_1)\int\limits_{t_1}^T \psi_2(t_2)dt_2dt_1=0,
$$

where

$$
\tilde{\psi}_1(t_1)=\psi_1(t_1)\int\limits_{t_1}^T \psi_2(t_2)dt_2, \tag{290}
$$

$$
\tilde{\psi}_3(t_3)=\psi_3(t_3)\int\limits_{t_3}^T \psi_2(t_2)dt_2. \tag{291}
$$

The relation (289) is proved. Theorem 3 is proved for the case $i_1,i_2,i_3=0,1,\ldots,m.$

Using standard relations between Ito and Stratonovich stochastic integrals (2), (3) of multiplicities 3 and 4, we have

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^s\int\limits_t^{s_1} ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)}+\right.\right.
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{s_2}\int\limits_t^{s_1} d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}\int\limits_t^{s_2} d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1+
$$

$$
\left.\left.+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1} ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_{s}^{(i_4)}-\right.\right.
$$

$$
-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2ds_1+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1} d\mathbf{w}_{s}^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+
$$

$$
+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2} d\mathbf{w}_{s}^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1} ds_2ds_1+
$$

$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\Biggr)^2\Biggr\}=$$

$$=\mathsf{M}\Biggl\{\Biggl(J[\psi^{(4)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-$$

$$-\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\Biggr)^2\Biggr\}=$$

$$=\mathsf{M}\Biggl\{\Biggl(J[\psi^{(4)}]_{T,t}-J[\psi^{(4)}]_{T,t}^{p,p,p,p}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}-S_1^{(i_3i_4)p}\right)+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-S_2^{(i_1i_4)p}\right)+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-S_3^{(i_1i_2)p}\right)-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\Biggl(\frac{1}{4}\int\limits_t^T\int\limits_t^{s_1}ds_2ds_1-$$

$$-\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_1)(s_1-t)ds_1ds\Biggr)-R_p\Biggr)^2\Biggr\},\tag{292}$$

where $S_1^{(i_3i_4)p}$, $S_2^{(i_1i_4)p}$, $S_3^{(i_1i_2)p}$ are the approximations of the iterated Stratonovich stochastic integrals

$$
\int\limits_t^{*T}\int\limits_t^{*s}\int\limits_t^{*s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)}, \quad \int\limits_t^{*T}\int\limits_t^{*s_2}\int\limits_t^{*s_1} d\mathbf{w}_s^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)}, \quad \int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2} d\mathbf{w}_s^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1,
$$

respectively (these approximations are obtained by the version of Theorem 3 for the case $i_1, i_2, i_3 = 0, 1, \ldots, m$); $J[\psi^{(4)}]_{T,t}^{p,p,p,p}$ is the approximation of the iterated Ito stochastic integral $J[\psi^{(4)}]_{T,t}$ obtained by Theorem 1 (see (12))

$$
J[\psi^{(4)}]_{T,t}^{p,p,p,p} = \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\neq 0\}} A_1^{(i_3 i_4)p} - \mathbf{1}_{\{i_1=i_3\neq 0\}} A_2^{(i_2 i_4)p} - \mathbf{1}_{\{i_1=i_4\neq 0\}} A_3^{(i_2 i_3)p} - \mathbf{1}_{\{i_2=i_3\neq 0\}} A_4^{(i_1 i_4)p} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\neq 0\}} A_5^{(i_1 i_3)p} - \mathbf{1}_{\{i_3=i_4\neq 0\}} A_6^{(i_1 i_2)p} + \mathbf{1}_{\{i_1=i_2\neq 0\}} \mathbf{1}_{\{i_3=i_4\neq 0\}} B_1^p +
$$

$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}} \mathbf{1}_{\{i_2=i_4\neq 0\}} B_2^p + \mathbf{1}_{\{i_1=i_4\neq 0\}} \mathbf{1}_{\{i_2=i_3\neq 0\}} B_3^p,
$$

where

$$
A_1^{(i_3 i_4)p} = \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad A_2^{(i_2 i_4)p} = \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_3} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)},
$$

$$
A_3^{(i_2 i_3)p} = \sum_{j_4,j_3,j_2=0}^{p} C_{j_4 j_3 j_2 j_4} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad A_4^{(i_1 i_4)p} = \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)},
$$

$$
A_5^{(i_1 i_3)p} = \sum_{j_4,j_3,j_1=0}^{p} C_{j_4 j_3 j_4 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \quad A_6^{(i_1 i_2)p} = \sum_{j_3,j_2,j_1=0}^{p} C_{j_3 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},
$$

$$
B_1^p = \sum_{j_1,j_4=0}^{p} C_{j_4 j_4 j_1 j_1}, \quad B_2^p = \sum_{j_4,j_3=0}^{p} C_{j_3 j_4 j_3 j_4},
$$

$$
B_3^p = \sum_{j_4,j_3=0}^{p} C_{j_4 j_3 j_3 j_4};
$$

R_p is the expression on the right-hand side of (119) before passing to the limits, i.e.

$$
R_p = -\mathbf{1}_{\{i_1=i_2\neq 0\}} \Delta_1^{(i_3 i_4)p} + \mathbf{1}_{\{i_1=i_3\neq 0\}} \left(-\Delta_2^{(i_2 i_4)p} + \Delta_1^{(i_2 i_4)p} + \Delta_3^{(i_2 i_4)p}\right) +
$$

$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}} \left(\Delta_4^{(i_2 i_3)p} - \Delta_5^{(i_2 i_3)p} + \Delta_6^{(i_2 i_3)p}\right) - \mathbf{1}_{\{i_2=i_3\neq 0\}} \Delta_3^{(i_1 i_4)p} +
$$

$$
+\mathbf{1}_{\{i_2=i_4\neq 0\}} \left(-\Delta_4^{(i_1 i_3)p} + \Delta_5^{(i_1 i_3)p} + \Delta_6^{(i_1 i_3)p}\right) - \mathbf{1}_{\{i_3=i_4\neq 0\}} \Delta_6^{(i_1 i_2)p} -
$$

$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\left(\sum_{j_3=0}^{p}a^p_{j_3j_3}+\sum_{j_3=0}^{p}c^p_{j_3j_3}-\sum_{j_3=0}^{p}b^p_{j_3j_3}\right)-$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\left(2\sum_{j_3=0}^{p}f^p_{j_3j_3}-\sum_{j_3=0}^{p}a^p_{j_3j_3}-\sum_{j_3=0}^{p}c^p_{j_3j_3}+\sum_{j_3=0}^{p}b^p_{j_3j_3}\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\sum_{j_3=0}^{p}a^p_{j_3j_3},$$

where

$$\Delta_1^{(i_3i_4)p}=\sum_{j_3,j_4=0}^{p}a^p_{j_4j_3}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},\quad \Delta_2^{(i_2i_4)p}=\sum_{j_4,j_2=0}^{p}b^p_{j_4j_2}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},$$

$$\Delta_3^{(i_2i_4)p}=\sum_{j_4,j_2=0}^{p}c^p_{j_4j_2}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)},\quad \Delta_4^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p}d^p_{j_3j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

$$\Delta_5^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p}e^p_{j_3j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},\quad \Delta_6^{(i_1i_3)p}=\sum_{j_3,j_1=0}^{p}f^p_{j_3j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)},$$

where $a^p_{j_4j_3}$, $b^p_{j_4j_2}$, $c^p_{j_4j_2}$, $d^p_{j_3j_1}$, $e^p_{j_3j_1}$, $f^p_{j_3j_1}$ are defined by the relations (102), (104), (105), (107)–(109).

From (292) and the elementary inequality $(a_1+\ldots+a_6)^2\le 6\left(a_1^2+\ldots+a_6^2\right)$ we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le$$

$$\le 6\left(Q_p^{(1)}+Q_p^{(2)}+Q_p^{(3)}+Q_p^{(4)}+Q_p^{(5)}+Q_p^{(6)}\right), \tag{293}$$

where

$$Q_p^{(1)}=\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}-J[\psi^{(4)}]_{T,t}^{p,p,p,p}\right)^3\right\},$$

$$Q_p^{(2)}=\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathsf{M}\left\{\left(\int_t^{*T}\int_t^{*s}\int_t^{*s_1}ds_2d\mathbf{w}_{s_1}^{(i_3)}d\mathbf{w}_s^{(i_4)}-S_1^{(i_3i_4)p}\right)^2\right\},$$

$$Q_p^{(3)}=\frac{1}{4}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathsf{M}\left\{\left(\int_t^{*T}\int_t^{*s_2}\int_t^{*s_1}d\mathbf{w}_s^{(i_1)}ds_1d\mathbf{w}_{s_2}^{(i_4)}-S_2^{(i_1i_4)p}\right)^2\right\},$$

$$Q_p^{(4)}=\frac{1}{4}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathsf{M}\left\{\left(\int\limits_t^{*T}\int\limits_t^{*s_1}\int\limits_t^{*s_2}d\mathbf{w}_s^{(i_1)}d\mathbf{w}_{s_2}^{(i_2)}ds_1-S_3^{(i_1 i_2)p}\right)^2\right\},$$

$$Q_p^{(5)}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\times$$

$$\times\left(\frac{1}{4}\int\limits_t^T(s_1-t)ds_1-\sum_{j_4=0}^{p}\frac{1}{2}\int\limits_t^T\phi_{j_4}(s)\int\limits_t^s\phi_{j_4}(s_1)(s_1-t)ds_1ds\right)^2,$$

$$Q_p^{(6)}=\mathsf{M}\left\{(R_p)^2\right\}.$$

From (263) we have

$$Q_p^{(1)}\le\frac{C_1}{p}, \tag{294}$$

where constant C_1 is independent of p.

Let us prove the version of Theorem 13 for the case $i_1,i_2,i_3=0,1,\ldots,m$. The case $i_1,i_2,i_3=1,\ldots,m$ has been proved in Theorem 13. It is easy to see that, in addition to the proof of Theorem 13, we need to prove the following inequalities

$$\left|\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_1)\psi_2(t_1)dt_1dt_3-\right.$$

$$\left.-\sum_{j_1=0}^{p}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3\right|\le\frac{C}{p}, \tag{295}$$

$$\left|\frac{1}{2}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)dt_1dt_3-\right.$$

$$\left.-\sum_{j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\psi_3(t_2)\int\limits_t^{t_2}\psi_1(t_1)dt_1dt_2dt_3\right|\le\frac{C}{p}, \tag{296}$$

$$\left|\sum_{j_1=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3\right|\le\frac{C}{p}, \tag{297}$$

where constant C is independent of p.

The inequalities (295) and (296) are equivalent to the following inequalities (see the proof of the cases (280), (281))

$$\left|\frac{1}{2}\int\limits_t^T \psi_1(t_2)\tilde{\psi}_2(t_2)dt_2 - \sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_2)\tilde{\psi}_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2\right| \le \frac{C}{p}, \tag{298}$$

$$\left|\frac{1}{2}\int\limits_t^T \psi_3(t_3)\bar{\psi}_2(t_3)dt_3 - \sum_{j_3=0}^{p}\int\limits_t^T \phi_{j_3}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\bar{\psi}_2(t_2)dt_2dt_3\right| \le \frac{C}{p}, \tag{299}$$

where $\tilde{\psi}_2(t_2)$, $\bar{\psi}_2(t_2)$ are defined by (285) and (288), respectively. The inequalities (298), (299) follow from (265), (267)–(269).

Let us prove (297). By analogy with the proof of (289) we have

$$\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_2dt_3 =$$

$$= \sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3 -$$

$$- \sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3 =$$

$$= \sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3 -$$

$$- \sum_{j_1=0}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3 -$$

$$- \sum_{j_1=p+1}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3 +$$

$$+ \sum_{j_1=p+1}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3 =$$

$$= - \sum_{j_1=p+1}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\psi_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\tilde{\psi}_1(t_1)dt_1dt_3 +$$

$$+ \sum_{j_1=p+1}^{\infty}\int\limits_t^T \phi_{j_1}(t_3)\tilde{\psi}_3(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\psi_1(t_1)dt_1dt_3, \tag{300}$$

where $\tilde{\psi}_1(t_1)$, $\tilde{\psi}_3(t_3)$ are defined by (290), (291), respectively.

Now the estimate (297) follows from (300) and (267)–(269). Theorem 13 is proved for the case $i_1, i_2, i_3 = 0, 1, \ldots, m$.

Using the version of Theorem 13 for the case $i_1, i_2, i_3 = 0, 1, \ldots, m$, we obtain the following estimates

$$Q_p^{(2)} \le \frac{C_2}{p}, \quad Q_p^{(3)} \le \frac{C_2}{p}, \quad Q_p^{(4)} \le \frac{C_2}{p}, \tag{301}$$

where constant C_2 does not depend on p.

From (265) we get

$$\frac{1}{2}\int\limits_t^T (s_1 - t)ds_1 - \sum_{j_4=0}^{p} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_4}(s_1)(s_1 - t)ds_1 ds =$$

$$= \sum_{j_4=p+1}^{\infty} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_4}(s_1)(s_1 - t)ds_1 ds. \tag{302}$$

Let us consider the case of Legendre polynomials. From (267) and (302) we have

$$\left| \sum_{j_4=p+1}^{\infty} \int\limits_t^T \phi_{j_4}(s) \int\limits_t^s \phi_{j_4}(s_1)(s_1 - t)ds_1 ds \right| \le \frac{C_3}{p}, \tag{303}$$

where constant C_3 is independent of p.

For the trigonometric case, the analogue of the inequality (303) can be obtained by analogy with (268) and (269). Then

$$Q_p^{(5)} \le \frac{C_4}{p^2}, \tag{304}$$

where constant C_4 does not depend on p.

Analyzing the proof of Theorem 4, we conclude that

$$Q_p^{(6)} \le \frac{C_5}{p} \tag{305}$$

for the polynomial and trigonometric cases; constant C_5 is independent of p. Combining (293)–(301), (304), (305), we get (279). Theorem 14 is proved.

12. Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4 in Modifications of Theorems 12-14 for the Case of Integration Interval $[t, s]$ $(s \in (t, T])$

Let us prove the following theorem.

Theorem 15 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover,* $\psi_1(\tau),\psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{s,t} = \int\limits_t^{*s} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{s,t} - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C(s)}{p} \tag{306}$$

is valid, where $s\in(t,T]$ $(s$ *is fixed*$)$, *constant* $C(s)$ *is independent of* p,

$$C_{j_2j_1}(s) = \int\limits_t^{s} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(\tau)d\mathbf{f}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Proof. The case $s=T$ has already been considered in Theorem 12. Below we consider the case $s\in(t,T)$. By analogy with (262) we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{s,t} - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(J[\psi^{(2)}]_{s,t} - J[\psi^{(2)}]_{s,t}^{p,p}\right)^2\right\} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^{s} \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1j_1}(s)\right)^2, \tag{307}$$

where (see (188))

$$J[\psi^{(2)}]_{s,t}^{p,p} = \sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s)\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right).$$

In [26] (Sect. 1.8) it is shown that

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}-J[\psi^{(k)}]_{s,t}^{p,\ldots,p}\right)^2\right\}\le\frac{k!P_k(s-t)^k}{p}, \tag{308}
$$

where $s\in(t,T]$ (s is fixed), $J[\psi^{(k)}]_{s,t}$ is defined by (185), $J[\psi^{(k)}]_{s,t}^{p,\ldots,p}$ is the expression on the right-hand side of (186) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=p$, $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, constant P_k depends only on k, $i_1,\ldots,i_k=1,\ldots,m$.

From (308) we get

$$
\mathsf{M}\left\{\left(J[\psi^{(2)}]_{s,t}-J[\psi^{(2)}]_{s,t}^{p,p}\right)^2\right\}\le\frac{C_1(s)}{p}, \tag{309}
$$

where constant $C_1(s)$ is independent of p.

Using (195), we obtain

$$
\frac{1}{2}\int\limits_t^s\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p}C_{j_1j_1}(s)=\sum_{j_1=p+1}^{\infty}C_{j_1j_1}(s). \tag{310}
$$

Consider the case of Legendre polynomials. By analogy with (42) we get for $n>m$ ($n,m\in\mathbb{N}$)

$$
\sum_{j_1=m+1}^{n}C_{j_1j_1}(s)=\sum_{j_1=m+1}^{n}\int\limits_t^s\psi_2(\theta)\phi_{j_1}(\theta)\int\limits_t^{\theta}\psi_1(\tau)\phi_{j_1}(\tau)d\tau d\theta=
$$

$$
=\frac{T-t}{4}\int\limits_{-1}^{z(s)}\psi_1(u(x))\psi_2(u(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx-
$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\frac{1}{2j_1+1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(u(y))\times
$$

$$
\times\Biggl(\left(P_{j_1+1}(z(s))-P_{j_1-1}(z(s))\right)\psi_2(s)-\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_2(u(y))-
$$

$$
-\frac{T-t}{2}\int\limits_{y}^{z(s)}\left(P_{j_1+1}(x)-P_{j_1-1}(x)\right)\psi_2'(u(x))dx\Biggr)dy, \tag{311}
$$

where

$$
u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t},
$$

and ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $u(y)$.

Applying the estimate (29) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we finally obtain

$$\left|\sum_{j_1=m+1}^{n} C_{j_1j_1}(s)\right| \le C_1\left(\frac{1}{n}+\frac{1}{m}\right)\int\limits_{-1}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/2}}+$$

$$+C_2\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/2}}+\frac{1}{\left(1-z^2(s)\right)^{1/4}}\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/4}}+\right.$$

$$\left.+\int\limits_{-1}^{z(s)}\frac{1}{\left(1-y^2\right)^{1/4}}\int\limits_{y}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/4}}dy\right), \tag{312}$$

where constants C_1, C_2 do not depend on n and m.

We assume that $s \in (t,T)$ $(z(s) \ne \pm 1)$ since the case $s = T$ has already been considered in Theorem 12. Then

$$\left|\sum_{j_1=m+1}^{n} C_{j_1j_1}(s)\right| \le C_3(s)\left(\frac{1}{n}+\frac{1}{m}+\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\right), \tag{313}$$

where constant $C_3(s)$ does not depend on n and m.

The relations (313) and (31) imply that

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_1}(s)\right| \le C_3(s)\left(\frac{1}{p}+\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\right) \le \frac{C_4(s)}{p}, \tag{314}$$

where constant $C_4(s)$ is independent of p.

For the trigonometric case, the analogue of the inequality (314) can be obtained by analogy with (268) and (269).

Combining (307), (309), (310), (314), we obtain the estimate (306). Theorem 15 is proved.

The arguments given earlier in this paper allow us to formulate the following two theorems.

Theorem 16 [26]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{s,t} = \int\limits_t^{*s}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{s,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}(s)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C(s)}{p}$$

is valid, where $s \in (t, T]$ *(*s *is fixed), constant* $C(s)$ *is independent of* p,

$$C_{j_3 j_2 j_1}(s) = \int\limits_t^s \psi_3(\tau)\phi_{j_3}(\tau) \int\limits_t^{\tau} \psi_2(s_1)\phi_{j_2}(s_1) \int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2) ds_2 ds_1 d\tau,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Theorem 17 [26]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{s,t} = \int\limits_t^{*s} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1, i_2, i_3, i_4 = 0, 1, \ldots, m)$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{s,t} - \sum_{j_1, j_2, j_3, j_4 = 0}^{p} C_{j_4 j_3 j_2 j_1}(s) \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C(s)}{p}$$

is valid, where $s \in (t, T]$ *(*s *is fixed), constant* $C(s)$ *is independent of* p,

$$C_{j_4 j_3 j_2 j_1}(s) = \int\limits_t^s \phi_{j_4}(s_4) \int\limits_t^{s_4} \phi_{j_3}(s_3) \int\limits_t^{s_3} \phi_{j_2}(s_2) \int\limits_t^{s_2} \phi_{j_1}(s_1) ds_1 ds_2 ds_3 ds_4,$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

13. Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$). Proof Under the Condition of Convergence of Trace Series

In this section, we prove the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) under the condition of convergence of trace series.

Let us introduce some notations and formulate some auxiliary results. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{315}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (315) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{316}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (316)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (7) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{317}$$

where $[x]$ is an integer part of a real number x, $\prod_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum_{\emptyset} \stackrel{\sf def}{=} 0$; another notations are the same as in Theorem 1.

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 18 [26] (Sect. 1.11), [36] (Sect. 15), [51]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{318}$$

converging in the mean-square sense is valid, where $\prod_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum_{\emptyset} \stackrel{\sf def}{=} 0$, $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 18 was considered in [63]. Note that we use another notations in comparison with [63]. Moreover, the proof of an analogue of Theorem 18 from [63] is somewhat different from the proof given in [26] (Sect. 1.11), [36] (Sect. 15), [51].

Denote

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{l}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}}\times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{319}$$

where

$$\mathrm{A}_{k,l} = \left\{(s_l,\ldots,s_1):\; s_l > s_{l-1}+1,\ldots,s_2 > s_1+1,\; s_l,\ldots,s_1 = 1,\ldots,k-1\right\}, \tag{320}$$

$$(s_l,\ldots,s_1)\in \mathrm{A}_{k,l},\quad l=1,\ldots,[k/2]\,,\quad i_s=0,1,\ldots,m,\quad s=1,\ldots,k,$$

$[x]$ is an integer part of a real number x, and $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Stratonovich and Ito stochastic integrals $J^*[\psi^{(k)}]_{T,t},\ J[\psi^{(k)}]_{T,t}$ of arbitrary multiplicity k, $k\in\mathbb{N}$ (see (2), (3)).

Theorem 19 [6] (1997), [12]-[19], [26]-[29]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous function at the interval $[t,T]$. Then, the following relation between iterated Stratonovich and Ito stochastic integrals*

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1} \tag{321}$$

is correct, where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Consider the Fourier coefficient

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{322}$$

corresponding to the function (4), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\frown(\cdot)} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k = \tag{323}$$

$$=\sqrt{T-t}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k =$$

$$=\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau),$ $\psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Let

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\curvearrowright j_m} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k = \tag{324}$$

$$= \bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1},$$

i.e. $\bar{C}_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \psi_{l-1}(\tau)\psi_l(\tau),\ \psi_{l+1}(\tau),\ \ldots,$ $\psi_k(\tau)$ (also we suppose that $\{l-1,l\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{325}$$

Introduce the following notation

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\sf def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{326}$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{327}$$

as follows: S_l multiplies (327) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{328}$$

Note that we write

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_2})\curvearrowright j_m,j_{g_1}=j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_m,j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}, \quad \ldots$$

Since (328) is again the Fourier coefficient, then the action of superposition S_lS_m on (328) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\} =$$

$$= \frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.$$

Theorem 20 [26], [34], [50], [72]. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty}\int\limits_t^s \Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2} \Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{329}$$

holds for all $s \in (t, T]$*, where the nonrandom functions* $\Phi_1(\tau)$*,* $\Phi_2(\tau)$ *are continuously differentiable on* $[t, T]$ *and the series on the right-hand side of* (329) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \quad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau} \Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s \in (t, T)$ *and for some* $\alpha, \beta > 0$*, where* $\Phi_1(\tau)$*,* $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$*,* $j, p \in \mathbb{N}$*, and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau < \infty, \quad \int\limits_t^T |\Psi_2(\tau)|\, d\tau < \infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (315)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$*,* $l_1 > l_2 > \ldots > l_d$*,* $d = 0, 1, 2, \ldots, r-1$*, where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\sf def}{=} \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$*.*

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{330}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{331}$$

that converges in the mean-square sense is valid, where

$$(332)\qquad C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. Note that (329) is true (see (195)). The proof of Theorem 20 will consist of several steps.

Step 1. Let us find a representation of the quantity

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

Note that (7) can be written as (see [26] or [29], Sect. 1.1.3)

$$(333)\qquad J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)},$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (164) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (2).

Let us consider the following multiple stochastic integral

$$(334)\qquad \underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J[\Phi]_{T,t}^{(i_1\ldots i_k)},$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Other notations are the same as in (164).

The stochastic integral with respect to the scalar standard Wiener process ($i_1=\ldots=i_k\neq 0$) and similar to (334) (the function $\Phi(t_1,\ldots,t_k)$ is assumed to be symmetric on the hypercube $[t,T]^k$) has been considered in literature (see, for example, Remark 1.5.7 [65]). The integral (334) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)}=J[\varphi_1]_{T,t}^{(i_1)}\ldots J[\varphi_k]_{T,t}^{(i_k)}\quad\text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k)=\varphi_1(t_1)\ldots\varphi_k(t_k)$ and

$$J[\varphi_l]_{T,t}^{(i_l)}=\int\limits_t^T\varphi_l(s)d\mathbf{w}_s^{(i_l)}\quad(l=1,\ldots,k).$$

Theorem 21 [26]–[29]. *Suppose that* $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ *is a continuous nonrandom function on* $[t,T]^k$*. Furthermore,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of functions in the* *space* $L_2([t,T])$, *each function* $\phi_j(x)$ *of which for finite* j *is continuous at the interval* $[t,T]$ *except may be for the finite number of points of the finite discontinuity as well as* $\phi_j(x)$ *right-continuous at the interval* $[t,T]$. *Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{335}$$

converging in the mean-square sense is valid, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is the multiple Wiener stochatic integral defined by* (164),

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{336}$$

is the Fourier coefficient. Other notations are the same as in Theorems 1, 18.

From (317) and (333) we conclude that

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{337}$$

w. p. 1, where notations are the same as in Theorems 1, 18 and $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (164). For a more detailed derivation of (337), see [51].

Using (337), we obtain

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}-$$

$$-\sum_{r=1}^{[k/2]}(-1)^r \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{338}$$

w. p. 1.

By iteratively applying the formula (338) (also see (10)–(14)), we obtain the following representation of the product

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\ \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1,}\tag{339}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$.

Multiplying both sides of the equality (339) by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get w. p. 1

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.}\tag{340}$$

Denote

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\tag{341}$$

$$K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}),\tag{342}$$

where $C_{j_k\ldots j_1}$ is defined by (332) and $\prod_{\emptyset}\stackrel{\sf def}{=}1$.

The equality (340) can be written as

$$J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\tag{343}$$

w. p. 1, where $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$ have the form (341), (342), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral defined by (334), $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined by (164).

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (340) or (343), we get w. p. 1 (see (333))

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{344}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (2).

If we prove that w. p. 1

$$\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}, \tag{345}$$

then (see (344), (345), and Theorem 19)

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{346}$$

w. p. 1, where notations in (346) are the same as in Theorem 19. Thus Theorem 20 will be proved.

From (343) we have that the multiple Stratonovich stochastic integral $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k is expressed as a sum of some constant value and multiple Wiener stochastic integrals $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ of multiplicities $k,\ k-2,\ k-4,\ \ldots,\ k-2[k/2]$ $(r=1,2,\ldots,[k/2])$.

The formulas (340), (343) can be considered as new representations of the Hu-Meyer formula for the case of a multidimensional Wiener process [66] (also see [65], [67]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (341)).

Note that the equality (343) can be obtained from (335) if we consider (335) for $\Phi(t_1,\ldots,t_k)=K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and without passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}$

For $k=2,3,4,5,6$ we have from (340) w. p. 1

$$
\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=J'[K_{p_1p_2}]_{T,t}^{(i_1i_2)}+\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}, \tag{347}
$$

$$
\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+
$$
$$
+\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}\Biggr), \tag{348}
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=J'[K_{p_1p_2p_3p_4}]_{T,t}^{(i_1i_2i_3i_4)}+
$$
$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+
$$
$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+
$$
$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{349}
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}=J'[K_{p_1p_2p_3p_4p_5}]_{T,t}^{(i_1i_2i_3i_4i_5)}+
$$
$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_3i_4i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_4i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_2i_3i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2i_3i_4)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_4i_5)}+
$$
$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_3i_5)}+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_3i_4)}+
$$
$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_1i_2i_5)}+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1i_2i_4)}+
$$
$$
+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_2i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}\Biggr), \tag{350}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}=J'[K_{p_1p_2p_3p_4p_5p_6}]_{T,t}^{(i_1i_2i_3i_4i_5i_6)}+$$
$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_3i_4i_5)}+$$
$$+\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_3i_4i_5)}+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_2i_4i_5)}+$$
$$+\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_2i_3i_5)}+\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_3i_4i_5i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_4i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_3i_5i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_2i_3i_4i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_4i_5i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_3i_5i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_3i_4i_6)}+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_2i_5i_6)}+$$
$$+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_2i_4i_6)}+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_1i_2i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2 i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2 i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3 i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr). \tag{351}$$

Note that the relation (349) can be written in the following form

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{p_4}\left(\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_4j_3j_1j_1}\right)J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_1,p_3\}}C_{j_4j_3j_2j_3}\right)J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_1,p_4\}}C_{j_4j_3j_2j_4}\right)J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_4j_3j_3j_1}\right)J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_2,p_4\}}C_{j_4j_3j_4j_1}\right)J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(\sum_{j_4=0}^{\min\{p_3,p_4\}}C_{j_4j_4j_2j_1}\right)J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{\min\{p_2,p_3\}}\sum_{j_4=0}^{\min\{p_1,p_4\}}C_{j_4j_2j_2j_4}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_3=0}^{\min\{p_1,p_3\}}\sum_{j_4=0}^{\min\{p_2,p_4\}}C_{j_4j_3j_4j_3}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_2=0}^{\min\{p_1,p_2\}}\sum_{j_4=0}^{\min\{p_3,p_4\}}C_{j_4j_4j_2j_2}\quad \text{w. p. 1.}$$

Further, we will use the representation (340) for $p_1=\ldots=p_k=p$, i.e.

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.}\tag{352}$$

Step 2. Let us prove that

$$\sum_{j_l=0}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=0\tag{353}$$

or

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1},\tag{354}$$

where $l-1\ge s+1$.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

We have

$$C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=\int_t^T\phi_{j_k}(t_k)\ldots\int_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int_t^{t_{l+1}}\phi_{j_l}(t_l)\int_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$
\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots
$$

$$
\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =
$$

$$
= \int\limits_t^{T} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s \times
$$

$$
\times \left(\int\limits_{t_{s+1}}^{T} \phi_{j_{s+2}}(t_{s+2}) \ldots \int\limits_{t_{l-2}}^{T} \phi_{j_{l-1}}(t_{l-1}) \int\limits_{t_{l-1}}^{T} \phi_{j_l}(t_l) \int\limits_{t_l}^{T} \phi_{j_{l+1}}(t_{l+1}) \ldots \right.
$$

$$
\left. \ldots \int\limits_{t_{k-1}}^{T} \phi_{j_k}(t_k) dt_k \ldots dt_{l+1} dt_l dt_{l-1} \ldots dt_{s+2} \right) dt_{s+1} =
$$

$$
= \int\limits_t^{T} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \underbrace{\int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)} dt_s \times
$$

$$
\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) \underbrace{\int\limits_{t_l}^{T} \phi_{j_{l+1}}(t_{l+1}) \ldots \int\limits_{t_{k-1}}^{T} \phi_{j_k}(t_k) dt_k \ldots dt_{l+1}}_{H_{j_k\ldots j_{l+1}}(t_l)} \times
$$

$$
\times \left(\underbrace{\int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1}}_{Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1})} dt_l \right) dt_{s+1} =
$$

$$
= \int\limits_t^{T} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1}\ldots j_1}(t_s) dt_s \times
$$

$$
\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k\ldots j_{l+1}}(t_l) Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1}) dt_l dt_{s+1}. \tag{355}
$$

Using the additive property of the integral, we obtain

$$
Q_{j_{l-1}\ldots j_{s+2}}(t_l, t_{s+1}) =
$$

$$=\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}=$$

$$=\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}}\phi_{j_{s+3}}(t_{s+3})\int\limits_{t}^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}dt_{s+3}\ldots dt_{l-1}-$$

$$-\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}}\phi_{j_{s+3}}(t_{s+3})dt_{s+3}\ldots dt_{l-1}\int\limits_{t}^{t_{s+1}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}=$$

$$\ldots$$

$$=\sum_{m=1}^{d}h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}),\quad d<\infty. \tag{356}$$

Combining (355) and (356), we have

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=\sum_{m=1}^{d}\left(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=0}^{p}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right). \tag{357}$$

Using the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l=$$

$$=\int\limits_{t}^{T}\mathbf{1}_{\{\tau<t_{s+1}\}}G_{j_{s-1}\ldots j_1}(\tau)\cdot\mathbf{1}_{\{\tau>t_{s+1}\}}H_{j_k\ldots j_{l+1}}(\tau)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(\tau)d\tau=0. \tag{358}$$

From (357) and (358) we get

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=-\sum_{m=1}^{d}\left(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=p+1}^{\infty}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\Biggr). \tag{359}$$

Combining Condition 2 of Theorem 20 and (355)–(357), (359), we have

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}=$$

$$=-\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\Biggl(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times$$

$$\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l dt_{s+1}\Biggr)=$$

$$=-\sum_{j_l=p+1}^{\infty}\int\limits_{t}^{T}\phi_{j_k}(t_k)\ldots\int\limits_{t}^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_{t}^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_{t}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots\int\limits_{t}^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_{t}^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots\int\limits_{t}^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k=$$

$$=-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}. \tag{360}$$

The equality (360) implies (353), (354).

Step 3. Using Conditions 1 and 2 of Theorem 20, we obtain

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}=$$

$$=\int\limits_{t}^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_{t}^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\sum_{j_l=0}^{p}\int\limits_{t}^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_{t}^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times$$

$$\times\int\limits_{t}^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k=$$

$$= \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \sum_{j_l=0}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(t_l)\phi_{j_l}(t_l) \int\limits_t^{t_l} \psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times$$

$$\times \int\limits_t^{t_{l-1}} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k -$$

$$- \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1} =$$

$$= \frac{1}{2} C_{j_k\ldots j_1}\Biggr|_{(j_l j_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}. \tag{361}$$

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (352), we have (see (333))

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+ \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{362}$$

Taking into account (354) and (361), we obtain for $r=1$

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =$$

$$= -\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2>g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} +$$

$$+ \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} \frac{1}{2} C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} -$$

$$-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_{g_1}=p+1}^{\infty}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_{g_1}=p+1}^{\infty}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+$$

$$+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} \frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}} \mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}= \tag{363}$$

$$=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{g_1}+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)1,g_1,g_2}\quad \text{w. p. 1}, \tag{364}$$

where $J[\psi^{(k)}]_{T,t}^{g_1}$ $(g_1=1,2,\ldots,k-1)$ is defined by (319),

$$R_{T,t}^{(p)1,g_1,g_2}=-\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\neq g_1,g_2} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}.$$

Let us explain the transition from (363) to (364). We have for $g_2=g_1+1$

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} \frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0,j_{g_1}=\ j_{g_2}}\times$$

$$\times\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\ \sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\zeta_{j_{m_1}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\ \sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times J'[\phi_{j_{m_1}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(0 i_{q_1}\ldots i_{q_{k-2}})}= \tag{365}
$$

$$
=\frac{1}{2}J[\psi^{(k)}]_{T,t}^{g_1}\quad \text{w. p. 1}, \tag{366}
$$

where

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown j_{m_1},j_{g_1}=\ j_{g_2},g_2=g_1+1}=
$$

$$
=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{g_1+3}}\psi_l(t_{g_1+2})\phi_{j_{g_1+2}}(t_{g_1+2})\int\limits_t^{t_{g_1+2}}\psi_{g_1+1}(t_{g_1})\psi_{g_1}(t_{g_1})\phi_{j_{m_1}}(t_{g_1})\times
$$

$$
\times\int\limits_t^{t_{g_1}}\psi_l(t_{g_1-1})\phi_{j_{g_1-1}}(t_{g_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{g_1-1}dt_{g_1}dt_{g_1+2}\ldots dt_k,
$$

$$
\zeta_{j_{m_1}}^{(0)}=\int\limits_t^T\phi_{j_{m_1}}(\tau)d\mathbf{w}_\tau^{(0)}=\int\limits_t^T\phi_{j_{m_1}}(\tau)d\tau=\begin{cases}\sqrt{T-t} & \text{if } j_{m_1}=0\\ 0 & \text{if } j_{m_1}\ne 0\end{cases}, \tag{367}
$$

$$
\phi_0(\tau)=\frac{1}{\sqrt{T-t}}. \tag{368}
$$

The transition from (365) to (366) is based on (333).

By Condition 3 of Theorem 20 we have (also see the property (165) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)1,g_1,g_2}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2}\right)^2=0,
$$

where constant K does not depend on p.

Thus

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{g_1}\quad \text{w. p. 1.}$$

Involving into consideration the second pair $\{g_3,g_4\}$ (the first pair is $\{g_1,g_2\}$), we obtain from (363) for $r=2$

$$\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{2}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=$$

$$=\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,g_3,g_4}}^{p}\left(\frac{1}{4}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}-\right.$$

$$-\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\mathbf{1}_{\{g_4=g_3+1\}}-$$

$$-\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\mathbf{1}_{\{g_2=g_1+1\}}+$$

$$\left.+\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\right)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=\tag{369}$$

$$=\frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_2,s_1}+\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}\tag{370}$$

w. p. 1, where $g_3\stackrel{\text{def}}{=}s_2$, $g_1\stackrel{\text{def}}{=}s_1$, $(s_2,s_1)\in\mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (319) and $\mathrm{A}_{k,2}$ is defined by (320),

$$R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,g_3,g_4}}^{p}\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,g_3,g_4}-\right.$$

$$\left.-S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,g_3,g_4}\right\}-S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,g_3,g_4}\right\}\right)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}.$$

Let us explain the transition from (369) to (370). We have for $g_2=g_1+1,\ g_4=g_3+1$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\frac{1}{4}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4}}\times$$

$$\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=$$

$$=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0(j_{g_4}j_{g_3})\curvearrowright 0,j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4}}\times$$

$$\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\zeta_0^{(0)}\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=$$

$$=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\sum_{j_{m_1},j_{m_3}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4}}\times$$

$$\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=$$

$$=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\sum_{j_{m_1},j_{m_3}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4}}\times$$

$$\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(00i_{q_1}\ldots i_{q_{k-4}})}= \tag{371}$$

$$=\frac{1}{4}J[\psi^{(k)}]_{T,t}^{s_2,s_1}\quad\text{w. p. 1.} \tag{372}$$

The transition from (371) to (372) is based on (333).

Note that

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\,j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\,j_{g_2}}$$

is the Fourier coefficient, where $g_2 = g_1 + 1$. Therefore, the value

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} =$$

$$= C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}(j_{g_3}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}$$

is determined recursively using (324) in an obvious way for $g_2 = g_1 + 1$ and $g_4 = g_3 + 1$.

By Condition 3 of Theorem 20 we have (also see the property (165) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \left(\left(\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right)^2 +\right.$$

$$\left. + \left(S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2 + \left(S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2\right) = 0,$$

where constant K is independent of p.

Thus

$$\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} = \frac{1}{4}\prod_{s=1}^{2} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_2,s_1} \quad \text{w. p. 1},$$

where $g_3 \stackrel{\sf def}{=} s_2$, $g_1 \stackrel{\sf def}{=} s_1$, $(s_2, s_1) \in \mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (319) and $\mathrm{A}_{k,2}$ is defined by (320).

Involving into consideration the third pair $\{g_6, g_5\}$ ($\{g_1, g_2\}$ is the first pair and $\{g_4, g_3\}$ is the second pair), we obtain from (369) for $r = 3$

$$\prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{3} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} = \prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4,g_5,g_6}}^{p} \left(\frac{1}{2^3} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}} \times\right.$$

$$\times \prod_{s=1}^{3} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} -$$

$$-\frac{1}{2^2}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_4=g_3+1\}}\mathbf{1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_5}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_6=g_5+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_2=g_1+1\}}-$$

$$\left.-\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigl|_{j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\right)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=$$

$$=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}+\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}$$

w. p. 1, where $g_{2i-1}\stackrel{\rm def}{=}s_i$; $i=1,2,3$, $(s_3,s_2,s_1)\in\mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (319) and $\mathrm{A}_{k,3}$ is defined by (320),

$$R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_5,g_6}}^{p}\left(-\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\neq g_1,g_2,\ldots,g_5,g_6}+\right.$$

$$+S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}+S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}+$$

$$+S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}-$$

$$
-S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-
$$

$$
-S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\Biggr)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}.
$$

By Condition 3 of Theorem 20 we have (also see the property (165) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p}\left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right)^2+\right.
$$

$$
+\left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+\left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$
+\left(S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$
+\left(S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+\left(S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$
\left.+\left(S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2\right)=0,
$$

where constant K does not depend on p.

Thus

$$
\underset{p\to\infty}{\text{l.i.m.}}\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{3}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}\quad\text{w. p. 1},
$$

where $g_{2i-1}\stackrel{\mathsf{def}}{=}s_i$; $i=1,2,3,$ $(s_3,s_2,s_1)\in\mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (319) and $\mathrm{A}_{k,3}$ is defined by (320).

Repeating the previous steps, we obtain for an arbitrary r ($r=1,2,\ldots,[k/2]$)

$$\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}= \tag{373}$$

$$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{374}$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2],\ (s_r,\ldots,s_1)\in \mathrm{A}_{k,r},\ J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (319) and $\mathrm{A}_{k,r}$ is defined by (320),

$$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl((-1)^r\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r}S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \tag{375}$$

Let us explain the transition from (373) to (374). We have for $g_2 = g_1 + 1, \ldots, g_{2r} = g_{2r-1} + 1$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}} \left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}} \times$$

$$\times C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \times$$

$$\times \zeta_{j_{m_1}}^{(0)} \zeta_{j_{m_3}}^{(0)} \ldots \zeta_{j_{m_{2r-1}}}^{(0)} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}} \times$$

$$\times C_{j_k\ldots j_1}\Bigl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}} \times$$

$$\times J'[\phi_{j_{m_1}} \phi_{j_{m_3}} \ldots \phi_{j_{m_{2r-1}}} \phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots 0 i_{q_1}\ldots i_{q_{k-2r}})} = \tag{376}$$

$$= \frac{1}{2^r} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1.} \tag{377}$$

The transition from (376) to (377) is based on (333).

Note that

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}, j_{g_1}=\ j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}, j_{g_1}=\ j_{g_2}}$$

is the Fourier coefficient, where $g_2 = g_1 + 1$. Therefore, the value

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}}, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}} =$$

$$= C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d-1}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}}, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}$$

is determined recursively using (324) in an obvious way for $g_2 = g_1 + 1, \ldots, g_{2d} = g_{2d-1} + 1$ and $d = 2, \ldots, r$.

By Condition 3 of Theorem 20 we have (also see the property (165) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\} \le$$

$$\le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\left(\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2 +\right.$$

$$+\sum_{l_1=1}^{r}\left(S_{l_1}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 +$$

$$+\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 +$$

$$\ldots$$

$$\left.+\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right) = 0,$$

where constant K does not depend on p.

So we have

$$\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$= \frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1,} \tag{378}$$

where $g_{2i-1} \stackrel{\sf def}{=} s_i;\ i = 1, 2, \ldots, r;\ r = 1, 2, \ldots, [k/2]\,,\ (s_r, \ldots, s_1) \in \mathrm{A}_{k,r},\ J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (319) and $\mathrm{A}_{k,r}$ is defined by (320).

Note that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \Biggl.\Biggr|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1} A_{g_1,g_3,\ldots,g_{2r-1}} =$$

$$= \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} A_{s_1,s_2,\ldots,s_r}, \tag{379}$$

where $A_{g_1,g_3,\ldots,g_{2r-1}},\ A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1} = s_i;\ i = 1, 2, \ldots, r;\ r = 1, 2, \ldots, [k/2]\,,\ \mathrm{A}_{k,r}$ is defined by (320):

$$\mathrm{A}_{k,r} = \bigl\{(s_r, \ldots, s_1):\ s_r > s_{r-1} + 1, \ldots, s_2 > s_1 + 1,\ s_r, \ldots, s_1 = 1, \ldots, k-1\bigr\}.$$

Using (362), (378), (379), and Theorem 19, we finally get

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{380}$$

w. p. 1, where (see (319))

$$J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{r}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}}\ \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_r+3}} \psi_{s_r+2}(t_{s_r+2}) \int\limits_t^{t_{s_r+2}} \psi_{s_r}(t_{s_r+1})\psi_{s_r+1}(t_{s_r+1})\times$$

$$\times \int\limits_t^{t_{s_r+1}} \psi_{s_r-1}(t_{s_r-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}dt_{s_r+1}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{381}$$

Theorem 20 is proved.

Let us make a number of remarks about Theorem 20. An expansion similar to (331) was obtained in [66], where the author used a definition of the Stratonovich stochastic integral, which differs from the definition from [2]. The proof from [66] is somewhat simpler than the proof proposed in this section. However, the results from [66] were obtained under the condition of convergence of trace series. The verification of this condition for the kernel (4) is a separate problem. In our proof we essentially use the structure of the Fourier coefficients (332) corresponding to the kernel (4). This circumstance actually made it possible to prove Theorem 20 using not the condition of finiteness of trace series, but using the condition of convergence to zero of explicit expressions for the remainders of the mentioned series. This leaves hope that it is possible to prove an analogue of Theorems 12–14 for the case of arbitrary k ($k\in\mathbb{N}$) (see Theorems 26–29 below).

Note that under the conditions of Theorem 20 (also see (354), (361)) the sequential order of the series

$$\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}$$

is not important.

We also note that the first and second conditions of Theorem 20 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ (see the proofs of Theorems 1–4 and Theorems 23–25 below). Moreover, (329) is true for an arbitrary basis in $L_2([t,T])$ (see (195)). It is easy to see that in the proofs of Theorems 1–4, 23–25 the conditions of Theorem 20 are verified for various special cases of iterated Stratonovich stochastic integrals of multiplicities 2–5 with respect to components of the multidimensional Wiener process.

Taking into account Theorem 5, we can formulate an analogue of Theorem 20 for the case of integration interval $[t,s]$ ($s\in(t,T]$) of iterated Stratonovich stochastic integrals of multiplicity k ($k\in\mathbb{N}$).

Denote

$$\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

and introduce the following notation

$$S_l\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

where $l=1,2,\ldots,r$,

$$
C_{j_k\ldots j_1}(s)\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot)}
$$

is defined by analogy with (323),

$$
C_{j_k\ldots j_1}(s)=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k. \tag{382}
$$

Theorem 22 [26], [34], [50], [72]. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied*:

1. *The equality*

$$
\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{383}
$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (383) converges absolutely.

2. *The estimates*

$$
\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_\tau^s\phi_j(\theta)\Phi_2(\theta)d\theta\right|\le\frac{\Psi_2(s,\tau)}{j^{1/2+\alpha}},
$$

$$
\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_3(s)}{p^{\beta}}
$$

hold for all s,τ such that $t<\tau<s<T$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$
\int\limits_t^s|\Psi_1(\tau)\Psi_2(s,\tau)|\,d\tau<\infty,\quad \int\limits_t^s|\Psi_3(\tau)|\,d\tau<\infty
$$

for all $s\in(t,T)$.

3. *The condition*

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}(s)\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0
$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (315)) and $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, where $r=1,2,\ldots,[k/2]$ and

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^{*s}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)} \qquad (384)$$

the following expansion

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (382), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$ $s\in(t,T)$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d{\bf w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\ne 0$), ${\bf w}_\tau^{(i)}={\bf f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* ${\bf w}_\tau^{(0)}=\tau$.

It is easy to see that the estimates (87), (95), (214), (217) and the results of Sect. 12 imply the fulfillment of Conditions 2 of Theorem 22 for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$.

Also the equality (195) guarantees the fulfillment of Condition 1 of Theorem 22 for these two systems of functions.

It should be noted that (see (375))

$$(-1)^r\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r}S_{l_1}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}S_{l_1}S_{l_2}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=$$

$$= \sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}, \tag{385}$$

where the meaning of the notations used in (375) is preserved.

For example, from (385) for the case $r=2$ we get

$$\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=\, j_{g_2},j_{g_3}=\, j_{g_4}} -$$

$$-\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},j_{g_3}=\, j_{g_4}} -$$

$$-\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},j_{g_3}=\, j_{g_4}} =$$

$$= \sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} -$$

$$-\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},j_{g_3}=\, j_{g_4}}.$$

As a result, Condition 3 of Theorem 20 can be replaced by a weaker condition

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}\Biggr)^2 = 0, \tag{386}$$

where $r=1,2,\ldots,[k/2]$.

However, Condition 3 of Theorem 20 itself contains a way of proving of the condition (386), which is partially realized in the proof of Theorems 23–25, 30 (see below).

In fact, when proving Theorem 25 (the case $r=3$ is proved in Theorem 30 for $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$), we proved the following equality

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}} =$$

$$= \frac{1}{4} \mathbf{1}_{\{g_2=g_1+1\}} \mathbf{1}_{\{g_4=g_3+1\}} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) (j_{g_4} j_{g_3}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, j_{g_3} = j_{g_4}}.$$

On the other hand, iterative application of (361) gives

$$\sum_{j_{g_1}=0}^{\infty} \ldots \sum_{j_{g_{2r-1}}=0}^{\infty} C_{j_k \ldots j_1} \Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1} \Bigg|_{(j_{g_2} j_{g_1}) \curvearrowright (\cdot) \ldots (j_{g_{2r}} j_{g_{2r-1}}) \curvearrowright (\cdot), j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}},$$

where $r = 1, 2, \ldots, [k/2]$.

14. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case $p_1 = p_2 = p_3 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

In this section, we present a simple proof of Theorem 3 based on Theorem 20. In this case, the conditions of Theorem 3 will be weakened.

First, consider the following equalities

$$\frac{1}{2} \int_{t_1}^{t_2} \Phi_1(\tau)\Phi_2(\tau) d\tau = \sum_{j=0}^{\infty} \int_{t_1}^{t_2} \Phi_2(\tau)\phi_j(\tau) \int_{t_1}^{\tau} \Phi_1(\theta)\phi_j(\theta) d\theta d\tau, \tag{387}$$

$$\frac{1}{2} \int_{t_1}^{t_2} \Phi_1(\tau)\Phi_2(\tau) d\tau = \sum_{j=0}^{\infty} \int_{t_1}^{t_2} \Phi_1(\theta)\phi_j(\theta) \int_{\theta}^{t_2} \Phi_2(\tau)\phi_j(\tau) d\tau d\theta \tag{388}$$

that will be used further, where $t \le t_1 < t_2 \le T$, $\Phi_1(\tau), \Phi_2(\tau) \in L_2([t,T])$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of funtions in $L_2([t,T])$. The equality (388) has already been proved (see (206)). Using (388) and Fubini's Theorem, we get (387).

Theorem 23 [26], [34], [50], [72]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \tag{389}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$,

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As follows from the previous sections, Conditions 1 and 2 of Theorem 20 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Let us verify Condition 3 of Theorem 20 for the iterated Stratonovich stochastic integral (389). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_3j_1j_1} \right)^2 = 0, \tag{390}$$

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1} \right)^2 = 0, \tag{391}$$

$$\lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_1} \right)^2 = 0. \tag{392}$$

We have

$$\sum_{j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_3j_1j_1} \right)^2 =$$

$$= \sum_{j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3 \right)^2 = \tag{393}$$

$$= \sum_{j_3=0}^{p} \left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3 \right)^2 \le \tag{394}$$

$$\le \sum_{j_3=0}^{\infty} \left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3 \right)^2 = \tag{395}$$

$$= \int\limits_t^T \psi_3^2(t_3) \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \right)^2 dt_3 \le \tag{396}$$

$$\leq \frac{K}{p^2} \ \to \ 0 \tag{397}$$

if $p \to \infty$, where constant K does not depend on p.

Note that the transition from (393) to (394) is based on the estimate (314) for the polynomial case and its analogue for the trigonometric case, the transition from (395) to (396) is based on the Parseval equality, and the transition from (396) to (397) is also based on the estimate (314) and its analogue for the trigonometric case.

By analogy with the previous case we have

$$\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1}\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 = \tag{398}$$

$$= \sum_{j_1=0}^{p}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 \leq \tag{399}$$

$$\leq \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 =$$

$$= \int\limits_t^T \psi_1^2(t_1)\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_1(t_3)\phi_{j_3}(t_3)dt_3dt_2\right)^2 dt_1 \leq \tag{400}$$

$$\leq \frac{K}{p^2} \ \to \ 0 \tag{401}$$

if $p \to \infty$, where constant K is independent of p.

The transition from (398) to (399) is based on an analogue of the estimate (314) for the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2\right|$$

for the polynomial and trigonometric cases, the transition from (400) to (401) is also based on the mentioned analogue of the estimate (314).

Further, we have

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_1}\right)^2 =$$

$$= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int_t^T \psi_3(t_3)\phi_{j_1}(t_3) \int_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3 \right)^2 =$$

$$= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2 \right)^2 = \tag{402}$$

$$= \sum_{j_2=0}^{p} \left(\int_t^T \psi_2(t_2)\phi_{j_2}(t_2) \sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2 \right)^2 \le \tag{403}$$

$$\le \sum_{j_2=0}^{\infty} \left(\int_t^T \psi_2(t_2)\phi_{j_2}(t_2) \sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2 \right)^2 =$$

$$= \int_t^T \psi_2^2(t_2) \left(\sum_{j_1=p+1}^{\infty} \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \int_{t_2}^{T} \psi_3(t_3)\phi_{j_1}(t_3)dt_3 \right)^2 dt_2. \tag{404}$$

The transition from (402) to (403) is based on the estimate (88) and its obvious analogue for the trigonometric case. However, the estimate (88) cannot be used to estimate the right-hand side of (404), since we get the divergent integral. For this reason, we will obtain a new estimate based on the relation (86).

From (29) and the estimate $|P_j(y)| \le 1$, $y \in [-1,1]$ we obtain

$$|P_j(y)| = |P_j(y)|^{\varepsilon} \cdot |P_j(y)|^{1-\varepsilon} \le |P_j(y)|^{1-\varepsilon} < \frac{C}{j^{1/2-\varepsilon/2}(1-y^2)^{1/4-\varepsilon/4}}, \tag{405}$$

where $y \in (-1,1)$, $j \in \mathbb{N}$, and ε is an arbitrary small positive real number.

Combining (86) and (405), we have the following estimate

$$\left| \int_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau \right| < \frac{C}{(j_1)^{1-\varepsilon/2}} \left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} + 1 \right), \tag{406}$$

where $s \in (t,T)$, $z(s)$ is defined by (26), constant C does not depend on j_1.

Similarly to (406) we obtain

$$\left| \int_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau \right| < \frac{C}{(j_1)^{1-\varepsilon/2}} \left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} + 1 \right), \tag{407}$$

where $s \in (t,T)$, constant C does not depend on j_1.

Combining (87) and (407), we have

$$\left| \int_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau \int_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau \right| <$$

$$< \frac{L}{(j_1)^{2-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+1\right), \tag{408}$$

where $s \in (t,T)$, $z(s)$ is defined by (26), constant L does not depend on j_1.

Observe that

$$\sum_{j_1=p+1}^{\infty}\frac{1}{(j_1)^{2-\varepsilon/2}} \le \int\limits_{p}^{\infty}\frac{dx}{x^{2-\varepsilon/2}} = \frac{1}{(1-\varepsilon/2)p^{1-\varepsilon/2}}. \tag{409}$$

Applying (408) and (409) to estimate the right-hand side of (404) gives

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{410}$$

if $p\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

The estimation of the right-hand side of (404) for the trigonometric case is carried out using the estimates (95), (96). At that we obtain the estimate (410) with $\varepsilon = 0$. Theorem 23 is proved.

15. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case $p_1 = \ldots = p_4 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\ldots$, $\psi_4(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 24 [26], [34], [50], [72]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{411}$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3,i_4 = 0,1,\ldots,m$,

$$C_{j_4j_3j_2j_1} = \int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\times$$

$$\times dt_2dt_3dt_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Proof. As follows from the previous sections, Conditions 1 and 2 of Theorem 20 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 20 for the iterated Stratonovich stochastic integral (411). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4j_3j_1j_1}\right)^2 = 0, \tag{412}$$

$$\lim_{p\to\infty} \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4j_1j_2j_1}\right)^2 = 0, \tag{413}$$

$$\lim_{p\to\infty} \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1j_3j_2j_1}\right)^2 = 0, \tag{414}$$

$$\lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 = 0, \tag{415}$$

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2 = 0, \tag{416}$$

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 = 0, \tag{417}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = 0, \tag{418}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = 0, \tag{419}$$

$$\lim_{p\to\infty} \left(\sum_{j_3=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 = 0, \tag{420}$$

$$\lim_{p\to\infty} \left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl.\Biggr|_{(j_1j_1)\frown(\cdot)}\right)^2 = 0, \tag{421}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{422}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{423}$$

where in (421)–(423) we use the notation (323).

Applying arguments similar to those we used in the proof of Theorem 23, we obtain for (412)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_4j_3j_1j_1}\right)^2=\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{424}$$

$$=\sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2\le \tag{425}$$

$$\le\sum_{j_3,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{426}$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_4\}}\psi_4^2(t_4)\psi_3^2(t_3)\times$$

$$\times\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3dt_4\le \tag{427}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{428}$$

if $p\to\infty$, where constant K is independent of p.

Note that the transition from (424) to (425) is based on the estimate (314) for the polynomial case and its analogue for the trigonometric case, the transition from (426) to (427) is based on the Parseval equality, and the transition from (427) to (428) is also based on the estimate (314) and its analogue for the trigonometric case.

Further, we have for (413)

$$\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{429}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2= \tag{430}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2\le$$

$$\le\sum_{j_2,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4^2(t_4)\psi_2^2(t_2)\times$$

$$\times\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2dt_2dt_4\le$$

$$\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{431}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The relation (431) was obtained by the same method as (428). Note that in obtaining (431) we used the estimates (87) and (214) for the polynomial case and their obvious analogues for the trigonometric case. We also used the integration order replacement in the iterated Riemann integrals (see (429), (430)).

Repeating the previous steps for (414) and (415), we get

$$\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2=$$

$$=\sum_{j_2,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2\le$$

$$\le\sum_{j_2,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2\le$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\psi_3^2(t_3)\psi_2^2(t_2)\times$$

$$\times\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4\right)^2dt_2dt_3\le$$

$$\le\frac{K}{p^2}\ \to\ 0 \qquad (432)$$

if $p\to\infty$, where constant K does not depend on p;

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 \le
$$

$$
\le \sum_{j_1,j_4=0}^{\infty} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_4\}} \psi_4^2(t_4)\psi_1^2(t_1) \times
$$

$$
\times \left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 \right)^2 dt_1 dt_4. \tag{433}
$$

Note that, by virtue of the additivity property of the integral, we have

$$
\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 = \tag{434}
$$

$$
= \sum_{j_2=p+1}^{\infty} \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) dt_2 dt_3 -
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_t^{t_1} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) dt_2 dt_3 -
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 \int\limits_t^{t_1} \psi_2(t_2)\phi_{j_2}(t_2) dt_2. \tag{435}
$$

However, all three series on the right-hand side of (435) have already been evaluated in (428) and (431). From (433) and (435) we finally obtain

$$
\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{436}
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

In complete analogy with (431), we have for (416)

$$
\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2 = \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.
$$

$$
\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.
$$

$$
\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2 =
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2 =
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2 \le
$$

$$
\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2 =
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_3\}}\psi_3^2(t_3)\psi_1^2(t_1)\times
$$

$$\times\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4\right)^2 dt_1dt_3 \le$$

$$\le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \qquad (437)$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

We have for (417)

$$\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t}^{T}\psi_4(t_4)\phi_{j_3}(t_4)\int\limits_{t}^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t}^{T}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2=$$

$$=\sum_{j_1,j_2=0}^{p}\left(\int\limits_{t}^{T}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2\le$$

$$\le\sum_{j_1,j_2=0}^{\infty}\left(\int\limits_{t}^{T}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\psi_1^2(t_1)\psi_2^2(t_2)\times$$

$$\times\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3\right)^2dt_2dt_1. \qquad (438)$$

It is easy to see that the integral (see (438))

$$\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3$$

is similar to the integral from the formula (434) if in the last integral we substitute $t_4 = T$. Therefore, by analogy with (436), we obtain

$$\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{439}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

Now consider (418)–(420). We have for (418) (see **Step 2** in the proof of Theorem 20)

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2. \tag{440}$$

Consider (416) and (437). We have

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2 = \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2\Biggr|_{j_1=j_3} \le$$

$$\le \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{441}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. Combining (440) and (441), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_1j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0$$

if $p\to\infty$, where constant K_1 does not depend on p.

Similarly for (419) we have (see (415), (436))

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2, \tag{442}$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2\Biggr|_{j_1=j_4}\le$$

$$\le\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}},\tag{443}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. Combining (442) and (443), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\right)^2\le\frac{(p+1)K}{p^{2-\varepsilon}}\le\frac{K_1}{p^{1-\varepsilon}}\ \to\ 0$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider (420). Using (361), we obtain

$$\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}=\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{\infty}C_{j_3j_3j_1j_1}-\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p}C_{j_3j_3j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}-\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p}C_{j_3j_3j_1j_1},\tag{444}$$

where (see (323))

$$C_{j_3j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2dt_3dt_4.$$

From the estimate (43) (polynomial case) and its analogue for the trigonometric case (see the proof of Lemma 1) we get

$$\left|\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}\right|\le\frac{C}{p},\tag{445}$$

where constant C is independent of p.

Further, we have (see (439))

$$\left(\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2\le(p+1)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=$$

$$=(p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2\Biggr|_{j_1=j_2}\le$$

$$\leq (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 \leq \frac{(p+1)K}{p^{2-\varepsilon}} \leq \frac{K_1}{p^{1-\varepsilon}}, \tag{446}$$

where constant K_1 does not depend on p.

Combining (444)–(446), we obtain

$$\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \leq \frac{K_2}{p^{1-\varepsilon}} \ \rightarrow\ 0$$

if $p\rightarrow\infty$, where constant K_2 does not depend on p.

Let us prove (421)–(423). It is not difficult to see that the estimate (445) proves (421).

Using the integration order replacement, we obtain

$$\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)} =$$

$$\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\psi_2(t_2)\int\limits_{t_2}^{T}\psi_4(t_4)\psi_3(t_4)dt_4\right)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2, \tag{447}$$

$$\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)} =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_3dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\left(\int\limits_t^{t_4}-\int\limits_t^{t_1}\right)\psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T\left(\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)dt_3\right)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_4- \tag{448}$$

$$- \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\left(\psi_1(t_1)\int\limits_t^{t_1}\psi_3(t_3)\psi_2(t_3)dt_3\right)\phi_{j_1}(t_1)dt_1dt_4. \tag{449}$$

Applying the estimate (43) (polynomial case) and its analogue for the trigonometric case (see the proof of Lemma 1) to the right-hand sides of (447)–(449), we get

$$\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{450}$$

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{451}$$

where constant C is independent of p. The estimates (450), (451) prove (422), (423).

The relations (412)–(423) are proved. Theorem 24 is proved.

16. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case $p_1 = \ldots = p_5 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\ldots$, $\psi_5(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 25 [26], [34], [50], [72]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{452}$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5 = 0,1,\ldots,m$,

$$C_{j_5\ldots j_1} = \int\limits_t^{T} \psi_5(t_5)\phi_{j_5}(t_5)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. Note that in this proof we write k instead of 5 when this is true for an arbitrary k ($k \in \mathbb{N}$). As follows from the previous sections, Conditions 1 and 2 of Theorem 20 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 20 for the iterated Stratonovich stochastic integral (452). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2}}\right)^2=0, \tag{453}$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0, \tag{454}$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=0, \tag{455}$$

where $(\{g_1,g_2\},\{g_3,g_4\},\{q_1\})$ and $(\{g_1,g_2\},\{q_1,q_2,q_3\})$ are partitions of the set $\{1,2,\ldots,5\}$ that is $\{g_1,g_2,g_3,g_4,q_1\}=\{g_1,g_2,q_1,q_2,q_3\}=\{1,2,\ldots,5\}$; braces mean an unordered set, and parentheses mean an ordered set.

Let us find a representation for $C_{j_k\ldots j_1}\bigr|_{j_{g_1}=j_{g_2},\ g_2>g_1+1}$ that will be convenient for further consideration.

Using the integration order replacement in Riemann integrals, we obtain

$$\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l)dt_l\times$$

$$\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\times$$

$$\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\times$$

$$\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots \int\limits_{t}^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}dt_{l+1} \ldots dt_k -$$

$$- \int\limits_{t}^{T} h_k(t_k) \ldots \int\limits_{t}^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_{t}^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int\limits_{t}^{t_{l-1}} h_l(t_l)dt_l \right) \int\limits_{t}^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots$$

$$\ldots \int\limits_{t}^{t_2} h_1(t_1)dt_1 \ldots dt_{l-2}dt_{l-1}dt_{l+1} \ldots dt_k, \tag{456}$$

where $1 < l < k$ and $h_1(\tau), \ldots, h_k(\tau)$ are continuous functions on the interval $[t, T]$. By analogy with (456) we have for $l = k$

$$\int\limits_{t}^{T} h_l(t_l) \int\limits_{t}^{t_l} h_{l-1}(t_{l-1}) \ldots \int\limits_{t}^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}dt_l =$$

$$= \int\limits_{t}^{T} h_1(t_1) \int\limits_{t_1}^{T} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{T} h_{l-1}(t_{l-1}) \int\limits_{t_{l-1}}^{T} h_l(t_l)dt_l dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_{t}^{T} h_l(t_l)dt_l \right) \int\limits_{t}^{T} h_1(t_1) \int\limits_{t_1}^{T} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{T} h_{l-1}(t_{l-1})dt_{l-1} \ldots dt_2 dt_1 -$$

$$- \int\limits_{t}^{T} h_1(t_1) \int\limits_{t_1}^{T} h_2(t_2) \ldots \int\limits_{t_{l-2}}^{T} h_{l-1}(t_{l-1}) \left(\int\limits_{t}^{t_{l-1}} h_l(t_l)dt_l \right) dt_{l-1} \ldots dt_2 dt_1 =$$

$$= \left(\int\limits_{t}^{T} h_l(t_l)dt_l \right) \int\limits_{t}^{T} h_{l-1}(t_{l-1}) \ldots \int\limits_{t}^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1} -$$

$$- \int\limits_{t}^{T} h_{l-1}(t_{l-1}) \left(\int\limits_{t}^{t_{l-1}} h_l(t_l)dt_l \right) \int\limits_{t}^{t_{l-1}} h_{l-2}(t_{l-2}) \ldots \int\limits_{t}^{t_2} h_1(t_1)dt_1 \ldots dt_{l-1}. \tag{457}$$

The formulas (456), (457) will be used further.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau), \ldots, \psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume for simplicity that $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$.

Let us continue the proof. Applying (456) to $C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$ (more precisely to $h_s(t_s) = \psi_s(t_s)\phi_{j_l}(t_s)$), we obtain for $l+1 \le k$, $s-1 \ge 1$, $l-1 \ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} = \tag{458}$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_s dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \left(\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) dt_s \right) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k -$$

$$- \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \left(\int\limits_t^{t_{s-1}} \phi_{j_l}(t_s) dt_s \right) \int\limits_t^{t_{s-1}} \phi_{j_{s-2}}(t_{s-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_{s-2} dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_l dt_{l+1} \ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} - \sum_{j_l=p+1}^{\infty} B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}.$$

Now we again apply the formula (456) to $A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$, $B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$ (more precisely to $h_l(t_l) = \psi_l(t_l)\phi_{j_l}(t_l)$). Then we have for $l+1 \le k$, $s-1 \ge 1$, $l-1 \ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) \times$$

$$\times \prod_{\substack{g=1 \\ g \ne l,s}}^{k} \psi_g(t_g) \phi_{j_g}(t_g) dt_1 \ldots dt_{s-1} dt_{s+1} \ldots dt_{l-1} dt_{l+1} \ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_{l+1} j_{l-1} \ldots j_{s+1} j_{s-1} \ldots j_1} = \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne l,s}, \tag{459}$$

where

$$F_p^{(1)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1 < \ldots < t_{s-1} < t_{s+1} < \ldots < t_{l-1} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{460}$$

$$F_p^{(2)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1 < \ldots < t_{s-1} < t_{s+1} < \ldots < t_{l-1} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{461}$$

$$F_p^{(3)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= -\mathbf{1}_{\{t_1 < \ldots < t_{s-1} < t_{s+1} < \ldots < t_{l-1} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{462}$$

$$F_p^{(4)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= -\mathbf{1}_{\{t_1 < \ldots < t_{s-1} < t_{s+1} < \ldots < t_{l-1} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau. \tag{463}$$

By analogy with (459) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1} \ldots j_2 j_l}, \tag{464}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_2 j_l} \quad (l+1 \le k), \tag{465}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} \quad (s-1 \ge 1). \tag{466}$$

Then we have for (464)–(466) (see (456), (457))

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1} \ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} G_p^{(d)}(t_2, \ldots, t_{k-1}) \prod_{g=2}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_2 \ldots dt_{k-1}, \tag{467}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_2j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} E_p^{(d)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times\prod_{\substack{g=2\\ g\ne l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{l-1}dt_{l+1}\ldots dt_k, \tag{468}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_lj_{k-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} D_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})\times$$

$$\times\prod_{\substack{g=1\\ g\ne s}}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{k-1}, \tag{469}$$

where

$$G_p^{(1)}(t_2,\ldots,t_{k-1}) = \mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^T \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$G_p^{(2)}(t_2,\ldots,t_{k-1}) = -\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(1)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(2)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{T} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{T} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau.$$

Let us now consider the value $C_{j_k\ldots j_1}\big|_{j_{g_1}=j_{g_2},\ g_2=g_1+1}$. To do this, we will make the following transformations

$$\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}}\right) h_l(t_{l-1}) \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-1}}\right) h_l(t_l)dt_l dt_{l-1}dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l \int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times$$

$$\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-$$

$$- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots$$

$$\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right) dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-$$

$$- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times$$

$$\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k+$$

$$+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)\times$$

$$\times\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k+$$

$$+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_ldt_{l-1}\right)\times$$

$$\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k, \qquad (470)$$

where $l+1\le k$, $l-2\ge 1$, and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$.

Applying (470) to $C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots\ldots j_1}$, we obtain for $l+1\le k$, $l-2\ge 1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots\ldots j_1}=$$

$$=\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)\times$$

$$\times\prod_{\substack{g=1\\ g\ne l-1,l}}^{k}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{l-2}dt_{l+1}\ldots dt_k=$$

$$= \sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1} = \sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_q\ldots j_1}\Big|_{q\ne l-1,l}, \tag{471}$$

where

$$H_p^{(1)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{472}$$

$$H_p^{(2)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-2}} \psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{473}$$

$$H_p^{(3)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_{l-1}(\tau)\phi_{j_l}(\tau) \int\limits_t^{\tau} \psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau, \tag{474}$$

$$H_p^{(4)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l-2}} \psi_{l-1}(\tau)\phi_{j_l}(\tau) \int\limits_t^{\tau} \psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau. \tag{475}$$

By analogy with (471) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_l}, \tag{476}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_lj_lj_{k-2}\ldots j_1}. \tag{477}$$

Then we have for (476), (477) (see (470) and its analogue for $t_{l+1}=T$)

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_l} = \int\limits_{[t,T]^{k-2}} L_p(t_3,\ldots,t_k) \prod_{g=3}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_3\ldots dt_k, \tag{478}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_lj_lj_{k-2}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} M_p^{(d)}(t_1,\ldots,t_{k-2}) \prod_{g=1}^{k-2} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{k-2}, \tag{479}$$

where

$$L_p(t_3,\ldots,t_k)=\mathbf{1}_{\{t_3<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(1)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(2)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(3)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(4)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau.$$

It is important to note that $C^{*(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$, $C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$ $(d=1,\ldots,4)$ are Fourier coefficients (see (459), (471)), that is, we can use Parseval's equality in the further proof.

Combining the equalities (459)–(463) (the case $g_2>g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 23, 24, we obtain for (459)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2}\right)^2\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\times\right.$$

$$\times \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\Biggr)^2 =$$

$$= \int\limits_{[t,T]^{k-2}} \left(\sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\right)^2 \times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k \le$$

$$\le 4\sum_{d=1}^{4} \int\limits_{[t,T]^{k-2}} \left(F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\right)^2 \times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k \le$$

$$\le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \qquad (480)$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (464)–(466) are considered analogously.

Absolutely similarly (see (480)) combining the equalities (471)–(475) (the case $g_2=g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 23, 24, we get for (471)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{d=1}^{4} C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\ne g_1,g_2}\right)^2 \le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty} \left(\sum_{d=1}^{4} C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\ne g_1,g_2}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty} \Biggl(\int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\times$$

$$\times \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\Biggr)^2 =$$

$$= \int\limits_{[t,T]^{k-2}} \left(\sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q) \right)^2 dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots\ldots dt_k \le$$

$$\le 4\sum_{d=1}^{4} \int\limits_{[t,T]^{k-2}} \left(H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q) \right)^2 dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k \le$$

$$\le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{481}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (476), (477) are considered analogously.

From (480), (481) and their analogues for the cases (464)–(466), (476), (477) we obtain

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}} \right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{482}$$

where constant K is independent of p. Thus the equality (453) is proved.

Let us prove the equality (454). Consider the following cases

1. $g_2>g_1+1,\ g_4=g_3+1,$ 2. $g_2=g_1+1,\ g_4>g_3+1,$

3. $g_2>g_1+1,\ g_4>g_3+1,$ 4. $g_2=g_1+1,\ g_4=g_3+1.$

The proof for Cases 1–3 will be similar. Consider, for example, Case 2. Using (360), we obtain

$$\sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1} \right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=0}^{p} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1} \right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_3}=0}^{p} \sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1} \right)^2 \le \tag{483}$$

$$\le (p+1)\sum_{j_{q_1}=0}^{p} \sum_{j_{g_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1} \right)^2 =$$

$$= (p+1)\sum_{j_{q_1}=0}^{p} \sum_{j_{g_3},j_{g_4}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1} \right)^2 \Bigg|_{j_{g_3}=j_{g_4}} \le$$

$$\leq (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2. \tag{484}$$

It is easy to see that the expression (484) (without the multiplier $p+1$) is a particular case ($g_4>g_3+1, g_2=g_1+1$) of the left-hand side of (482). Combining (482) and (484), we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2\leq$$

$$\leq \frac{(p+1)K}{p^{2-\varepsilon}}\leq \frac{K_1}{p^{1-\varepsilon}}\ \to\ 0 \tag{485}$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider Case 4 ($g_2=g_1+1$, $g_4=g_3+1$). We have (see (361))

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\left(\sum_{j_{g_3}=0}^{\infty}-\sum_{j_{g_3}=0}^{p}\right)C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}-\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2\leq$$

$$\leq\frac{1}{2}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2+ \tag{486}$$

$$+2\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2. \tag{487}$$

An expression similar to (487) was estimated (see (483)–(485)). Let us estimate (486). We have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright(\cdot)}\right)^2=$$

$$=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright 0}\right)^2\leq$$

$$\leq (T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\curvearrowright j_{g_3}}\right)^2, \tag{488}$$

where the notations are the same as in the proof of Theorem 20.

The expression (488) without the multiplier $T-t$ is an expression of type (412)–(417) before passing to the limit $\lim\limits_{p\to\infty}$ (the only difference is the replacement of one of the weight functions $\psi_1(\tau),\ldots,\psi_4(\tau)$ in (412)–(417) by the product $\psi_{l+1}(\tau)\psi_l(\tau)$ $(l=1,\ldots,4)$. Therefore, for Case 4 $(g_2=g_1+1,\ g_4=g_3+1)$, we obtain the estimate

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1,g_2=g_1+1}\right)^2 \leq$$

$$\leq \frac{K}{p^{1-\varepsilon}}, \tag{489}$$

where constant K is independent of p.

The estimates (485), (489) prove (454). Let us prove (455). By analogy with (488) we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 =$$

$$= (T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright 0,j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \leq$$

$$\leq (T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{g_1},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2. \tag{490}$$

Thus, we obtain the estimate (see (488) and the proof of Theorem 24)

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \leq$$

$$\leq \frac{K}{p^{2-\varepsilon}}, \tag{491}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The estimate (491) proves (455). Theorem 25 is proved.

17. Estimates for the Mean-Square Approximation Error of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicity k in Theorems 20, 22

In this section, we estimate the mean-square approximation error for iterated Stratonovich stochastic integrals of multiplicity k ($k\in\mathbb{N}$) in Theorems 20, 22.

Theorem 26 [26], [34], [50], [72]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuously differentiable nonrandom function at the interval $[t,T]$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then the following estimates*

$$
{\sf M}\left\{\left(J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\le
$$

$$
\le K_1\left(\frac{1}{p}+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}{\sf M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right), \tag{492}
$$

$$
{\sf M}\left\{\left(J^{*}[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\le
$$

$$
\le K_2(s)\left(\frac{1}{p}+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}{\sf M}\left\{\left(R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right) \tag{493}
$$

hold, where $s\in(t,T]$ (s is fixed), $i_1,\ldots,i_k=1,\ldots,m$,

$$
R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Biggr|_{T=s},
$$

$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ is defined by (375), $J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ and $J^{*}[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}$ are iterated Stratonovich stochastic integrals (330) and (384), $C_{j_k\ldots j_1}$ and $C_{j_k\ldots j_1}(s)$ are Fourier coefficients (322) and (382), constants K_1 and $K_2(s)$ are independent of p; another notations are the same as in Theorems 1, 20, 22.

Proof. Note that Conditions 1 and 2 of Theorems 20, 22 are satisfied under the conditions of Theorem 26 (see Remark 2.4 in [26]). Then from the proof of Theorem 20 it follows that the expression (380) before passing to limit $\underset{p\to\infty}{\text{l.i.m.}}$ has the form

$$
\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}+
$$

$$+\sum_{r=1}^{[k/2]}\Biggl(\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Biggr), \tag{494}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the approximation for the iterated Ito stochastic integral (2), which is obtained using Theorem 18, i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{495}$$

$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}$ is the approximation obtained using (495) for the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ (see (381)).

Using (494) and Theorem 19, we have

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}+$$

$$+\Bigl(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\Bigr)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\frac{1}{2^r}\Biggl(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\Biggr)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=$$

$$=J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\Bigl(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\Bigr)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\frac{1}{2^r}\Biggl(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\Biggr)+$$

$$
+\sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{496}
$$

w. p. 1, where we denote $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$.

Applying (263), we obtain the following estimates

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le\frac{C}{p}, \tag{497}
$$

$$
\mathsf{M}\left\{\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)^2\right\}\le
$$

$$
\le\frac{C}{p}, \tag{498}
$$

where constant C does not depend on p.

From (496)–(498) and the elementary inequality

$$
(a_1+a_2+\ldots+a_n)^2\le n\left(a_1^2+a_2^2+\ldots+a_n^2\right),\quad n\in\mathbb{N}
$$

we obtain (492).

The estimate (493) is obtained similarly to the estimate (492) using Theorems 5, 22 and (308). Theorem 26 is proved.

18. Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3–5 in Theorems 23–25

In this section, we consider the rate of convergence of approximations of iterated Stratonovich stochastic integrals in Theorems 23–25. It is easy to see that in Theorems 23–25 the second term in parentheses on the right-hand side of (492) is estimated. Combining these results with Theorem 26, we obtain the following theorems.

Theorem 27 [26], [34], [50], [72]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}
$$

the following estimate

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p}
$$

is fulfilled, where $i_1, i_2, i_3 = 1, \ldots, m$*, constant* C *is independent of* p*,*

$$
C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3
$$

and

$$
\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j*.*

Theorem 28 [26], [34], [50], [72]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} d\mathbf{f}_{t_4}^{(i_4)}
$$

the following estimate

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}}
$$

holds, where $i_1, i_2, i_3, i_4 = 1, \ldots, m$*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$
C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \times
$$

$$
\times dt_2 dt_3 dt_4;
$$

another notations are the same as in Theorem 27*.*

Theorem 29 [26], [34], [50], [72]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_5}^{(i_5)}
$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

is valid, where $i_1,\ldots,i_5=1,\ldots,m$, *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorem 27, 28.

19. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case $p_1=\ldots=p_6\to\infty$ and $\psi_1(\tau),\ \ldots,\ \psi_6(\tau)\equiv 1$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 30 [26], [34], [72], [73]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)}\qquad(499)$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. As noted in Sect. 13, Conditions 1 and 2 of Theorem 20 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 20 for the iterated Stratonovich stochastic integral (499). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3},j_{q_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2=0, \tag{500}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0, \tag{501}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1}\right)^2=0, \tag{502}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right)^2=0, \tag{503}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_6=g_5+1}\right)^2=0, \tag{504}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_4=g_3+1,g_6=g_5+1}\right)^2=0, \tag{505}$$

where the expressions

$$(\{g_1,g_2\},\{g_3,g_4\},\{g_5,g_6\}\}),\quad (\{g_1,g_2\},\{g_3,g_4\},\{q_1,q_2\}\}),\quad (\{g_1,g_2\},\{q_1,q_2,q_3,q_4\})$$

are partitions of the set $\{1,2,\ldots,6\}$ that is $\{g_1,g_2,g_3,\,g_4,g_5,g_6\}=\{g_1,g_2,g_3,g_4,q_1,q_2\}=\{g_1,g_2,q_1,$ $q_2,q_3,q_4\}=\{1,2,\,\ldots,6\}$; braces mean an unordered set, and parentheses mean an ordered set.

The equalities (500), (502) were proved earlier (see the proof of equalities (482), (488)). The relation (505) follows from the estimate (43) for the polynomial case and its analogue for the trigonometric case. It is easy to see that the equalities (501) and (504) are proved in complete analogy with the proof of (454), (488).

Thus, we have to prove the relation (503). The equality (503) is equivalent to the following equalities

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=0, \tag{506}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_3j_2j_3j_2j_1}=0, \tag{507}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=0, \tag{508}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=0, \tag{509}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_2j_3j_3j_1}=0, \tag{510}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=0, \tag{511}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=0, \tag{512}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=0, \tag{513}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=0, \tag{514}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0, \tag{515}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_1j_3j_3j_2j_1}=0, \tag{516}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0, \tag{517}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0, \tag{518}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0, \tag{519}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0. \tag{520}$$

Consider in detail the case of Legendre polynomials (the case of trigonometric functions is considered in complete analogy).

First, we prove the following equality for the Fourier coefficients for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

$$C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$

$$+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}. \tag{521}$$

Using the integration order replacement, we have

$$C_{j_6j_5j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6=$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6+$$

$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)dt_5dt_6\ C_{j_4j_3j_2j_1}+$$

$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$

$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$

$$\ldots$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}-$$

$$-\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\ldots\int\limits_{t_2}^T \phi_{j_1}(t_1)dt_1\ldots dt_5dt_6 =$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} -$$

$$-C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} - C_{j_1j_2j_3j_4j_5j_6}. \tag{522}$$

The equality (522) completes the proof of the relation (521).

Let us consider (506). From (354) we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}. \tag{523}$$

Applying (521), we get

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_1j_2j_3} = 2\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1} - C_{j_2j_3}C_{j_1j_3j_2j_1} + C_{j_1j_2j_3}C_{j_3j_2j_1} -$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1} + C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr). \tag{524}$$

Recall that the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j=0,1,2,\ldots,$$

where

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j$$

is the Legendre polynomial.

Note that

$$C_{j_2j_1} = \int\limits_t^T \phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(\theta)d\theta d\tau =$$

$$
= \frac{T-t}{2} \begin{cases} 1/\sqrt{(2j_1+1)(2j_1+3)} & \text{if } j_2 = j_1+1,\ j_1 = 0,1,2,\ldots \\ -1/\sqrt{4j_1^2-1} & \text{if } j_2 = j_1-1,\ j_1 = 1,2,\ldots \\ 1 & \text{if } j_1 = j_2 = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{525}
$$

$$
C_{j_1} = \int\limits_t^T \phi_{j_1}(\tau)d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_1 = 0 \\ 0 & \text{if } j_1 \neq 0 \end{cases}. \tag{526}
$$

Moreover, the generalized Parseval equality gives

$$
\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3}C_{j_3j_2j_1} =
$$

$$
= \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3 \times
$$

$$
\times \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3 =
$$

$$
= \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_1)dt_1dt_2dt_3 \times
$$

$$
\times \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3 =
$$

$$
= \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3 \times
$$

$$
\times \int\limits_{[t,T]^3} \mathbf{1}_{\{t_1<t_2<t_3\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3 =
$$

$$
= \int\limits_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}}\mathbf{1}_{\{t_1<t_2<t_3\}}dt_1dt_2dt_3 = 0. \tag{527}
$$

Using the above arguments and also (354), (523), and (524), we get

$$-\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1j_3j_2j_1}=$$

$$=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_2j_3}C_{j_1j_3j_2j_1}-$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1}+C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr)=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_3j_1j_2j_3}C_{j_2j_1}\Biggr)=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}-\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2j_3}C_{j_2j_1}=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}+\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}. \tag{528}$$

By analogy with the proof of (418) (see the proof of Theorem 24) we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}=\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_10j_2j_1}=0, \tag{529}$$

where we used the following representation

$$C_{j_2j_10j_2j_1}=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}dt_3dt_2dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)(t_4-t)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5+$$

$$+\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\bar C_{j_2j_1j_2j_1}+\tilde C_{j_2j_1j_2j_1}.$$

Further, we have (see (525))

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1} = \lim_{p\to\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{00}C_{j_300j_3}+$$

$$+\sum_{j_1=1}^{p}C_{j_1-1,j_1}C_{j_3j_1,j_1-1,j_3}+\sum_{j_1=1}^{p-1}C_{j_1+1,j_1}C_{j_3j_1,j_1+1,j_3}+C_{1,0}C_{j_301j_3}\Biggr). \tag{530}$$

Observe that

$$|C_{j_1-1,j_1}|+|C_{j_1+1,j_1}|\le\frac{K}{j_1}\quad (j_1=1,\ldots,p), \tag{531}$$

$$|C_{j_300j_3}|+|C_{j_3j_1,j_1-1,j_3}|+|C_{j_3j_1,j_1+1,j_3}|+|C_{j_301j_3}|\le\frac{K_1}{j_3^2}\quad (j_3\ge p+1), \tag{532}$$

where constants K, K_1 do not depend on j_1, j_3.

The estimate (531) follow from (525). At the same time, the estimate (532) can be obtained using the following reasoning. First note that the integration order replacement gives

$$C_{j_3j_1j_2j_3}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1\right)dt_2\left(\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4\right)dt_3. \tag{533}$$

Note analogues of the estimate (87)

$$\left|\int\limits_t^x\phi_{j_1}(s)ds\right|<\frac{C}{j_1(1-(z(x))^2)^{1/4}},\quad \left|\int\limits_x^T\phi_{j_1}(s)ds\right|<\frac{C}{j_1(1-(z(x))^2)^{1/4}},\quad x\in(t,T), \tag{534}$$

where $j_1>0$, constant C does not depend on j_1.

Applying the estimates (89) and (534) to (533) gives the estimate (532). Using (530), (531), and (532), we obtain

$$\left|\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}\right|\le K\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\left(1+\sum_{j_1=1}^{p}\frac{1}{j_1}\right)\le$$

$$\le K\int\limits_p^{\infty}\frac{dx}{x^2}\left(2+\int\limits_1^p\frac{dx}{x}\right)=\frac{K(2+lnp)}{p}\to 0 \tag{535}$$

if $p\to\infty$, where constant K is independent of p. Thus, the equality (506) is proved (see (528), (529), (535)).

The relation (507) is proved in complete analogy with the proof of equality (506). For (507) we have (see (521))

$$\lim_{p\to\infty}\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1}+\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_2j_3j_1}\right)=2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_1}C_{j_3j_2j_3j_2j_1}-C_{j_3j_1}C_{j_2j_3j_2j_1}+C_{j_2j_3j_1}C_{j_3j_2j_1}-$$

$$-C_{j_3j_2j_3j_1}C_{j_2j_1}+C_{j_2j_3j_2j_3j_1}C_{j_1}\Bigr)=$$

$$=2\lim_{p\to\infty}\left(\sqrt{T-t}\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_20}-\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}\right)=$$

$$=-2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}.$$

To estimate the Fourier coefficient $C_{j_3j_2j_3j_1}$, we use the following (see the proof of (506) for more details)

$$C_{j_3j_2j_3j_1}=\int_t^T\phi_{j_3}(t_4)\int_t^{t_4}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_3}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int_t^T\phi_{j_3}(t_4)\int_t^{t_4}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_1)\int_{t_1}^{t_3}\phi_{j_3}(t_2)dt_2dt_1dt_3dt_4=$$

$$=\int_t^T\phi_{j_3}(t_4)\int_t^{t_4}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_4-$$

$$-\int_t^T\phi_{j_3}(t_4)\int_t^{t_4}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_1)\left(\int_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1dt_3dt_4=$$

$$=\int_t^T\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int_t^{t_3}\phi_{j_1}(t_1)dt_1\left(\int_{t_3}^T\phi_{j_3}(t_4)dt_4\right)dt_3-$$

$$-\int_t^T\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_1)\left(\int_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1\left(\int_{t_3}^T\phi_{j_3}(t_4)dt_4\right)dt_3.$$

Let us prove (508). From (354) we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_3j_1j_2j_1}. \tag{536}$$

Applying (521) and (536), we get (we replaced j_3 by j_4)

$$\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}+\sum_{j_1,j_2,j_4=0}^{p}C_{j_1j_2j_1j_4j_2j_4}=2\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}=$$

$$=\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_4}C_{j_2j_4j_1j_2j_1}-C_{j_2j_4}C_{j_4j_1j_2j_1}+C_{j_4j_2j_4}C_{j_1j_2j_1}-$$

$$-C_{j_1j_4j_2j_4}C_{j_2j_1}+C_{j_2j_1j_4j_2j_4}C_{j_1}\Biggr)=$$

$$=2\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Biggr)+$$

$$+\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4}C_{j_1j_2j_1}. \tag{537}$$

Further, we have (see (354))

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4}C_{j_1j_2j_1}=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=$$

$$=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=0, \tag{538}$$

where we applied the equality (392).

Furthermore, by analogy with the proof of (506), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Biggr)=0. \tag{539}$$

To estimate the Fourier coefficient $C_{j_1j_4j_2j_4}$ in (539), we use the following (see the proof of (506) for more details)

$$C_{j_1j_4j_2j_4}=\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_3dt_4=$$

$$= \int_t^T \phi_{j_1}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) \left(\int_t^{t_2} \phi_{j_4}(t_1) dt_1 \right) \int_{t_2}^{t_4} \phi_{j_4}(t_3) dt_3 dt_2 dt_4 =$$

$$= \int_t^T \phi_{j_1}(t_4) \left(\int_t^{t_4} \phi_{j_4}(t_3) dt_3 \right) \int_t^{t_4} \phi_{j_2}(t_2) \left(\int_t^{t_2} \phi_{j_4}(t_1) dt_1 \right) dt_2 dt_4 -$$

$$- \int_t^T \phi_{j_1}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) \left(\int_t^{t_2} \phi_{j_4}(t_3) dt_3 \right) \left(\int_t^{t_2} \phi_{j_4}(t_1) dt_1 \right) dt_2 dt_4.$$

The relations (536)–(539) complete the proof of equality (508).
Let us prove (509). Using (354), we get

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_1 j_2 j_3 j_3 j_2 j_1} = \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_1 j_2 j_3 j_3 j_2 j_1}. \tag{540}$$

Applying (521) and (540), we obtain

$$2 \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_1 j_2 j_3 j_3 j_2 j_1} =$$

$$= \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Biggl(C_{j_1} C_{j_2 j_3 j_3 j_2 j_1} - C_{j_2 j_1} C_{j_3 j_3 j_2 j_1} + \left(C_{j_3 j_2 j_1}\right)^2 -$$

$$- C_{j_3 j_3 j_2 j_1} C_{j_2 j_1} + C_{j_2 j_3 j_3 j_2 j_1} C_{j_1} \Biggr) =$$

$$= 2 \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \left(C_{j_1} C_{j_2 j_3 j_3 j_2 j_1} - C_{j_2 j_1} C_{j_3 j_3 j_2 j_1} \right) +$$

$$+ \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \left(C_{j_3 j_2 j_1}\right)^2. \tag{541}$$

In [26] (Sect. 1.7.2) the following estimate

$$\sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1}^2 \le$$

$$\le L_k \sum_{j_s=p+1}^{\infty} \frac{1}{j_s^2} \le L_k \int_p^{\infty} \frac{dx}{x^2} = \frac{L_k}{p} \tag{542}$$

is proved for the polynomial and trigonometric cases, where $s = 1, \ldots, k$, constant L_k depends on k and $T - t$.

Using the estimate (542), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2 = 0. \tag{543}$$

By analogy with the proof of (506), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_1}C_{j_2j_3j_3j_2j_1} - C_{j_2j_1}C_{j_3j_3j_2j_1}\Bigr) = 0, \tag{544}$$

where we applied the equality (419). To estimate the Fourier coefficient $C_{j_3j_3j_2j_1}$ in (544), we used the following (see the proof of (506) for more details)

$$C_{j_3j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\left(\int\limits_{t_2}^{T}\phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \tag{545}$$

Combining the equalities (540)–(544), we obtain (509).

Let us prove (510) (we replace j_2 by j_4 and j_3 by j_2 in (510)). As noted in Sect. 13, the sequential order of the series

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}$$

is not important. This follows directly from the formulas (361) and (354).

Applying the mentioned property and (354), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{546}$$

Observe that (see the above reasoning)

$$\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = \sum_{j_4=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{547}$$

Using (521) and (547), we obtain

$$\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1j_4j_4j_2j_2j_1}+C_{j_1j_2j_2j_4j_4j_1}\Bigr)=2\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}+C_{j_4j_4j_1}C_{j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)+$$

$$+\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2. \tag{548}$$

The equality

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2=0 \tag{549}$$

follows from the relation (391).

By analogy with the proof of equality (506) we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)=0, \tag{550}$$

where we applied the equality (420). To estimate the Fourier coefficient $C_{j_2j_4j_4j_1}$ in (550), we used the following (see the proof of (506) for more details)

$$C_{j_2j_4j_4j_1}=\int_t^T\phi_{j_2}(t_4)\int_t^{t_4}\phi_{j_4}(t_3)\int_t^{t_3}\phi_{j_4}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int_t^T\phi_{j_2}(t_4)\int_t^{t_4}\phi_{j_1}(t_1)\int_{t_1}^{t_4}\phi_{j_4}(t_2)\int_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_1dt_4=$$

$$=\frac{1}{2}\int_t^T\phi_{j_2}(t_4)\int_t^{t_4}\phi_{j_1}(t_1)\left(\int_{t_1}^{t_4}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4+$$

$$+\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4-$$

$$-\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)dt_1dt_4.$$

The relation (510) follows from (546), (548)–(550).

Consider (511). Using the integration order replacement, we obtain

$$C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{4}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3. \tag{551}$$

Applying the estimates (534) to (551) gives the following estimate

$$\left|C_{j_3j_3j_2j_2j_1j_1}\right|\le\frac{K}{j_1^2j_3^2}\quad(j_1,j_3>0,\ j_2\ge 0), \tag{552}$$

where constant K does not depend on j_1,j_2,j_3.

Further, we get (see (361))

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\Bigg|_{(j_2j_2)\frown(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}, \tag{553}$$

where

$$C_{j_3j_3j_2j_2j_1j_1}\Bigg|_{(j_2j_2)\frown(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\int\limits_t^{t_4}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_5}dt_4dt_2dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)(t_5-t)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6+$$

$$+\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6\stackrel{\sf def}{=}$$

(554) $$\stackrel{\sf def}{=}C'_{j_3j_3j_1j_1}+C''_{j_3j_3j_1j_1}.$$

Let us substitute (554) into (553)

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}+$$

(555) $$+\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}.$$

The relation (420) implies that

(556) $$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}=0,\quad \lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}=0.$$

From the estimate (552) we get

$$\left|\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\right|\le K(p+1)\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\le$$

(557) $$\le K(p+1)\left(\int\limits_p^{\infty}\frac{dx}{x^2}\right)^2\le\frac{K(p+1)}{p^2}\ \to\ 0$$

if $p\to\infty$, where constant K is independent of p.

The relations (555)–(557) complete the proof of (511).

Let us prove (512). Using the integration order replacement, we get

$$C_{j_2j_3j_3j_2j_1j_1}=$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6dt_5dt_4dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)dt_5dt_3-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)dt_5dt_3. \tag{558}
$$

Applying (354) and (361), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=-\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggr|_{(j_3j_3)\frown(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggr|_{(j_3j_3)\frown(\cdot)}-\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}-
$$

$$
-\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}-\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}-
$$

$$
-\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}. \tag{559}
$$

The equality

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggr|_{(j_3j_3)\frown(\cdot)}=0 \tag{560}
$$

follows from the inequality similar to (446) (see the proof of Theorem 24), where we used the following representation

$$C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\frown(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{t_6}dt_4dt_3dt_6=$$

$$+\int\limits_t^T\phi_{j_2}(t_6)(t_6-t)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6+$$

$$+\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)(t-t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6\stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=}C^{*}_{j_2j_2j_1j_1}+C^{**}_{j_2j_2j_1j_1}. \tag{561}$$

Applying the estimates (534) and (406) ($\varepsilon=1/2$) to (558) gives the following estimates

$$|C_{j_2j_3j_3j_2j_1j_1}|\le\frac{K}{j_1^2j_2j_3^{3/4}}\quad(j_1,j_2,j_3>0), \tag{562}$$

$$|C_{j_200j_2j_1j_1}|\le\frac{K}{j_1^2j_2}\quad(j_1,j_2>0), \tag{563}$$

$$|C_{0j_3j_30j_1j_1}|\le\frac{K}{j_1^2j_3}\quad(j_1,j_3>0), \tag{564}$$

$$|C_{0000j_1j_1}|\le\frac{K}{j_1^2}\quad(j_1>0). \tag{565}$$

Using the estimate (562), we have

$$\left|\sum_{j_2=1}^p\sum_{j_3=1}^p\sum_{j_1=p+1}^\infty C_{j_2j_3j_3j_2j_1j_1}\right|\le K\sum_{j_1=p+1}^\infty\frac{1}{j_1^2}\sum_{j_2=1}^p\frac{1}{j_2}\sum_{j_3=1}^p\frac{1}{j_3^{3/4}}\le$$

$$\le K\int\limits_p^\infty\frac{dx}{x^2}\left(1+\int\limits_1^p\frac{dx}{x}\right)\left(1+\int\limits_1^p\frac{dx}{x^{3/4}}\right)\le K_1\frac{1+lnp}{p^{3/4}}\ \to 0 \tag{566}$$

if $p\to\infty$, where constants K,K_1 do not depend on p.

Similarly we get (see (563)–(565))

$$
\left|\sum_{j_1=p+1}^{\infty} C_{0000j_1j_1}\right| + \left|\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{0j_3j_30j_1j_1}\right| + \left|\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_200j_2j_1j_1}\right| \to 0 \tag{567}
$$

if $p\to\infty$.

The relations (559), (560), (566), (567) prove (512).

Consider (513). Using the integration order replacement, we get

$$
C_{j_3j_2j_3j_2j_1j_1} =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_2}(t_5)\int\limits_t^{t_5} \phi_{j_3}(t_4)\int\limits_t^{t_4} \phi_{j_2}(t_3)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)^2 \int\limits_{t_3}^T \phi_{j_3}(t_4)\int\limits_{t_4}^T \phi_{j_2}(t_5)\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6dt_5dt_4dt_3 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)^2 \int\limits_{t_3}^T \phi_{j_2}(t_5)\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6\int\limits_{t_3}^{t_5} \phi_{j_3}(t_4)dt_4dt_5dt_3 =
$$

$$
= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)^2 \int\limits_{t_3}^T \phi_{j_2}(t_5)\left(\int\limits_t^{t_5} \phi_{j_3}(t_4)dt_4\right)\left(\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6\right)dt_5dt_3 -
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)^2 \left(\int\limits_t^{t_3} \phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T \phi_{j_2}(t_5)\left(\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6\right)dt_5dt_3. \tag{568}
$$

Applying (354), we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1} = \sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1} =
$$

$$
= -\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1}. \tag{569}
$$

Further proof of the equality (513) is based on the relations (568), (569) and is similar to the proof of the formula (512).

Let us prove (514). Applying the integration order replacement, we obtain

$$
C_{j_3j_3j_2j_1j_2j_1} =
$$

$$
=\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$
=\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_4)\int\limits_{t_4}^{T}\phi_{j_3}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1 =
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1 =
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4 =
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_1}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)dt_2dt_4-
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)^2 dt_2dt_4. \tag{570}
$$

Using (354), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1} = \sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1} =
$$

$$
= -\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}. \tag{571}
$$

Further proof of the equality (514) is based on the relations (570), (571) and is similar to the proof of the relations (512), (513).

Consider (515). Using the integration order replacement, we have

$$
C_{j_3j_3j_1j_2j_2j_1} =
$$

$$
=\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =
$$

$$=\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_5)\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4} \phi_{j_2}(t_3)dt_3dt_2dt_4 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \left(\int\limits_t^{t_4} \phi_{j_2}(t_3)dt_3\right)\int\limits_t^{t_4} \phi_{j_2}(t_2)\left(\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\right)dt_2dt_4-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 \int\limits_t^{t_4} \phi_{j_2}(t_2)\left(\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_2} \phi_{j_2}(t_3)dt_3\right)dt_2dt_4. \tag{572}$$

Applying (354) and (361), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1} = -\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_3j_1j_2j_2j_1} =$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_1j_2j_2j_1} = \sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_1j_2j_2j_1} =$$

$$=\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\frown(\cdot)} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1}. \tag{573}$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\frown(\cdot)} = 0 \tag{574}$$

follows from the inequality (446), where we proceed similarly to the proof of equality (560) (see (561)).

The relation

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1} = 0 \tag{575}$$

is proved on the basis of (572) and similarly with the proof of (512). The equalities (573)–(575) prove (515).

Let us prove (516). Using (354) and (361), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_1j_3j_3j_2j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1} =$$

$$= \frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}. \tag{576}$$

Using the equality (418) we have

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)} = 0, \tag{577}$$

where we proceed similarly to the proof of equality (560) (see (561)).

Further, we will prove the following relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1} = 0 \tag{578}$$

using the equality (521). From (521) we have

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1} = \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2j_1j_3j_3j_2j_1} + C_{j_1j_2j_3j_3j_1j_2}\Bigr) =$$

$$= \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2}C_{j_1j_3j_3j_2j_1} - C_{j_1j_2}C_{j_3j_3j_2j_1} + C_{j_3j_1j_2}C_{j_3j_2j_1} -$$

$$-C_{j_3j_3j_1j_2}C_{j_2j_1} + C_{j_2j_3j_3j_1j_2}C_{j_1}\Bigr) =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2j_3j_3j_1j_2}C_{j_1} - C_{j_3j_3j_1j_2}C_{j_2j_1}\Bigr) +$$

$$+\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}. \tag{579}$$

The generalized Parseval equality gives (by analogy with (527))

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1} = 0. \tag{580}$$

Let us prove the following equality

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr)=0. \tag{581}$$

The relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_3j_3j_1j_2}C_{j_1}=0 \tag{582}$$

is proved by the same methods as in the proof of equality (506) and also using Theorem 24 and (361).
Further, we have (see (361))

$$\sum_{j_3=0}^{p}C_{j_3j_3j_1j_2}=\frac{1}{2}C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\frown(\cdot)}-\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}. \tag{583}$$

Moreover,

$$C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\frown(\cdot)}=\int\limits_t^T\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3=$$

$$=\int\limits_t^T\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1\int\limits_{t_2}^T dt_3dt_2=\int\limits_t^T(T-t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2=$$

$$=\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^T(T-t_2)\phi_{j_1}(t_2)dt_2dt_1=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^T(T-t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}(T-t_1)\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\stackrel{\mathsf{def}}{=}\tilde C_{j_2j_1}. \tag{584}$$

Using (583), (584), and the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_3j_1j_2}C_{j_2j_1}=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\tilde C_{j_2j_1}-$$

$$-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}C_{j_2j_1}=-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}C_{j_2j_1}. \tag{585}$$

We have (see (545))

$$C_{j_3j_3j_1j_2}=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^T\phi_{j_1}(t_2)\Biggl(\int\limits_{t_2}^T\phi_{j_3}(t_3)dt_3\Biggr)^2dt_2dt_1. \tag{586}$$

By analogy with (535) and also using (586), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}C_{j_2j_1}=0. \tag{587}$$

Combining (585) and (587), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_3j_1j_2}C_{j_2j_1}=0. \tag{588}$$

The relation (581) follows from (582) and (588). From (579)–(581) we get (578). The equalities (576)–(578) complete the proof of (516).

For the proof of (517)–(520) we will use a new idea. More precisely, we will consider the sums of expressions (517)–(520) with the expressions already studied throughout this proof.

Let us begin from (517). Applying the integration order replacement, we obtain

$$C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_2}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)^2\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_2}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5-$$

$$-\int\limits_t^T\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)^2\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5. \tag{589}$$

Using (354), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=$$

(590)
$$=\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right).$$

Further, by analogy with the proof of equality (512) and using (589), we obtain

(591)
$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0.$$

From (590) and (591) we get

(592)
$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0.$$

Moreover (see (506)),

(593)
$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_1j_2}=0.$$

Combining (592) and (593), we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0.$$

The equality (517) is proved.

Consider (518). Using the integration order replacement, we have

$$C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$
-\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_5dt_6 =
$$

$$
=\int\limits_t^T \phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-
$$

$$
-\int\limits_t^T \phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5. \tag{594}
$$

Using (354), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=
$$

$$
=\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr). \tag{595}
$$

By analogy with the proof of (512) and applying (594), we get

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=0. \tag{596}
$$

From (595) and (596) we have

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=0. \tag{597}
$$

Moreover (see (507)),

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_1j_2}=0. \tag{598}
$$

Combining (597) and (598), we finally obtain

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0.
$$

The equality (518) is proved.

Now consider (519). Using the integration order replacement, we obtain

$$
C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}=
$$

$$= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) dt_3dt_4dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) \int\limits_{t_3}^{t_5} \phi_{j_3}(t_4)dt_4dt_3dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \left(\int\limits_t^{t_5} \phi_{j_3}(t_4)dt_4\right) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) dt_3dt_5dt_6 -$$

$$- \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) \left(\int\limits_t^{t_3} \phi_{j_3}(t_4)dt_4\right) dt_3dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_1}(t_5) \left(\int\limits_t^{t_5} \phi_{j_3}(t_4)dt_4\right) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6)dt_6\right) dt_5 -$$

(599)
$$- \int\limits_t^T \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right) \left(\int\limits_t^{t_3} \phi_{j_3}(t_4)dt_4\right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6)dt_6\right) dt_5.$$

Applying (354) and (361), we obtain

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3j_1j_3j_2j_2j_1} + C_{j_3j_1j_3j_2j_1j_2}\Bigr) =$$

$$= -\sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_3j_1j_3j_2j_2j_1} + C_{j_3j_1j_3j_2j_1j_2}\Bigr) =$$

$$= \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3j_1j_3j_2j_2j_1} + C_{j_3j_1j_3j_2j_1j_2}\Bigr) -$$

(600)
$$-\frac{1}{2} \sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}\Bigr|_{(j_2j_2)\frown(\cdot)}.$$

The equality

(601)
$$\lim_{p\to\infty} \frac{1}{2} \sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}\Bigr|_{(j_2j_2)\frown(\cdot)} = 0$$

follows from the equality (418), where we proceed similarly to the proof of equality (560) (see (561)).

By analogy with the proof of (512) and applying (599), we get

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{602}$$

From (600)–(602) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{603}$$

Moreover (see (508)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_1j_2}=0. \tag{604}$$

Combining (603) and (604), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0.$$

The equality (519) is proved.

Finally consider (520). Using the integration order replacement, we have

$$C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-$$

(605)
$$-\int\limits_t^T \phi_{j_3}(t_5)\int\limits_t^{t_5} \phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5.$$

Using (354) and (361), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)-$$

$$-\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)-$$

$$-\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}. \tag{606}$$

The equalities

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)=0, \tag{607}$$

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\lim_{p\to\infty}\frac{1}{4}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)(j_3j_3)\curvearrowright(\cdot)}-$$

$$-\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_1j_1)\curvearrowright(\cdot)}=0 \tag{608}$$

follows from the equalities (418), (419), where we used the same technique as in (561). When proving (608), we also applied (361) and (43).

By analogy with the proof of (512) and applying (605), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=0. \tag{609}$$

From (606)–(609) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=0. \tag{610}$$

Furthermore (see (510)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}=0. \tag{611}$$

Combining (610) and (611), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0.$$

The equality (520) is proved. Theorem 30 is proved.

20. Generalization of Theorem 23. The Case $p_1,\ p_2,\ p_3\to\infty$ and Continuously Differetiable Weight Functions (The Cases of Legendre Polynomials and Trigonometric Functions)

This section is devoted to the following theorem.

Theorem 31 [26], [34], [72]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau),\ \psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{612}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m$,

$$C_{j_3j_2j_1}=\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Proof. Let us consider the case of Legendre polynomials (the trigonometric case is simpler and can be considered similarly). Applying (348), we obtain

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}} C_{j_1j_2j_1} J'[\phi_{j_2}]_{T,t}^{(i_2)} \tag{613}$$

w. p. 1, where notations are the same as in (348).

Using Theorem 19 (see (321) for the case $k=3$), Theorem 1 (see (333)) as well as (366) (see the derivation of (366)) and (361), we get

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} = J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2 d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3 =$$

$$= J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} + \frac{1}{2}J[\psi^{(3)}]_{T,t}^{1} + \frac{1}{2}J[\psi^{(3)}]_{T,t}^{2} =$$

$$= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Bigg|_{(j_2j_1)\curvearrowright(\cdot),\, j_1=j_2} J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\Bigg|_{(j_3j_2)\curvearrowright(\cdot),\, j_2=j_3} J'[\phi_{j_1}]_{T,t}^{(i_1)} =$$

$$
= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)} \tag{614}
$$

w. p. 1.

Using (613), (614) and the elementary inequality

$$
(a+b+c+d)^2\le 4\left(a^2+b^2+c^2+d^2\right),
$$

we obtain

$$
\mathsf{M}\left\{\left(J^{*}[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le 4\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}-J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathsf{M}\left\{\left(\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)}\right)^2\right\}=
$$

$$
=4A_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}B_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}C_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}D_{p_1p_2p_3}. \tag{615}
$$

Theorem 1 gives (see (333))

$$\lim_{p_1,p_2,p_3\to\infty} A_{p_1p_2p_3}=0. \tag{616}$$

Further, in complete analogy with (410) and using (354), we obtain

$$D_{p_1p_2p_3}=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}\right)^2=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le$$

$$\le\sum_{j_2=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le\frac{K}{\left(\min\{p_1,p_3\}\right)^{2-\varepsilon}}\ \to\ 0 \tag{617}$$

if $p_1,p_2,p_3\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

We have

$$B_{p_1p_2p_3}=\mathsf{M}\left\{\left(\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)\right)^2\right\}\le$$

$$\le 2E_{p_3}+2F_{p_1p_2p_3}, \tag{618}$$

where

$$E_{p_3}=\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\},$$

$$F_{p_1p_2p_3}=\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}=$$

$$=\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2. \tag{619}$$

By analogy with (397) we get

$$\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1}\right)^2 \le \sum_{j_3=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1}\right)^2 \le$$

$$\le \frac{K}{\left(\min\{p_1,p_2\}\right)^2} \ \to\ 0 \tag{620}$$

if $p_1, p_2, p_3 \to \infty$, where constant K does not depend on p.

Moreover,

$$\lim_{p_3\to\infty} E_{p_3} = \lim_{p_1,p_2,p_3\to\infty} E_{p_3} = 0. \tag{621}$$

Combining (618)–(621), we obtain

$$\lim_{p_1,p_2,p_3\to\infty} B_{p_1p_2p_3} = 0. \tag{622}$$

Consider $C_{p_1p_2p_3}$. We have

$$C_{p_1p_2p_3} = \mathsf{M}\left\{\left(\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)\right)^2\right\} \le$$

$$\le 2G_{p_1} + 2H_{p_1p_2p_3}, \tag{623}$$

where

$$G_{p_1} = \mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\},$$

$$H_{p_1p_2p_3} = \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\} =$$

$$= \sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty} C_{j_3j_3j_1}\right)^2. \tag{624}$$

By analogy with (401) we get

$$
\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le\sum_{j_1=0}^{\infty}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le
$$

$$
\le\frac{K}{\left(\min\{p_2,p_3\}\right)^2}\ \to\ 0 \tag{625}
$$

if $p_1,p_2,p_3\to\infty$, where constant K does not depend on p.
Moreover,

$$
\lim_{p_1\to\infty}G_{p_1}=\lim_{p_1,p_2,p_3\to\infty}G_{p_1}=0. \tag{626}
$$

Combining (623)–(626), we obtain

$$
\lim_{p_1,p_2,p_3\to\infty}C_{p_1p_2p_3}=0. \tag{627}
$$

The relations (615)–(617), (622), (627) complete the proof of Theorem 31. Theorem 31 is proved.

21. Modification of Condition 3 of Theorem 20 Using Parseval's Equality

First, note that (see the proof of Theorem 20 and (377))

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-\right.
$$

$$
\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)\times
$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}}\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}}\right)\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\frac{1}{2^r}\prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1.} \tag{628}$$

Using (628) and the condition (386), we obtain (378). This means that we get (380). Thus the expansion (331) is proved.

Analyzing the proof of Theorems 20, 19 and taking into account the above arguments, it is easy to see that the following theorem is true.

Theorem 32 [26], [34], [72]. *Assume that the continuous functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following condition*

$$\lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \times$$

$$\times\left(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}}\right)^2 = 0 \tag{629}$$

is satisfied for all $r = 1, 2, \ldots, [k/2]$. *Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity* k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Let us consider the special case $k = 2$ of Theorem 32 in more detail. In this case, the condition (629) takes the following form (compare with (53))

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1. \qquad (630)$$

It is easy to see that the condition $\phi_0(x) = 1/\sqrt{T-t}$ can be omitted in Theorems 32 for the case $k = 2$ (see the proof of Theorem 20).

Summing up the above arguments, we obtain the following generalization of Theorem 2.

Theorem 33. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t, T])$. *Moreover,* $\psi_1(\tau), \psi_2(\tau)$ *are continuous functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorem 2.

The condition of continuity of the functions $\psi_1(\tau), \psi_2(\tau)$ is related to the definition [2] of the Stratonovich stochastic integral that we use.

Theorem 33 can be generalized to the case $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T]$ if instead of the definition from [2] we will use another definition of the Stratonovich stochastic integral (see [26] (Sect. 2.18, Theorem 2.44) for details).

Let us make some remarks about the development of the approach based on Theorem 20 and describe the algorithm of the verification of Condition 3 of Theorem 20. First, consider the case $k = 2n + 1,\ n = 3, 4, \ldots$ (k is the multiplicity of the iterated Stratonovich stochastic integral (330)). Let Conditions 1 and 2 of Theorem 20 be satisfied. Consider the equality (385). The right-hand side of (385) has the form

$$\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -$$

$$-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}.$$

Iterated application of the formulas (456), (457), (470) separately to the values

$$\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

and

$$\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}$$

($g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (315), $r = 1, 2, \ldots, [k/2]$, $2r < k$) gives the following representation (see (386))

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}\right)^2 \le$$

$$\le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\sum_{j_{g_1}=0}^{p} \ldots \sum_{j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k \ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}} j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\, j_{g_2},\ldots,j_{g_{2r-1}}=\, j_{g_{2r}}}\right)^2 =$$

$$
= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty} \Biggl(\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times
$$

$$
\times \prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k \Biggr)^2, \tag{631}
$$

where

$$
R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) =
$$

$$
= \sum_{d=1}^{4^r} \bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) -
$$

$$
- \sum_{d=1}^{2^r} \tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) \in L_2([t,T]^{k-2r})
$$

and

$$
\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times
$$

$$
\times \prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k
$$

is the Fourier coefficient of

$$
\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) =
$$

$$
= R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k} \psi_q(t_q).
$$

Also note that some of the functions

$$
\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)
$$

and

$$
\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)
$$

can be identically equal to zero.

Obviously, we could use another representation for the function

$$R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) \tag{632}$$

based on the left-hand side of the equality (385) and (456), (457), (470) (see Sect. 13, 16 for details). In Sect. 16, we considered the function (632) in detail for the case $k\ge 5$, $r=1$.

Parseval's equality gives

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\Biggl(\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times$$

$$\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\Biggr)^2=$$

$$=\int\limits_{[t,T]^{k-2r}}\left(\hat R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\right)^2\times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k=$$

$$=\left\|\hat R_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{633}$$

Combining (631) and (633), we obtain

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2\le$$

$$\le\left\|\hat R_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{634}$$

Assume that we have succeeded in proving the following equality

$$\lim_{p\to\infty}\left\|\hat R_p\right\|^2_{L_2([t,T]^{k-2r})}=0. \tag{635}$$

Applying (634) and (635), we get (compare with (386))

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\left.\vphantom{\sum}\right)^2=0. \tag{636}$$

As noted in Sect. 13, Condition 3 of Theorem 20 can be replaced by a weaker condition (386) (or (636)). Also Condition 3 of Theorem 20 can be replaced by (635). From (636) we obviously obtain

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}. \tag{637}$$

According to (385), the equality (637) will be satisfied if

$$\lim_{p\to\infty}S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=0, \tag{638}$$

where $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (315), $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, $r=1,2,\ldots,[k/2]$,

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\mathsf{def}}{=}\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$, where

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}},\quad S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}$$

are defined by (325), (326), $l=1,2,\ldots,r$ (see Sect. 13 for details).

Let us make some remarks about the function (632) for the case $k>5$, $r=2$. In this case, using the left-hand side of the equality (385) and (456), (457), (470), we represent the function (632) as the sum of several functions. In particular, among these functions will be the following functions

$$Q_p(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_{q-1},t_{q+1},\ldots,t_{g-1},t_{g+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_{q-1}<t_{q+1}<\ldots<t_{g-1}<t_{g+1}<\ldots<t_k\}}\times$$

$$\times \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \times$$

$$\times \sum_{j_q=p+1}^{\infty} \int\limits_t^{t_{q+1}} \psi_q(\tau)\phi_{j_q}(\tau)d\tau \int\limits_t^{t_{g-1}} \psi_g(\tau)\phi_{j_q}(\tau)d\tau, \tag{639}$$

$$\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k) = \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times \sum_{j_l=p+1}^{\infty} \left(\int\limits_t^{t_{l-2}} \psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times \sum_{j_q=p+1}^{\infty} \left(\int\limits_t^{t_{l-2}} \psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta} \psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right), \tag{640}$$

$$\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k) = \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times \sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty} \int\limits_t^{t_{l+3}} \psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau} \psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times \int\limits_t^{\tau} \psi_{l+2}(u)\phi_{j_q}(u)dud\tau, \tag{641}$$

$$\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{l-1}<t_{l+2}<\ldots<t_{q-1}<t_{q+2}<\ldots<t_k\}}\times$$

$$\times \sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty} \left(\int\limits_t^{t_{l+2}} \psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times \left(\int\limits_t^{t_{q+2}} \psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta} \psi_q(u)\phi_{j_l}(u)dud\theta\right). \tag{642}$$

Note that the pairs (g_1,g_2), (g_3,g_4) for the functions (640) and (641) have the property: $g_2 = g_1+1$, $g_4 = g_3+1$, $g_3 = g_2+1$. At the same time, the pairs (g_1,g_2), (g_3,g_4) for the function (639) have the following property: $g_2 > g_1+1$, $g_4 > g_3+1$, $g_3 \ge g_2+1$. For the function (642), the pairs (g_1,g_2), (g_3,g_4) chosen as follows: $g_2 > g_1+1$, $g_4 > g_3+1$, $g_4 = g_2+1$, $g_3 = g_1+1$. Generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered. We consider the functions (639)–(642) only as an example.

Suppose that $s+1=l-1$, $l+1=q-1$, $q+1=g-1$ in (639). Let us show that (we consider the case of Legendre polynomials; the trigonometric case is simpler and can be considered similarly)

$$\lim_{p\to\infty}\left\|Q_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{643}$$

$$\lim_{p\to\infty}\left\|\bar Q_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{644}$$

$$\lim_{p\to\infty}\left\|\tilde Q_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{645}$$

$$\lim_{p\to\infty}\left\|\hat Q_p\right\|^2_{L_2([t,T]^{k-4})}=0. \tag{646}$$

First consider the proof of (643). We have ($s+1=l-1$, $l+1=q-1$, $q+1=g-1$)

$$\left(Q_p(t_1,\ldots,t_{l-3},t_{l-1},t_{l+1},t_{l+3},t_{l+5},\ldots,t_k)\right)^2=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-3}<t_{l-1}<t_{l+1}<t_{l+3}<t_{l+5}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2. \tag{647}$$

Using the estimate (406), we obtain

$$\left|\int\limits_t^{s}\psi(\tau)\phi_j(\tau)d\tau\right|<\frac{K}{j^{1-\varepsilon/2}(1-z^2(s))^{1/4-\varepsilon/4}}, \tag{648}$$

where $j\in\mathbb{N}$, $s\in(t,T)$, $z(s)$ is defined by (26), $\varepsilon\in(0,1)$, constant K does not depend on j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, $\psi(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$.

Applying (648) and (409) (we take ε instead of $\varepsilon/2$ in (409)), we get

$$\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2\le$$

$$\leq \frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-1}))^{1-\varepsilon}(1-z^2(t_{l+3}))^{1-\varepsilon}}, \tag{649}$$

where $t_{l-1}, t_{l+3} \in (t, T)$, constant K_1 is independent of p. Combining (647) and (649), we have (643).

Let us prove (644). Applying the estimate (405) in (311) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we obtain

$$\left|\sum_{j=m+1}^{n} C_{jj}(s)\right| \leq C_1\left(\frac{1}{n^{1-\varepsilon}}+\frac{1}{m^{1-\varepsilon}}\right)\int\limits_{-1}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/2-\varepsilon/2}}+$$

$$+C_2\sum_{j=m+1}^{n}\frac{1}{j^{2-\varepsilon}}\left(\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/2-\varepsilon/2}}+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/4-\varepsilon/4}}+\right.$$

$$\left.+\int\limits_{-1}^{z(s)}\frac{1}{\left(1-y^2\right)^{1/4-\varepsilon/4}}\int\limits_{y}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/4-\varepsilon/4}}dy\right), \tag{650}$$

where

$$C_{jj}(s)=\int\limits_{t}^{s}\psi_2(\tau)\phi_j(\tau)\int\limits_{t}^{\tau}\psi_1(\theta)\phi_j(\theta)d\theta d\tau,$$

$s \in (t, T)$, constants C_1, C_2 do not depend on n and m.

From (650) we have

$$\left|\sum_{j=m+1}^{\infty} C_{jj}(s)\right| \leq \frac{K_1}{m^{1-\varepsilon}}+K_2\sum_{j=m+1}^{\infty}\frac{1}{j^{2-\varepsilon}}\left(1+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\right), \tag{651}$$

where $s \in (t, T)$, constants K_1, K_2 do not depend on m.

Applying (409) (we take ε instead of $\varepsilon/2$ in (409)) in (651), we get

$$\left|\sum_{j=m+1}^{\infty} C_{jj}(s)\right| \leq \frac{K}{m^{1-\varepsilon}\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}, \tag{652}$$

where $s \in (t, T)$, constant K is independent of m.

Using the estimate (652), we obtain (see (640))

$$\left(\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2 = \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\left(\int\limits_{t}^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_{t}^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.$$

$$\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right)\Biggr)^2\le$$

$$\le\frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-2}))^{1-\varepsilon}},\qquad (653)$$

where $t_{l-2}\in(t,T)$, constant K_1 is independent of p. The inequality (653) completes the proof of (644).

Let us prove (645). Applying (310) in (641), we get

$$\left(\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2\le$$

$$\le\Biggl(\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\Biggr)^2=$$

$$=\Biggl(\frac{1}{2}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\psi_{l+2}(\tau)d\tau-$$

$$-\sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\Biggr)^2=$$

$$=(a-b)^2\le 2(|a|^2+|b|^2).\qquad (654)$$

Further, we have

$$|a|\le\frac{1}{2}\int\limits_t^{t_{l+3}}|\psi_{l+1}(\tau)|\left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right||\psi_{l+2}(\tau)|\,d\tau,\qquad (655)$$

$$|b|\le\sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}|\psi_{l+1}(\tau)\phi_{j_q}(\tau)|\left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right|\times$$

$$\times \left| \int\limits_t^{\tau} \psi_{l+2}(u)\phi_{j_q}(u)du \right| d\tau. \tag{656}$$

Combining (652) and (655), we obtain

$$|a| \le \frac{C}{p^{1-\varepsilon}}, \tag{657}$$

where constant C is independent of p.

Separating in (656) the term with the number $j_q = 0$ and then applying (89), (87), (652), we obtain

$$|b| \le \frac{K}{p^{1-\varepsilon}} \left(\int\limits_t^{t_{l+3}} \frac{d\tau}{\left(1-z^2(\tau)\right)^{1/2-\varepsilon/4}} + \sum_{j_q=1}^{p} \frac{1}{j_q} \int\limits_t^{t_{l+3}} \frac{d\tau}{\left(1-z^2(\tau)\right)^{3/4-\varepsilon/4}} \right) \le$$

$$\le \frac{K_1}{p^{1-\varepsilon}} \left(1 + \sum_{j_q=1}^{p} \frac{1}{j_q} \right) \le \frac{K_1}{p^{1-\varepsilon}} \left(2 + \int\limits_1^p \frac{dx}{x} \right) =$$

$$= \frac{K_1\left(2 + lnp\right)}{p^{1-\varepsilon}} \ \to \ 0 \tag{658}$$

if $p \to \infty$. The estimates (654), (657), (658) complete the proof of (645).

Finally, consider the proof of (646). Using the elementary inequality $|ab| \le (a^2+b^2)/2$ and Parseval's equality, we have

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2 \le$$

$$\le \left(\sum_{j_l=p+1}^{\infty} \sum_{j_{l+1}=p+1}^{\infty} \left| \int\limits_t^{t_{l+2}} \psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta \right| \times \right.$$

$$\left. \times \left| \int\limits_t^{t_{q+2}} \psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_q(u)\phi_{j_l}(u)dud\theta \right| \right)^2 \le$$

$$\le \frac{1}{4} \left(\sum_{j_l=p+1}^{\infty} \sum_{j_{l+1}=p+1}^{\infty} \left(\int\limits_t^{t_{l+2}} \psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u)dud\theta \right)^2 + \right.$$

$$\left. + \sum_{j_l=p+1}^{\infty} \sum_{j_{l+1}=p+1}^{\infty} \left(\int\limits_t^{t_{q+2}} \psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_q(u)\phi_{j_l}(u)dud\theta \right)^2 \right)^2 \le$$

$$\leq \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\leq$$

$$\leq \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+2}}\psi_{l+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)du\right)^2d\theta+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{q+2}}\psi_{q+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)du\right)^2d\theta\right)^2. \qquad (659)$$

From (659) and (31), (87) we obtain

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2\leq$$

$$\leq \frac{K}{p^2}\ \to\ 0$$

if $p\to\infty$, where constant K does not depend on p. Thus the equalities (643)–(646) are proved.

Recall that the function (632) (this function is defined using the left-hand side of the equality (385)) for the case $k>5,\ r=2$ is represented as the sum of several functions. Four of them, namely Q_p, $\bar{Q}_p$, $\tilde{Q}_p$, $\hat{Q}_p$ (these functions correspond to the particular case of choosing the pairs (g_1,g_2), (g_3,g_4); generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered), have been studied above. Absolutely similarly, we can consider the remaining functions (for all possible pairs (g_1,g_2), (g_3,g_4)) whose sum is the function (632) for the case $k>5,\ r=2$. As a result, we will have

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=2).$$

After that, we can go to the function (632) for the case $k>5,\ r=3,\ 2r<k$ (this function is defined using the left-hand side of the equality (385)) and follow the same steps as above. This will lead us to the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=3,\ 2r<k).$$

Then we can move on to the next step and so on. As a result, we get the equality (635) ($r=1,2,\ldots,[k/2]$). Thus the condition (386) is satisfied for the case $k=2n+1,\ n=3,4,\ldots$ (recall that the condition (386) is weaker than Condition 3 of Theorem 20 and the condition (386) can be used in Theorem 20 instead of Condition 3).

For the case $k=2n,\ n=3,4,\ldots$ we follow the above steps for $r=1,2,\ldots,[k/2]-1$ $(2r\leq k-2)$. For $2r=k$ we use the same technique as in the proof of the equalities (418)–(420). Recall that we used (354), (361) and Parseval's equality in the proof of (418)–(420).

The obvious disadvantage of the proposed algorithm is the drastic increase of complexity of the proof when moving from $r = 1$ to $r = 2$, $r = 2$ to $r = 3$ and so on.

The proofs of Theorems 24 and 25 contain a rather simple trick of passing from $r = 1$ to $r = 2$. Unfortunately, this procedure cannot be applied already at the transition from $r = 2$ to $r = 3$.

Note that the case $k = 6$, $r = 3$ was successfully considered in Theorem 30 under the following simplifying assumption: $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$.

Nevertheless, the results obtained in this paper are quite sufficient for practical needs (see Chapters 4 and 5 [26] for details).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[6] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[7] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[8] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[9] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[15] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2010), A.1-A.257.

DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[18] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[20] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[21] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[23] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N.Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[25] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[26] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184v46 [math.PR]. 2023, 998 pp.

[27] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[28] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[29] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs (Third Edition). Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[30] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2023, 222 pp.

[31] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[33] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series, [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[34] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.

[35] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.

[36] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.

[37] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[38] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604

[39] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[41] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[42] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Pringsheim method [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[43] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[44] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[45] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier-Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[46] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[47] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[48] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier-Legendre series. Journal of Physics: Conference Series. 2021, Vol. 1925, article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[49] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[50] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2022), 83-186. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[51] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR], 2023, 58 pp.

[52] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[53] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[54] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.

[55] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[56] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[57] Itô, K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.

[58] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[59] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[60] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[61] Hobson E.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge University Press, Cambridge, 1931, 502 pp.

[62] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[63] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[64] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

[65] Budhiraja A. Multiple stochastic integrals and Hilbert space valued traces with applications to asymptotic statistics and non-linear filtering. Ph. D. Thesis, The University of North Caroline at Chapel Hill, 1994, VII+132 pp.

[66] Rybakov K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2021), 81–115. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.5.html

[67] Johnson G.W., Kallianpur G. Homogeneous chaos, p-forms, scaling and the Feynman integral. Transactions of the American Mathematical Society, 340 (1993), 503-548.

[68] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[69] Kuznetsov D.F. The three-step strong numerical methods of the orders of accuracy 1.0 and 1.5 for Ito stochastic differential equations. [In English]. Journal of Automation and Information Sciences (Begell House), 2002, 34 (Issue 12), 14 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v34.i12.30
Available at: http://www.sde-kuznetsov.spb.ru/02a.pdf

[70] Kuznetsov D.F. Finite-difference strong numerical methods of order 1.5 and 2.0 for stochastic differential Ito equations with nonadditive multidimensional noise. [In English]. Journal of Automation and Information Sciences (Begell House), 2001, 33 (Issue 5-8), 13 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v33.i5-8.180
Available at: http://www.sde-kuznetsov.spb.ru/01c.pdf

[71] Kuznetsov D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing, Saarbrucken, 2012, 409 pp. Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[72] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2023, 158 pp.

[73] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[74] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional

Wiener process based on Multiple Fourier-Legendre series. MATEC Web of Conferences, 362 (2022), article id: 01014, 10 pp. DOI: http://doi.org/10.1051/matecconf/202236201014

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITY 3 BASED ON GENERALIZED MULTIPLE FOURIER SERIES CONVERGING IN THE MEAN: GENERAL CASE OF SERIES SUMMATION

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the development of the method of expansion and mean-square approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series converging in the mean. We adapt this method for iterated Stratonovich stochastic integrals of multiplicity 3 from the Taylor–Stratonovich expansion. The main result of the article has been derived using the triple Fourier–Legendre series and triple trigonometric Fourier series for the general case of series summation. Some recent results on the expansion of iterated Stratonovich stochastic integrals of multiplicities 3 to 6 are given. The results of the article can be applied to the numerical integration of Ito stochastic differential equations in accordance with the strong criterion of convergence.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
Keywords: Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Legendre Polynomial, Mean-square approximation, Expansion.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The non-random functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[5]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, and

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively; $i_1, \ldots, i_k = 0, 1, \ldots, m$. In this paper we use the definition of the Stratonovich stochastic integral from [2].

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[5]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [6]-[27].

The construction of effective expansions (that converge in the mean-square sense) for the iterated Stratonovich stochastic integrals (3) of multiplicity 3 composes the subject of this article.

The problem of effective jointly numerical modeling (in accordance with the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[61]. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case allows the investigation with using the Ito formula [2]-[5].

Seems that iterated stochastic integrals can be approximated by multiple integral sums of different types [3], [5], [58]. However, this approach implies partitioning of the integration interval $[t,T]$ of iterated stochastic integrals (the length $T - t$ of this interval is a rather small value, because it is the integration step of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to unacceptably high computational cost and accumulation of computation errors [10].

In [3] (also see [2], [4], [5], [59], [60]) Milstein G.N. proposed to expand (2), (3) into iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as the trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple stochastic integrals (3) were presented in [2], [4], [59], [60] ($k = 1, 2, 3$) and in [3], [5] ($k = 1, 2$) for the case $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$.

Moreover, the authors of the works [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [59] (pp. 438-439), [60] (pp. 263-264) use the Wong–Zakai approximation [62]-[64] (without rigorous proof) within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process. See discussion in Sect. 9 of this paper for detail.

Note that in [61] the method (similar to the Milstein approach) of expansion of double Ito stochastic integrals (2) ($k = 2$; $\psi_1(\tau), \psi_2(\tau) \equiv 1$; $i_1, i_2 = 1, \ldots, m$) based on the series expansion of the Wiener process [65] using Haar basis functions and trigonometric basis functions has been considered.

It is necessary to note that the approach based on the Karhunen–Loeve expansion [3] excelled in several times (or even in several orders) the methods of integral sums [3], [5], [58] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [6], [7] (also see [14]-[19], [22], [24], [25]-[27]), where $J^*[\psi^{(k)}]_{T,t}$ was represented as the multiple stochastic integral from the certain discontinuous non-random function of k variables, and the function was then expressed as the generalized iterated Fourier series by complete systems of continuously differentiable functions that are orthonormal in the space $L_2([t, T])$. As a result, the general iterated series expansion of products of standard Gaussian random variables was obtained in [6], [7] (also see [14]-[19], [22], [24], [25]-[27]) for (3) with an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series. It was shown [6], [7] (also see [14]-[19], [22], [24], [25]-[27]) that the method of generalized iterated Fourier series leads to the approach based on the Karhunen–Loeve expansion [3] in the case of trigonometric system of functions and to a substantially simpler expansion of (3) in the case of Legendre polynomial system.

Obviously, the approach based on the Karhunen–Loeve expansion [3] and the method of generalized iterated Fourier series [6], [7] (also see [14]-[19], [22], [24], [25]-[27]) lead to iterated application of the operation of limit transition. So, these methods may not converge in the mean-square sense to appropriate integrals (3) for some methods of series summation. The mentioned problem not appears in the method, which is proposed for (2) in Theorems 1, 2 (see below).

2. Method of Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

Let us consider another approach to the expansion of iterated Ito stochastic integrals (2) [10]-[22], [24]-[57] (the so-called method of generalized multiple Fourier series). The idea of this method is as follows: the iterated Ito stochastic integral (2) of the multiplicity k is represented as the multiple stochastic integral from the certain discontinuous non-random function of k variables defined on the hypercube $[t, T]^k$, where $[t, T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated non-random function of k variables is expanded in the hypercube $[t, T]^k$ into the generalized multiple Fourier series that converges in the mean-square sense in the space $L_2([t, T]^k)$.

After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. Coefficients of this series are coefficients of the generalized multiple Fourier series for the mentioned non-random function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2).

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{4}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{5}$$

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

Theorem 1 [10] (2006) (also see [11]-[22], [24]-[57]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

where $J[\psi^{(k)}]_{T,t}$ *is defined by* (2),

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1\},$$

$$\mathrm{L}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i *or* j (*if* $i \neq 0$), $C_{j_k \ldots j_1}$ *is the Fourier coefficient* (5), $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ *is the partition of the interval* $[t, T]$, *which satisfies the condition* (6).

In [12]-[19], [22], [24]-[27], [35] it was shown that Theorem 1 is valid for convergence in the mean of degree $2n$ $(n \in \mathbb{N})$. The convergence with probability 1 in Theorem 1 is proved in [25]-[28] for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t, T])$ can also be applied in Theorem 1 [10]-[19], [22], [24]-[27], [35]. The modification of Theorem 1 for complete orthonormal with weigth $r(t_1) \ldots r(t_k) \geq 0$ systems of functions in the space $L_2([t, T]^k)$ can be found in [24]-[27], [36]. Note that Theorem 1 and Theorem 2 (see below) have been applied to the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process in [25]-[27] (Chapter 7), [54]-[57].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [10]-[22], [24]-[57]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{10}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1, \ldots, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{11}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1, \ldots, p_4 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_4=0}^{p_4} C_{j_4 \ldots j_1} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$\left.+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\right), \tag{12}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{13}$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \qquad (14)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Thus, we obtain the following useful possibilities and advantages of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (5)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) (see [20], [22], [24]-[27], [34]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [2]-[5] but Legendre polynomials.

4. As it turned out (see [6]-[22], [24]-[57]), it is more convenient to work with Legendre polynomials for construction the approximations of iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions (see [6]-[22], [24]-[57]). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [25]-[27] (Sect. 5.3), [39], [40].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see [61]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1 and Theorem 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$; $i_1,i_2,i_3=0,1,\ldots,m$) of iterated stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [59] (pp. 438-439), [60] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [62]-[64] (see discussion in Sect. 9 of this paper for detail).

Note that the correctness of formulas (9)–(14) can be verified by the fact that if $i_1=\ldots=i_6=i=1,\ldots,m$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv\psi(\tau)$, then we can derive from (9)–(14) the well known equalities, which be fulfilled w. p. 1 [11]-[19], [22], [24]-[27]

$$J[\psi^{(1)}]_{T,t}=\frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t}=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}=\frac{1}{4!}\left(\delta_{T,t}^4-6\delta_{T,t}^2\Delta_{T,t}+3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}=\frac{1}{5!}\left(\delta_{T,t}^5-10\delta_{T,t}^3\Delta_{T,t}+15\delta_{T,t}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t} = \frac{1}{6!}\left(\delta_{T,t}^6 - 15\delta_{T,t}^4\Delta_{T,t} + 45\delta_{T,t}^2\Delta_{T,t}^2 - 15\Delta_{T,t}^3\right),$$

where

$$\delta_{T,t} = \int\limits_t^T \psi(\tau)d\mathbf{f}_\tau^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(\tau)d\tau.$$

The above relations can be independently obtained using the Ito formula and Hermite polynomials.

3. Generalization of Theorem 1 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$

For further consideration, let us consider the generalization of formulas (9)–(14) for the case of an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{15}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\} = \{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (15) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{16}$$

Below there are several examples of sums in the form (16)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$
$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$
$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (7) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times \right.$$

$$\left. \times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \right), \tag{17}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (17) for $k = 5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_5=0}^{p_5} C_{j_5\ldots j_1} \left(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} - \right.$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1} = i_{g_2} \neq 0\}} \mathbf{1}_{\{j_{g_1} = j_{g_2}\}} \prod_{l=1}^{3} \zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$\left. + \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1} = i_{g_2} \neq 0\}} \mathbf{1}_{\{j_{g_1} = j_{g_2}\}} \mathbf{1}_{\{i_{g_3} = i_{g_4} \neq 0\}} \mathbf{1}_{\{j_{g_3} = j_{g_4}\}} \zeta_{j_{q_1}}^{(i_{q_1})} \right).$$

The last equality obviously agrees with (13).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [25] (Sect. 1.11), [35] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (18)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [67]. Note that we use another notations [25] (Sect. 1.11), [35] (Sect. 15) in comparison with [67]. Moreover, the proof of an analogue of Theorem 2 from [67] is somewhat different from the proof given in [25] (Sect. 1.11), [35] (Sect. 15).

As it turned out, the adaptation of the method of generalized multiple Fourier series (Theorems 1, 2) to the iterated Stratonovich stochastic integrals (3) leads simpler expansions than (9)–(14). The article is devoted to deriving the analogues of Theorems 1, 2 for triple Stratonovich stochastic integrals from the so-called Taylor–Stratonovich expansion [2]. In this work, we use triple Fourier–Legendre series as well as triple trigonometric Fourier series for construction of expansions of the iterated Stratonovich stochastic integrals (3). At that, we consider the general case of series summation (Sect. 4–6).

The rest of the article is organized as follows. In Sect. 4, we formulate and prove Theorem 3 on expansion of iterated Stratonovich stochastic integrals (3) of third multiplicity with constant weight functions using triple Fourier–Legendre series. Sect. 5 is devoted to the generalization of Theorem 3 for the case of binomial weight functions. In Sect. 6, we obtain an analogue of Theorem 3 using triple trigonometric Fourier series. Sect. 7 is devoted to modifications of Theorems 3–5. In Sect. 8, we consider some recent results on expansions of iterated Stratonovich stochastic integrals of multiplicities 3 to 6. Sect. 9 is devoted to the discussion of main results of this article from point of view of the Wong–Zakai approximation [62]-[64].

4. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case of Legendre Polynomials

Theorem 3 [15]-[19], [22], [24]-[27]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

(19) $$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(s)\int\limits_t^s \phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds.$$

Proof. If we prove w. p. 1 the following equalities

(20) $$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right),$$

(21) $$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

(22) $$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0,$$

then in accordance with Theorems 1, 2 (see (11)), formulas (20)–(22), standard relations between iterated Ito and Stratonovich stochastic integrals as well as in accordance with the formulas (they also follow from Theorems 1, 2)

$$\frac{1}{2}\int\limits_t^T\int\limits_t^\tau ds d\mathbf{f}_\tau^{(i_3)} = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right) \quad \text{w. p. 1},$$

$$\frac{1}{2}\int\limits_t^T\int\limits_t^\tau d\mathbf{f}_s^{(i_1)} d\tau = \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right) \quad \text{w. p. 1}$$

we will have

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^\tau ds d\mathbf{f}_\tau^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^\tau d\mathbf{f}_s^{(i_1)}d\tau.$$

It means that the expansion (19) will be proved.

First let us prove that

$$\sum_{j_1=0}^{\infty} C_{0j_1j_1} = \frac{1}{4}(T-t)^{3/2}, \tag{23}$$

$$\sum_{j_1=0}^{\infty} C_{1j_1j_1} = \frac{1}{4\sqrt{3}}(T-t)^{3/2}. \tag{24}$$

We have

$$C_{000} = \frac{(T-t)^{3/2}}{6},$$

$$C_{0j_1j_1} = \int\limits_t^T \phi_0(s) \int\limits_t^s \phi_{j_1}(s_1) \int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds =$$

$$= \frac{1}{2} \int\limits_t^T \phi_0(s) \left(\int\limits_t^s \phi_{j_1}(s_1) ds_1 \right)^2 ds, \quad j_1 \ge 1. \tag{25}$$

Here $\phi_j(s)$ looks as follows

$$\phi_j(s) = \sqrt{\frac{2j+1}{T-t}} P_j \left(\left(s - \frac{T+t}{2} \right) \frac{2}{T-t} \right), \quad j \ge 0, \tag{26}$$

where $P_j(x)$ is the Legendre polynomial.

Let us substitute (26) into (25) and calculate $C_{0j_1j_1}$ $(j_1 \ge 1)$

$$C_{0j_1j_1} = \frac{2j_1+1}{2(T-t)^{3/2}} \int\limits_t^T \left(\int\limits_{-1}^{z(s)} P_{j_1}(y) \frac{T-t}{2} dy \right)^2 ds =$$

$$= \frac{(2j_1+1)\sqrt{T-t}}{8} \int\limits_t^T \left(\int\limits_{-1}^{z(s)} \frac{1}{2j_1+1} \left(P'_{j_1+1}(y) - P'_{j_1-1}(y) \right) dy \right)^2 ds =$$

$$= \frac{\sqrt{T-t}}{8(2j_1+1)} \int\limits_t^T \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right)^2 ds, \tag{27}$$

where here and further

$$z(s) = \left(s - \frac{T+t}{2} \right) \frac{2}{T-t},$$

and we used the following well-known properties of the Legendre polynomials

$$P_j(y) = \frac{1}{2j+1} \left(P'_{j+1}(y) - P'_{j-1}(y) \right), \quad P_j(-1) = (-1)^j, \quad j \ge 1.$$

Also we denote

$$\frac{dP_j}{dy}(y) \stackrel{\rm def}{=} P_j^{'}(y).$$

From (27) using the property of orthogonality of the Legendre polynomials, we get the following relation

$$C_{0j_1j_1} = \frac{(T-t)^{3/2}}{16(2j_1+1)}\int\limits_{-1}^{1}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy =$$

$$= \frac{(T-t)^{3/2}}{8(2j_1+1)}\left(\frac{1}{2j_1+3}+\frac{1}{2j_1-1}\right),$$

where we used the property

$$\int\limits_{-1}^{1}P_j^2(y)dy = \frac{2}{2j+1}, \quad j\ge 0.$$

Then

$$\sum_{j_1=0}^{\infty}C_{0j_1j_1} = \frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{(2j_1+1)(2j_1+3)}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}-\frac{1}{3}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right) =$$

$$= \frac{(T-t)^{3/2}}{6}+\frac{(T-t)^{3/2}}{8}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right) = \frac{(T-t)^{3/2}}{4}.$$

The relation (23) is proved.

Let us check the correctness of (24). Let us represent $C_{1j_1j_1}$ in the form

$$C_{1j_1j_1} = \frac{1}{2}\int\limits_t^T\phi_1(s)\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2 ds =$$

$$= \frac{(T-t)^{3/2}(2j_1+1)\sqrt{3}}{16}\int\limits_{-1}^{1}P_1(y)\left(\int\limits_{-1}^{y}P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1\ge 1.$$

Since the functions

$$\left(\int\limits_{-1}^{y}P_{j_1}(y_1)dy_1\right)^2, \quad j_1\ge 1$$

are even, then, correspondently the functions

$$P_1(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy, \quad j_1 \ge 1$$

are uneven.

It means that $C_{1j_1j_1}=0$ $(j_1\ge 1)$. From the other hand

$$C_{100}=\frac{\sqrt{3}(T-t)^{3/2}}{16}\int\limits_{-1}^{1} y(y+1)^2dy=\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_1=0}^{\infty} C_{1j_1j_1}=C_{100}+\sum_{j_1=1}^{\infty} C_{1j_1j_1}=\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (24) is proved.

Let us prove the equality (20). Using (24), we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_{0}^{(i_3)}+\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_{1}^{(i_3)}+\sum_{j_1=0}^{p_1}\sum_{j_3=2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=$$

$$=\sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_{0}^{(i_3)}+\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_{1}^{(i_3)}+\sum_{j_1=0}^{p_1}\ \sum_{j_3=2, j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{28}$$

Since

$$C_{j_3j_1j_1}=\frac{(T-t)^{3/2}(2j_1+1)\sqrt{2j_3+1}}{16}\int\limits_{-1}^{1} P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy$$

and degree of the polynomial

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

equals to $2j_1+2$, then $C_{j_3j_1j_1}=0$ for $j_3>2j_1+2$. It explains the circumstance that we put $2j_1+2$ instead of p_3 on the right-hand side of the formula (28).

Moreover, the function

$$\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$$

is uneven for uneven j_3. It means that $C_{j_3j_1j_1}=0$ for uneven j_3. That is why we summarize using even j_3 on the right-hand side of the formula (28).

Then we have

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2,j_3-\text{even}}^{2j_1+2}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\sum_{j_1=(j_3-2)/2}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=$$

$$=\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{29}$$

We replaced $(j_3-2)/2$ by zero on the right-hand side of the formula (29), since $C_{j_3j_1j_1}=0$ for $0\le j_1<(j_3-2)/2$.

Let us substitute (29) into (28)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\sum_{j_1=0}^{p_1}C_{0j_1j_1}\zeta_0^{(i_3)}+\frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_3)}+$$

$$+\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{30}$$

It is easy to see that the right-hand side of the formula (30) does not depend on p_3.

If we prove that

$$\lim_{p_1\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\right)^2\right\}=0, \tag{31}$$

then the relaion (20) will be proved.

Using (30) and (23), we can write the left-hand side of (31) in the following form

$$\lim_{p_1\to\infty}\mathsf{M}\left\{\left(\left(\sum_{j_1=0}^{p_1}C_{0j_1j_1}-\frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_3)}+\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}=$$

$$=\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1}C_{0j_1j_1}-\frac{(T-t)^{3/2}}{4}\right)^2+\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=$$

$$=\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2. \tag{32}$$

If we prove that

$$\lim_{p_1\to\infty}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=0, \tag{33}$$

then the relation (20) will be proved.

We have

$$\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\left((s-t)-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2\right)ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\phi_{j_3}(s)\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2\le$$

$$\le\frac{1}{4}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds\right)^2. \tag{34}$$

Obtaining (34), we used the Parseval equality in the form

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2=\int\limits_t^T\left(\mathbf{1}_{\{s_1<s\}}\right)^2ds_1=s-t \tag{35}$$

and a property of othogonality of the Legendre polynomials

$$\int\limits_t^T\phi_{j_3}(s)(s-t)ds=0,\quad j_3\ge 2. \tag{36}$$

Then we have for $j_1\in\mathbb{N}$

$$\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2=\frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)}P_{j_1}(y)dy\right)^2=$$

$$=\frac{T-t}{4(2j_1+1)}\left(\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^{'}(y)-P_{j_1-1}^{'}(y)\right)dy\right)^2=$$

$$=\frac{T-t}{4(2j_1+1)}\left(P_{j_1+1}\left(z(s)\right)-P_{j_1-1}\left(z(s)\right)\right)^2\le$$

$$\le\frac{T-t}{2(2j_1+1)}\left(P_{j_1+1}^2\left(z(s)\right)+P_{j_1-1}^2\left(z(s)\right)\right). \tag{37}$$

For the Legendre polynomials the following well-known estimate is correct

$$|P_n(y)|<\frac{K}{\sqrt{n+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad n\in\mathbb{N}, \tag{38}$$

where constant K does not depend on y and n.

The estimate (38) can be written for the function $\phi_n(s)$ in the following form

$$|\phi_n(s)|<\sqrt{\frac{2n+1}{n+1}}\frac{K}{\sqrt{T-t}}\frac{1}{\left(1-z^2(s)\right)^{1/4}}<$$

$$<\frac{K_1}{\sqrt{T-t}}\frac{1}{\left(1-z^2(s)\right)^{1/4}}, \tag{39}$$

where $n\in\mathbb{N}$, $K_1=K\sqrt{2}$, $s\in(t,T)$.

Let us estimate the right-hand side of (37) using the estimate (38)

$$\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2<\frac{T-t}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\frac{1}{(1-(z(s))^2)^{1/2}}<$$

$$<\frac{(T-t)K^2}{2j_1^2}\frac{1}{(1-(z(s))^2)^{1/2}}, \tag{40}$$

where $j_1\in\mathbb{N}$, $s\in(t,T)$.

Substituting the estimate (40) into the relation (34) and using in (34) the estimate (39) for $|\phi_{j_3}(s)|$, we obtain

$$\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\right)^2<$$

$$<\frac{(T-t)K^4K_1^2}{16}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T\frac{ds}{\left(1-(z(s))^2\right)^{3/4}}\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2=$$

$$= \frac{(T-t)^3 K^4 K_1^2 (p_1+1)}{64} \left(\int_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} \right)^2 \left(\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \right)^2. \tag{41}$$

Since

$$\int_{-1}^{1} \frac{dy}{(1-y^2)^{3/4}} < \infty \tag{42}$$

and

$$\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \le \int_{p_1}^{\infty} \frac{dx}{x^2} = \frac{1}{p_1}, \tag{43}$$

then from (41) we obtain

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} \right)^2 < \frac{C(T-t)^3 (p_1+1)}{p_1^2} \ \to\ 0 \quad \text{if} \ \ p_1 \to \infty, \tag{44}$$

where constant C does not depend on p_1 and $T-t$. From (44) it follows (33), and the relation (33) implies the formula (20).

Let us prove the equaity (21). First let us prove that

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = \frac{1}{4}(T-t)^{3/2}, \tag{45}$$

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 j_1} = -\frac{1}{4\sqrt{3}}(T-t)^{3/2}. \tag{46}$$

We have

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = C_{000} + \sum_{j_3=1}^{\infty} C_{j_3 j_3 0},$$

$$C_{000} = \frac{(T-t)^{3/2}}{6},$$

$$C_{j_3 j_3 0} = \frac{(T-t)^{3/2}}{16(2j_3+1)} \int_{-1}^{1} \left(P_{j_3+1}^2(y) + P_{j_3-1}^2(y) \right) dy =$$

$$= \frac{(T-t)^{3/2}}{8(2j_3+1)} \left(\frac{1}{2j_3+3} + \frac{1}{2j_3-1} \right), \quad j_3 \ge 1.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8} \left(\sum_{j_3=1}^{\infty} \frac{1}{(2j_3+1)(2j_3+3)} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1} \right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8} \left(\sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1} - \frac{1}{3} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1} \right) =$$

$$= \frac{(T-t)^{3/2}}{6} + \frac{(T-t)^{3/2}}{8} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2} \right) = \frac{(T-t)^{3/2}}{4}.$$

The relation (45) is proved. Let us check the equality (46). We have

$$C_{j_3 j_3 j_1} = \int\limits_t^T \phi_{j_3}(s) \int\limits_t^s \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds =$$

$$= \int\limits_t^T \phi_{j_1}(s_2) ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \int\limits_{s_1}^T \phi_{j_3}(s) ds =$$

$$= \frac{1}{2} \int\limits_t^T \phi_{j_1}(s_2) \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \right)^2 ds_2 =$$

$$= \frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16} \int\limits_{-1}^{1} P_{j_1}(y) \left(\int\limits_y^1 P_{j_3}(y_1) dy_1 \right)^2 dy, \quad j_3 \ge 1. \tag{47}$$

Since the functions

$$\left(\int\limits_y^1 P_{j_3}(y_1) dy_1 \right)^2, \quad j_3 \ge 1$$

are even, then the functions

$$P_1(y) \left(\int\limits_y^1 P_{j_3}(y_1) dy_1 \right)^2 dy, \quad j_3 \ge 1$$

are uneven. It means that $C_{j_3 j_3 1} = 0$ $(j_3 \ge 1)$.

Moreover,

$$C_{001} = \frac{\sqrt{3}(T-t)^{3/2}}{16} \int\limits_{-1}^{1} y(1-y)^2 dy = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 1} = C_{001} + \sum_{j_3=1}^{\infty} C_{j_3 j_3 1} = -\frac{(T-t)^{3/2}}{4\sqrt{3}}.$$

The relation (46) is proved.

Using the obtained results, we have

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\sum_{j_1=2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} =$$

$$= \sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} + \sum_{j_3=0}^{p_3}\ \sum_{j_1=2, j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{48}$$

Since

$$C_{j_3j_3j_1} = \frac{(T-t)^{3/2}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1} P_{j_1}(y)\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2 dy, \quad j_3 \ge 1,$$

and degree of the polynomial

$$\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2$$

equals to $2j_3+2$, then $C_{j_3j_3j_1}=0$ for $j_1 > 2j_3+2$. It explains the circumstance that we put $2j_3+2$ instead of p_1 on the right-hand side of the formula (48).

Moreover, the function

$$\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2$$

is even. It means that the function

$$P_{j_1}(y)\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2$$

is uneven for uneven j_1. It means that $C_{j_3j_3j_1}=0$ for uneven j_1. It explains the summation with respect to even j_1 on the right-hand side of (48).

Then we have

$$\sum_{j_3=0}^{p_3}\ \sum_{j_1=2, j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=(j_1-2)/2}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} =$$

$$= \sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{49}$$

We replaced $(j_1-2)/2$ by zero on the right-hand side of (49), since $C_{j_3j_3j_1}=0$ for $0 \le j_3 < (j_1-2)/2$.

Let us substitute (49) into (48)

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)} - \frac{(T-t)^{3/2}}{4\sqrt{3}}\zeta_1^{(i_1)} +$$

$$+\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{50}$$

It is easy to see that the right-hand side of the formula (50) does not depend on p_1.

If we prove that

$$\lim_{p_3\to\infty} \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} - \frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\right)^2\right\} = 0, \tag{51}$$

then (21) will be proved.

Using (50) and (45), (46), we can write the left-hand side of the formula (51) in the following form

$$\lim_{p_3\to\infty} \mathsf{M}\left\{\left(\left(\sum_{j_3=0}^{p_3} C_{j_3j_30} - \frac{(T-t)^{3/2}}{4}\right)\zeta_0^{(i_1)} + \sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\} =$$

$$= \lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_1} C_{j_3j_30} - \frac{(T-t)^{3/2}}{4}\right)^2 + \lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 =$$

$$= \lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2.$$

If we prove that

$$\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 = 0, \tag{52}$$

then the relation (21) will be proved.

From (47) we obtain

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 =$$

$$= \frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \phi_{j_1}(s_2)\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2 =$$

$$
= \frac{1}{4} \sum_{j_1=2, j_1-\text{even}}^{2p_3+2} \left(\int\limits_t^T \phi_{j_1}(s_2) \left((T-s_2) - \sum_{j_3=p_3+1}^{\infty} \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \right)^2 \right) ds_2 \right)^2 =
$$

$$
= \frac{1}{4} \sum_{j_1=2, j_1-\text{even}}^{2p_3+2} \left(\int\limits_t^T \phi_{j_1}(s_2) \sum_{j_3=p_3+1}^{\infty} \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \right)^2 ds_2 \right)^2 \le
$$

$$
\le \frac{1}{4} \sum_{j_1=2, j_1-\text{even}}^{2p_3+2} \left(\int\limits_t^T |\phi_{j_1}(s_2)| \sum_{j_3=p_3+1}^{\infty} \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \right)^2 ds_2 \right)^2. \tag{53}
$$

In order to get (53) we used the Parseval equality in the form

$$
\sum_{j_1=0}^{\infty} \left(\int\limits_s^T \phi_{j_1}(s_1) ds_1 \right)^2 = \int\limits_t^T \left(\mathbf{1}_{\{s<s_1\}} \right)^2 ds_1 = T - s \tag{54}
$$

and a property of othogonality of the Legendre polynomials

$$
\int\limits_t^T \phi_{j_3}(s)(T-s) ds = 0, \quad j_3 \ge 2. \tag{55}
$$

Then we have for $j_3 \in \mathbb{N}$

$$
\left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \right)^2 = \frac{(T-t)}{4(2j_3+1)} \left(P_{j_3+1}(z(s_2)) - P_{j_3-1}(z(s_2)) \right)^2 \le
$$

$$
\le \frac{T-t}{2(2j_3+1)} \left(P_{j_3+1}^2(z(s_2)) + P_{j_3-1}^2(z(s_2)) \right) <
$$

$$
< \frac{T-t}{2(2j_3+1)} \left(\frac{K^2}{j_3+2} + \frac{K^2}{j_3} \right) \frac{1}{(1-(z(s_2))^2)^{1/2}} <
$$

$$
< \frac{(T-t)K^2}{2j_3^2} \frac{1}{(1-(z(s_2))^2)^{1/2}}, \quad s \in (t, T). \tag{56}
$$

In order to get (56) we used the estimate (38).

Substituting the estimate (56) into the relation (53) and using in (53) the estimate (39) for $|\phi_{j_1}(s_2)|$, we obtain

$$
\sum_{j_1=2, j_1-\text{even}}^{2p_3+2} \left(\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1} \right)^2 <
$$

$$< \frac{(T-t)K^4K_1^2}{16}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T\frac{ds_2}{(1-z^2(s_2))^{3/4}}\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2 =$$

$$= \frac{(T-t)^3K^4K_1^2(p_3+1)}{64}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}\right)^2\left(\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2. \tag{57}$$

Using (42) and (43) in (57), we get

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2 < \frac{C(T-t)^3(p_3+1)}{p_3^2} \quad\to\ 0 \quad \text{with} \quad p_3\to\infty, \tag{58}$$

where constant C does not depend on p_3 and $T-t$.

From (58) it follows (52), and the relation (52) implies the formula (21). The relation (21) is proved.

Let us prove the equality (22). Since $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)\equiv 1$, then the following relation for the Fourier coefficients is correct

$$C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3},$$

where $C_j=0$ for $j\ge 1$ and $C_0=\sqrt{T-t}$. Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \tag{59}$$

Therefore, considering (20) and (21), we can write w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}C_0^3\zeta_0^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)-$$

$$-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)=0. \tag{60}$$

The relation (22) is proved. Theorem 3 is proved.

It is easy to see that the formula (19) can be proved for the case $i_1=i_2=i_3$ using the Ito formula

$$\int_t^{*T}\int_t^{*t_3}\int_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)} = \frac{1}{6}\left(\int_t^T d\mathbf{f}_s^{(i_1)}\right)^3 = \frac{1}{6}\left(C_0\zeta_0^{(i_1)}\right)^3 =$$

$$= C_{000}\zeta_0^{(i_1)}\zeta_0^{(i_1)}\zeta_0^{(i_1)},$$

where the equality is fulfilled w. p. 1.

5. Generalization of Theorem 3

Let us consider the following generalization of Theorem 3.

Theorem 4 [15]-[19], [22], [24]-[27]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \int_t^{*T}(t-t_3)^{l_3}\int_t^{*t_3}(t-t_2)^{l_2}\int_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{61}$$

converging in the mean-square sense is valid for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
2. $i_1=i_2\ne i_3$ *and* $l_1=l_2\ne l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
3. $i_1\ne i_2=i_3$ *and* $l_1\ne l_2=l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
4. $i_1,i_2,i_3=1,\ldots,m$; $l_1=l_2=l_3=l$ *and* $l=0,1,2,\ldots,$

where

$$C_{j_3 j_2 j_1} = \int_t^T (t-s)^{l_3}\phi_{j_3}(s)\int_t^s (t-s_1)^{l_2}\phi_{j_2}(s_1)\int_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2 ds_1 ds.$$

Proof. Case 1 directly follows from (11).

Let us consider Case 2 ($i_1=i_2\ne i_3,\ l_1=l_2=l\ne l_3$ and $l_1,l_3=0,1,2,\ldots$). So, we prove the following expansion

$$I_{l_1 l_1 l_3 T,t}^{*(i_1 i_1 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m), \tag{62}$$

where $l,l_3=0,1,2,\ldots$, and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^l\phi_{j_1}(s_2)ds_2ds_1ds. \tag{63}$$

If we prove w. p. 1 the formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)}, \tag{64}$$

where coefficients $C_{j_3j_1j_1}$ has the form (63), then using Theorems 1, 2 and standard relations between iterated Ito and Stratonovich stochastic integrals we obtain the expansion (62).

Using Theorems 1 and 2, we can write

$$\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}\quad \text{w. p. 1},$$

where

$$\tilde{C}_{j_3}=\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1ds.$$

Then

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}=$$

$$=\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}+\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}.$$

Therefore,

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)}\right)^2\right\}=$$

$$=\lim_{p_1\to\infty}\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{65}$$

Let us prove that

$$\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=0. \tag{66}$$

We have

$$\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds - \frac{1}{2}\int_t^T \phi_{j_3}(s)(t-s)^{l_3}\int_t^s (t-s_1)^{2l} ds_1 ds\right)^2$$

$$= \frac{1}{4}\left(\int_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\sum_{j_1=0}^{p_1}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 - \int_t^s (t-s_1)^{2l} ds_1\right) ds\right)^2 =$$

$$= \frac{1}{4}\left(\int_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int_t^s (t-s_1)^{2l} ds_1 - \sum_{j_1=p_1+1}^{\infty}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 - \right.\right.$$

$$\left.\left. - \int_t^s (t-s_1)^{2l} ds_1\right) ds\right)^2 =$$

$$= \frac{1}{4}\left(\int_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \tag{67}$$

In order to get (67) we used the Parseval equality, which looks as follows

$$\sum_{j_1=0}^{\infty}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 = \int_t^T K^2(s,s_1) ds_1, \tag{68}$$

where

$$K(s,s_1) = (t-s_1)^l \mathbf{1}_{\{s_1<s\}}, \quad s, s_1 \in [t,T].$$

Taking into account the nondecreasing of the functional sequence

$$u_n(s) = \sum_{j_1=0}^{n}\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2,$$

continuity of its members and continuity of the limit function

$$u(s) = \int_t^s (t-s_1)^{2l} ds_1$$

at the interval $[t,T]$ in accordance with the Dini Theorem we have uniform convergence of the functional sequences $u_n(s)$ to the limit function $u(s)$ at the interval $[t,T]$.

From (67) using the inequality of Cauchy–Bunyakovsky, we obtain

$$\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)^2 \le$$

$$\le \frac{1}{4}\int\limits_t^T \phi_{j_3}^2(s)(t-s)^{2l_3}ds \int\limits_t^T \left(\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right)^2 ds \le$$

$$\le \frac{1}{4}\varepsilon^2 (T-t)^{2l_3}\int\limits_t^T \phi_{j_3}^2(s)ds(T-t) = \frac{1}{4}(T-t)^{2l_3+1}\varepsilon^2 \tag{69}$$

when $p_1 > N(\varepsilon)$, where $N(\varepsilon)$ exists for any $\varepsilon > 0$. From (69) it follows (66).

Further,

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{70}$$

We put $2(j_1+l+1)+l_3$ instead of p_3, since $C_{j_3j_1j_1} = 0$ for $j_3 > 2(j_1+l+1)+l_3$. This conclusion follows from the relation

$$C_{j_3j_1j_1} = \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)Q_{2(j_1+l+1)+l_3}(s)ds,$$

where $Q_{2(j_1+l+1)+l_3}(s)$ is a polynomial of the degree $2(j_1+l+1)+l_3$.

It is easy to see that

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{71}$$

Note that we included some zero coefficients $C_{j_3j_1j_1}$ into the sum $\sum\limits_{j_1=0}^{p_1}$. From (70) and (71) we have

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} = \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 =$$

$$=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2=$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s(t-s_1)^{2l}ds_1-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right)ds\right)^2$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \tag{72}$$

In order to get (72) we used the Parseval equality (68) and the following relation

$$\int\limits_t^T\phi_{j_3}(s)Q_{2l+1+l_3}(s)ds=0;\ \ j_3>2l+1+l_3,$$

where $Q_{2l+1+l_3}(s)$ is a polynomial of degree $2l+1+l_3$.

Further, we have for $j_1\in\mathbb{N}$

$$\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2=\frac{(T-t)^{2l+1}(2j_1+1)}{2^{2l+2}}\left(\int\limits_{-1}^{z(s)}P_{j_1}(y)(1+y)^l dy\right)^2=$$

$$=\frac{(T-t)^{2l+1}}{2^{2l+2}(2j_1+1)}\left((1+z(s))^l R_{j_1}(s)-l\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2\le$$

$$\le\frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_1+1)}\left(\left(\frac{2(s-t)}{T-t}\right)^{2l}R_{j_1}^2(s)+l^2\left(\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)(1+y)^{l-1}dy\right)^2\right)\le$$

$$\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\left(2^{2l+1}Z_{j_1}(s)+l^2\int\limits_{-1}^{z(s)}(1+y)^{2l-2}dy\int\limits_{-1}^{z(s)}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)^2dy\right)\le$$

$$\le\frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)}\left(2^{2l+1}Z_{j_1}(s)+\frac{2l^2}{2l-1}\left(\frac{2(s-t)}{T-t}\right)^{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right)\le$$

$$\leq \frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(2Z_{j_1}(s)+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\right), \tag{73}$$

where

$$R_{j_1}(s)=P_{j_1+1}(z(s))-P_{j_1-1}(z(s)),$$

$$Z_{j_1}(s)=P_{j_1+1}^2(z(s))+P_{j_1-1}^2(z(s)).$$

Let us estimate the right-hand side of (73) using (38)

$$\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2<$$

$$<\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\left(\frac{2}{(1-(z(s))^2)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{1/2}}\right)<$$

$$<\frac{(T-t)^{2l+1}K^2}{2j_1^2}\left(\frac{2}{(1-(z(s))^2)^{1/2}}+\frac{l^2\pi}{2l-1}\right), \quad s\in(t,T). \tag{74}$$

From (72) and (74) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}\leq$$

$$\leq\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2\leq$$

$$\leq\frac{1}{4}(T-t)^{2l_3}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T|\phi_{j_3}(s)|\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2<$$

$$<\frac{(T-t)^{4l+2l_3+1}K^4K_1^2}{16}\times$$

$$\times\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\left(\int\limits_t^T\frac{2ds}{\left(1-(z(s))^2\right)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds}{\left(1-(z(s))^2\right)^{1/4}}\right)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2\leq$$

$$\le \frac{(T-t)^{4l+2l_3+3}K^4K_1^2}{64}\cdot\frac{2p_1+1}{p_1^2}\left(\int\limits_{-1}^{1}\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2\le$$

$$\le (T-t)^{4l+2l_3+3}C\frac{2p_1+1}{p_1^2}\to 0\quad\text{when}\ \ p_1\ \to\ \infty, \tag{75}$$

where constant C does not depend on p_1 and $T-t$.

From (65), (66), and (75) it follows (64), and the relation (64) implies the formula (62).

Let us consider Case 3 ($i_2=i_3\ne i_1$, $l_2=l_3=l\ne l_1$, and $l_1,l_3=0,1,2,\ldots$). So, we prove the following expansion

$$I_{l_1l_3l_3T,t}^{*(i_1i_3i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_3)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{76}$$

where $l,l_1=0,1,2,\ldots$, and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2ds_1ds. \tag{77}$$

If we prove w. p. 1 the formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds, \tag{78}$$

where coefficients $C_{j_3j_3j_1}$ has the form (77), then using Theorems 1, 2 and standard relations between iterated Ito and Stratonovich stochastic integrals we obtain the expansion (76).

Using Theorems 1, 2 and the Ito formula we can write

$$\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\int\limits_t^T(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsd\mathbf{f}_{s_1}^{(i_1)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)}\quad\text{w. p. 1},$$

where

$$\tilde{C}_{j_1}=\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsds_1.$$

Then

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)}=$$

$$
=\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)\zeta_{j_1}^{(i_1)}+\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}.
$$

Therefore,

$$
\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds\right)^2\right\}=
$$

$$
=\lim_{p_3\to\infty}\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}. \tag{79}
$$

Let us prove that

$$
\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2=0. \tag{80}
$$

We have

$$
\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2=
$$

$$
=\left(\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}ds_2\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^{l}ds_1\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^{l}ds-\right.
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsds_1\right)^2=
$$

$$
=\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^{l}ds_1\right)^2ds_2-\right.
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsds_1\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\left(\sum_{j_3=0}^{p_3}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^{l}ds\right)^2-\int\limits_{s_1}^T(t-s)^{2l}ds\right)ds_1\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\left(\int\limits_{s_1}^T(t-s)^{2l}ds-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^{l}ds\right)^2-\int\limits_{s_1}^T(t-s)^{2l}ds\right)ds_1\right)^2
$$

$$
= \frac{1}{4}\left(\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2 ds_1\right)^2. \tag{81}
$$

In order to get (81) we used the Parseval equality, which looks as follows

$$
\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2 = \int\limits_t^T K^2(s,s_1)ds, \tag{82}
$$

where

$$
K(s,s_1)=(t-s)^l\mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].
$$

Taking into account nondecreasing of the functional sequence

$$
u_n(s_1)=\sum_{j_3=0}^{n}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2,
$$

continuity of its members and continuity of the limit function

$$
u(s_1)=\int\limits_{s_1}^T (t-s)^{2l}ds
$$

at the interval $[t,T]$ according to the Dini Theorem we have uniform convergence of the functional sequence $u_n(s_1)$ to the limit function $u(s_1)$ at the interval $[t,T]$.

From (81) using the inequality of Cauchy–Bunyakovsky, we obtain

$$
\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2\le
$$

$$
\le \frac{1}{4}\int\limits_t^T \phi_{j_1}^2(s_1)(t-s_1)^{2l_1}ds_1\int\limits_t^T\left(\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^l ds\right)^2\right)^2 ds_1\le
$$

$$
\le \frac{1}{4}\varepsilon^2(T-t)^{2l_1}\int\limits_t^T \phi_{j_1}^2(s_1)ds_1(T-t)=\frac{1}{4}(T-t)^{2l_1+1}\varepsilon^2 \tag{83}
$$

when $p_3>N(\varepsilon)$, where $N(\varepsilon)$ exists for any $\varepsilon>0$.

From (83) it follows (80).

We have

$$
\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{84}
$$

We put $2(j_3+l+1)+l_1$ instead of p_1, since $C_{j_3j_3j_1}=0$ when $j_1>2(j_3+l+1)+l_1$. This conclusion follows from the relation

$$C_{j_3j_3j_1}=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)Q_{2(j_3+l+1)+l_1}(s_2)ds_2,$$

where $Q_{2(j_3+l+1)+l_1}(s)$ is a polynomial of degree $2(j_3+l+1)+l_1$.

It is easy to see that

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \tag{85}$$

Note that we included some zero coefficients $C_{j_3j_3j_1}$ into the sum $\sum\limits_{j_3=0}^{p_3}$.

From (84) and (85) we have

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=$$

$$=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T(t-s_1)^{2l}ds_1-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2\right)ds_2\right)^2$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2. \tag{86}$$

In order to get (86) we used the Parseval equality (82) and the following relation

$$\int\limits_t^T \phi_{j_1}(s)Q_{2l+1+l_1}(s)ds = 0, \quad j_1 > 2l+1+l_1,$$

where $Q_{2l+1+l_1}(s)$ is a polynomial of degree $2l+1+l_1$.

Further, we have for $j_3 \in \mathbb{N}$

$$\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 = \frac{(T-t)^{2l+1}(2j_3+1)}{2^{2l+2}}\left(\int\limits_{z(s_2)}^1 P_{j_3}(y)(1+y)^l dy\right)^2 =$$

$$= \frac{(T-t)^{2l+1}}{2^{2l+2}(2j_3+1)}\left((1+z(s_2))^l Q_{j_3}(s_2) - l\int\limits_{z(s_2)}^1 \left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\right)^2 \le$$

$$\le \frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_3+1)}\left(\left(\frac{2(s_2-t)}{T-t}\right)^{2l} Q_{j_3}^2(s_2)) + l^2\left(\int\limits_{z(s_2)}^1 \left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\right)^2\right) \le$$

$$\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\left(2^{2l+1}H_{j_3}(s_2) + l^2\int\limits_{z(s_2)}^1 (1+y)^{2l-2}dy \int\limits_{z(s_2)}^1 \left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)^2 dy\right) \le$$

$$\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\left(2^{2l+1}H_{j_3}(s_2) + \frac{2^{2l}l^2}{2l-1}\left(1-\left(\frac{(s_2-t)}{T-t}\right)^{2l-1}\right)\int\limits_{z(s_2)}^1 \left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right) \le$$

$$\le \frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(2H_{j_3}(s_2) + \frac{l^2}{2l-1}\int\limits_{z(s_2)}^1 \left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\right), \tag{87}$$

where

$$Q_{j_3}(s_2) = P_{j_3-1}(z(s_2)) - P_{j_3+1}(z(s_2)),$$

$$H_{j_3}(s_2) = P_{j_3-1}^2(z(s_2)) + P_{j_3+1}^2(z(s_2)).$$

Let us estimate the right-hand side of (87) using (38)

$$\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 <$$

$$< \frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\frac{dy}{(1-y^2)^{1/2}}\right) <$$

$$< \frac{(T-t)^{2l+1}K^2}{2j_3^2}\left(\frac{2}{(1-(z(s_2))^2)^{1/2}}+\frac{l^2\pi}{2l-1}\right), \quad s\in(t,T). \tag{88}$$

From (86) and (88) we obtain

$$\mathrm{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\le$$

$$\le \frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T|\phi_{j_1}(s_2)|(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2\le$$

$$\le \frac{1}{4}(T-t)^{2l_1}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T|\phi_{j_1}(s_2)|\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2<$$

$$< \frac{(T-t)^{4l+2l_1+1}K^4K_1^2}{16}\times$$

$$\times\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\left(\int\limits_t^T\frac{2ds_2}{(1-(z(s_2))^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds_2}{(1-(z(s_2))^2)^{1/4}}\right)\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2\le$$

$$\le \frac{(T-t)^{4l+2l_1+3}K^4K_1^2}{64}\cdot\frac{2p_3+1}{p_3^2}\left(\int\limits_{-1}^1\frac{2dy}{(1-y^2)^{3/4}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^1\frac{dy}{(1-y^2)^{1/4}}\right)^2\le$$

$$\le (T-t)^{4l+2l_1+3}C\frac{2p_3+1}{p_3^2}\to 0 \quad \text{when} \quad p_3\ \to\ \infty, \tag{89}$$

where constant C does not depend on p_3 and $T-t$.

From (79), (80), and (89) it follows (78), and the relation (78) implies the expansion (76).

Let us consider Case 4 ($l_1=l_2=l_3=l=0,1,2,\ldots$ and $i_1,i_2,i_3=1,\ldots,m$). So, we will prove the following expansion for iterated Stratonovich stochastic integral of third multiplicity

$$I_{lll_{T,t}}^{*(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m), \tag{90}$$

where the series converges in the mean-square sense, $l=0,1,2,\ldots$, and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^l\phi_{j_1}(s_2)ds_2ds_1ds. \tag{91}$$

If we prove w. p. 1 the following formula

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{92}$$

where coefficients $C_{j_3j_2j_1}$ have the form (91), then using Theorems 1, 2, relations (64), (78) when $l_1=l_3=l$ and standard relations between iterated Ito and Stratonovich stochastic integrals we will have the expansion (90).

Since $\psi_1(s),\ \psi_2(s),\ \psi_3(s)\equiv(t-s)^l$, then the following equality for the Fourier coefficients takes place

$$C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3},$$

where $C_{j_3j_2j_1}$ has the form (91) and

$$C_{j_1}=\int\limits_t^T\phi_{j_1}(s)(t-s)^lds.$$

Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \tag{93}$$

Taking into account (64) and (78) when $l_3=l_1=l$ and the Ito formula, we have w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\sum_{j_3=0}^{l}C_{j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^ld\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_t^T(t-s)^l\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_2)}-$$

$$-\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^ld\mathbf{f}_{s_1}^{(i_2)}ds=$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}+\frac{1}{2(2l+1)}\int\limits_t^T(t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}-
$$

$$
-\frac{1}{2}\int\limits_t^T(t-s_1)^l\int\limits_{s_1}^T(t-s)^{2l}dsd\mathbf{f}_{s_1}^{(i_2)}=
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}+\frac{1}{2(2l+1)}\int\limits_t^T(t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}-
$$

$$
-\frac{1}{2(2l+1)}\left((T-t)^{2l+1}\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}+\int\limits_t^T(t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}\right)=
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{l}C_{j_1}^2\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}-\frac{(T-t)^{2l+1}}{2(2l+1)}\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}=
$$

$$
=\frac{1}{2}\left(\sum_{j_1=0}^{l}C_{j_1}^2-\int\limits_t^T(t-s)^{2l}ds\right)\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}=0.
$$

Here, the Parseval equality looks as follows

$$
\sum_{j_1=0}^{\infty}C_{j_1}^2=\sum_{j_1=0}^{l}C_{j_1}^2=\int\limits_t^T(t-s)^{2l}ds=\frac{(T-t)^{2l+1}}{2l+1}
$$

and

$$
\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_2)}=\sum_{j_3=0}^{l}C_{j_3}\zeta_{j_3}^{(i_2)}\quad \text{w. p. 1.}
$$

The expansion (90) is proved. Theorem 4 is proved.

It is easy to see that using the Ito formula, we obtain for the case $i_1=i_2=i_3$

$$
\int\limits_t^{*T}(t-s)^l\int\limits_t^{*s}(t-s_1)^l\int\limits_t^{*s_1}(t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)}d\mathbf{f}_{s_1}^{(i_1)}d\mathbf{f}_s^{(i_1)}=
$$

$$
=\frac{1}{6}\left(\int\limits_t^T(t-s)^l d\mathbf{f}_s^{(i_1)}\right)^3=\frac{1}{6}\left(\sum_{j_1=0}^{l}C_{j_1}\zeta_{j_1}^{(i_1)}\right)^3=
$$

$$
=\sum_{j_1,j_2,j_3=0}^{l}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}\quad \text{w. p. 1.} \tag{94}
$$

6. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case of Trigonometric Functions

In this section we will prove the following theorem.

Theorem 5 [15]-[19], [22], [24]-[27]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{95}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds.$$

Proof. If we prove w. p. 1 the following formulas

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}, \tag{96}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)}d\tau, \tag{97}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{98}$$

then from the equalities (96)–(98), Theorems 1, 2, and standard relations between iterated Ito and Stratonovich stochastic integrals we will obtain the expansion (95).

We have

$$S_{p_1,p_3}\stackrel{\mathsf{def}}{=}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{(T-t)^{3/2}}{6}\zeta_0^{(i_3)}+$$

$$+\sum_{j_1=1}^{p_1} C_{0,2j_1,2j_1}\zeta_0^{(i_3)}+\sum_{j_1=1}^{p_1} C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{p_1} C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3} C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)}, \tag{99}$$

where the summation is stopped, when $2j_1,\ 2j_1-1>p_1$ or $2j_3,\ 2j_3-1>p_3$ and

$$C_{0,2l,2l}=\frac{(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{0,2l-1,2l-1}=\frac{3(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{2l,0,0}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 l^2}, \tag{100}$$

$$C_{2r-1,2l,2l}=0,\quad C_{2l-1,0,0}=-\frac{\sqrt{2}(T-t)^{3/2}}{4\pi l},\quad C_{2r-1,2l-1,2l-1}=0, \tag{101}$$

$$C_{2r,2l,2l}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ \\ 0, & r\neq 2l\end{cases}, \tag{102}$$

$$C_{2r,2l-1,2l-1}=\begin{cases}\sqrt{2}(T-t)^{3/2}/(16\pi^2l^2), & r=2l\\ \\ -\sqrt{2}(T-t)^{3/2}/(4\pi^2l^2), & r=l\\ \\ 0, & r\neq l,\ r\neq 2l\end{cases}. \tag{103}$$

Let us show that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3}. \tag{104}$$

We have

$$S_{2p_1,2p_3}=S_{2p_1,2p_3-1}+\sum_{j_1=0}^{2p_1} C_{2p_3,j_1,j_1}\zeta_{2p_3}^{(i_3)}. \tag{105}$$

Using the relations (100), (102), and (103), we obtain

$$\sum_{j_1=0}^{2p_1} C_{2p_3,j_1,j_1}=C_{2p_3,0,0}+\sum_{j_1=1}^{2p_1} C_{2p_3,j_1,j_1}=$$

$$=C_{2p_3,0,0}+\sum_{j_1=1}^{p_1}\Bigl(C_{2p_3,2j_1-1,2j_1-1}+C_{2p_3,2j_1,2j_1}\Bigr)=$$

$$=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2p_3^2}\left(1-\mathbf{1}_{\{p_1\ge p_3\}}\right). \tag{106}$$

From (105), (106) we get

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1}. \tag{107}$$

Further, we have (see (100)–(102))

$$S_{2p_1,2p_3-1}=S_{2p_1-1,2p_3-1}+\sum_{j_3=0}^{2p_3-1}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}, \tag{108}$$

$$\sum_{j_3=0}^{2p_3-1}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}=C_{0,2p_1,2p_1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{2p_3}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}-C_{2p_3,2p_1,2p_1}\zeta_{2p_3}^{(i_3)}=$$

$$=C_{0,2p_1,2p_1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{p_3}\left(C_{2j_3-1,2p_1,2p_1}\zeta_{2j_3-1}^{(i_3)}+C_{2j_3,2p_1,2p_1}\zeta_{2j_3}^{(i_3)}\right)-C_{2p_3,2p_1,2p_1}\zeta_{2p_3}^{(i_3)}=$$

$$=\frac{(T-t)^{3/2}}{8\pi^2p_1^2}\zeta_0^{(i_3)}+\frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2p_1^2}\left(\mathbf{1}_{\{p_3=2p_1\}}-\mathbf{1}_{\{p_3\ge 2p_1\}}\right)\zeta_{4p_1}^{(i_3)}. \tag{109}$$

From (108), (109) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3-1}. \tag{110}$$

Further, we have

$$S_{2p_1,2p_3}=S_{2p_1-1,2p_3}+\sum_{j_3=0}^{2p_3}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}, \tag{111}$$

$$\sum_{j_3=0}^{2p_3}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}=C_{0,2p_1,2p_1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{2p_3}C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)}=$$

$$=C_{0,2p_1,2p_1}\zeta_0^{(i_3)}+\sum_{j_3=1}^{p_3}\left(C_{2j_3-1,2p_1,2p_1}\zeta_{2j_3-1}^{(i_3)}+C_{2j_3,2p_1,2p_1}\zeta_{2j_3}^{(i_3)}\right). \tag{112}$$

From (112), (100)–(102) we obtain

$$\sum_{j_3=0}^{2p_3} C_{j_3,2p_1,2p_1}\zeta_{j_3}^{(i_3)} = \frac{(T-t)^{3/2}}{8\pi^2 p_1^2}\zeta_0^{(i_3)} - \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_1^2}\mathbf{1}_{\{p_3\ge 2p_1\}}\zeta_{4p_1}^{(i_3)}. \tag{113}$$

The relations (111), (113) mean that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S_{2p_1-1,2p_3}. \tag{114}$$

The equalities (107), (110), and (114) imply (104). This means that instead of (96) it is enough to prove the following equality

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T\int\limits_t^\tau ds d\mathbf{f}_\tau^{(i_3)} \quad \text{w. p. 1.} \tag{115}$$

We have

$$S_{2p_1,2p_3} = \sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{(T-t)^{3/2}}{6}\zeta_0^{(i_3)} +$$

$$+\sum_{j_1=1}^{p_1} C_{0,2j_1,2j_1}\zeta_0^{(i_3)} + \sum_{j_1=1}^{p_1} C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)} + \sum_{j_3=1}^{p_3} C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)} +$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)} + \sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)} + \sum_{j_3=1}^{p_3} C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)} +$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)} + \sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1} C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)}. \tag{116}$$

After substituting (100)–(103) into (116), we obtain

$$\sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = (T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_3)} + \frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\ \zeta_0^{(i_3)} -\right.$$

$$\left. -\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)} - \frac{\sqrt{2}}{4\pi^2}\sum_{j_3=1}^{\min\{p_1,p_3\}}\frac{1}{j_3^2}\zeta_{2j_3}^{(i_3)} + \frac{\sqrt{2}}{4\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\zeta_{2j_3}^{(i_3)}\right). \tag{117}$$

From (117) we have w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = (T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_3)} + \frac{1}{2\pi^2}\sum_{j_1=1}^{\infty}\frac{1}{j_1^2}\ \zeta_0^{(i_3)} -\right.$$

$$
-\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\Biggr). \tag{118}
$$

Using Theorems 1, 2 and the system of trigonometric functions, we get w. p. 1

$$
\frac{1}{2}\int\limits_t^T\int\limits_t^s d\tau d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\int\limits_t^T(s-t)d\mathbf{f}_s^{(i_3)}=
$$

$$
=\frac{(T-t)^{3/2}}{4}\underset{p_3\to\infty}{\text{l.i.m.}}\left(\zeta_0^{(i_3)}-\frac{\sqrt{2}}{\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\right). \tag{119}
$$

From (118) and (119) it follows that

$$
\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{2p_3}\sum_{j_1=0}^{2p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=
$$

$$
=(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{12}\zeta_0^{(i_3)}-\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\right)=
$$

$$
=(T-t)^{3/2}\left(\frac{1}{4}\zeta_0^{(i_3)}-\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\right)=
$$

$$
=\frac{1}{2}\int\limits_t^T\int\limits_t^s d\tau d\mathbf{f}_s^{(i_3)},
$$

where the equality is fulfilled w. p. 1.

So, the relations (115) and (96) are proved for the case of trigonometric system of functions.

Let us prove the relation (97). We have

$$
S'_{p_1,p_3}\stackrel{\mathsf{def}}{=}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{(T-t)^{3/2}}{6}\zeta_0^{(i_1)}+
$$

$$
+\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,0}\zeta_0^{(i_1)}+\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+
$$

$$
+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)}+
$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)}+\sum_{j_1=1}^{p_1} C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}, \tag{120}$$

where the summation is stopped, when $2j_3,\ 2j_3-1>p_3$ or $2j_1,\ 2j_1-1>p_1$ and

$$C_{2l,2l,0}=\frac{(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{2l-1,2l-1,0}=\frac{3(T-t)^{3/2}}{8\pi^2 l^2},\quad C_{0,0,2r}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 r^2}, \tag{121}$$

$$C_{2l-1,2l-1,2r-1}=0,\quad C_{0,0,2r-1}=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi r},\quad C_{2l,2l,2r-1}=0, \tag{122}$$

$$C_{2l,2l,2r}=\begin{cases}-\sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ \\ 0, & r\ne 2l\end{cases}, \tag{123}$$

$$C_{2l-1,2l-1,2r}=\begin{cases}\sqrt{2}(T-t)^{3/2}/(16\pi^2 l^2), & r=2l\\ \\ -\sqrt{2}(T-t)^{3/2}/(4\pi^2 l^2), & r=l\\ \\ 0, & r\ne l,\ r\ne 2l\end{cases}. \tag{124}$$

Let us show that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3-1}=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3}. \tag{125}$$

We have

$$S'_{2p_1,2p_3}=S'_{2p_1-1,2p_3}+\sum_{j_3=0}^{2p_3} C_{j_3,j_3,2p_1}\zeta_{2p_1}^{(i_1)}. \tag{126}$$

Using the relations (121), (123), and (124), we obtain

$$\sum_{j_1=0}^{2p_3} C_{j_3,j_3,2p_1}=C_{0,0,2p_1}+\sum_{j_3=1}^{2p_3} C_{j_3,j_3,2p_1}=$$

$$=C_{0,0,2p_1}+\sum_{j_3=1}^{p_3}\Bigl(C_{2j_3-1,2j_3-1,2p_1}+C_{2j_3,2j_3,2p_1}\Bigr)=$$

$$=\frac{\sqrt{2}(T-t)^{3/2}}{4\pi^2 p_1^2}\bigl(1-\mathbf{1}_{\{p_3\ge p_1\}}\bigr). \tag{127}$$

From (126), (127) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3}. \tag{128}$$

Further, we get (see (121)–(123))

$$S'_{2p_1-1,2p_3} = S'_{2p_1-1,2p_3-1} + \sum_{j_1=0}^{2p_1-1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)}, \tag{129}$$

$$\sum_{j_1=0}^{2p_1-1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} - C_{2p_3,2p_3,2p_1}\zeta_{2p_1}^{(i_1)} =$$

$$= C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\left(C_{2p_3,2p_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + C_{2p_3,2p_3,2j_1}\zeta_{2j_1}^{(i_1)}\right) - C_{2p_3,2p_3,2p_1}\zeta_{2p_1}^{(i_1)} =$$

$$= \frac{(T-t)^{3/2}}{8\pi^2 p_3^2}\zeta_0^{(i_1)} + \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_3^2}\left(\mathbf{1}_{\{p_1=2p_3\}} - \mathbf{1}_{\{p_1\ge 2p_3\}}\right)\zeta_{4p_3}^{(i_1)}. \tag{130}$$

From (129), (130) we obtain

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1-1,2p_3-1}. \tag{131}$$

Further, we have

$$S'_{2p_1,2p_3} = S'_{2p_1,2p_3-1} + \sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)}, \tag{132}$$

$$\sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} =$$

$$= C_{2p_3,2p_3,0}\zeta_0^{(i_1)} + \sum_{j_1=1}^{p_1}\left(C_{2p_3,2p_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + C_{2p_3,2p_3,2j_1}\zeta_{2j_1}^{(i_1)}\right). \tag{133}$$

From (133), (121)–(123) we obtain

$$\sum_{j_1=0}^{2p_1} C_{2p_3,2p_3,j_1}\zeta_{j_1}^{(i_1)} = \frac{(T-t)^{3/2}}{8\pi^2 p_3^2}\zeta_0^{(i_1)} - \frac{\sqrt{2}(T-t)^{3/2}}{16\pi^2 p_3^2}\mathbf{1}_{\{p_1\ge 2p_3\}}\zeta_{4p_3}^{(i_1)}. \tag{134}$$

The relations (132), (134) mean that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3} = \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ S'_{2p_1,2p_3-1}. \tag{135}$$

The equalities (128), (131), and (135) imply (125). This means that instead of (97) it is enough to prove the following equality

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau\quad\text{w. p. 1.}\tag{136}$$

We have

$$S'_{2p_1,2p_3}=\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{(T-t)^{3/2}}{6}\zeta_0^{(i_1)}+$$

$$+\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,0}\zeta_0^{(i_1)}+\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3}C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)}+\sum_{j_1=1}^{p_1}C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}.\tag{137}$$

After substituting (121)–(124) into (137), we obtain

$$\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\ \zeta_0^{(i_1)}+$$

$$+\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}-\frac{\sqrt{2}}{4\pi^2}\sum_{j_1=1}^{\min\{p_1,p_3\}}\frac{1}{j_1^2}\zeta_{2j_1}^{(i_1)}+\frac{\sqrt{2}}{4\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\zeta_{2j_1}^{(i_1)}\Biggr).\tag{138}$$

From (138) we have w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\pi^2}\sum_{j_3=1}^{\infty}\frac{1}{j_3^2}\ \zeta_0^{(i_1)}+$$

$$+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\Biggr).\tag{139}$$

Using the Ito formula and Theorems 1, 2 for the case of trigonometric system of functions, we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau=\frac{1}{2}\left((T-t)\int\limits_t^T d\mathbf{f}_s^{(i_1)}+\int\limits_t^T (t-s)d\mathbf{f}_s^{(i_1)}\right)=$$

$$=\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\right). \qquad (140)$$

From (139) and (140) it follows that

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{2p_1}\sum_{j_3=0}^{2p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=$$

$$=(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{12}\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\right)=$$

$$=(T-t)^{3/2}\left(\frac{1}{4}\zeta_0^{(i_1)}+\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\right)=$$

$$=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau,$$

where the equality is fulfilled w. p. 1.

So, the relations (136) and (97) are proved for the case of trigonometric system of functions.

Let us prove the equality (98). Since $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)\equiv 1$, then the following relation for the Fourier coefficients is correct

$$C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}=\frac{1}{2}C_{j_1}^2C_{j_3}.$$

Then w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \qquad (141)$$

Taking into account (96) and (97), we can write w. p. 1

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$
=\frac{1}{2}C_0^3\zeta_0^{(i_2)}-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-
$$

$$
-\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=
$$

$$
=\frac{1}{2}(T-t)^{3/2}\zeta_0^{(i_2)}-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}+\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_2)}\right)-
$$

$$
-\frac{1}{4}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\underset{p_1\to\infty}{\text{l.i.m.}}\frac{\sqrt{2}}{\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_2)}\right)=0.
$$

From Theorems 1, 2 and (96)–(98) we obtain the expansion (95). Theorem 5 is proved.

7. Modifications of Theorems 3–5

Let us consider the following modification of Theorem 4.

Theorem 6 [17]-[19], [22], [24]-[27]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)$ are continuously differentiable functions at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)
$$

the following expansion

$$
J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{142}
$$

converging in the mean-square sense is valid for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3,$
2. $i_1=i_2\ne i_3$ *and* $\psi_1(\tau)\equiv\psi_2(\tau),$
3. $i_1\ne i_2=i_3$ *and* $\psi_2(\tau)\equiv\psi_3(\tau),$
4. $i_1,i_2,i_3=1,\ldots,m$ *and* $\psi_1(\tau)\equiv\psi_2(\tau)\equiv\psi_3(\tau),$

where

$$
C_{j_3j_2j_1}=\int\limits_t^T\psi_3(s)\phi_{j_3}(s)\int\limits_t^s\psi_2(s_1)\phi_{j_2}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.
$$

Proof. Case 1 directly follows from Theorems 1, 2. Let us consider Case 2. We will prove w. p. 1 the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 d\mathbf{f}_s^{(i_3)},$$

where

$$C_{j_3j_1j_1} = \int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi(s_1)\phi_{j_1}(s_1)\int\limits_t^{s_1}\psi(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.$$

Using Theorems 1, 2 we can write the following

$$\frac{1}{2}\int\limits_t^T \psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1ds.$$

We have

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} = \sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\frac{1}{2}\sum_{j_1=0}^{p}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds - \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\int\limits_t^s \psi^2(s_1)ds_1ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\left(\sum_{j_1=0}^{p}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 - \int\limits_t^s \psi^2(s_1)ds_1\right)ds\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=0}^{p}\left(\int\limits_t^T \phi_{j_3}(s)\psi_3(s)\sum_{j_1=p+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 ds\right)^2. \tag{143}$$

In order to get (143) we used the Parseval equality in the form

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)\psi(s_1)ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1 = \int\limits_t^s \psi^2(s_1)ds_1,$$

where

$$K(s,s_1) = \psi(s_1)\mathbf{1}_{\{s_1<s\}},\quad s,s_1\in[t,T].$$

We have for $j_1 \in \mathbb{N}$

$$\left(\int\limits_t^s \psi(s_1)\phi_{j_1}(s_1)ds_1\right)^2 =$$

$$= \frac{(T-t)(2j_1+1)}{4}\left(\int\limits_{-1}^{z(s)} P_{j_1}(y)\psi\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2 =$$

$$= \frac{T-t}{4(2j_1+1)}\Biggl(\left(P_{j_1+1}(z(s))-P_{j_1-1}(z(s))\right)\psi(s)-$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(s)}\left(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi'\left(\frac{T-t}{2}y+\frac{T+t}{2}\right)dy\right)^2, \tag{144}$$

where

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ' is a derivative of the function $\psi(s)$ with respect to the variable

$$\frac{T-t}{2}y + \frac{T+t}{2}.$$

Further consideration is similar to the proof of Case 2 from Theorem 4. Finally, from (143) and (144) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_3j_1j_1} - \frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\} <$$

$$< K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}} + \int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2 \le$$

$$\le \frac{K_1}{p} \to 0 \quad \text{if} \quad p \to \infty,$$

where K, K_1 are constants. Case 2 is proved.

Let us consider Case 3. In this case we will prove w. p. 1 the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T \psi^2(s)\int\limits_t^s \psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds,$$

where

$$C_{j_3j_3j_1} = \int\limits_t^T \psi(s)\phi_{j_3}(s)\int\limits_t^s \psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1}\psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.$$

Using the Ito formula, we obtain w. p. 1

$$\frac{1}{2}\int\limits_t^T \psi^2(s)\int\limits_t^s \psi_1(s_1)d\mathbf{f}_{s_1}^{(i_1)}ds = \frac{1}{2}\int\limits_t^T \psi_1(s_1)\int\limits_{s_1}^T \psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)}. \tag{145}$$

Applying Theorems 1 and 2, we have

$$\frac{1}{2}\int\limits_t^T \psi_1(s_1)\int\limits_{s_1}^T \psi^2(s)dsd\mathbf{f}_{s_1}^{(i_1)} = \frac{1}{2}\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1} C_{j_1}^*\zeta_{j_1}^{(i_1)}, \tag{146}$$

where

$$C_{j_1}^* = \int\limits_t^T \phi_{j_1}(s_1)\psi_1(s_1)\int\limits_{s_1}^T \psi^2(s)dsds_1.$$

Moreover,

$$C_{j_3j_3j_1} = \int\limits_t^T \psi(s)\phi_{j_3}(s)\int\limits_t^s \psi(s_1)\phi_{j_3}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds =$$

$$= \int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2)\int\limits_{s_2}^T \psi(s_1)\phi_{j_3}(s_1)\int\limits_{s_1}^T \psi(s)\phi_{j_3}(s)dsds_1ds_2 =$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(s_2)\phi_{j_1}(s_2)\left(\int\limits_{s_2}^T \psi(s_1)\phi_{j_3}(s_1)ds_1\right)^2 ds_2. \tag{147}$$

From (145)–(147) we obtain

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1} - \frac{1}{2}C_{j_1}^*\right)\zeta_{j_1}^{(i_1)}\right)^2\right\} = \sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p} C_{j_3j_3j_1} - \frac{1}{2}C_{j_1}^*\right)^2 =$$

$$= \frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(s_1)\psi_1(s_1)\left(\sum_{j_3=0}^{p}\left(\int\limits_{s_1}^T \phi_{j_3}(s)\psi(s)ds_1\right)^2 - \int\limits_{s_1}^T \psi^2(s)ds\right)ds_1\right)^2$$

$$= \frac{1}{4}\sum_{j_1=0}^{p}\left(\int\limits_t^T \phi_{j_1}(s_1)\psi_1(s_1)\sum_{j_3=p+1}^{\infty}\left(\int\limits_{s_1}^T \phi_{j_3}(s)\psi(s)ds\right)^2 ds_1\right)^2. \tag{148}$$

In order to get (148) we used the Parseval equality in the form

$$\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^{T}\phi_{j_3}(s)\psi(s)ds\right)^2=\int\limits_{t}^{T}K^2(s,s_1)ds=\int\limits_{s_1}^{T}\psi^2(s)ds,$$

where

$$K(s,s_1)=\psi(s)\mathbf{1}_{\{s>s_1\}},\quad s,s_1\in[t,T].$$

Further consideration is similar to the proof of Case 3 from Theorem 4. Finally, from (148) we get

$$\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C^*_{j_1}\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}<$$

$$<K\frac{p}{p^2}\left(\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{3/4}}+\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{1/4}}\right)^2\le$$

$$\le\frac{K_1}{p}\quad\to 0\quad \text{if } p\ \to\infty,$$

where K,K_1 are constants. Case 3 is proved.

Let us consider Case 4. We will prove w. p. 1 the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0\quad(\psi_1(s),\psi_2(s),\psi_3(s)\equiv\psi(s)).$$

In Case 4 we obtain w. p. 1

$$\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_3=0}^{p}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_3=0}^{p}\left(\frac{1}{2}C_{j_1}^2C_{j_3}-C_{j_1j_1j_3}-C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\sum_{j_3=0}^{p}C_{j_3}\zeta_{j_3}^{(i_2)}-\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_3=0}^{p}C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)}-$$

$$-\underset{p\to\infty}{\text{l.i.m.}}\ \sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{\infty}C_{j_1}^2\int\limits_{t}^{T}\psi(s)d\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_{t}^{T}\psi^2(s)\int\limits_{t}^{s}\psi(s_1)d\mathbf{f}_{s_1}^{(i_2)}ds-$$

$$-\frac{1}{2}\int\limits_{t}^{T}\psi(s)\int\limits_{t}^{s}\psi^2(s_1)ds_1d\mathbf{f}_s^{(i_2)}=\frac{1}{2}\int\limits_{t}^{T}\psi^2(s)ds\int\limits_{t}^{T}\psi(s)d\mathbf{f}_s^{(i_2)}-$$

$$
-\frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_{s_1}^T \psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}-\frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_t^{s_1} \psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}=
$$

$$
=\frac{1}{2}\int\limits_t^T \psi^2(s)ds\int\limits_t^T \psi(s)d\mathbf{f}_s^{(i_2)}-\frac{1}{2}\int\limits_t^T \psi(s_1)\int\limits_t^T \psi^2(s)dsd\mathbf{f}_{s_1}^{(i_2)}=0,
$$

where we used the Parseval equality in the form

$$
\sum_{j_1=0}^{\infty}C_j^2=\sum_{j=0}^{\infty}\left(\int\limits_t^T \psi(s)\phi_j(s)ds\right)^2=\int\limits_t^T \psi^2(s)ds.
$$

Case 4 and Theorem 6 are proved.

Let us consider the trigonometric version of Theorem 6.

Theorem 7 [19], [22], [24]-[27]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable functions at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)
$$

the following expansion

$$
J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}
$$

converging in the mean-square sense is valid for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3,$
2. $i_1=i_2\ne i_3$ *and* $\psi_1(\tau)\equiv\psi_2(\tau),$
3. $i_1\ne i_2=i_3$ *and* $\psi_2(\tau)\equiv\psi_3(\tau),$
4. $i_1,i_2,i_3=1,\ldots,m$ *and* $\psi_1(\tau)\equiv\psi_2(\tau)\equiv\psi_3(\tau),$

where

$$
C_{j_3j_2j_1}=\int\limits_t^T \psi_3(s)\phi_{j_3}(s)\int\limits_t^s \psi_2(s_1)\phi_{j_2}(s_1)\int\limits_t^{s_1} \psi_1(s_2)\phi_{j_1}(s_2)ds_2ds_1ds.
$$

Proof. We have

$$
\int\limits_t^s \phi_{j_1}(\theta)\psi(\theta)d\theta=\frac{\sqrt{2}}{\sqrt{T-t}}\int\limits_t^s \begin{cases}\psi(\theta)\,\sin\left((2\pi j_1(\theta-t))/(T-t)\right)d\theta\\ \\ \psi(\theta)\,\cos\left((2\pi j_1(\theta-t))/(T-t)\right)d\theta\end{cases}=
$$

$$
=\sqrt{\frac{T-t}{2}}\frac{1}{\pi j_1}\left(\begin{cases}-\psi(s)\ \cos\left((2\pi j_1(s-t))/(T-t)\right)+\psi(t)\\ \\ \psi(s)\ \sin\left((2\pi j_1(s-t))/(T-t)\right)\end{cases}\right. +
$$

$$
\left.+\int\limits_t^s\begin{cases}\psi'(\theta)\ \cos\left((2\pi j_1(\theta-t))/(T-t)\right)d\theta\\ \\ -\psi'(\theta)\ \sin\left((2\pi j_1(\theta-t))/(T-t)\right)d\theta\end{cases}\right),
$$

where $j_1\ne 0$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in $L_2([t,T])$.

Then

$$
\left|\int\limits_t^s\phi_{j_1}(\theta)\psi(\theta)d\theta\right|\le\frac{K}{j_1}\quad (j_1\ne 0). \tag{149}
$$

Analogously, we get

$$
\left|\int\limits_s^T\phi_{j_1}(\theta)\psi(\theta)d\theta\right|\le\frac{K}{j_1}\quad (j_1\ne 0). \tag{150}
$$

Using (143), (148)–(150), we obtain

$$
\mathsf{M}\left\{\left(\sum_{j_3=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_3j_1j_1}-\frac{1}{2}\tilde C_{j_3}\right)\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{K_1}{p}\ \to 0\quad\text{if}\quad p\ \to\infty,
$$

$$
\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\left(\sum_{j_3=0}^{p}C_{j_3j_3j_1}-\frac{1}{2}C^{*}_{j_1}\right)\zeta_{j_1}^{(i_1)}\right)^2\right\}\le\frac{K_1}{p}\ \to 0\quad\text{if}\quad p\ \to\infty,
$$

where constant K_1 does not depend on p.

The consideration of Case 4 is similar to the case of Legendre polynomials (see Theorem 6). Theorem 7 is proved.

Note that the analogues of Theorems 6 and 7 have been proved in [29] without the restrictions 1–4 (see the formulations of Theorems 6 and 7). However, in [29] the additional smoothness assumptions were used.

8. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 6. Some Recent Results

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [25] (Sect. 2.10–2.19), [29] (Sect. 13–21), [33] (Sect. 5–12), [45] (Sect. 7–14), [74], [75]. Let us formulate five theorems that were proved using this approach.

Theorem 8 [25], [29], [33], [45], [74]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{151}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{152}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (151) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (152)*, constant* C *is independent of* p,

$$C_{j_3j_2j_1} = \int\limits_t^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorems* 1, 2*.*

Theorem 9 [25], [29], [33], [45], [74]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \tag{153}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \tag{154}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{155}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (153), (154) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (155)*, constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 8*.*

Theorem 10 [25], [29], [33], [45], [74]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{156}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{157}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{158}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (156), (157) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (158)*, constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 8, 9*.*

Theorem 11 [25], [29], [33], [45], [74]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{159}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^{T}\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 8–10.

Theorem 12 [25], [29], [33], [45]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m,$

$$C_{j_3j_2j_1}=\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

another notations are the same as in Theorems 8–11.

Obviously, Theorem 12 generalizes the main results of this article for iterated Stratonovich stochastic integrals of third multiplicity.

9. Theorems 1–12 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [62], [63], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [62]-[64] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [65], [66]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{160}$$

where $\tau\in[t,T]$, $t\geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (160) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{161}$$

From (161) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{162}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{163}$$

where $p_1,\ldots,p_k\in\mathbb{N},\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$
d\mathbf{w}_{\tau}^{(i)p}=\begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for} \quad i=1,\ldots,m\\ \\ d\tau & \text{for} \quad i=0\end{cases}, \tag{164}
$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (162).

Let us substitute (162) into (163)

$$
\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{165}
$$

where

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$
C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k
$$

is the Fourier coefficient.

To best of our knowledge [62]-[64] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [64] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (161) were not considered in [62], [63] (also see [64], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [64] for approximations of the Wiener process based on its series expansion (160) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (165) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [62], [63] (also see [64], Theorems 7.1, 7.2).

From the other hand, Theorems 1, 2 and Theorems 3–12 from this paper (also see Chapters 1 and 2 from [25]) can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the approximation (161) of the Wiener process. At that, the iterated Riemann–Stieltjes integrals (163) converge (according to Theorems 1–12 from this article and Chapters 1, 2 from [25]) to the appropriate iterated Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (160), (161), and Theorems 3–12) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [62]-[64]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{166}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (166) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{167}$$

Using (167) and standard relations between Ito and Stratonovich stochastic integrals, it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{168}$$

where $\Delta \to 0$ if $N \to \infty$ $(N\Delta = T)$.

Obviously, (168) agrees with Theorem 7.1 (see [64], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (160) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0, T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{169}$$

where $d\mathbf{f}_\tau^{(i)p}$ is defined by the relation (162).

Let us substitute (162) into (169)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{170}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (165).

As we noted above, approximations of the Wiener process that are similar to (161) were not considered in [62], [63] (also see Theorems 7.1, 7.2 in [64]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [64] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [25]-[27]. More precisely, using Theorem 2.2 [25], we obtain from (170) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{171}$$

From the other hand, by Theorems 1, 2 (see (10)) for the case $k = 2$ we obtain from (170) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2\}} \mathbf{1}_{\{j_1 = j_2\}} \right) + \mathbf{1}_{\{i_1 = i_2\}} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{172}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then using standard relations between Ito and Stratonovich stochastic integrals and (172) we obtain (171).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[6] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[7] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[8] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[9] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786

pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[15] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[18] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[20] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[21] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[23] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N.Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[25] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR]. 2022, 947 pp.

[26] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [in English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[27] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[28] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[29] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 219 pp.

[30] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[31] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[32] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[33] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 145 pp.

[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.

[35] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [in English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[36] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [in English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[38] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[39] Kuznetsov D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[41] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[42] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [in English]. arXiv:1712.08991 [math.PR]. 2017, 57 pp.

[43] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combained approach based on generalized multiple and iterated Fourier series. [in English]. arXiv:1801.05654 [math.PR]. 2018, 46 pp.

[44] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. [in English]. arXiv:1801.07248 [math.PR]. 2018, 20 pp.

[45] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 155 pp.

[46] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [in English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.

[47] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. [in English]. arXiv:1806.10705 [math.PR]. 2018, 29 pp.

[48] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Ito expansion. arXiv:1805.12527 [math.PR]. 2018, 29 pp. [In English].

[49] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. [in English]. arXiv: 1802.04844 [math.PR]. 2018, 37 pp.

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.

[51] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor-Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. [In English]. arXiv:1801.08862 [math.PR], 2018, 36 p.

[52] Kuznetsov D.F. New simple method of expansion of iterated Ito stochasic integrals of multiplicity 2 based on the expansion of the Brownian motion using Legendre polynomials and trigonometric functions. [In English]. arXiv:1807.00409 [math.PR], 2019, 23 pp.

[53] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. [In English]. arXiv:2001.10192 [math.PR], 2020, 90 pp.

[54] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic

partial differential equations. [in English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[55] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2020, 32 pp.

[56] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [in English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[57] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[58] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham) 17 (2013), 355-366.

[59] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.

[60] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010. 868 pp.

[61] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. [In Russian]. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[62] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[63] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[64] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[65] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[66] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[67] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differential Equations and Control Processes, 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[68] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[69] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [in English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[70] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [in English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[71] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[72] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[73] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[74] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[75] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic

Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF ARBITRARY MULTIPLICITY BASED ON GENERALIZED ITERATED FOURIER SERIES CONVERGING POINTWISE

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on iterated trigonometric Fourier series converging pointwise. The case of iterated Fourier–Legendre series is considered in details for $k = 2$. The obtained expansions provide a possibility to represent the iterated Stratonovich stochastic integral in the form of iterated series of products of standard Gaussian random variables. Convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) of the expansions is proved. Some recent results on the expansion of iterated Stratonovich stochastic integrals of multiplicities 3 to 6 are given. The results of the article can be applied to the numerical solution of Ito stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.
Keywords: Iterated Stratonovich stochastic integral, Iterated Ito stochastic integral, Generalized iterated Fourier series, Generalized multiple Fourier series, Fourier–Legendre series, Trigonometric Fourier series, Approximation, Expansion.

1. Introduction

The idea of representing of iterated Ito and Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific discontinuous nonrandom functions of several variables and following expansion of these functions using generalized iterated and multiple Fourier series in order to get effective mean-square approximations of the mentioned stochastic integrals was proposed and developed in a lot of publications of the author [1]-[41]. The terms "generalized iterated Fourier series" and "generalized multiple Fourier series" mean that these series are constructed using various complete orthonormal systems of functions in the space $L_2([t,T])$, and not only using the trigonometric system of functions. Here $[t,T]$ is an interval of integration of iterated Ito and Stratonovich stochastic integrals. For the first time approach of generalized iterated and multiple Fourier series is considered in [1] (1997), [2] (1998), and [4] (2006) (also see references to early publications (1994-1996) in [1], [2], [4], [18]-[21]). Usage of the Fourier–Legendre series for approximation of iterated Ito and Stratonovich stochastic integrals took place for the first time in [1] (1997) (also see [2]-[41]). The results from [1]-[41] and this work convincingly testify that there is a doubtless relation between the multiplier factor $1/2$, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Ito stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its generalized Fourier series converges to the value $(f(x+0)+f(x-0))/2$. In addition, as it is demonstrated in [1]-[41], the final formulas for expansions of iterated Stratonovich stochastic integrals based on the Fourier–Legendre series are essentially simpler than its analogues based on the trigonometric Fourier series. Note that another approaches to approximation of iterated Ito and Stratonovich stochastic integrals can be found in [42]-[58]. For example, in [4]-[40] the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series is proposed and developed. The ideas underlying this method are close to the ideas of the method considered in this article.

2. Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Consider the following iterated Stratonovich and Ito stochastic integrals

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1}$$

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\int^{*}\ \text{and}\ \int$$

denote Stratonovich and Ito stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [43]).

Further, we will denote the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ as $\{\phi_j(x)\}_{j=0}^{\infty}$. We will also pay attention on the following well-known facts about these two systems of functions.

Suppose that the function $f(x)$ is bounded at the interval $[t,T]$. Moreover, its derivative $f'(x)$ is continuous function at the interval $[t,T]$ except may be the finite number of points of the finite discontinuity. Then the generalized Fourier series

$$\sum_{j=0}^{\infty} C_j \phi_j(x)$$

with the Fourier coefficients

$$C_j = \int\limits_t^T f(x)\phi_j(x)dx$$

converges at any internal point x of the interval $[t,T]$ to the value $\left(f(x+0)+f(x-0)\right)/2$ and converges uniformly to $f(x)$ on any closed interval of continuity of the function $f(x)$ laying inside $[t,T]$. At the same time the Fourier–Legendre series converges if $x=t$ and $x=T$ to $f(t+0)$ and $f(T-0)$ correspondently, and the trigonometric Fourier series converges if $x=t$ and $x=T$ to $\left(f(t+0)+f(T-0)\right)/2$ in the case of periodic continuation of the function $f(x)$.

Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k \\ \\ 0,\ \text{otherwise} \end{cases} = \prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{3}$$

where $t_1,\ldots,t_k \in [t,T]$ $(k \ge 2)$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Let us formulate the following statement.

Theorem 1 [18] (Sect. 2.4) (also see [1] (1997), [2], [10]-[13], [16], [17], [19]-[21], [41]). *Suppose that every function $\psi_l(\tau)$ $(l=1,\ldots,k)$ is twice continuously differentiable at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ defined by (1) the following expansion*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{4}$$

converging in the mean of degree $2n$ $(n \in \mathbb{N})$ is valid, where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i \ne 0$) and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient.

Note that (4) means the following

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\;\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^{2n}\right\}=0, \tag{6}$$

where $\overline{\lim}$ means lim sup.

Proof. Let us condider several lemmas. Define the function $K^*(t_1,\ldots,t_k)$ on the hypercube $[t,T]^k$ as follows

$$K^*(t_1,\ldots,t_k)=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right) \tag{7}$$

for $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K^*(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, where $\mathbf{1}_A$ is the indicator of the set A.

Lemma 1 [1] (1997), [2], [10]-[13], [16]-[21], [41]. *Under the conditions of Theorem* 1 *the function* $K^*(t_1,\ldots,t_k)$ *is represented in any internal point of the hypercube* $[t,T]^k$ *by the generalized iterated Fourier series*

$$K^*(t_1,\ldots,t_k)=\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=}\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\quad (t_1,\ldots,t_k)\in(t,T)^k, \tag{8}$$

where $C_{j_k\ldots j_1}$ *has the form* (5). *At that, the iterated series* (8) *converges at the boundary of the hypercube* $[t,T]^k$ *(not necessarily to the function* $K^*(t_1,\ldots,t_k)$*).*

Proof. We will perform the proof using induction. Consider the case $k=2$. Let us expand the function $K^*(t_1,t_2)$ using the variable t_1, when t_2 is fixed, into the generalized Fourier series at the interval (t,T)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2)\phi_{j_1}(t_1)\quad (t_1\ne t,T), \tag{9}$$

where

$$C_{j_1}(t_2)=\int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1=\int\limits_t^T K(t_1,t_2)\phi_{j_1}(t_1)dt_1=$$

$$=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The equality (9) is fulfilled pointwise at each point of the interval (t,T) with respect to the variable t_1, when $t_2\in[t,T]$ is fixed, due to the piecewise smoothness of the function $K^*(t_1,t_2)$ with respect to the variable $t_1\in[t,T]$ (t_2 is fixed).

Note also that due to the well-known properties of the Fourier series, the series (9) converges when $t_1=t$ and $t_1=T$ (not necessarily to the function $K^*(t_1,t_2)$).

Obtaining (9) we also used the fact that the right-hand side of (9) converges when $t_1=t_2$ (point of a finite discontinuity of the function $K(t_1,t_2)$) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

The function $C_{j_1}(t_2)$ is a continuously differentiable one at the interval $[t,T]$. Let us expand it into the generalized Fourier series at the interval (t,T)

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\ne t,T), \tag{10}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and the equality (10) is fulfilled pointwise at any point of the interval (t,T). The right-hand side of (10) converges when $t_2=t$ and $t_2=T$ (not necessarily to $C_{j_1}(t_2)$).

Let us substitute (10) into (9)

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2. \tag{11}$$

Note that the series on the right-hand side of (11) converges at the boundary of the square $[t,T]^2$ (not necessarily to $K^*(t_1,t_2)$). Lemma 1 is proved for the case $k=2$.

Note that proving Lemma 1 for the case $k=2$, we get the following equality (see (9))

$$\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)=\sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\cdot\phi_{j_1}(t_1), \tag{12}$$

which is fulfilled pointwise at the interval (t,T), besides the series on the right-hand side of (12) converges when $t_1=t$ and $t_1=T$.

Let us introduce the assumption of induction

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots$$

$$\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{13}$$

Then

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\psi_{k}(t_{k})\int\limits_t^{t_{k}}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}\prod_{l=1}^{k-1}\phi_{j_l}(t_l)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{14}$$

On the other hand, the left-hand side of (14) can be represented in the following form

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_k(t_k)\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}$$

into the generalized Fourier series at the interval (t,T) using the variable t_k. Lemma 1 is proved.

Let us introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1}\ \stackrel{\rm def}{=}\ \prod_{p=1}^{l}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}}\ \times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\ \times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\ \times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{15}$$

where

$$\mathrm{A}_{k,l}=\left\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1,\ s_l,\ldots,s_1=1,\ldots,k-1\right\}, \tag{16}$$

$$(s_l,\ldots,s_1)\in\mathrm{A}_{k,l},\quad l=1,\ldots,[k/2],\quad i_s=0,1,\ldots,m,\quad s=1,\ldots,k,$$

$[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on relation between iterated Ito and Stratonovich stochastic integrals $J^*[\psi^{(k)}]_{T,t}$, $J[\psi^{(k)}]_{T,t}$ of fixed multiplicity k (see (1), (2)).

Lemma 2 [18] (Sect. 2.4) (also see [1] (1997), [2], [10]-[13], [16], [17], [19]-[21]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the following relation between iterated Ito and Stratonovich stochastic integrals is correct*

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1}, \tag{17}$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero; *hereinafter w. p. 1 means "with probability 1".*

Proof. Let us prove the equality (17) using induction. The case $k=1$ is obvious. If $k=2$, then from (17) we get

$$J^*[\psi^{(2)}]_{T,t}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}J[\psi^{(2)}]^1_{T,t}\quad \text{w. p. 1.} \tag{18}$$

Let us demonstrate that the equality (18) is correct w. p. 1. In order to do it let us consider the function $F(x,\tau)=x\psi_2(\tau)$ and the process $F(\eta_{\tau,t},\tau)$, where $\eta_{\tau,t}=J[\psi^{(1)}]_{\tau,t}$, $\tau\in[t,T]$. Then

$$\frac{\partial F}{\partial x}(x,\tau)=\psi_2(\tau),\quad d\eta_{\tau,t}=\psi_1(\tau)d\mathbf{w}_{\tau}^{(i_1)}. \tag{19}$$

From (19) we obtain that the diffusion coefficient of the process $\eta_{\tau,t}$, $\tau\in[t,T]$ equals to $\mathbf{1}_{\{i_1\ne 0\}}\psi_1(\tau)$. Further, using the standard relations between Stratonovich and Ito stochastic integrals [43] (also see [18] (Sect. 2.4)), we obtain the relation (18). Thus, the statement of Lemma 2 is proved for $k=1$ and $k=2$.

Assume that the statement of Lemma 2 is correct for some integer k ($k>2$), and let us prove its correctness when the value k is greater per unit. Using the assumption of induction, we obtain w. p. 1

$$J^*[\psi^{(k+1)}]_{T,t}=$$

$$=\int\limits_t^{*T}\psi_{k+1}(\tau)\left(J[\psi^k]_{\tau,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]^{s_r,\ldots,s_1}_{\tau,t}\right)d\mathbf{w}_{\tau}^{(i_{k+1})}=$$

$$=\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}+$$

$$+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]^{s_r,\ldots,s_1}_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}. \tag{20}$$

Applying the Ito formula and the standard relation between Stratonovich and Ito stochastic integrals, we get w. p. 1

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}=J[\psi^{(k+1)}]_{T,t}+\frac{1}{2}J[\psi^{(k+1)}]^k_{T,t}, \tag{21}$$

$$\int\limits_t^{*T}\psi_{k+1}(\tau)J[\psi^{(k)}]^{s_r,\ldots,s_1}_{\tau,t}d\mathbf{w}_{\tau}^{(i_{k+1})}=$$

$$
(22) \qquad = \begin{cases} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} & \text{if} \ \ s_r = k-1 \\ \\ J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} + J[\psi^{(k+1)}]_{T,t}^{k,s_r,\ldots,s_1}/2 & \text{if} \ \ s_r < k-1 \end{cases}.
$$

After substituting (21) and (22) into (20) and regrouping of summands we pass to the following relations, which are valid w. p. 1

$$
(23) \qquad J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}
$$

when k is even and

$$
(24) \qquad J^*[\psi^{(k'+1)}]_{T,t} = J[\psi^{(k'+1)}]_{T,t} + \sum_{r=1}^{[k'/2]+1} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k'+1,r}} J[\psi^{(k'+1)}]_{T,t}^{s_r,\ldots,s_1}
$$

when $k' = k+1$ is uneven.

From (23) and (24) we have w. p. 1

$$
(25) \qquad J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{[(k+1)/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}.
$$

Lemma 2 is proved.

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t,T]$ such that

$$
(26) \qquad t=\tau_0<\ldots<\tau_N=T, \quad \Delta_N = \max_{0\le j\le N-1} \Delta\tau_j \to 0 \ \ \text{if} \ \ N\to\infty, \quad \Delta\tau_j = \tau_{j+1}-\tau_j.
$$

Lemma 3. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Then*

$$
(27) \qquad J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1},
$$

where $J[\psi^{(k)}]_{T,t}$ is the iterated Ito stochastic integral (2), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$ satisfying the condition (26).

Proof. It is easy to notice that using the additive property of stochastic integrals we can write the following

(28) $$J[\psi^{(k)}]_{T,t}=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}+\varepsilon_N\qquad \hbox{w. p. 1},$$

where

$$\varepsilon_N=\sum_{j_k=0}^{N-1}\int\limits_{\tau_{j_k}}^{\tau_{j_k+1}}\psi_k(s)\int\limits_{\tau_{j_k}}^{s}\psi_{k-1}(\tau)J[\psi^{(k-2)}]_{\tau,t}d{\bf w}_{\tau}^{(i_{k-1})}d{\bf w}_{s}^{(i_k)}+$$

$$+\sum_{r=1}^{k-3}G[\psi_{k-r+1}^{(k)}]_N\sum_{j_{k-r}=0}^{j_{k-r+1}-1}\int\limits_{\tau_{j_{k-r}}}^{\tau_{j_{k-r}+1}}\psi_{k-r}(s)\int\limits_{\tau_{j_{k-r}}}^{s}\psi_{k-r-1}(\tau)J[\psi^{(k-r-2)}]_{\tau,t}d{\bf w}_{\tau}^{(i_{k-r-1})}d{\bf w}_{s}^{(i_{k-r})}+$$

$$+G[\psi_3^{(k)}]_N\sum_{j_2=0}^{j_3-1}J[\psi^{(2)}]_{\tau_{j_2+1},\tau_{j_2}},$$

$$G[\psi_m^{(k)}]_N=\sum_{j_k=0}^{N-1}\sum_{j_{k-1}=0}^{j_k-1}\ldots\sum_{j_m=0}^{j_{m+1}-1}\prod_{l=m}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$J[\psi_l]_{s,\theta}=\int\limits_{\theta}^{s}\psi_l(\tau)d{\bf w}_{\tau}^{(i_l)},$$

$$(\psi_m,\psi_{m+1},\ldots,\psi_k)\stackrel{\rm def}{=}\psi_m^{(k)},\quad (\psi_1,\ldots,\psi_k)\stackrel{\rm def}{=}\psi_1^{(k)}=\psi^{(k)}.$$

Using the standard estimates (38), (39) for the moments of stochastic integrals, we obtain w. p. 1

(29) $$\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr{\rm l.i.m.}\cr\noalign{\vskip1pt}}}}_{N\to\infty}\ \varepsilon_N=0.$$

Comparing (28) and (29), we get

(30) $$J[\psi^{(k)}]_{T,t}=\mathop{\rm l.i.m.}_{N\to\infty}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}\qquad \hbox{w. p. 1}.$$

Let us write $J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}$ in the form

$$J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}=\psi_l(\tau_{j_l})\Delta{\bf w}_{\tau_{j_l}}^{(i_l)}+\int\limits_{\tau_{j_l}}^{\tau_{j_l+1}}(\psi_l(\tau)-\psi_l(\tau_{j_l}))d{\bf w}_{\tau}^{(i_l)}\quad \hbox{w. p. 1}$$

and substitute it into (30). Then, due to the moment properties of stochastic integrals and continuity (which means uniform continuity) of the functions $\psi_l(s)$ $(l=1,\ldots,k)$ it is easy to see that the prelimit expression on the right-hand side of (30) is a sum of the prelimit expression on the right-hand side of (27) and the value which tends to zero in the mean-square sense if $N\to\infty$. Lemma 3 is proved.

Remark 1. *It is easy to see that if* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (27) *for some* $l \in \{1,\ldots,k\}$ *is replaced with* $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p=2,\ i_l\ne 0)$, *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\psi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p=3,4,\ldots,$ *then the right-hand side of the formula* (27) *will become zero w. p. 1. If we replace* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (27) *for some* $l\in\{1,\ldots,k\}$ *with* $(\Delta\tau_{j_l})^p$ $(p=2,3,\ldots)$, *then the right-hand side of the formula* (27) *also will be equal to zero w. p. 1.*

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J[\Phi]_{T,t}^{(k)}, \tag{31}$$

where $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a nonrandom function (the properties of this function will be specified further).

Denote

$$D_k=\{(t_1,\ldots,t_k):\ t\le t_1<\ldots<t_k\le T\}. \tag{32}$$

We will use the same symbol D_k to denote the open and closed domains corresponding to the domain D_k defined by (32). However, we always specify what domain we consider (open or closed). Also we will write $\Phi(t_1,\ldots,t_k)\in C(D_k)$ if $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function of k variables in the closed domain D_k.

Let us consider the iterated Ito stochastic integral

$$I[\Phi]_{T,t}^{(k)}\stackrel{\text{def}}{=}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{33}$$

where $\Phi(t_1,\ldots,t_k)\in C(D_k)$.

Using the arguments which similar to the arguments used in the proof of Lemma 3 it is easy to demonstrate that if $\Phi(t_1,\ldots,t_k)\in C(D_k)$, then the following equality is fulfilled

$$I[\Phi]_{T,t}^{(k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\Phi(\tau_{j_1},\ldots,\tau_{j_k})\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\quad\text{w. p. 1.} \tag{34}$$

In order to explain this, let us check the correctness of the equality (34) when $k=3$. For definiteness we will suppose that $i_1,i_2,i_3=1,\ldots,m$. We have

$$I[\Phi]_{T,t}^{(3)}\stackrel{\text{def}}{=}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\int\limits_t^{\tau_{j_3}}\int\limits_t^{t_2}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_t^{t_2}\Phi(t_1,t_2,\tau_{j_3})d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}=$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \left(\int\limits_{t}^{\tau_{j_2}} + \int\limits_{\tau_{j_2}}^{t_2} \right) \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (35)$$

Let us demonstrate that the second limit on the right-hand side of (35) equals to zero. Actually, for the second moment of its prelimit expression we get

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_2}}^{t_2} \Phi^2(t_1,t_2,\tau_{j_3}) dt_1 dt_2 \Delta\tau_{j_3} \le M^2 \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1} \frac{1}{2}\left(\Delta\tau_{j_2}\right)^2 \Delta\tau_{j_3} \to 0$$

when $N\to\infty$. Here M is a constant, which restricts the module of the function $\Phi(t_1,t_2,t_3)$ due to its continuity, $\Delta\tau_j = \tau_{j+1}-\tau_j$.

Considering the obtained conclusions, we have

$$I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,\tau_{j_2},\tau_{j_3}) - \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3})\right) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3}) \Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (36)$$

In order to get the sought result, we just have to demonstrate that the first two limits on the right-hand side of (36) equal to zero. Let us prove that the first one of them equals to zero (proof for the second limit is similar).

The second moment of prelimit expression of the first limit on the right-hand side of (36) equals to the following expression

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\left(\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1dt_2\Delta\tau_{j_3}. \tag{37}$$

Since the function $\Phi(t_1,t_2,t_3)$ is continuous in the closed bounded domain D_3, then it is uniformly continuous in this domain. Therefore, if the distance between two points of the domain D_3 is less than $\delta(\varepsilon)$ ($\delta(\varepsilon)>0$ exists for any $\varepsilon>0$ and it does not depend on mentioned points), then the corresponding oscillation of the function $\Phi(t_1,t_2,t_3)$ for these two points of the domain D_3 is less than ε.

If we assume that $\Delta\tau_j<\delta(\varepsilon)$ ($j=0,1,\ldots,N-1$), then the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$ is obviously less than $\delta(\varepsilon)$. In this case

$$|\Phi(t_1,t_2,\tau_{j_3})-\Phi(t_1,\tau_{j_2},\tau_{j_3})|<\varepsilon.$$

Consequently, when $\Delta\tau_j<\delta(\varepsilon)$ ($j=0,\ 1,\ldots,N-1$) the expression (37) is estimated by the following value

$$\varepsilon^2\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3}<\varepsilon^2\frac{(T-t)^3}{6}.$$

Therefore, the first limit on the right-hand side of (36) equals to zero. Similarly, we can prove that the second limit on the right-hand side of (36) equals to zero.

Consequently, the equality (34) is proved for $k=3$. The cases $k=2$ and $k>3$ are analyzed absolutely similarly.

It is necessary to note that the proof of correctness of (34) is similar when the nonrandom function $\Phi(t_1,\ldots,t_k)$ is continuous in the open domain D_k and bounded at its boundary.

Let us consider the class $\mathrm{M}_2([0,T])$ of functions $\xi:[0,T]\times\Omega\rightarrow\mathbb{R}$, which are measurable with respect to the variables (t,ω) and F_t-measurable for all $t\in[0,T]$. Moreover, $\xi(\tau,\omega)$ is independent with increments $\mathbf{f}_{t+\Delta}-\mathbf{f}_t$ for $t\ge\tau$ ($\Delta>0$),

$$\int\limits_0^T\mathsf{M}\left\{\xi^2(t,\omega)\right\}dt<\infty,$$

and $\mathsf{M}\left\{\xi^2(t,\omega)\right\}<\infty$ for all $t\in[0,T]$.

It is well-known [43], [60] that the Ito stochastic integral exists in the mean-square sense for any $\xi\in\mathrm{M}_2([0,T])$. Further, we will denote $\xi(\tau,\omega)$ as ξ_τ.

Lemma 4. *Suppose that $\Phi(t_1,\ldots,t_k)\in C(D_k)$ or $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function in the open domain D_k and bounded at its boundary. Then*

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^{2n}\right\}\le C_k\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi^{2n}(t_1,\ldots,t_k)dt_1\ldots dt_k,\quad C_k<\infty,$$

where $I[\Phi]_{T,t}^{(k)}$ is defined by the formula (33).

Proof. Using standard estimates for moments of stochastic integrals, we have [60]

(38) $$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau df_\tau\right|^{2n}\right\} \le (T-t)^{n-1}\left(n(2n-1)\right)^n \int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau,$$

(39) $$\mathsf{M}\left\{\left|\int\limits_t^T \xi_\tau d\tau\right|^{2n}\right\} \le (T-t)^{2n-1}\int\limits_t^T \mathsf{M}\left\{|\xi_\tau|^{2n}\right\}d\tau,$$

where the process ξ_τ is such that $(\xi_\tau)^n \in \mathrm{M}_2([t,T])$ and f_t is a scalar standard Wiener process, $n=1,2,\ldots$

Let us denote

$$\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t} = \int\limits_t^{t_{l+1}}\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_l)}_{t_l},$$

where $l=1,\ldots,\ k-1$ and $\xi[\Phi]^{(0)}_{t_1,\ldots,t_k,t} \stackrel{\rm def}{=} \Phi(t_1,\ldots,t_k)$.

By induction it is easy to demonstrate that $\left(\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t}\right)^n \in \mathrm{M}_2([t,T])$ with respect to the variable t_{l+1}. Further, using the estimates (38) and (39) repeatedly we obtain the statement of Lemma 4. Lemma 4 is proved.

Lemma 5 [1] (1997), [2], [10]-[13], [16]-[21]. *Suppose that every $\varphi_l(s)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$. Then*

(40) $$\prod_{l=1}^k J[\varphi_l]_{T,t} = J[\Phi]^{(k)}_{T,t}\quad \text{w. p. 1},$$

where

$$J[\varphi_l]_{T,t} = \int\limits_t^T \varphi_l(s)d\mathbf{w}^{(i_l)}_s,\quad \Phi(t_1,\ldots,t_k) = \prod_{l=1}^k \varphi_l(t_l),$$

and the integral $J[\Phi]^{(k)}_{T,t}$ is defined by the equality (31).

Proof. Let at first $i_l \ne 0$ $(l=1,\ldots,k)$. Denote

$$J[\varphi_l]_N \stackrel{\rm def}{=} \sum_{j=0}^{N-1}\varphi_l(\tau_j)\Delta\mathbf{w}^{(i_l)}_{\tau_j}.$$

Since

$$\prod_{l=1}^k J[\varphi_l]_N - \prod_{l=1}^k J[\varphi_l]_{T,t} =$$

$$= \sum_{l=1}^k\left(\prod_{g=1}^{l-1}J[\varphi_g]_{T,t}\right)\left(J[\varphi_l]_N - J[\varphi_l]_{T,t}\right)\left(\prod_{g=l+1}^{k}J[\varphi_g]_N\right),$$

then due to the Minkowski inequality and the inequality of Cauchy-Bunyakovsky we obtain

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k} J[\varphi_l]_N - \prod_{l=1}^{k} J[\varphi_l]_{T,t}\right|^2\right\}\right)^{1/2} \le C_k \sum_{l=1}^{k} \left(\mathsf{M}\left\{\left|J[\varphi_l]_N - J[\varphi_l]_{T,t}\right|^4\right\}\right)^{1/4}, \tag{41}$$

where C_k is a constant.

Note that

$$J[\varphi_l]_N - J[\varphi_l]_{T,t} = \sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}, \quad J[\Delta\varphi_l]_{\tau_{j+1},\tau_j} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))\, d\mathbf{w}_s^{(i_l)}.$$

Since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ are independent for various j, then [61]

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} \sum_{q=0}^{j-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \tag{42}$$

Moreover, since $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ is a Gaussian random variable, we have

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds,$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds\right)^2.$$

Using these relations and continuity (which means uniform continuity) of the functions $\varphi_l(s)$, we obtain

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le$$

$$\le \varepsilon^4 \left(3\sum_{j=0}^{N-1} (\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1} \Delta\tau_j \sum_{q=0}^{j-1} \Delta\tau_q\right) < 3\varepsilon^4 \left(\delta(\varepsilon)(T-t) + (T-t)^2\right),$$

where $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ which does not depend on points of the interval $[t, T]$ and such that $|\varphi_l(\tau_j) - \varphi_l(s)| < \varepsilon$, $s \in [\tau_j, \tau_{j+1}]$). Then the right-hand side of the formula (42) tends to zero when $N \to \infty$.

Taking into account this fact as well as (41), we obtain (40). If $\mathbf{w}_{t_l}^{(i_l)}=t_l$ for some $l\in\{1,\ldots,k\}$, then the proof of Lemma 5 becomes obviously simpler and it is performed similarly. Lemma 5 is proved.

Using Lemma 2 and (34), we obtain w. p. 1

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=J[K^*]_{T,t}^{(k)},\tag{43}$$

where the stochastic integral $J[K^*]_{T,t}^{(k)}$ is defined in accordance with (31).

Let us subsitute the relation

$$K^*(t_1,\ldots,t_k)=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)+K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

into (43) (here we suppose that $p_1,\ldots,p_k<\infty$).

Then using Lemma 5, we obtain

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\quad\text{w. p. 1},\tag{44}$$

where the stochastic integral $J[R_{p_1\ldots p_k}]_{T,t}^{(k)}$ is defined in accordance with (31) and

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)=K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\tag{45}$$

$$\zeta_{j_l}^{(i_l)}=\int\limits_t^T\phi_{j_l}(s)d\mathbf{w}_s^{(i_l)}.$$

According to Lemma 1, we obtain

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}R_{p_1\ldots p_k}(t_1,\ldots,t_k)=0\quad\text{when}\quad(t_1,\ldots,t_k)\in(t,T)^k,\tag{46}$$

where the left-hand side of (46) is bounded on $[t,T]^k$.

Lemma 6. *Under the conditions of Theorem 1 the following equality is correct*

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}=0,\quad n\in\mathbb{N}.$$

Proof. At first let us analize in detail the cases $k=2,3,4$. Using (80) (see below), we have w. p. 1

$$J[R_{p_1p_2}]_{T,t}^{(2)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} + \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1=0}^{N-1} R_{p_1p_2}(\tau_{l_1},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)} =$$

$$= \int_t^T\int_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)} + \int_t^T\int_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\int_t^T R_{p_1p_2}(t_1,t_1)dt_1, \tag{47}$$

where

$$R_{p_1p_2}(t_1,t_2) = K^*(t_1,t_2) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad p_1,\ p_2 < \infty. \tag{48}$$

Using Lemma 4, we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} \le C_n\left(\int_t^T\int_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 + \right.$$

$$\left. + \int_t^T\int_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1 + \mathbf{1}_{\{i_1=i_2\neq 0\}}\int_t^T\left(R_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1\right), \tag{49}$$

where constant $C_n < \infty$ depends on n and $T-t$ $(n = 1, 2, \ldots)$.

Further, we have

$$\int_t^T\int_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 + \int_t^T\int_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1 =$$

$$= \int_t^T\int_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 + \int_t^T\int_{t_2}^T\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 = \int_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2. \tag{50}$$

Combining (49) and (50), we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2}]_{T,t}^{(2)}\right|^{2n}\right\} \le$$

$$\leq C_n \left(\int\limits_{[t,T]^2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1dt_2 + \mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \left(R_{p_1p_2}(t_1,t_1)\right)^{2n} dt_1 \right), \tag{51}$$

where constant $C_n < \infty$ depends on n and $T-t$ ($n = 1, 2, \ldots$).

Since the integrals on the right-hand side of (51) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty} \lim_{p_2\to\infty} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} = 0 \quad \text{when} \quad (t_1,t_2) \in (t,T)^2, \tag{52}$$

where $n \in \mathbb{N}$, the left-hand side is bounded on $[t,T]^2$.

According to (9)–(11) and (48), we obtain

$$R_{p_1p_2}(t_1,t_2) = \left(K^*(t_1,t_2) - \sum_{j_1=0}^{p_1} C_{j_1}(t_2)\phi_{j_1}(t_1) \right) +$$

$$+ \left(\sum_{j_1=0}^{p_1} \left(C_{j_1}(t_2) - \sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_2}(t_2) \right) \phi_{j_1}(t_1) \right). \tag{53}$$

Then, applying two times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty} \overline{\lim\limits_{p_2\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \int\limits_{[t,T]^2} \left(R_{p_1p_2}(t_1,t_2)\right)^{2n} dt_1dt_2 = 0, \qquad \lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \int\limits_t^T \left(R_{p_1p_2}(t_1,t_1)\right)^{2n} dt_1 = 0. \tag{54}$$

We will discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem when we consider the case of arbitrary $k \in \mathbb{N}$ later in this section.

From (51) and (54) we obtain

$$\lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \mathsf{M}\left\{ \left| J[R_{p_1p_2}]_{T,t}^{(2)} \right|^{2n} \right\} = 0, \quad n \in \mathbb{N}.$$

Let us consider the case $k = 3$. Using (81) (see below), we have w. p. 1

$$J[R_{p_1p_2p_3}]_{T,t}^{(3)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{l_3-1} \sum_{l_1=0}^{l_2-1} \Biggl(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} +$$

$$
+R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+
$$

$$
\left.+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+
$$

$$
\left.+R_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}\Biggl(R_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+
$$

$$
\left.+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}\right)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}R_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}+\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_1)}+\int_t^T\int_t^{t_3}\int_t^{t_2}R_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\int_t^T\int_t^{t_3}R_{p_1p_2p_3}(t_2,t_2,t_3)dt_2d\mathbf{w}_{t_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\int_t^T\int_t^{t_3}R_{p_1p_2p_3}(t_2,t_3,t_2)dt_2d\mathbf{w}_{t_3}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\int_t^T\int_t^{t_3}R_{p_1p_2p_3}(t_3,t_2,t_2)dt_2d\mathbf{w}_{t_3}^{(i_1)}+\mathbf{1}_{\{i_2=i_3\neq 0\}}\int_t^T\int_t^{t_3}R_{p_1p_2p_3}(t_1,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}dt_3+
$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{p_1p_2p_3}(t_3,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}dt_3+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{p_1p_2p_3}(t_3,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}dt_3. \tag{55}$$

Applying Lemma 4, we obtain

$$\begin{aligned}
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}\le C_n\Biggl(\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Bigl(&\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+\\
&+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+\\
&+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Bigr)dt_1dt_2dt_3+\\
&+\int\limits_t^T\int\limits_t^{t_3}\Bigl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\Bigl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Bigr)+\\
&+\mathbf{1}_{\{i_1=i_3\ne 0\}}\Bigl(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\Bigr)+\\
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\Bigl(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\Bigr)dt_2dt_3\Bigr),\quad C_n<\infty.
\end{aligned} \tag{56}$$

Further, we have

$$\begin{aligned}
&\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Bigl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n}+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n}+\\
&+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Bigr)dt_1dt_2dt_3=\\
&=\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3,
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\int\limits_t^T\int\limits_t^{t_3}\Bigl(\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}+\left(R_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Bigr)dt_2dt_3=\\
&=\int\limits_t^T\int\limits_t^{t_3}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3+\int\limits_t^T\int\limits_{t_3}^{T}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=
\end{aligned}$$

$$= \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n} dt_2 dt_3, \tag{58}$$

$$\int\limits_t^T \int\limits_t^{t_3} \left(\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} + \left(R_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n} \right) dt_2 dt_3 =$$

$$= \int\limits_t^T \int\limits_t^{t_3} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2 dt_3 + \int\limits_t^T \int\limits_{t_3}^{T} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2 dt_3 =$$

$$= \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2 dt_3, \tag{59}$$

$$\int\limits_t^T \int\limits_t^{t_3} \left(\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} + \left(R_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n} \right) dt_2 dt_3 =$$

$$= \int\limits_t^T \int\limits_t^{t_3} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2 dt_3 + \int\limits_t^T \int\limits_{t_3}^{T} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2 dt_3 =$$

$$= \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2 dt_3. \tag{60}$$

Combining (56) and (57)–(60), we get

$$\mathsf{M}\left\{ \left| J[R_{p_1p_2p_3}]_{T,t}^{(3)} \right|^{2n} \right\} \le C_n \left(\int\limits_{[t,T]^3} \left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n} dt_1 dt_2 dt_3 + \right.$$

$$+ \mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n} dt_2 dt_3 +$$

$$+ \mathbf{1}_{\{i_1=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} dt_2 dt_3 +$$

$$\left. + \mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} dt_2 dt_3 \right), \quad C_n < \infty. \tag{61}$$

Since the integrals on the right-hand side of (61) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}R_{p_1p_2p_3}(t_1,t_2,t_3)=0\quad\hbox{when}\quad (t_1,t_2,t_3)\in(t,T)^3,$$

where the left-hand side is bounded on $[t,T]^3$.

According to the proof of Lemma 1 and (45) for $k=3$, we have

$$R_{p_1p_2p_3}(t_1,t_2,t_3)=\left(K^{*}(t_1,t_2,t_3)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,t_3)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,t_3)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}(t_3)-\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\phi_{j_3}(t_3)\right)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right), \tag{62}$$

where

$$C_{j_1}(t_2,t_3)=\int\limits_t^T K^{*}(t_1,t_2,t_3)\phi_{j_1}(t_1)dt_1,\quad C_{j_2j_1}(t_3)=\int\limits_{[t,T]^2}K^{*}(t_1,t_2,t_3)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2.$$

Then, applying three times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\;\overline{\lim\limits_{p_3\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\;\overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n}dt_1dt_2dt_3=0, \tag{63}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\;\overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n}dt_2dt_3=0, \tag{64}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\;\overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n}dt_2dt_3=0, \tag{65}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\;\overline{\lim_{p_3\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n}dt_2dt_3=0. \tag{66}$$

From (61)–(66) we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\;\overline{\lim_{p_3\to\infty}}\mathsf{M}\left\{\left|J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right|^{2n}\right\}=0,\quad n\in\mathbb{N}.$$

Let us consider the case $k=4$. Using (82) (see below), we have w. p. 1

$$J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{(l_2,l_2,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_1=0}^{l_3-1}\sum_{(l_1,l_3,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{(l_3,l_3,l_3,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_3},\tau_{l_3},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{l_4-1}\sum_{(l_2,l_2,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{l_4-1}\sum_{(l_1,l_4,l_4,l_4)}\left(R_{p_1p_2p_3p_4}(\tau_{l_1},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\right)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}R_{p_1p_2p_3p_4}(\tau_{l_4},\tau_{l_4},\tau_{l_4},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_4}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=$$

$$=\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\sum_{(t_1,t_3,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_1,t_3,t_4)dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_1,t_4)dt_1d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_1)dt_1 d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_4)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_2,t_4)d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_4}^{(i_4)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_3}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\sum\limits_{(t_1,t_2,t_3)}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)dt_2dt_4+\right.$$

$$\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_4,t_2,t_2)dt_2dt_4\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)dt_2dt_4+\right.$$

$$\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_2,t_4,t_2)dt_2dt_4\right)+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)dt_2dt_4+\right.$$

$$\left.+\int\limits_t^T\int\limits_t^{t_4}R_{p_1p_2p_3p_4}(t_4,t_2,t_2,t_4)dt_2dt_4\right), \tag{67}$$

where the expression

$$\sum\limits_{(a_1,\ldots,a_k)}$$

means the sum with respect to all possible permutations $(a_1,\ldots,a_k)$. Note that an analogue of (67) was obtained in [32], Sect. 6 (also see [18]-[21]) with using a different approach.

By analogy with (61) we obtain

$$\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\} \le C_n \left(\int\limits_{[t,T]^4} \left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n} dt_1dt_2dt_3dt_4 + \right.$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_2,t_3,t_4)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_3,t_2,t_4)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}} \int\limits_{[t,T]^3} \left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n} dt_2dt_3dt_4 +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n} dt_2dt_4 +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n} dt_2dt_4 +$$

$$\left. +\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}} \int\limits_{[t,T]^2} \left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n} dt_2dt_4 \right), \quad C_n < \infty. \tag{68}$$

Since the integrals on the right-hand side of (68) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\lim_{p_4\to\infty} R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4) = 0 \quad \text{when} \quad (t_1,t_2,t_3,t_4) \in (t,T)^4,$$

where the left-hand side is bounded on $[t,T]^4$.

According to the proof of Lemma 1 and (45) for $k=4$, we have

$$R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4) = \left(K^*(t_1,t_2,t_3,t_4) - \sum_{j_1=0}^{p_1} C_{j_1}(t_2,t_3,t_4)\phi_{j_1}(t_1)\right) +$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,t_3,t_4)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,t_4)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_2j_1}(t_3,t_4)-\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(t_4)\phi_{j_3}(t_3)\right)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(C_{j_3j_2j_1}(t_4)-\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\phi_{j_4}(t_4)\right)\phi_{j_3}(t_3)\phi_{j_2}(t_2)\phi_{j_1}(t_1)\right),$$

where

$$C_{j_1}(t_2,t_3,t_4)=\int\limits_t^T K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2j_1}(t_3,t_4)=\int\limits_{[t,T]^2} K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$C_{j_3j_2j_1}(t_4)=\int\limits_{[t,T]^3} K^*(t_1,t_2,t_3,t_4)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)dt_1dt_2dt_3.$$

Then, applying four times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ \overline{\lim\limits_{p_3\to\infty}}\ \overline{\lim\limits_{p_4\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^4}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n}dt_1dt_2dt_3dt_4=0, \tag{69}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_3,t_4)\right)^{2n}dt_2dt_3dt_4=0, \tag{70}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_2,t_4)\right)^{2n}dt_2dt_3dt_4=0, \tag{71}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_3,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \tag{72}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_2,t_4)\right)^{2n}dt_2dt_3dt_4=0, \tag{73}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_2,t_4,t_2)\right)^{2n}dt_2dt_3dt_4=0, \tag{74}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_3,t_4,t_2,t_2)\right)^{2n}dt_2dt_3dt_4=0, \tag{75}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_2,t_4,t_4)\right)^{2n}dt_2dt_4=0, \tag{76}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_2,t_4)\right)^{2n}dt_2dt_4=0, \tag{77}$$

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}(t_2,t_4,t_4,t_2)\right)^{2n}dt_2dt_4=0. \tag{78}$$

Combaining (68) with (69)–(78), we get

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ \overline{\lim_{p_3\to\infty}}\ \overline{\lim_{p_4\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\}=0,\quad n\in\mathbb{N}.$$

Lemma 6 is proved for the case $k=4$.

Let us consider the case of arbitrary k ($k\in\mathbb{N}$). Let us analyze the stochastic integral defined by (31) and find its representation convenient for the following consideration. In order to do it we introduce several notations. Suppose that

$$S_N^{(k)}(a)=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\sum_{(j_1,\ldots,j_k)}a_{(j_1,\ldots,j_k)},$$

$$\mathrm{C}_{s_r}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=$$

$$=\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}a_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)},$$

where

$$\prod_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)\ \stackrel{\rm def}{=}\ \mathbf{I}_{j_{s_r},j_{s_r+1}}\ldots\mathbf{I}_{j_{s_1},j_{s_1+1}}(j_1,\ldots,j_k),$$

$$\mathrm{C}_{s_0}\ldots\mathrm{C}_{s_1}S_N^{(k)}(a)=S_N^{(k)}(a),\qquad \prod_{l=1}^{0}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)=(j_1,\ldots,j_k),$$

$$\mathbf{I}_{j_l,j_{l+1}}(j_{q_1},\ldots,j_{q_2},j_l,j_{q_3},\ldots,j_{q_{k-2}},j_l,j_{q_{k-1}},\ldots,j_{q_k})\ \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} (j_{q_1},\ldots,j_{q_2},j_{l+1},j_{q_3},\ldots,j_{q_{k-2}},j_{l+1},j_{q_{k-1}},\ldots,j_{g_k}),$$

where $l \in \mathbb{N}$, $l \ne q_1,\ldots,q_2,q_3,\ldots,q_{k-2},q_{k-1},\ldots,q_k$, $s_1,\ldots,s_r = 1,\ldots,k-1$, $s_r > \ldots > s_1$, $a_{(j_{q_1},\ldots,j_{q_k})}$ is a scalar value, $q_1,\ldots,q_k = 1,\ldots,k$, the expression

$$\sum_{(j_{q_1},\ldots,j_{q_k})}$$

means the sum with respect to all possible permutations $(j_{q_1},\ldots,j_{q_k})$.

Using induction it is possible to prove the following equality

$$\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{N-1} a_{(j_1,\ldots,j_k)} = \sum_{r=0}^{k-1}\ \sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1} {\rm C}_{s_r}\ldots {\rm C}_{s_1} S_N^{(k)}(a), \tag{79}$$

where $k=2,3,\ldots$

Hereinafter in this section, we will identify the following records $a_{(j_1,\ldots,j_k)} = a_{(j_1\ldots j_k)} = a_{j_1\ldots j_k}$. In particular, from (79) for $k=2,3,4$ we get the following formulas

$$\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2)} = S_N^{(2)}(a) + {\rm C}_1 S_N^{(2)}(a) =$$

$$= \sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2)} a_{(j_1 j_2)} + \sum_{j_2=0}^{N-1} a_{(j_2 j_2)} = \sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2}+a_{j_2j_1}) +$$

$$+\sum_{j_2=0}^{N-1} a_{j_2j_2}, \tag{80}$$

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1} a_{(j_1,j_2,j_3)} = S_N^{(3)}(a) + {\rm C}_1 S_N^{(3)}(a) + {\rm C}_2 S_N^{(3)}(a) + {\rm C}_2{\rm C}_1 S_N^{(3)}(a) =$$

$$= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3)} a_{(j_1j_2j_3)} + \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3)} a_{(j_2j_2j_3)} +$$

$$+\sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3)} a_{(j_1j_3j_3)} + \sum_{j_3=0}^{N-1} a_{(j_3j_3j_3)} =$$

$$= \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2j_3}+a_{j_1j_3j_2}+a_{j_2j_1j_3}+a_{j_2j_3j_1}+a_{j_3j_2j_1}+a_{j_3j_1j_2}) +$$

$$+\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\left(a_{j_2j_2j_3}+a_{j_2j_3j_2}+a_{j_3j_2j_2}\right)+\sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}\left(a_{j_1j_3j_3}+a_{j_3j_1j_3}+a_{j_3j_3j_1}\right)+$$

$$+\sum_{j_3=0}^{N-1}a_{j_3j_3j_3}, \tag{81}$$

$$\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1}a_{(j_1,j_2,j_3,j_4)}=S_N^{(4)}(a)+\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_2S_N^{(4)}(a)+$$

$$+\mathrm{C}_3S_N^{(4)}(a)+\mathrm{C}_2\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_2S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_2\mathrm{C}_1S_N^{(4)}(a)=$$

$$=\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3,j_4)}a_{(j_1j_2j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3,j_4)}a_{(j_2j_2j_3j_4)}$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3,j_4)}a_{(j_1j_3j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_4,j_4)}a_{(j_1j_2j_4j_4)}+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{(j_3,j_3,j_3,j_4)}a_{(j_3j_3j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)}a_{(j_2j_2j_4j_4)}+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_4,j_4,j_4)}a_{(j_1j_4j_4j_4)}+\sum_{j_4=0}^{N-1}a_{j_4j_4j_4j_4}=$$

$$=\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\left(a_{j_1j_2j_3j_4}+a_{j_1j_2j_4j_3}+a_{j_1j_3j_2j_4}+a_{j_1j_3j_4j_2}+\right.$$

$$+a_{j_1j_4j_3j_2}+a_{j_1j_4j_2j_3}+a_{j_2j_1j_3j_4}+a_{j_2j_1j_4j_3}+a_{j_2j_4j_1j_3}+a_{j_2j_4j_3j_1}+a_{j_2j_3j_1j_4}+$$

$$+a_{j_2j_3j_4j_1}+a_{j_3j_1j_2j_4}+a_{j_3j_1j_4j_2}+a_{j_3j_2j_1j_4}+a_{j_3j_2j_4j_1}+a_{j_3j_4j_1j_2}+a_{j_3j_4j_2j_1}+$$

$$\left.+a_{j_4j_1j_2j_3}+a_{j_4j_1j_3j_2}+a_{j_4j_2j_1j_3}+a_{j_4j_2j_3j_1}+a_{j_4j_3j_1j_2}+a_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\left(a_{j_2j_2j_3j_4}+a_{j_2j_2j_4j_3}+a_{j_2j_3j_2j_4}+a_{j_2j_4j_2j_3}+a_{j_2j_3j_4j_2}+a_{j_2j_4j_3j_2}+\right.$$

$$\left.+a_{j_3j_2j_2j_4}+a_{j_4j_2j_2j_3}+a_{j_3j_2j_4j_2}+a_{j_4j_2j_3j_2}+a_{j_4j_3j_2j_2}+a_{j_3j_4j_2j_2}\right)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\left(a_{j_3j_3j_1j_4}+a_{j_3j_3j_4j_1}+a_{j_3j_1j_3j_4}+a_{j_3j_4j_3j_1}+a_{j_3j_4j_1j_3}+a_{j_3j_1j_4j_3}+\right.$$

$$\left.+a_{j_1j_3j_3j_4}+a_{j_4j_3j_3j_1}+a_{j_4j_3j_1j_3}+a_{j_1j_3j_4j_3}+a_{j_1j_4j_3j_3}+a_{j_4j_1j_3j_3}\right)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}(a_{j_4j_4j_1j_2}+a_{j_4j_4j_2j_1}+a_{j_4j_1j_4j_2}+a_{j_4j_2j_4j_1}+a_{j_4j_2j_1j_4}+a_{j_4j_1j_2j_4}+$$

$$+a_{j_1j_4j_4j_2}+a_{j_2j_4j_4j_1}+a_{j_2j_4j_1j_4}+a_{j_1j_4j_2j_4}+a_{j_1j_2j_4j_4}+a_{j_2j_1j_4j_4})+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}(a_{j_3j_3j_3j_4}+a_{j_3j_3j_4j_3}+a_{j_3j_4j_3j_3}+a_{j_4j_3j_3j_3})+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}(a_{j_2j_2j_4j_4}+a_{j_2j_4j_2j_4}+a_{j_2j_4j_4j_2}+a_{j_4j_2j_2j_4}+a_{j_4j_2j_4j_2}+a_{j_4j_4j_2j_2})+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}(a_{j_1j_4j_4j_4}+a_{j_4j_1j_4j_4}+a_{j_4j_4j_1j_4}+a_{j_4j_4j_4j_1})+$$

$$+\sum_{j_4=0}^{N-1}a_{j_4j_4j_4j_4}. \tag{82}$$

Perhaps, the formula (79) for any k $(k\in\mathbb{N})$ was found by the author for the first time [1] (1997). Assume that

$$a_{(j_1,\ldots,j_k)}=\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)},$$

where $\Phi\left(t_1,\ldots,t_k\right)$ is a nonrandom function of k variables. Then from (31) and (79) we have

$$J[\Phi]_{T,t}^{(k)}=\sum_{r=0}^{[k/2]}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\times$$

$$\times\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)}\times$$

$$\times\Biggl[\Phi\biggl(\tau_{j_1},\ldots,\tau_{j_{s_1-1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+2}},\ldots,\tau_{j_{s_r-1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+2}},\ldots,\tau_{j_k}\biggr)\times$$

$$\times\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_{s_1-1}}}^{(i_{s_1-1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1+1})}\Delta\mathbf{w}_{\tau_{j_{s_1+2}}}^{(i_{s_1+2})}\ldots$$

$$\ldots\Delta\mathbf{w}_{\tau_{j_{s_r-1}}}^{(i_{s_r-1})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r+1})}\Delta\mathbf{w}_{\tau_{j_{s_r+2}}}^{(i_{s_r+2})}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_k)}\Biggr]=$$

$$= \sum_{r=0}^{[k/2]} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r} \quad \text{w. p. 1,} \tag{83}$$

where

$$I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r} = \int\limits_t^T \ldots \int\limits_t^{t_{s_r+3}} \int\limits_t^{t_{s_r+2}} \int\limits_t^{t_{s_r}} \ldots \int\limits_t^{t_{s_1+3}} \int\limits_t^{t_{s_1+2}} \int\limits_t^{t_{s_1}} \ldots \int\limits_t^{t_2} \sum_{\prod\limits_{l=1}^{r} \mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)} \times$$

$$\times \Biggl[\Phi\Bigl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\Bigr) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1})} d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1+1})} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})} \ldots$$

$$\ldots d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})} d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r})} d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r+1})} d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})} \ldots d\mathbf{w}_{t_k}^{(i_k)} \Biggr], \tag{84}$$

where $k \ge 2$, the set $\mathrm{A}_{k,r}$ is defined by (16). We suppose that the right-hand side of (84) exists as the Ito stochastic integral.

Remark 2. *The summands on the right-hand side of* (84) *should be understood as follows: for each permutation from the set*

$$\prod_{l=1}^{r} \mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k) = (t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k)$$

it is necessary to perform replacement on the right-hand side of (84) *of all pairs* (*their number is equals to* r) *of differentials* $d\mathbf{w}_{t_p}^{(i)} d\mathbf{w}_{t_p}^{(j)}$ *with similar lower indexes by the values* $\mathbf{1}_{\{i=j\ne 0\}} dt_p$.

Note that the term in (83) for $r = 0$ should be understood as follows

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \Bigl(\Phi\left(t_1,\ldots,t_k\right) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \Bigr), \tag{85}$$

where

$$\sum_{(t_1,\ldots,t_k)}$$

means the sum with respect to all possible permutations $(t_1,\ldots,t_k)$. At the same time permutations $(t_1,\ldots,t_k)$ when summing are performed in (85) only in the expression, which is enclosed in parentheses (see [18], Sect. 1.1.3 for details).

Using Lemma 4, we get

$$\mathsf{M}\left\{\left|J[\Phi]_{T,t}^{(k)}\right|^{2n}\right\} \le$$

(86) $$\le C_{nk} \sum_{r=0}^{[k/2]} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} \mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\},$$

where

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\} \le$$

$$\le C_{nk}^{s_1\ldots s_r} \int\limits_t^T \ldots \int\limits_t^{t_{s_r+3}} \int\limits_t^{t_{s_r+2}} \int\limits_t^{t_{s_r}} \ldots \int\limits_t^{t_{s_1+3}} \int\limits_t^{t_{s_1+2}} \int\limits_t^{t_{s_1}} \ldots \int\limits_t^{t_2} \sum_{\prod\limits_{l=1}^{r} \mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)} \times$$

$$\times \Phi^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr)\times$$

(87) $$\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k,$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing in (87) are performed only in the value

$$\Phi^{2n}\biggl(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\biggr).$$

Consider (86), (87) for $\Phi(t_1,\ldots,t_k)=R_{p_1\ldots p_k}(t_1,\ldots,t_k)$

$$\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} \le$$

(88) $$\le C_{nk} \sum_{r=0}^{[k/2]} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} \mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\},$$

$$\mathsf{M}\left\{\left|I[R_{p_1\ldots p_k}]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\} \le$$

$$\leq C_{nk}^{s_1\ldots s_r} \int\limits_t^T \ldots \int\limits_t^{t_{s_r+3}} \int\limits_t^{t_{s_r+2}} \int\limits_t^{t_{s_r}} \ldots \int\limits_t^{t_{s_1+3}} \int\limits_t^{t_{s_1+2}} \int\limits_t^{t_{s_1}} \ldots \int\limits_t^{t_2} \sum_{\prod\limits_{l=1}^{r} \mathbf{I}_{t_{s_l}, t_{s_l+1}}(t_1,\ldots,t_k)} \times$$

$$\times R_{p_1\ldots p_k}^{2n}\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\right)\times$$

$$\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k, \tag{89}$$

where C_{nk} and $C_{nk}^{s_1\ldots s_r}$ are constants and permutations when summing in (89) are performed only in the value

$$R_{p_1\ldots p_k}^{2n}\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},t_{s_1+2},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},t_{s_r+2},\ldots,t_k\right).$$

From the other hand, we can consider the generalization of the formulas (51), (61), (68) for the case of arbitrary k ($k \in \mathbb{N}$). In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{90}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (90) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}}, \tag{91}$$

where $a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}} \in \mathbb{R}$.

Below there are several examples of sums in the form (91)

$$\sum_{\substack{(\{g_1,g_2\}) \\ \{g_1,g_2\}=\{1,2\}}} a_{g_1 g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\}) \\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1 g_2,g_3 g_4} = a_{12,34} + a_{13,24} + a_{23,14}, \tag{92}$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

(93) $$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135}+$$

$$+a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4}+$$

$$+a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2}+$$

$$+a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can generalize the formulas (51), (61), (68) for the case of arbitrary k ($k \in \mathbb{N}$)

$$\mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} \le C_{nk}\left(\int\limits_{[t,T]^k} \left(R_{p_1\ldots p_k}(t_1,\ldots,t_k)\right)^{2n} dt_1\ldots dt_k+\right.$$

$$+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\ldots\mathbf{1}_{\{i_{g_{2r-1}}=i_{g_{2r}}\ne 0\}}\times$$

$$\times\int\limits_{[t,T]^{k-r}}\left(R_{p_1\ldots p_k}\left(t_1,\ldots,t_k\right)\Bigg|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}\right)^{2n}\times$$

(94) $$\left.\times\left(dt_1\ldots dt_k\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}\right),$$

where C_{nk} is a constant,

$$\left(t_1,\ldots,t_k\right)\Bigg|_{t_{g_1}=t_{g_2},\ldots,t_{g_{2r-1}}=t_{g_{2r}}}$$

means the ordered set $(t_1,\ldots,t_k)$ where we put $t_{g_1}=t_{g_2},\ \ldots,\ t_{g_{2r-1}}=t_{g_{2r}}$.

Moreover,

$$\left(dt_1\ldots dt_k\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\ldots,\left(dt_{g_{2r-1}}dt_{g_{2r}}\right)\curvearrowright dt_{g_{2r-1}}}$$

means the product $dt_1\ldots dt_k$ where we replace all pairs $dt_{g_1}dt_{g_2},\ldots,dt_{g_{2r-1}}dt_{g_{2r}}$ by $dt_{g_1},\ldots,dt_{g_{2r-1}}$ correspondingly.

Note that the estimate like (94), where all indicators $\mathbf{1}_{\{\cdot\}}$ must be replaced with 1, can be obtained from the estimates (88), (89). The comparison of (94) with the relation (5.36) in [17] (Theorem 5.2, p. A.273) or with the relation (1.54) in [18] (Theorem 1.2, p. 60) shows a similar structure of these formulas.

Let us consider the particular case of (94) for $k=4$

$$\begin{aligned}
\mathsf{M}\left\{\left|J[R_{p_1p_2p_3p_4}]_{T,t}^{(4)}\right|^{2n}\right\}\le C_{n4}\Biggl(&\int\limits_{[t,T]^4}\left(R_{p_1p_2p_3p_4}(t_1,t_2,t_3,t_4)\right)^{2n}dt_1dt_2dt_3dt_4+\\
&+\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}}\mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\int\limits_{[t,T]^3}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigg|_{t_{g_1}=t_{g_2}}\right)^{2n}\times\\
&\times\left(dt_1dt_2dt_3dt_4\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1}}+\\
&+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}}\mathbf{1}_{\{i_{g_1}=i_{g_2}\ne 0\}}\mathbf{1}_{\{i_{g_3}=i_{g_4}\ne 0\}}\int\limits_{[t,T]^2}\left(R_{p_1p_2p_3p_4}\left(t_1,t_2,t_3,t_4\right)\Bigg|_{t_{g_1}=t_{g_2},t_{g_3}=t_{g_4}}\right)^{2n}\times\\
&\times\left(dt_1dt_2dt_3dt_4\right)\Bigg|_{\left(dt_{g_1}dt_{g_2}\right)\curvearrowright dt_{g_1},\left(dt_{g_3}dt_{g_4}\right)\curvearrowright dt_{g_3}}\Biggr).
\end{aligned}\tag{95}$$

It is not difficult to notice that (95) is consistent with (68) (see (92), (93)).

According to (7), we have the following expression for all internal points of the hypercube $[t,T]^k$

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right)-$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l). \tag{96}$$

Due to (96) the function $R_{p_1\ldots p_k}(t_1,\ldots,t_k)$ is continuous in the open domains of integration of integrals on the right-hand side of (89) and it is bounded at the boundaries of these domains for $p_1,\ldots,p_k<\infty$.

Let us perform the iterated passage to the limit

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}$$

under the integral signs in the estimate (94) (like it was performed for the 2-dimensional, 3-dimentional, and 4-dimensional cases (see above)). Then, taking into account (46), we obtain the required result. More precisely, since the integrals on the right-hand side of (94) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}R_{p_1\ldots p_k}(t_1,\ldots,t_k)=0,\quad \text{when}\quad (t_1,\ldots,t_k)\in(t,T)^k,$$

where the left-hand side is bounded on $[t,T]^k$.

According to the proof of Lemma 1 and (45), we have

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)=$$

$$=\left(K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right)+$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2,\ldots,t_k)-\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right)+$$

$$\ldots$$

$$+\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}}\left(C_{j_{k-1}\ldots j_1}(t_k)-\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)\right)\phi_{j_{k-1}}(t_{k-1})\ldots\phi_{j_1}(t_1)\right), \tag{97}$$

where

$$C_{j_1}(t_2,\ldots,t_k)=\int\limits_t^T K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)dt_1,$$

$$C_{j_2 j_1}(t_3,\ldots,t_k)=\int\limits_{[t,T]^2} K^*(t_1,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2,$$

$$\ldots$$

$$C_{j_{k-1}\ldots j_1}(t_k)=\int\limits_{[t,T]^{k-1}} K^*(t_1,\ldots,t_k)\prod_{l=1}^{k-1}\phi_{j_l}(t_l)dt_1\ldots dt_{k-1}.$$

Then, applying k times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}\ldots\overline{\lim\limits_{p_k\to\infty}}$) the Lebesgue's Dominated Convergence Theorem in the integrals on the right-hand side of (94), we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\ \mathsf{M}\left\{\left|J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\}=0,\quad n\in\mathbb{N}.$$

Let us discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in (94).

It is well known that [62]

$$\left|\sum_{k=1}^{N}\frac{\sin kx}{k}\right|\le C \tag{98}$$

for all N and x, where constant C does not depend on N and x.

Moreover,

$$\sum_{j=1}^{N}\frac{1}{j^2}\le\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}. \tag{99}$$

Applying double integration by parts (as in (2.28), Sect. 2.1.1 [18]), we estimate the partial sums of one-dimensional trigonometric Fourier series

$$\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1),\quad \sum_{j_2=0}^{p_2}C_{j_2 j_1}(t_3,\ldots,t_k)\phi_{j_2}(t_2),\quad\ldots\quad \sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)$$

in (97) using (99) and (see (98))

$$\left|\sum_{k=1}^{N}\frac{1}{k}\sin\frac{2\pi k(x-y)}{T-t}\right|\le C,\quad \left|\sum_{k=1}^{N}\frac{1}{k}\sin\frac{2\pi k(x-t)}{T-t}\right|\le C$$

(here $N\in\mathbb{N}$ and $x,y\in\mathbb{R}$, constant C does not depend on N and x,y) as follows

$$\left|\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right|\le C_1,\quad \left|\sum_{j_1=0}^{p_1}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)\right|\le C_2,\ \ldots\ \left|\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_k}(t_k)\right|\le C_k,$$

where constant C_1 does not depend on p_1, constant C_2 does not depend on p_2, etc.

Moreover,

$$|K^*(t_1,\ldots,t_k)| \le \tilde{C}_1, \quad |C_{j_1}(t_2,\ldots,t_k)| \le \tilde{C}_2, \quad \ldots \quad \left|C_{j_{k-1}\ldots j_1}(t_k)\right| \le \tilde{C}_k,$$

where constant $\tilde{C}_1$ does not depend on p_1, constant $\tilde{C}_2$ does not depend on p_2, etc.

Further, the construction of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in (94) is obvious.

For example, to pass to the limit $\overline{\lim\limits_{p_k\to\infty}}$, the integrable majorant has the form (it is constructed on the base of (97))

$$\left(R_{p_1\ldots p_k}(t_1,\ldots,t_k)\right)^{2n} \le$$

$$\le \Biggl(\left(\tilde{C}_1 + C_1\right) +$$

$$+ \sum_{j_1=0}^{p_1}\left(\tilde{C}_2 + C_2\right)|\phi_{j_1}(t_1)| + \ldots$$

$$\ldots + \sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}}\left(\tilde{C}_k + C_k\right)\left|\phi_{j_{k-1}}(t_{k-1})\ldots\phi_{j_1}(t_1)\right|\Biggr)^{2n} \le$$

$$\le \Biggl(\left(\tilde{C}_1 + C_1\right) +$$

$$+\sqrt{\frac{2}{T-t}}\,(p_1+1)\left(\tilde{C}_2 + C_2\right) + \ldots$$

$$\ldots + \left(\sqrt{\frac{2}{T-t}}\right)^{k-1}(p_1+1)\ldots(p_{k-1}+1)\left(\tilde{C}_k + C_k\right)\Biggr)^{2n}, \tag{100}$$

where $n \in \mathbb{N}$, the numbers $p_1,\ldots,p_{k-1}$ are fixed and the right-hand side of (100) is independent of p_k.

Theorem 1 is proved.

It easy to notice that if we expand the function $K^*(t_1,\ldots,t_k)$ into the generalized Fourier series at the interval (t,T) at first with respect to the variable t_k, after that with respect to the variable t_{k-1}, etc., then we will have the expansion

$$K^*(t_1,\ldots,t_k) = \sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l) \tag{101}$$

instead of the expansion (8).

Let us prove the expansion (101). Similarly with (12) we have the following equality

$$\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}} + \frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right) = \sum_{j_k=0}^{\infty}\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\phi_{j_k}(t_k), \tag{102}$$

which is fulfilled pointwise at the interval (t,T), besides the series on the right-hand side of (102) converges when $t_1=t$ and $t_1=T$.

Let us introduce the assumption of induction

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{103}$$

Then

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\sum_{j_2=0}^{\infty}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2\prod_{l=2}^{k}\phi_{j_l}(t_l)=$$

$$=\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\sum_{j_k=0}^{\infty}\ldots\sum_{j_3=0}^{\infty}\psi_2(t_2)\times$$

$$\times\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_3\prod_{l=3}^{k}\phi_{j_l}(t_l)=$$

$$=\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\prod_{l=2}^{k}\psi_l(t_l)\prod_{l=2}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{104}$$

From the other hand, the left-hand side of (104) can be represented in the following form

$$\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

by expanding the function

$$\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2$$

into the generalized Fourier series at the interval (t,T) using the variable t_1. Here we applied the following replacement of integration order

$$\int\limits_{t}^{T}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\ldots\int\limits_{t_{k-1}}^{T}\psi_k(t_k)\phi_{j_k}(t_k)dt_k\ldots dt_2dt_1=$$

$$=\int\limits_{t}^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_{t}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\ldots dt_k=$$

$$=C_{j_k\ldots j_1}.$$

The expansion (101) is proved. So, we can formulate the following theorem.

Theorem 2 [18] (Sect. 2.4) (also see [1] (1997), [12], [13], [16], [17], [19]-[21]. *Suppose that the conditions of Theorem* 1 *are fulfilled. Then*

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{105}$$

where notations are the same as in Theorem 1.

Note that (105) means the following

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^{2n}\right\}=0,$$

where $n\in\mathbb{N}$.

Let us make a remark about how one can obtain an analogue of Theorem 1 for the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $n=1$ (the case of mean-square convergence), $k=2$.

First note the well known estimate for Legendre polynomials [63]

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{106}$$

where constant K does not depend on y and j.

By analogy with (51) we have

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\}\le$$

$$\leq 2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2 + \mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \tag{107}$$

From Remark 1.6, Sect. 1.7.2 [18] and (1.72), (2.103) [18] we obtain for the case of Legendre polynomials

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2 dt_1dt_2 = 0.$$

Further, we have (see (53))

$$R_{p_1p_2}(t_1,t_1) = \left(K^*(t_1,t_1) - \sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1)\right) +$$

$$+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_1) - \sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_2}(t_1)\right)\phi_{j_1}(t_1)\right). \tag{108}$$

Then, taking into account (52), (108) and applying two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\overline{\lim\limits_{p_2\to\infty}}$) the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0.$$

Let us discuss the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem in our case.

Using double integration by parts (as in (2.22), Sect. 2.1.1 [18]), we estimate the partial sums of one-dimensional Fourier–Legendre series

$$\sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1), \qquad \sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_2}(t_1)$$

in (108) using (106) and (99) as follows

$$\left|\sum_{j_1=0}^{p_1} C_{j_1}(t_1)\phi_{j_1}(t_1)\right| \leq K_1\left(1 + \frac{1}{\left(1-(z(t_1))^2\right)^{1/2}} + \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \tag{109}$$

$$\left|\sum_{j_2=0}^{p_2} C_{j_2j_1}\phi_{j_2}(t_1)\right| \leq K_2\left(1 + \frac{1}{\left(1-(z(t_1))^2\right)^{1/4}}\right), \tag{110}$$

where

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t},$$

constant K_1 does not depend on p_1, and constant K_2 does not depend on p_2.

Thus, integrable majorants in our case can be easily constracted using (108), (109) and (110) (see the proof of Theorem 1 for details).

An analogue of Theorem 1 for the case of Legendre polynomials and $n = 1$ (the case of mean-square convergence), $k = 2$ is obtained.

3. Examples. The Case of Legendre Polynomials

In this section, we provide some practical material (based on an analogue of Theorem 1 for the case of Legendre polynomials and $k = 2$, $n = 1$) on expansions of iterated Stratonovich stochastic integrals of the following form [18]-[21]

$$I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)T,t} = \int\limits_t^{*T} (t-t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}^{(i_1)}_{t_1} \ldots d\mathbf{f}^{(i_k)}_{t_k}, \tag{111}$$

where $i_1, \ldots, i_k = 1, \ldots, m,\ \ l_1, \ldots, l_k = 0, 1, \ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \tag{112}$$

where $P_j(x)$ is the Legendre polynomial.

Using an analogue of Theorem 1 for the system of functions (112) and $k = 2$, $n = 1$, we obtain the following expansions of iterated Stratonovich stochastic integrals [1]–[21], [24], [25], [27], [29]-[40]

$$I^{*(i_1)}_{(0)T,t} = \sqrt{T-t}\zeta^{(i_1)}_0,$$

$$I^{*(i_1)}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2}\left(\zeta^{(i_1)}_0 + \frac{1}{\sqrt{3}}\zeta^{(i_1)}_1\right), \tag{113}$$

$$I^{*(i_1)}_{(2)T,t} = \frac{(T-t)^{5/2}}{3}\left(\zeta^{(i_1)}_0 + \frac{\sqrt{3}}{2}\zeta^{(i_1)}_1 + \frac{1}{2\sqrt{5}}\zeta^{(i_1)}_2\right), \tag{114}$$

$$I^{*(i_1 i_2)}_{(00)T,t} = \frac{T-t}{2}\left(\zeta^{(i_1)}_0\zeta^{(i_2)}_0 + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta^{(i_1)}_{i-1}\zeta^{(i_2)}_i - \zeta^{(i_1)}_i\zeta^{(i_2)}_{i-1}\right)\right), \tag{115}$$

$$I^{*(i_1 i_2)}_{(01)T,t} = -\frac{T-t}{2} I^{*(i_1 i_2)}_{(00)T,t} - \frac{(T-t)^2}{4}\left(\frac{\zeta^{(i_1)}_0\zeta^{(i_2)}_1}{\sqrt{3}} + \right.$$

$$\left. + \sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta^{(i_1)}_i\zeta^{(i_2)}_{i+2} - (i+1)\zeta^{(i_1)}_{i+2}\zeta^{(i_2)}_i}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta^{(i_1)}_i\zeta^{(i_2)}_i}{(2i-1)(2i+3)}\right)\right),$$

$$I_{(10)T,t}^{*(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_2)}\zeta_1^{(i_1)}}{\sqrt{3}}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),$$

$$I_{(02)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-(T-t)I_{(01)T,t}^{*(i_1 i_2)}+\frac{(T-t)^3}{8}\left(\frac{2\zeta_2^{(i_2)}\zeta_0^{(i_1)}}{3\sqrt{5}}+\right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.\left.+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(20)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-(T-t)I_{(10)T,t}^{*(i_1 i_2)}+\frac{(T-t)^3}{8}\left(\frac{2\zeta_0^{(i_2)}\zeta_2^{(i_1)}}{3\sqrt{5}}+\right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.\left.+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(11)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1 i_2)}+I_{(01)T,t}^{*(i_1 i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\left(\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.\right.$$

$$\left.\left.+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(3)T,t}^{*(i_1)}=-\frac{(T-t)^{7/2}}{4}\left(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\right),$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)} \tag{116}$$

are independent standard Gaussian random variables for various i or j $(i = 1, \ldots, m)$.

4. Examples. The Case of Trigonometric Functions

Let us consider the Milstein expansions of the integrals $I_{(1)T,t}^{(i_1)}$, $I_{(00)T,t}^{*(i_1 i_2)}$, $I_{(2)T,t}^{*(i_1)}$ (see [43]-[45]) based on the trigonometric Fourier expansion of the Brownian Bridge process (the version of the so-called Karhunen–Loeve expansion)

$$I_{(1)T,t}^{*(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{117}$$

$$I_{(2)T,t}^{*(i_1)} = (T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)} - \frac{1}{\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{118}$$

$$I_{(00)T,t}^{*(i_1 i_2)} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} +$$

$$+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{119}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Gaussian random variables defined by the relation (116) in which $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonornal system of trigonometric functions in $L_2([t, T])$.

It is obviously that at least (117)–(119) are significantly more complicated in comparison with (113)–(115). Note that (117)–(119) also can be obtained using Theorem 1 [1], [2], [4]-[13], [16]-[40].

5. Further Remarks

In this section, we consider some approaches on the base of Theorem 1 for the case $k = 2$. Moreover, we explain the potential difficulties associated with the use of generalized multiple Fourier series converging almost everywhere in the hypercube $[t, T]^k$ in the proof of Theorem 1.

First, we show how iterated series can be replaced by multiple one in Theorem 1 (the case $k = 2$ and $n = 1$) and in analogue of Theorem 1 for the case of Legendre polynomials (the case $k = 2$ and $n = 1$).

We have

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \lim_{p\to\infty} \overline{\lim_{q\to\infty}} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le$$

$$\le \lim_{p\to\infty} \overline{\lim_{q\to\infty}} \left(2\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{q} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} +\right.$$

$$\left. +2\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}\right) =$$

$$= \lim_{p\to\infty} \overline{\lim_{q\to\infty}} 2\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =$$

$$= \lim_{p\to\infty} \overline{\lim_{q\to\infty}} 2\sum_{j_1=0}^{p}\sum_{j_1'=0}^{p}\sum_{j_2=p+1}^{q}\sum_{j_2'=p+1}^{q} C_{j_2j_1}C_{j_2'j_1'}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_1'}^{(i_1)}\right\}\mathsf{M}\left\{\zeta_{j_2}^{(i_2)}\zeta_{j_2'}^{(i_2)}\right\} =$$

$$= 2\lim_{p\to\infty}\lim_{q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{q} C_{j_2j_1}^2 =$$

$$= 2\lim_{p\to\infty}\lim_{q\to\infty}\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{q} C_{j_2j_1}^2 - \sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}^2\right) = \tag{120}$$

$$= 2\left(\lim_{p,q\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{q} C_{j_2j_1}^2 - \lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p} C_{j_2j_1}^2\right) = \tag{121}$$

$$= \int\limits_{[t,T]^2} K^2(t_1,t_2)dt_1dt_2 - \int\limits_{[t,T]^2} K^2(t_1,t_2)dt_1dt_2 = 0, \tag{122}$$

where the function $K(t_1,t_2)$ is defined by (3) for $k=2$.

Note that the transition from (120) to (121) is based on the theorem on reducing of a limit to iterated one. Moreover, the transition from (121) to (122) is based on the Parseval equality.

Thus, we obtain the following Theorem.

Theorem 3. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$ and $\psi_1(\tau)$ is twice continuously differentiable nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral* (1) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_2j_1}$ has the form

$$C_{j_2j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i=1,\ldots,m)$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Note that Theorem 3 is a modification (for the case $p_1=p_2=p$ of series summation) of Theorem 2.1 [18].

From the other hand, Theorem 1 implies the following

$$0\le\left|\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\,\mathsf{M}\left\{\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}-J^*[\psi^{(k)}]_{T,t}\right\}\right|\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left|\mathsf{M}\left\{\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}-J^*[\psi^{(k)}]_{T,t}\right\}\right|\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\,\mathsf{M}\left\{\left|J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}\right|\right\}\le$$

$$\le\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}\right)^2\right\}\right)^{1/2}=0. \tag{123}$$

Moreover,

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\left(\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}\right)=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}-\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}. \tag{124}$$

Combining (123) and (124), we obtain

$$\mathsf{M}\left\{J^*[\psi^{(k)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\mathsf{M}\left\{\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right\}. \tag{125}$$

The relation (125) with $k=2$ implies the following

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau=$$

$$=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}, \tag{126}$$

where $\mathbf{1}_A$ is the indicator of the set A.
Since

$$\mathsf{M}\left\{\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right\}=\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}},$$

then from (126) we obtain

$$\mathsf{M}\left\{J^*[\psi^{(2)}]_{T,t}\right\}=\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}=$$

$$=\mathbf{1}_{\{i_1=i_2\neq 0\}}\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}=\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}, \tag{127}$$

where $C_{j_1j_1}$ is defined by (5) for $k=2$ and $j_1=j_2$, i.e.

$$C_{j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$

From (126) and (127) we obtain the following relation

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau. \tag{128}$$

Note that the equality (128) and existence of the limit on the left-hand side of (128) are proved in [18] (Sect. 2.1.2, 2.1.4), [23] for the polynomial and trigonometric cases ($\psi_1(\tau), \psi_2(\tau)$ are continuously differentiable functions on $[t, T]$) as well as for an arbitrary complete orthonormal system of functions in $L_2([t, T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$.

Let us address now to the following theorem on expansion of iterated Ito stochastic integrals (2).

Theorem 4 [4] (2006), [5]-[40]. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{129}$$

where

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g\ne r);\ g,r = 1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1, \ldots, i_k = 0, 1, \ldots, m$,*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{130}$$

are independent standard Gaussian random variables for various i or j (if $i \ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (5), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is the partition of the interval $[t, T]$, which satisfies the condition (26).

Consider transformed particular cases for $k = 1, \ldots, 5$ of Theorem 4 [4] (2006), [5]-[40]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{131}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{132}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{133}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{134}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{135}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Note that in [18], [22], [80] Theorem 4 is generalized to the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ (see Theorem 11 below).

From (132) for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$, $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ and (128) we obtain

$$
J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) =
$$

$$
= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty} C_{j_1j_1} =
$$

$$
= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau. \tag{136}
$$

Since

$$
J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T \psi_1(\tau)\psi_2(\tau)d\tau \quad \text{w. p. 1}, \tag{137}
$$

where $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$ (this condition is related to the definition of the Stratonovich stochastic integral that we use [43] (also see [18] (Sect. 2.1.1))), then from (136) we finally get the following expansion

$$
J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}.
$$

Thus, we obtain the following theorem.

Theorem 5 [18] (Sect. 2.1.4). *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau),\psi_2(\tau)$ are continuous nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral (1) of multiplicity* 2

$$
J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1,i_2=0,1,\ldots,m)
$$

the following expansion

$$
J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}
$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_2j_1}$ has the form

$$
C_{j_2j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2
$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i = 1, \ldots, m)$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Note that analogues of Theorem 5 for the multiplicities 3 to 6 of the iterated Stratonovich stochastic integrals (1) and the systems of Legendre polynomials and trigonometric functions have been formulated and proved in [18], [23], [26], [78], [79] (see Theorems 12–15 below).

We have

$$J^*[\psi^{(2)}]_{T,t}^{p_1,p_2} \stackrel{\text{def}}{=} J[\psi^{(2)}]_{T,t}^{p_1,p_2} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \psi_1(s)\psi_2(s) ds =$$

$$= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}} \int\limits_t^T \psi_1(s)\psi_2(s) ds =$$

$$= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T \psi_1(s)\psi_2(s) ds - \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1 j_1}\right), \qquad (138)$$

where

$$J[\psi^{(2)}]_{T,t}^{p_1,p_2} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right)$$

is the approximation of iterated Ito stochastic integral (2) ($k = 2$) based on Theorem 4 (see (132)).

Moreover, from (137) and (138) we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - J^*[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\} = \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\} \ \to\ 0 \qquad (139)$$

if $p_1, p_2 \to \infty$ $(n \in \mathbb{N})$, where the relation

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\} \ \to\ 0$$

if $p_1, p_2 \to \infty$ $(n \in \mathbb{N})$ is proved in [18] (see Sect. 1.1.9, 1.12).

Further (see (138)),

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^{2n}\right\} =$$

$$= \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - J^*[\psi^{(2)}]_{T,t}^{p_1,p_2} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T \psi_1(s)\psi_2(s) ds - \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_1 j_1}\right)\right)^{2n}\right\} \le$$

(140)

$$\leq K_n\left(\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-J^*[\psi^{(2)}]_{T,t}^{p_1,p_2}\right)^{2n}\right\}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\left(\frac{1}{2}\int\limits_t^T\psi_1(s)\psi_2(s)ds-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}\right)^{2n}\right),$$

where constant $K_n<\infty$ depends on n.

Taking into account (128), (139), and (140), we get

$$\lim_{p_1,p_2\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^{2n}\right\}=0. \tag{141}$$

Thus, we obtain the following theorem.

Theorem 6 [18] (Sect. 2.4.2). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau),\psi_2(\tau)$ are continuous nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral* (1) *of multiplicity* 2

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean of degree $2n$, $n\in\mathbb{N}$ (see (141)) *is valid, where the Fourier coefficient $C_{j_2j_1}$ is defined by* (5) *and*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i=1,\ldots,m)$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Let us consider some other approaches close to the approaches outlined in this section. Now we turn to multiple trigonometric Fourier series converging almost everywhere. Let us formulate the well-known result from the theory of multiple trigonometric Fourier series.

Theorem 7 [64]. *Suppose that*

$$\int\limits_{[0,2\pi]^k}|f(x_1,\ldots,x_k)|\left(\log^+|f(x_1,\ldots,x_k)|\right)^k\log^+\log^+|f(x_1,\ldots,x_k)|dx_1\ldots dx_k<\infty. \tag{142}$$

Then, for the square partial sums

$$\sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(x_l)$$

of the multiple trigonometric Fourier series we have

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(x_l) = f(x_1,\ldots,x_k)$$

almost everywhere in $[0,2\pi]^k$, *where* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of trigonometric functions in the space* $L_2([0,2\pi])$,

$$C_{j_k \ldots j_1} = \int\limits_{[0,2\pi]^k} f(x_1,\ldots,x_k) \prod_{l=1}^{k} \phi_{j_l}(x_l) dx_1 \ldots dx_k$$

is the Fourier coefficient of the function $f(x_1,\ldots,x_k)$, *and* $\log^+ x = \log\max\{1,\ x\}$.

Obviously, Theorem 7 can be reformulated for the hypercube $[t,T]^k$ instead of the hypercube $[0,2\pi]^k$.

If we tried to apply Theorem 7 in the proof of Theorem 1, then we would encounter the following difficulties. Note that the right-hand side of (94) contains multiple integrals over hypercubes of various dimensions, namely over hypercubes $[t,T]^k$, $[t,T]^{k-1}$, etc. Obviously, the convergence almost everywhere in $[t,T]^k$ does not mean the convergence almost everywhere in $[t,T]^{k-1}$, $[t,T]^{k-2}$, etc. This means that we could not apply the Lebesgue's Dominated Convergence Theorem in the proof of Lemma 6 and thus could not complete the proof of Theorem 1. Although multiple series are more convenient for approximation than iterated series as in Theorem 1.

Suppose that $\psi_1(\tau), \psi_2(\tau)$ are continuously differentiable functions on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. In [18] (Sect. 2.1.2) it was shown that

$$\lim_{p_1,p_2\to\infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1) = K^*(t_1,t_1), \quad t_1 \in (t,T), \tag{143}$$

where $C_{j_2 j_1}$ is defined by (5) ($k=2$).

This means that we can repeat the proof of Theorem 1 for the case $k=2$ and apply the Lebesgue's Dominated Convergence Theorem in the formula (94), since Theorem 7 and (143) implies the convergence almost everywhere in $[t,T]^2$ and almost everywhere in $[t,T]$ ($t_1 = t_2 \in [t,T]$) of the multiple trigonometric Fourier series

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} C_{j_2 j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad t_1, t_2 \in [t,T]^2 \tag{144}$$

to the function $K^*(t_1,t_2)$ (the question of finding an integrable majorant for Lebesgue's Dominated Convergence Theorem is omitted here). So, we can obtain the particular case of Theorem 6.

Let us consider the another approach. The following fact is well-known.

Proposition 1. *Let* $\left\{x_{n_1,\ldots,n_k}\right\}_{n_1,\ldots,n_k=1}^{\infty}$ *be a multi-index sequence and let there exists the limit*

$$\lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k} < \infty.$$

Moreover, let there exists the limit

$$\lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = y_{n_1,\ldots,n_{k-1}} < \infty \quad \textit{for any} \quad n_1,\ldots,n_{k-1}.$$

Then there exists the iterated limit

$$\lim_{n_1,\ldots,n_{k-1}\to\infty}\ \lim_{n_k\to\infty} x_{n_1,\ldots,n_k},$$

and moreover,

$$\lim_{n_1,\ldots,n_{k-1}\to\infty}\ \lim_{n_k\to\infty} x_{n_1,\ldots,n_k} = \lim_{n_1,\ldots,n_k\to\infty} x_{n_1,\ldots,n_k}.$$

Denote

$$C_{j_s\ldots j_1}(t_{s+1},\ldots,t_k) = \int\limits_{[t,T]^s} K(t_1,\ldots,t_k)\prod_{l=1}^{s}\phi_{j_l}(t_l)dt_1\ldots dt_s \quad (s=1,\ldots,k-1).$$

where $K(t_1,\ldots,t_k)$ has the form (3). For $s=k$ we suppose that $C_{j_k\ldots j_1}$ is defined by (5).

Consider the following Fourier series

$$\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{145}$$

$$\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3), \tag{146}$$

$$\ldots$$

$$\lim_{p_1,\ldots,p_{k-1}\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_{k-1}=0}^{p_{k-1}} C_{j_{k-1}\ldots j_1}(t_k)\phi_{j_1}(t_1)\ldots\phi_{j_{k-1}}(t_{k-1}), \tag{147}$$

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k), \tag{148}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

The author does not know the answere to the question on existence of the limits (145)–(148) even for the case $p_1=\ldots=p_k$ and trigonometric Fourier series. Obviously, at least for the case $k=2$ and $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ the answere to the above question is positive for the Fourier–Legendre series as well as for the trigonometric Fourier series.

If we suppose the existence of the limits (145)–(148), then combining Proposition 1 and the proof of Lemma 1 we obtain

$$K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2,\ldots,t_k)\phi_{j_1}(t_1)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)= \tag{149}$$

$$=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}(t_3,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)=$$

$$=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{150}$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_3j_2j_1}(t_4,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)= \tag{151}$$

$$=\lim_{p_1,p_2,p_3\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{\infty}C_{j_4\ldots j_1}(t_5,\ldots,t_k)\phi_{j_1}(t_1)\phi_{j_3}(t_4)= \tag{152}$$

$$=\ldots=$$

$$=\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k). \tag{153}$$

Note that the transition from (150) to (151) is based on (149) and the proof of Lemma 1. The transition from (151) to (152) is based on (150) and the proof of Lemma 1.

Using (153) we could get the version of Theorem 1 with multiple series instead of iterated ones.

6. Refinement of Theorems 1 and 2 for Iterated Stratonovich Stochastic Integrals of Multiplicities 2 and 3 ($i_1,i_2,i_3=1,\ldots,m$). The Case of Mean-Square Convergence

In this section, it will be shown that the upper limits in Theorems 1 and 2 (the cases $k=2$, $k=3$ and $n=1$) can be replaced by the usual limits.

Theorem 8 [18] (Sect. 2.4). *Suppose that every $\psi_l(\tau)$ $(l=1,2,3)$ is twice continuously differentiable function at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, the iterated Stratonovich stochastic integrals $J^*[\psi^{(2)}]_{T,t}$ and*

$J^*[\psi^{(3)}]_{T,t}$ $(i_1, i_2, i_3 = 1, \ldots, m)$ *defined by* (1) *are expanded into the converging in the mean-square sense iterated series*

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} = 0, \tag{154}$$

$$\lim_{p_2\to\infty}\lim_{p_1\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_2=0}^{p_2}\sum_{j_1=0}^{p_1} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} = 0, \tag{155}$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} = 0, \tag{156}$$

$$\lim_{p_3\to\infty}\lim_{p_2\to\infty}\lim_{p_1\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_3=0}^{p_3}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} = 0, \tag{157}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)} \quad (i = 1, \ldots, m, \quad j = 0, 1, \ldots)$$

are independent standard Gaussian random variables for various i or j and $C_{j_2j_1}$, $C_{j_3j_2j_1}$ are defined by (5).

Proof. We will prove the equalities (154) and (156) (the equalities (155) and (157) can be proved similarly using the expansion (101) instead of the expansion (8)).

From (47) we have w. p. 1

$$J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} = J[R_{p_1p_2}]_{T,t}^{(2)} =$$

$$= \int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1, \tag{158}$$

where we used the same notations as in (47).

Uning (158), we obtain

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} = \int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}^2(t_1,t_2)dt_1dt_2 + \int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}^2(t_1,t_2)dt_2dt_1 +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(2\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\right)=$$

$$=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+\int\limits_t^T\int\limits_{t_2}^T R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T\int\limits_{t_1}^T R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_2dt_1\right)+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2=$$

$$=\int\limits_{[t,T]^2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T\int\limits_{t_2}^T R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2\right)+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2=$$

$$=\int\limits_{[t,T]^2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\left(\int\limits_{[t,T]^2}R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2+\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\right). \tag{159}$$

Since the integrals on the right-hand side of (159) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}R_{p_1p_2}(t_1,t_2)=0\quad\text{when}\quad(t_1,t_2)\in(t,T)^2,$$

where the left-hand side is bounded on $[t,T]^2$ (see (46)).

Then, applying two times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem and taking into account (9), (10), and (53), we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_{[t,T]^2}R_{p_1p_2}^2(t_1,t_2)dt_1dt_2=0, \tag{160}$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_{[t,T]^2} R_{p_1p_2}(t_1,t_2)R_{p_1p_2}(t_2,t_1)dt_1dt_2=0, \tag{161}$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=0. \tag{162}$$

The relations (159)–(162) imply the following equality

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2}]^{(2)}_{T,t}\right)^2\right\}=0.$$

The relation (154) is proved.

Let us prove the relation (156). Using (55) and the integration order replacement technique for iterated Ito stochastic integrals (see Chapter 3 in [18]-[21]), we get w. p. 1

$$J^*[\psi^{(3)}]_{T,t}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}=J[R_{p_1p_2p_3}]^{(3)}_{T,t}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{f}^{(i_1)}_{t_1}d\mathbf{f}^{(i_2)}_{t_2}d\mathbf{f}^{(i_3)}_{t_3}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{f}^{(i_1)}_{t_1}d\mathbf{f}^{(i_3)}_{t_2}d\mathbf{f}^{(i_2)}_{t_3}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{f}^{(i_2)}_{t_1}d\mathbf{f}^{(i_1)}_{t_2}d\mathbf{f}^{(i_3)}_{t_3}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{f}^{(i_3)}_{t_1}d\mathbf{f}^{(i_1)}_{t_2}d\mathbf{f}^{(i_2)}_{t_3}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{f}^{(i_3)}_{t_1}d\mathbf{f}^{(i_2)}_{t_2}d\mathbf{f}^{(i_1)}_{t_3}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{f}^{(i_2)}_{t_1}d\mathbf{f}^{(i_3)}_{t_2}d\mathbf{f}^{(i_1)}_{t_3}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_2,t_2,t_3)dt_2\right)d\mathbf{f}^{(i_3)}_{t_3}+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_1,t_2,t_2)dt_2\right)d\mathbf{f}^{(i_1)}_{t_1}+$$

$$+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T\left(\int\limits_t^T R_{p_1p_2p_3}(t_3,t_2,t_3)dt_3\right)d\mathbf{f}_{t_2}^{(i_2)}. \tag{163}$$

Let us calculate the second moment of $J[R_{p_1p_2p_3}]_{T,t}^{(3)}$ using (163). We have

$$\mathsf{M}\left\{\left(J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right)^2\right\}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\left(\sum_{(t_1,t_2,t_3)}R_{p_1p_2p_3}^2(t_1,t_2,t_3)\right)dt_1dt_2dt_3+ \tag{164}$$

$$+2\left(\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(1)}(t_1,t_2,t_3)dt_1dt_2dt_3+\right.$$

$$+\mathbf{1}_{\{i_1=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(2)}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(3)}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$\left.+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}^{(4)}(t_1,t_2,t_3)dt_1dt_2dt_3\right)+$$

$$+\int\limits_{[t,T]^3}\left(\mathbf{1}_{\{i_1=i_2\}}R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_2,t_3)+\right.$$

$$+\mathbf{1}_{\{i_2=i_3\}}R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_3,t_2,t_2)+$$

$$+\mathbf{1}_{\{i_1=i_3\}}R_{p_1p_2p_3}(t_1,t_3,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)+$$

$$+2\cdot\mathbf{1}_{\{i_1=i_2=i_3\}}\Biggl(R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_2)+$$

$$+R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_2)+$$

$$\left.\left.+R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)\right)\right)dt_1dt_2dt_3, \tag{165}$$

where permutation (t_1,t_2,t_3) when summing in (164) are performed only in the value $R_{p_1p_2p_3}^2(t_1,t_2,t_3)$ and the functions $G_{p_1p_2p_3}^{(i)}(t_1,t_2,t_3)$ $(i=1,\ldots,4)$ are defined by the following relations

$$G_{p_1p_2p_3}^{(1)}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_2,t_1,t_3)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_3,t_1,t_2)+$$

$$+R_{p_1p_2p_3}(t_2,t_3,t_1)R_{p_1p_2p_3}(t_3,t_2,t_1),$$

$$G_{p_1p_2p_3}^{(2)}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_3,t_1,t_2),$$

$$G_{p_1p_2p_3}^{(3)}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_1,t_3,t_2)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_3,t_2,t_1)R_{p_1p_2p_3}(t_3,t_1,t_2),$$

$$G_{p_1p_2p_3}^{(4)}(t_1,t_2,t_3)=R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_2,t_3,t_1)+$$

$$+R_{p_1p_2p_3}(t_1,t_2,t_3)R_{p_1p_2p_3}(t_3,t_1,t_2)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_2,t_1,t_3)+$$

$$+R_{p_1p_2p_3}(t_1,t_3,t_2)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_1)+$$

$$+R_{p_1p_2p_3}(t_2,t_3,t_1)R_{p_1p_2p_3}(t_3,t_1,t_2).$$

Further,

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\left(\sum_{(t_1,t_2,t_3)}R_{p_1p_2p_3}^2(t_1,t_2,t_3)\right)dt_1dt_2dt_3=$$

(166)
$$=\int\limits_{[t,T]^3}R_{p_1p_2p_3}^2(t_1,t_2,t_3)dt_1dt_2dt_3.$$

We will say that the function $\Phi(t_1,t_2,t_3)$ is symmetric if

$$\Phi(t_1,t_2,t_3)=\Phi(t_1,t_3,t_2)=\Phi(t_2,t_1,t_3)=\Phi(t_2,t_3,t_1)=$$

$$=\Phi(t_3,t_1,t_2)=\Phi(t_3,t_2,t_1).$$

For the symmetric function $\Phi(t_1,t_2,t_3)$, we have

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\left(\sum_{(t_1,t_2,t_3)}\Phi(t_1,t_2,t_3)\right)dt_1dt_2dt_3=$$

$$=6\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)dt_1dt_2dt_3=$$

$$=\int\limits_{[t,T]^3}\Phi(t_1,t_2,t_3)dt_1dt_2dt_3. \tag{167}$$

The relation (167) implies that

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)dt_1dt_2dt_3=\frac{1}{6}\int\limits_{[t,T]^3}\Phi(t_1,t_2,t_3)dt_1dt_2dt_3. \tag{168}$$

It is easy to check that the functions $G^{(i)}_{p_1p_2p_3}(t_1,t_2,t_3)$ $(i=1,\ldots,4)$ are symmetric. Using this property as well as (165), (166), and (168), we obtain

$$\mathsf{M}\left\{\left(J[R_{p_1p_2p_3}]^{(3)}_{T,t}\right)^2\right\}=\int\limits_{[t,T]^3}R^2_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$+\frac{1}{3}\int\limits_{[t,T]^3}\Biggl(\mathbf{1}_{\{i_1=i_2\}}G^{(1)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$+\mathbf{1}_{\{i_1=i_3\}}G^{(2)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$+\mathbf{1}_{\{i_2=i_3\}}G^{(3)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}G^{(4)}_{p_1p_2p_3}(t_1,t_2,t_3)dt_1dt_2dt_3\Biggr)dt_1dt_2dt_3+$$

$$+\int\limits_{[t,T]^3}\Biggl(\mathbf{1}_{\{i_1=i_2\}}R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_2,t_3)+$$

$$+\mathbf{1}_{\{i_2=i_3\}}R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_3,t_2,t_2)+$$

$$+\mathbf{1}_{\{i_1=i_3\}}R_{p_1p_2p_3}(t_1,t_3,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)+$$

$$+2\cdot\mathbf{1}_{\{i_1=i_2=i_3\}}\Biggl(R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_3,t_2,t_2)+$$

$$+R_{p_1p_2p_3}(t_1,t_1,t_3)R_{p_1p_2p_3}(t_2,t_3,t_2)+$$

$$+R_{p_1p_2p_3}(t_3,t_1,t_1)R_{p_1p_2p_3}(t_2,t_3,t_2)\Biggr)\Biggr)dt_1dt_2dt_3. \tag{169}$$

Since the integrals on the right-hand side of (169) exist as Riemann integrals, then they are equal to the corresponding Lebesgue integrals. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}R_{p_1p_2p_3}(t_1,t_2,t_3)=0\quad\text{when}\quad (t_1,t_2,t_3)\in(t,T)^3,$$

where the left-hand side is bounded on $[t,T]^3$ (see (46)).

Using (62) and applying three times (we mean an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}\lim\limits_{p_3\to\infty}$) the Lebesgue's Dominated Convergence Theorem in the equality (169), we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\lim_{p_3\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2p_3}]_{T,t}^{(3)}\right)^2\right\}=0.$$

The relation (156) is proved. Theorem 8 is proved.

Developing the approach used in the proof of Theorem 8, we can in principle prove the following formulas

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}=0,$$

$$\lim_{p_k\to\infty}\ldots\lim_{p_1\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}=0,$$

which are correct under the conditions of Theorem 1.

7. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity k. The Case $i_1=\ldots=i_k\ne 0$ and Different Weight Functions $\psi_1(\tau),\ldots,\psi_k(\tau)$

In this section, we generalize the approach considered in [34] (also see [18], Sect. 2.1.2) to the case $i_1=\ldots=i_k\ne 0$ and different weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ $(k>2)$.

Let us formulate the following theorem.

Theorem 9 [18] (Sect. 2.22). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau),\ldots,\psi_k(\tau)$ $(k\ge 2)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_1)} \quad (i_1 = 1, \ldots, m)$$

the following equality

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_1)}\right)^{2n}\right\} = 0$$

is valid, where $n \in \mathbb{N}$,

$$C_{j_k\ldots j_1} = \int\limits_t^{T} \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient and

$$\zeta_j^{(i_1)} = \int\limits_t^{T} \phi_j(\tau) d\mathbf{f}_\tau^{(i_1)} \quad (i_1 = 1, \ldots, m)$$

are independent standard Gaussian random variables for various j.

Proof. The case $k = 2$ is proved in Theorem 6. Consider the case $k > 2$. First, consider the case $k = 3$ in detail. Define the auxiliary function

$$K'(t_1, t_2, t_3) = \frac{1}{6}\begin{cases} \psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1 \le t_2 \le t_3 \\ \psi_1(t_1)\psi_2(t_3)\psi_3(t_2), & t_1 \le t_3 \le t_2 \\ \psi_1(t_2)\psi_2(t_1)\psi_3(t_3), & t_2 \le t_1 \le t_3 \\ \psi_1(t_2)\psi_2(t_3)\psi_3(t_1), & t_2 \le t_3 \le t_1 \\ \psi_1(t_3)\psi_2(t_2)\psi_3(t_1), & t_3 \le t_2 \le t_1 \\ \psi_1(t_3)\psi_2(t_1)\psi_3(t_2), & t_3 \le t_1 \le t_2 \end{cases}, \quad t_1, t_2, t_3 \in [t, T].$$

Using Lemma 3, Remark 1, and (17), we obtain w. p. 1

$$J[K']_{T,t}^{(3)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} K'(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}) \Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_3=0}^{N-1} \sum_{l_2=0}^{l_3-1} \sum_{l_1=0}^{l_2-1} K'(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}) \Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)} + \right.$$

$$+ \sum_{l_3=0}^{N-1} \sum_{l_1=0}^{l_3-1} \sum_{l_2=0}^{l_1-1} K'(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}) \Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)} \Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)} +$$

$$+\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}\sum_{l_3=0}^{l_1-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+$$

$$+\sum_{l_2=0}^{N-1}\sum_{l_3=0}^{l_2-1}\sum_{l_1=0}^{l_3-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+$$

$$+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\sum_{l_3=0}^{l_2-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+$$

$$+\sum_{l_1=0}^{N-1}\sum_{l_3=0}^{l_1-1}\sum_{l_2=0}^{l_3-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+$$

$$+\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_1})\left(\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}K'(\tau_{l_1},\tau_{l_3},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}+$$

$$+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}K'(\tau_{l_1},\tau_{l_2},\tau_{l_2})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}+\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}K'(\tau_{l_3},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}+$$

$$\left.+\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}K'(\tau_{l_2},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}+\sum_{l_2=0}^{N-1}\sum_{l_3=0}^{l_2-1}K'(\tau_{l_2},\tau_{l_2},\tau_{l_3})\left(\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\right)^2\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\right)=$$

$$=\frac{1}{6}\left(\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T\psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_1)\int\limits_t^{t_1}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}+\right.$$

$$+\int\limits_t^T\psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}+\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\int\limits_t^{t_1}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\int\limits_t^T\psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}+\int\limits_t^T\psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T\psi_3(t_2)\int\limits_t^{t_2}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_2}^{(i_1)}+\int\limits_t^T\psi_3(t_1)\int\limits_t^{t_1}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3+$$

$$+\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_2)d\mathbf{f}_{t_2}^{(i_1)}dt_3+\int\limits_t^T \psi_3(t_2)\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_3)d\mathbf{f}_{t_3}^{(i_1)}dt_2\Biggr)=$$

$$=\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}+$$

$$+\frac{1}{2}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_3}^{(i_1)}+\frac{1}{2}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3=$$

$$=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_1)}\stackrel{\sf def}{=}J^*[\psi^{(3)}]_{T,t}, \tag{170}$$

where the multiple stochastic integral $J[K']_{T,t}^{(3)}$ is defined by (31) and $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (26).

For each $\delta>0$ let us call the exact upper edge of difference $|f(\mathbf{t}')-f(\mathbf{t}'')|$ in the set of all points $\mathbf{t}'$, $\mathbf{t}''$ which belong to the domain D as the module of continuity of the function $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ in the k-dimentional domain D $(k\ge 1)$ if the distance between $\mathbf{t}',\mathbf{t}''$ satisfies the condition $\rho(\mathbf{t}',\mathbf{t}'')<\delta$.

We will say that the function of k $(k\ge 1)$ variables $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ belongs to the Hölder class with the parameter $\alpha\in(0,1]$ $(f(\mathbf{t})\in C^{\alpha}(D))$ in the domain D if the module of continuity of the function $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ in the domain D have the orders $o(\delta^{\alpha})$ $(\alpha\in(0,1))$ and $O(\delta)$ $(\alpha=1)$.

In 1967, Zhizhiashvili L.V. proved that the rectangular sums of multiple trigonometric Fourier series of the function of k variables in the hypercube $[t,T]^k$ converge uniformly to this function in the hypercube $[t,T]^k$ if the function belongs to $C^{\alpha}([t,T]^k)$, $\alpha>0$ (definition of the Hölder class with any parameter $\alpha>0$ can be found in the well known mathematical analysis tutorials [76]).

More precisely, the following statement is correct.

Theorem 10 [76]. *If the function $f(x_1,\ldots,x_n)$ is periodic with period 2π with respect to each variable and belongs in $\mathbb{R}^n$ to the Hölder class $C^{\alpha}(\mathbb{R}^n)$ for any $\alpha>0$, then the rectangular partial sums of multiple trigonometric Fourier series of the function $f(x_1,\ldots,x_n)$ converge to this function uniformly in $\mathbb{R}^n$.*

In [34] (also see [18], Sect. 2.1.2) it was shown that the following function

$$K'(t_1,t_2)=\begin{cases}\psi_1(t_1)\psi_2(t_2), & t_1\le t_2\\ \\ \psi_1(t_2)\psi_2(t_1), & t_2\le t_1\end{cases},\qquad t_1,t_2\in[t,T]$$

belongs to the class $C^1([t,T]^2)$. Moreover, the following Fourier–Legendre expansion

$$K'(t_1,t_2)=\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$= \lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\left(C_{j_2j_1}+C_{j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2) \tag{171}$$

is valid for $(t_1,t_2)\in(t,T)^2$.

Using Theorem 10 for $n=3$ and generalizing the Fourier–Legendre expansion (171) for the function $K'(t_1,t_2,t_3)$, we obtain

$$K'(t_1,t_2,t_3)=\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3), \tag{172}$$

where the multiple Fourier series (172) converges to the function $K'(t_1,t_2,t_3)$ in $(t,T)^3$ and the partial sums of the series (172) have an integrable majorant on $[t,T]^3$ that does not depend on p. For the trigonomertic case, the above statement follows from Theorem 10 (the proof that the function $K'(t_1,t_2,t_3)$ belongs to the Hölder class with parameter 1 in $[t,T]^3$ is omitted and can be carried out in the same way as for the function $K'(t_1,t_2)$ in the two-dimensional case [34] (also see [18], Sect. 2.1.2)). The proof of generalization of the Fourier–Legendre expansion (171) to the three-dimensional case (see (172)) is omitted. The proof that the partial sums of the series (172) have an integrable majorant on $[t,T]^3$ is also omitted.

Denote

$$R'_{ppp}(t_1,t_2,t_3)=K'(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3).$$

Using Lemma 5 and (170), we get w. p. 1

$$J^*[\psi^{(3)}]_{T,t}=J[K']_{T,t}^{(3)}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\frac{1}{6}\Biggl(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+$$

$$+C_{j_2j_3j_1}+C_{j_1j_2j_3}+C_{j_1j_3j_2}\Biggr)\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}+J[R'_{ppp}]_{T,t}^{(3)}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}+J[R'_{ppp}]_{T,t}^{(3)}.$$

Then

$$\mathsf{M}\left\{\left(J[R'_{ppp}]_{T,t}^{(3)}\right)^{2n}\right\}=\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}\right)^{2n}\right\},$$

where $n\in\mathbb{N}$.

Applying (we mean here the passage to the limit $\lim\limits_{p\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (94) for $k = 3$ and $R'_{ppp}(t_1, t_2, t_3)$ instead of $R_{p_1p_2p_3}(t_1, t_2, t_3)$, we obtain

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J[R'_{ppp}]^{(3)}_{T,t}\right)^{2n}\right\} = 0.$$

Theorems 9 is proved for the case $k = 3$.

To prove Theorem 9 for the case $k > 3$, consider the auxiliary function

$$K'(t_1,\ldots,t_k) = \frac{1}{k!}\begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 \le \ldots \le t_k \\ \ldots & \\ \psi_1(t_{g_1})\ldots\psi_k(t_{g_k}), & t_{g_1} \le \ldots \le t_{g_k} \\ \ldots & \\ \psi_1(t_k)\ldots\psi_k(t_1), & t_k \le \ldots \le t_1 \end{cases}, \quad t_1,\ldots,t_k \in [t,T], \tag{173}$$

where $\{g_1,\ldots,g_k\} = \{1,\ldots,k\}$ and we take into account all possible permutations $(g_1,\ldots,g_k)$ on the right-hand side of the formula (173).

Further, we have w. p. 1

$$J[K']^{(k)}_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]^{s_r,\ldots,s_1}_{T,t}, \tag{174}$$

where the function $K'(t_1,\ldots,t_k)$ is defined by (173); another notations are the same as in (15) and Lemma 2 ($i_1 = \ldots = i_k \ne 0$ in (15)).

From (174) and Lemma 2 we obtain w. p. 1

$$J^*[\psi^{(k)}]_{T,t} = J[K']^{(k)}_{T,t}, \tag{175}$$

where $i_1 = \ldots = i_k \ne 0$.

Generalizing the above reasoning to the case $k > 3$ and taking into account (175), we get

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} \frac{1}{k!}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right)\zeta^{(i_1)}_{j_1}\ldots\zeta^{(i_1)}_{j_k} + J[R'_{p\ldots p}]^{(k)}_{T,t} =$$

$$= \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\zeta^{(i_1)}_{j_1}\ldots\zeta^{(i_1)}_{j_k} + J[R'_{p\ldots p}]^{(k)}_{T,t},$$

where

$$R'_{p\ldots p}(t_1,\ldots,t_k) \stackrel{\mathsf{def}}{=} K'(t_1,\ldots,t_k) -$$

$$
-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}\frac{1}{k!}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right)\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k),
$$

the expression

$$
\sum_{(j_1,\ldots,j_k)}
$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

Further,

$$
\mathsf{M}\left\{\left(J[R'_{p\ldots p}]^{(k)}_{T,t}\right)^{2n}\right\}=\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\zeta^{(i_1)}_{j_1}\ldots\zeta^{(i_1)}_{j_k}\right)^{2n}\right\},
$$

where $n\in\mathbb{N}$.

Applying (we mean here the passage to the limit $\lim\limits_{p\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integrals on the right-hand side of (94) for the function $R'_{p\ldots p}(t_1,\ldots,t_k)$ instead of the function $R_{p_1\ldots,p_k}(t_1,\ldots,t_k)$, we obtain

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(J[R'_{p\ldots p}]^{(k)}_{T,t}\right)^{2n}\right\}=0.
$$

Theorems 9 is proved.

8. Recent Results on Expansion of Iterated Ito and Stratonovich Stochastic Integrals

Using (91), we can write (129) as

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta^{(i_l)}_{j_l}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta^{(i_{q_l})}_{j_{q_l}}\Biggr),\tag{176}
$$

where $[x]$ is an integer part of a real number x, $\prod\limits_{\emptyset}\stackrel{\sf def}{=}1$, $\sum\limits_{\emptyset}\stackrel{\sf def}{=}0$; another notations are the same as in Theorem **4**.

In particular, from (176) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (135).

Let us consider the generalization of Theorem 4 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 11 [18] (Sect. 1.11), [22] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x, $\prod_{\emptyset}\stackrel{\sf def}{=}1$, $\sum_{\emptyset}\stackrel{\sf def}{=}0$; another notations are the same as in Theorem 4.

It should be noted that an analogue of Theorem 11 was considered in [77]. Note that we use another notations [18] (Sect. 1.11), [22] (Sect. 15) in comparison with [77]. Moreover, the proof of an analogue of Theorem 11 from [77] is different from the proof given in [18] (Sect. 1.11), [22] (Sect. 15).

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [18] (Sect. 2.10–2.16), [23] (Sect. 13–19), [26] (Sect. 5–11), [78] (Sect. 7–13), [79] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 12 [18], [23], [26], [78], [79]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

(177) $$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

(178) $$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (177) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (178)*, constant* C *is independent of* p,

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*); another notations are the same as in Theorems* 4, 11.

Theorem 13 [18], [23], [26], [78], [79]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

(179) $$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following relations

(180) $$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

(181) $$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (179), (180) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (181)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 12.

Theorem 14 [18], [23], [26], [78], [79]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{182}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{183}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{184}$$

are fulfilled, where $i_1,\ldots,i_5 = 0,1,\ldots,m$ *in* (182), (183) *and* $i_1,\ldots,i_5 = 1,\ldots,m$ *in* (184), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 12, 13.

Theorem 15 [18], [23], [26], [78]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{185}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 12–14.

9. Theorems 3–5, 12–15 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [66], [67], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [66]-[68] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [69], [70]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^\tau\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{186}$$

where $\tau\in[t,T]$, $t\geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (186) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^\tau\phi_j(s)ds\ \zeta_j^{(i)}. \tag{187}$$

From (187) we obtain

$$d\mathbf{f}_{\tau}^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau)\zeta_j^{(i)} d\tau. \tag{188}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{189}$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m,\ \ p_1, \ldots, p_k \in \mathbb{N}$,

$$d\mathbf{w}_{\tau}^{(i)p} = \begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for} \quad i = 1, \ldots, m \\ \\ d\tau & \text{for} \quad i = 0 \end{cases}, \tag{190}$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (188).

Let us substitute (188) into (189)

$$\int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{191}$$

where

$$\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_s^{(0)} = s$,

$$C_{j_k \ldots j_1} = \int_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [66]-[68] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [68] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (187) were not considered in [66], [67] (also see [68], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [68] for approximations of the Wiener process based on its series expansion (186) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (191) to the iterated Stratonovich stochastic integral (1) does not follow from the results of the papers [66], [67] (also see [68], Theorems 7.1, 7.2).

Nevertheless, the authors of the works [43] (Sect. 5.8, pp. 202–204), [46] (pp. 438-439), [47] (pp. 82-84), [54] (pp. 263-264) use the Wong–Zakai approximation [66]-[68] (without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals based on the trigonometric series expansion of the Brownian bridge process (version of the so-called Karhunen–Loeve expansion).

From the other hand, Theorems 3–5, 12–15 from this paper can be considered as the proof of the Wong–Zakai approximation based on the iterated Riemann–Stieltjes integrals (189) of multiplicities 1 to 6 and the Wiener process approximation (187) on the base of its series expansion. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 3–5, 12–15) to the appropriate Stratonovich stochastic integrals (1) of multiplicities 1 to 6. Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (186), (187), and Theorems 3, 12–15) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [66]-[68]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{192}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (192) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{193}
$$

Using (193) and standard relation between Stratonovich and Ito stochastic integrals, it is not difficult to show that

$$
\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=
$$

$$
=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{194}
$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (194) agrees with Theorem 7.1 (see [68], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (186) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \tag{195}
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (188).

Let us substitute (188) into (195)

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{196}
$$

where

$$
C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (191).

As we noted above, approximations of the Wiener process that are similar to (187) were not considered in [66], [67] (also see Theorems 7.1, 7.2 in [68]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [68] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [18]-[21]. More precisely, using Theorems 3, 5 from this paper, we obtain from (196) the desired result

$$
\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{197}$$

From the other hand, by Theorem 4 (see (132)) for the case $k = 2$ we obtain from (196) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right) + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{198}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from (198) we obtain (197).

References

[1] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[2] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publ., 204 pp. (ISBN 5-7422-0045-5)

[3] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 32, Issue 12, (2000), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[4] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[5] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[6] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[7] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[8] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[9] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at:
http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[10] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at:
http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[11] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, St.-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at:
http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[12] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[13] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2017, no. 1, 385 pp. (A.1–A.385).
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[14] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[15] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[18] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [In English]. arXiv:2003.14184v45 [math.PR], 2023, 996 pp.

[19] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[20] Kuznetsov D.F. Mean-square approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical integration of Itô SDEs and semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[21] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Differential Equations and Control Processes, 1 (2023), A.1-A.947. Available at:
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[22] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.

[23] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2023, 222 pp.
[24] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.
[25] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations, based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.
[26] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.
[27] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 70 pp.
[28] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604
[29] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[30] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[31] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[32] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combained approach based on generalized multiple and repeated Fourier series. [In English]. arXiv:1801.05654 [math.PR], 2018, 46 pp.
[33] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[34] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.
[35] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. [in English]. arXiv:1801.07248 [math.PR]. 2018, 20 pp.
[36] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.
[37] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html
[38] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes, 3 (2020), 129-162. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html
[39] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100
[40] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878
[41] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [in English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.
[42] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2019), 32-52. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[43] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.
[44] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.
[45] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.
[46] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.
[47] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.
[48] Averina T.A., Prigarin S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.
[49] Prigarin S.M., Belov S.M. One application of series expansions of Wiener process. [In Russian]. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.
[50] Wiktorsson M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487,
[51] Ryden T., Wiktorsson M. On the simulation of iterated Ito integrals. Stochastic Processes and their Applications, 91, 1 (2001), 151-168.
[52] Gaines J. G., Lyons, T. J. Random generation of stochastic area integrals. SIAM J. Appl. Math. 54 (1994), 1132-1146.
[53] Gilsing H., Shardlow T. SDELab: A package for solving stochastic differential equations in MATLAB. Journal of Computational and Applied Mathematics. 2, 205 (2007), 1002-1018.
[54] Platen E., Bruti–Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.
[55] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.
[56] Rybakov K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2019), 1-31. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html
[57] Tang X., Xiao A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics. 45 (2019), 813-846.
[58] Zahri M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science. 8, 2 (2014), 186-198.
[59] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[60] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982. 612 pp.
[61] Skhorohod A.V. Stochastc Processes with Independent Increments. Nauka, Moscow, 1964. 280 pp.
[62] Bari, N.K. Trigonometric Series. Fiz.– Mat. Lit., Moscow, 1961, 936 pp.
[63] Hobson, E.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge University Press, Cambridge, 1931, 502 pp.
[64] Sjölin P. Convergence almost everywhere of certain singular integrals and multiple Fourier series Ark. Mat. 9, 1-2 (1971), 65-90.
[65] Fefferman C. On the convergence of multiple Fourier series. Bulletin of the American Mathematical Society. 77, 5 (1971), 744-745.
[66] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[67] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[68] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[69] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.
[70] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.
[71] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[72] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.
[73] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.
[74] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier-Legendre series. Journal of Physics: Conference Series. 2021, Vol. 1925, article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010
[75] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2
[76] Ilin V.A., Poznyak E.G. Foundations of mathematical analysis. Part II. Nauka, Moscow, 1973, 448 pp.
[77] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[78] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2023, 158 pp. [in English].
[79] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html
[80] Kuznetsov, D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. arXiv:2307.11006 [math.PR], 2023, 58 pp. [In English].

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITIES 1 TO 4. COMBAINED APPROACH BASED ON GENERALIZED MULTIPLE AND ITERATED FOURIER SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4 on the base of the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series that are converge in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k=1,2,3,4$. The mean-square convergence of the second part is proved on the base of generalized iterated Fourier series that are converge pointwise. At that, we do not use the iterated Ito stochastic integrals as a tool of the proof and directly consider the iterated Stratonovich stochastic integrals. The cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series are considered in detail. The considered expansions contain only one operation of the limit transition in contrast to its existing analogues. This property is very important for the mean-square approximation of iterated stochastic integrals. The results of the article can be applied to the numerical integration of Ito stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.

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1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[5]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, and

$$\int \text{ and } \int^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively; $i_1, \ldots, i_k = 0, 1, \ldots, m$. In this paper we use the definition of the Stratonovich stochastic integral from [2].

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[5]. At the same time $\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [6]-[24].

The construction of effective expansions (converging in the mean-square sense) for collections of the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 composes the subject of this article.

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[50]. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)$. This case allows the investigation with using of the Ito formula [2]-[5].

Consider a brief review of the mean-square approximation methods for the iterated stochastic integrals (2) and (3).

Seems that iterated stochastic integrals can be approximated by multiple integral sums of different types [3], [5], [47]. However, this approach implies partitioning of the interval of integration $[t,T]$ of iterated stochastic integrals (the length $T-t$ of this interval is a small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant computational costs [10].

In [3] (also see [2], [4], [5], [48], [49]) Milstein G.N. proposed to expand (2) or (3) into iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as a trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals (2), (3) were presented in [2], [4], [48], [49] ($k=1,2,3$) and in [3], [5] ($k=1,2$) for the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$; $i_1,i_2,i_3=0,1,\ldots,m$.

Moreover, the authors of the works [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [48] (pp. 438-439), [49] (pp. 263-264) use the Wong–Zakai approximation [51]-[53] (without rigorous proof) within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process. See discussion in Sect. 7 of this paper for details.

Note that in [50] the method (similar to the Milstein approach) of expansion of iterated (double) Ito stochastic integrals (2) ($k=2$; $\psi_1(s),\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered.

It is necessary to note that the approach based on the Karhunen–Loeve expansion [3] excelled in several times (or even in several orders) the methods of multiple integral sums [3], [5], [47] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [6], [7] (also see [14]-[19], [22], [24]-[27]), where $J^*[\psi^{(k)}]_{T,t}$ was represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables, and the function was then expressed as the generalized iterated Fourier series by complete systems of continuously differentiable functions that are orthonormal in the space $L_2([t,T])$. As a result, the general iterated series expansion of products of standard Gaussian random variables was obtained in [6], [7] (also see [14]-[19], [22], [24]-[27]) for (3) with an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series. It was shown [6], [7] (also see [14]-[19], [22], [24]-[27]) that the method of generalized iterated Fourier series leads to the Milstein approach based on the Karhunen–Loeve expansion [3] in the case of trigonometric system of functions and to a substantially simpler expansions of (3) in the case of Legendre polynomial system.

As we mentioned above, the Milstein approach based on the Karhunen–Loeve expansion [3] and the method of generalized iterated Fourier series [6], [7] (also see [14]-[19], [22], [24]-[27]) lead to iterated application of the operation of limit transition. So, these methods may not converge in the mean-square sense to the appropriate iterated Stratonovich stochastic integrals (3) for some methods of series summation.

The mentioned problem (iterated application of the operation of limit transition) not appears in the method, which is proposed for (2) in Theorems 1, 2 (see below) [10]-[22], [24]-[44]. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we obtain (see Theorems 1, 2 below) the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2). We will call this method as the method of generalized multiple Fourier series.

2. Method of Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ & \\ 0 & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2, \tag{4}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{5}$$

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

Theorem 1 [10] (2006), [11]-[22], [24]-[44]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\right.$$

$$\left.-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\text{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{7}$$

where $J[\psi^{(k)}]_{T,t}$ *is defined by* (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (5), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (6).

It was shown [12]-[19], [22], [24]-[27], [35] that Theorem 1 is valid for convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) and for convergence with probability 1 [25]-[28].

Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorem 1 [12]-[19], [22], [24]-[27], [35]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [24]-[27], [36].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [10]-[22], [24]-[44]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{10}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{11}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{12}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{13}$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_6\neq 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\neq 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_6\neq 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\neq 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_5=i_6\neq 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr),
\end{aligned}
\tag{14}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (5)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square approximation error for the iterated Ito stochastic integrals (2) (see [20], [22], [24]-[27], [34]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [2]-[5] but Legendre polynomials.

4. As it turned out (see [6]-[22], [24]-[44]), it is more convenient to work with the Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions (see [6]-[22], [24]-[44]). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [25]-[27], [39], [40].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] (also see [50]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1 and Theorem 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1;\ i_1,i_2,i_3=1,\ldots,m$) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [48] (pp. 438-439), [49] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [51]-[53] (see discussion in Sect. 7 of this paper for details).

For further consideration, let us consider the generalization of formulas (9)–(14) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{15}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (15) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{16}$$

Below there are several examples of sums in the form (16)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (7) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),\qquad(17)$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (17) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$
+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).
$$

The last equality obviously agrees with (13).

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [25] (Sect. 1.11), [35] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{18}
$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [54]. Note that we use another notations [25] (Sect. 1.11), [35] (Sect. 15) in comparison with [54]. Moreover, the proof of an analogue of Theorem 2 from [54] is somewhat different from the proof given in [25] (Sect. 1.11), [35] (Sect. 15).

3. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4. Some Old Results

As it turned out, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3) at least for multiplicities 1 to 6 (the case $k=1$ obviously corresponds to (9)). Expansions of the mentioned iterated Stratonovich stochastic integrals turned out simpler than the appropriate expansions for the iterated Ito stochastic integrals (2) based on Theorems 1, 2. Let us formulate some theorems on expansions of the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 4.

Theorem 3 [17]-[19], [22], [24]-[27], [43]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable function on $[t,T]$ and $\psi_1(\tau)$ is twice continuously differentiable function on $[t,T]$. Then, the iterated Stratonovich stochastic integral of second multiplicity*

$$
J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad(i_1,i_2=0,1,\ldots,m)
$$

is expanded into the multiple series

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

converging in the mean-square sense, where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \neq 0$*),*

$$C_{j_2j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

is the Fourier coefficient.

Note that in [24]-[27], [41], [44] Theorem 3 is proved under weaker conditions.

Theorem 4 [24]-[27], [41], [44]. *Suppose that the following conditions are fulfilled*:
1. *The functions* $\psi_1(\tau)$ *and* $\psi_2(\tau)$ *are continuously differentiable at the interval* $[t,T]$.
2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the multiple series

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

converging in the mean-square sense, where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \neq 0$*),*

$$C_{j_2j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

is the Fourier coefficient.

Theorem 5 [18], [19], [22], [24]-[27], [43]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function at the interval* $[t,T]$ *and* $\psi_1(\tau),\psi_3(\tau)$ *are twice continuously differentiable functions at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{19}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3,$$

another notations are the same as in Theorems 1, 2.

Theorem 6 [17]-[19], [22], [24]-[27], [43]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

converging in the mean-square sense is valid, where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\neq 0$*)*,

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4,$$

$\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

Note that in [17]-[19], [22], [24]-[27], [43] the expansions (9)–(12) have been applied for the proof of Theorems 3–6. In this article, we will prove Theorems 4–6 by an another approach. This approach will be called as the combined approach. More precisely, we will use the scheme of the proof of Theorem 1 from this paper (see [10]–[19], [22], [24]-[27], [35] for details) for the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 4. As a result, we will obtain two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part will be proved on the base of generalized multiple Fourier series converging in $L_2([t,T]^k)$ $(k=2,3,4)$. At the same time, the mean-square convergence of the second part will be proved on the base of generalized iterated Fourier series converging pointwise. At that, we do not use the iterated Ito stochastic integrals (2) as a tool of the proof and directly consider the iterated Stratonovich stochastic integrals (3).

4. Auxiliary Lemmas

In this section, we collected several lemmas, which will be used for the proof of Theorems 4–6.

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\ \infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{20}$$

Lemma 1 [10]–[19], [22], [24]-[27], [35]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^k\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\quad w.\ p.\ 1, \tag{21}$$

where $J[\psi^{(k)}]_{T,t}$ has the form (2), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition (20); hereinafter w. p. 1 means with probability 1.

Remark 1. *It is easy to see that if $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (21) for some $l\in\{1,\ldots,k\}$ is replaced with $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$ $(p=2,\ i_l\ne 0)$, then the differential $d\mathbf{w}_{t_l}^{(i_l)}$ in the integral $J[\psi^{(k)}]_{T,t}$ will be replaced with dt_l. If $p=3,4,\ldots,$ then the right-hand side of the formula (21) will become zero w. p. 1. If we replace $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ in (21) for some $l\in\{1,\ldots,k\}$ with $(\Delta\tau_{j_l})^p$ $(p=2,3,\ldots)$, then the right-hand side of the formula (21) will also be equal to zero w. p. 1.*

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^k\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J[\Phi]_{T,t}^{(k)}, \tag{22}$$

where $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a nonrandom function (the properties of this function will be specified further), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition (20).

Denote

$$D_k=\{(t_1,\ldots,t_k):\ t\le t_1<\ldots<t_k\le T\}. \tag{23}$$

We will use the same symbol D_k to denote the open and closed domains corresponding to the domain D_k defined by (23). However, we always specify what domain we consider (open or closed).

Also we will write $\Phi(t_1,\ldots,t_k)\in C(D_k)$ if $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function of k variables in the closed domain D_k.

Let us consider the iterated Ito stochastic integral

$$I[\Phi]_{T,t}^{(k)} \stackrel{\rm def}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\Phi(t_1,\ldots,t_k) \in C(D_k)$.

Lemma 2 [10]–[19], [22], [24]-[27], [35]. *Suppose that $\Phi(t_1,\ldots,t_k) \in C(D_k)$ or $\Phi(t_1,\ldots,t_k)$ is a continuous nonrandom function in the open domain D_k and bounded at its boundary. Then*

$$I[\Phi]_{T,t}^{(k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \quad \text{w. p. 1}, \tag{24}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (20).

Lemma 3 [10]–[19], [22], [24]-[27], [35]. *Suppose that every $\varphi_i(\tau)$ $(i=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$\prod_{l=1}^{k} J[\varphi_l]_{T,t} = J[\Phi]_{T,t}^{(k)} \quad \text{w. p. 1}, \tag{25}$$

where

$$J[\varphi_l]_{T,t} = \int\limits_t^T \varphi_l(s) d\mathbf{w}_s^{(i_l)}, \quad \Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} \varphi_l(t_l)$$

and the integral $J[\Phi]_{T,t}^{(k)}$ is defined by the equality (22).

Let us introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1}) \times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1}) \ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1}) \times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1}) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})} \ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})} dt_{s_l+1} d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{26}$$

where

$$\mathrm{A}_{k,l} = \Bigl\{(s_l,\ldots,s_1):\ s_l > s_{l-1}+1,\ldots,s_2 > s_1+1,\ s_l,\ldots,s_1 = 1,\ldots,k-1\Bigr\}, \tag{27}$$

$(s_l,\ldots,s_1)\in \mathrm{A}_{k,l},\ \ l=1,\ldots,[k/2],\ \ i_s=0,1,\ldots,m,\ \ s=1,\ldots,k,\ \ [x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Lemma 4 [6] (1997), [7], [10]-[19], [22], [24]-[27], [30]. *Suppose that every $\psi_l(\tau)(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the following relation between iterated Stratonovich and Ito stochastic integrals is correct*

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad w.\ p.\ 1, \tag{28}$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Let us define the function $K^*(t_1,\ldots,t_k)$ on the hypercube $[t,T]^k$ $(k\ge 2)$ by the following relation

$$K^*(t_1,\ldots,t_k)=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right), \tag{29}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Lemma 5 [6], [7], [14]-[19], [22], [24]-[27], [30]. *Under the conditions of Lemma 4 the following relation is correct*

$$J[K^*]_{T,t}^{(k)}=J^*[\psi^{(k)}]_{T,t}\quad w.\ p.\ 1, \tag{30}$$

where $J[K^*]_{T,t}^{(k)}$ is defined by the equality (22).

Proof. Substituting (29) into (22) and using Lemmas 1, 2, 4 with Remark 1, it is easy to notice that w. p. 1

$$J[K^*]_{T,t}^{(k)}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=J^*[\psi^{(k)}]_{T,t}. \tag{31}$$

Let us consider the following generalized multiple Fourier sum

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),$$

where $C_{j_k\ldots j_1}$ is the Fourier coefficient of the form

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K^*(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k. \tag{32}$$

Let us subsitute the relation

$$K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)+K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)$$

into $J[K^*]_{T,t}^{(k)}$ (here $p_1,\ldots,p_k<\infty$).

Then, using Lemma 3, we obtain

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+J[R_{p_1\ldots p_k}]_{T,t}^{(k)}\quad \text{w. p. 1,} \tag{33}$$

where the stochastic integral $J[R_{p_1\ldots p_k}]_{T,t}^{(k)}$ is defined in accordance with (22) and

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)=K^*(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{34}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}.$$

5. Proof of Theorem 4 Using the Combined Approach

From (33) we obtain

$$J^*[\psi^{(2)}]_{T,t}=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+J[R_{p_1p_2}]_{T,t}^{(2)}\quad \text{w. p. 1,} \tag{35}$$

where

$$J[R_{p_1p_2}]_{T,t}^{(2)}=\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1,$$

$$R_{p_1p_2}(t_1,t_2)=K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\quad (p_1,p_2<\infty),$$

$$K^*(t_1,t_2)=K(t_1,t_2)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_1),$$

where

$$K(t_1,t_2)=\begin{cases}\psi_1(t_1)\psi_2(t_2), & t_1<t_2\\ \\ 0, & \text{otherwise}\end{cases},\quad t_1,t_2\in[t,T].$$

Let us consider the case $i_1,i_2\ne 0$ (another cases can be considered absolutely analogously). Using standard estimates for moments of stochastic integrals [1], we get

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]^{(2)}_{T,t}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\int\limits_t^T\int\limits_t^{t_2}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}\right)^2\right\}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\le$$

$$\le 2\left(\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2+\int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_2dt_1\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2=$$

$$=2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \tag{36}$$

Moreover, we have

$$\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2dt_1dt_2.$$

The function $K(t_1,t_2)$ is piecewise continuous in the square $[t,T]^2$. At this situation it is well known that the generalized multiple Fourier series of the function $K(t_1,t_2)\in L_2([t,T]^2)$ is converging to this function in the square $[t,T]^2$ in the mean-square sense, i.e.

$$\lim_{p_1,p_2\to\infty}\left\|K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_k}C_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^2)}=0,$$

where

$$\|f\|_{L_2([t,T]^2)}=\left(\int\limits_{[t,T]^2}f^2(t_1,t_2)dt_1dt_2\right)^{1/2}.$$

So, we obtain

$$\lim_{p_1,p_2\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2=0. \tag{37}$$

Note that

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=$$

$$=\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\psi_2(t_1)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}. \tag{38}$$

In [19] (Theorem 3, p. A.59), [24] (Theorem 5.3, p. A.294), [25]-[27] (Theorems 2.1, 2.2), [43] (Theorem 2), [44] (Theorem 6) the following equality

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1} \tag{39}$$

is proved. Note that the existence of the limit on the right-hand side of (39) is proved in [25]-[27], [43] for the polynomial and trigonometric cases.

From (36)–(39) it follows that

$$\lim_{p_1,p_2\to\infty}\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\}=0.$$

Theorem 4 is proved.

6. Proof of Theorem 5 Using the Combined Approach

Let us consider (33) for $k=3$ and $p_1=p_2=p_3=p$

$$J^*[\psi^{(3)}]_{T,t}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+J[R_{ppp}]_{T,t}^{(3)}\quad \text{w. p. 1}, \tag{40}$$

where

$$J[R_{ppp}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{ppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)},$$

$$R_{ppp}(t_1,t_2,t_3)\stackrel{\text{def}}{=}K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3),$$

$$K^*(t_1,t_2,t_3)=\prod_{l=1}^{3}\psi_l(t_l)\Biggl(\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2<t_3\}}+$$

$$+\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2=t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2=t_3\}}\Biggr).$$

Furthermore, we have w. p. 1

$$J[R_{ppp}]_{T,t}^{(3)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{ppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\Biggl(R_{ppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{ppp}(\tau_{l_1},\tau_{l_3},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_3)}+$$

$$+R_{ppp}(\tau_{l_2},\tau_{l_1},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}+$$

$$+R_{ppp}(\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_3)}+$$

$$+R_{ppp}(\tau_{l_3},\tau_{l_2},\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_3)}+$$

$$
+R_{ppp}(\tau_{l_3},\tau_{l_1},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_3)}\Biggr)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\Biggl(R_{ppp}(\tau_{l_2},\tau_{l_2},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{ppp}(\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_3)}+
$$

$$
+R_{ppp}(\tau_{l_3},\tau_{l_2},\tau_{l_2})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_2}}^{(i_3)}\Biggr)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1}\Biggl(R_{ppp}(\tau_{l_1},\tau_{l_3},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{ppp}(\tau_{l_3},\tau_{l_1},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}+
$$

$$
+R_{ppp}(\tau_{l_3},\tau_{l_3},\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_1}}^{(i_3)}\Biggr)+
$$

$$
+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}R_{ppp}(\tau_{l_3},\tau_{l_3},\tau_{l_3})\Delta\mathbf{f}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{f}_{\tau_{l_3}}^{(i_3)}=
$$

$$
=R_{T,t}^{(1)ppp}+R_{T,t}^{(2)ppp},
$$

where

$$
R_{T,t}^{(1)ppp}=
$$

$$
=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_2,t_3)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_1,t_3,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_2,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_2,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_1)}d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_3,t_2,t_1)d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}R_{ppp}(t_3,t_1,t_2)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_3)}d\mathbf{f}_{t_3}^{(i_1)},
$$

$$
R_{T,t}^{(2)ppp}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_2,t_3)dt_2d\mathbf{f}_{t_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_3,t_2)dt_2d\mathbf{f}_{t_3}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_2,t_2)dt_2d\mathbf{f}_{t_3}^{(i_1)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3.
$$

Moreover, we obtain

$$
\mathsf{M}\left\{\left(J[R_{ppp}]_{T,t}^{(3)}\right)^2\right\}\le 2\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}+2\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\}. \tag{41}
$$

Now, using standard estimates for moments of stochastic integrals [1], we obtain the following inequality

$$
\mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}\le
$$

$$
\le 6\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Biggl(\left(R_{p_1p_2p_3}(t_1,t_2,t_3)\right)^2+\left(R_{p_1p_2p_3}(t_1,t_3,t_2)\right)^2+\left(R_{p_1p_2p_3}(t_2,t_1,t_3)\right)^2+
$$

$$
+\left(R_{p_1p_2p_3}(t_2,t_3,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_2,t_1)\right)^2+\left(R_{p_1p_2p_3}(t_3,t_1,t_2)\right)^2\Biggr)dt_1dt_2dt_3=
$$

$$
=6\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2dt_1dt_2dt_3.
$$

We have

$$
\int\limits_{[t,T]^3}\left(R_{ppp}(t_1,t_2,t_3)\right)^2dt_1dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^3}\left(K^*(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2dt_1dt_2dt_3=
$$

$$
=\int\limits_{[t,T]^3}\left(K(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2dt_1dt_2dt_3,
$$

where

$$K(t_1,t_2,t_3)=\begin{cases}\psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1<t_2<t_3\\ \\ 0, & \text{otherwise}\end{cases},\quad t_1,t_2,t_3\in[t,T].$$

So, we get

$$\lim_{p\to\infty}\ \mathsf{M}\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\}\le$$

$$\le 6\lim_{p\to\infty}\int\limits_{[t,T]^3}\left(K(t_1,t_2,t_3)-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1dt_2dt_3=0, \tag{42}$$

where $K(t_1,t_2,t_3)\in L_2([t,T]^3)$.

After the integration order replacement in iterated Ito stochastic integrals [46] (also see [19], [24] or Chapter 3 in [25]-[27]) from $R_{T,t}^{(2)ppp}$ we obtain w. p. 1

$$R_{T,t}^{(2)ppp}=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_2,t_3)dt_2d\mathbf{f}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_3,t_1)d\mathbf{f}_{t_1}^{(i_3)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_2,t_2)dt_2d\mathbf{f}_{t_3}^{(i_1)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_1,t_3,t_3)d\mathbf{f}_{t_1}^{(i_1)}dt_3\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_2,t_3,t_2)dt_2d\mathbf{f}_{t_3}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_3}R_{ppp}(t_3,t_1,t_3)d\mathbf{f}_{t_1}^{(i_2)}dt_3\right)=$$

$$=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_2,t_2,t_1)dt_2d\mathbf{f}_{t_1}^{(i_3)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_2,t_2,t_1)dt_2d\mathbf{f}_{t_1}^{(i_3)}\right)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_1,t_2,t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_1,t_2,t_2)dt_2d\mathbf{f}_{t_1}^{(i_1)}\right)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\left(\int\limits_t^T\int\limits_t^{t_1}R_{ppp}(t_2,t_1,t_2)dt_2d\mathbf{f}_{t_1}^{(i_2)}+\int\limits_t^T\int\limits_{t_1}^{T}R_{ppp}(t_2,t_1,t_2)dt_2d\mathbf{f}_{t_1}^{(i_2)}\right)=$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_2,t_2,t_3)dt_2\right)d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_1,t_2,t_2)dt_2\right)d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\left(\int\limits_t^T R_{ppp}(t_3,t_2,t_3)dt_3\right)d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\left(\frac{1}{2}\mathbf{1}_{\{t_2<t_3\}}+\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\right)\psi_1(t_2)\psi_2(t_2)\psi_3(t_3)-
$$

$$
-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_2)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\Biggr)dt_2d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\left(\frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\right)\psi_1(t_1)\psi_2(t_2)\psi_3(t_2)-
$$

$$
-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_2)\Biggr)dt_2d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^T\Biggl(\frac{1}{4}\mathbf{1}_{\{t_2=t_3\}}\psi_1(t_3)\psi_2(t_2)\psi_3(t_3)-
$$

$$
-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_1}\phi_{j_1}(t_3)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\Biggr)dt_3d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_3(t_3)\int\limits_t^{t_3}\psi_1(t_2)\psi_2(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_1j_1}\phi_{j_3}(t_3)\right)d\mathbf{f}_{t_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\psi_3(t_2)dt_2-\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_3j_1}\phi_{j_1}(t_1)\right)d\mathbf{f}_{t_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T(-1)\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}C_{j_1j_2j_1}\phi_{j_2}(t_2)d\mathbf{f}_{t_2}^{(i_2)}=
$$

$$
= \mathbf{1}_{\{i_1=i_2\ne 0\}} \left(\frac{1}{2}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3} \psi_1(t_2)\psi_2(t_2)dt_2 d\mathbf{f}_{t_3}^{(i_3)} - \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right) +
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}} \left(\frac{1}{2}\int\limits_t^T \psi_1(t_1)\int\limits_{t_1}^{T} \psi_2(t_2)\psi_3(t_2)dt_2 d\mathbf{f}_{t_1}^{(i_1)} - \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right) -
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}} \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}.
$$

From [19] (Theorem 6, pp. A.116–A.117), [24] (Theorem 5.5', p. A.371), [25]-[27] (Chapter 2), [43] (Theorem 3) we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{(2)ppp}\right)^2\right\} \le
$$

$$
\le 3\left(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3} \psi_1(t_2)\psi_2(t_2)dt_2 d\mathbf{f}_{t_3}^{(i_3)} - \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} +\right.
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathsf{M}\left\{\left(\sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_1 j_3 j_1}\zeta_{j_3}^{(i_2)}\right)^2\right\} +
$$

$$
\left. +\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T \psi_1(t_1)\int\limits_{t_1}^{T} \psi_2(t_2)\psi_3(t_2)dt_2 d\mathbf{f}_{t_1}^{(i_1)} - \sum_{j_1=0}^{p}\sum_{j_3=0}^{p} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\right) \to 0 \tag{43}
$$

if $p \to \infty$. Using (40)–(43), we obtain the expansion (19). Theorem 5 is proved.

7. Proof of Theorem 6 Using the Combined Approach

Let us consider (33) for the case $k = 4$, $p_1 = p_2 = p_3 = p_4 = p$, and $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$, $\psi_4(\tau) \equiv 1$

$$
\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} =
$$

$$
= \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p} C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} + J[R_{pppp}]_{T,t}^{(4)} \quad \text{w. p. 1,} \tag{44}
$$

where

$$J[R_{pppp}]_{T,t}^{(4)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{pppp}(t_1,t_2,t_3,t_4)\stackrel{\text{def}}{=}K^*(t_1,t_2,t_3,t_4)-$$

$$-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_4=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\phi_{j_4}(t_4),$$

$$K^*(t_1,t_2,t_3,t_4)\stackrel{\text{def}}{=}\prod_{l=1}^{3}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1=t_2<t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2=t_3<t_4\}}+$$

$$+\frac{1}{4}\mathbf{1}_{\{t_1=t_2=t_3<t_4\}}+\frac{1}{2}\mathbf{1}_{\{t_1<t_2<t_3=t_4\}}+\frac{1}{4}\mathbf{1}_{\{t_1=t_2<t_3=t_4\}}+$$

$$+\frac{1}{4}\mathbf{1}_{\{t_1<t_2=t_3=t_4\}}+\frac{1}{8}\mathbf{1}_{\{t_1=t_2=t_3=t_4\}}.$$

Moreover, we have

$$J[R_{pppp}]_{T,t}^{(4)}=\sum_{i=0}^{7}R_{T,t}^{(i)pppp}\quad\text{w. p. 1,}\tag{45}$$

where

$$R_{T,t}^{(0)pppp}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{l_4-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1}\sum_{(l_1,l_2,l_3,l_4)}\Biggl(R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_4})\times$$

$$\times\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}\Biggr),$$

where permutations (l_1,l_2,l_3,l_4) when summing are performed only in the expression, which is enclosed in parentheses,

$$R_{T,t}^{(1)pppp}=\mathbf{1}_{\{i_1=i_2\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_1\neq l_3,l_1\neq l_4,l_3\neq l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(2)pppp}=\mathbf{1}_{\{i_1=i_3\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\neq l_2,l_1\neq l_4,l_2\neq l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(3)pppp}=\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(4)pppp}=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)},$$

$$R_{T,t}^{(5)pppp}=\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$R_{T,t}^{(6)pppp}=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3},$$

$$R_{T,t}^{(7)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\lim_{N\to\infty}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}R_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}.$$

The relations (44) and (45) imply that Theorem 6 will be proved if

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(i)pppp}\right)^2\right\}=0,\quad i=0,1,\ldots,7.$$

We have (see (24))

$$R_{T,t}^{(0)pppp}=\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\sum_{(t_1,t_2,t_3,t_4)}\left(R_{pppp}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\right),$$

where permutations (t_1,t_2,t_3,t_4) when summing are performed only in the expression, which is enclosed in parentheses.

From the other hand [19], [24]-[27], [35]

$$R_{T,t}^{(0)pppp}=\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}R_{pppp}(t_1,t_2,t_3,t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)},$$

where permutations $(t_1,\ldots,t_4)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_4}^{(i_4)}$. At the same time, the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_4)$, then i_r swapped with i_q in the permutations $(i_1,\ldots,i_4)$.

So, we obtain

$$
\mathsf{M}\left\{\left(R_{T,t}^{(0)pppp}\right)^2\right\}\le 24\sum_{(t_1,t_2,t_3,t_4)}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}\left(R_{pppp}(t_1,t_2,t_3,t_4)\right)^2 dt_1dt_2dt_3dt_4=
$$

$$
=24\int\limits_{[t,T]^4}\left(R_{pppp}(t_1,t_2,t_3,t_4))\right)^2 dt_1dt_2dt_3dt_4\ \to 0
$$

if $p\to\infty$, where $K^*(t_1,t_2,t_3,t_4)\in L_2([t,T]^4)$.

Let us consider $R_{T,t}^{(1)pppp}$

$$
R_{T,t}^{(1)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_1\ne l_3,l_1\ne l_4,l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}R_{pppp}(\tau_{l_1},\tau_{l_1},\tau_{l_3},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_3}<\tau_{l_4}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}=\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_3}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_3,l_1=0\\ l_3\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_3}<\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1}\Biggl(0-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_3,j_1=0}^{p}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}\quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_2}dt_1dt_2,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_t^{t_2}dt_1dt_2\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(1)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(2)pppp}$

$$
R_{T,t}^{(2)pppp}=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_4,l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_1},\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_2\ne l_4}}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} =
$$

$$
= \mathbf{1}_{\{i_1=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_1})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_1}\Delta\tau_{l_4} =
$$

$$
= -\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}\quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} = 0\quad \text{w. p. 1},
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1} = 0.
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(2)pppp}\right)^2\right\} = 0.
$$

Let us consider $R_{T,t}^{(3)pppp}$

$$
R_{T,t}^{(3)pppp} = \mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =
$$

$$
= \mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =
$$

$$
= \mathbf{1}_{\{i_1=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_3,l_2,l_1=0\\ l_2\ne l_3}}^{N-1} \Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}} -
$$

$$
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Biggr)\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =
$$

$$
= \mathbf{1}_{\{i_1=i_4\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}(-1) \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times \phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1) \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times \phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_3})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_1})\Delta\tau_{l_1}\Delta\tau_{l_3} =
$$

$$
= -\mathbf{1}_{\{i_1=i_4\ne 0\}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}} \sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4} \quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_4,j_3,j_2=0}^{p} C_{j_4j_3j_2j_4}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = 0 \quad \text{w. p. 1,}
$$

$$
\lim_{p\to\infty} \sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4} = 0.
$$

Then

$$
\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(3)pppp}\right)^2\right\} = 0.
$$

Let us consider $R_{T,t}^{(4)pppp}$

$$
R_{T,t}^{(4)pppp} = \mathbf{1}_{\{i_2=i_3\ne 0\}} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_2, l_1\ne l_4, l_2\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)} =
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_2},\tau_{l_4})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2,l_1=0\\ l_1\ne l_4}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_4}}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_4})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_2=i_3\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}\quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0,
$$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} = \frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}dt_2 d\mathbf{w}_{t_4}^{(i_4)} \quad \text{w. p. 1.}$$

Then

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(4)pppp}\right)^2\right\}=0.$$

Let us consider $R_{T,t}^{(5)pppp}$

$$R_{T,t}^{(5)pppp} = \mathbf{1}_{\{i_2=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_2,l_1\neq l_3,l_2\neq l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_2=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_2=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_3}}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-$$

$$-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= \mathbf{1}_{\{i_2=i_4\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\neq l_3}}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\tau_{l_2}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= -\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\times$$

$$\times\phi_{j_1}(\tau_{l_3})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_3} =$$

$$= -\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}\quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_3,j_1=0}^{p} C_{j_4j_3j_4j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}=0\quad \text{w. p. 1,}
$$

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0.
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(5)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(6)pppp}$

$$
R_{T,t}^{(6)pppp}=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2,l_1\ne l_3,l_2\ne l_3}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1} G_{pppp}(\tau_{l_1},\tau_{l_2},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}+
$$

$$
+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}<\tau_{l_3}\}}+\frac{1}{4}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}=\tau_{l_3}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_1}=\tau_{l_2}=\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_3,l_2,l_1=0\\ l_1\ne l_2}}^{N-1}\Biggl(\frac{1}{2}\mathbf{1}_{\{\tau_{l_1}<\tau_{l_2}<\tau_{l_3}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Biggr)\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\tau_{l_3}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3-\sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}\sum_{l_1=0}^{N-1}(-1)\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_3})\Delta\tau_{l_1}\Delta\tau_{l_3}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3-\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}=
$$

$$
=\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3-\right.
$$

$$
\left.-\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}-\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\right)\quad \text{w. p. 1.}
$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$
\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3,
$$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_4,j_2,j_1=0}^{p}C_{j_4j_4j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{4}\int\limits_t^T\int\limits_t^{t_3}dt_1dt_3\quad \text{w. p. 1.}
$$

Then

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(6)pppp}\right)^2\right\}=0.
$$

Let us consider $R_{T,t}^{(7)pppp}$

$$
R_{T,t}^{(7)pppp}=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_4,l_2=0\\ l_2\ne l_4}}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_2},\tau_{l_4},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_2},\tau_{l_4})\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}G_{pppp}(\tau_{l_2},\tau_{l_4},\tau_{l_4},\tau_{l_2})\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{4}\mathbf{1}_{\{\tau_{l_2}<\tau_{l_4}\}}+\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_2})\phi_{j_4}(\tau_{l_4})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_4=0}^{N-1}\sum_{l_2=0}^{N-1}\Biggl(\frac{1}{8}\mathbf{1}_{\{\tau_{l_2}=\tau_{l_4}\}}-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}\times
$$

$$
\times\phi_{j_1}(\tau_{l_2})\phi_{j_2}(\tau_{l_4})\phi_{j_3}(\tau_{l_4})\phi_{j_4}(\tau_{l_2})\Biggr)\Delta\tau_{l_2}\Delta\tau_{l_4}=
$$

$$
=\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\left(\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4}dt_2dt_4-\sum_{j_4,j_1=0}^{p}C_{j_4j_4j_1j_1}\right)-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_4,j_1=0}^{p}C_{j_4j_1j_4j_1}-
$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}.$$

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

$$\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_4j_1j_1}=\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4} dt_2dt_4,$$

$$\lim_{p\to\infty}\sum_{j_4,j_1=0}^{p} C_{j_4j_1j_4j_1}=0,$$

$$\lim_{p\to\infty}\sum_{j_4,j_2=0}^{p} C_{j_4j_2j_2j_4}=0.$$

Then

$$\lim_{p\to\infty} R_{T,t}^{(7)pppp}=0.$$

Theorem 6 is proved.

8. Theorems 1–6 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [51], [52], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [51]-[53] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [55], [56]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)},\tag{46}$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (46) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{47}$$

From (47) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{48}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{49}$$

where $i_1,\ldots,i_k=0,1,\ldots,m,\ \ p_1,\ldots,p_k\in\mathbb{N}$,

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{50}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (48).

Let us substitute (48) into (49)

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{51}$$

where

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [51]-[53] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [53] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (47) were not considered in [51], [52] (also see [53], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [53] for approximations of the Wiener process based on its series expansion (46) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (51) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [51], [52] (also see [53], Theorems 7.1, 7.2).

From the other hand, Theorems 1–6 and Theorems 7–10 (see below) from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the Riemann–Stieltjes integrals (49) and approximation (47) of the Wiener process. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 1–6 and Theorems 7–10 (see below)) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (46), (47), and Theorems 3–10) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [51]-[53]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{52}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (52) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=
$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{53}
$$

Using (53), it is not difficult to show that

$$
\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=
$$

$$
=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{54}
$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (54) agrees with Theorem 7.1 (see [53], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (46) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \tag{55}
$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (48).

Let us substitute (48) into (55)

$$
\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{56}
$$

where

$$
C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds
$$

is the Fourier coefficient; another notations are the same as in (51).

As we noted above, approximations of the Wiener process that are similar to (47) were not considered in [51], [52] (also see Theorems 7.1, 7.2 in [53]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [53] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [25]-[27]. More precisely, using Theorems 3, 4 from this paper, we obtain from (56) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \qquad (57)$$

From the other hand, by Theorems 1, 2 (see (10)) for the case $k=2$ we obtain from (56) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \qquad (58)$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from standard relation between Ito and Stratonovich stochastic integrals and (58) we obtain (57).

9. Recent Results on Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [25] (Sect. 2.10–2.16), [32] (Sect. 5–11), [33] (Sect. 7–13), [43] (Sect. 13–19), [63] (Sect. 4–9), [64]. Let us formulate four theorems that were obtained using this approach.

Theorem 7 [25], [32], [33], [43], [63]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{59}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{60}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (59) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (60), *constant* C *is independent of* p,

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 8 [25], [32], [33], [43], [63]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4)\int\limits_t^{*t_4} \psi_3(t_3)\int\limits_t^{*t_3} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{61}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{62}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{63}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (61), (62) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (63), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4 j_3 j_2 j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;$$

another notations are the same as in Theorem 7.

Theorem 9 [25], [32], [33], [43], [63]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{64}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}, \tag{65}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{66}$$

are fulfilled, where $i_1, \ldots, i_5 = 0, 1, \ldots, m$ *in* (64), (65) *and* $i_1, \ldots, i_5 = 1, \ldots, m$ *in* (66), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_5;$$

another notations are the same as in Theorems 7, 8.

Theorem 10 [25], [32], [33], [43], [64]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{67}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 7–9.

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[6] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[7] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at:
http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[8] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[9] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at:
http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at:
http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[15] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at:
http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at:
http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at:
http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[18] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[20] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[21] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[23] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N.Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[25] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR]. 2022, 923 pp.

[26] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[27] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[28] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[29] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[30] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[31] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.
[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 129 pp.
[33] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.
[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.
[35] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.
[36] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.
[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604
[38] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060
[39] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.
[41] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[42] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[43] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [in English]. arXiv:1712.09516 [math.PR]. 2022, 204 pp.
[44] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.
[45] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100
[46] Kuznetsov D.F. Integration replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [in English]. arXiv:1801.04634 [math.PR], 2018, 28 pp.
[47] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.
[48] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.
[49] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.
[50] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].
[51] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[52] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[53] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[54] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[55] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[56] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[57] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[58] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[59] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].

[60] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[61] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[62] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[63] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[64] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITY 2. COMBINED APPROACH BASED ON GENERALIZED MULTIPLE AND ITERATED FOURIER SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansion of iterated Stratonovich stochastic integrals of multiplicity 2 on the base of the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^2)$. The mean-square convergence of the second part is proved on the base of generalized iterated (double) Fourier series converging pointwise. At that, we prove the iterated limit transition for the second part of the expansion on the base of Lebesgue's Dominated Convergence Theorem. The results of the article can be applied to the numerical integration of Ito stochastic differential equations.

Contents

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Let us consider the following collections of iterated Stratonovich and Ito stochastic integrals

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$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \tag{1}$$

$$J[\psi^{(2)}]_{T,t} = \int\limits_t^{T} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \tag{2}$$

where every $\psi_l(\tau)$ $(l = 1,\ 2)$ is a nonrandom function at the interval $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau,\ \ i_1, \ldots, i_k = 0,\ 1, \ldots, m,$

$$\int\limits^{*} \text{ and } \int$$

denote Stratonovich and Ito stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [1]).

Further, we will denote as $\{\phi_j(x)\}_{j=0}^{\infty}$ the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Also we will pay a special attention on the following well-known facts connecting to these two systems of functions [2].

Suppose that the function $f(x)$ is bounded at the interval $[t, T]$. Moreover, its derivative $f'(x)$ is a continuous function at the interval $[t, T]$ except may be the finite number of points of the finite discontinuity. Then the Fourier series

$$\sum_{j=0}^{\infty} C_j \phi_j(x), \quad C_j = \int\limits_t^T f(x)\phi_j(x)dx$$

converges at any internal point x of the interval $[t, T]$ to the value $\left(f(x + 0) + f(x - 0)\right)/2$ and converges uniformly to $f(x)$ on any closed interval of continuity of the function $f(x)$ laying inside $[t, T]$. At the same time, the Fourier–Legendre series converges if $x = t$ and $x = T$ to $f(t + 0)$ and $f(T - 0)$ correspondently, and the trigonometric Fourier series converges if $x = t$ and $x = T$ to $\left(f(t + 0) + f(T - 0)\right)/2$ in the case of periodic continuation of the function $f(x)$.

2. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2

The use of generalized multiple and iterated Fourier series by various complete orthonormal systems of functions in the space $L_2([t, T])$ for the expansion of iterated Ito and Stratonovich stochastic integrals is reflected in a number of works of the author [3]-[44]. In these papers, several new approaches to the mean-square approximation of iterated stochastic integrals were proposed and developed. One of the mentioned approaches (the so-called combined approach) for the expansion of iterated Stratonovich stochastic integrals of multiplicities 1 to 4 based on generalized multiple and iterated Fourier series has been considered in [4]. In this article, we consider the case of second multiplicity of iterated Stratonovich stochastic integrals. At that, we prove the mean-square convergence of the expansion of iterated Stratonovich stochastic integrals using the another method in comparison with the method from [4].

Theorem 1 [3] (2013) (also see [8] (Sect. 2.1.1) and references therein). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable nonrandom function on $[t, T]$ and*

$\psi_1(\tau)$ is twice continuously differentiable nonrandom function on $[t,T]$. Then the iterated Stratonovich stochastic integral of the second multiplicity

$$J^*[\psi^{(2)}]_{T,t}={\int\limits_t^{*}}^{T}\psi_2(t_2){\int\limits_t^{*}}^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the multiple series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense, where l.i.m. *is a limit in the mean-square sense,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i\neq 0$),

$$C_{j_2j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

is the Fourier coefficient.

Remark 1. *It should be noted that Theorem* 1 *is proved in* [3] (2013) (*also see* [8] (*Sect.* 2.1.1) *and references therein*). *The proof from* [3], [8] (*Sect.* 2.1.1) *is based on double integration by parts. Below we consider another proof of Theorem* 1.

Proof. Let us consider some auxiliary lemmas from [3] (also see [8] and references therein). At that, we will consider the particular case of these lemmas for $k=2$.

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{3}$$

Lemma 1 [3] (also see [8] and references therein). *Suppose that every $\psi_l(\tau)$ $(l=1,\ 2)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$J[\psi^{(2)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}\psi_1(\tau_{j_1})\psi_2(\tau_{j_2})\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)}\quad \text{w. p. 1}, \tag{4}$$

where $J[\psi^{(2)}]_{T,t}$ is the iterated Ito stochastic integral (2), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, *$\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$ satisfying the condition* (3); *hereinafter w. p.* 1 *means with probability* 1.

Let us define the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{N-1}\Phi\left(\tau_{j_1},\tau_{j_2}\right)\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)}\stackrel{\text{def}}{=}J[\Phi]_{T,t}^{(2)}, \tag{5}$$

where $\Phi(t_1, t_2): [t, T]^2 \to \mathbb{R}$ is a nonrandom function (the properties of this function will be specified further), $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ is the partition of the interval $[t, T]$ satisfying the condition (3).

Denote

$$D_2 = \{(t_1, t_2): \ t \le t_1 < t_2 \le T\}. \tag{6}$$

We will use the same symbol D_2 to denote the open and closed domains corresponding to the domain D_2 defined by (6). However, we always specify what domain we consider (open or closed).

Also we will write $\Phi(t_1, t_2) \in C(D_2)$ if $\Phi(t_1, t_2)$ is a continuous nonrandom function of two variables in the closed domain D_2.

Let us consider the iterated Ito stochastic integral

$$I[\Phi]_{T,t}^{(2)} \stackrel{\rm def}{=} \int\limits_t^T \int\limits_t^{t_2} \Phi(t_1, t_2) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)},$$

where $\Phi(t_1, t_2) \in C(D_2)$.

Lemma 2 [3] (also see [8] and references therein). *Suppose that $\Phi(t_1, t_2) \in C(D_2)$ or $\Phi(t_1, t_2)$ is a continuous nonrandom function in the open domain D_2 and bounded at its boundary. Then*

$$I[\Phi]_{T,t}^{(2)} = \underset{N \to \infty}{\rm l.i.m.} \sum_{j_k=0}^{N-1} \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1}, \tau_{j_2}) \Delta \mathbf{w}_{\tau_{j_1}}^{(i_1)} \Delta \mathbf{w}_{\tau_{j_2}}^{(i_2)} \quad \text{w. p. 1}, \tag{7}$$

where $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ is the partition of the interval $[t, T]$ satisfying the condition (3).

Lemma 3 [3] (also see [8] and references therein). *Suppose that every $\varphi_l(\tau)$ $(l = 1, \ 2)$ is a continuous nonrandom function at the interval $[t, T]$. Then*

$$J[\varphi_1]_{T,t} J[\varphi_2]_{T,t} = J[\Phi]_{T,t}^{(2)} \quad w. \ p. \ 1, \tag{8}$$

where

$$\Phi(t_1, t_2) = \varphi_1(t_1)\varphi_2(t_2), \qquad J[\varphi_l]_{T,t} = \int\limits_t^T \varphi_l(\tau) d\mathbf{w}_{\tau}^{(i_l)} \quad (l = 1, \ 2)$$

and the stochastic integral $J[\Phi]_{T,t}^{(2)}$ is defined by the equality (5), $i_1, i_2 = 0, 1, \ldots, m$.

In accordance to the standard relations between Stratonovich and Ito stochastic integrals we have w. p. 1 [1]

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \mathbf{1}_{\{i_1 = i_2 \ne 0\}} \int\limits_t^T \psi_1(t_1)\psi_2(t_1) dt_1, \tag{9}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us define the function $K^*(t_1, t_2)$ at the square $[t, T]^2$ as follows

$$K^*(t_1, t_2) = \psi_1(t_1)\psi_2(t_2) \left(\mathbf{1}_{\{t_1 < t_2\}} + \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \right) = K(t_1, t_2) + \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \psi_1(t_1)\psi_2(t_2), \tag{10}$$

where

$$K(t_1,t_2)=\begin{cases}\psi_1(t_1)\psi_2(t_2), & t_1<t_2\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,t_2\in[t,T]$$

and $\mathbf{1}_A$ is the indicator of the set A.

Lemma 4 [3] (also see [8] and references therein). *Under the conditions of Theorem* 1 *the following relation*

$$J[K^*]_{T,t}^{(2)}=J^*[\psi^{(2)}]_{T,t} \tag{11}$$

is valid w. p. 1*, where* $J[K^*]_{T,t}^{(2)}$ *is defined by the equality* (5).

Proof. Substituting (10) into (5) and using Lemmas 1 and 2, it is easy to see that

$$J[K^*]_{T,t}^{(2)}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=J^*[\psi^{(2)}]_{T,t}\quad \text{w. p. 1.} \tag{12}$$

Let us consider the following generalized double Fourier sum

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),$$

where $C_{j_2j_1}$ is the Fourier coefficient of the form

$$C_{j_2j_1}=\int\limits_{[t,T]^2}K^*(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2. \tag{13}$$

Substitute the relation

$$K^*(t_1,t_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)+K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)$$

with finite p_1 and p_2 into $J[K^*]_{T,t}^{(2)}$. Then, using Lemma 3, we obtain

$$J^*[\psi^{(2)}]_{T,t}=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+J[R_{p_1p_2}]_{T,t}^{(2)}\quad \text{w. p. 1,} \tag{14}$$

where the stochastic integral $J[R_{p_1p_2}]_{T,t}^{(2)}$ is defined in accordance with (5) and

$$R_{p_1p_2}(t_1,t_2)=K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \tag{15}$$

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{w}_{\tau}^{(i)},$$

$$J[R_{p_1p_2}]_{T,t}^{(2)}=\int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1.$$

Let us consider the case $i_1,i_2\ne 0$ (another cases can be considered absolutely analogously). Using standard estimates for moments of stochastic integrals [23], we obtain

$$\mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\int\limits_t^T\int\limits_t^{t_2} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1} R_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}\right)^2\right\}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2\le$$

$$\le 2\left(\int\limits_t^T\int\limits_t^{t_2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2+\int\limits_t^T\int\limits_t^{t_1}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_2dt_1\right)+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2=$$

$$=2\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2+\mathbf{1}_{\{i_1=i_2\ne 0\}}\left(\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1\right)^2. \tag{16}$$

We have

$$\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}\left(K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\right)^2dt_1dt_2.$$

The function $K(t_1,t_2)$ is piecewise continuous in the square $[t,T]^2$. At this situation it is well-known that the generalized multiple Fourier series of the function $K(t_1,t_2)\in L_2([t,T]^2)$ is converging to this function in the square $[t,T]^2$ in the mean-square sense, i.e.

$$\lim_{p_1,p_2\to\infty}\left\|K(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\prod_{l=1}^{2}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^2)}=0,$$

where

$$\|f\|_{L_2([t,T]^2)}=\left(\int\limits_{[t,T]^2}f^2(t_1,t_2)dt_1dt_2\right)^{1/2}.$$

So, we obtain

$$\lim_{p_1,p_2\to\infty}\int\limits_{[t,T]^2}\left(R_{p_1p_2}(t_1,t_2)\right)^2dt_1dt_2=0. \tag{17}$$

Note that

$$\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=$$

$$=\int\limits_t^T\left(\frac{1}{2}\psi_1(t_1)\psi_2(t_1)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\mathbf{1}_{\{j_1=j_2\}}=$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_1j_1}. \tag{18}$$

From (18) we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1= \tag{19}$$

$$=\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1-\lim_{p_1\to\infty}\sum_{j_1=0}^{p_1}C_{j_1j_1}=$$

$$= \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1. \tag{20}$$

Note that the existence of the limit

$$\lim_{p_1\to\infty}\sum_{j_1=0}^{p_1} C_{j_1j_1}$$

is proved in [8] (Sect. 2.1.1, 2.1.2) for the polynomial and trigonometric cases.

If we prove the following relation

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0, \tag{21}$$

then from (20) we get

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}, \tag{22}$$

$$\lim_{p_1,p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1 = 0. \tag{23}$$

From (16), (17), and (23) we obtain

$$\lim_{p_1,p_2\to\infty} \mathsf{M}\left\{\left(J[R_{p_1p_2}]_{T,t}^{(2)}\right)^2\right\} = 0$$

and Theorem 1 will be proved.

Let us expand the function $K^*(t_1,t_2)$ (see (10)) using the variable t_1, when t_2 is fixed, into the generalized Fourier series at the interval (t,T)

$$K^*(t_1,t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2)\phi_{j_1}(t_1) \quad (t_1\neq t,T), \tag{24}$$

where

$$C_{j_1}(t_2) = \int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1 = \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

The equality (24) is satisfied pointwise in each point of the interval (t, T) with respect to the variable t_1, when $t_2 \in [t, T]$ is fixed, due to a piecewise smoothness of the function $K^*(t_1, t_2)$ with respect to the variable $t_1 \in [t, T]$ (t_2 is fixed).

Note also that due to well-known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (24) converges when $t_1 = t$ and $t_1 = T$.

Obtaining (24), we also used the fact that the right-hand side of (24) converges when $t_1 = t_2$ (point of a finite discontinuity of the function $K(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K(t_2 - 0, t_2) + K(t_2 + 0, t_2)\right) = \frac{1}{2}\psi_1(t_2)\psi_2(t_2) = K^*(t_2, t_2).$$

The function $C_{j_1}(t_2)$ is a continuously differentiable one at the interval $[t, T]$. Let us expand it into the generalized Fourier series at the interval (t, T)

$$C_{j_1}(t_2) = \sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_2}(t_2) \quad (t_2 \neq t, T), \tag{25}$$

where

$$C_{j_2 j_1} = \int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2 = \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2$$

and the equality (25) is satisfied pointwise at any point of the interval (t, T) (the right-hand side of (25) converges when $t_2 = t$ and $t_1 = T$).

Let us substitute (25) into (24)

$$K^*(t_1, t_2) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad (t_1, t_2) \in (t, T)^2. \tag{26}$$

Futhermore, the series on the right-hand side of (26) converges at the boundary of the square $[t, T]^2$.

From (15) and (26) we obtain

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty} R_{p_1 p_2}(t_1, t_1) = 0 \quad \text{when} \quad t_1 \in (t, T). \tag{27}$$

Since the integral

$$\int\limits_t^T R_{p_1 p_2}(t_1, t_1)dt_1 \tag{28}$$

exists as Riemann integral, then this integral equals to the corresponding Lebesgue integral. Moreover,

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty} R_{p_1 p_2}(t_1, t_1) = 0 \quad \text{when} \quad t_1 \in (t, T), \tag{29}$$

where the left-hand side of (29) is bounded on $[t, T]$.

According to (15), (24)–(26), we have

$$
R_{p_1p_2}(t_1,t_2)=\left(K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}C_{j_1}(t_2)\phi_{j_1}(t_1)\right)+
$$

$$
+\left(\sum_{j_1=0}^{p_1}\left(C_{j_1}(t_2)-\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_2}(t_2)\right)\phi_{j_1}(t_1)\right).
$$

Then, appling two times (we mean here an iterated passage to the limit $\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$) the Lebesgue's Dominated Convergence Theorem to the integral (28), we obtain

$$
\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T R_{p_1p_2}(t_1,t_1)dt_1=0.
$$

For a discussion of the choice of integrable majorants when applying Lebesgue's Dominated Convergence Theorem to the integral (28) for the polynomial and trigonometric cases, see [8] (Sect. 2.4.1), [43] (Sect. 2).

Note that the developement of the approach from this article can be found in [8] (Sect. 2.4), [43].

3. Some Recent Results on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [8] (Sect. 2.10–2.16), [13] (Sect. 13–19), [34] (Sect. 5–11), [35] (Sect. 7–13), [64] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 2 [8], [13], [34], [35], [64]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)
$$

the following relations

$$
J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{30}
$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{31}
$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (30) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (31)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorem* 1*.*

Theorem 3 [8], [13], [34], [35], [64]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t, T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \qquad (32)$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \qquad (33)$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \qquad (34)$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (32), (33) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (34)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t, T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t, T])$*,*

$$C_{j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;$$

another notations are the same as in Theorem 2*.*

Theorem 4 [8], [13], [34], [35], [64]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{35}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{36}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{37}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (35), (36) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (37), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^{T}\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 2, 3.

Theorem 5 [8], [13], [34], [35]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{38}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^{T}\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 2–4.

Recently the equality (22) was proved in [66] (also see [8] (Sect. 2.1.4)) for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. This means that we have the following generalizaion of Theorem 1.

Theorem 6 [8] (Sect. 2.1.4). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \psi_2(\tau)$ *are continuous functions on* $[t,T]$. *Then the iterated Stratonovich stochastic integral of the second multiplicity*

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \quad (i_1, i_2 = 0, 1, \ldots, m)$$

is expanded into the multiple series

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

that converges in the mean-square sense; *where notations are the same as in Theorem* 1.

The condition of continuity of the functions $\psi_1(\tau), \psi_2(\tau)$ is related to the definition of the Stratonovich stochastic integral that we use (see [1]).

4. Theorems 1–6 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s \in [0, T]$. Let $\mathbf{f}_s^{(i)p}$, $p \in \mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i = 1, \ldots, m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [50], [51], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [50]-[52] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s \in [0, T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [53], [54]

$$\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^{\tau} \phi_j(s) ds \ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}, \tag{39}$$

where $\tau \in [t, T]$, $t \geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (39) converges for any $\tau \in [t, T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{40}$$

From (40) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{41}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{42}$$

where $i_1,\ldots,i_k=0,1,\ldots,m,\ \ p_1,\ldots,p_k\in\mathbb{N}$,

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{43}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (41).

Let us substitute (41) into (42)

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{44}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

Consider the following iterated Stratonovich stochastic integrals

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{45}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuously differentiable nonrandom function at the interval $[t,T]$; another notations are the same as in (1).

To best of our knowledge [50]-[52] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [52] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (40) were not considered in [50], [51] (also see [52], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [52] for approximations of the Wiener process based on its series expansion (39) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (44) to the appropriate iterated Stratonovich stochastic integral (45) does not follow from the results of the papers [50], [51] (also see [52], Theorems 7.1, 7.2).

However, in [1] (Sect. 5.8, pp. 202–204), [55] (pp. 82-84), [56] (pp. 438-439), [57] (pp. 263-264) the authors use (without rigorous proof) the Wong–Zakai approximation [50]-[52] within the frames of the method of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process [58].

From the other hand, Theorems 1–6 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (45) of multiplicities 2 to 6 based on the approximation (40) of the Wiener process. At that, the Riemann–Stieltjes integrals (42) of multiplicities 2 to 6 converge in the mean-square sense to the appropriate Stratonovich stochastic integrals (45). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (39), (40), and Theorems 1–5) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [50]-[52]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{46}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (46) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$= \sum_{l=0}^{N-1} \int\limits_{l\Delta}^{(l+1)\Delta} \left(\sum_{q=0}^{l-1} \int\limits_{q\Delta}^{(q+1)\Delta} \frac{\Delta \mathbf{f}_{q\Delta}^{(i_1)}}{\Delta} d\tau + \int\limits_{l\Delta}^{s} \frac{\Delta \mathbf{f}_{l\Delta}^{(i_1)}}{\Delta} d\tau \right) \frac{\Delta \mathbf{f}_{l\Delta}^{(i_2)}}{\Delta} ds =$$

$$= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta \mathbf{f}_{q\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{\Delta^2} \sum_{l=0}^{N-1} \Delta \mathbf{f}_{l\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} \int\limits_{l\Delta}^{(l+1)\Delta} \int\limits_{l\Delta}^{s} d\tau ds =$$

$$= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta \mathbf{f}_{q\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta \mathbf{f}_{l\Delta}^{(i_1)} \Delta \mathbf{f}_{l\Delta}^{(i_2)}. \tag{47}$$

Using (47), it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau) d\mathbf{b}_{\Delta}^{(i_2)}(s) = \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} + \frac{1}{2} \mathbf{1}_{\{i_1=i_2\}} \int\limits_0^T ds = \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)}, \tag{48}$$

where $\Delta \to 0$ if $N \to \infty$ $(N\Delta = T)$.

Obviously, (48) agrees with Theorem 7.1 (see [52], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (39) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{49}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (41).

Let us substitute (41) into (49)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{50}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (44).

As we noted above, approximations of the Wiener process that are similar to (40) were not considered in [50], [51] (also see Theorems 7.1, 7.2 in [52]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [52] to the case under consideration is not obvious.

On the other hand, we can apply Theorem 1 from this paper and obtain from (50) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} = \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{51}$$

References

[1] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.

[2] Hobson E.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge, Cambridge Univ. Press, 1931. 502 pp.

[3] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[4] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.05654 [math.PR]. 2019, 46 pp.

[5] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[6] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[7] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[8] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [In English]. arXiv:2003.14184v45 [math.PR], 2023, 996 pp.

[9] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[10] Kuznetsov D.F. Mean-Square Approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical integration of Itô SDEs and semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[11] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[12] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[13] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2023, 222 pp.

[14] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier–Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR]. 2023, 49 pp.

[15] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations

and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[16] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: St.-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[21] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[22] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.

[23] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[24] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[25] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[26] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[27] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations, based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[28] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[29] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[30] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[31] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[32] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 32, Issue 12, (2000), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[33] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[34] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.
[35] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2023, 158 pp.
[36] Kuznetsov D.F. Exact calculation of mean-square error in the method of approximation of iterated Ito stochastic integrals based on the generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2023, 71 pp.
[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604
[38] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.
[39] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html
[40] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html
[41] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100
[42] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878
[43] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [in English]. arXiv:1801.00784v20 [math.PR]. 2023, 80 pp.
[44] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2019), 32-52. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html
[45] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.
[46] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor–Stratonovich expansion. [In English]. arXiv:1806.10705 [math.PR]. 2018, 29 pp.
[47] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor–Ito expansion. [In English]. arXiv:1805.12527 [math.PR]. 2018, 29 pp.
[48] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [In English]. arXiv:1801.01564 [math.PR]. 2018, 66 pp.
[49] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[50] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[51] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[52] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[53] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. Moscow, Nauka, 1974. 696 pp.
[54] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.
[55] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.
[56] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.
[57] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.

[58] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.

[59] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[60] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[61] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[62] Kuznetsov, D.F., Kuznetsov, M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[63] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[64] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[65] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[66] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF FIFTH AND SIXTH MULTIPLICITY BASED ON GENERALIZED MULTIPLE FOURIER SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the construction of expansions of iterated Stratonovich stochastic integrals of fifth and sixth multiplicities based on the method of generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k \in \mathbb{N}$. Specifically, we use multiple Fourier–Legendre series and multiple trigonometric Fourier series. Expansions of iterated Sratonovich stochastic integrals of multiplicities 1–4 were obtained in previous works of the author. The considered expansions converges in the mean-square sense and contain only one operation of the limit transition in contrast to its existing analogues. Expansions of iterated Stratonovich stochastic integrals turned out much simpler than appropriate expansions of iterated Ito stochastic integrals. We use expansions of the latter as a tool of the proof of expansions for iterated Stratonovich stochastic integrals. Iterated Stratonovich stochastic integrals are the part of the Taylor–Stratonovich expansion of solutions of Ito stochastic differential equations. That is why the results of the article can be applied to the numerical integrations of Ito stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
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1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent.

Let us consider the following iterated Ito and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$,

$$\int \ \text{and} \ \int^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively; $i_1, \ldots, i_k = 0, 1, \ldots, m$. Note that in this paper we use the definition of the Stratonovich stochastic integral from [1].

The problem of effective jointly numerical modeling (in accordance to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (1) and (2) arises when solving the problem of numerical integration of Ito stochastic differential equations (SDEs) [1]-[4]. It is well known that this problem is difficult from theoretical and computing point of view [1]-[54]. The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case allows the investigation with using the Ito formula [1]-[4]. Note that even for the mentioned coincidence $(i_1 = \ldots = i_k \neq 0)$, but for different functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ the mentioned difficulties persist, and relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be represented effectively in a finite form (for the mean-square approximation) using the system of standard Gaussian random variables.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated stochastic integrals can be simplified but cannot be solved. The equations with additive vector noise, with additive scalar noise, with non-additive scalar noise, with a small parameter are related to such types of equations [1]-[4]. For the mentioned types of equations, simplifications are connected with the fact that some coefficient functions from stochastic analogues of the Taylor formula identically equal to zero or due to the presence of a small parameter we may neglect some members from stochastic analogues of the Taylor formula, which include difficult for approximation iterated stochastic integrals [1]-[3].

There are several approaches to solution of the problem of jointly numerical modeling (in accordance to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (1) and (2) [1]-[55].

One of the most effective methods of this problem solving is the method based on generalized multiple Fourier series, which is proposed and developed by the author in a lot of publications [7]-[45] (see Theorems 1, 2 below). It is important to note that the operation of limit transition is implemented only once in the method [7]-[45]. At the same time the existing analogues of the method [7]-[45] lead to iterated application of the operation of limit transition [1]-[6], [54].

For example, the authors of the works [1] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) use the Wong–Zakai approximation [58]-[60] (without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals [2] (1988) based on the series expansion of the Brownian bridge process (version of the so-called Karhunen-Loeve expansion). See discussion in Sect. 13 of this paper for details.

The idea of the method [7]-[45] (see Theorems 1, 2 below) is as follows: the iterated Ito stochastic integral (1) of multiplicity k ($k \in \mathbb{N}$) is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k = [t,T] \times \ldots \times [t,T]$ (k times), where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (1). Then, the mentioned nonrandom function is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (1) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of the generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (1). Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

As it turned out [8]-[15], [20]-[22], [28], [32], [35]-[42] the adaptation of Theorem 1 for the iterated Stratonovich stochastic integrals (2) of multiplicities 1 to 4 leads to relatively simple expansions compared to expansions for the appropriate iterated Ito stochastic integrals (1) (see (8)–(13) below). The developement of the mentioned adaptation for the iterated Stratonovich stochastic integrals (2) of multiplicities 5 and 6 composes the subject of this article.

In Sect. 2, we formulate Theorem 1 on expansion of iterated Ito stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on generalized multiple Fourier series [7] (2006) (see also [8]-[45]). The particular cases $k = 5, 6$ of Theorem 1 will be used for the proof of main results (Theorems 17, 22 (Sect. 8, 11)). Sect. 3 is devoted to the hypothesis (Hypothesis 1) on expansion of the iterated Stratonovich stochastic integrals (2) of arbitrary multiplicity k [12]-[15], [36]. As mentioned above, the proof of Hypothesis 1 for the cases $k = 5, 6$ composes the subject of the article. In Sect. 4, we consider several theorems (some old results), which were formulated and proved by the author. These theorems are particular cases of Hypothesis 1 for $k = 2, 3, 4$ [8]-[15], [20]-[22], [28], [32], [35]-[42]. In Sect 5, we give the proof of Hypothesis 1 under the condition of convergence of trace series. Expansions of iterated Stratonovich stochastic integrals of multiplicities 3 and 4 are considered in Sect. 6, 7. Rate of the mean-square convergence of expansions of iterated Stratonovich stochastic integrals is considered in Sect. 9, 10. Sect. 13 is devoted to a discussion of the connection between Theorems 1, 2, 5-12, 15-17, 22 and the Wong–Zakai approximation of the iterated Stratonovich stochastic integrals (2) based on the series expansion of the Wiener process with using the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. In Sect. 12, 14 we consider generalizations of the results from Sect. 5, 6.

2. Expansion of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity k Based on Generalized Multiple Fourier Series Converging in the Mean

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Theorem 2 (see below)). Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{3}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{4}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{5}$$

Theorem 1 [7] (2006) [8]-[45]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (6)$$

where $J[\psi^{(k)}]_{T,t}$ *is defined by* (1),

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g\ne r);\ g,r = 1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)} \qquad (7)$$

are independent standard Gaussian random variables for various i *or* j (*if* $i\ne 0$), $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (4), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (5).

It was shown in [17]-[22] that Theorem 1 is valid for convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) and for convergence with probability 1 [12]-[15], [45]. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorem 1 [7]-[22]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [11]-[15], [43]. The generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ is given in [12] (Sect. 1.11), [33] (Sect. 15).

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (4)) for calculation of expansion coefficients of the iterated Ito stochastic integral (1) with any fixed multiplicity k.

2. We have possibilities for explicit calculation of the mean-square approximation error of the iterated Ito stochastic integral (1) (see [10]-[15], [23], [34]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [1]-[3] but Legendre polynomials.

4. As it turned out (see [7]-[51]), it is more convenient to work with Legendre polynomials for constructing the approximations of iterated Ito and Stratonovich stochastic integrals. Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [12]-[15], [27], [31].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process [1], [2] (also see [54]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$; $i_1,i_2,i_3=1,\ldots,m$) of

the iterated Ito and Stratonovich stochastic integrals (1), (2). Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition) since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [1] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [2] together with the Wong–Zakai approximation [58]-[60].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [7]-[43]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{8}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{9}$$

$$\begin{aligned} J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \\ -\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \end{aligned} \tag{10}$$

$$\begin{aligned} J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} - \\ -\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\ +\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \\ +\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} + \\ +\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \end{aligned} \tag{11}$$

$$\begin{aligned} J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} - \\ -\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \\ -\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \end{aligned}$$

$$
\begin{gathered}
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{gathered}
\tag{12}
$$

$$
\begin{gathered}
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
\end{gathered}
$$

$$
\begin{gathered}
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
\left.-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\right), \qquad (13)
\end{gathered}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

For further consideration, let us consider the generalization of formulas (8)–(13) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (1). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{14}
$$

where

$$
\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (14) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{15}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (15)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2,g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (6) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{16}$$

where $[x]$ is an integer part of a real number x and $\prod_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum_{\emptyset} \stackrel{\sf def}{=} 0$; another notations are the same as in Theorem 1.

In particular, from (16) for $k=5$ we obtain

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -
$$

$$
- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +
$$

$$
+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).
$$

The last equality obviously agrees with (12).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [12] (Sect. 1.11), [33] (Sect. 15), [55]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansion*

$$
J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{17}
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x *and* $\prod_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum_{\emptyset} \stackrel{\sf def}{=} 0$; *another notations are the same as in Theorem* 1*.*

It should be noted that an analogue of Theorem 2 was considered in [56]. Note that we use another notations [12] (Sect. 1.11), [33] (Sect. 15) in comparison with [56]. Moreover, the proof of an analogue of Theorem 2 from [56] is somewhat different from the proof given in [12] (Sect. 1.11), [33] (Sect. 15).

Note that for the integrals $J[\psi^{(k)}]_{T,t}$ defined by (1) the mean-square approximation error can be calculated exactly and estimated efficiently.

Assume that $J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}$ is the approximation of (1), which is the expression on the right-hand side of (17) before passing to the limit

$$
J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),
$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$E_k^{p_1,\ldots,p_k} \stackrel{\rm def}{=} {\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\},$$

$$E_k^{p_1,\ldots,p_k} \stackrel{\rm def}{=} E_k^p \ \ \text{if} \ \ p_1=\ldots=p_k=p,$$

$$I_k \stackrel{\rm def}{=} \|K\|^2_{L_2([t,T]^k)}=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k.$$

In [10]-[15], [33], [34] it was shown that

$$E_k^q \le k!\left(I_k-\sum_{j_1,\ldots,j_k=0}^{q} C^2_{j_k\ldots j_1}\right) \tag{18}$$

for the following two cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$,
2. $i_1,\ldots,i_k=0,1,\ldots,m$ and $T-t\in(0,1)$.

The value E_k^p can be calculated exactly.

Theorem 3 [12] (Sect. 1.12), [34] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$E_k^p = I_k-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}{\sf M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right\}, \tag{19}$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$${\sf M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Then from Theorem 3 for pairwise different $i_1,\ldots,i_k$ and for $i_1=\ldots=i_k$ we obtain

$$E_k^p = I_k-\sum_{j_1,\ldots,j_k=0}^{p} C^2_{j_k\ldots j_1},$$

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\Biggr).$$

Another examples of the calculation of E_k^p can be found in [12], [34].

3. The Hypothesis on Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k

Note that three hypotheses on expansion of the iterated Stratonovich stochastic integrals (2) of arbitrary multiplicity k has been formulated by the author in [8]-[15], [36]. Let us consider one of the mentioned hypotheses.

Hypothesis 1 [8]-[15], [36]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l=1,2,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral* (2) *of kth multiplicity*

$$J^*[\psi^{(k)}]_{T,t} = {\int\limits_t^{*T}} \psi_k(t_k)\ldots {\int\limits_t^{*t_2}} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1,\ldots,i_k=0,1,\ldots,m) \tag{20}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{21}$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_k\ldots j_1}$ has the form

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

l.i.m. *is a limit in the mean-square sense,*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i\neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i=1,\ldots,m)$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Hypothesis 1 allows us to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{22}$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\}=0.$$

The iterated Stratonovich stochastic integrals (20) are part of the Taylor–Stratonovich expansion [1]-[3] (also see [7]-[15], [57]). It means that the approximations (22) can be useful for the numerical integration of Ito SDEs.

The expansion (21) has only one operation of the limit transition and by this reason is suitable for approximation of iterated Stratonovich stochastic integrals.

Let us consider the idea of the proof of Hypothesis 1. Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1}\stackrel{\rm def}{=}\prod_{p=1}^{l}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}}\times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)},\qquad (23)$$

where $(s_l,\ldots,s_1)\in\mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l}=\left\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;\ s_l,\ldots,s_1=1,\ldots,k-1\right\},\qquad (24)$$

where $l=1,\ldots,[k/2]$, $i_s=0,1,\ldots,m$, $s=1,\ldots,k$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Ito and Stratonovich stochastic integrals (1) and (2) of arbitrary multiplicity k.

Theorem 4 [46] (1997) (also see [7]-[15]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the following relation between iterated Ito and Stratonovich stochastic integrals (1) and (2) is correct*

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1},\qquad (25)$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero, here and further w. p. 1 *means with probability* 1.

Note that the condition of continuity of the functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ is related to the definition [1] of the Stratonovich stochastic integral that we use.

According to (6), we have

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)}=J[\psi^{(k)}]_{T,t}+$$

$$+\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}. \tag{26}$$

From (3) and (25) it follows that

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)} \tag{27}$$

if

$$\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=$$

$$=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}\quad \text{w. p. 1.}$$

In the following section we consider some theorems prooving Hypothesis 1 for the cases $k=2,3,4$. The case $k=1$ obviously follows from Theorem 1 (see (8)). The cases $k=5,6$ (see Theorems 17, 22) will be proved in Sect. 8, 11.

4. Expansions of Iterated Stratonovich Stochastc Integrals of Multiplicities 2 to 4. Some Old Results

As it turned out, approximations of the iterated Stratonovich stochastic integrals (2) (see Theorems 5–11 below) are essentially simpler than their analogues for the iterated Ito stochastic integrals (1) based on Theorems 1, 2. For the first time this fact was mentioned in [7] (2006).

We begin the consideration from the multiplicity $k=2$ since according to (8) the expansions for iterated Ito and Stratonovich stochastic integrals (1), (2) of first multiplicity are equal to each other w. p. 1.

The following theorems adapt Theorems 1, 2 for the integrals (2) of multiplicity 2 (Hypothesis 1 for the case $k=2$).

Theorem 5 [8]-[15], [20]-[22], [37]. *Suppose that the following conditions are fulfulled:*

1. *The function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the function $\psi_1(\tau)$ is twice continuously differentiable at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.*

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the converging in the mean-square sense double series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where the meaning of the notations introduced in the formulation of Theorem 1 *is saved.*

Prooving Theorem 5 [8]-[15], [20]-[22], [37] we used Theorem 1 and double integration by parts. This procedure leads to the condition of double continuously differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$. The mentioned condition can be weakened. As a result, we have the following theorem.

Theorem 6 [11]-[15], [28], [40]. *Suppose that the following conditions are fulfilled*:
1. *Every* $\psi_l(\tau)$ $(l=1,2)$ *is a continuously differentiable function at the interval* $[t,T]$.
2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the converging in the mean-square sense double series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where the meaning of the notations introduced in the formulation of Theorem 1 *is saved.*

Note that the another approaches to the proof of Theorem 6 can be found in the monographs [12]-[15] (see Chapter 2).

The following four theorems (Theorems 7–10) adapt Theorems 1, 2 for the iterated Stratonovich stochastic integrals (2) of multiplicity 3 (Hypothesis 1 for the case $k=3$). The notations used in Theorems 7–10 are the same as in Theorems 1, 2.

Theorem 7 [8]-[15], [20]-[22], [39]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$
\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}
$$

that is converges in the mean-square sense is valid, where

$$
C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3.
$$

Theorem 8 [8]-[15], [20]-[22], [39]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)} = \int\limits_t^{*T}(t-t_3)^{l_3}\int\limits_t^{*t_3}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)
$$

the following expansion

$$
I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}
$$

that is converges in the mean-square sense is valid for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
2. $i_1=i_2\ne i_3$ *and* $l_1=l_2\ne l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
3. $i_1\ne i_2=i_3$ *and* $l_1\ne l_2=l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
4. $i_1,i_2,i_3=1,\ldots,m;\ l_1=l_2=l_3=l$ *and* $l=0,1,2,\ldots,$

where

$$
C_{j_3j_2j_1} = \int\limits_t^T (t-t_3)^{l_3}\phi_{j_3}(t_3)\int\limits_t^{t_3}(t-t_2)^{l_2}\phi_{j_2}(t_2)\int\limits_t^{t_2}(t-t_1)^{l_1}\phi_{j_1}(t_1)dt_1dt_2dt_3.
$$

Theorem 9 [8]-[15], [20]-[22]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_l(\tau)$ $(l=1,2,3)$ are continuously differentiable functions at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)
$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{28}$$

that is converges in the mean-square sense is valid for each of the following cases

1. $i_1 \neq i_2,\ i_2 \neq i_3,\ i_1 \neq i_3$,
2. $i_1 = i_2 \neq i_3$ *and* $\psi_1(\tau) \equiv \psi_2(\tau)$,
3. $i_1 \neq i_2 = i_3$ *and* $\psi_2(\tau) \equiv \psi_3(\tau)$,
4. $i_1, i_2, i_3 = 1, \ldots, m$ *and* $\psi_1(\tau) \equiv \psi_2(\tau) \equiv \psi_3(\tau)$,

where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3.$$

Theorem 10 [9]-[15], [22], [37]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space $L_2([t,T])$. Furthermore, let the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the functions $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{29}$$

that is converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3.$$

The following theorem adapts Theorems 1, 2 for the iterated Stratonovich stochastic integrals (2) of multiplicity 4 (Hypothesis 1 for the case $k=4$).

Theorem 11 [9]-[15], [22], [37]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1i_2i_3i_4)} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$I_{T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}$$

that is converges in the mean-square sense is valid, where

$$C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4,$$

$\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ *are independent standard Wiener processes and* $\mathbf{w}_\tau^{(0)} = \tau$; *another notations are the same as in Theorems* 1, 2.

5. Proof of Hypothesis 1 Under the Condition of Convergence of Trace Series

In this section, we prove the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k $(k \in \mathbb{N})$ under the condition of convergence of trace series. Let us recall some notations.

Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{30}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

Consider the sum with respect to all possible partitions (30)

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r}, q_1 \ldots q_{k-2r}}$$

and the Fourier coefficient

$$C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k \tag{31}$$

corresponding to the function (3), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. At that we suppose $\phi_0(x) = 1/\sqrt{T-t}$.

Denote

$$C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots j_1} \Bigg|_{(j_l j_l) \frown (\cdot)} \stackrel{\mathsf{def}}{=} \tag{32}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k=$$

$$=\sqrt{T-t}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k=$$

$$=\sqrt{T-t}\hat C_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat C_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Let

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Biggr|_{(j_lj_l)\curvearrowright j_m}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k= \qquad (33)$$

$$=\bar C_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1},$$

i.e. $\bar C_{j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}j_mj_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ldots,\psi_{l-2}(\tau),\psi_{l-1}(\tau)\psi_l(\tau),\psi_{l+1}(\tau),\ldots,$ $\psi_k(\tau)$ (also we suppose that $\{l-1,l\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$).

Denote

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (34)$$

Introduce the following notation

$$S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}{\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}\sum\limits_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum\limits_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum\limits_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum\limits_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum\limits_{j_{g_{3}}=p+1}^{\infty}\ \sum\limits_{j_{g_{1}}=p+1}^{\infty}C_{j_k\ldots j_1}\biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\frown(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{35}$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{36}$$

as follows: S_l multiplies (36) by ${\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum\limits_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k\ldots j_1}\biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\frown(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{37}$$

Note that we write

$$C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_2})\frown(\cdot),j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_1})\frown(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_2})\frown j_m,j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_1})\frown j_m,j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_2})\frown(\cdot),(j_{g_3}j_{g_4})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=C_{j_k\ldots j_1}\biggr|_{(j_{g_1}j_{g_1})\frown(\cdot)(j_{g_3}j_{g_3})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}},\quad\ldots$$

Since (37) is again the Fourier coefficient, then the action of superposition S_lS_m on (37) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^3}\prod_{s=1}^{3}{\bf 1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot)(j_{g_6}j_{g_5})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3 S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_6}j_{g_5})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.$$

Theorem 12 [12], [36], [37], [51]. *Assume that the continuously differentiable functions* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions* $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \qquad (38)$$

holds for all $s\in(t,T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t,T]$ *and the series on the right-hand side of* (38) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s\in(t,T)$ *and for some* $\alpha,\beta>0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$, $j,p\in\mathbb{N}$, *and*

$$\int\limits_t^T\Psi_1^2(\tau)d\tau<\infty,\quad \int\limits_t^T\left|\Psi_2(\tau)\right|d\tau<\infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (30)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} \stackrel{\text{def}}{=} \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]^{(i_1 \ldots i_k)}_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k} \tag{39}$$

the following expansion

$$J^*[\psi^{(k)}]^{(i_1 \ldots i_k)}_{T,t} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta^{(i_l)}_{j_l} \tag{40}$$

that converges in the mean-square sense is valid, where

$$C_{j_k \ldots j_1} = \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k \tag{41}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta^{(i)}_j = \int\limits_t^T \phi_j(\tau) d\mathbf{w}^{(i)}_\tau$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}^{(i)}_\tau = \mathbf{f}^{(i)}_\tau$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}^{(0)}_\tau = \tau$.

Proof. First note that (38) is fulfilled (see [12], Sect. 2.1.4 or [75]). The proof of Theorem 12 will consist of several steps.

Step 1. Let us find a representation of the quantity

$$\sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta^{(i_l)}_{j_l}$$

that will be convenient for further consideration.

Let us consider the following multiple stochastic integral

$$\underset{N \to \infty}{\text{l.i.m.}} \sum_{\substack{j_1, \ldots, j_k = 0 \\ j_q \ne j_r;\ q \ne r;\ q, r = 1, \ldots, k}}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta \mathbf{w}^{(i_l)}_{\tau_{j_l}} \stackrel{\text{def}}{=} J'[\Phi]^{(i_1 \ldots i_k)}_{T,t}, \tag{42}$$

where for simplicity we assume that $\Phi(t_1, \ldots, t_k) : [t, T]^k \to \mathbb{R}$ is a continuous nonrandom function on $[t, T]^k$. Moreover, $\Delta \mathbf{w}^{(i)}_{\tau_j} = \mathbf{w}^{(i)}_{\tau_{j+1}} - \mathbf{w}^{(i)}_{\tau_j}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t, T]$, which satisfies the condition (5), $i_1, \ldots, i_k = 0, 1, \ldots, m$.

The stochastic integral with respect to the scalar standard Wiener process ($i_1 = \ldots = i_k \neq 0$) and similar to (42) was considered in [67] (1951) and is called the multiple Wiener stochastic integral [67].

Note that the following well known estimate

$$\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le C_k \int\limits_{[t,T]^k} \Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k \tag{43}$$

is true for the multiple Wiener stochastic integral, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (42) and C_k is a constant.

From the proof of Theorem 1 (see the proof of Theorem 5.1 in the original paper [7] (2006) in Russian or proof of Theorems 1.1, 1.16 in the monograph [12] in English) it follows that (6), (17) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{44}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (42) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (1), i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}.$$

Consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^k \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J[\Phi]_{T,t}^{(i_1\ldots i_k)}, \tag{45}$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Another notations are the same as in (42).

The stochastic integral with respect to the scalar standard Wiener process ($i_1 = \ldots = i_k \neq 0$) and similar to (45) (the function $\Phi(t_1,\ldots,t_k)$ is assumed to be symmetric on the hypercube $[t,T]^k$) has been considered in the literature (see, for example, Remark 1.5.7 [68]). The integral (45) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)} = J[\varphi_1]_{T,t}^{(i_1)}\ldots J[\varphi_k]_{T,t}^{(i_k)} \quad \text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k)=\varphi_1(t_1)\ldots\varphi_k(t_k)$ and

$$J[\varphi_l]_{T,t}^{(i_l)} = \int\limits_t^T \varphi_l(\tau)d\mathbf{w}_\tau^{(i_l)} \quad (l=1,\ldots,k).$$

Theorem 13 [12], [14]. *Suppose that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Furthermore, let $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j is continuous at the interval $[t,T]$ except* *may be for the finite number of points of the finite discontinuity as well as $\phi_j(x)$ is right-continuous at the interval $[t,T]$. Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{46}$$

converging in the mean-square sense is valid, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (42),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{47}$$

is the Fourier coefficient. Another notations are the same as in Theorems 1, 2.

From (17) and (44) we conclude that

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+$$

$$+\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{48}$$

w. p. 1, where notations are the same as in Theorem 2 and $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (42).

Using (48), we obtain

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}-$$

$$-\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{49}$$

w. p. 1.

By iteratively applying the formula (49) (also see (9)–(13)), we obtain the following representation of the product

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1,}\tag{50}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$.

Multiplying both sides of the equality (50) by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get w. p. 1

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.}\tag{51}$$

Denote

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\tag{52}$$

$$K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}),\tag{53}$$

where $C_{j_k\ldots j_1}$ is defined by (41) and $\prod_{\emptyset}\stackrel{\sf def}{=}1$.

The equality (51) can be written as

$$J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} = J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\ J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{54}$$

w. p. 1, where $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$ have the form (52), (53), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral defined by (45), $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined by (42).

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (51) or (54), we get w. p. 1 (see (44))

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} = \tag{55}$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \underset{p\to\infty}{\text{l.i.m.}} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{56}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (1).

If we prove that w. p. 1

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}, \tag{57}$$

then (see (55), (57), and Theorem 4)

(58)

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$$

w. p. 1, where notations in (58) are the same as in Theorem 4. Thus Theorem 12 will be proved.

From (54) we have that the multiple Stratonovich stochastic integral $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k is expressed as a sum of some constant value and multiple Wiener stochastic integrals $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ of multiplicities $k,\ k-2,\ k-4,\ \ldots,\ k-2[k/2]$ $(r=1,2,\ldots,[k/2])$.

The formulas (51), (54) can be considered as new representations of the Hu-Meyer formula for the case of a multidimensional Wiener process [69] (also see [68], [70]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (52)).

Note that the equality (54) can be obtained from (46) if we consider (46) for $\Phi(t_1,\ldots,t_k) = K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and without passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$

For $k=2,3,4,5,6$ from (51) we have w. p. 1

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} = J'[K_{p_1p_2}]_{T,t}^{(i_1i_2)} + \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}, \tag{59}$$

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)} +$$
$$+\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}]_{T,t}^{(i_3)} + \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}]_{T,t}^{(i_1)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}\Biggr), \tag{60}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J'[K_{p_1p_2p_3p_4}]_{T,t}^{(i_1i_2i_3i_4)} +$$
$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)} +$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)} +$$
$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{61}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} = J'[K_{p_1p_2p_3p_4p_5}]_{T,t}^{(i_1i_2i_3i_4i_5)}+$$
$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_3i_4i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_4i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_2i_3i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2i_3i_4)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_4i_5)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_3i_5)}+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_3i_4)}+$$
$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_1i_2i_5)}+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1i_2i_4)}+$$
$$+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_2i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}\Biggr), \tag{62}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6} C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} = J'[K_{p_1p_2p_3p_4p_5p_6}]_{T,t}^{(i_1i_2i_3i_4i_5i_6)}+$$
$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6} C_{j_6j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_3i_4i_5)}+$$
$$+\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_3i_4i_5)}+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_2i_4i_5)}+$$
$$+\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_2i_3i_5)}+\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_3i_4i_5i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_4i_5i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_3i_5i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_2i_3i_4i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_4i_5i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_3i_5i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_3i_4i_6)}+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_2i_5i_6)}+$$
$$+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_2i_4i_6)}+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_1i_2i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\neq 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\neq 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\neq 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\neq 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\neq 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\neq 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_3=i_6\neq 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr). \tag{63}$$

Note that the relation (61) can be written in the following form

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}=$$

$$
=\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4j_3j_2j_1}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{p_4}\left(\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_4j_3j_1j_1}\right)J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_1,p_3\}}C_{j_4j_3j_2j_3}\right)J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_1,p_4\}}C_{j_4j_3j_2j_4}\right)J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_4j_3j_3j_1}\right)J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_2,p_4\}}C_{j_4j_3j_4j_1}\right)J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(\sum_{j_4=0}^{\min\{p_3,p_4\}}C_{j_4j_4j_2j_1}\right)J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{\min\{p_2,p_3\}}\sum_{j_4=0}^{\min\{p_1,p_4\}}C_{j_4j_2j_2j_4}+
$$

$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_3=0}^{\min\{p_1,p_3\}}\sum_{j_4=0}^{\min\{p_2,p_4\}}C_{j_4j_3j_4j_3}+
$$

$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_2=0}^{\min\{p_1,p_2\}}\sum_{j_4=0}^{\min\{p_3,p_4\}}C_{j_4j_4j_2j_2}\quad \text{w. p. 1.}
$$

Step 2. Let us prove that

$$
\sum_{j_l=0}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=0 \tag{64}
$$

or

$$
\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}, \tag{65}
$$

where $l-1\ge s+1$.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

We have

$$C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_sdt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s\times$$

$$\times\Biggl(\int\limits_{t_{s+1}}^T\phi_{j_{s+2}}(t_{s+2})\ldots\int\limits_{t_{l-2}}^T\phi_{j_{l-1}}(t_{l-1})\int\limits_{t_{l-1}}^T\phi_{j_l}(t_l)\int\limits_{t_l}^T\phi_{j_{l+1}}(t_{l+1})\ldots$$

$$\ldots\int\limits_{t_{k-1}}^T\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_ldt_{l-1}\ldots dt_{s+2}\Biggr)dt_{s+1}=$$

$$=\int\limits_t^T\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\underbrace{\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)}dt_s\times$$

$$\times\int\limits_{t_{s+1}}^T\phi_{j_l}(t_l)\underbrace{\int\limits_{t_l}^T\phi_{j_{l+1}}(t_{l+1})\ldots\int\limits_{t_{k-1}}^T\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}}_{H_{j_k\ldots j_{l+1}}(t_l)}\times$$

$$\times\left(\underbrace{\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}}_{Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})}dt_l\right)dt_{s+1}=$$

$$=\int\limits_t^T\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times$$

$$\times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k \ldots j_{l+1}}(t_l) Q_{j_{l-1} \ldots j_{s+2}}(t_l, t_{s+1}) dt_l dt_{s+1}. \tag{66}$$

Using the additive property of the integral, we obtain

$$Q_{j_{l-1} \ldots j_{s+2}}(t_l, t_{s+1}) =$$

$$= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} \ldots dt_{l-1} =$$

$$= \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3}) \int\limits_{t}^{t_{s+3}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} dt_{s+3} \ldots dt_{l-1} -$$

$$- \int\limits_{t_{s+1}}^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots \int\limits_{t_{s+1}}^{t_{s+4}} \phi_{j_{s+3}}(t_{s+3}) dt_{s+3} \ldots dt_{l-1} \int\limits_{t}^{t_{s+1}} \phi_{j_{s+2}}(t_{s+2}) dt_{s+2} =$$

$$\ldots$$

$$= \sum_{m=1}^{d} h^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_l) q^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_{s+1}), \quad d < \infty. \tag{67}$$

Combining (66) and (67), we have

$$\sum_{j_l=0}^{p} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} =$$

$$= \sum_{m=1}^{d} \left(\int\limits_{t}^{T} \phi_{j_{s+1}}(t_{s+1}) q^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_{s+1}) \sum_{j_l=0}^{p} \int\limits_{t}^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1} \ldots j_1}(t_s) dt_s \times \right.$$

$$\left. \times \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k \ldots j_{l+1}}(t_l) h^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_l) dt_l dt_{s+1} \right). \tag{68}$$

Using the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty} \int\limits_{t}^{t_{s+1}} \phi_{j_l}(t_s) G_{j_{s-1} \ldots j_1}(t_s) dt_s \int\limits_{t_{s+1}}^{T} \phi_{j_l}(t_l) H_{j_k \ldots j_{l+1}}(t_l) h^{(m)}_{j_{l-1} \ldots j_{s+2}}(t_l) dt_l =$$

$$= \int\limits_{t}^{T} \mathbf{1}_{\{\tau < t_{s+1}\}} G_{j_{s-1} \ldots j_1}(\tau) \cdot \mathbf{1}_{\{\tau > t_{s+1}\}} H_{j_k \ldots j_{l+1}}(\tau) h^{(m)}_{j_{l-1} \ldots j_{s+2}}(\tau) d\tau = 0. \tag{69}$$

From (68) and (69) we get

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} =$$

$$= -\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right). \tag{70}$$

Combining Condition 2 of Theorem 12 and (66)–(68), (70), we have

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} =$$

$$= -\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\left(\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right) =$$

$$= -\sum_{j_l=p+1}^{\infty}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_sdt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k =$$

$$= -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}. \tag{71}$$

The equality (71) implies (64), (65).

Step 3. Using Conditions 1 and 2 of Theorem 12, we obtain

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1} =$$

$$
=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times\sum_{j_l=0}^{p}\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times\int\limits_t^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k=
$$

$$
=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\times
$$

$$
\times\sum_{j_l=0}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times\int\limits_t^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k-
$$

$$
-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=
$$

$$
=\frac{1}{2}C_{j_k\ldots j_1}\Biggr|_{(j_lj_l)\curvearrowright(\cdot)}-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}. \tag{72}
$$

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (51), we have (see (44))

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+
$$

$$
+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \tag{73}
$$

Taking into account (65) and (72), we obtain for $r=1$

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2>g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+$$

$$+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}-$$

$$-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=-\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+$$

$$+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}= \tag{74}$$

$$=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{g_1}+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)1,g_1,g_2}\quad \text{w. p. 1}, \tag{75}$$

where $J[\psi^{(k)}]_{T,t}^{g_1}$ $(g_1=1,2,\ldots,k-1)$ is defined by (23),

$$R_{T,t}^{(p)1,g_1,g_2}=-\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl.\Biggr|_{q\ne g_1,g_2}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}.$$

Let us explain the transition from (74) to (75). We have for $g_2=g_1+1$

$$
\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright 0,j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times\zeta_{j_{m_1}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=
$$

$$
=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times
$$

$$
\times J'[\phi_{j_{m_1}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(0 i_{q_1}\ldots i_{q_{k-2}})}= \tag{76}
$$

$$
=\frac{1}{2}J[\psi^{(k)}]_{T,t}^{g_1}\quad \hbox{w. p. 1}, \tag{77}
$$

where

$$
C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2},g_2=g_1+1}=
$$

$$
=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{g_1+3}}\psi_l(t_{g_1+2})\phi_{j_{g_1+2}}(t_{g_1+2})\int\limits_t^{t_{g_1+2}}\psi_{g_1+1}(t_{g_1})\psi_{g_1}(t_{g_1})\phi_{j_{m_1}}(t_{g_1})\times
$$

$$
\times\int\limits_t^{t_{g_1}}\psi_l(t_{g_1-1})\phi_{j_{g_1-1}}(t_{g_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{g_1-1}dt_{g_1}dt_{g_1+2}\ldots dt_k,
$$

$$
\zeta_{j_{m_1}}^{(0)}=\int\limits_t^T\phi_{j_{m_1}}(\tau)d\mathbf{w}_{\tau}^{(0)}=\int\limits_t^T\phi_{j_{m_1}}(\tau)d\tau=\begin{cases}\sqrt{T-t} & \hbox{if } j_{m_1}=0\\ 0 & \hbox{if } j_{m_1}\ne 0\end{cases},
$$

$$\phi_0(\tau)=\frac{1}{\sqrt{T-t}}.$$

The transition from (76) to (77) is based on (44).

By Condition 3 of Theorem 12 we have (also see the property (43) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)1,g_1,g_2}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2}\right)^2=0,$$

where constant K does not depend on p.

Thus

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{g_1}\quad \text{w. p. 1.}$$

Involving into consideration the second pair $\{g_3,g_4\}$ (the first pair is $\{g_1,g_2\}$), we obtain from (74) for $r=2$

$$\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{2}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=$$

$$=\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\left(\frac{1}{4}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}-\right.$$

$$-\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\mathbf{1}_{\{g_4=g_3+1\}}-$$

$$-\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\mathbf{1}_{\{g_2=g_1+1\}}+$$

$$\left.+\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\right)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}= \tag{78}$$

$$
= \frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_2,s_1} + \prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)2,g_1,g_2,g_3,g_4} \tag{79}
$$

w. p. 1, where $g_3 \stackrel{\textsf{def}}{=} s_2$, $g_1 \stackrel{\textsf{def}}{=} s_1$, $(s_2,s_1)\in \mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (23) and $\mathrm{A}_{k,2}$ is defined by (24),

$$
R_{T,t}^{(p)2,g_1,g_2,g_3,g_4} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4} -\right.
$$

$$
\left. -S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\} - S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}.
$$

Let us explain the transition from (78) to (79). We have for $g_2=g_1+1$, $g_4=g_3+1$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\frac{1}{4}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright 0(j_{g_4}j_{g_3})\curvearrowright 0,j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\zeta_0^{(0)}\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\sum_{j_{m_1},j_{m_3}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\frac{1}{4}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\ \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(00i_{q_1}\ldots i_{q_{k-4}})}= \tag{80}
$$

$$
=\frac{1}{4}J[\psi^{(k)}]_{T,t}^{s_2,s_1}\quad \text{w. p. 1.} \tag{81}
$$

The transition from (80) to (81) is based on (44).

Note that

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}
$$

is the Fourier coefficient, where $g_2=g_1+1$. Therefore, the value

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}=
$$

$$
=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}(j_{g_3}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}
$$

is determined recursively using (33) in an obvious way for $g_2=g_1+1$ and $g_4=g_3+1$.

By Condition 3 of Theorem 12 we have (also see the property (43) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right)^2+\right.
$$

$$
\left.+\left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2+\left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2\right)=0,
$$

where constant K is independent of p.

Thus

$$
\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{2}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=\frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_2,s_1}\quad \text{w. p. 1,}
$$

where $g_3 \stackrel{\sf def}{=} s_2,\ g_1 \stackrel{\sf def}{=} s_1,\ (s_2,s_1)\in {\rm A}_{k,2},\ J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (23) and ${\rm A}_{k,2}$ is defined by (24).

Involving into consideration the third pair $\{g_6,g_5\}$ ($\{g_1,g_2\}$ is the first pair and $\{g_4,g_3\}$ is the second pair), we obtain from (78) for $r=3$

$$\prod_{s=1}^{3}{\bf 1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr{\rm l.i.m.}\cr$\scriptstyle p\to\infty$\cr}}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{3}{\bf 1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=\prod_{s=1}^{3}{\bf 1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times$$

$$\times\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr{\rm l.i.m.}\cr$\scriptstyle p\to\infty$\cr}}}\sum_{\stackrel{j_1,\ldots,j_q,\ldots,j_k=0}{q\ne g_1,g_2,g_3,g_4,g_5,g_6}}^{p}\left(\frac{1}{2^3}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}\times\right.$$

$$\times\prod_{s=1}^{3}{\bf 1}_{\{g_{2s}=g_{2s-1}+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_4=g_3+1\}}{\bf 1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_2=g_1+1\}}{\bf 1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_2=g_1+1\}}{\bf 1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_6=g_5+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}{\bf 1}_{\{g_2=g_1+1\}}-$$

$$\left.-\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Bigl.\Bigr|_{j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4},j_{g_5}=~j_{g_6}}\right)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=$$

$$
=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}+\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}
$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,3,\ (s_3,s_2,s_1)\in \mathrm{A}_{k,3},\ J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (23) and $\mathrm{A}_{k,3}$ is defined by (24),

$$
R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p}\left(-\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}+\right.
$$

$$
+S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}+S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}+
$$

$$
+S_3\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-
$$

$$
-S_3S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-S_3S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}-
$$

$$
\left.-S_2S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}.
$$

By Condition 3 of Theorem 12 we have (also see the property (43) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p}\left(\left(\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right)^2+\right.
$$

$$
+\left(S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+\left(S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$
+\left(S_3\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$
+\left(S_3S_1\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+\left(S_3S_2\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2+
$$

$$+\left(S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2\Biggr)=0,$$

where constant K does not depend on p.

Thus

$$\underset{p\to\infty}{\text{l.i.m.}}\ \prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{3}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}\quad\text{w. p. 1},$$

where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,3,\ (s_3,s_2,s_1)\in \mathrm{A}_{k,3},\ J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (23) and $\mathrm{A}_{k,3}$ is defined by (24).

Repeating the previous steps, we obtain for an arbitrary r $(r=1,2,\ldots,[k/2])$

$$\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

(82) $$+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=$$

(83) $$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}+\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2],\ (s_r,\ldots,s_1)\in\mathrm{A}_{k,r},\ J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (23) and $\mathrm{A}_{k,r}$ is defined by (24),

$$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl((-1)^r\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r}S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \tag{84}$$

Let us explain the transition from (82) to (83). We have for $g_2=g_1+1,\ldots,g_{2r}=g_{2r-1}+1$

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\times$$

$$\times C_{j_k\ldots j_1}\Bigr|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times$$

$$\times\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}\ldots\zeta_{j_{m_{2r-1}}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown j_{m_{2r-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots 0i_{q_1}\ldots i_{q_{k-2r}})}= \tag{85}$$

$$=\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1.} \tag{86}$$

The transition from (85) to (86) is based on (44).

Note that

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown j_{m_1},j_{g_1}=\ j_{g_2}}=C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\frown j_{m_1},j_{g_1}=\ j_{g_2}}$$

is the Fourier coefficient, where $g_2=g_1+1$. Therefore, the value

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown j_{m_1}\ldots(j_{g_{2d}}j_{g_{2d-1}})\frown j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}=$$

$$=C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\frown j_{m_1}\ldots(j_{g_{2d-1}}j_{g_{2d-1}})\frown j_{m_{2d-1}},j_{g_1}=\ j_{g_2},\ldots,j_{g_{2d-1}}=\ j_{g_{2d}}}$$

is determined recursively using (33) in an obvious way for $g_2=g_1+1,\ \ldots,\ g_{2d}=g_{2d-1}+1$ and $d=2,\ldots,r$.

By Condition 3 of Theorem 12 we have (also see the property (43) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\le$$

$$\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2+\right.$$

$$+\sum_{l_1=1}^{r}\left(S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$+\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2+$$

$$\ldots$$

$$\left.+\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right)=0,$$

where constant K does not depend on p.

So we have

$$\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\text{w. p. 1,}\tag{87}$$

where $g_{2i-1}\stackrel{\text{def}}{=}s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2]\,,\ (s_r,\ldots,s_1)\in \mathrm{A}_{k,r},\ J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (23) and $\mathrm{A}_{k,r}$ is defined by (24).

Note that

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\Bigg|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1}A_{g_1,g_3,\ldots,g_{2r-1}}=$$

$$=\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}A_{s_1,s_2,\ldots,s_r},\tag{88}$$

where $A_{g_1,g_3,\ldots,g_{2r-1}}$, $A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1}=s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2]\,,\ \mathrm{A}_{k,r}$ is defined by (24):

$$\mathrm{A}_{k,r}=\big\{(s_r,\ldots,s_1):\ s_r>s_{r-1}+1,\ldots,s_2>s_1+1,\ s_r,\ldots,s_1=1,\ldots,k-1\big\}.$$

Using (73), (87), (88), and Theorem 4, we finally get

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{89}$$

w. p. 1, where (see (23))

$$J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\rm def}{=} \prod_{p=1}^{r} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_r+3}} \psi_{s_r+2}(t_{s_r+2}) \int\limits_t^{t_{s_r+2}} \psi_{s_r}(t_{s_r+1})\psi_{s_r+1}(t_{s_r+1})\times$$

$$\times \int\limits_t^{t_{s_r+1}} \psi_{s_r-1}(t_{s_r-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots\ d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}dt_{s_r+1}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{90}$$

Theorem 12 is proved.

Let us make a number of remarks about Theorem 12. An expansion similar to (40) was obtained in [69], where the author used the definition (387) of the Stratonovich stochastic integral, which differs from the definition we use in this article [1]. The proof from [69] is somewhat simpler than the proof proposed in this work. However, the results from [69] were obtained under the condition of convergence of trace series. The verification of this condition for the kernel (3) is a separate problem. In our proof, we essentially use the structure of the Fourier coefficients (31) corresponding to the kernel $K(t_1,\ldots,t_k)$ of the form (3). This circumstance actually made it possible to prove Theorem 12 using not the condition of finiteness of trace series, but using the condition of convergence to zero of explicit expressions for the remainders of the mentioned series. This leaves hope that it is possible to prove analog of Theorems 2.35–2.37 [12], [14] on the rate of the mean-square convergence of approximations of iterated Stratonovich stochastic integrals for the case of arbitrary k ($k\in\mathbb{N}$).

Note that under the conditions of Theorem 12 (also see (65), (72)) the sequential order of the series

$$\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}$$

is not important.

We also note that the first and second conditions of Theorem 12 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ (see the proofs of Theorems 5–11 (Theorems 2.1–2.8 in [12]-[15])). It is easy to see that in the proofs of Theorems 5–11 (Theorems 2.1–2.8 in [12]-[15]) the conditions of Theorem 12 are verified for various special cases of iterated Stratonovich stochastic integrals of multiplicities 2–4 with respect to components of the multidimensional Wiener process.

It should be noted that (see (84))

$$(-1)^r \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=$$

$$=\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},\qquad (91)$$

where the meaning of the notations used in (84) is preserved.

For example, from (91) for the case $r=2$ we get

$$\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\,j_{g_3}=j_{g_4}}-$$

$$-\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\,j_{g_3}=j_{g_4}}-$$

$$-\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\,j_{g_3}=j_{g_4}}=$$

$$=\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\,j_{g_3}=j_{g_4}}-$$

$$-\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),\,j_{g_1}=j_{g_2},\,j_{g_3}=j_{g_4}}.$$

As a result, Condition 3 of Theorem 12 can be replaced by a weaker condition

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=0, \tag{92}$$

where $r=1,2,\ldots,[k/2]$.

However, Condition 3 of Theorem 12 itself contains a way of proving of the condition (92), which is partially realized in the proof of Theorems 15–17, 22 (see below).

In fact, when proving Theorem 17 (the case $r=3$ is proved in Theorem 22 for $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$), we proved the following equality

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=$$

$$=\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4}}.$$

On the other hand, iterative application of (72) gives

$$\sum_{j_{g_1}=0}^{\infty}\ldots\sum_{j_{g_{2r-1}}=0}^{\infty}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}},$$

where $r=1,2,\ldots,[k/2]$.

Taking into account the generalization of Theorem 1 for the case of integration interval $[t,s]$ ($s\in(t,T]$) of iterated Ito stochastic integrals (see Theorems 1.11, 1.24 in [12]), we can formulate an analogue of Theorem 12 for the case of integration interval $[t,s]$ ($s\in(t,T]$) of iterated Stratonovich stochastic integrals of multiplicity k ($k\in\mathbb{N}$).

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

and introduce the following notation

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}{\bf 1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}(s)\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

where $l=1,2,\ldots,r$,

$$C_{j_k\ldots j_1}(s)\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot)}$$

is defined by analogy with (32),

$$C_{j_k\ldots j_1}(s)=\int\limits_t^s\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k. \tag{93}$$

Theorem 14 [12], [36], [37], [51]. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{94}$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (94) converges absolutely.

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_\tau^s\phi_j(\theta)\Phi_2(\theta)d\theta\right|\le\frac{\Psi_2(s,\tau)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_3(s)}{p^{\beta}}$$

hold for all s,τ such that $t<\tau<s<T$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$\int\limits_t^s|\Psi_1(\tau)\Psi_2(s,\tau)|\,d\tau<\infty,\quad \int\limits_t^s|\Psi_3(\tau)|\,d\tau<\infty$$

for all $s\in(t,T)$.

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (30)*) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^{*s}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\tag{95}$$

the following expansion

$$J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}(s)$ *is the Fourier coefficient* (93), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m$, $s\in(t,T)$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

In Sect. 2.1.2 of the monograpths [12]–[15], the following formula is proved

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1},\tag{96}$$

where

$$C_{j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$.

Moreover (see Sect. 2.1.2 of the monograpths [12]–[15]), the following estimate

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}\right| \le \frac{C}{p}, \tag{97}$$

holds under the above assumptions, where constant C does not depend on p.

The relations (96) and (97) have been modified for the Legendre polynomial system as follows (see Sect. 2.8, 2.13 of the monograpth [14])

$$\frac{1}{2}\int_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}(s), \tag{98}$$

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le \frac{C}{p}\left(\frac{1}{\left(1-z^2(s)\right)^{1/4}}+1\right), \tag{99}$$

where $s \in (t,T)$ (s is fixed, the case $s=T$ corresponds to (96) and (97)), constant C does not depend on p, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$,

$$C_{j_1 j_1}(s) = \int_t^s \psi_2(t_2)\phi_{j_1}(t_2)\int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2,$$

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}. \tag{100}$$

For the trigonometric case, the estimate (99) is replaced by

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le \frac{C}{p}, \tag{101}$$

where $s \in [t,T]$, constant C does not depend p.

Note the well known estimate for the Legendre polynomials

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1,1), \quad j \in \mathbb{N}, \tag{102}$$

where $P_j(y)$ is the Legendre polynomial, constant K does not depend on y and j.

We also note the following useful estimates for the case of Legendre polynomials ([12]-[15], Chapters 1, 2)

$$\left|\int_t^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{103}$$

$$\left|\int_x^T \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{104}$$

$$\left|\int\limits_{v}^{x}\psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+1\right), \tag{105}$$

where $j\in\mathbb{N}$, $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t,T]$, constant C does not depend on j.

For the case of trigonometric functions we note the following obvious estimates

$$\left|\int\limits_{t}^{x}\psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{106}$$

$$\left|\int\limits_{x}^{T}\psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{107}$$

$$\left|\int\limits_{v}^{x}\psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{108}$$

where $j\in\mathbb{N}$, $x,v\in[t,T]$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t,T]$, constant C is independent of j.

It is easy to see that the estimates (99), (101), (103), (105), (106), (108) imply the fulfillment of Condition 2 of Theorem 14 for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Also the equality (98) and its analogue for the trigonometric case as well as the equatily (96) guarantee the fulfillment of Condition 1 of Theorems 12, 14 for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ (see the proof of Theorems 2.27, 2.38 [14]). Furthermore, Condition 2 of Theorem 12 follow from (97), (103), (104), (106), (107).

Recently, the equality (96) is proved for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ in [75] or [12] (Sect. 2.1.4).

6. Weakening of the Conditions of Theorem 10. Simple Proof Based on Theorem 12

In this section, we present a simple proof of Theorem 10 based on Theorem 12. In this case, the conditions of Theorem 10 will be weakened.

First, consider the following equalities

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int\limits_{t_1}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau, \tag{109}$$

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_1(\theta)\phi_j(\theta)\int\limits_{\theta}^{t_2}\Phi_2(\tau)\phi_j(\tau)d\tau d\theta \tag{110}$$

that will be used further, where $t \le t_1 < t_2 \le T$, $\Phi_1(\tau), \Phi_2(\tau) \in L_2([t,T])$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of funtions in $L_2([t,T])$.

The equality (110) is proved in Sect. 2.7.2 [12]. Using (110) and Fubini's Theorem, we get (109).

Theorem 15 [12], [36], [37], [51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \tag{111}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1, i_3, i_3 = 0, 1, \ldots, m$,

$$C_{j_3 j_2 j_1} = \int\limits_t^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^{T} \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As noted above (see Sect. 5), Conditions 1 and 2 of Theorem 12 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 12 for the iterated Stratonovich stochastic integral (111). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_3 j_1 j_1} \right)^2 = 0, \tag{112}$$

$$\lim_{p\to\infty} \sum_{j_1=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1} \right)^2 = 0, \tag{113}$$

$$\lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1} \right)^2 = 0. \tag{114}$$

We have

$$\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_3 j_1 j_1}\right)^2 =$$

$$= \sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 = \tag{115}$$

$$= \sum_{j_3=0}^{p}\left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 \le \tag{116}$$

$$\le \sum_{j_3=0}^{\infty}\left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 = \tag{117}$$

$$= \int\limits_t^T \psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2 dt_3 \le \tag{118}$$

$$\le \frac{K}{p^2} \ \to\ 0 \tag{119}$$

if $p\to\infty$, where constant K does not depend on p.

Note that the transition from (115) to (116) is based on the estimate (99) for the polynomial case and its analogue (101) for the trigonometric case, the transition from (117) to (118) is based on the Parseval equality, and the transition from (118) to (119) is also based on the estimate (99) and its analogue (101) for the trigonometric case.

By analogy with the previous case we have

$$\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1}\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 =$$

$$= \sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} \int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 = \tag{120}$$

$$= \sum_{j_1=0}^{p}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 \le \tag{121}$$

$$\le \sum_{j_1=0}^{\infty}\left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2dt_1\right)^2 =$$

$$=\int\limits_t^T \psi_1^2(t_1)\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^T \psi_1(t_3)\phi_{j_3}(t_3)dt_3dt_2\right)^2 dt_1 \le \tag{122}$$

$$\le \frac{K}{p^2} \ \to \ 0 \tag{123}$$

if $p\to\infty$, where constant K is independent of p.

The transition from (120) to (121) is based on analogues of the estimates (99), (101) for the value

$$\left|\sum_{j_3=p+1}^{\infty}\int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2)\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3)dt_3dt_2\right|$$

for the polynomial and trigonometric cases, the transition from (122) to (123) is also based on the mentioned analogues of the estimates (99), (101).

Further, we have

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_1}\right)^2 =$$

$$=\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_3(t_3)\phi_{j_1}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2 =$$

$$=\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2 = \tag{124}$$

$$=\sum_{j_2=0}^{p}\left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2 \le \tag{125}$$

$$\le\sum_{j_2=0}^{\infty}\left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2 =$$

$$=\int\limits_t^T \psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2 dt_2. \tag{126}$$

The transition from (124) to (125) is based on the estimates (103), (104) and its obvious analogues (106), (107) for the trigonometric case. However, the estimates (103), (104) cannot be used to estimate the right-hand side of (126), since we get the divergent integral. For this reason, we will obtain new estimate based on the relation [12]-[15]

$$\int\limits_t^x \psi(s)\phi_{j_1}(s)ds = \frac{\sqrt{T-t}\sqrt{2j_1+1}}{2}\int\limits_{-1}^{z(x)} P_{j_1}(y)\psi(u(y))dy =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\Biggl((P_{j_1+1}(z(x)) - P_{j_1-1}(z(x)))\psi(x) -$$

$$-\frac{T-t}{2}\int\limits_{-1}^{z(x)}((P_{j_1+1}(y) - P_{j_1-1}(y))\psi'(u(y))dy\Biggr), \tag{127}$$

where $x \in (t,T)$, $j_1 \geq p+1$, $z(x)$ is defined by (100), $P_j(x)$ is the Legendre polynomial, ψ' is a derivative of the continuously differentiable function $\psi(\tau)$ with respect to the variable $u(y)$,

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}.$$

From (102) and the estimate $|P_j(y)| \leq 1$, $y \in [-1,1]$ we obtain

$$|P_j(y)| = |P_j(y)|^{\varepsilon}\cdot|P_j(y)|^{1-\varepsilon} \leq |P_j(y)|^{1-\varepsilon} < \frac{C}{j^{1/2-\varepsilon/2}(1-y^2)^{1/4-\varepsilon/4}}, \tag{128}$$

where $y \in (-1,1)$, $j \in \mathbb{N}$, and ε is an arbitrary small positive real number.

Combining (127) and (128), we have the following estimate

$$\left|\int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau\right| < \frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} + 1\right), \tag{129}$$

where $s \in (t,T)$, $z(s)$ is defined by (100), constant C does not depend on j_1.

Similarly to (129) we obtain

$$\left|\int\limits_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau\right| < \frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}} + 1\right), \tag{130}$$

where $s \in (t,T)$, constant C is independent of j_1.

Combining (103) and (130), we have

$$\left|\int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau\int\limits_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau\right| <$$

$$
(131)\qquad < \frac{L}{(j_1)^{2-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+1\right),
$$

where $s\in(t,T)$, $z(s)$ is defined by (100), constant L does not depend on j_1.

Observe that

$$
(132)\qquad \sum_{j_1=p+1}^{\infty}\frac{1}{(j_1)^{2-\varepsilon/2}}\le\int\limits_p^{\infty}\frac{dx}{x^{2-\varepsilon/2}}=\frac{1}{(1-\varepsilon/2)p^{1-\varepsilon/2}}.
$$

Applying (131) and (132) to estimate the right-hand side of (126) gives

$$
(133)\qquad \sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0
$$

if $p\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

The estimation of the right-hand side of (126) for the trigonometric case is carried out using the estimates (106), (107). At that we obtain the estimate (133) with $\varepsilon=0$. Theorem 15 is proved.

7. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4 for the Case of Smooth Weight Functions $\psi_1(\tau),\ \ldots,\ \psi_4(\tau)$. Simple Proof Based on Theorem 12

Theorem 16 [12], [36], [37], [51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
(134)\qquad J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}
$$

the following expansion

$$
J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}
$$

that converges in the mean-square sense is valid, where $i_1,i_3,i_3,i_4=0,1,\ldots,m,$

$$
C_{j_4j_3j_2j_1}=\int\limits_t^{T}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4
$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Proof. As noted above (see Sect. 5), Conditions 1 and 2 of Theorem 12 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 12 for the iterated Stratonovich stochastic integral (134). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_3 j_1 j_1} \right)^2 = 0, \tag{135}$$

$$\lim_{p\to\infty} \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_1 j_2 j_1} \right)^2 = 0, \tag{136}$$

$$\lim_{p\to\infty} \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_3 j_2 j_1} \right)^2 = 0, \tag{137}$$

$$\lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4 j_2 j_2 j_1} \right)^2 = 0, \tag{138}$$

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2 j_3 j_2 j_1} \right)^2 = 0, \tag{139}$$

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_1} \right)^2 = 0, \tag{140}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2 j_1 j_2 j_1} \right)^2 = 0, \tag{141}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1} \right)^2 = 0, \tag{142}$$

$$\lim_{p\to\infty} \left(\sum_{j_3=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_3 j_3 j_1 j_1} \right)^2 = 0, \tag{143}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Big|_{(j_1j_1)\curvearrowright(\cdot)}\right)^2=0, \tag{144}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\Big|_{(j_3j_3)\curvearrowright(\cdot)}\right)^2=0, \tag{145}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Big|_{(j_2j_2)\curvearrowright(\cdot)}\right)^2=0, \tag{146}$$

where in (144)–(146) we use the notation (32).

Applying arguments similar to those we used in the proof of Theorem 15, we obtain for (135)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_4j_3j_1j_1}\right)^2=$$

$$=\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{147}$$

$$=\sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2\le \tag{148}$$

$$\le\sum_{j_3,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{149}$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_4\}}\psi_4^2(t_4)\psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3dt_4\le \tag{150}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{151}$$

if $p\to\infty$, where constant K is independent of p.

Note that the transition from (147) to (148) is based on the estimate (99) for the polynomial case and its analogue for the trigonometric case, the transition from (149) to (150) is based on the Parseval equality, and the transition from (150) to (151) is also based on the estimate (99) and its analogue for the trigonometric case.

Further, we have for (136)

$$\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=$$

$$=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{152}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2= \tag{153}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2\le$$

$$\le\sum_{j_2,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4^2(t_4)\psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2dt_2dt_4\le$$

$$\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{154}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The relation (154) was obtained by the same method as (151). Note that in obtaining (154) we used the estimates (105), (129) for the polynomial case and (106), (108) for the trigonometric case. We also used the integration order replacement in the iterated Riemann integrals (see (152), (153)).

Repeating the previous steps for (137) and (138), we get

$$\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_3 j_2 j_1}\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 =$$

$$= \sum_{j_2,j_3=0}^{p}\left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 \le$$

$$\le \sum_{j_2,j_3=0}^{\infty}\left(\int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2 =$$

$$= \int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\psi_3^2(t_3)\psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4\right)^2 dt_2dt_3 \le$$

$$\le \frac{K}{p^2} \ \rightarrow\ 0 \tag{155}$$

if $p\to\infty$, where constant K does not depend on p;

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4 j_2 j_2 j_1}\right)^2 =$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \times \right.
$$

$$
\left. \times \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 =
$$

$$
= \sum_{j_1,j_4=0}^{p} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 \le
$$

$$
\le \sum_{j_1,j_4=0}^{\infty} \left(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \times \right.
$$

$$
\left. \times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 dt_1 dt_4 \right)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_4\}} \psi_4^2(t_4)\psi_1^2(t_1) \left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 \right)^2 dt_1 dt_4. \tag{156}
$$

Note that, by virtue of the additivity property of the integral, we have

$$
\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 dt_2 = \tag{157}
$$

$$
= \sum_{j_2=p+1}^{\infty} \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) dt_2 dt_3 -
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_t^{t_1} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) dt_2 dt_3 -
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) dt_3 \int\limits_t^{t_1} \psi_2(t_2)\phi_{j_2}(t_2) dt_2. \tag{158}
$$

However, all three series on the right-hand side of (158) have already been evaluated in (151) and (154). From (156) and (158) we finally obtain

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{159}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

In complete analogy with (154), we have for (139)

$$\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$\left.\times\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=$$

$$=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2\le$$

$$\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2=$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_3\}} \psi_3^2(t_3)\psi_1^2(t_1) \left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2 \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_2}(t_4)dt_4 \right)^2 dt_1 dt_3 \le$$

$$\le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \qquad (160)$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

We have for (140)

$$\sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_1} \right)^2 =$$

$$= \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} \int\limits_{t}^{T} \psi_4(t_4)\phi_{j_3}(t_4) \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \times \right.$$

$$\left. \times \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 \right)^2 =$$

$$= \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} \int\limits_{t}^{T} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.$$

$$\left. \times \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \right)^2 =$$

$$= \sum_{j_1,j_2=0}^{p} \left(\int\limits_{t}^{T} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.$$

$$\left. \times \sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \right)^2 \le$$

$$\le \sum_{j_1,j_2=0}^{\infty} \left(\int\limits_{t}^{T} \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^{T} \psi_2(t_2)\phi_{j_2}(t_2) \times \right.$$

$$\left. \times \sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 \right)^2 =$$

$$= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_2\}} \psi_1^2(t_1)\psi_2^2(t_2) \left(\sum_{j_3=p+1}^{\infty} \int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4) dt_4 dt_3 \right)^2 dt_2 dt_1. \qquad (161)$$

It is easy to see that the integral (see (161))

$$\int\limits_{t_2}^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_{t_3}^{T} \psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3$$

is similar to the integral from the formula (157) if in the last integral we substitute $t_4 = T$. Therefore, by analogy with (159), we obtain

$$\sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1} \right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \tag{162}$$

if $p \to \infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

Now consider (141)–(143). We have for (141) (see **Step 2** in the proof of Theorem 12)

$$\left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1} \right)^2 = \left(\sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1} \right)^2 \le$$

$$\le (p+1) \sum_{j_1=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1} \right)^2. \tag{163}$$

Consider (139) and (160). We have

$$\sum_{j_1=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1} \right)^2 = \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1} \right)^2 \Biggr|_{j_1=j_3} \le$$

$$\le \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1} \right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{164}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (163) and (164), we obtain

$$\left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1} \right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to \ 0$$

if $p \to \infty$, where constant K_1 does not depend on p.

Similarly for (142) we have (see (138), (159))

$$\left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1} \right)^2 = \left(\sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1} \right)^2 \le$$

$$\le (p+1) \sum_{j_1=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1} \right)^2, \tag{165}$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2\Biggl|_{j_1=j_4} \le$$

$$\le \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{166}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (165) and (166), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider (143). Using (72), we obtain

$$\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{\infty} C_{j_3j_3j_1j_1} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1} =$$

$$= \frac{1}{2}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)} - \sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p} C_{j_3j_3j_1j_1}, \tag{167}$$

where (see (32))

$$C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2dt_3dt_4.$$

From the estimate (97) for the polynomial and trigonometric cases we get

$$\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)}\right| \le \frac{C}{p}, \tag{168}$$

where constant C is independent of p.

Further, we have (see (162))

$$\left(\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 =$$

$$= (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2\Biggl|_{j_1=j_2} \le$$

(169)
$$\le (p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}},$$

where constant K_1 does not depend on p.

Combining (167)–(169), we obtain

$$\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\right)^2 \le \frac{K_2}{p^{1-\varepsilon}} \ \to\ 0$$

if $p\to\infty$, where constant K_2 does not depend on p.

Let us prove (144)–(146). It is not difficult to see that the estimate (168) proves (144).

Using the integration order replacement, we obtain

$$\sum_{j_1=p+1}^{\infty} C_{j_3j_3j_1j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)} =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\psi_3(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_4 =$$

(170)
$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \left(\psi_2(t_2)\int\limits_{t_2}^{T}\psi_4(t_4)\psi_3(t_4)dt_4\right)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

$$\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)} =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_3dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\left(\int\limits_t^{t_4}-\int\limits_t^{t_1}\right)\psi_3(t_3)\psi_2(t_3)dt_3dt_1dt_4 =$$

(171)
$$= \sum_{j_1=p+1}^{\infty}\int\limits_t^T \left(\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\psi_2(t_3)dt_3\right)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_4-$$

$$
-\sum_{j_1=p+1}^{\infty}\int\limits_t^T \psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\left(\psi_1(t_1)\int\limits_t^{t_1}\psi_3(t_3)\psi_2(t_3)dt_3\right)\phi_{j_1}(t_1)dt_1dt_4. \tag{172}
$$

Applying the estimate (97) (polynomial and trigonometric cases) to the right-hand sides of (170)–(172), we get

$$
\left|\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{173}
$$

$$
\left|\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\Bigg|_{(j_2j_2)\curvearrowright(\cdot)}\right| \le \frac{C}{p}, \tag{174}
$$

where constant C is independent of p. The estimates (173), (174) prove (145), (146).

The relations (135)–(146) are proved. Theorem 16 is proved.

8. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case $p_1 = \ldots = p_5 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\ldots$, $\psi_5(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 17 [12], [36], [37], [51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{175}
$$

the following expansion

$$
J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5 = 0,1,\ldots,m,$

$$
C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5
$$

and

$$
\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Proof. Note that in this proof we write k instead of 5 when this is true for an arbitrary k ($k \in \mathbb{N}$). As noted before (see Sect. 5), Conditions 1 and 2 of Theorem 12 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Let us verify Condition 3 of Theorem 12 for the iterated Stratonovich stochastic integral (175). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}, j_{q_3}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}} \right)^2 = 0, \tag{176}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}} \right)^2 = 0, \tag{177}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}=0}^{p} \left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, g_2=g_1+1} \right)^2 = 0, \tag{178}$$

where $(\{g_1, g_2\}, \{g_3, g_4\}, \{q_1\})$ and $(\{g_1, g_2\}, \{q_1, q_2, q_3\})$ are partitions of the set $\{1, 2, \ldots, 5\}$ that is $\{g_1, g_2, g_3, g_4, q_1\} = \{g_1, g_2, q_1, q_2, q_3\} = \{1, 2, \ldots, 5\}$; braces mean an unordered set, and parentheses mean an ordered set.

Let us find a representation for $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2>g_1+1}$ that will be convenient for further consideration.

Using the integration order replacement in the Riemann integrals, we obtain

$$\int_t^T h_k(t_k)\ldots \int_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int_t^{t_{l+1}} h_l(t_l) \int_t^{t_l} h_{l-1}(t_{l-1})\ldots \int_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \int_t^T h_k(t_k)\ldots \int_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int_t^{t_{l+1}} h_1(t_1) \int_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \int_{t_{l-1}}^{t_{l+1}} h_l(t_l)dt_l \times$$

$$\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k =$$

$$= \int_t^T h_k(t_k)\ldots \int_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int_t^{t_{l+1}} h_l(t_l)dt_l \right) \int_t^{t_{l+1}} h_1(t_1) \int_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \times$$

$$\times dt_{l-1}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k -$$

$$- \int_t^T h_k(t_k)\ldots \int_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int_t^{t_{l+1}} h_1(t_1) \int_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1}) \left(\int_t^{t_{l-1}} h_l(t_l)dt_l \right) \times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k -$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (179)$$

where $1 < l < k$ and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$. By analogy with (179) we have for $l = k$

$$\int\limits_t^T h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_l =$$

$$= \int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^T h_l(t_l)dt_ldt_{l-1}\ldots dt_2dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1}\ldots dt_2dt_1 -$$

$$-\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)dt_{l-1}\ldots dt_2dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1} -$$

$$-\int\limits_t^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}. \qquad (180)$$

The formulas (179), (180) will be used further.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume for simplicity that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

Let us continue the proof. Applying (179) to $C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}$ (more precisely to $h_s(t_s) = \psi_s(t_s)\phi_{j_l}(t_s)$), we obtain for $l+1\le k$, $s-1\ge 1$, $l-1\ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = \tag{181}$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_l}(t_s) \int\limits_t^{t_s} \phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_sdt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \left(\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s)dt_s\right) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k -$$

$$- \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \left(\int\limits_t^{t_{s-1}} \phi_{j_l}(t_s)dt_s\right) \int\limits_t^{t_{s-1}} \phi_{j_{s-2}}(t_{s-2})\ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{s-2}dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} A_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} - \sum_{j_l=p+1}^{\infty} B_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}.$$

Now we apply the formula (179) to $A_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}$, $B_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}$ (more precisely to $h_l(t_l)=\psi_l(t_l)\phi_{j_l}(t_l)$). Then we have for $l+1\le k$, $s-1\ge 1$, $l-1\ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=1\\ g\neq l,s}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{l-1}dt_{l+1}\ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_{l+1}j_{l-1}\ldots j_{s+1}j_{s-1}\ldots j_1} = \sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\neq l,s}, \tag{182}$$

where

$$F_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{183}$$

$$F_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{184}$$

$$F_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{185}$$

$$F_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau. \tag{186}$$

By analogy with (182) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l}, \tag{187}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_2 j_l} \quad (l+1\le k), \tag{188}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} \quad (s-1\ge 1). \tag{189}$$

Then we have for (187)–(189) (see (179), (180))

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} G_p^{(d)}(t_2,\ldots,t_{k-1}) \prod_{g=2}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{k-1}, \tag{190}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} E_p^{(d)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=2\\ g\neq l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{l-1}dt_{l+1}\ldots dt_k, \tag{191}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} D_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})\times$$

$$\times \prod_{\substack{g=1\\ g\neq s}}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{k-1}, \tag{192}$$

where

$$G_p^{(1)}(t_2,\ldots,t_{k-1}) = \mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^T \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$G_p^{(2)}(t_2,\ldots,t_{k-1}) = -\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(1)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(2)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_2} \psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^T \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =$$

$$
= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^T \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =
$$

$$
= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau,
$$

$$
D_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1}) =
$$

$$
= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{k-1}} \psi_k(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau.
$$

Now let us consider the value $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2=g_1+1}$. To do this, we will make the following transformations

$$
\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots
$$

$$
\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}}\right) h_l(t_{l-1}) \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-1}}\right) h_l(t_l)dt_l dt_{l-1}dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k =
$$

$$
= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l \int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times
$$

$$
\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k -
$$

$$
- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots
$$

$$
\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right) dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k -
$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)\int\limits_t^{t_{l+1}} h_1(t_1)\times
$$

$$
\times\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k+
$$

$$
+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)dt_{l-2}\ldots dt_2 dt_1 dt_{l+1}\ldots dt_k=
$$

$$
=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-
$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times
$$

$$
\times\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right)\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k-
$$

$$
-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)\times
$$

$$
\times\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k+
$$

$$
+\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})\int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right)\times
$$

$$
\times\int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-3}dt_{l-2}dt_{l+1}\ldots dt_k, \tag{193}
$$

where $l+1\le k$, $l-2\ge 1$, and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$.

Applying (193) to $C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots\ldots j_1}$, we obtain for $l+1\le k$, $l-2\ge 1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l j_{l-2}\ldots\ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} H_p^{(d)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=1\\ g\neq l-1,l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{l-2}dt_{l+1}\ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1} = \sum_{d=1}^{4} C^{**(d)}_{j_k\ldots j_q\ldots j_1}\Big|_{q\neq l-1,l}, \tag{194}$$

where

$$H_p^{(1)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{195}$$

$$H_p^{(2)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-2}} \psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{196}$$

$$H_p^{(3)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_{l-1}(\tau)\phi_{j_l}(\tau) \int\limits_t^{\tau} \psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau, \tag{197}$$

$$H_p^{(4)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l-2}} \psi_{l-1}(\tau)\phi_{j_l}(\tau) \int\limits_t^{\tau} \psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau. \tag{198}$$

By analogy with (194) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_l}, \tag{199}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_l j_{k-2}\ldots j_1}. \tag{200}$$

Then we have for (199), (200) (see (193) and its analogue for $t_{l+1} = T$)

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_l} = \int\limits_{[t,T]^{k-2}} L_p(t_3,\ldots,t_k)\prod_{g=3}^{k}\psi_g(t_g)\phi_{j_g}(t_g)dt_3\ldots dt_k, \tag{201}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_lj_lj_{k-2}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} M_p^{(d)}(t_1,\ldots,t_{k-2})\prod_{g=1}^{k-2}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{k-2}, \tag{202}$$

where

$$L_p(t_3,\ldots,t_k) = \mathbf{1}_{\{t_3<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(1)}(t_1,\ldots,t_{k-2}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(2)}(t_1,\ldots,t_{k-2}) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(3)}(t_1,\ldots,t_{k-2}) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(4)}(t_1,\ldots,t_{k-2}) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau.$$

It is important to note that $C^{*(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$, $C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$ $(d=1,\ldots,4)$ are Fourier coefficients (see (182), (194)), that is, we can use Parseval's equality in the further proof.

Combining the equalities (182)–(186) (the case $g_2 > g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 15, 16, we obtain for (182)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\,g_2>g_1+1}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2 \le \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty} \left(\sum_{d=1}^{4} C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty} \left(\int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\times\right.$$

$$\left.\times \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\right)^2 =$$

$$= \int\limits_{[t,T]^{k-2}} \left(\sum_{d=1}^{4} F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\right)^2 \times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k \le$$

$$\le 4\sum_{d=1}^{4} \int\limits_{[t,T]^{k-2}} \left(F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k) \prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k} \psi_q(t_q)\right)^2 \times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k \le$$

$$\le \frac{K}{p^{2-\varepsilon}} \ \to\ 0 \tag{203}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (187)–(189) are considered analogously.

Absolutely similarly (see (203)) combining the equalities (194)–(198) (the case $g_2=g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 15, 16, we get for (194)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2 =$$

$$= \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2 =$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggl|_{q\ne g_1,g_2}\right)^2\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggl|_{q\ne g_1,g_2}\right)^2=
$$

$$
=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\times\right.
$$

$$
\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\right)^2=
$$

$$
=\int\limits_{[t,T]^{k-2}}\left(\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\right)^2dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots\ldots dt_k\le
$$

$$
\le 4\sum_{d=1}^{4}\int\limits_{[t,T]^{k-2}}\left(H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\right)^2dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\le
$$

$$
\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{204}
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (199), (200) are considered analogously.

From (203), (204) and their analogues for the cases (187)–(189), (199), (200) we obtain

$$
\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2}}\right)^2\le\frac{K}{p^{2-\varepsilon}}, \tag{205}
$$

where constant K is independent of p. Thus the equality (176) is proved.

Let us prove the equality (177). Consider the following cases

1. $g_2>g_1+1,\ g_4=g_3+1,$ 2. $g_2=g_1+1,\ g_4>g_3+1,$
3. $g_2>g_1+1,\ g_4>g_3+1,$ 4. $g_2=g_1+1,\ g_4=g_3+1.$

The proof for Cases 1–3 will be similar. Consider, for example, Case 2. Using (71), we obtain

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=
$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=0}^{p} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 \le \tag{206}$$

$$\le (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 =$$

$$= (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2\Bigg|_{j_{g_3}=j_{g_4}} \le$$

$$\le (p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2. \tag{207}$$

It is easy to see that the expression (207) (without the multiplier $p+1$) is a particular case ($g_4 > g_3+1, g_2 = g_1+1$) of the left-hand side of (205). Combining (205) and (207), we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \to 0 \tag{208}$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider Case 4 ($g_2 = g_1+1,\ g_4 = g_3+1$). We have (see (72))

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\left(\sum_{j_{g_3}=0}^{\infty}-\sum_{j_{g_3}=0}^{p}\right) C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 =$$

$$= \sum_{j_{q_1}=0}^{p}\left(\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)} - \sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 \le$$

$$\le \frac{1}{2}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)}\right)^2 + \tag{209}$$

$$+2\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2. \tag{210}$$

An expression similar to (210) was estimated (see (206)–(208)). Let us estimate (209). We have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)}\right)^2=$$

$$=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown 0}\right)^2\le$$

$$\le(T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown j_{g_3}}\right)^2, \tag{211}$$

where the notations are the same as in the proof of Theorem 12.

The expression (211) without the multiplier $T-t$ is an expression of type (135)–(140) before passing to the limit $\lim\limits_{p\to\infty}$ (the only difference is the replacement of one of the weight functions $\psi_1(\tau),\ldots,\psi_4(\tau)$ in (135)–(140) by the product $\psi_{l+1}(\tau)\psi_l(\tau)$ $(l=1,\ldots,4)$. Therefore, for Case 4 $(g_2=g_1+1,\ g_4=g_3+1)$, we obtain the estimate

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1,g_2=g_1+1}\right)^2\le\frac{K}{p^{1-\varepsilon}}, \tag{212}$$

where constant K is independent of p.

The estimates (208), (212) prove (177). Let us prove (178). By analogy with (211) we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\frown(\cdot),j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=$$

$$=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\frown 0,j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2\le$$

$$\le(T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\frown j_{g_1},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2. \tag{213}$$

Thus, we obtain the estimate (see (211) and the proof of Theorem 16)

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{214}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

The estimate (214) proves (178). Theorem 17 is proved.

9. Estimates for the Mean-Square Approximation Error of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicity k in Theorems 12, 14

In this section, we estimate the mean-square approximation error for iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbb{N}$) in Theorems 12, 14.

Theorem 18 [12], [36], [37], [51]. *Suppose that every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuously differentiable nonrandom function at the interval $[t,T]$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then the following estimates*

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le$$

$$\le K_1\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right), \tag{215}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\} \le$$

$$\le K_2(s)\left(\frac{1}{p} + \sum_{r=1}^{[k/2]}\ \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right) \tag{216}$$

hold, where $s \in (t,T]$ (s is fixed), $i_1,\ldots,i_k = 1,\ldots,m$,

$$R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} = R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Biggl|_{T=s},$$

$R_{T,t}^{(p)r}$ is defined by (84), $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ and $J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}$ are iterated Stratonovich stochastic integrals (39) and (95), $C_{j_k\ldots j_1}$ and $C_{j_k\ldots j_1}(s)$ are Fourier coefficients (31) and (93), constants K_1 and $K_2(s)$ are independent of p; another notations are the same as in Theorems 1, 12, 14.

Proof. Note that Conditions 1 and 2 of Theorems 12, 14 are satisfied under the conditions of Theorem 18 (see Remark 2.4 in [12] or see Sect. 5 from this paper). Then from the proof of Theorem 12 it follows that the expression (89) before passing to limit $\underset{p\to\infty}{\text{l.i.m.}}$ has the form

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}+$$

$$+\sum_{r=1}^{[k/2]}\Biggl(\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Biggr), \qquad (217)$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the approximation for the iterated Ito stochastic integral (1), which is obtained using Theorem 1 (see (16)), i.e. (see Theorem 1.2 in [12]-[15] for details)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \qquad (218)$$

$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}$ is the approximation obtained using (218) for the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ (see (23)).

Using (217) and Theorem 4, we have

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}+$$

$$+\biggl(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\biggr)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}\frac{1}{2^r}\biggl(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\biggr)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} =$$

$$= J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p} - I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{219}$$

w. p. 1, where we denote $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$.

In [12] (Sect. 1.7.2, Remark 1.7) it is shown that under the conditions of Theorem 18 the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}\right)^2\right\} \le \frac{C}{p} \tag{220}$$

holds, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (1), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ has the form (218), $i_1,\ldots,i_k=0,1,\ldots,m$, constant C depends only on k and $T-t$.

Applying (220), we obtain the following estimates

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le \frac{C}{p}, \tag{221}$$

$$\mathsf{M}\left\{\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p} - I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)^2\right\} \le$$

$$\le \frac{C}{p}, \tag{222}$$

where constant C does not depend on p.

From (219)–(222) and the elementary inequality

$$(a_1+a_2+\ldots+a_n)^2 \le n\left(a_1^2+a_2^2+\ldots+a_n^2\right), \quad n\in\mathbb{N}$$

we obtain (215).

The estimate (216) is obtained similarly to the estimate (215) using Theorem 1.11 in [12], Theorem 14 and the estimate [12] (Sect. 1.8.1, Remark 1.12)

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}\right)^2\right\} \le \frac{C}{p},$$

where

$$
J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)},
$$

$$
J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\stackrel{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})}{\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r{\bf 1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}{\bf 1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),
$$

where $s\in(t,T]$ (s is fixed), $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (93), $i_1,\ldots,i_k=0,1,\ldots,m$, constant C depends only on k and $s-t$; another notations are the same as in Theorems 2, 13.

Theorem 18 is proved.

10. Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3–5 in Theorems 15–17

In this section, we consider the rate of convergence of approximations of iterated Stratonovich stochastic integrals in Theorems 15–17. It is easy to see that in Theorems 15–17 the second term in parentheses on the right-hand side of (215) is estimated. Combining these results with Theorem 18, we obtain the following theorems.

Theorem 19 [12], [36], [37], [51]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^{*}[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}
$$

the following estimate

$$
{\sf M}\left\{\left(J^{*}[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p}
$$

is fulfilled, where $i_1,i_2,i_3=1,\ldots,m$, constant C is independent of p,

$$
C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3
$$

and

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{w}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i or j.

Theorem 20 [12], [36], [37], [51]. *Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\ \psi_4(\tau)$ be continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}d\mathbf{f}_{t_4}^{(i_4)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

holds, where $i_1,i_2,i_3,i_4=1,\ldots,m$, constant C does not depend on p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\times$$

$$\times dt_2dt_3dt_4;$$

another notations are the same as in Theorem 19.

Theorem 21 [12], [36], [37], [51]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\ \psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_5}^{(i_5)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

is valid, where $i_1,\ldots,i_5=1,\ldots,m$, constant C is independent of p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$

and $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorem 19, 20.

11. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case $p_1 = \ldots = p_6 \to \infty$ and $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 22 [12], [36], [37], [61]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{223}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As noted in Sect. 5, Conditions 1 and 2 of Theorem 12 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 12 for the iterated Stratonovich stochastic integral (223). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3},j_{q_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_6\ldots j_1}\Big|_{j_{g_1}=j_{g_2}}\right)^2 = 0, \tag{224}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0, \tag{225}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1}\right)^2=0, \tag{226}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right)^2=0, \tag{227}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_6}j_{g_5})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_6=g_5+1}\right)^2=0, \tag{228}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\frown(\cdot)(j_{g_6}j_{g_5})\frown(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_4=g_3+1,g_6=g_5+1}\right)^2=0, \tag{229}$$

where the expressions

$$(\{g_1,g_2\},\{g_3,g_4\},\{g_5,g_6\}),\quad (\{g_1,g_2\},\{g_3,g_4\},\{q_1,q_2\}),\quad (\{g_1,g_2\},\{q_1,q_2,q_3,q_4\})$$

are partitions of the set $\{1,2,\ldots,6\}$ that is $\{g_1,g_2,g_3,\,g_4,g_5,g_6\}=\{g_1,g_2,g_3,g_4,q_1,q_2\}=\{g_1,g_2,q_1,$ $q_2,q_3,q_4\}=\{1,2,\,\ldots,6\}$; braces mean an unordered set, and parentheses mean an ordered set.

The equalities (224), (226) were proved earlier (see the proof of equalities (205), (211)). The relation (229) follows from the estimate (97) for the polynomial case and its analogue for the trigonometric case. It is easy to see that the equalities (225) and (228) are proved in complete analogy with the proof of (177), (211).

Thus, we have to prove the relation (227). The equality (227) is equivalent to the following equalities

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=0, \tag{230}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_3j_2j_3j_2j_1}=0, \tag{231}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=0, \tag{232}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=0, \tag{233}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_2j_2j_3j_3j_1}=0, \tag{234}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1}=0, \tag{235}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_3j_2j_1j_1}=0, \tag{236}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_2j_1j_1}=0, \tag{237}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1}=0, \tag{238}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1}=0, \tag{239}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_1j_3j_3j_2j_1}=0, \tag{240}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_2j_3j_2j_1}=0, \tag{241}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_1j_3j_2j_1}=0, \tag{242}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}=0, \tag{243}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_3j_1j_2j_1}=0. \tag{244}$$

Consider in detail the case of Legendre polynomials (the case of trigonometric functions is considered in complete analogy).

First, we prove the following equality for the Fourier coefficients for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

$$C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$

$$+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}. \tag{245}$$

Using the integration order replacement, we have

$$C_{j_6j_5j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_5 dt_6 =$$

$$= \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{T} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_4 dt_5 dt_6 -$$

$$- \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_4 dt_5 dt_6 =$$

$$= C_{j_6} C_{j_5j_4j_3j_2j_1} -$$

$$- \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \int\limits_t^{T} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_3 dt_4 dt_5 dt_6 +$$

$$+ \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \int\limits_{t_5}^{T} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_3 dt_4 dt_5 dt_6 =$$

$$= C_{j_6} C_{j_5j_4j_3j_2j_1} -$$

$$- \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) dt_5 dt_6 \ C_{j_4j_3j_2j_1} +$$

$$+ \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \int\limits_{t_5}^{T} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_3 dt_4 dt_5 dt_6 =$$

$$= C_{j_6} C_{j_5j_4j_3j_2j_1} - C_{j_5j_6} C_{j_4j_3j_2j_1} +$$

$$+ \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \int\limits_{t_5}^{T} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_3 dt_4 dt_5 dt_6 =$$

$$\ldots$$

$$= C_{j_6} C_{j_5j_4j_3j_2j_1} - C_{j_5j_6} C_{j_4j_3j_2j_1} + C_{j_4j_5j_6} C_{j_3j_2j_1} - C_{j_3j_4j_5j_6} C_{j_2j_1} + C_{j_2j_3j_4j_5j_6} C_{j_1} -$$

$$- \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^{T} \phi_{j_5}(t_5) \ldots \int\limits_{t_2}^{T} \phi_{j_1}(t_1) dt_1 \ldots dt_5 dt_6 =$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} -$$

(246) $$-C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} - C_{j_1j_2j_3j_4j_5j_6}.$$

The equality (246) completes the proof of the relation (245).
Let us consider (230). From (65) we obtain

(247) $$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}.$$

Applying (245), we get

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_1j_2j_3} = 2\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1} - C_{j_2j_3}C_{j_1j_3j_2j_1} + C_{j_1j_2j_3}C_{j_3j_2j_1} -$$

(248) $$-C_{j_3j_1j_2j_3}C_{j_2j_1} + C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr).$$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

(249) $$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0,1,2,\ldots,$$

where

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j$$

is the Legendre polynomial.
Note that

$$C_{j_2j_1} = \int\limits_t^T \phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(\theta)d\theta d\tau =$$

(250) $$= \frac{T-t}{2}\begin{cases} 1/\sqrt{(2j_1+1)(2j_1+3)} & \text{if } j_2 = j_1+1,\ j_1 = 0,1,2,\ldots \\ -1/\sqrt{4j_1^2-1} & \text{if } j_2 = j_1-1,\ j_1 = 1,2,\ldots \\ 1 & \text{if } j_1 = j_2 = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$C_{j_1}=\int\limits_t^T\phi_{j_1}(\tau)d\tau=\begin{cases}\sqrt{T-t} & \text{if } j_1=0\\ \\ 0 & \text{if } j_1\ne 0\end{cases}. \tag{251}$$

Moreover, the generalized Parseval equality gives

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_2j_3}C_{j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_3<t_2<t_1\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3\times$$

$$\times\int\limits_{[t,T]^3}\mathbf{1}_{\{t_1<t_2<t_3\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3=$$

$$=\int\limits_{[t,T]^3}\mathbf{1}_{\{t_3<t_2<t_1\}}\mathbf{1}_{\{t_1<t_2<t_3\}}dt_1dt_2dt_3=0. \tag{252}$$

Using the above arguments and also (65), (247), and (248), we get

$$-\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1j_3j_2j_1}=$$

$$=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_2j_3}C_{j_1j_3j_2j_1}-$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1}+C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr)=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_3j_1j_2j_3}C_{j_2j_1}\biggr)=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}-\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2j_3}C_{j_2j_1}=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}+\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}. \tag{253}$$

By analogy with the proof of (141) (see the proof of Theorem 16) we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}=\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_10j_2j_1}=0, \tag{254}$$

where we used the following representation

$$C_{j_2j_10j_2j_1}=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}dt_3dt_2dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)(t_4-t)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5+$$

$$+\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\bar C_{j_2j_1j_2j_1}+\tilde C_{j_2j_1j_2j_1}.$$

Further, we have (see (250))

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}=\lim_{p\to\infty}\sum_{j_3=p+1}^{\infty}\biggl(C_{00}C_{j_300j_3}+$$

$$+\sum_{j_1=1}^{p}C_{j_1-1,j_1}C_{j_3j_1,j_1-1,j_3}+\sum_{j_1=1}^{p-1}C_{j_1+1,j_1}C_{j_3j_1,j_1+1,j_3}+C_{1,0}C_{j_301j_3}\biggr). \tag{255}$$

Observe that

$$|C_{j_1-1,j_1}|+|C_{j_1+1,j_1}|\le \frac{K}{j_1}\quad (j_1=1,\ldots,p), \tag{256}$$

$$|C_{j_300j_3}|+|C_{j_3j_1,j_1-1,j_3}|+|C_{j_3j_1,j_1+1,j_3}|+|C_{j_301j_3}|\le \frac{K_1}{j_3^2}\quad (j_3\ge p+1), \tag{257}$$

where constants K, K_1 do not depend on j_1, j_3.

The estimate (256) follow from (250). At the same time, the estimate (257) can be obtained using the following reasoning. First note that the integration order replacement gives

$$C_{j_3j_1j_2j_3}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1\right)dt_2\left(\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4\right)dt_3. \tag{258}$$

Consider the well-known estimate for Legendre polynomials

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{259}$$

where constant K does not depend on y and j.

The estimate (259) can be rewritten for the function $\phi_j(x)$ (see (249)) in the following form

$$|\phi_j(x)|<\sqrt{\frac{2j+1}{j+1}}\frac{K}{\sqrt{T-t}}\frac{1}{\left(1-z^2(x)\right)^{1/4}}<\frac{K_1}{\sqrt{T-t}}\frac{1}{\left(1-z^2(x)\right)^{1/4}}, \tag{260}$$

where $K_1=K\sqrt{2}$, $x\in(t,T)$, $j\in\mathbb{N}$,

$$z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

Note analogues of the estimates (103), (104)

$$\left|\int\limits_t^x\phi_{j_1}(s)ds\right|<\frac{C}{j_1(1-(z(x))^2)^{1/4}},\quad \left|\int\limits_x^T\phi_{j_1}(s)ds\right|<\frac{C}{j_1(1-(z(x))^2)^{1/4}},\quad x\in(t,T), \tag{261}$$

where $j_1>0$, constant C does not depend on j_1.

Applying the estimates (260) and (261) to (258) gives the estimate (257). Using (255), (256), and (257), we obtain

$$\left|\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}\right|\le K\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\left(1+\sum_{j_1=1}^{p}\frac{1}{j_1}\right)\le$$

$$\leq K\int\limits_p^{\infty}\frac{dx}{x^2}\left(2+\int\limits_1^p\frac{dx}{x}\right)=\frac{K(2+lnp)}{p}\to 0 \qquad (262)$$

if $p\to\infty$, where constant K is independent of p. Thus, the equality (230) is proved (see (253), (254), (262)).

The relation (231) is proved in complete analogy with the proof of equality (230). For (231) we have (see (245))

$$\lim_{p\to\infty}\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_3j_2j_3j_2j_1}+\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_2j_3j_2j_3j_1}\right)=2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1j_3j_2j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_1}C_{j_3j_2j_3j_2j_1}-C_{j_3j_1}C_{j_2j_3j_2j_1}+C_{j_2j_3j_1}C_{j_3j_2j_1}-$$

$$-C_{j_3j_2j_3j_1}C_{j_2j_1}+C_{j_2j_3j_2j_3j_1}C_{j_1}\Biggr)=$$

$$=2\lim_{p\to\infty}\left(\sqrt{T-t}\sum_{j_2,j_3=0}^{p}C_{j_3j_2j_3j_20}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1}C_{j_3j_2j_3j_1}\right)=$$

$$=-2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1}C_{j_3j_2j_3j_1}.$$

To estimate the Fourier coefficient $C_{j_3j_2j_3j_1}$, we use the following (see the proof of (230) for more details)

$$C_{j_3j_2j_3j_1}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_3}\phi_{j_3}(t_2)dt_2dt_1dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_4-$$

$$-\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\left(\int\limits_{t_3}^T\phi_{j_3}(t_4)dt_4\right)dt_3-$$

$$
-\int\limits_t^T \phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1\left(\int\limits_{t_3}^T\phi_{j_3}(t_4)dt_4\right)dt_3.
$$

Let us prove (232). From (65) we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}C_{j_3j_2j_3j_1j_2j_1}. \tag{263}
$$

Applying (245) and (263), we get (we replaced j_3 by j_4)

$$
\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}+\sum_{j_1,j_2,j_4=0}^{p}C_{j_1j_2j_1j_4j_2j_4}=2\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4j_1j_2j_1}=
$$

$$
=\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_4}C_{j_2j_4j_1j_2j_1}-C_{j_2j_4}C_{j_4j_1j_2j_1}+C_{j_4j_2j_4}C_{j_1j_2j_1}-
$$

$$
-C_{j_1j_4j_2j_4}C_{j_2j_1}+C_{j_2j_1j_4j_2j_4}C_{j_1}\Biggr)=
$$

$$
=2\sum_{j_1,j_2,j_4=0}^{p}\Bigl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Bigr)+
$$

$$
+\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4}C_{j_1j_2j_1}. \tag{264}
$$

Further, we have (see (65))

$$
\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}C_{j_4j_2j_4}C_{j_1j_2j_1}=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p}C_{j_1j_2j_1}\right)^2=
$$

$$
=\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=0, \tag{265}
$$

where we applied the equality (114).

Furthermore, by analogy with the proof of (230), we have

$$
\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\Bigl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Bigr)=0. \tag{266}
$$

To estimate the Fourier coefficient $C_{j_1j_4j_2j_4}$ in (266), we use the following (see the proof of (230) for more details)

$$C_{j_1j_4j_2j_4}=\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4-$$

$$-\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_3)dt_3\right)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4.$$

The relations (263)–(266) complete the proof of equality (232).
Let us prove (233). Using (65), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}. \tag{267}$$

Applying (245) and (267), we obtain

$$2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}+\left(C_{j_3j_2j_1}\right)^2-$$

$$-C_{j_3j_3j_2j_1}C_{j_2j_1}+C_{j_2j_3j_3j_2j_1}C_{j_1}\Biggr)=$$

$$=2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}\right)+$$

$$+\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2. \tag{268}$$

In [12] (Sect. 1.7.2) the following estimate

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\leq L_k \sum_{j_s=p+1}^{\infty} \frac{1}{j_s^2} \leq L_k \int\limits_{p}^{\infty} \frac{dx}{x^2} = \frac{L_k}{p} \tag{269}$$

is proved for the polynomial and trigonometric cases, where $s = 1, \ldots, k$, constant L_k depends on k and $T - t$.

Using the estimate (269), we get

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \left(C_{j_3j_2j_1}\right)^2 = 0. \tag{270}$$

By analogy with the proof of (230), we have

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1} - C_{j_2j_1}C_{j_3j_3j_2j_1}\Biggr) = 0, \tag{271}$$

where we applied the equality (142). To estimate the Fourier coefficient $C_{j_3j_3j_2j_1}$ in (271), we used the following (see the proof of (230) for more details)

$$C_{j_3j_3j_2j_1} = \int\limits_{t}^{T} \phi_{j_3}(t_4) \int\limits_{t}^{t_4} \phi_{j_3}(t_3) \int\limits_{t}^{t_3} \phi_{j_2}(t_2) \int\limits_{t}^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{t}^{T} \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2) \int\limits_{t_2}^{T} \phi_{j_3}(t_3) \int\limits_{t_3}^{T} \phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 =$$

$$= \frac{1}{2} \int\limits_{t}^{T} \phi_{j_1}(t_1) \int\limits_{t_1}^{T} \phi_{j_2}(t_2) \left(\int\limits_{t_2}^{T} \phi_{j_3}(t_3) dt_3\right)^2 dt_2 dt_1. \tag{272}$$

Combining the equalities (267)–(271), we obtain (233).

Let us prove (234) (we replace j_2 by j_4 and j_3 by j_2 in (234)). As noted in Sect. 5, the sequential order of the series

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_4=p+1}^{\infty}$$

is not important. This follows directly from the formulas (72) and (65).

Applying the mentioned property and (65), we get

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = -\sum_{j_1=0}^{p} \sum_{j_2=p+1}^{\infty} \sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{273}$$

Observe that (see the above reasoning)

$$\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=\sum_{j_4=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}. \tag{274}$$

Using (245) and (274), we obtain

$$\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1j_4j_4j_2j_2j_1}+C_{j_1j_2j_2j_4j_4j_1}\Bigr)=2\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}C_{j_1j_4j_4j_2j_2j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}+C_{j_4j_4j_1}C_{j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)+$$

$$+\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2. \tag{275}$$

The equality

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_2j_1}\right)^2=0 \tag{276}$$

follows from the relation (113).

By analogy with the proof of equality (230) we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr)=0, \tag{277}$$

where we applied the equality (143). To estimate the Fourier coefficient $C_{j_2j_4j_4j_1}$ in (277), we used the following (see the proof of (230) for more details)

$$C_{j_2j_4j_4j_1}=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4+$$

$$+\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4-$$

$$-\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)dt_1dt_4.$$

The relation (234) follows from (273), (275)–(277).

Consider (235). Using the integration order replacement, we obtain

$$C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{4}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3. \qquad (278)$$

Applying the estimates (261) to (278) gives the following estimate

$$|C_{j_3j_3j_2j_2j_1j_1}|\le\frac{K}{j_1^2j_3^2}\quad (j_1,j_3>0,\ j_2\ge 0), \qquad (279)$$

where constant K does not depend on j_1,j_2,j_3.

Further, we get (see (72))

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\Biggl|_{(j_2j_2)\frown(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}, \qquad (280)$$

where

$$C_{j_3j_3j_2j_2j_1j_1}\Biggl|_{(j_2j_2)\frown(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\int\limits_t^{t_4}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_5}dt_4dt_2dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)(t_5-t)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6+$$

$$+\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}C'_{j_3j_3j_1j_1}+C''_{j_3j_3j_1j_1}. \tag{281}$$

Let us substitute (281) into (280)

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}+$$

$$+\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}. \tag{282}$$

The relation (143) implies that

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}=0,\quad \lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}=0. \tag{283}$$

From the estimate (279) we get

$$\left|\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\right|\le K(p+1)\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\le$$

$$\le K(p+1)\left(\int\limits_p^{\infty}\frac{dx}{x^2}\right)^2\le\frac{K(p+1)}{p^2}\ \to\ 0 \tag{284}$$

if $p\to\infty$, where constant K is independent of p.

The relations (282)–(284) complete the proof of (235).

Let us prove (236). Using the integration order replacement, we get

$$C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)dt_5dt_3-$$

$$-\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)dt_5dt_3. \tag{285}$$

Applying (65) and (72), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=-\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\frown(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\frown(\cdot)}-\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}-$$

$$-\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}-\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}-$$

$$-\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}. \tag{286}$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=0 \tag{287}$$

follows from the inequality similar to (169) (see the proof of Theorem 16), where we used the following representation

$$C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^{t_6}dt_4dt_3dt_6=$$

$$+\int\limits_t^T\phi_{j_2}(t_6)(t_6-t)\int\limits_t^{t_6}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6+$$

$$+\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_3)(t-t_3)\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}C^{*}_{j_2j_2j_1j_1}+C^{**}_{j_2j_2j_1j_1}. \tag{288}$$

Applying the estimates (261) and (129) ($\varepsilon=1/2$) to (285) gives the following estimates

$$|C_{j_2j_3j_3j_2j_1j_1}|\le\frac{K}{j_1^2j_2j_3^{3/4}}\quad (j_1,j_2,j_3>0), \tag{289}$$

$$|C_{j_200j_2j_1j_1}|\le\frac{K}{j_1^2j_2}\quad (j_1,j_2>0), \tag{290}$$

$$|C_{0j_3j_30j_1j_1}|\le\frac{K}{j_1^2j_3}\quad (j_1,j_3>0), \tag{291}$$

$$|C_{0000j_1j_1}|\le\frac{K}{j_1^2}\quad (j_1>0). \tag{292}$$

Using the estimate (289), we have

$$\left|\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\right|\le K\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_2=1}^{p}\frac{1}{j_2}\sum_{j_3=1}^{p}\frac{1}{j_3^{3/4}}\le$$

$$\leq K\int\limits_{p}^{\infty}\frac{dx}{x^2}\left(1+\int\limits_{1}^{p}\frac{dx}{x}\right)\left(1+\int\limits_{1}^{p}\frac{dx}{x^{3/4}}\right)\leq K_1\frac{1+lnp}{p^{3/4}}\ \to 0 \tag{293}$$

if $p\to\infty$, where constants K, K_1 do not depend on p.

Similarly we get (see (290)–(292))

$$\left|\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}\right|+\left|\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}\right|+\left|\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}\right|\to 0 \tag{294}$$

if $p\to\infty$.

The relations (286), (287), (293), (294) prove (236).

Consider (237). Using the integration order replacement, we get

$$C_{j_3j_2j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_{t}^{T}\phi_{j_3}(t_6)\int\limits_{t}^{t_6}\phi_{j_2}(t_5)\int\limits_{t}^{t_5}\phi_{j_3}(t_4)\int\limits_{t}^{t_4}\phi_{j_2}(t_3)\left(\int\limits_{t}^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_{t}^{T}\phi_{j_2}(t_3)\left(\int\limits_{t}^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{2}\int\limits_{t}^{T}\phi_{j_2}(t_3)\left(\int\limits_{t}^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=$$

$$=\frac{1}{2}\int\limits_{t}^{T}\phi_{j_2}(t_3)\left(\int\limits_{t}^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\left(\int\limits_{t}^{t_5}\phi_{j_3}(t_4)dt_4\right)\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5dt_3-$$

$$-\frac{1}{2}\int\limits_{t}^{T}\phi_{j_2}(t_3)\left(\int\limits_{t}^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_{t}^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5dt_3. \tag{295}$$

Applying (65), we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=$$

$$=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}. \tag{296}$$

Further proof of the equality (237) is based on the relations (295), (296) and is similar to the proof of the formula (236).

Let us prove (238). Applying the integration order replacement, we obtain

$$C_{j_3j_3j_2j_1j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_1}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)dt_2dt_4-$$

$$-\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)^2dt_2dt_4. \tag{297}$$

Using (65), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=$$

$$=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}. \tag{298}$$

Further proof of the equality (238) is based on the relations (297), (298) and is similar to the proof of the relations (236), (237).

Consider (239). Using the integration order replacement, we have

$$C_{j_3j_3j_1j_2j_2j_1}=$$

$$=\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_2}(t_3)\int\limits_{t_3}^{T}\phi_{j_1}(t_4)\int\limits_{t_4}^{T}\phi_{j_3}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_2}(t_3)\int\limits_{t_3}^{T}\phi_{j_1}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_4=$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_2}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)dt_2dt_4-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^{T}\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_2}\phi_{j_2}(t_3)dt_3\right)dt_2dt_4. \qquad (299)$$

Applying (65) and (72), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=-\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}. \qquad (300)$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}=0 \qquad (301)$$

follows from the inequality (169), where we proceed similarly to the proof of equality (287) (see (288)).

The relation

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2j_2j_1}=0 \tag{302}$$

is proved on the basis of (299) and similarly with the proof of (236). The equalities (300)–(302) prove (239).

Let us prove (240). Using (65) and (72), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_1j_3j_3j_2j_1}=\sum_{j_3=p+1}^{\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1}=$$

$$=\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}. \tag{303}$$

Using the equality (141) we have

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_3j_3j_2j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}=0, \tag{304}$$

where we proceed similarly to the proof of equality (287) (see (288)).

Further, we will prove the following relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}=0 \tag{305}$$

using the equality (245). From (245) we have

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2j_1j_3j_3j_2j_1}+C_{j_1j_2j_3j_3j_1j_2}\Bigr)=$$

$$=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2}C_{j_1j_3j_3j_2j_1}-C_{j_1j_2}C_{j_3j_3j_2j_1}+C_{j_3j_1j_2}C_{j_3j_2j_1}-$$

$$-C_{j_3j_3j_1j_2}C_{j_2j_1}+C_{j_2j_3j_3j_1j_2}C_{j_1}\Bigr)=$$

$$=\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Bigr)+$$

$$+\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}. \tag{306}$$

The generalized Parseval equality gives (by analogy with (252))

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}=0. \tag{307}$$

Let us prove the following equality

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr)=0. \tag{308}$$

The relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2}C_{j_1}=0 \tag{309}$$

is proved by the same methods as in the proof of equality (230) and also using Theorem 16 and (72).

Further, we have (see (72))

$$\sum_{j_3=0}^{p} C_{j_3j_3j_1j_2}=\frac{1}{2}C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\frown(\cdot)}-\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}. \tag{310}$$

Moreover,

$$C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\frown(\cdot)}=\int\limits_t^T\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3=$$

$$=\int\limits_t^T\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1\int\limits_{t_2}^{T}dt_3dt_2=\int\limits_t^T(T-t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2=$$

$$=\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^{T}(T-t_2)\phi_{j_1}(t_2)dt_2dt_1=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^{T}(T-t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}(T-t_1)\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\stackrel{\sf def}{=}\tilde{C}_{j_2j_1}. \tag{311}$$

Using (310), (311), and the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1}=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\tilde{C}_{j_2j_1}-$$

$$-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}C_{j_2j_1}=-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2}C_{j_2j_1}. \tag{312}$$

We have (see (272))

$$C_{j_3j_3j_1j_2} = \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_1)\int\limits_{t_1}^T \phi_{j_1}(t_2)\left(\int\limits_{t_2}^T \phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \tag{313}$$

By analogy with (262) and also using (313), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \tag{314}$$

Combining (312) and (314), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \tag{315}$$

The relation (308) follows from (309) and (315). From (306)–(308) we get (305). The equalities (303)–(305) complete the proof of (240).

For the proof of (241)–(244) we will use a new idea. More precisely, we will consider the sums of expressions (241)–(244) with the expressions already studied throughout this proof.

Let us begin from (241). Applying the integration order replacement, we obtain

$$C_{j_3j_1j_2j_3j_2j_1} + C_{j_3j_1j_2j_3j_1j_2} =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\int\limits_t^{t_5} \phi_{j_2}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\int\limits_t^{t_5} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5} \phi_{j_2}(t_4)dt_4dt_3dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\left(\int\limits_t^{t_5} \phi_{j_2}(t_4)dt_4\right)\int\limits_t^{t_5} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$- \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\int\limits_t^{t_5} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)^2\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_1}(t_5)\left(\int\limits_t^{t_5} \phi_{j_2}(t_4)dt_4\right)\int\limits_t^{t_5} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6\right)dt_5-$$

$$-\int\limits_t^T \phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)^2\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5. \tag{316}$$

Using (65), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr). \tag{317}$$

Further, by analogy with the proof of equality (236) and using (316), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=0. \tag{318}$$

From (317) and (318) we get

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=0. \tag{319}$$

Moreover (see (230)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_1j_2}=0. \tag{320}$$

Combining (319) and (320), we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0.$$

The equality (241) is proved.

Consider (242). Using the integration order replacement, we have

$$C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_5dt_6=$$

$$=\int\limits_t^T \phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-$$

$$-\int\limits_t^T \phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5. \tag{321}$$

Using (65), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=$$

$$=\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr). \tag{322}$$

By analogy with the proof of (236) and applying (321), we get

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=0. \tag{323}$$

From (322) and (323) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Bigr)=0. \tag{324}$$

Moreover (see (231)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_1j_2}=0. \tag{325}$$

Combining (324) and (325), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0.$$

The equality (242) is proved.

Now consider (243). Using the integration order replacement, we obtain

$$C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}=$$

$$=\int_t^T\phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_3}(t_4)\int_t^{t_4}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int_t^T\phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int_t^T\phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\left(\int_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int_t^{t_5}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int_t^T\phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=$$

$$=\int_t^T\phi_{j_1}(t_5)\left(\int_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int_t^{t_5}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int_{t_5}^T\phi_{j_3}(t_6)dt_6\right)dt_5-$$

(326)

$$-\int_t^T\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_2}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int_{t_5}^T\phi_{j_3}(t_6)dt_6\right)dt_5.$$

Applying (65) and (72), we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=$$

$$=-\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)-$$

$$
-\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}. \tag{327}
$$

The equality

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)} = 0 \tag{328}
$$

follows from the equality (141), where we proceed similarly to the proof of equality (287) (see (288)).

By analogy with the proof of (236) and applying (326), we get

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{329}
$$

From (327)–(329) we have

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{330}
$$

Moreover (see (232)),

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_1j_2}=0. \tag{331}
$$

Combining (330) and (331), we finally obtain

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_3j_2j_2j_1}=0.
$$

The equality (243) is proved.

Finally consider (244). Using the integration order replacement, we have

$$
C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}=
$$

$$
=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-
$$

$$
-\int\limits_t^T \phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T \phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-
$$

(332)

$$
-\int\limits_t^T \phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5.
$$

Using (65) and (72), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=
$$

$$
=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}\Biggr)-
$$

$$
-\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=
$$

$$
=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}\Biggr)+
$$

$$
+\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)-
$$

$$
-\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Bigl.\Bigr|_{(j_1j_1)\frown(\cdot)}. \tag{333}
$$

The equalities

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Bigl.\Bigr|_{(j_3j_3)\frown(\cdot)}\Biggr)=0, \tag{334}
$$

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\lim_{p\to\infty}\frac{1}{4}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)(j_3j_3)\curvearrowright(\cdot)}-$$

$$-\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=0 \qquad (335)$$

follows from the equalities (141), (142), where we used the same technique as in (288). When proving (335), we also applied (72) and (97).

By analogy with the proof of (236) and applying (332), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=0. \qquad (336)$$

From (333)–(336) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Bigr)=0. \qquad (337)$$

Furthermore (see (234)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}=0. \qquad (338)$$

Combining (337) and (338), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0.$$

The equality (244) is proved. Theorem 22 is proved.

12. Generalization of Theorem 15. The Case p_1, p_2, $p_3\to\infty$ and Continuously Differetiable Weight Functions (The Cases of Legendre Polynomials and Trigonometric Functions)

This section is devoted to the following theorem.

Theorem 23 [12], [36], [37]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{339}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m,$

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. Let us consider the case of Legendre polynomials (the trigonometric case is simpler and can be considered similarly). Applying (60), we obtain

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}=J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)} \tag{340}$$

w. p. 1, where notations are the same as in (60).

Using Theorem 4 (see (25) for the case $k=3$), Theorem 1 (see (44)) as well as (77) (see the derivation of (77)) and (72), we get

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}=J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3=$$

$$= J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} + \frac{1}{2} J[\psi^{(3)}]_{T,t}^{1} + \frac{1}{2} J[\psi^{(3)}]_{T,t}^{2} =$$

$$= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1 i_2 i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{p_3\to\infty}{\text{l.i.m.}} \ \frac{1}{2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl|_{(j_2j_1)\curvearrowright(\cdot),\, j_1=j_2} J'[\phi_{j_3}]_{T,t}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \underset{p_1\to\infty}{\text{l.i.m.}} \ \frac{1}{2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\Biggl|_{(j_3j_2)\curvearrowright(\cdot),\, j_2=j_3} J'[\phi_{j_1}]_{T,t}^{(i_1)} =$$

$$= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1 i_2 i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}} \underset{p_3\to\infty}{\text{l.i.m.}} \ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}} \underset{p_1\to\infty}{\text{l.i.m.}} \ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} \tag{341}$$

w. p. 1.

Using (340), (341) and the elementary inequality

$$(a+b+c+d)^2 \le 4\left(a^2+b^2+c^2+d^2\right),$$

we obtain

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le$$

$$\le 4\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} - J'[K_{p_1p_2p_3}]_{T,t}^{(i_1 i_2 i_3)}\right)^2\right\} +$$

$$+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}\times$$

$$\times\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}} \ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)} - \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\} +$$

$$+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}\times$$

$$\times\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}+$$

$$+4\cdot\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathsf{M}\left\{\left(\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)}\right)^2\right\}=$$

$$=4A_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_2\neq 0\}}B_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_2=i_3\neq 0\}}C_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_3\neq 0\}}D_{p_1p_2p_3}. \tag{342}$$

Theorem 1 gives (see (44))

$$\lim_{p_1,p_2,p_3\to\infty}A_{p_1p_2p_3}=0. \tag{343}$$

Further, in complete analogy with (133) and using (65), we obtain

$$D_{p_1p_2p_3}=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1j_2j_1}\right)^2=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le$$

$$\le\sum_{j_2=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1j_2j_1}\right)^2\le\frac{K}{\left(\min\{p_1,p_3\}\right)^{2-\varepsilon}}\ \to\ 0 \tag{344}$$

if $p_1,p_2,p_3\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

We have

$$B_{p_1p_2p_3}=\mathsf{M}\left\{\left(\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)\right)^2\right\}\le$$

$$\le 2E_{p_3}+2F_{p_1p_2p_3}, \tag{345}$$

where

$$E_{p_3}=\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\},$$

$$F_{p_1p_2p_3} = \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)} - \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\} =$$

$$= \sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1}\right)^2. \tag{346}$$

By analogy with (119) we get

$$\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1}\right)^2 \le \sum_{j_3=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty} C_{j_3j_1j_1}\right)^2 \le$$

$$\le \frac{K}{\left(\min\{p_1,p_2\}\right)^2} \ \to\ 0 \tag{347}$$

if $p_1, p_2, p_3 \to \infty$, where constant K does not depend on p.

Moreover,

$$\lim_{p_3\to\infty} E_{p_3} = \lim_{p_1,p_2,p_3\to\infty} E_{p_3} = 0. \tag{348}$$

Combining (345)–(348), we obtain

$$\lim_{p_1,p_2,p_3\to\infty} B_{p_1p_2p_3} = 0. \tag{349}$$

Consider $C_{p_1p_2p_3}$. We have

$$C_{p_1p_2p_3} = \mathsf{M}\left\{\left(\left(\underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} - \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3j_3j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)\right)^2\right\} \le$$

$$\le 2G_{p_1} + 2H_{p_1p_2p_3}, \tag{350}$$

where

$$G_{p_1}=\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\},$$

$$H_{p_1p_2p_3}=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}=$$

$$=\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2. \tag{351}$$

By analogy with (123) we get

$$\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le\sum_{j_1=0}^{\infty}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le$$

$$\le\frac{K}{\left(\min\{p_2,p_3\}\right)^2}\ \to\ 0 \tag{352}$$

if $p_1,p_2,p_3\to\infty$, where constant K does not depend on p.

Moreover,

$$\lim_{p_1\to\infty}G_{p_1}=\lim_{p_1,p_2,p_3\to\infty}G_{p_1}=0. \tag{353}$$

Combining (350)–(353), we obtain

$$\lim_{p_1,p_2,p_3\to\infty}C_{p_1p_2p_3}=0. \tag{354}$$

The relations (342)–(344), (349), (354) complete the proof of Theorem 23. Theorem 23 is proved.

13. Theorems 1, 2, 5-12, 15-17, 22, 23 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [58], [59], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to the iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [58]-[60] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [62], [63]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{355}$$

where $\tau\in[t,T]$, $t\geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (355) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{356}$$

From (356) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{357}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{358}$$

where $p_1,\ldots,p_k\in\mathbb{N}$, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{359}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (357).

Let us substitute (357) into (358)

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{360}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)} = s$,

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [58]-[60] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [60] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (356) were not considered in [58], [59] (also see [60], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [60] for approximations of the Wiener process based on its series expansion (355) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (360) to the iterated Stratonovich stochastic integral (2) does not follow from the results of the papers [58], [59] (also see [60], Theorems 7.1, 7.2).

From the other hand, Theorems 1, 2, 5-12, 15-17, 22, 23 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (2) of multiplicities 1 to 6 (or of multiplicity k under the condition of convergence of trace series (Theorem 12)) based on the approximation (356) of the Wiener process. At that, the Riemann–Stieltjes integrals (358) converge (according to Theorems 5-12, 15-17, 22, 23) to the appropriate Stratonovich stochastic integrals (2). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (355), (356), and Theorems 5-12, 15-17, 22, 23) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [58]-[60]).

Let $\mathbf{b}_\Delta^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_\Delta^{(i)}(t) = \mathbf{f}_{k\Delta}^{(i)} + \frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)} = \mathbf{f}_{(k+1)\Delta}^{(i)} - \mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

(361) $$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (361) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

(362) $$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}.$$

Using (362) and Theorem 4, it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=$$

(363) $$=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)},$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (363) agrees with Theorem 7.1 (see [60], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (355) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{364}$$

where $d\mathbf{f}_\tau^{(i)p}$ is defined by the relation (357).

Let us substitute (357) into (364)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{365}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (360).

As we noted above, approximations of the Wiener process that are similar to (356) were not considered in [58], [59] (also see Theorems 7.1, 7.2 in [60]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [60] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [12]-[15]. More precisely, using Theorems 5, 6, we obtain from (365) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{366}$$

From the other hand, by Theorems 1, 2 (see (9)) for the case $k = 2$ we obtain from (365) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2\}} \mathbf{1}_{\{j_1 = j_2\}} \right) + \mathbf{1}_{\{i_1 = i_2\}} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1 = i_2\}} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1}. \tag{367}$$

Since

$$\sum_{j_1 = 0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1 = 0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from Theorem 4 ($k=2$) and (367) we obtain (366).

14. Generalization of Theorem 12 for Complete Orthonormal Systems of Functions in $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$ such that the Condition (369) is Satisfied

First, note that (see the proof of Thorem 12 and (86))

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-\right.$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1.} \tag{368}$$

Using (368) and the condition (92), we obtain (87). This means that we get (89). Thus the expansion (40) is proved.

Analyzing the proof of Theorems 12 and 4 and taking into account the above arguments, it is easy to see that the following theorem is true.

Theorem 24 [12], [36]. *Assume that the continuous functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following condition*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=0 \tag{369}$$

is satisfied for all $r=1,2,\ldots,[k/2]$. Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Further in this section, we generalize Theorems 12, 24 to the case of complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$ such that the condition (369) is satisfied.

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a complete probability space and let $f(t,\omega)\stackrel{\mathsf{def}}{=}f_t:[0,T]\times\Omega\rightarrow\mathbb{R}$ be the standard Wiener process defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s\subset\mathrm{F}_t\subset\mathrm{F}$ for $s<t$.
2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.
3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0,\ \Delta>0$ is independent with the events of σ-algebra F_t.

Let $\xi(\tau,\omega)\stackrel{\mathsf{def}}{=}\xi_\tau:[0,T]\times\Omega\rightarrow\mathbb{R}$ be some random process, which is measurable with respect to the pair of variables (τ,ω) and satisfies to the following condition

$$\int\limits_t^T|\xi_\tau|d\tau<\infty\quad\text{w. p. 1}\quad(t\ge 0).$$

Let $\tau_j^{(N)},\ j=0,1,\ldots,N$ be a partition of the interval $[t,T],\ t\ge 0$ such that

$$t=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad\max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\rightarrow 0\ \ \text{if}\ \ N\rightarrow\infty.\tag{370}$$

Further, for simplicity, we write τ_j instead of $\tau_j^{(N)}$.

Consider the definition of the Stratonovich stochastic integral, which differs from the definition given in [1] (recall that we use definition [1] above in this article).

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\frac{1}{\tau_{j+1}-\tau_j}\int\limits_{\tau_j}^{\tau_{j+1}}\xi_s ds\ \left(f_{\tau_{j+1}}-f_{\tau_j}\right)\stackrel{\mathsf{def}}{=}\int\limits_t^T\xi_\tau\circ df_\tau\tag{371}$$

is called [71] the Stratonovich stochastic integral of the process $\xi_\tau,\ \tau\in[t,T]$, where $\tau_j,\ j=0,1,\ldots,N$ is a partition of the interval $[t,T]$ satisfying the condition (370).

We also denote by

$$\int\limits_t^{\tau} \xi_s \circ df_s$$

the Stratonovich stochastic integral like (371) (if it exists) of $\xi_s \mathbf{1}_{\{s\in[t,\tau]\}}$ for $\tau \in [t,T]$, $t \ge 0$.

It is known [71] (Lemma A.2) that the following iterated Stratonovich stochastic integral

$$J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)} = \int\limits_t^{\tau} \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\ldots \circ d\mathbf{w}_{t_k}^{(i_k)} \tag{372}$$

exists for the case $i_1 = \ldots = i_k \ne 0$, where $\tau \in [t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $i_1,\ldots,i_k = 0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$, $\mathbf{f}_{\tau}^{(i)}$ $(i = 1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

In [72] (2021) an analogue of Theorem 4 (1997) is proved for the case $i_1 = \ldots = i_k \ne 0$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$.

Let us denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\mathsf{def}}{=} \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}, \tag{373}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2,3,\ldots,k)$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (376), $\sum\limits_{\emptyset}$ is supposed to be equal to zero; another notations are the same as in Theorem 4.

Further, by analogy with (51), (54) and using the version of (48) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ (see [12] or [15], Sect. 1.11) instead of (48), we obtain the following generalization of (51) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau)$, $\ldots$, $\psi_k(\tau) \in L_2([t,T])$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1}, \tag{374}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$, $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined as in [67] (1951). Note that in [67] the case of a scalar Wiener process has been considered.

It should be noted that Theorem 1.16 [12] (Sect. 1.11) and Theorem 2 can be reformulated as follows (also see [33], Sect. 15)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr{\rm l.i.m.}\cr}}}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad\hbox{w. p. 1,}\tag{375}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral defined as in [67] (1951) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)};\tag{376}$$

another notations are the same as in Theorem 2.

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}$ in (374) and using the equality (375), we get w. p. 1 the following equality

$$\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\ \sum_{\stackrel{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})}{\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\ \prod_{s=1}^r{\bf 1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p_1,\ldots,p_k\to\infty}{\rm l.i.m.}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^r{\bf 1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})},\tag{377}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ is the multiple Wiener stochastic integral defined as in [67] (1951) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (376).

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$. Then we have

$$\sum_{j=0}^{\infty}\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le$$

$$\le\frac{1}{2}\sum_{j=0}^{\infty}\left(\left(\int\limits_t^T{\bf 1}_{\{\tau<s\}}\phi_j(\tau)\Phi_1(\tau)d\tau\right)^2+\left(\int\limits_t^T{\bf 1}_{\{\tau>s\}}\phi_j(\tau)\Phi_2(\tau)d\tau\right)^2\right)=$$

$$=\frac{1}{2}\left(\int\limits_t^s\Phi_1^2(\tau)d\tau+\int\limits_s^T\Phi_2^2(\tau)d\tau\right)\le\frac{1}{2}\left(\|\Phi_1\|_{L_2([t,T])}^2+\|\Phi_2\|_{L_2([t,T])}^2\right)<\infty,\tag{378}$$

i.e. the series

$$\sum_{j=0}^{\infty}\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau$$

converges absolutely.

By interpreting the integrals in (66)–(69) as Lebesgue integrals, using Fubini's theorem in (66) and Lebesgue's Dominated Convergence Theorem in (68), we obtain (64) (see (69), (378)) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Using the equality (38) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$ as well as absolute convergence of the series on the right-hand side of (38) for this case (see [75] or [12], Sect. 2.1.4), we obtain [73] (Sect. 3.5.2, Theorem 3.5.2) the generalization of (72) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Repeating the steps of the proof of Theorem 12 below the formula (73) using (373), (377) or steps of the proof of Theorem 24 using (373), (377), we obtain for complete orthonormal systems $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ (for which the condition (369) is satisfied) the following equality

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=$$

$$=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=\bar J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{379}$$

w. p. 1, where notations in (379) are the same as in Theorem 4 and $\bar J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (373).

Thus the following two theorems are proved.

Theorem 25 [12], [15], [36]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$ *are such that the folowing condition*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)^2=0 \tag{380}$$

is satisfied for all $r=1,2,\ldots,[k/2]$. *Then, for the sum* $\bar J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Ito stochastic integrals defined by* (373) *the following expansion*

(381)
$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Theorem 26 [12], [36]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^\infty$ $(\phi_0(x) = 1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$*,* $\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2,3,\ldots,k)$ *are such that the condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (30)) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d \in \{1,2,\ldots,r\}$*,* $l_1 > l_2 > \ldots > l_d$*,* $d = 0,1,2,\ldots,r-1$*, where* $r = 1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\text{def}}{=} \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$*. Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Ito stochastic integrals defined by (373) the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_{\tau}^{(i)} = \mathbf{f}_{\tau}^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Note that in Theorems 25, 26 (the case $k = 2$) the condition $\psi_1(\tau)\psi_2(\tau) \in L_2([t,T])$ can be omitted.

Using Theorem 4 together with Proposition 3.1 [72] and the proof of Lemma A.2 [71], we can write $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ w. p. 1 and reformulate Theorems 25, 26 for $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ ($J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (372)).

Let us consider the special case $k = 2$ of Theorem 25 in more detail. In this case, the condition (380) takes the following form (compare with (96))

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1. \tag{382}$$

Recall that the equality (382) is valid for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$ (see [75] or [12], Sect. 2.1.4).

From Proposition 3.1 [72] for the case $k = 2$ we obtain

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)} = \int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)} +$$

$$+\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 \tag{383}$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, $i = 1, \ldots, m$,

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)}$$

is defined by (371), (372) and

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)}$$

is the iterated Ito stochastic integral of the form (1) ($k = 2$).

On the other hand, it is not difficult to show that

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(j)} = \int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(j)} \tag{384}$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, $i \neq j$ $(i, j = 1, \ldots, m)$, another notations are the same as in (383).

Combining (383) and (384), we get (see (373))

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\circ d\mathbf{w}_{t_2}^{(i_2)}=\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1\ \stackrel{\sf def}{=}\ \bar J^{*}[\psi^{(2)}]_{T,t}^{(i_1 i_2)} \tag{385}$$

w. p. 1, where $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, $i_1,i_2=1,\ldots,m.$

It is easy to see that the condition $\phi_0(x)=1/\sqrt{T-t}$ can be omitted in Theorems 25, 26 for the case $k=2$ (see the proof of Theorem 12).

Summing up the above arguments, we obtain the following generalization of Theorem 5 to the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

Theorem 27 [12]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral*

$$J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{f}_{t_1}^{(i_1)}\circ d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{386}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 5, 6 *and $J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ is defined by* (372).

In this section, it is also appropriate to mention the so-called multiple Stratonovich stochastic integral [71] (also see [68]).

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}\ldots\sum_{l_k=0}^{N-1}\frac{1}{\Delta\tau_{l_1}\ldots\Delta\tau_{l_k}}\int\limits_{[\tau_{l_1},\tau_{l_1+1}]\times\ldots\times[\tau_{l_k},\tau_{l_k+1}]}K(t_1,\ldots,t_k)dt_1\ldots dt_k\ \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\ \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\ \bar J^S[K]_{T,t}^{(i_1\ldots i_k)} \tag{387}$$

is called [71] the multiple Stratonovich stochastic integral of the function $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$, where $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\Delta\tau_j=\tau_{j+1}-\tau_j$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$ satisfying the condition (370), $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$, $\mathbf{f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

Note that in [71] the case $i_1=\ldots=i_k\ne 0$ was considered. We also denote by $\bar J^S[K]_{s,t}^{(i_1\ldots i_k)}$ the multiple Stratonovich stochastic integral (387) (if it exists) of the function $K(t_1,\ldots,t_k)\mathbf{1}_{\{(t_1,\ldots,t_k)\in[t,s]^k\}}$, where $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$, $s\in[t,T]$, $t\ge 0$.

Let the function $K(t_1,\ldots,t_k)$ be chosen as follows

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1\le\ldots\le t_k\\ \\ 0, & \text{otherwise}\end{cases},\qquad (388)$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

We will denote the multiple Stratonovich stochastic integral (387) of the function (388) as follows $\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$. It is known [71] (Lemma A.2) that the Stratonovich stochastic integrals $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ and $\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ exist for the case $i_1=\ldots=i_k\ne 0$. Moreover, $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ w. p. 1 for this case [71] (Lemma A.2).

Recall that an expansion similar to (40) was obtained in [69] for the multiple Stratonovich stochastic integral (387) under the condition of convergence of trace series.

Recently, another approach to the expansion of integral (387) has been proposed (assuming that the integral (387) exists), where multiple Fourier–Walsh and Fourier–Haar series ($k\in\mathbb{N}$) have been applied [77]. The convergence was proved with respect to the special subsequence ($p_1=\ldots=p_k=p=2^m$, $m\to\infty$ in a formula similar to (381) [77]).

15. Modification of Condition 3 of Theorem 12 Using Parseval's Equality

Let us make some remarks about the development of the approach based on Theorem 12 and describe the algorithm of the verification of Condition 3 of Theorem 12. First, consider the case $k=2n+1$, $n=3,4,\ldots$ (k is the multiplicity of the iterated Stratonovich stochastic integral (39)). Let Conditions 1 and 2 of Theorem 12 be satisfied. Consider the equality (91). The right-hand side of (91) has the form

$$\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}.$$

Iterated application of the formulas (179), (180), (193) separately to the values

$$\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

and

$$\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}$$

($g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (30), $r=1,2,\ldots,[k/2]$, $2r<k$) gives the following representation (see (92))

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2\le$$

$$\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\int\limits_{[t,T]^{k-2r}}R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\;dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\right)^2, \tag{389}$$

where

$$R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=$$

$$=\sum_{d=1}^{4^r}\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)-$$

$$-\sum_{d=1}^{2^r}\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\in L_2([t,T]^{k-2r})$$

and

$$\int\limits_{[t,T]^{k-2r}}R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times$$

$$\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\;dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k$$

is the Fourier coefficient of

$$\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=$$

$$=R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q).$$

Also note that some of the functions

$$\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$$

and

$$\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$$

can be identically equal to zero.

Obviously, we could use another representation for the function

(390) $$R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$$

based on the left-hand side of the equality (91) and (179), (180), (193) (see Sect. 5, 8 for details). In Sect. 8, we considered the function (390) in detail for the case $k\ge 5$, $r=1$.

Parseval's equality gives

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\int\limits_{[t,T]^{k-2r}}R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\right)^2=$$

$$=\int\limits_{[t,T]^{k-2r}}\left(\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\right)^2\times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k=$$

(391) $$=\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}.$$

Combining (389) and (391), we obtain

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)^2\le$$

$$\le\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{392}$$

Assume that we have succeeded in proving the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0. \tag{393}$$

Applying (392) and (393), we get (compare with (92))

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)^2=0. \tag{394}$$

As noted in Sect. 5, Condition 3 of Theorem 12 can be replaced by a weaker condition (92) (or (394)). Also Condition 3 of Theorem 12 can be replaced by (393). From (394) we obviously obtain

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{395}$$

According to (91), the equality (395) will be satisfied if

$$\lim_{p\to\infty}S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=0, \tag{396}$$

where $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (30), $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, $r=1,2,\ldots,[k/2]$,

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\rm def}{=}\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$, where

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}},\quad S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}$$

are defined by (34), (35), $l=1,2,\ldots,r$ (see Sect. 5 for details).

Let us make some remarks about the function (390) for the case $k>5$, $r=2$. In this case, using the left-hand side of the equality (91) and (179), (180), (193), we represent the function (390) as the sum of several functions. In particular, among these functions will be the following functions

$$Q_p(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_{q-1},t_{q+1},\ldots,t_{g-1},t_{g+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_{q-1}<t_{q+1}<\ldots<t_{g-1}<t_{g+1}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times$$

$$\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{q+1}}\psi_q(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{g-1}}\psi_g(\tau)\phi_{j_q}(\tau)d\tau, \tag{397}$$

$$\bar Q_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right), \tag{398}$$

$$\tilde Q_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau, \tag{399}$$

$$\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-1}<t_{l+2}<\ldots<t_{q-1}<t_{q+2}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right). \qquad (400)$$

Note that the pairs (g_1,g_2), (g_3,g_4) for the functions (398) and (399) have the property: $g_2=g_1+1$, $g_4=g_3+1$, $g_3=g_2+1$. At the same time, the pairs (g_1,g_2), (g_3,g_4) for the function (397) have the following property: $g_2>g_1+1$, $g_4>g_3+1$, $g_3\ge g_2+1$. For the function (400), the pairs (g_1,g_2), (g_3,g_4) chosen as follows: $g_2>g_1+1$, $g_4>g_3+1$, $g_4=g_2+1$, $g_3=g_1+1$. Generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered. We consider the functions (397)–(400) only as an example.

Suppose that $s+1=l-1$, $l+1=q-1$, $q+1=g-1$ in (397). Let us show that (we consider the case of Legendre polynomials; the trigonometric case is simpler and can be considered similarly)

$$\lim_{p\to\infty}\left\|Q_p\right\|^2_{L_2([t,T]^{k-4})}=0, \qquad (401)$$

$$\lim_{p\to\infty}\left\|\bar{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0, \qquad (402)$$

$$\lim_{p\to\infty}\left\|\tilde{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0, \qquad (403)$$

$$\lim_{p\to\infty}\left\|\hat{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0. \qquad (404)$$

First consider the proof of (401). We have $(s+1=l-1,\ l+1=q-1,\ q+1=g-1)$

$$\left(Q_p(t_1,\ldots,t_{l-3},t_{l-1},t_{l+1},t_{l+3},t_{l+5},\ldots,t_k)\right)^2=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-3}<t_{l-1}<t_{l+1}<t_{l+3}<t_{l+5}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2. \qquad (405)$$

Using the estimate (129), we obtain

$$\left|\int\limits_t^s \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{K}{j^{1-\varepsilon/2}(1-z^2(s))^{1/4-\varepsilon/4}}, \tag{406}$$

where $j\in\mathbb{N}$, $s\in(t,T)$, $z(s)$ is defined by (100), $\varepsilon\in(0,1)$, constant K does not depend on j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, $\psi(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$.

Applying (406) and (132) (we take ε instead of $\varepsilon/2$ in (132)), we get

$$\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2\le$$

$$\le\frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-1}))^{1-\varepsilon}(1-z^2(t_{l+3}))^{1-\varepsilon}}, \tag{407}$$

where $t_{l-1},t_{l+3}\in(t,T)$, constant K_1 is independent of p. Combining (405) and (407), we have (401).

Let us prove (402). The following equality is proved in Sect. 12 [37] (also see Sect. 2.9 [12]) for the case of Legendre polynomials ($n>m$; $n,m\in\mathbb{N}$)

$$\sum_{j=m+1}^{n}C_{jj}(s)=\sum_{j=m+1}^{n}\int\limits_t^s\psi_2(\theta)\phi_j(\theta)\int\limits_t^{\theta}\psi_1(\tau)\phi_j(\tau)d\tau d\theta=$$

$$=\frac{T-t}{4}\int\limits_{-1}^{z(s)}\psi_1(u(x))\psi_2(u(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx-$$

$$-\frac{(T-t)^2}{8}\sum_{j=m+1}^{n}\frac{1}{2j+1}\int\limits_{-1}^{z(s)}\left(P_{j+1}(y)-P_{j-1}(y)\right)\psi_1'(u(y))\times$$

$$\times\Biggl(\left(P_{j+1}(z(s))-P_{j-1}(z(s))\right)\psi_2(s)-\left(P_{j+1}(y)-P_{j-1}(y)\right)\psi_2(u(y))-$$

$$\left.-\frac{T-t}{2}\int\limits_y^{z(s)}\left(P_{j+1}(x)-P_{j-1}(x)\right)\psi_2'(u(x))dx\right)dy, \tag{408}$$

where $s\in(t,T)$,

$$C_{jj}(s)=\int\limits_t^s\psi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_j(\theta)d\theta d\tau,$$

$$u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $u(y)$.

Applying the estimate (128) in (408) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we obtain

$$\left|\sum_{j=m+1}^{n} C_{jj}(s)\right|\le C_1\left(\frac{1}{n^{1-\varepsilon}}+\frac{1}{m^{1-\varepsilon}}\right)\int\limits_{-1}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/2-\varepsilon/2}}+$$

$$+C_2\sum_{j=m+1}^{n}\frac{1}{j^{2-\varepsilon}}\left(\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/2-\varepsilon/2}}+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/4-\varepsilon/4}}+\right.$$

$$\left.+\int\limits_{-1}^{z(s)}\frac{1}{\left(1-y^2\right)^{1/4-\varepsilon/4}}\int\limits_{y}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/4-\varepsilon/4}}dy\right),\tag{409}$$

where $s\in(t,T)$, constants C_1, C_2 do not depend on n and m.

From (409) we have

$$\left|\sum_{j=m+1}^{\infty} C_{jj}(s)\right|\le\frac{K_1}{m^{1-\varepsilon}}+K_2\sum_{j=m+1}^{\infty}\frac{1}{j^{2-\varepsilon}}\left(1+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\right),\tag{410}$$

where $s\in(t,T)$, constants K_1, K_2 do not depend on m.

Applying (132) (we take ε instead of $\varepsilon/2$ in (132)) in (410), we get

$$\left|\sum_{j=m+1}^{\infty} C_{jj}(s)\right|\le\frac{K}{m^{1-\varepsilon}\left(1-z^2(s)\right)^{1/4-\varepsilon/4}},\tag{411}$$

where $s\in(t,T)$, constant K is independent of m.

Using the estimate (411), we obtain (see (398))

$$\left(\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right)\right)^2\le$$

$$\leq \frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-2}))^{1-\varepsilon}}, \tag{412}$$

where $t_{l-2} \in (t,T)$, constant K_1 is independent of p. The inequality (412) completes the proof of (402).

Let us prove (403). The following equality is proved in Sect. 12 [37] (also see Sect. 2.9 [12]) for the cases of Legendre polynomials and trigonometric functions

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1 j_1}(s) = \sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s), \tag{413}$$

where $s \in (t,T)$ and

$$C_{jj}(s) = \int\limits_t^s \psi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau} \psi_1(\theta)\phi_j(\theta)d\theta d\tau.$$

Applying (413) in (399), we get

$$\left(\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2 \leq$$

$$\leq \left(\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}} \psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.$$

$$\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 =$$

$$= \left(\frac{1}{2}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\psi_{l+2}(\tau)d\tau-\right.$$

$$-\sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 =$$

$$= (a-b)^2 \leq 2(|a|^2+|b|^2). \tag{414}$$

Further, we have

$$|a| \le \frac{1}{2} \int\limits_t^{t_{l+3}} |\psi_{l+1}(\tau)| \left| \sum_{j_l=p+1}^{\infty} \int\limits_t^{\tau} \psi_{l-1}(\theta)\phi_{j_l}(\theta) \int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u) du d\theta \right| |\psi_{l+2}(\tau)| \, d\tau, \tag{415}$$

$$|b| \le \sum_{j_q=0}^{p} \int\limits_t^{t_{l+3}} |\psi_{l+1}(\tau)\phi_{j_q}(\tau)| \left| \sum_{j_l=p+1}^{\infty} \int\limits_t^{\tau} \psi_{l-1}(\theta)\phi_{j_l}(\theta) \int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u) du d\theta \right| \times$$

$$\times \left| \int\limits_t^{\tau} \psi_{l+2}(u)\phi_{j_q}(u) du \right| d\tau. \tag{416}$$

Combining (411) and (415), we obtain

$$|a| \le \frac{C}{p^{1-\varepsilon}}, \tag{417}$$

where constant C is independent of p.

Separating in (416) the term with the number $j_q = 0$ and then applying (260), (103), (411), we obtain

$$|b| \le \frac{K}{p^{1-\varepsilon}} \left(\int\limits_t^{t_{l+3}} \frac{d\tau}{\left(1 - z^2(\tau)\right)^{1/2-\varepsilon/4}} + \sum_{j_q=1}^{p} \frac{1}{j_q} \int\limits_t^{t_{l+3}} \frac{d\tau}{\left(1 - z^2(\tau)\right)^{3/4-\varepsilon/4}} \right) \le$$

$$\le \frac{K_1}{p^{1-\varepsilon}} \left(1 + \sum_{j_q=1}^{p} \frac{1}{j_q} \right) \le \frac{K_1}{p^{1-\varepsilon}} \left(2 + \int\limits_1^{p} \frac{dx}{x} \right) =$$

$$= \frac{K_1 \left(2 + lnp\right)}{p^{1-\varepsilon}} \ \to \ 0 \tag{418}$$

if $p \to \infty$. The estimates (414), (417), (418) complete the proof of (403).

Finally, consider the proof of (404). Using the elementary inequality $|ab| \le (a^2 + b^2)/2$ and Parseval's equality, we have

$$\left(\hat{Q}_p(t_1, \ldots, t_{l-1}, t_{l+2}, \ldots, t_{q-1}, t_{q+2}, \ldots, t_k) \right)^2 \le$$

$$\le \left(\sum_{j_l=p+1}^{\infty} \sum_{j_{l+1}=p+1}^{\infty} \left| \int\limits_t^{t_{l+2}} \psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_l(u)\phi_{j_l}(u) du d\theta \right| \times \right.$$

$$\left. \times \left| \int\limits_t^{t_{q+2}} \psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta) \int\limits_t^{\theta} \psi_q(u)\phi_{j_l}(u) du d\theta \right| \right)^2 \le$$

$$\leq \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\leq$$

$$\leq \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\leq$$

$$\leq \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+2}}\psi_{l+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)du\right)^2d\theta+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{q+2}}\psi_{q+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)du\right)^2d\theta\right)^2. \tag{419}$$

Note that

$$\sum_{j=p+1}^{\infty}\frac{1}{j^2}\leq\int\limits_p^{\infty}\frac{dx}{x^2}=\frac{1}{p}. \tag{420}$$

From (419) and (420), (103) we obtain

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2\leq$$
$$\leq\frac{K}{p^2}\ \to\ 0$$

if $p\to\infty$, where constant K does not depend on p. Thus the equalities (401)–(404) are proved.

Recall that the function (390) (this function is defined using the left-hand side of the equality (91)) for the case $k>5$, $r=2$ is represented as the sum of several functions. Four of them, namely Q_p, $\bar{Q}_p$, $\tilde{Q}_p$, $\hat{Q}_p$ (these functions correspond to the particular case of choosing the pairs (g_1,g_2), (g_3,g_4); generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered), have been studied above. Absolutely similarly, we can consider the remaining functions (for all possible pairs (g_1,g_2), (g_3,g_4)) whose sum is the function (390) for the case $k>5$, $r=2$. As a result, we will have

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=2).$$

After that, we can go to the function (390) for the case $k > 5$, $r = 3$, $2r < k$ (this function is defined using the left-hand side of the equality (91)) and follow the same steps as above. This will lead us to the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})} = 0 \quad (k > 5,\ r = 3,\ 2r < k).$$

Then we can move on to the next step and so on. As a result, we get the equality (393) ($r = 1, 2, \ldots, [k/2]$). Thus the condition (92) is satisfied for the case $k = 2n + 1$, $n = 3, 4, \ldots$ (recall that the condition (92) is weaker than Condition 3 of Theorem 12 and the condition (92) can be used in Theorem 12 instead of Condition 3).

For the case $k = 2n$, $n = 3, 4, \ldots$ we follow the above steps for $r = 1, 2, \ldots, [k/2] - 1$ ($2r \le k - 2$). For $2r = k$ we use the same technique as in the proof of the equalities (141)–(143). Recall that we used (65), (72) and Parseval's equality in the proof of (141)–(143).

The obvious disadvantage of the proposed algorithm is the drastic increase of complexity of the proof when moving from $r = 1$ to $r = 2$, $r = 2$ to $r = 3$ and so on.

The proofs of Theorems 16 and 17 contain a rather simple trick of passing from $r = 1$ to $r = 2$. Unfortunately, this procedure cannot be applied already at the transition from $r = 2$ to $r = 3$.

Note that the case $k = 6$, $r = 3$ was successfully considered in Theorem 22 under the following simplifying assumption: $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$.

Nevertheless, the results obtained in this paper are quite sufficient for practical needs (see Chapters 4 and 5 [12] for details).

References

[1] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992, 632 pp.

[2] Milstein G.N. Numerical Integration of Stochastic Differential Equations. [In Russian]. Ural University Press, Sverdlovsk, 1988, 225 pp.

[3] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[5] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl., 10, 4 (1992), 431-441.

[6] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010, 868 pp.

[7] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[8] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[9] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1-A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[12] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184v45 [math.PR]. 2023, 998 pp.

[13] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[14] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[15] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs (Third Edition). Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[20] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[21] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[22] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[23] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[24] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[25] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.

[26] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[27] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[28] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier–Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[29] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[30] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[31] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604

[33] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [in English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.

[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [in English]. arXiv:1801.01079 [math.PR]. 2023, 71 pp.

[35] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. [in English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[36] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2023, 160 pp.

[37] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [in English]. [in English]. arXiv:1712.09516 [math.PR]. 2023, 223 pp.

[38] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.

[39] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [in English]. arXiv:1801.01564 [math.PR]. 2018, 66 pp.

[40] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier–Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.

[41] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [in English]. arXiv:1712.08991 [math.PR]. 2013, 58 pp.

[42] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [in English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[43] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [in English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[44] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[45] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[46] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[47] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[48] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[49] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2023, 80 pp.

[51] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[52] Kuznetsov D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [In English]. arXiv:1801.04634 [math.PR]. 2018, 28 pp.

[53] Allen E. Approximation of triple stochastic integrals through region subdivision. Communicat. in Appl. Anal. (Special Tribute Issue to Prof. V. Lakshmikantham), 17 (2013), 355-366.

[54] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. [In Russian]. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[55] Kuznetsov, D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR], 2023, 58 pp.

[56] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[57] Kuznetsov D.F. New representations of the Taylor–Stratonovich expansion. J. Math. Sci. (N.Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[58] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat. 5, 36 (1965), 1560-1564.

[59] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci. 3 (1965), 213-229.

[60] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[61] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[62] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Nauka, Moscow, 1974, 696 pp.

[63] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. Ph.D. Thesis, California Inst. of Technology, 2006, 225 pp.

[64] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[65] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[66] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[67] Itô K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.

[68] Budhiraja A. Multiple stochastic integrals and Hilbert space valued traces with applications to asymptotic statistics and non-linear filtering. Ph. D. Thesis, The University of North Caroline at Chapel Hill, 1994, VII+132 pp.

[69] Rybakov K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), 81-115. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.5.html

[70] Johnson G.W., Kallianpur G. Homogeneous chaos, p-forms, scaling and the Feynman integral. Transactions of the American Mathematical Society, 340 (1993), 503-548.

[71] Bardina X., Jolis M. Weak convergence to the multiple Stratonovich integral. Stochastic Processes and their Applications, Elsevier, 90, 2 (2000), 277-300.

[72] Bardina X., Rovira C. On the strong convergence of multiple ordinary integrals to multiple Stratonovich integrals. Publicacions Matemàtiques, 65 (2021), 859-876. DOI: http://doi.org/10.5565/PUBLMAT6522114

[73] Pugachev V.S. Lectures on Fuctional Analysis. MAI, Moscow, 1996, 744 pp.

[74] Hairer, M. On Malliavin's proof of Hörmander's theorem. Bulletin Des Sciences Mathématiques, 135, 6-7 (2011), 650-666.

[75] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

[76] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional Wiener process based on Multiple Fourier-Legendre series. MATEC Web of Conferences, 362 (2022), article id: 01014, 10 pp. DOI: http://doi.org/10.1051/matecconf/202236201014

[77] Rybakov K.A. Features of the expansion of multiple stochastic Stratonovich integrals using Walsh and Haar functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), 137-150. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.9.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITY 2 BASED ON DOUBLE FOURIER–LEGENDRE SERIES SUMMARIZED BY PRINGSHEIM METHOD

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansion of iterated Stratonovich stochastic integrals of second multiplicity into the double series of products of standard Gaussian random variables. The proof of expansion is based on the application of double Fourier–Legendre series and double trigonometric Fourier series summarized by Pringsheim method. The results of the article are generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$, where $\psi_1(\tau),\psi_2(\tau)$ are weight functions of the iterated Stratonovich stochastic integral of second multiplicity. The considered expansion can be applied to the numerical integration of Ito stochastic differential equations. Some recent results on the expansion of iterated Stratonovich stochastic integrals of multiplicities 3 to 6 are given.

Contents

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1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the Ito SDE (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[5]. Moreover, one of the most important features of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a nonrandom function at the interval $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively. In this paper we use the definition of the Stratonovich stochastic integral from [3].

Note that usually in applications the functions $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ are equal to 1 or have a binomial form. More precisely, $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[6]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k,\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [7]-[42].

Effective solution of the problem of combined mean-square approximation for collections of iterated Stratonovich stochastic integrals (3) of second multiplicity composes the subject of this article.

It is well known that the mean-square approximation of iterated Ito and Stratonovich stochastic integrals (2), (3) using multiple integral sums requires significant computational costs [11] since this approach implies the partitioning of the integration interval $[t,T]$ of the iterated stochastic integrals

(2), (3) into parts ($T-t$ is already a sufficiently small value since $T-t$ plays the role of an integration step in numerical methods for solving Ito SDEs).

More efficient approximation methods for the iterated stochastic integrals (2), (3) use Fourier series, and they do not require the interval $[t,T]$ to be subdivided into smaller parts. One such method was proposed in [2] and elaborated in [3], [4]. This method, which received widespread use, is based on the Karhunen–Loeve expansion of the Brownian bridge process [2] in the eigenfunctions of its covariance, which form a complete orthonormal trigonometric basis in the space $L_2([t,T])$.

Note that in [11] (2006) the more general and effective method (the so-called method of generalized multiple Fourier series) for the mean-square approximation of iterated Ito stochastic integrals (2) was proposed. This method is based on the generalized multiple Fourier series that converge in the sense of norm in Hilbert space $L_2([t,T]^k)$, where $[t,T]^k$ is the hypercube $[t,T]\times\ldots\times[t,T]$ (k times) and k is the multiplicity of the iterated Ito stochastic integral. The method of generalized multiple Fourier series was developed in [12]-[41], [43]-[61].

An important feature of the method of generalized multiple Fourier series is that various complete orthonormal systems of functions in the space $L_2(t,T])$ can be used (the method proposed in [2] admits only the trigonometric system of functions). Hence, we can state the problem of comparing the efficiency of using different complete orthonormal systems of functions in the space $L_2(t,T])$ in the context of numerical solution of Ito SDEs. This problem has been solved in [37], [38] (also see [25]-[28]). In particular, in [25]-[28], [37], [38] it was shown that the optimal system of basis functions in the framework of numerical solution of Ito SDEs is the system of Legendre polynomials. This fact is true at least for high-order strong numerical methods with orders of convergence $1.5, 2.0, \ldots$ That is why the part of this article is devoted to the expansions of iterated Stratonovich stochastic integrals with multiplicity 2 based on multiple Fourier–Legendre series.

Usage of Fourier series with respect to the system of Legendre polynomials for approximation of iterated stochastic integrals took place for the first time in [7] (1997), [8] (1998), [9] (also see [10]-[41], [43]-[60]).

The results of [7] (also see [8]-[41], [43]-[59]) convincingly testify that there is a doubtless relation between multiplier factor $1/2$, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Ito stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its Fourier series converges to the value

$$\frac{f(x+0)+f(x-0)}{2}.$$

In addition, in [7], [8], [16]-[20], [23]-[29], [31], [33], [35], [39], [49]-[51], [54], [58] several theorems on expansion of iterated Stratonovich stochastic integrals were formulated and proved. As shown in these papers, the final formulas for expansions of iterated Stratonovich stochastic integrals are more compact than their analogues for iterated Ito stochastic integrals.

This paper continues the study of the relationships between generalized multiple Fourier series and iterated stochastic integrals. We use the double Fourier–Legendre series and double trigonometric Fourier series (summarized by Pringsheim method) for the proof of Theorem 5.3 [24] or Theorem 2.1 [25] (also see [26], [28]). As shown below the conditions of these theorems can be weakened.

Moreover, the mentioned theorems are generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

2. Method of Generalized Multiple Fourier Series

Let us consider an approach to the expansion of iterated Ito stochastic integrals [11]-[41], [43]-[61] (the so-called method of generalized multiple Fourier series).

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$
K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\quad k\ge 2, \tag{4}
$$

and $K(t_1)\equiv\psi_1(t_1),\ t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well-known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$
\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,
$$

where

$$
C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{5}
$$

$$
\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.
$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ such that

$$
t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}
$$

Theorem 1 [11] (2006) (also see [12]-[41], [43]-[61]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in $L_2([t,T])$. Then*

$$
J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -
$$

$$
-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}
$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$
\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},
$$

$$
\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\bigr\},
$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (5), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (6).

Note that the condition of continuity of the functions $\phi_j(x)$ $(j=0,1,\ldots)$ can be weakened (see [11]-[20], [23]-[28]). Another versions and generalizations of Theorem 1 can be found in [12]-[41], [43]-[61].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [11]-[41], [43]-[59]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{10}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{11}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^4\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{12}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^5\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{13}$$

where $\mathbf{1}_A$ is the indicator of the set A.

For further consideration, let us consider the generalization of formulas (9)–(13) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{14}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (14) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{15}$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}$.

Below there are several examples of sums in the form (15)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$

$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$

$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$

$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (7) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr), \tag{16}$$

where $[x]$ is an integer part of a real number x, $\prod\limits_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\sf def}{=} 0$; another notations are the same as in Theorem **1**.

In particular, from (16) for $k = 5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_5=0}^{p_5} C_{j_5\ldots j_1} \Biggl(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} -$$

$$
-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+
$$

$$
+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).
$$

The last equality obviously agrees with (13).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [25] (Sect. 1.11), [46] (Sect. 15), [61]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x, $\prod_{\emptyset}\stackrel{\sf def}{=}1$, $\sum_{\emptyset}\stackrel{\sf def}{=}0$; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [62] using Hermite polynomials. Note that we use another notations [25] (Sect. 1.11), [46] (Sect. 15), [61] in comparison with [62]. Moreover, the proof from [62] is different from the proof given in [25] (Sect. 1.11), [46] (Sect. 15), [61]. The results of [62], as well as the results of [25] (Sect. 1.11), [46] (Sect. 15), [61] are based on our idea [11] (2006) on the expansion of the kernel (4) (or $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$) into a generalized multiple Fourier series (see [11], Chapter 5, Theorem 5.1, pp. 235-245 or [25], Chapter 1 for details).

3. Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Second Multiplicity. Some Old Results

In a number of works of the author [16]-[20], [23]-[29], [31], [33], [35], [49], [51], [54], [58] Theorems 1, 2 have been adapted for iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6 (also see the case of multiplicity k ($k\in\mathbb{N}$) in [25] (Sect. 2.10), [29], [33], [47]). For example, we can formulate the following theorem for iterated Stratonovich stochastic integrals of second multiplicity.

Theorem 3 [16]-[20], [23]-[29], [49], [51], [54], [58]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function on* $[t,T]$ *and* $\psi_1(\tau)$ *is twice continuously differentiable functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m), \tag{17}$$

where $J^*[\psi^{(2)}]_{T,t}$ *is defined by* (3); *another notations are the same as in Theorem* 1.

Note that the proof of Theorem 3 is based on the proof of the following equality

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}, \tag{18}$$

where $C_{j_1j_1}$ is defined by (5) for $k=2$ and $j_1=j_2$; $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

According to the standard relation between Ito and Stratonovich stochastic integrals, we can write w. p. 1 (with probability 1)

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1, \tag{19}$$

where we assume that the functions $\psi_1(\tau),\psi_2(\tau)$ are continuous at the interval $[t,T]$. This condition is related to the definition of the Stratonovich stochastic integral that we use [3] (also see Sect. 2.1.1 [25]).

From the other hand according to (10), we obtain

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{20}$$

From (18)–(20) we get (17). Note that the existence of the limit on the right-hand side of (18) will be proved below (see Lemma 2 and Theorem 7).

The proof of Theorem 3 [16]-[20], [23]-[29], [31], [33], [35], [49], [51], [54], [58] is based on double (iterated) Fourier–Legendre series and analogous trigonometric Fourier series. This proof uses double integration by parts, which leads to the requirement of double continuous differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$.

In this article, we formulate and prove the analogue of Theorem 3 (Theorem 6, see below) but under the weakened conditions: the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are assumed to be continuously differentiable only one time at the interval $[t,T]$. At that we will use double Fourier–Legendre series and double trigonometric Fourier series summarized by Pringsheim method for the proof of Theorem 6 (see below).

In Sect. 5 (see Theorem 7), we generalize the equality (18) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$.

4. Proof of the Equality (18). The Case of Legendre Polynomials and Trigonometric Functions as well as Continuously Differentiable Functions $\psi_1(\tau),\ \psi_2(\tau)$

Let $P_j(x)$ $(j=0,1,2,\ldots)$ be the Legendre polynomial. Consider the function $f(x,y)$ defined for $(x,y)\in[-1,1]^2$.

Consider the double Fourier–Legendre series summarized by Pringsheim method and corresponding to the function $f(x,y)$

$$\lim_{n,m\to\infty}\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{\sqrt{(2j+1)(2i+1)}}{2}C_{ij}^{*}P_i(x)P_j(y)\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\sum_{i,j=0}^{\infty}\frac{\sqrt{(2j+1)(2i+1)}}{2}C_{ij}^{*}P_i(x)P_j(y), \tag{21}$$

where

$$C_{ij}^{*}=\frac{\sqrt{(2j+1)(2i+1)}}{2}\int\limits_{[-1,1]^2}f(x,y)P_i(x)P_j(y)dxdy. \tag{22}$$

Let us consider the generalization for the case of two variables [63] of the theorem on equiconvergence for the Fourier–Legendre series [64].

Theorem 4 [63]. *Let $f(x,y)\in L_2([-1,1]^2)$ and the function*

$$f(x,y)(1-x^2)^{-1/4}(1-y^2)^{-1/4}$$

is integrable on the square $[-1,1]^2$. Moreover, let

$$|f(x,y)-f(u,v)|\le G(y)|x-u|+H(x)|y-v|,$$

where $G(y),H(x)$ are bounded functions on the square $[-1,1]^2$. Then for all $(x,y)\in(-1,1)^2$ the following equality is satisfied

$$\lim_{n,m\to\infty}\Biggl(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{\sqrt{(2j+1)(2i+1)}}{2}C_{ij}^{*}P_i(x)P_j(y)-$$

$$-(1-x^2)^{-1/4}(1-y^2)^{-1/4}S_{nm}(\arccos x,\arccos y,F)\Biggr)=0, \tag{23}$$

where $S_{nm}(\theta,\varphi,F)$ is a partial sum of the double trigonometric Fourier series of the auxiliary function

$$F(\theta,\varphi)=\sqrt{|\sin\theta|}\sqrt{|\sin\varphi|}f(\cos\theta,\cos\varphi),\quad \theta,\varphi\in[0,\pi],$$

the Fourier coefficient C^*_{ij} *has the form* (22). *At that, the convergence in* (23) *is uniform on the rectangle*

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \textit{for any} \quad \varepsilon, \delta > 0.$$

From Theorem 4, for example, follows that for all $(x, y) \in (-1, 1)^2$ the following equality is fulfilled

$$\lim_{n,m\to\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{\sqrt{(2j+1)(2i+1)}}{2}C^*_{ij}P_i(x)P_j(y) - f(x,y)\right) = 0 \tag{24}$$

and convergence in (24) is uniform on the rectangle

$$[-1+\varepsilon, 1-\varepsilon] \times [-1+\delta, 1-\delta] \quad \text{for any} \quad \varepsilon, \delta > 0$$

if the corresponding conditions of convergence of the double trigonometric Fourier series of the auxiliary function

$$g(x, y) = f(x, y)(1-x^2)^{1/4}(1-y^2)^{1/4} \tag{25}$$

are satisfied.

Note that Theorem 4 does not imply any conclusions on the behavior of the double Fourier–Legendre series on the boundary of the square $[-1, 1]^2$.

For each $\delta > 0$ *let us call the exact upper edge of the difference* $|f(\mathbf{t}') - f(\mathbf{t}'')|$ *in the set of all points* $\mathbf{t}'$, $\mathbf{t}''$ (*which belong to the domain* D) *as the module of continuity of the function* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *in the* k*-dimentional domain* D ($k \ge 1$) *if the distance* $\rho(\mathbf{t}', \mathbf{t}'')$ *between* $\mathbf{t}$ *and* $\mathbf{t}''$ *satisfies the condition* $\rho(\mathbf{t}', \mathbf{t}'') < \delta$.

We will say that the function of k ($k \ge 1$) *variables* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *belongs to the Hölder class with the parameter* $\alpha \in (0, 1]$ ($f(\mathbf{t}) \in C^{\alpha}(D)$) *in the domain* D *if the module of continuity of the function* $f(\mathbf{t})$ ($\mathbf{t} = (t_1, \ldots, t_k)$) *in the domain* D *has orders* $o(\delta^{\alpha})$ ($\alpha \in (0, 1)$) *and* $O(\delta)$ ($\alpha = 1$).

In 1967, Zhizhiashvili L.V. proved that the rectangular sums of multiple trigonometric Fourier series of the function of k variables in the hypercube $[t, T]^k$ converge uniformly in the hypercube $[t, T]^k$ to this function if it belongs to the class $C^{\alpha}([t, T]^k)$, $\alpha > 0$ (definition of the Holder class with the parameter $\alpha > 0$ can be found in the well-known mathematical analysis tutorials; see, for example, [65]). More precisely, the following theorem is correct.

Theorem 5 [65]. *If the function* $f(x_1, \ldots, x_n)$ *is periodic with period* 2π *with respect to each variable and belongs in* $\mathbb{R}^n$ *to the Holder class* C^{α} *for any* $\alpha > 0$, *then the rectangular partial sums of the multiple trigonometric Fourier series of the function* $f(x_1, \ldots, x_n)$ *converge to this function uniformly in* $\mathbb{R}^n$.

Lemma 1. *Let the function* $f(x, y)$ *satisfies to the following condition*

$$|f(x, y) - f(x_1, y_1)| \le C_1|x - x_1| + C_2|y - y_1|,$$

where $C_1, C_2 < \infty$ *and* (x, y), $(x_1, y_1) \in [-1, 1]^2$. *Then the following inequality is fulfilled*

$$|g(x, y) - g(x_1, y_1)| \le K\rho^{1/4}, \tag{26}$$

where $g(x,y)$ *has the form* (25),

$$\rho=\sqrt{(x-x_1)^2+(y-y_1)^2},$$

(x,y) *and* $(x_1,y_1)\in[-1,1]^2$, $K<\infty$.

Proof. First, we assume that $x\neq x_1$, $y\neq y_1$. In this case we have

$$|g(x,y)-g(x_1,y_1)|=|(1-x^2)^{1/4}(1-y^2)^{1/4}(f(x,y)-f(x_1,y_1))+$$

$$+f(x_1,y_1)((1-x^2)^{1/4}(1-y^2)^{1/4}-(1-x_1^2)^{1/4}(1-y_1^2)^{1/4})|\le C_1|x-x_1|+C_2|y-y_1|+$$

$$+C_3|(1-x^2)^{1/4}(1-y^2)^{1/4}-(1-x_1^2)^{1/4}(1-y_1^2)^{1/4}|,\quad C_3<\infty. \tag{27}$$

Moreover,

$$|(1-x^2)^{1/4}(1-y^2)^{1/4}-(1-x_1^2)^{1/4}(1-y_1^2)^{1/4}|=$$

$$=|(1-x^2)^{1/4}((1-y^2)^{1/4}-(1-y_1^2)^{1/4})+(1-y_1^2)^{1/4}((1-x^2)^{1/4}-(1-x_1^2)^{1/4})|\le$$

$$\le|(1-y^2)^{1/4}-(1-y_1^2)^{1/4}|+|(1-x^2)^{1/4}-(1-x_1^2)^{1/4}|, \tag{28}$$

$$|(1-x^2)^{1/4}-(1-x_1^2)^{1/4}|=$$

$$=|((1-x)^{1/4}-(1-x_1)^{1/4})(1+x)^{1/4}+(1-x_1)^{1/4}((1+x)^{1/4}-(1+x_1)^{1/4})|\le$$

$$\le K_1(|(1-x)^{1/4}-(1-x_1)^{1/4}|+|(1+x)^{1/4}-(1+x_1)^{1/4}|),\quad K_1<\infty. \tag{29}$$

It is not difficult to see that

$$|(1\pm x)^{1/4}-(1\pm x_1)^{1/4}|=$$

$$=\frac{|(1\pm x)-(1\pm x_1)|}{((1\pm x)^{1/2}+(1\pm x_1)^{1/2})((1\pm x)^{1/4}+(1\pm x_1)^{1/4})}=$$

$$=|x_1-x|^{1/4}\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}}\cdot\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}}\le|x_1-x|^{1/4}. \tag{30}$$

The last inequality follows from the obvious inequalities

$$\frac{|x_1-x|^{1/2}}{(1\pm x)^{1/2}+(1\pm x_1)^{1/2}}\le 1,$$

$$\frac{|x_1-x|^{1/4}}{(1\pm x)^{1/4}+(1\pm x_1)^{1/4}}\le 1.$$

From (27)–(30) we obtain

$$|g(x,y)-g(x_1,y_1)|\le C_1|x-x_1|+C_2|y-y_1|+C_4(|x_1-x|^{1/4}+|y_1-y|^{1/4})\le$$

$$\le C_5\rho+C_6\rho^{1/4}\le K\rho^{1/4},$$

where $C_5, C_6, K<\infty$.

The cases $x=x_1, y\ne y_1$ and $x\ne x_1, y=y_1$ can be considered analogously to the case $x\ne x_1, y\ne y_1$. At that, the consideration begins from the inequalities

$$|g(x,y)-g(x_1,y_1)|\le K_2|(1-y^2)^{1/4}f(x,y)-(1-y_1^2)^{1/4}f(x_1,y_1)|$$

$(x=x_1,\ y\ne y_1)$ and

$$|g(x,y)-g(x_1,y_1)|\le K_2|(1-x^2)^{1/4}f(x,y)-(1-x_1^2)^{1/4}f(x_1,y_1)|$$

$(x\ne x_1,\ y=y_1)$, where $K_2<\infty$. Lemma 1 is proved.

Lemma 1 and Theorem 5 imply that rectangular partial sums of the double trigonometric Fourier series of the function $g(x,y)$ (in the case of periodic continuation of the function $g(x,y)$) converge uniformly in the square $[-1,1]^2$ to the function $g(x,y)$. This means that the equality (24) holds.

Theorem 6 [25]-[28], [39], [50]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable functions on $[t,T]$. Then for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{31}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorem 3.

Proof. Let us prove the equality

$$\frac{1}{2}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}C_{j_1j_1},$$

where $C_{j_1 j_1}$ is defined by the formula (5) for $k=2$ and $j_1=j_2$. At that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

Consider the auxiliary function

$$K'(t_1,t_2)=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases},\quad t_1,t_2\in[t,T]$$

and prove that

$$|K'(t_1,t_2)-K'(t_1^*,t_2^*)|\le L(|t_1-t_1^*|+|t_2-t_2^*|), \tag{32}$$

where $L<\infty$, and (t_1,t_2), $(t_1,t_2)\in[t,T]^2$.

By the Lagrange formula for the functions $\psi_1(t_1^*)$, $\psi_2(t_1^*)$ at the interval $[\min\{t_1,t_1^*\},\max\{t_1,t_1^*\}]$ and for the functions $\psi_1(t_2^*)$, $\psi_2(t_2^*)$ at the interval $[\min\{t_2,t_2^*\},\max\{t_2,t_2^*\}]$ we obtain

$$|K'(t_1,t_2)-K'(t_1^*,t_2^*)|\le\left|\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}\right|+$$

$$+L_1|t_1-t_1^*|+L_2|t_2-t_2^*|,\quad L_1,L_2<\infty. \tag{33}$$

We have

$$\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}=$$

$$=\begin{cases}0, \quad t_1\ge t_2,\ t_1^*\ge t_2^* \quad\text{or}\quad t_1\le t_2,\ t_1^*\le t_2^*\\ \\ \psi_2(t_1)\psi_1(t_2)-\psi_1(t_1)\psi_2(t_2), \quad t_1\ge t_2,\ t_1^*\le t_2^*.\\ \\ \psi_1(t_1)\psi_2(t_2)-\psi_2(t_1)\psi_1(t_2), \quad t_1\le t_2,\ t_1^*\ge t_2^*\end{cases} \tag{34}$$

By the Lagrange formula for the functions $\psi_1(t_2)$, $\psi_2(t_2)$ at the interval $[\min\{t_1,t_2\},\max\{t_1,t_2\}]$ we get the estimate

$$\left|\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}\right|\le$$

$$\leq L_3|t_2-t_1|\begin{cases}0, & t_1\geq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\leq t_2,\ t_1^*\leq t_2^*\\ \\ 1, & t_1\leq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\geq t_2,\ t_1^*\leq t_2^*\end{cases},\quad L_3<\infty. \tag{35}$$

Let us show that if $t_1\leq t_2,\ t_1^*\geq t_2^*$ or $t_1\geq t_2,\ t_1^*\leq t_2^*$, then the following inequality is satisfied

$$|t_2-t_1|\leq|t_1^*-t_1|+|t_2^*-t_2|. \tag{36}$$

First, consider the case $t_1\geq t_2,\ t_1^*\leq t_2^*$. For this case

$$t_2+(t_1^*-t_2^*)\leq t_2\leq t_1.$$

Then

$$(t_1^*-t_1)-(t_2^*-t_2)\leq t_2-t_1\leq 0$$

and (36) is satisfied.

For the case $t_1\leq t_2,\ t_1^*\geq t_2^*$ we have

$$t_1+(t_2^*-t_1^*)\leq t_1\leq t_2.$$

Then

$$(t_1-t_1^*)-(t_2-t_2^*)\leq t_1-t_2\leq 0$$

and (36) is also satisfied.

From (35) and (36) we obtain

$$\left|\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\geq t_2\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1\leq t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\geq t_2^*\\ \\ \psi_1(t_1)\psi_2(t_2), & t_1^*\leq t_2^*\end{cases}\right|\leq$$

$$\leq L_3(|t_1^*-t_1|+|t_2^*-t_2|)\begin{cases}0, & t_1\geq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\leq t_2,\ t_1^*\leq t_2^*\\ \\ 1, & t_1\leq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\geq t_2,\ t_1^*\leq t_2^*\end{cases}\leq$$

$$\leq L_3(|t_1^*-t_1|+|t_2^*-t_2|)\begin{cases}1, & t_1\geq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\leq t_2,\ t_1^*\leq t_2^*\\ \\ 1, & t_1\leq t_2,\ t_1^*\geq t_2^* \quad \text{or} \quad t_1\geq t_2,\ t_1^*\leq t_2^*\end{cases}=$$

$$=L_3(|t_1^*-t_1|+|t_2^*-t_2|). \tag{37}$$

From (33) and (37) we get (32). Let

$$t_1 = \frac{T-t}{2}x + \frac{T+t}{2}, \quad t_2 = \frac{T-t}{2}y + \frac{T+t}{2},$$

where $x, y \in [-1, 1]$. Then

$$K'(t_1, t_2) \equiv K^*(x, y) = \begin{cases} \psi_2\left(h(x)\right)\psi_1\left(h(y)\right), & x \ge y \\ \\ \psi_1\left(h(x)\right)\psi_2\left(h(y)\right), & x \le y \end{cases},$$

where $x, y \in [-1, 1]$ and

$$h(x) = \frac{T-t}{2}x + \frac{T+t}{2}. \tag{38}$$

Inequality (32) can be written in the form

$$|K^*(x,y) - K^*(x^*,y^*)| \le L^*(|x - x^*| + |y - y^*|), \tag{39}$$

where $L^* < \infty$ and (x, y), $(x^*, y^*) \in [-1, 1]^2$.

Thus, the function $K^*(x, y)$ satisfies the conditions of Lemma 1 and hence for the function

$$K^*(x,y)(1-x^2)^{1/4}(1-y^2)^{1/4}$$

the inequality (26) is fulfilled.

Due to the continuous differentiability of the functions $\psi_1\left(h(x)\right)$ and $\psi_2\left(h(x)\right)$ at the interval $[-1, 1]$ we have $K^*(x, y) \in L_2([-1, 1]^2)$. In addition

$$\int\limits_{[-1,1]^2} \frac{K^*(x,y)dxdy}{(1-x^2)^{1/4}(1-y^2)^{1/4}} \le C\left(\int\limits_{-1}^{1}\frac{1}{(1-x^2)^{1/4}}\int\limits_{-1}^{x}\frac{1}{(1-y^2)^{1/4}}dydx + \right.$$

$$\left. + \int\limits_{-1}^{1}\frac{1}{(1-x^2)^{1/4}}\int\limits_{x}^{1}\frac{1}{(1-y^2)^{1/4}}dydx\right) < \infty, \quad C < \infty.$$

Thus, the conditions of Theorem 4 are fulfilled for the function $K^*(x, y)$. Note that the mentioned properties of the function $K^*(x, y)$, $x, y \in [-1, 1]$ also correct for the function $K'(t_1, t_2)$, $t_1, t_2 \in [t, T]$.

Remark 1. *On the basis of* (32) *it can be argued that the function* $K'(t_1, t_2)$ *belongs to the Holder class with parameter* 1 *in* $[t, T]^2$*. Hence by Theorem* 5 *this function can be expanded into the uniformly convergent double trigonometric Fourier series in the square* $[t, T]^2$*, which summarized by Pringsheim method. However, the expansions of iterated stochastic integrals obtained by using the system of Legendre polynomials are essentially simpler than their analogues obtained by using the trigonometric system of functions* (*see Sect.* 7)*.*

Let us expand the function $K'(t_1,t_2)$ into a double Fourier–Legendre series or double trigonometric Fourier series in the square $[t,T]^2$. This series is summable by the method of rectangular sums (Pringsheim method), i.e.

$$K'(t_1,t_2)=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\left(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2+\right.$$

$$\left.+\int\limits_t^T\psi_1(t_2)\phi_{j_2}(t_2)\int\limits_{t_2}^T\psi_2(t_1)\phi_{j_1}(t_1)dt_1\right)dt_2\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\left(C_{j_2j_1}+C_{j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2),\quad (t_1,t_2)\in(t,T)^2. \tag{40}$$

Moreover, the convergence of the series (40) is uniform on the rectangle

$$[t+\varepsilon,T-\varepsilon]\times[t+\delta,T-\delta]\quad\text{for any}\quad\varepsilon,\delta>0\quad(\text{in particular, we can choose }\varepsilon=\delta).$$

In addition, the series (40) converges to $K'(t_1,t_2)$ at any inner point of the square $[t,T]^2$. Note that Theorem 4 does not answer the question of convergence of the series (40) on a boundary of the square $[t,T]^2$. In obtaining (40) we replaced the order of integration in the second iterated integral.

Let us substitute $t_1=t_2$ into (40). After that, let us rewrite the limit on the right-hand side of (40) as two limits. Let us replace j_1 with j_2, j_2 with j_1, n_1 with n_2, and n_2 with n_1 in the second limit. Thus, we get

$$\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)=\frac{1}{2}\psi_1(t_1)\psi_2(t_1),\quad t_1\in(t,T). \tag{41}$$

According to the above reasoning, the equality (41) holds uniformly on the interval $[t+\varepsilon,T-\varepsilon]$ for any $\varepsilon>0$. Additionally, (41) holds at each interior point of the interval $[t,T]$.

Let us fix $\varepsilon>0$ and integrate the equality (41) at the interval $[t+\varepsilon,T-\varepsilon]$. Due to the uniform convergence of the series (41) we can swap the series and the integral

$$\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=\frac{1}{2}\int\limits_{t+\varepsilon}^{T-\varepsilon}\psi_1(t_1)\psi_2(t_1)dt_1. \tag{42}$$

Lemma 2. *Under the conditions of Theorem* 6 *the following limit*

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}C_{j_1j_1}$$

exists and is finite, where $C_{j_1j_1}$ *is defined by* (5) *for* $k=2$ *and* $j_1=j_2$, *i.e.*

$$C_{j_1j_1}=\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$

The proof of Lemma 2 will be given further in this section. Using the equality (42) for $n_1=n_2=n$ and Lemma 2, we get

$$\frac{1}{2}\int\limits_{t+\varepsilon}^{T-\varepsilon}\psi_1(t_1)\psi_2(t_1)dt_1=\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1=$$

$$=\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\left(\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1-\int\limits_t^{t+\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1-\right.$$

$$\left.-\int\limits_{T-\varepsilon}^{T}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1\right)=$$

$$=\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\left(\mathbf{1}_{\{j_1=j_2\}}-\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)\varepsilon\right)=$$

$$=\lim_{n\to\infty}\sum_{j_1=0}^{n}C_{j_1j_1}-\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr),\tag{43}$$

where $\theta\in[t,t+\varepsilon]$, $\lambda\in[T-\varepsilon,T]$. In obtaining (43) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j=0,1,2\ldots$

Applying (43), we obtain

$$\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)=$$

$$=\lim_{n\to\infty}\sum_{j_1=0}^{n}C_{j_1j_1}-\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1,$$

where the limits

$$\lim_{n\to\infty}\sum_{j_1=0}^{n}C_{j_1j_1},\quad \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\int\limits_{t+\varepsilon}^{T-\varepsilon}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1$$

exist and are finite (see Lemma 2 and the equality (42)). This means that the limit

$$\varepsilon\lim_{n\to\infty}\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)$$

also exists and is finite.

Suppose that the following relations

$$\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(T)\phi_{j_1}(T)\right| \le K < \infty, \quad \left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(t)\phi_{j_1}(t)\right| \le K < \infty \tag{44}$$

are satisfied (the relations (44) will be proved further in this section); constant K does not depend on n.

Note that

$$\left|\varepsilon \lim_{n\to\infty}\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\Bigl(\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\Bigr)\right| =$$

$$= \lim_{n\to\infty}\varepsilon\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right|. \tag{45}$$

Using (41) $(n_1=n_2=n)$ and (44), we obtain

$$\varepsilon\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)+\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right| \le$$

$$\le \varepsilon\left(\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\theta)\phi_{j_2}(\theta)\right|+\left|\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_1}(\lambda)\phi_{j_2}(\lambda)\right|\right) \le 2\varepsilon K_1 \ \to\ 0 \tag{46}$$

if $\varepsilon\to +0$, where $\theta\in[t,t+\varepsilon]$, $\lambda\in[T-\varepsilon,T]$, constant K_1 is independent on n.

Performing the passage to the limit $\lim\limits_{\varepsilon\to+0}$ in the equality (43) and taking into account (45), (46), we get

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{47}$$

Thus, to complete the proof of Theorem 6, it is necessary to prove (44) and Lemma 2. To prove (44) and Lemma 2, as well as for further consideration, we need some well known properties of the Legendre polynomials [64], [67].

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j=0,1,2,\ldots, \tag{48}$$

where $P_j(x)$ is the Legendre polynomial.

It is known that the Legendre polynomial $P_j(x)$ is represented, for example, as

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

At the boundary points of the interval of orthogonality the Legendre polynomials satisfy the following relations

$$P_j(1) = 1, \quad P_j(-1) = (-1)^j,$$

$$P_{j+1}(1) - P_{j-1}(1) = 0, \quad P_{j+1}(-1) - P_{j-1}(-1) = 0,$$

$$P_{j+1}(1) - P_j(1) = 0, \quad P_{j+1}(-1) + P_j(-1) = 0,$$

where $j = 0, 1, 2, \ldots$

Relation of the Legendre polynomial $P_j(x)$ with derivatives of the Legendre polynomials $P_{j+1}(x)$ and $P_{j-1}(x)$ is expressed by the following equality

$$P_j(x) = \frac{1}{2j+1}\left(P^{'}_{j+1}(x) - P^{'}_{j-1}(x)\right), \quad j = 1, 2, \ldots \tag{49}$$

The recurrent relation has the form

$$xP_j(x) = \frac{(j+1)P_{j+1}(x) + jP_{j-1}(x)}{2j+1}, \quad j = 1, 2, \ldots$$

Orthogonality of Legendre polynomial $P_j(x)$ to any polynomial $Q_k(x)$ of lesser degree we write in the following form

$$\int\limits_{-1}^{1} Q_k(x)P_j(x)dx = 0, \quad k = 0, 1, 2, \ldots, j-1.$$

From the property

$$\int\limits_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if } \ k \neq j \\ 2/(2j+1) & \text{if } \ k = j \end{cases}$$

it follows that the orthonormal on the interval $[-1, 1]$ Legendre polynomials determined by the relation

$$P_j^*(x) = \sqrt{\frac{2j+1}{2}}P_j(x), \quad j = 0, 1, 2, \ldots$$

It is well known that there is an estimate

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j = 1, 2, \ldots, \tag{50}$$

where constant K does not depends on y and j.

Moreover,

$$|P_j(x)| \le 1, \quad x \in [-1, 1], \quad j = 0,\ 1, \ldots \tag{51}$$

The Christoffel–Darboux formula has the form

$$\sum_{j=0}^{n}(2j+1)P_j(x)P_j(y)=(n+1)\frac{P_n(x)P_{n+1}(y)-P_{n+1}(x)P_n(y)}{y-x}. \tag{52}$$

Let us prove (44). From (52) for $x=\pm 1$ we obtain

$$\sum_{j=0}^{n}(2j+1)P_j(y)=(n+1)\frac{P_{n+1}(y)-P_n(y)}{y-1}, \tag{53}$$

$$\sum_{j=0}^{n}(2j+1)(-1)^jP_j(y)=(n+1)(-1)^n\frac{P_{n+1}(y)+P_n(y)}{y+1}. \tag{54}$$

From the other hand (see (49))

$$\sum_{j=0}^{n}(2j+1)P_j(y)=1+\sum_{j=1}^{n}(2j+1)P_j(y)=$$

$$=1+\sum_{j=1}^{n}(P_{j+1}^{'}(y)-P_{j-1}^{'}(y))=1+\left(\sum_{j=1}^{n}(P_{j+1}(y)-P_{j-1}(y))\right)^{'}=$$

$$=1+(P_{n+1}(x)+P_n(x)-x-1)'=(P_n(x)+P_{n+1}(x))' \tag{55}$$

and

$$\sum_{j=0}^{n}(2j+1)(-1)^jP_j(y)=1+\sum_{j=1}^{n}(-1)^j(2j+1)P_j(y)=$$

$$=1+\sum_{j=1}^{n}(-1)^j(P_{j+1}^{'}(y)-P_{j-1}^{'}(y))=1+\left(\sum_{j=1}^{n}(-1)^j(P_{j+1}(y)-P_{j-1}(y))\right)^{'}=$$

$$=1+((-1)^n(P_{n+1}(x)-P_n(x))-x+1)'=(-1)^n(P_{n+1}(x)-P_n(x))'. \tag{56}$$

Applying (53)–(56), we get

$$(n+1)\frac{P_{n+1}(y)-P_n(y)}{y-1}=(P_n(x)+P_{n+1}(x))', \tag{57}$$

$$(n+1)\frac{P_{n+1}(y)+P_n(y)}{y+1}=(P_{n+1}(x)-P_n(x))'. \tag{58}$$

Let us prove the boundedness of the first sum in (44). We have

$$\sum_{j_1,j_2=0}^{n}C_{j_2j_1}\phi_{j_2}(T)\phi_{j_1}(T)=$$

$$= \frac{1}{4}\sum_{j_2=0}^{n}\sum_{j_1=0}^{n}(2j_2+1)(2j_1+1)\int\limits_{-1}^{1}\psi_2(h(y))P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))P_{j_1}(y_1)dy_1dy =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\sum_{j_2=0}^{n}(2j_2+1)P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))\sum_{j_1=0}^{n}(2j_1+1)P_{j_1}(y_1)dy_1dy =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)) =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_1(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y))+$$

$$+\frac{1}{4}\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)+P_n(y_1))\right)d(P_{n+1}(y)+P_n(y)) =$$

$$= \frac{1}{4}I_1 + \frac{1}{4}I_2,$$

where

$$\Delta(h(y)) = \psi_2(h(y)) - \psi_1(h(y)), \quad h(y) = \frac{T-t}{2}y + \frac{T+t}{2}. \tag{59}$$

Further,

$$I_1 = \frac{1}{2}\Biggl(\int\limits_{-1}^{1}\psi_1(h(y))d(P_{n+1}(y)+P_n(y))\Biggr)^2 =$$

$$= \frac{1}{2}\Biggl(2\psi_1(T) - \int\limits_{-1}^{1}(P_{n+1}(y)+P_n(y))\psi_1'(h(y))\frac{T-t}{2}dy\Biggr)^2 < C_1 < \infty,$$

where ψ_1' is a derivative of the function ψ_1 with respect to the variable y, constant C_1 does not depend on n.

By the Lagrange formula we obtain

$$\Delta(h(y)) = \psi_2\biggl(\frac{1}{2}(T-t)(y-1)+T\biggr) - \psi_1\biggl(\frac{1}{2}(T-t)(y-1)+T\biggr) =$$

$$= \psi_2(T) - \psi_1(T) + (y-1)\Bigl(\psi_2'(\xi_y) - \psi_1'(\theta_y)\Bigr)\frac{1}{2}(T-t) =$$

$$= C_1 + \alpha_y(y-1), \tag{60}$$

where $|\alpha_y| < \infty$ and $C_1 = \psi_2(T) - \psi_1(T)$.

Let us substitute (60) into the integral I_2

$$
I_2 = I_3 + I_4,
$$

where

$$
I_3 = \int\limits_{-1}^{1} \alpha_y (y-1) \left(\int\limits_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)),
$$

$$
I_4 = C_1 \int\limits_{-1}^{1} \left(\int\limits_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)).
$$

Integrating by parts and using (57), we obtain

$$
I_3 = \int\limits_{-1}^{1} \frac{\alpha_y (y-1)(n+1)(P_{n+1}(y) - P_n(y))}{y-1} \Biggl(\psi_1(h(y))(P_{n+1}(y) + P_n(y)) -
$$

$$
- \int\limits_{-1}^{y} (P_{n+1}(y_1) + P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t) dy_1 \Biggr) dy.
$$

Applying the etimate (50) and taking into account the boundedness of α_y and $\psi_1'(h(y_1))$, we have that $|I_3| < \infty$.

Using the integration order replacement in I_4, we get

$$
I_4 = C_1 \int\limits_{-1}^{1} \psi_1(h(y_1)) \left(\int\limits_{y_1}^{1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) =
$$

$$
= C_1 \int\limits_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \int\limits_{-1}^{1} d(P_{n+1}(y) + P_n(y)) -
$$

$$
- C_1 \int\limits_{-1}^{1} \psi_1(h(y_1)) \left(\int\limits_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) =
$$

$$
= I_5 - I_6.
$$

Consider I_5

$$
I_5 = 2C_1 \int\limits_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) =
$$

$$
= 2C_1 \left(2\psi_1(T) - \int\limits_{-1}^{1} (P_{n+1}(y_1) + P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t) dy_1 \right).
$$

Applying the estimate (51) and using the boundedness of $\psi_1'(h(y_1))$, we obtain that $|I_5| < \infty$.

Since (see (60))

$$\psi_1(h(y)) = \psi_1\left(\frac{1}{2}(T-t)(y-1)+T\right) =$$

$$= \psi_1(T) + (y-1)\psi_1'(\theta_y)\frac{1}{2}(T-t) = C_2 + \beta_y(y-1),$$

where $|\beta_y| < \infty$ and $C_2 = \psi_1(T)$, then

$$I_6 = C_3\int\limits_{-1}^{1}\left(\int\limits_{-1}^{y_1} d(P_{n+1}(y)+P_n(y))\right)d(P_{n+1}(y_1)+P_n(y_1))+$$

$$+C_1\int\limits_{-1}^{1}\beta_{y_1}(y_1-1)\left(\int\limits_{-1}^{y_1} d(P_{n+1}(y)+P_n(y))\right)d(P_{n+1}(y_1)+P_n(y_1)) =$$

$$= \frac{C_3}{2}\left(\int\limits_{-1}^{1} d(P_{n+1}(y)+P_n(y))\right)^2 +$$

$$+C_1\int\limits_{-1}^{1}\frac{\beta_{y_1}(y_1-1)(n+1)(P_{n+1}(y_1)-P_n(y_1))}{y_1-1}\left(\int\limits_{-1}^{y_1} d(P_{n+1}(y)+P_n(y))\right)dy_1 =$$

$$= 2C_3 + C_1\int\limits_{-1}^{1}\beta_{y_1}(n+1)(P_{n+1}(y_1)-P_n(y_1))(P_{n+1}(y_1)+P_n(y_1))dy_1.$$

Using the estimate (50) and taking into account the bounedness of β_{y_1}, we obtain that $|I_6| < \infty$. Thus, the boundedness of the first sum in (44) is proved.

Let us prove the boundedness of the second sum in (44). We have

$$\sum_{j_1,j_2=0}^{n} C_{j_2j_1}\phi_{j_2}(t)\phi_{j_1}(t) =$$

$$= \frac{1}{4}\sum_{j_2=0}^{n}\sum_{j_1=0}^{n}(2j_2+1)(2j_1+1)(-1)^{j_1+j_2}\int\limits_{-1}^{1}\psi_2(h(y))P_{j_2}(y)\int\limits_{-1}^{y}\psi_1(h(y_1))P_{j_1}(y_1)dy_1dy =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_2(h(y))\sum_{j_2=0}^{n}(2j_2+1)P_{j_2}(y)(-1)^{j_2}\int\limits_{-1}^{y}\psi_1(h(y_1))\sum_{j_1=0}^{n}(2j_1+1)P_{j_1}(y_1)(-1)^{j_1}dy_1dy =$$

$$= \frac{(-1)^{2n}}{4}\int\limits_{-1}^{1}\psi_2(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y)) =$$

$$= \frac{1}{4}\int\limits_{-1}^{1}\psi_1(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y))+$$

$$+\frac{1}{4}\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{-1}^{y}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y))=$$

$$=\frac{1}{4}J_1+\frac{1}{4}J_2,$$

where $\Delta(h(y))$, $h(y)$ are defined by (59).

Further,

$$J_1=\frac{1}{2}\left(\int\limits_{-1}^{1}\psi_1(h(y))d(P_{n+1}(y)-P_n(y))\right)^2=$$

$$=\frac{1}{2}\left(2(-1)^n\psi_1(t)-\int\limits_{-1}^{1}(P_{n+1}(y)-P_n(y))\psi_1'(h(y))\frac{T-t}{2}dy\right)^2<K_1<\infty, \tag{61}$$

where ψ_1' is a derivative of the function ψ_1 with respect to the variable y, constant K_1 is independent of n.

By the Lagrange formula we obtain

$$\Delta(h(y))=\psi_2\left(\frac{1}{2}(T-t)(y+1)+t\right)-\psi_1\left(\frac{1}{2}(T-t)(y+1)+t\right)=$$

$$=\psi_2(t)-\psi_1(t)+(y+1)\Bigl(\psi_2'(\mu_y)-\psi_1'(\rho_y)\Bigr)\frac{1}{2}(T-t)=$$

$$=K_2+\gamma_y(y+1), \tag{62}$$

where $|\gamma_y|<\infty$ and $K_2=\psi_2(t)-\psi_1(t)$.

Consider J_2

$$J_2=\int\limits_{-1}^{1}\Delta(h(y))d(P_{n+1}(y)-P_n(y))\int\limits_{-1}^{1}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))-$$

$$-\int\limits_{-1}^{1}\Delta(h(y))\left(\int\limits_{y}^{1}\psi_1(h(y_1))d(P_{n+1}(y_1)-P_n(y_1))\right)d(P_{n+1}(y)-P_n(y))=$$

$$=J_3J_4-J_5.$$

The integral J_4 was considered earlier (see J_1 and (61)), i.e. it has already been shown that $|J_4|<\infty$. Analogously, we have that $|J_3|<\infty$.

Let us substitute (62) into the integral J_5

$$J_5=J_6+J_7,$$

where

$$J_6 = \int_{-1}^{1} \gamma_y (y+1) \left(\int_{y}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)),$$

$$J_7 = K_2 \int_{-1}^{1} \left(\int_{y}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)).$$

Integrating by parts and using (58), we get

$$J_6 = \int_{-1}^{1} \frac{\gamma_y (y+1)(n+1)(P_{n+1}(y) + P_n(y))}{y+1} \Biggl(-\psi_1(h(y))(P_{n+1}(y) - P_n(y)) -$$

$$- \int_{y}^{1} (P_{n+1}(y_1) - P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t) dy_1 \Biggr) dy.$$

Applying the etimate (50) and taking into account the boundedness of γ_y and $\psi_1'(h(y_1))$, we have that $|J_6| < \infty$.

Using the integration order replacement in J_7, we obtain

$$J_7 = K_2 \int_{-1}^{1} \psi_1(h(y_1)) \left(\int_{-1}^{y_1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) =$$

$$= K_2 \int_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \int_{-1}^{1} d(P_{n+1}(y) - P_n(y)) - K_2 J_8 =$$

$$= K_2 J_4 2(-1)^n - K_2 J_8,$$

where

$$J_8 = \int_{-1}^{1} \psi_1(h(y_1)) \left(\int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)).$$

Since (see (62))

$$\psi_1(h(y)) = \psi_1 \left(\frac{1}{2}(T-t)(y+1) + t \right) =$$

$$= \psi_1(t) + (y+1)\psi_1'(\rho_y) \frac{1}{2}(T-t) = K_3 + \varepsilon_y (y+1), \tag{63}$$

where $|\varepsilon_y| < \infty$ and $K_3 = \psi_1(t)$, then

$$J_8 = K_3 \int_{-1}^{1} \left(\int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) +$$

$$+\int\limits_{-1}^{1}\varepsilon_y(y+1)\left(\int\limits_{y_1}^{1}d(P_{n+1}(y)-P_n(y))\right)d(P_{n+1}(y_1)-P_n(y_1))=$$

$$=\frac{K_3}{2}\left(\int\limits_{-1}^{1}d(P_{n+1}(y)-P_n(y))\right)^2+$$

$$+\int\limits_{-1}^{1}\frac{\varepsilon_{y_1}(y_1+1)(n+1)(P_{n+1}(y_1)+P_n(y_1))}{y_1+1}(P_n(y_1)-P_{n+1}(y_1))dy=$$

$$=2K_3+\int\limits_{-1}^{1}\varepsilon_{y_1}(n+1)(P_{n+1}(y_1)+P_n(y_1))(P_n(y_1)-P_{n+1}(y_1))dy. \tag{64}$$

When obtaining the equality (64), we used (58). Applying the estimate (50) and taking into account the bounedness of ε_{y_1}, we obtain that $|J_8|<\infty$. Thus, the boundedness of the second sum in (44) is proved. The relations (44) are proved.

Let us prove Lemma 2. We will prove that

$$\sum_{j_1=0}^{n}C_{j_1j_1}$$

is the Cauchy sequence for the cases of Legendre polynomials and trigonometric functions.

Consider the case of Legendre polynomials and fix $n>m$ $(n,m\in\mathbb{N})$. We have

$$\sum_{j_1=m+1}^{n}C_{j_1j_1}=\sum_{j_1=m+1}^{n}\int\limits_t^T\psi_2(s)\phi_{j_1}(s)\int\limits_t^s\psi_1(\tau)\phi_{j_1}(\tau)d\tau ds=$$

$$=\frac{T-t}{4}\sum_{j_1=m+1}^{n}(2j_1+1)\int\limits_{-1}^{1}\psi_2(h(x))P_{j_1}(x)\int\limits_{-1}^{x}\psi_1(h(y))P_{j_1}(y)dydx=$$

$$=\frac{T-t}{4}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-$$

$$-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\psi_2(h(x))P_{j_1}(x)\int\limits_{-1}^{x}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))dydx=$$

$$=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\sum_{j_1=m+1}^{n}\left(P_{j_1+1}(x)P_{j_1}(x)-P_{j_1}(x)P_{j_1-1}(x)\right)dx-$$

$$
-\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\int\limits_{y}^{1}P_{j_1}(x)\psi_2(h(x))dxdy=
$$

$$
=\frac{T-t}{4}\int\limits_{-1}^{1}\psi_1(h(x))\psi_2(h(x))\left(P_{n+1}(x)P_n(x)-P_{m+1}(x)P_m(x)\right)dx+
$$

$$
+\frac{(T-t)^2}{8}\sum_{j_1=m+1}^{n}\frac{1}{2j_1+1}\int\limits_{-1}^{1}\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_1'(h(y))\times
$$

$$
\times\Biggl(\left(P_{j_1+1}(y)-P_{j_1-1}(y)\right)\psi_2(h(y))+
$$

$$
+\frac{T-t}{2}\int\limits_{y}^{1}\left(P_{j_1+1}(x)-P_{j_1-1}(x)\right)\psi_2'(h(x))dx\Biggr)dy, \tag{65}
$$

where ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $h(y)$ (see (38)).

Applying the estimate (50) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we finally obtain

$$
\left|\sum_{j_1=m+1}^{n}C_{j_1j_1}\right|\le C_1\left(\frac{1}{n}+\frac{1}{m}\right)\int\limits_{-1}^{1}\frac{dx}{\left(1-x^2\right)^{1/2}}+
$$

$$
+C_2\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\left(\int\limits_{-1}^{1}\frac{dy}{\left(1-y^2\right)^{1/2}}+\int\limits_{-1}^{1}\frac{1}{\left(1-y^2\right)^{1/4}}\int\limits_{y}^{1}\frac{dx}{\left(1-x^2\right)^{1/4}}dy\right)\le
$$

$$
\le C_3\left(\frac{1}{n}+\frac{1}{m}+\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\right)\to 0 \tag{66}
$$

if $n,m\to\infty$ $(n>m)$, where constants C_1,C_2,C_3 do not depend on n and m.

Consider the trigonometric case. Below in this section we write $\lim\limits_{n,m\to\infty}$ instead of $\lim\limits_{\substack{n,m\to\infty\\ n>m}}$. Fix $n>m$ $(n,m\in\mathbf{N})$. Denote

$$
S_{n,m}\stackrel{\sf def}{=}\sum_{j_1=m+1}^{n}C_{j_1j_1}=\sum_{j_1=m+1}^{n}\int\limits_{t}^{T}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_{t}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.
$$

By analogy with (65) we obtain

$$
S_{2n,2m}=\sum_{j_1=2m+1}^{2n}\int\limits_t^T\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2=
$$

$$
=\frac{2}{T-t}\sum_{j_1=m+1}^{n}\Biggl(\int\limits_t^T\psi_2(t_2)\sin\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2+
$$

$$
+\int\limits_t^T\psi_2(t_2)\cos\frac{2\pi j_1(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi j_1(t_1-t)}{T-t}dt_1dt_2\Biggr)=
$$

$$
=\frac{T-t}{2\pi^2}\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\Biggl(\psi_1(t)\Biggl(\psi_2(t)-\psi_2(T)+\int\limits_t^T\psi_2'(t_2)\cos\frac{2\pi j_1(t_2-t)}{T-t}dt_2\Biggr)-
$$

$$
-\int\limits_t^T\psi_1'(t_1)\cos\frac{2\pi j_1(t_1-t)}{T-t}\Biggl(\psi_2(T)-\psi_2(t_1)\cos\frac{2\pi j_1(t_1-t)}{T-t}-
$$

$$
-\int\limits_{t_1}^T\psi_2'(t_2)\cos\frac{2\pi j_1(t_2-t)}{T-t}dt_2\Biggr)dt_1+
$$

$$
+\int\limits_t^T\psi_1'(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}\Biggl(\psi_2(t_1)\sin\frac{2\pi j_1(t_1-t)}{T-t}+
$$

$$
+\int\limits_{t_1}^T\psi_2'(t_2)\sin\frac{2\pi j_1(t_2-t)}{T-t}dt_2\Biggr)dt_1\Biggr), \tag{67}
$$

where $\psi_1'(\tau)$, $\psi_2'(\tau)$ are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable τ.

From (67) we get

$$
|S_{2n,2m}|\le C\sum_{j_1=m+1}^{n}\frac{1}{j_1^2}\to 0 \tag{68}
$$

if $n,m\to\infty$ $(n>m)$, where constant C does not depend on n and m.

Further,

$$
S_{2n-1,2m}=S_{2n,2m}-
$$

$$
-\frac{2}{T-t}\int\limits_t^T\psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2, \tag{69}
$$

$$S_{2n,2m-1} = S_{2n,2m}+$$

$$+\frac{2}{T-t}\int\limits_t^T \psi_2(t_2)\cos\frac{2\pi m(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2, \tag{70}$$

$$S_{2n-1,2m-1} = S_{2n,2m-1}-$$

$$-\frac{2}{T-t}\int\limits_t^T \psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2 =$$

$$= S_{2n,2m} + \frac{2}{T-t}\int\limits_t^T \psi_2(t_2)\cos\frac{2\pi m(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi m(t_1-t)}{T-t}dt_1dt_2-$$

$$-\frac{2}{T-t}\int\limits_t^T \psi_2(t_2)\cos\frac{2\pi n(t_2-t)}{T-t}\int\limits_t^{t_2}\psi_1(t_1)\cos\frac{2\pi n(t_1-t)}{T-t}dt_1dt_2. \tag{71}$$

Integrating by parts in (69)–(71), we obtain

$$|S_{2n-1,2m}| \le |S_{2n,2m}| + \frac{C_1}{n}, \tag{72}$$

$$|S_{2n,2m-1}| \le |S_{2n,2m}| + \frac{C_1}{m}, \tag{73}$$

$$|S_{2n-1,2m-1}| \le |S_{2n,2m}| + C_1\left(\frac{1}{m}+\frac{1}{n}\right), \tag{74}$$

where constant C_1 does not depend on n and m.

The relations (68), (72)–(74) imply that

$$\lim_{n,m\to\infty}|S_{2n,2m}| = \lim_{n,m\to\infty}|S_{2n-1,2m}| = \lim_{n,m\to\infty}|S_{2n,2m-1}| = \lim_{n,m\to\infty}|S_{2n-1,2m-1}| = 0. \tag{75}$$

From (75) we get

$$\lim_{n,m\to\infty}|S_{n,m}| = 0. \tag{76}$$

Lemma 2 is proved. Theorem 6 is proved.

5. Proof of the Equality (18). The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$

Theorem 7. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$. Then the following equality*

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=\frac{1}{2}\int\limits_t^T\psi_1(\tau)\psi_2(\tau)d\tau \tag{77}$$

is fulfilled.

Proof. First consider the case $\psi_1(\tau)\equiv\psi_2(\tau)$ or

$$\psi_1(\tau)=\psi_2(\tau)\int\limits_t^{\tau}g(\theta)d\theta, \tag{78}$$

where $\tau\in[t,T]$ and $\psi_1(\tau),\psi_2(\tau),g(\tau)\in L_2([t,T])$.

First suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ such that $\phi_j(x)$ for $j<\infty$ is continuous at the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity. Furthermore, let $\psi_1(\tau)\equiv\psi_2(\tau)$ or the equality (78) is satisfied. Here we suppose that $\psi_1(\tau),\psi_2(\tau),g(\tau)$ are continuous functions at the interval $[t,T]$.

Using the integration order replacement and the Parseval equality, we have (see (78))

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_j(t_1)dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2}\psi_2(t_1)\phi_j(t_1)\int\limits_t^{t_1}g(\tau)d\tau dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\int\limits_{\tau}^{T}\psi_2(t_1)\phi_j(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_j(t_2)dt_2dt_1d\tau=$$

$$=\frac{1}{2}\sum_{j=0}^{\infty}\int\limits_t^T g(\tau)\left(\int\limits_{\tau}^{T}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2d\tau= \tag{79}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\sum_{j=0}^{\infty}\left(\int\limits_t^T \mathbf{1}_{\{\tau<t_1\}}\psi_2(t_1)\phi_j(t_1)dt_1\right)^2d\tau= \tag{80}$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\int\limits_t^T \mathbf{1}_{\{\tau<t_1\}}\psi_2^2(t_1)dt_1d\tau=$$

$$=\frac{1}{2}\int\limits_t^T g(\tau)\int\limits_\tau^T \psi_2^2(t_1)dt_1d\tau=$$

$$=\frac{1}{2}\int\limits_t^T \psi_2^2(t_1)\int\limits_t^{t_1} g(\tau)d\tau dt_1= \tag{81}$$

$$=\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1, \tag{82}$$

where the transition from (79) to (80) is based on the Dini Theorem (using the continuity of the functions $u_q(\tau)$ (see below), the nondecreasing property of the functional sequence

$$u_q(\tau)=\sum_{j=0}^{q}\left(\int\limits_\tau^T \psi_2(t_1)\phi_j(t_1)dt_1\right)^2,$$

and the continuity of the limit function

$$u(\tau)=\int\limits_\tau^T \psi_2^2(t_1)dt_1$$

according to Dini's Theorem, we have the uniform convergence $u_q(\tau)$ to $u(\tau)$ at the interval $[t,T]$).

From the other hand, using the integration order replacement and the generalized Parseval equality as well as (81), we get

$$\sum_{j=0}^{\infty}\int\limits_t^T \psi_1(t_2)\phi_j(t_2)\int\limits_t^{t_2} \psi_2(t_1)\phi_j(t_1)dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} g(\tau)d\tau\int\limits_t^{t_2} \psi_2(t_1)\phi_j(t_1)dt_1dt_2=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)\int\limits_{t_1}^{T} \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} g(\tau)d\tau dt_2dt_1=$$

$$=\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)dt_1\int\limits_t^{T} \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} g(\tau)d\tau dt_2-$$

$$-\sum_{j=0}^{\infty}\int\limits_t^T \psi_2(t_1)\phi_j(t_1)\int\limits_t^{t_1} \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} g(\tau)d\tau dt_2dt_1=$$

$$= \int\limits_t^T \psi_2(t_1) \cdot \psi_2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 - \frac{1}{2} \int\limits_t^T \psi_2^2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_2^2(t_1) \int\limits_t^{t_1} g(\tau)d\tau dt_1 = \frac{1}{2} \int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1. \tag{83}$$

In addition, for the case $\psi_1(\tau) \equiv \psi_2(\tau)$, using the Parseval equality, we obtain

$$\sum_{j=0}^{\infty} \int\limits_t^T \psi_1(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1)dt_1 dt_2 =$$

$$= \frac{1}{2} \sum_{j=0}^{\infty} \left(\int\limits_t^T \psi_1(t_1)\phi_j(t_1)dt_1 \right)^2 =$$

$$= \frac{1}{2} \int\limits_t^T \psi_1^2(t_1)dt_1. \tag{84}$$

By interpreting the integrals in the above formulas as Lebesgue integrals, using Fubini's theorem and Lebesgue's Dominated Convergence Theorem in the above reasoning, we get the equality (77) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau), g(\tau) \in L_2([t,T])$.

Suppose that

$$\psi_2(\tau) = (\tau - t)^l, \quad g(\tau) = k(\tau - t)^{k-1},$$

where $l = 0, 1, 2, \ldots, k = 1, 2, \ldots$

From (78) we have

$$\psi_1(\tau) = \psi_2(\tau) \int\limits_t^{\tau} g(\theta)d\theta = k(\tau - t)^l \int\limits_t^{\tau} (\theta - t)^{k-1}d\theta = (\tau - t)^{l+k}.$$

Taking into account (82)–(84), we obtain

$$\sum_{j=0}^{\infty} \int\limits_t^T (t_2 - t)^l \phi_j(t_2) \int\limits_t^{t_2} (t_1 - t)^{l+k}\phi_j(t_1)dt_1 dt_2 =$$

$$= \sum_{j=0}^{\infty} \int\limits_t^T (t_2 - t)^{l+k} \phi_j(t_2) \int\limits_t^{t_2} (t_1 - t)^l \phi_j(t_1)dt_1 dt_2 =$$

$$= \frac{1}{2} \int\limits_t^T (\tau - t)^{2l+k} d\tau, \tag{85}$$

where $k, l = 0, 1, 2, \ldots$

The equality similar to (85) was obtained in [68] using other arguments. In addition, the formula similar to (85) was used in [68] to generalize the equality (77) to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. Consider this approach [68] in more detail.

Let us rewrite the equality (85) in the following form

$$\sum_{j=0}^{\infty} \int\limits_t^T (t_2-t)^l \phi_j(t_2) \int\limits_t^{t_2} (t_1-t)^m \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int\limits_t^T (\tau-t)^l (\tau-t)^m d\tau, \tag{86}$$

where $l, m = 0, 1, 2, \ldots$

Since the equality (86) is valid for monomials with respect to $\tau - t$ $(\tau \in [t,T])$, it will obviously also be valid for Legendre polynomials that form a complete orthonormal system of functions in the space $L_2([t,T]$ and finite linear combinations of Legendre polynomials.

Let $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$ and $\psi_1^{(p)}(\tau), \psi_2^{(q)}(\tau)$ be approximations of the functions $\psi_1(\tau), \psi_2(\tau)$, respectively, which are partial sums of the corresponding Fourier–Legendre series. Then we have (see (86))

$$\sum_{j=0}^{\infty} \int\limits_t^T \psi_2^{(q)}(t_2) \phi_j(t_2) \int\limits_t^{t_2} \psi_1^{(p)}(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int\limits_t^T \psi_1^{(p)}(\tau) \psi_2^{(q)}(\tau) d\tau, \tag{87}$$

where $p, q \in \mathbb{N}$, the series converges absolutly and its sum does not depend on a basis system $\{\phi_j(x)\}_{j=0}^{\infty}$.

Let us fix q in (87). The right-hand side of (87) for a fixed q defines (as a scalar product in $L_2([t,T])$) a linear bounded (and therefore continuous) functional in $L_2([t,T])$, which is given by the function $\psi_2^{(q)}$. The left-hand side of the equality (87) has the same properties. Let us implement the passage to the limit $\lim\limits_{p\to\infty}$ in (87)

$$\sum_{j=0}^{\infty} \int\limits_t^T \psi_2^{(q)}(t_2) \phi_j(t_2) \int\limits_t^{t_2} \psi_1(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int\limits_t^T \psi_1(\tau) \psi_2^{(q)}(\tau) d\tau, \tag{88}$$

where $q \in \mathbb{N}$. The equality (88) defines a linear bounded functional in $L_2([t,T])$ given by the function ψ_1. Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (88)

$$\sum_{j=0}^{\infty} \int\limits_t^T \psi_2(t_2) \phi_j(t_2) \int\limits_t^{t_2} \psi_1(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int\limits_t^T \psi_1(\tau) \psi_2(\tau) d\tau,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$.

Thus we have the following theorem.

Theorem 8 [25]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. Then for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \tag{89}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorem 3.

6. Some Recent Results on Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [25] (Sect. 2.10–2.16), [29] (Sect. 13–19), [33] (Sect. 5–11), [47] (Sect. 7–13), [48]. Let us formulate four theorems that were obtained using this approach.

Theorem 9 [25], [29], [33], [47]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\psi_2(\tau),$ $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{90}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{91}$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (90) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (91)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 10 [25], [29], [33], [47]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \tag{92}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \tag{93}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{94}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (92), (93) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (94)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$C_{j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;$$

another notations are the same as in Theorem 9.

Theorem 11 [25], [29], [33], [47]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{95}$$

the following relations

(96) $$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)},$$

(97) $$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}}$$

are fulfilled, where $i_1,\ldots,i_5 = 0,1,\ldots,m$ *in* (95), (96) *and* $i_1,\ldots,i_5 = 1,\ldots,m$ *in* (97), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 9, 10.

Theorem 12 [25], [29], [33], [47]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

(98) $$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 9–11.

7. Expansions of Iterated Stratonovich Stochastic Integrals of First and Second Multiplicity Based on Multiple Fourier–Legendre Series

We will use the following notations for iterated Stratonovich stochastic integrals of first and second multiplicities

(99)
$$I_{(l_1)T,t}^{*(i_1)}=\int\limits_t^{*T}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)},$$

(100)
$$I_{(l_1 l_2)T,t}^{*(i_1 i_2)}=\int\limits_t^{*T}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)},$$

where $l_1, l_2 = 0, 1, \ldots;\ i_1, \ldots, i_k = 1, \ldots, m.$

Note that together with the iterated Stratonovich stochastic integrals of higher multiplicities than the second, the stochastic integrals (99) and (100) are included in the so-called unified Taylor–Stratonovich expansion [42] (also see [25]-[26]). This expansion can be used for construction of high-order strong numerical methods for Ito SDEs (definition of a strong numerical method see, for example, in [3]).

Consider the expansions of some stochastic integrals (99) and (100) obtained by using Theorems 6, 8 (see (31) or (89))

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

(101)
$$I_{(1)T,t}^{*(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

(102)
$$I_{(2)T,t}^{*(i_1)}=\frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

(103)
$$I_{(00)T,t}^{*(i_1 i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

$$I_{(01)T,t}^{*(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_1)}\zeta_1^{(i_2)}}{\sqrt{3}}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),$$

$$I_{(10)T,t}^{*(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_2)}\zeta_1^{(i_1)}}{\sqrt{3}}+\right.$$

(104)
$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),$$

$$I_{(02)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-(T-t)I_{(01)T,t}^{*(i_1 i_2)}+\frac{(T-t)^3}{8}\left(\frac{2\zeta_2^{(i_2)}\zeta_0^{(i_1)}}{3\sqrt{5}}+\right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.\left.+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(20)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-(T-t)I_{(10)T,t}^{*(i_1 i_2)}+\frac{(T-t)^3}{8}\left(\frac{2\zeta_0^{(i_2)}\zeta_2^{(i_1)}}{3\sqrt{5}}+\right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.\left.+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(11)T,t}^{*(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1 i_2)}+I_{(01)T,t}^{*(i_1 i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\left(\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.\right.$$

$$\left.\left.+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right),$$

$$I_{(3)T,t}^{*(i_1)}=-\frac{(T-t)^{7/2}}{4}\left(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\right),$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}\quad(i=1,\ldots,m)\tag{105}$$

are independent standard Gaussian random variables for various i or j.

Note the simplicity of the formulas (101), (102). For comparison, we present analogs of the formulas (101), (102) obtained in [4] (also see [3]) using the method proposed in [2]

$$I_{(1)T,t}^{(i_1)q} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{106}$$

$$I_{(2)T,t}^{(i_1)q} = (T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)} + \sqrt{\beta_q}\mu_q^{(i_1)}\right) - \right.$$

$$\left. -\frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{107}$$

where $\zeta_j^{(i)}$ is defined by the formula (105), $\phi_j(s)$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$, and $\zeta_0^{(i)},\ \zeta_{2r}^{(i)},\ \zeta_{2r-1}^{(i)},\ \xi_q^{(i)},\ \mu_q^{(i)}$ $(r=1,\ldots,q,\ \ i=1,\ldots,m)$ are independent standard Gaussian random variables, $i_1=1,\ldots,m$,

$$\xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)}, \qquad \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2},$$

$$\mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)}, \qquad \beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}.$$

Another example of obvious advantage of the Legendre polynomials over the trigonometric functions (in the framework of the considered problem) is the truncated expansion of the iterated Stratonovich stochastic integral $I_{(10)T,t}^{*(i_1i_2)}$ obtained by Theorems 6, 8 in which instead of the double Fourier–Legendre series is taken the double trigonometric Fourier series

$$I_{(10)T,t}^{*(i_1i_2)q} = -(T-t)^2\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)} + \right.$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)} - 2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right) +$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)} + \frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)} - 2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right) -$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)} + \frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right) +$$

$$+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \right.$$

$$\left.\left. +\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} + \zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\right)\right), \tag{108}$$

where the meaning of notations included in (106), (7) is saved.

An analogue of the formula (108) (for the case of Legendre polynomials) is (according to (103) and (104)) the following representation

$$I_{(10)T,t}^{*(i_1 i_2)q} = -\frac{T-t}{2} I_{(00)T,t}^{*(i_1 i_2)q} - \frac{(T-t)^2}{4}\left(\frac{\zeta_0^{(i_2)}\zeta_1^{(i_1)}}{\sqrt{3}} + \right.$$

$$\left. + \sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{109}$$

where

$$I_{(00)T,t}^{*(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

which is obviously substantially simpler than (108).

Here it is necessary to pay a special attention on the fact that the representation (109) includes a single sum with the upper summation limit q while the representation (108) includes the double sum with the same summation limit. In numerical simulation, obviously, the formula (109) is more economical in terms of computational cost than its analogue (108).

There is another feature that should be noted in connection with the formula (108). This formula was first obtained in [4] by the method from [2]. As we noted in Sect. 1, the method [2] of approximation of iterated stochastic integrals is based on the trigonometric series expansion of the Brownian bridge process. So, this method leads to iterated application of the operation of limit transition (in contrast to Theorems 1, 2, 6, and 8–12 in which limit transition is performed only once). This means, generally speaking, that the mean-square convergence of $I_{(10)T,T}^{*(i_1 i_2)q}$ (see (108)) to $I_{(10)T,T}^{*(i_1 i_2)}$ does not follow if $q \to \infty$ for the method [2]. The same applies to some others approximations of iterated Stratonovich stochastic integrals obtained in [4] by the method [2] (see discussion in Sect. 8 for details).

The validity of the formula

$$\lim_{q\to\infty} \mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)} - I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\} = 0,$$

where $I_{(10)T,T}^{*(i_1 i_2)q}$ is defined by (108), follows from Theorems 3, 6, and 8.

8. Theorems 1, 2, 6, 8–12 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i = 1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s \in [0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p \in \mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i = 1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i = 1,\ldots,m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i = 1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i = 1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [69], [70], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [69]-[71] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [72], [73]

$$\mathbf{f}_{\tau}^{(i)}-\mathbf{f}_{t}^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{110}$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (110) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_{\tau}^{(i)p}-\mathbf{f}_{t}^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_{\tau}^{(i)}-\mathbf{f}_{t}^{(i)}$, which has the following form

$$\mathbf{f}_{\tau}^{(i)p}-\mathbf{f}_{t}^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{111}$$

From (111) we obtain

$$d\mathbf{f}_{\tau}^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{112}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{113}$$

where $p_1,\ldots,p_k\in\mathbb{N},\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$d\mathbf{w}_{\tau}^{(i)p}=\begin{cases}d\mathbf{f}_{\tau}^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{114}$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (112).

Let us substitute (112) into (113)

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{115}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [69]-[71] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [71] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (111) were not considered in [69], [70] (also see [71], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [71] for approximations of the Wiener process based on its series expansion (110) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (115) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [69], [70] (also see [71], Theorems 7.1, 7.2).

From the other hand, Theorems 1, 2, 6, 8–12 can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1–5 and k ($k\in\mathbb{N}$) based on the approximation (111) of the Wiener process. At that, the Riemann–Stieltjes integrals (113) converge (according to Theorems 1, 2, 6, 8–12) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (110), (111), and Theorems 6, 9–12) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [69]-[71]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{116}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s), \quad i_1,i_2=1,\ldots,m.$$

Using (116) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{117}$$

Using (117) and (19) it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{118}$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (118) agrees with Theorem 7.1 (see [71], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (110) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}, \quad i_1,i_2=1,\ldots,m, \tag{119}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (112).

Let us substitute (112) into (119)

$$\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{120}$$

where

$$C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient; another notations are the same as in (115).

As we noted above, approximations of the Wiener process that are similar to (111) were not considered in [69], [70] (also see Theorems 7.1, 7.2 in [71]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [71] to the case under consideration is not obvious.

However, the authors of the works [3] (Sect. 5.8, pp. 202–204), [4] (pp. 438-439), [74] (pp. 82-84), [75] (pp. 263-264) use the Wong–Zakai approximation [69]-[71] (without rigorous proof) within the frames of the approach [2] based on the series expansion of the Brownian bridge process.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [25]-[26]. More precisely, using Theorem 6 from this paper we obtain from (120) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{121}$$

From the other hand, by Theorems 1, 2 (see (10)) for the case $k=2$ we obtain from (120) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{122}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=$$

$$=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from (19) and (122) we obtain (121).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[2] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer-Verlag, Berlin, 1992, 632 pp.

[4] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl. 10, 4 (1992), 431-441.

[5] Milstein G.N., Tretyakov M.V. Stochastic numerics for mathematical physics. Springer-Verlag, Berlin, 2004, 596 pp.

[6] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.

[7] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[8] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html
Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[9] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[10] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[11] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227
Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228
Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[16] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7
Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[18] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233
Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[20] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2017), A.1-A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[21] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[22] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[23] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2018), A.1-A.1073.
Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[25] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2023, 996 pp.

[26] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2020), A.1-A.606.
Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[27] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Differential Equations and Control Processes, 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[28] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Differential Equations and Control Processes, 1 (2023), A.1-A.947. Available at:
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[29] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 221 pp.

[30] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2022, 106 pp.

[31] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2019, 77 pp.

[32] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 42 pp.

[33] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2023, 148 pp.

[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 67 pp.

[35] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604

[36] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[37] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[38] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.
[39] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[40] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[41] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.
[42] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N.Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[43] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Ito expansion. [In English]. arXiv:1805.12527 [math.PR]. 2018, 29 pp.
[44] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 29 pp.
[45] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html
[46] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2023, 144 pp.
[47] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [In English]. arXiv:1801.03195 [math.PR]. 2023, 159 pp.
[48] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2022), 83-186. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html
[49] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [In English]. arXiv:1801.01564 [math.PR]. 2018, 64 pp.
[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR]. 2018, 40 pp.
[51] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.07248 [math.PR]. 2018, 20 pp.
[52] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [In English]. arXiv:1712.08991 [math.PR]. 2017, 56 pp.
[53] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.
[54] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. [In English]. arXiv:1801.05654 [math.PR]. 2018, 46 pp.
[55] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html
[56] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:2006.16040 [math.PR]. 2020, 33 pp.

[57] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. [In English]. arXiv:1806.10705 [math.PR]. 2018, 29 pp.

[58] Kuznetsov D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing: Saarbrucken, 2012, 409 pp.
Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[59] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[60] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[61] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR]. 2023, 58 pp.

[62] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2021), 109-140. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[63] Starchenko T.K. About conditions of convergence of double Fourier-Legendre series. [In Russian]. Proc. of the Mathematical inst. of NAS of Belarus. Analitical methods of analysis and differential equations. Minsk, 2005, no. 5, pp. 124-126.

[64] Suetin P.K. Classical Orthogonal Polynomials. 3rd Edition. Fizmatlit, Moscow, 2005, 480 p.

[65] Ilin V.A., Poznyak E.G. Foundations of Mathematical Analysis. Part II. Nauka, Moscow, 1973, 448 p.

[66] Zhizhiashvili L.V. Conjugate Functions and Trigonometric Series. Tbil. Univ. Press, Tbilisi, 1969, 271 pp.

[67] Hobson E.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge Univ. Press, Cambridge, 1931, 502 pp.

[68] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

[69] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[70] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[71] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[72] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Nauka, Moscow, 1974, 696 pp.

[73] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. Ph. D. Thesis, California Inst. of Technology, 2006, 225 pp.

[74] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[75] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010, 868 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

THE HYPOTHESES ON EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF ARBITRARY MULTIPLICITY AND THEIR PARTIAL PROOF

DMITRIY F. KUZNETSOV

Abstract. In this article we collected more than ten theorems on expansions of iterated Ito and Stratonovich stochastic integrals, which have been formulated and proved by the author. These theorems open up a new direction for study of iterated Ito and Stratonovich stochastic integrals. The expansions based on multiple Fourier–Legendre series as well as on multiple trigonomectic Fourier series are presented in the article. Some of these theorems are connected with the iterated stochastic integrals of multiplicities 1 to 6. Also we consider two theorems on expansions of iterated Ito stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ as well as two theorems on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on iterated trigonometric Fourier series converging pointwise. On the base of the presented theorems we formulate 3 hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) based on generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$. The proof of one of these hypotheses (Hypothesis 2) is given under the condition of convergence of trace series. The mentioned iterated Stratonovich stochastic integrals are part of the Taylor–Stratonovich expansion. Moreover, most of the considered expansions of iterated Stratonovich stochastic integrals contain only one operation of the limit transition and substantially simpler than their analogues for iterated Ito stochastic integrals. Therefore, the results of the article can be applied to numerical integration of Ito stochastic differential equations. Also, the results of the article were reformulated in the form of theorems of the Wong–Zakai type for iterated Stratonovich stochastic integrals.

Contents

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1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as the Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems of various physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[18]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively. In this paper we use the definition of the Stratonovich stochastic integral from [2].

Note that $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ in [2]-[7]. At the same time $\psi_l(\tau)\equiv(t-\tau)^{q_l}$ $(l=1,\ldots,k,\ q_1,\ldots,q_k=0,1,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ in [8]-[18].

Effective solution of the problem of combined mean-square approximation for collections of iterated Ito and Stratonovich stochastic integrals (2) and (3) composes the subject of the article.

We want to mention in short that there are two main criteria of numerical methods convergence for Ito SDEs [2]-[4]: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution.

Using the strong numerical methods, we may build sample pathes of Ito SDEs numerically. These methods require the combined mean-square approximation for collections of iterated Ito and Stratonovich stochastic integrals (2) and (3). The strong numerical methods are used when building new mathematical models on the basis of Ito SDEs and solving various mathematical problems connected with Ito SDEs. Among these problems we mention signal filtering in the background of random noise, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems, etc. [2]-[5].

The problem of effective jointly numerical modeling (in accordance to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[5], [10]-[65].

The only exception is connected with a narrow particular case, when $i_1=\ldots=i_k\neq 0$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv\psi(\tau)$. This case allows the investigation with using of the Ito formula [2]-[4].

Note that even for the mentioned coincidence $(i_1=\ldots=i_k\neq 0)$, but for different functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ the mentioned difficulties persist, and relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be represented effectively in a finite form (within the framework of the mean-square approximation) using the system of standard Gaussian random variables.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated Ito and Stratonovich stochastic integrals may be simplified but cannot be solved. The equations with additive scalar noise, with additive vector noise, with non-additive scalar noise, with a small parameter are related to such types of equations [2]-[4]. For the mentioned types of equations, simplifications are connected to the fact that some members from stochastic Taylor expansions (Taylor–Ito and Taylor–Stratonovich expansions) are equal to zero or we may neglect some members (which include difficult for approximation iterated stochastic integrals) from these expansions due to the presence of a small parameter [2]-[4]. In this article, we consider Ito SDEs with multidimentional and non-additive noise (non-commutative case).

Consider a brief overview of existing methods of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals.

Seems that iterated stochastic integrals may be approximated by multiple integral sums [3], [4], [64]. However, this approach implies the partitioning of the interval of integration $[t,T]$ for iterated stochastic integrals. The length $T-t$ of this interval is already fairly small (because it is a step of integration of numerical methods for Ito SDEs) and does not need to be partitioned. Computational experiments show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors [10].

In [3] (also see [2], [4]) Milstein G.N. proposed to expand the integral (2) of multiplicity 2 ($\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ and $i_1,i_2=1,\ldots,m$) into the iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as the trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion for the Brownian bridge process). To obtain the Milstein expansion of (2) or (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to general expansions of the integrals (2), (3) of arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals were presented in [2] ($k=1,2,3$) and in [3], [4] ($k=1,2$) for the simplest case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv 1$ and $i_1,i_2,i_3=0,1,\ldots,m$. Moreover, the authors of the works [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [67] (pp. 438-439), [68] (pp. 263-264) use the Wong–Zakai approximation [69], [70], [74] (without rigorous proof) within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process. See discussion in Sect. 15 of this paper for details.

Note that in [65] the method of approximation of the double Ito stochastic integral (2) ($\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ and $i_1,i_2=1,\ldots,m$) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered. The restrictions of the method [65] as well as the Milstein approach [3] are connected with the iterated application of the operation of limit transition at least starting from the second (in general case) and third multiplicity of iterated stochastic integrals.

It is necessary to note that the Milstein approach [3] excelled in several times or even in several orders the methods of multiple integral sums [3], [4], [64] (we mean here the diminishing of computational costs).

An alternative strong approximation method (see Theorems 1 and 2 below) was proposed for (3) in [15] (Sect. 2.4) (also see [14], [16]-[18], [22]-[25], [36] (1997), [37] (1998), [52]), where $J^*[\psi^{(k)}]_{T,t}$ was represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables, and the function was then expressed as the generalized iterated Fourier series in a complete systems of continuous functions that are orthonormal in the space $L_2([t,T])$. As a result, an iterated series expansion of the integral (3) in terms of products of standard Gaussian random variables was obtained in [15] (Sect. 2.4) (also see [14], [16]-[18], [22]-[25], [36] (1997), [37] (1998), [52]) for an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series.

It was shown in [15] (also see [14], [16]-[18], [22]-[25], [36] (1997), [37] (1998), [52]) that the method of generalized iterated Fourier series leads to the Milstein expansion [3] of the integral (3) in the case of trigonometric system and to the substantially simpler expansion of the integral (3) in the case of Legendre polynomials system (at least for the case of multiplicity $k=2$ of the integral (3), $i_1\neq i_2$).

Note that the method of generalized iterated Fourier series as well as the Milstein approach [3] leads to iterated application of the operation of limit transition. As mentioned above, this problem appears for triple stochastic integrals ($i_1,i_2,i_3=1,\ldots,m$) or even for some double stochastic integrals in the case, when $\psi_1(\tau)$, $\psi_2(\tau)\not\equiv 1$ ($i_1,i_2=1,\ldots,m$) [10].

The mentioned problem (iterated application of the operation of limit transition) not appears in the method, which is considered for the integrals (2) in Theorems 3, 4 (see below) [10] (2006) [11]-[34], [40]-[49], [51], [53]-[55]. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated nonrandom function is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 3, 4 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2). Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (13)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) [12]-[18], [26], [44].

3. Since the used multiple Fourier series is generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only the trigonometric functions as in [2]-[4] but the Legendre polynomials as well as the systems of Haar and Rademacher–Walsh functions.

4. As it turned out [10]-[34], [40]-[49], [51], [53]-[55] it is more convenient to work with the Legendre polynomials for approximation of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned direction are considered in [31], [40] (also see [15]-[18]).

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see [65]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 3, 4 (see below)) starting from the second or third multiplicity of the iterated Ito stochastic integrals (2). Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [67] (pp. 438-439), [68] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [69], [70], [74] (see Sect. 15 for details).

6. As it turned out, the method of generalized multiple Fourier series can be adapted for the iterated Stratonovich stochastic integrals (3) at least for multiplicities 1 to 6 [11]-[18], [23]-[25], [32], [36], [37], [41], [47]-[49], [52], [55]. Expansions of these iterated Stratonovich stochastic integrals turned out to be simpler (see Theorems 6–13, 15–18, 23 below) than the appropriate expansions of the iterated Ito stochastic integrals (2) from Theorems 3, 4.

In this article, we collect more than ten theorems formulated and proved by the author that develop the mentioned direction of investigations. Moreover, on the base of the presented theorems, we formulate 3 hypotheses (Hypotheses 1–3) on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k. The results of the article prove (the cases of Legendre polynomials and trigonometric functions) Hypothesis 1 for $k=1,\ldots,6$, Hypothesis 2 for $k=1,\ldots,5$ and Hypothesis 3 for $k=1,\ldots,3$. Moreover, the proof of Hypotheses 2 and 3 is given for an arbitrary k under the condition of convergence of trace series.

2. Hypotheses on Expansions of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k

Taking into account Theorems 1–13, 15–18, 23 (see below), let us formulate the following hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k.

Hypothesis 1 [11]-[18]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of kth multiplicity*

$$
I_{T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4}
$$

$$
(i_1,\ldots,i_k=0,1,\ldots,m)
$$

the following expansion

$$
I_{T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{5}
$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_k\ldots j_1}$ has the form

$$
C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k,
$$

l.i.m. *is a limit in the mean-square sense,*

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i=1,\ldots,m)$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral $I_{T,t}^{*(i_1\ldots i_k)}$ by the sum

$$
I_{T,t}^{*(i_1\ldots i_k)p}=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{6}
$$

where

$$
\lim_{p\to\infty}\mathsf{M}\left\{\left(I_{T,t}^{*(i_1\ldots i_k)}-I_{T,t}^{*(i_1\ldots i_k)p}\right)^2\right\}=0.
$$

The integrals (4) are integrals from the Taylor–Stratonovich expansion [2]. It means that the approximations (6) can be very useful for the numerical integration of Ito SDEs. The expansion (5) contains only one operation of the limit transition and by this reason is convenient for approximation of iterated Stratonovich stochastic integrals. Moreover, the author supposes that the analogue of Hypothesis 1 will be valid for the iterated Stratonovich stochastic integrals (3).

Hypothesis 2 [14]-[18]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral* (3) *of* k*th multiplicity*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

$$(i_1,\ldots,i_k=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{7}$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form*

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

l.i.m. *is a limit in the mean-square sense,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\neq 0$*),* $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ *are independent standard Wiener processes* $(i=1,\ldots,m)$ *and* $\mathbf{w}_{\tau}^{(0)}=\tau$*.*

Hypothesis 2 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{8}$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\}=0.$$

Let us consider the more general statement, then Hypotheses 1 and 2.

Hypothesis 3 [15]-[18]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* (3) *of* k*th multiplicity*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

$$(i_1,\ldots,i_k=0,1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{9}$$

converging in the mean-square sense is valid, where the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form*

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

l.i.m. *is a limit in the mean-square sense,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*if* $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i=1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

In the next section, we consider two theorems on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity k. Expansions from Theorems 1 and 2 (see below) contain an iterated operation of the limit transition in comparison with Hypotheses 1–3. This feature creates some difficulties when estimating the mean-square approximation error of iterated Stratonovich stochastic integrals. On the other hand, Theorems 1 and 2 contain the same expansion terms of iterated Stratonovich stochastic integrals as in Hypotheses 1–3.

3. Expansions of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity Based on Iterated Fourier Series Converging Pointwise

Let us formulate the following theorem.

Theorem 1 [15] (Sect. 2.4) (also see [14], [16]-[18], [22]-[25], [36] (1997), [37] (1998), [52]). *Suppose that the functions* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$. *Then, the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1, \ldots, i_k = 0, 1, \ldots, m)$$

is expanded into the converging in the mean of degree $2n$ $(n \in \mathbb{N})$ *iterated series*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}, \tag{10}$$

i.e.

$$\lim_{p_1\to\infty} \overline{\lim_{p_2\to\infty}} \ldots \overline{\lim_{p_k\to\infty}} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}\right)^{2n}\right\} = 0,$$

where $\overline{\lim}$ *means* $\limsup$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \neq 0$*), the Fourier coefficient* $C_{j_k \ldots j_1}$ *has the form*

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K^*(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k =$$

$$= \int\limits_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k,$$

where

$$K^*(t_1, \ldots, t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \left(\mathbf{1}_{\{t_l < t_{l+1}\}} + \frac{1}{2} \mathbf{1}_{\{t_l = t_{l+1}\}}\right), \quad t_1, \ldots, t_k \in [t, T]$$

for $k \geq 2$ *and* $K^*(t_1) \equiv \psi_1(t_1)$ *for* $t_1 \in [t, T]$*. Here* $\mathbf{1}_A$ *is the indicator of the set* A*.*

Let us consider the following theorem.

Theorem 2 [15] (Sect. 2.4) (also see [14], [16]-[18], [25] (2013), [52]). *Suppose that the functions* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *are twice continuously differentiable at the interval* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of trigonometric functions in the space* $L_2([t, T])$*. Then, the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1, \ldots, i_k = 0, 1, \ldots, m)$$

is expanded into the converging in the mean of degree $2n$ $(n \in \mathbb{N})$ *iterated series*

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{11}$$

i.e.

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\,\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_k=0}^{p_k}\ldots\sum_{j_1=0}^{p_1}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^{2n}\right\}=0;$$

another notations are the same as in Theorem 1.

It is not difficult to see that the members of expansions (7), (10), and (11) are the same. However, as mentioned before, the expansion (7) contains only one operation of the limit transition. At the same time the expansions (10), (11) contain an iterated operation of the limit transiton.

In [15] (Sect. 2.4.1) it is shown that Theorems 1 and 2 will remain valid for the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, $k=2$, and $2n=2$ (the case of mean-square convergence). In this case the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the function $\psi_1(\tau)$ is twice continuously differentiable at the interval $[t,T]$.

As it tured out, the approach considered in the next section gives the key to the proof of Hypotheses 1–3.

4. Expansion of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity k Based on Generalized Multiple Fourier Series Converging in the Mean

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Theorem 4 (see below)). Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2, \tag{12}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{13}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of $[t, T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} \Delta\tau_j \to 0 \ \text{ if } \ N \to \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \tag{14}$$

Theorem 3 [10] (2006), [11]-[34], [40]-[49], [51], [53]-[55]. *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \right.$$

$$\left. - \underset{N \to \infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k) \in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \right), \tag{15}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1, \ldots, l_k): \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1\},$$

$$\mathrm{L}_k = \{(l_1, \ldots, l_k): \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \ne l_r \ (g \ne r); \ g, r = 1, \ldots, k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i \ne 0$), $C_{j_k \ldots j_1}$ is the Fourier coefficient (13), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition (14).

Let us consider the transformed particular cases of Theorem 3 (see (15)) for $k = 1$–6 [10]-[34], [40]-[49], [51], [53]-[55] (the case $k = 7$ can be found in [11]-[18], [42])

$$J[\psi^{(1)}]_{T,t} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{16}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \ne 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{17}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{18}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{19}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{20}$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$
\begin{gathered}
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
\end{gathered}
$$

$$-\mathbf{1}_{\{i_6=i_2\neq 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\neq 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\neq 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\neq 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\neq 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{21}$$

where $\mathbf{1}_A$ is the indicator of the set A.

It was shown that Theorem 3 is valid for convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [11]-[25], [42]. Moreover, the convergence with probability 1 (further w. p. 1) is proved in Theorem 3 for iterated Ito stochastic integrals of multiplicity k for the cases of Legendre polynomials and trigonometric functions [42]-[45], [57], [58] (also see [15]-[18]).

As it turned out, Theorem 3 remains valid for some discontinuous complete orthonormal systems of functions in the space $L_2([t,T])$. For example, Theorem 3 is true for the system of Haar functions as well as for the system of Rademacher–Walsh functions [10]-[25], [42].

In [11]-[25], [54] we demonstrate that approach to expansion of iterated Ito stochastic integrals considered in Theorem 3 is essentially general and allows some modifications for other types of iterated stochastic integrals. Versions of Theorem 3 for iterated stochastic integrals with respect to martingale Poisson measures and for iterated stochastic integrals with respect to martingales are obtained in [11]-[25], [54]. The mentioned theorems are sufficiently natural according to general properties of martingales. Another modification of Theorem 3 can be found in [15]-[18], [42], [54], where complete orthonormal with weight $r(x) \geq 0$ systems of functions in the space $L_2([t,T])$ were considered.

A generalization of Theorem 3 (see Theorem 4 below) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ is given in [15] (Sect. 1.11), [42] (Sect. 15), [43]. Moreover, Theorems 3 and 4 allow us to calculate exactly the mean-square approximation error for the iterated Ito stochastic integral (2) of arbitrary multiplicity k (see [14], [15]-[18], [44]). Here we consider an approxination as the expression before passing to the limit in (15) or (24) (see below).

Application of Theorem 3 and Theorem 4 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [15]-[18] (Chapter 7) and in [33]-[35].

Consider the generalization of formulas (16)–(21) for the case of an arbitrary multiplicity k of the stochastic integral $J[\psi^{(k)}]_{T,t}$ as well as for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{22}$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (22) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{23}$$

where $a_{g_1 g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}} \in \mathbb{R}$.

Below there are several examples of sums in the form (23)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1 g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1 g_2,q_1 q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1 g_2,q_1 q_2 q_3} = a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$

$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1 g_2,g_3 g_4,q_1} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$

$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can generalize Theorem 3.

Theorem 4 [15] (Sect. 1.11), [42] (Sect. 15), [43]. *Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} =\ i_{g_{2s}} \ne 0\}} \mathbf{1}_{\{j_{g_{2s-1}} =\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr) \tag{24}$$

that converges in the mean-square sense is valid, where $[x]$ is an integer part of a real number x and $\prod\limits_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\sf def}{=} 0$; another notations are the same as in Theorem 3.

In particular from (24) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (20).

It should be noted that an analogue of Theorem 4 for multiple Ito stochastic integrals was considered in [66]. Note that we use another notations in comparison with [66]. Moreover, the proof of an analogue of Theorem 4 from [66] is different from the proof given in [15] (Sect. 1.11), [42] (Sect. 15), [43].

5. The Idea of the Proof of Hypotheses 1, 2, and 3

Let us consider the idea of the proof of Hypotheses 1–3. Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1}\ \stackrel{\text{def}}{=}\ \prod_{p=1}^{l}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}}\ \times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{25}$$

where $(s_l,\ldots,s_1)\in \mathrm{A}_{k,l}$,

$$\mathrm{A}_{k,l}=\left\{(s_l,\ldots,s_1):\ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;\ s_l,\ldots,s_1=1,\ldots,k-1\right\},\tag{26}$$

$l=1,2,\ldots,[k/2],\ \ i_s=0,1,\ldots,m,\ \ s=1,\ldots,k,\ \ [x]$ is an integer part of a real number $x,\ \ \mathbf{1}_A$ is the indicator of the set A.

Let us formulate the statement on connection between iterated Ito and Stratonovich stochastic integrals (2) and (3) of arbitrary multiplicity k.

Theorem 5 [36] (1997) (also see [10]-[18]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the following relation between iterated Ito and Stratonovich stochastic integrals (2) and (3) is correct*

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad \text{w. p. 1}, \tag{27}$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Note that the condition of continuity of the functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ is related to the definition [2] of the Stratonovich stochastic integral that we use.

According to (15), we have

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)}=J[\psi^{(k)}]_{T,t}+$$

$$+\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}. \tag{28}$$

From (5) and (27) it follows that

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{g=1}^{k}\zeta_{j_g}^{(i_g)} \tag{29}$$

if

$$\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=$$

$$=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\prod_{g=1}^{k}\phi_{j_g}(\tau_{l_g})\Delta\mathbf{w}_{\tau_{l_g}}^{(i_g)}\quad \text{w. p. 1}.$$

In the case $p_1=\ldots=p_k=p$ and $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ from (29) we obtain the statement of Hypothesis 1 (see (5)).

If $p_1=\ldots=p_k=p$ and every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$, then from (29) we obtain the statement of Hypothesis 2 (see (7)).

In the case when every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$ from (29) we obtain the statement of Hypothesis 3 (see (9)).

In the following sections we consider some theorems proving Hypothesis 1 for $k=1,\ldots,6$, Hypothesis 2 for $k=1,\ldots,5$ and Hypothesis 3 for $k=1,\ldots,3$. Moreover, the proof of Hypotheses 2 and 3 is given for an arbitrary k under the condition of convergence of trace series. The case $k=1$ obviously directly follows from Theorem 3 (see (16)).

6. Expansions of Iterated Stratonovich Stochastc Integrals of Multiplicity 2–4. Some Old Results

As it turned out, approximations of the iterated Stratonovich stochastic integrals (3) are essentially simpler than the appropriate approximations of the iterated Ito stochastic integrals (2) based on Theorems 3 and 4. For the first time this fact was mentioned in [10] (2006).

According to the standard connection between Ito and Stratonovich stochastic integrals, the iterated Ito and Stratonovich stochastic integrals (2) and (3) of first multiplicity are equal to each other w. p. 1. So, we begin the consideration from the multiplicity $k=2$.

The following theorems adapt Theorems 3, 4 for the integrals (3) of multiplicities 2–4.

Theorem 6 [11]-[18], [23]-[25], [47]. *Suppose that the following conditions are fulfilled*:

1. *The function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the function $\psi_1(\tau)$ is twice continuously differentiable at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.*

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the converging in the mean-square sense double series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where notations are the same as in Theorems 3, 4.

Proving Theorem 6 [11]-[18], [23]-[25], [47], we used Theorem 3 and double integration by parts. This procedure leads to the condition of double continuously differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$. The mentioned condition can be weakened, but the proof becomes more complicated. As a result, we have the following theorem.

Theorem 7 [15]-[18], [32], [49]. *Suppose that the following conditions are fulfilled*:

1. *Every $\psi_l(\tau)$ $(l=1,2)$ is a continuously differentiable function at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.*

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

is expanded into the converging in the mean-square sense double series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where notations are the same as in Theorems 3, 4.

The proof of Theorem 7 [15]-[18], [32], [49] is based on Theorem 3 and double Fourier–Legendre series as well as double trigonometric Fourier series summarized by Pringsheim method at the square $[t,T]^2$.

Recently, Theorem 7 has been generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ (see [15], Sect. 2.18) or Theorem 42 below.

The following 4 theorems (Theorems 8–11) adapt Theorems 3, 4 for the integrals (3) of multiplicity 3.

Theorem 8 [11]-[18], [23]-[25], [48]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{30}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^{T}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 3, 4.

Obviously, that Theorem 8 proves Hypothesis 3 for the case $k=3$ and $\psi_l(\tau)\equiv 1$ $(l=1,2,3)$.

Let us consider the generalization of Theorem 8 (the case of Legendre polynomials) for the binomial weight functions.

Theorem 9 [11]-[18], [23]-[25], [48]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\int\limits_t^{*T}(t-t_3)^{l_3}\int\limits_t^{*t_3}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \stackrel{\text{def}}{=}$$

$$\stackrel{\rm def}{=} \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \qquad (31)$$

that converges in the mean-square sense is valid for each of the following cases

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
2. $i_1 = i_2 \ne i_3$ *and* $l_1 = l_2 \ne l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
3. $i_1 \ne i_2 = i_3$ *and* $l_1 \ne l_2 = l_3$ *and* $l_1, l_2, l_3 = 0, 1, 2, \ldots$
4. $i_1, i_2, i_3 = 1, \ldots, m;\ l_1 = l_2 = l_3 = l$ *and* $l = 0, 1, 2, \ldots,$

where

$$C_{j_3j_2j_1} = \int\limits_t^T (t-t_3)^{l_3}\phi_{j_3}(t_3)\int\limits_t^{t_3}(t-t_2)^{l_2}\phi_{j_2}(t_2)\int\limits_t^{t_2}(t-t_1)^{l_1}\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 3, 4.

We can introduce the weight functions $\psi_l(\tau)$ $(l = 1, 2, 3)$ with some properties of smoothness. However, we consider in this case the more specific method of series summation $(p_1 = p_2 = p_3 = p \to \infty)$.

Theorem 10 [11]-[18], [23]-[25]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_l(\tau)$ $(l = 1, 2, 3)$ *are continuously differentiable functions at the interval* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\quad (i_1, i_2, i_3 = 1, \ldots, m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\rm l.i.m.}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \qquad (32)$$

that converges in the mean-square sense is valid for each of the following cases

1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3,$
2. $i_1 = i_2 \ne i_3$ *and* $\psi_1(\tau) \equiv \psi_2(\tau),$
3. $i_1 \ne i_2 = i_3$ *and* $\psi_2(\tau) \equiv \psi_3(\tau),$
4. $i_1, i_2, i_3 = 1, \ldots, m$ *and* $\psi_1(\tau) \equiv \psi_2(\tau) \equiv \psi_3(\tau),$

where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 3, 4.

We can omit Cases 1–4 in Theorem 10 in the case when the functions $\psi_1(\tau)$ and $\psi_3(\tau)$ are twice continuously differentiable.

Theorem 11 [12]-[18], [25], [47]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space* $L_2([t,T])$. *Furthermore, let the function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{33}$$

that converges in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 3, 4.

The following theorem adapts Theorems 3, 4 for the integrals (3) ($\psi_l(\tau)\equiv 1,\ l=1,\ldots,4$) of multiplicity 4.

Theorem 12 [12]-[18], [25], [47]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following expansion

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where

$$C_{j_4j_3j_2j_1}=\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4,$$

$\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ *are independent standard Wiener processes and* $\mathbf{w}_{\tau}^{(0)}=\tau$.

It is obvious that Theorem 12 prove Hypothesis 1 for the case $k=4$.

7. Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$). Proof of Hypothesis 2 Under the Condition of Convergence of Trace Series

In this section, we prove the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) under the condition of convergence of trace series.

Let us introduce some notations and formulate some auxiliary results.

Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{34}
$$

where

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

Recall that the expression (34) is called the partition. Let us consider the sum with respect to all possible partitions

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}},
$$

where $a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}} \in \mathbb{R}$.

Consider the Fourier coefficient

$$
C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{35}
$$

corresponding to the function (12), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. At that we suppose $\phi_0(x) = 1/\sqrt{T-t}$.

Denote

$$
C_{j_k\ldots j_{l+1} j_l j_l j_{l-2}\ldots j_1}\Bigg|_{(j_l j_l)\frown(\cdot)} \stackrel{\sf def}{=}
$$

$$
\stackrel{\sf def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\times
$$

$$
\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k = \tag{36}
$$

$$
= \sqrt{T-t}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times
$$

$$\times \int_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2}) \ldots \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_{l-2}dt_l t_{l+1} \ldots dt_k =$$

$$= \sqrt{T-t}\hat{C}_{j_k \ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat{C}_{j_k \ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k \ldots j_1}$ but with a new shorter multi-index $j_k \ldots j_{l+1}0j_{l-2} \ldots j_1$ and new weight functions $\psi_1(\tau), \ldots, \psi_{l-2}(\tau), \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau), \ldots, \psi_k(\tau)$ (also we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$).

Let

$$C_{j_k \ldots j_{l+1}j_l j_l j_{l-2}\ldots j_1}\Bigg|_{(j_l j_l)\frown j_m} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \int_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_{j_m}(t_l)\times$$

$$\times \int_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2}) \ldots \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_{l-2}dt_l t_{l+1} \ldots dt_k = \tag{37}$$

$$= \bar{C}_{j_k \ldots j_{l+1}j_m j_{l-2}\ldots j_1},$$

i.e. $\bar{C}_{j_k \ldots j_{l+1}j_m j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k \ldots j_1}$ but with a new shorter multi-index $j_k \ldots j_{l+1}j_m j_{l-2} \ldots j_1$ and new weight functions $\psi_1(\tau), \ldots, \psi_{l-2}(\tau), \psi_{l-1}(\tau)\psi_l(\tau), \psi_{l+1}(\tau), \ldots$, $\psi_k(\tau)$ (also we suppose that $\{l-1, l\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$).

Denote

$$\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}. \tag{38}$$

Introduce the following notation

$$S_l \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} \stackrel{\sf def}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty} \sum_{j_{g_{2l-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}\Bigg|_{(j_{g_{2l}} j_{g_{2l-1}})\frown(\cdot), j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}. \tag{39}$$

Note that the operation S_l $(l = 1, 2, \ldots, r)$ acts on the value

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{40}$$

as follows: S_l multiplies (40) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{41}$$

Note that we write

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright j_m,j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright j_m,j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=C_{j_k\ldots j_1}\Biggr|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}},\quad\ldots$$

Since (41) is again the Fourier coefficient, then the action of superposition S_lS_m on (41) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.$$

Theorem 13 [15], [46], [47], [55], [62]. *Assume that the continuously differentiable functions* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions* $(\phi_0(x)=1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}\int\limits_t^s\Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{42}$$

holds for all $s\in(t,T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t,T]$ *and the series on the right-hand side of* (42) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\qquad\left|\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s\in(t,T)$ *and for some* $\alpha,\beta>0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$, $j,p\in\mathbb{N}$, *and*

$$\int\limits_t^T\Psi_1^2(\tau)d\tau<\infty,\qquad\int\limits_t^T\left|\Psi_2(\tau)\right|d\tau<\infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (34)*) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, *where* $r=1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{43}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{44}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{45}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Proof. First note that (42) is fulfilled (see [15], Sect. 2.1.4 or [94]). The proof of Theorem 13 will consist of several steps.

Step 1. Let us find a representation of the quantity

$$\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that will be convenient for further consideration.

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_k=0\\ j_q\neq j_r;\ q\neq r;\ q,r=1,\ldots,k}}^{N-1}\Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\stackrel{\text{def}}{=}J'[\Phi]_{T,t}^{(i_1\ldots i_k)}, \tag{46}$$

where for simplicity we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Moreover, $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (14), $i_1,\ldots,i_k=0,1,\ldots,m.$

The stochastic integral with respect to the scalar standard Wiener process $(i_1=\ldots=i_k\neq 0)$ and similar to (46) was considered in [85] (1951) and is called the multiple Wiener stochastic integral [85].

Note that the following well known estimate

$$\mathsf{M}\left\{\left(J'[\Phi]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le C_k\int\limits_{[t,T]^k}\Phi^2(t_1,\ldots,t_k)dt_1\ldots dt_k \tag{47}$$

is true for the multiple Wiener stochastic integral, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (46) and C_k is a constant.

From the proof of Theorem 3 (see the proof of Theorem 5.1 in the original paper [10] (2006) in Russian or proof of Theorem 1.1 in the monographs [15]-[18] in English) it follows that (15) can be written as

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{48}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochatic integral defined by (46) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (2), i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{49}$$

Let us consider the following multiple stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^k \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J[\Phi]_{T,t}^{(i_1\ldots i_k)}, \tag{50}$$

where we assume that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Another notations are the same as in (46).

The stochastic integral with respect to the scalar standard Wiener process ($i_1=\ldots=i_k\neq 0$) and similar to (50) (the function $\Phi(t_1,\ldots,t_k)$ is assumed to be symmetric on the hypercube $[t,T]^k$) has been considered in literature (see, for example, Remark 1.5.7 [86]). The integral (50) is sometimes called the multiple Stratonovich stochastic integral. This is due to the fact that the following rule of the classical integral calculus holds for this integral

$$J[\Phi]_{T,t}^{(i_1\ldots i_k)} = J[\varphi_1]_{T,t}^{(i_1)}\ldots J[\varphi_k]_{T,t}^{(i_k)} \quad \text{w. p. 1},$$

where $\Phi(t_1,\ldots,t_k)=\varphi_1(t_1)\ldots\varphi_k(t_k)$ and

$$J[\varphi_l]_{T,t}^{(i_l)} = \int\limits_t^T \varphi_l(s)d\mathbf{w}_s^{(i_l)} \quad (l=1,\ldots,k).$$

Theorem 14 [15], [18]. *Suppose that $\Phi(t_1,\ldots,t_k):\ [t,T]^k\to\mathbb{R}$ is a continuous nonrandom function on $[t,T]^k$. Furthermore, $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t,T])$, each function $\phi_j(x)$ of which for finite j is continuous at the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity as well as $\phi_j(x)$ right-continuous at the interval $[t,T]$. Then the following expansion*

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^r \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{51}$$

converging in the mean-square sense is valid, where $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is the multiple Wiener stochatic integral defined by* (46),

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}\Phi(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{52}$$

is the Fourier coefficient. Another notations are the same as in Theorems 3, 4.

From (24) and (48) we conclude that

$$J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+$$

$$+\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{53}$$

w. p. 1, where notations are the same as in Theorem 4 and $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral (46). For a more detailed derivation of (53), see [43] (also see [15])).

Using (53), we obtain

$$\prod_{l=1}^k\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}-$$

$$-\sum_{r=1}^{[k/2]}(-1)^r\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})} \tag{54}$$

w. p. 1.

By iteratively applying the formula (54) (also see (17)–(21)), we obtain the following representation of the product

$$\prod_{l=1}^k\zeta_{j_l}^{(i_l)}$$

as the sum of some constant value and multiple Wiener stochastic integrals of multiplicities not exceeding k

$$\prod_{l=1}^k\zeta_{j_l}^{(i_l)}=J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad\text{w. p. 1}, \tag{55}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\stackrel{\sf def}{=}1$ for $k=2r$.

Multiplying both sides of the equality (55) by $C_{j_k\ldots j_1}$ and summing over $j_1,\ldots,j_k$, we get w. p. 1

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \tag{56}$$

Denote

$$K_{p_1\ldots p_k}(t_1,\ldots,t_k)=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{57}$$

$$K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\phi_{j_{q_l}}(t_{q_l}), \tag{58}$$

where $C_{j_k\ldots j_1}$ is defined by (45) and $\prod_{\emptyset}\stackrel{\text{def}}{=}1$.

The equality (56) can be written as

$$J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\ J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{59}$$

w. p. 1, where $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}(t_{q_1},\ldots,t_{q_{k-2r}})$ have the form (57), (58), $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Stratonovich stochastic integral defined by (50), $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined by (46).

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ $(p_1=\ldots=p_k=p)$ in (56) or (59), we get w. p. 1 (see (48))

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}= \tag{60}$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \underset{p\to\infty}{\text{l.i.m.}} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times J'[K_{p\ldots p}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \tag{61}$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (49).

If we prove that w. p. 1

$$\sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}, \tag{62}$$

then (see (60), (62), and Theorem 5)

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} =$$

$$= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{63}$$

w. p. 1, where notations in (63) are the same as in Theorem 5. Thus Theorem 13 will be proved.

From (59) we have that the multiple Stratonovich stochastic integral $J[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ of multiplicity k is expressed as a sum of some constant value and multiple Wiener stochastic integrals $J'[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ and $J'[K_{p_1\ldots p_k}^{g_1\ldots g_{2r},q_1\ldots q_{k-2r}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ of multiplicities $k,\ k-2,\ k-4,\ \ldots,\ k-2[k/2]$ $(r=1,2,\ldots,[k/2])$.

The formulas (56), (59) can be considered as new representations of the Hu-Meyer formula for the case of a multidimensional Wiener process [87] (also see [86], [88]) and kernel $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ (see (57)).

Note that the equality (59) can be obtained from (51) if we consider (51) for $\Phi(t_1,\ldots,t_k) = K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ and without passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$

For example, for $k=2,3,4,5,6$ we have from (56) w. p. 1

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} = J'[K_{p_1p_2}]_{T,t}^{(i_1i_2)} + \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}, \tag{64}$$

$$
\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} = J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+
$$
$$
+\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}\Biggr), \tag{65}
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = J'[K_{p_1p_2p_3p_4}]_{T,t}^{(i_1i_2i_3i_4)}+
$$
$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+
$$
$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+
$$
$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{66}
$$

$$
\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} = J'[K_{p_1p_2p_3p_4p_5}]_{T,t}^{(i_1i_2i_3i_4i_5)}+
$$
$$
+\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_3i_4i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_4i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_2i_3i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_2i_3i_4)}+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_4i_5)}+
$$
$$
+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_3i_5)}+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_3i_4)}+
$$
$$
+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_1i_2i_5)}+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_1i_2i_4)}+
$$
$$
+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_1i_2i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}]_{T,t}^{(i_5)}+
$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}]_{T,t}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}]_{T,t}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}+$$
$$\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right), \tag{67}$$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}=J'[K_{p_1p_2p_3p_4p_5p_6}]_{T,t}^{(i_1i_2i_3i_4i_5i_6)}+$$
$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_2i_3i_4i_5)}+$$
$$+\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_3i_4i_5)}+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_1i_2i_4i_5)}+$$
$$+\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_1i_2i_3i_5)}+\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_3i_4i_5i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_4i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_2i_3i_5i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_2i_3i_4i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_4i_5i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_3i_5i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_3i_4i_6)}+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_1i_2i_5i_6)}+$$
$$+\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_1i_2i_4i_6)}+\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_1i_2i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_5}\phi_{j_6}]_{T,t}^{(i_5i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_6}]_{T,t}^{(i_4i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_6}]_{T,t}^{(i_3i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_6}]_{T,t}^{(i_2i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_6}]_{T,t}^{(i_1i_6)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_4}\phi_{j_5}]_{T,t}^{(i_4i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_5}]_{T,t}^{(i_1i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_5}]_{T,t}^{(i_2i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_5}]_{T,t}^{(i_3i_5)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr). \tag{68}$$

Note that the relation (66) can be written in the following form

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1} J'[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1i_2i_3i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_3=0}^{p_3}\sum_{j_4=0}^{p_4}\left(\sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_4j_3j_1j_1}\right) J'[\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_3i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_1,p_3\}} C_{j_4j_3j_2j_3}\right) J'[\phi_{j_2}\phi_{j_4}]_{T,t}^{(i_2i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_1,p_4\}} C_{j_4j_3j_2j_4}\right) J'[\phi_{j_2}\phi_{j_3}]_{T,t}^{(i_2i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_4=0}^{p_4}\left(\sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_4j_3j_3j_1}\right) J'[\phi_{j_1}\phi_{j_4}]_{T,t}^{(i_1i_4)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}\left(\sum_{j_4=0}^{\min\{p_2,p_4\}} C_{j_4j_3j_4j_1}\right) J'[\phi_{j_1}\phi_{j_3}]_{T,t}^{(i_1i_3)}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(\sum_{j_4=0}^{\min\{p_3,p_4\}} C_{j_4j_4j_2j_1}\right) J'[\phi_{j_1}\phi_{j_2}]_{T,t}^{(i_1i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\sum_{j_2=0}^{\min\{p_2,p_3\}}\ \sum_{j_4=0}^{\min\{p_1,p_4\}} C_{j_4j_2j_2j_4}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\sum_{j_3=0}^{\min\{p_1,p_3\}}\sum_{j_4=0}^{\min\{p_2,p_4\}} C_{j_4j_3j_4j_3}+$$

$$+\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_2=0}^{\min\{p_1,p_2\}}\sum_{j_4=0}^{\min\{p_3,p_4\}} C_{j_4j_4j_2j_2}\quad \text{w. p. 1.}$$

Further, we will use the representation (56) for $p_1 = \ldots = p_k = p$, i.e.

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad \text{w. p. 1.} \tag{69}$$

Step 2. Let us prove that

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = 0 \tag{70}$$

or

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = -\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}, \tag{71}$$

where $l-1\ge s+1$.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

We have

$$C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} =$$

$$=\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_sdt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k =$$

$$
=\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s\times
$$

$$
\times\left(\int\limits_{t_{s+1}}^T \phi_{j_{s+2}}(t_{s+2})\ldots\int\limits_{t_{l-2}}^T\phi_{j_{l-1}}(t_{l-1})\int\limits_{t_{l-1}}^T\phi_{j_l}(t_l)\int\limits_{t_l}^T\phi_{j_{l+1}}(t_{l+1})\ldots\right.
$$

$$
\left.\ldots\int\limits_{t_{k-1}}^T\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}dt_ldt_{l-1}\ldots dt_{s+2}\right)dt_{s+1}=
$$

$$
=\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\underbrace{\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}}_{G_{j_{s-1}\ldots j_1}(t_s)}dt_s\times
$$

$$
\times\int\limits_{t_{s+1}}^T\phi_{j_l}(t_l)\underbrace{\int\limits_{t_l}^T\phi_{j_{l+1}}(t_{l+1})\ldots\int\limits_{t_{k-1}}^T\phi_{j_k}(t_k)dt_k\ldots dt_{l+1}}_{H_{j_k\ldots j_{l+1}}(t_l)}\times
$$

$$
\times\left(\underbrace{\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}}_{Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})}dt_l\right)dt_{s+1}=
$$

$$
=\int\limits_t^T \phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times
$$

$$
\times\int\limits_{t_{s+1}}^T\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})dt_ldt_{s+1}. \tag{72}
$$

Using the additive property of the integral, we obtain

$$
Q_{j_{l-1}\ldots j_{s+2}}(t_l,t_{s+1})=
$$

$$
=\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}\ldots dt_{l-1}=
$$

$$
=\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}}\phi_{j_{s+3}}(t_{s+3})\int\limits_t^{t_{s+3}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}dt_{s+3}\ldots dt_{l-1}-
$$

$$-\int\limits_{t_{s+1}}^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots\int\limits_{t_{s+1}}^{t_{s+4}}\phi_{j_{s+3}}(t_{s+3})dt_{s+3}\ldots dt_{l-1}\int\limits_{t}^{t_{s+1}}\phi_{j_{s+2}}(t_{s+2})dt_{s+2}=$$

$$\ldots$$

(73)
$$=\sum_{m=1}^{d}h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1}),\quad d<\infty.$$

Combining (72) and (73), we have

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=\sum_{m=1}^{d}\left(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=0}^{p}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

(74)
$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right).$$

Using the generalized Parseval equality, we obtain

$$\sum_{j_l=0}^{\infty}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_l=$$

(75)
$$=\int\limits_{t}^{T}\mathbf{1}_{\{\tau<t_{s+1}\}}G_{j_{s-1}\ldots j_1}(\tau)\cdot\mathbf{1}_{\{\tau>t_{s+1}\}}H_{j_k\ldots j_{l+1}}(\tau)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(\tau)d\tau=0.$$

From (74) and (75) we get

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$=-\sum_{m=1}^{d}\left(\int\limits_{t}^{T}\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\sum_{j_l=p+1}^{\infty}\int\limits_{t}^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.$$

(76)
$$\left.\times\int\limits_{t_{s+1}}^{T}\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right).$$

Combining Condition 2 of Theorem 13 and (72)–(74), (76), we have

$$\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}=$$

$$
=-\sum_{j_l=p+1}^{\infty}\sum_{m=1}^{d}\left(\int\limits_t^T\phi_{j_{s+1}}(t_{s+1})q^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)G_{j_{s-1}\ldots j_1}(t_s)dt_s\times\right.
$$

$$
\left.\times\int\limits_{t_{s+1}}^T\phi_{j_l}(t_l)H_{j_k\ldots j_{l+1}}(t_l)h^{(m)}_{j_{l-1}\ldots j_{s+2}}(t_l)dt_ldt_{s+1}\right)=
$$

$$
=-\sum_{j_l=p+1}^{\infty}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots
$$

$$
\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots
$$

$$
\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_sdt_{s+1}\ldots dt_{l-1}dt_ldt_{l+1}\ldots dt_k=
$$

$$
=-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}. \tag{77}
$$

The equality (77) implies (70), (71).

Step 3. Using Conditions 1 and 2 of Theorem 13, we obtain

$$
\sum_{j_l=0}^{p}C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=
$$

$$
=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\sum_{j_l=0}^{p}\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times\int\limits_t^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_ldt_{l+1}\ldots dt_k=
$$

$$
=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\sum_{j_l=0}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(t_l)\phi_{j_l}(t_l)\int\limits_t^{t_l}\psi_{l-1}(t_{l-1})\phi_{j_l}(t_{l-1})\times
$$

$$
\times\int\limits_t^{t_{l-1}}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_ldt_{l+1}\ldots dt_k-
$$

$$
-\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}=
$$

$$= \frac{1}{2} C_{j_k\ldots j_1}\Biggl|_{(j_l j_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1} j_l j_l j_{l-2}\ldots j_1}. \tag{78}$$

Step 4. Passing to the limit $\underset{p\to\infty}{\text{l.i.m.}}$ in (69), we have (see (48))

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} +$$

$$+ \sum_{r=1}^{[k/2]} \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \ne 0\}} \times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} \quad \text{w. p. 1.} \tag{79}$$

Taking into account (71) and (78), we obtain for $r = 1$

$$\mathbf{1}_{\{i_{g_1} = i_{g_2} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \mathbf{1}_{\{j_{g_1} = j_{g_2}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =$$

$$= -\mathbf{1}_{\{i_{g_1} = i_{g_2} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1} = j_{g_2}} \mathbf{1}_{\{g_2 > g_1+1\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} +$$

$$+ \mathbf{1}_{\{i_{g_1} = i_{g_2} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\ne g_1,g_2}}^{p} \frac{1}{2} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2} j_{g_1})\curvearrowright(\cdot),\, j_{g_1} = j_{g_2}} \mathbf{1}_{\{g_2 = g_1+1\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} -$$

$$- \mathbf{1}_{\{i_{g_1} = i_{g_2} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1} = j_{g_2}} \mathbf{1}_{\{g_2 = g_1+1\}} \times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =$$

$$= -\mathbf{1}_{\{i_{g_1} = i_{g_2} \ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\ne g_1,g_2}}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1} = j_{g_2}} \times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}+$$

$$+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\mathbf{1}_{\{g_2=g_1+1\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}= \tag{80}$$

$$=\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}J[\psi^{(k)}]_{T,t}^{g_1}+\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)1,g_1,g_2}\quad \text{w. p. 1}, \tag{81}$$

where $J[\psi^{(k)}]_{T,t}^{g_1}$ $(g_1=1,2,\ldots,k-1)$ is defined by (25),

$$R_{T,t}^{(p)1,g_1,g_2}=-\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}.$$

Let us explain the transition from (80) to (81). We have for $g_2=g_1+1$

$$\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\frac{1}{2}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0,j_{g_1}=\ j_{g_2}}\times$$

$$\times\zeta_0^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times$$

$$\times\zeta_{j_{m_1}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})}=$$

$$=\frac{1}{2}\mathbf{1}_{\{i_{g_1}=\ i_{g_2}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\sum_{j_{m_1}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}\times$$

$$\times J'[\phi_{j_{m_1}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2}}}]_{T,t}^{(0i_{q_1}\ldots i_{q_{k-2}})}= \tag{82}$$

$$
= \frac{1}{2} J[\psi^{(k)}]_{T,t}^{g_1} \quad \text{w. p. 1,} \tag{83}
$$

where

$$
C_{j_k \ldots j_1} \Big|_{(j_{g_2} j_{g_1}) \curvearrowright j_{m_1}, j_{g_1} = j_{g_2}, g_2 = g_1 + 1} =
$$

$$
= \int_t^T \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int_t^{t_{g_1+3}} \psi_l(t_{g_1+2})\phi_{j_{g_1+2}}(t_{g_1+2}) \int_t^{t_{g_1+2}} \psi_{g_1+1}(t_{g_1})\psi_{g_1}(t_{g_1})\phi_{j_{m_1}}(t_{g_1}) \times
$$

$$
\times \int_t^{t_{g_1}} \psi_l(t_{g_1-1})\phi_{j_{g_1-1}}(t_{g_1-1}) \ldots \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_{g_1-1} dt_{g_1} dt_{g_1+2} \ldots dt_k,
$$

$$
\zeta_{j_{m_1}}^{(0)} = \int_t^T \phi_{j_{m_1}}(\tau) d\mathbf{w}_\tau^{(0)} = \int_t^T \phi_{j_{m_1}}(\tau) d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_{m_1} = 0 \\ 0 & \text{if } j_{m_1} \neq 0 \end{cases},
$$

$$
\phi_0(\tau) = \frac{1}{\sqrt{T-t}}.
$$

The transition from (82) to (83) is based on (48).

By Condition 3 of Theorem 13 we have (also see the property (47) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)1,g_1,g_2}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0 \\ q\neq g_1,g_2}}^{p} \left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\neq g_1,g_2}\right)^2 = 0,
$$

where constant K does not depend on p.

Thus

$$
\mathbf{1}_{\{i_{g_1} = i_{g_2} \neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \mathbf{1}_{\{j_{g_1} = j_{g_2}\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2}})} =
$$

$$
= \frac{1}{2} \mathbf{1}_{\{g_2 = g_1+1\}} J[\psi^{(k)}]_{T,t}^{g_1} \quad \text{w. p. 1.}
$$

Involving into consideration the second pair $\{g_3, g_4\}$ (the first pair is $\{g_1, g_2\}$), we obtain from (80) for $r = 2$

$$
\prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}} = i_{g_{2s}} \neq 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{2} \mathbf{1}_{\{j_{g_{2s-1}} = j_{g_{2s}}\}} \times
$$

$$
\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =
$$

$$
=\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\times
$$

$$
\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\Biggl(\frac{1}{4}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4}}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}-
$$

$$
-\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4}}\mathbf{1}_{\{g_4=g_3+1\}}-
$$

$$
-\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4}}\mathbf{1}_{\{g_2=g_1+1\}}+
$$

$$
+\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4}}\Biggr)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}= \tag{84}
$$

$$
=\frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_2,s_1}+\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)2,g_1,g_2,g_3,g_4} \tag{85}
$$

w. p. 1, where $g_3\stackrel{\sf def}{=}s_2$, $g_1\stackrel{\sf def}{=}s_1$, $(s_2,s_1)\in\mathrm{A}_{k,2}$, $J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (25) and $\mathrm{A}_{k,2}$ is defined by (26),

$$
R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\Biggl(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,g_3,g_4}-
$$

$$
-S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,g_3,g_4}\right\}-S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,g_3,g_4}\right\}\Biggr)\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}.
$$

Let us explain the transition from (84) to (85). We have for $g_2=g_1+1$, $g_4=g_3+1$

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\frac{1}{4}C_{j_k\ldots j_1}\Biggr|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=~j_{g_2},j_{g_3}=~j_{g_4}}\times
$$

$$
\times\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=
$$

$$= \frac{1}{4} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright 0(j_{g_4}j_{g_3})\curvearrowright 0,j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times$$

$$\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \zeta_0^{(0)}\zeta_0^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =$$

$$= \frac{1}{4} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times$$

$$\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)} J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})} =$$

$$= \frac{1}{4} \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p} \sum_{j_{m_1},j_{m_3}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} \times$$

$$\times \prod_{s=1}^{2} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(00i_{q_1}\ldots i_{q_{k-4}})} = \tag{86}$$

$$= \frac{1}{4} J[\psi^{(k)}]_{T,t}^{s_2,s_1} \quad \text{w. p. 1.} \tag{87}$$

The transition from (86) to (87) is based on (48).

Note that

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}} = C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=\ j_{g_2}}$$

is the Fourier coefficient, where $g_2 = g_1 + 1$. Therefore, the value

$$C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}(j_{g_4}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}} =$$

$$= C_{j_k\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}(j_{g_3}j_{g_3})\curvearrowright j_{m_3},j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}$$

is determined recursively using (37) in an obvious way for $g_2 = g_1 + 1$ and $g_4 = g_3 + 1$.

By Condition 3 of Theorem 13 we have (also see the property (47) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{(p)2,g_1,g_2,g_3,g_4}\right)^2\right\}\le K\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4}}^{p}\left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right)^2+\right.$$

$$\left.+\left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2+\left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,g_3,g_4}\right\}\right)^2\right)=0,$$

where constant K is independent of p.

Thus

$$\prod_{s=1}^{2}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{2}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-4}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-4}})}=\frac{1}{4}\prod_{s=1}^{2}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_2,s_1}\quad \text{w. p. 1},$$

where $g_3\overset{\mathsf{def}}{=}s_2,\ g_1\overset{\mathsf{def}}{=}s_1,\ (s_2,s_1)\in\mathrm{A}_{k,2},\ J[\psi^{(k)}]_{T,t}^{s_2,s_1}$ is defined by (25) and $\mathrm{A}_{k,2}$ is defined by (26).

Involving into consideration the third pair $\{g_6,g_5\}$ ($\{g_1,g_2\}$ is the first pair and $\{g_4,g_3\}$ is the second pair), we obtain from (84) for $r=3$

$$\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{3}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\times$$

$$\times\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,g_3,g_4,g_5,g_6}}^{p}\left(\frac{1}{2^3}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\times\right.$$

$$\times\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_4=g_3+1\}}\mathbf{1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_6=g_5+1\}}-$$

$$-\frac{1}{2^2}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},j_{g_3}=\,j_{g_4},j_{g_5}=\,j_{g_6}}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_6}j_{g_5})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_6=g_5+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_4=g_3+1\}}+$$

$$+\frac{1}{2}\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}} \mathbf{1}_{\{g_2=g_1+1\}}-$$

$$\left.-\sum_{j_{g_5}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4},j_{g_5}=\ j_{g_6}}\right)\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}=$$

$$=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}+\prod_{s=1}^{3}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}$$

w. p. 1, where $g_{2i-1}\stackrel{\sf def}{=}s_i$; $i=1,2,3$, $(s_3,s_2,s_1)\in \mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (25) and $\mathrm{A}_{k,3}$ is defined by (26),

$$R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_5,g_6}}^{p}\left(-\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}+\right.$$

$$+S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}+S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}+$$

$$+S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}-$$

$$-S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}-S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}-$$

$$\left.-S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_5,g_6}\right\}\right)J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})}.$$

By Condition 3 of Theorem 13 we have (also see the property (47) of multiple Wiener stochastic integral)

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)3,g_1,g_2,\ldots,g_5,g_6}\right)^2\right\} \le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_5,g_6}}^{p} \left(\left(\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right)^2 + \right.$$

$$+\left(S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$+\left(S_3\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$+\left(S_3S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 + \left(S_3S_2\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2 +$$

$$\left. +\left(S_2S_1\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}\right)^2\right) = 0,$$

where constant K does not depend on p.

Thus

$$\underset{p\to\infty}{\text{l.i.m.}} \prod_{s=1}^{3} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{3} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-6}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-6}})} = \frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1} \quad \text{w. p. 1},$$

where $g_{2i-1} \stackrel{\mathsf{def}}{=} s_i$; $i=1,2,3$, $(s_3,s_2,s_1)\in \mathrm{A}_{k,3}$, $J[\psi^{(k)}]_{T,t}^{s_3,s_2,s_1}$ is defined by (25) and $\mathrm{A}_{k,3}$ is defined by (26).

Repeating the previous steps, we obtain for an arbitrary r $(r=1,2,\ldots,[k/2])$

$$\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{s=1}^{r} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\times$$

$$\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} = \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\times$$

$$\times \underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\, j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times \prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} = \tag{88}$$

$$= \frac{1}{2^r}\prod_{s=1}^{r} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} + \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} \underset{p\to\infty}{\text{l.i.m.}}\ R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{89}$$

w. p. 1, where $g_{2i-1} \overset{\text{def}}{=} s_i;\ i=1,2,\ldots,r;\ r=1,2,\ldots,[k/2],\ (s_r,\ldots,s_1)\in \mathrm{A}_{k,r},\ J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (25) and $\mathrm{A}_{k,r}$ is defined by (26),

$$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} = \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \Biggl((-1)^r \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} +$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\Biggr)\times$$

$$\times J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}. \tag{90}$$

Let us explain the transition from (88) to (89). We have for $g_2=g_1+1,\ldots,g_{2r}=g_{2r-1}+1$

$$\underset{p\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \frac{1}{2^r}C_{j_k\ldots j_1}\biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=~j_{g_2},\ldots,j_{g_{2r-1}}=~j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\ne 0\}} J'[\phi_{j_{q_1}} \ldots \phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =$$

$$
=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright 0\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright 0,j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times
$$

$$
\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\left(\zeta_0^{(0)}\right)^r J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times
$$

$$
\times\zeta_{j_{m_1}}^{(0)}\zeta_{j_{m_3}}^{(0)}\ldots\zeta_{j_{m_{2r-1}}}^{(0)}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=
$$

$$
=\frac{1}{2^r}\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{m_1},j_{m_3}\ldots,j_{m_{2r-1}}=0}^{p}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=i_{g_{2s}}\ne 0\}}\times
$$

$$
\times C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright j_{m_{2r-1}},j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\times
$$

$$
\times J'[\phi_{j_{m_1}}\phi_{j_{m_3}}\ldots\phi_{j_{m_{2r-1}}}\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(00\ldots 0i_{q_1}\ldots i_{q_{k-2r}})}= \tag{91}
$$

$$
=\frac{1}{2^r}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\text{w. p. 1.} \tag{92}
$$

The transition from (91) to (92) is based on (48).

Note that

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1},j_{g_1}=j_{g_2}}
$$

is the Fourier coefficient, where $g_2=g_1+1$. Therefore, the value

$$
C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=j_{g_2},\ldots,j_{g_{2d-1}}=j_{g_{2d}}}=
$$

$$
= C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright j_{m_1}\ldots(j_{g_{2d-1}}j_{g_{2d-1}})\curvearrowright j_{m_{2d-1}},j_{g_1}=j_{g_2},\ldots,j_{g_{2d-1}}=j_{g_{2d}}}
$$

is determined recursively using (37) in an obvious way for $g_2 = g_1 + 1, \ldots, g_{2d} = g_{2d-1} + 1$ and $d = 2, \ldots, r$.

By Condition 3 of Theorem 13 we have (also see the property (47) of multiple Wiener stochastic integral)

$$
\lim_{p\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\} \le
$$

$$
\le K \lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(\left(\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2 + \right.
$$

$$
+ \sum_{l_1=1}^{r}\left(S_{l_1}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 +
$$

$$
+ \sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r}\left(S_{l_1}S_{l_2}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 +
$$

$$
\ldots
$$

$$
\left. + \sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r}\left(S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2\right) = 0,
$$

where constant K does not depend on p.

So we have

$$
\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\times
$$

$$
\times J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})} =
$$

$$
= \frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \quad \text{w. p. 1}, \tag{93}
$$

where $g_{2i-1} \stackrel{\mathsf{def}}{=} s_i$; $i = 1, 2, \ldots, r$; $r = 1, 2, \ldots, [k/2]$, $(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}$, $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ is defined by (25) and $\mathrm{A}_{k,r}$ is defined by (26).

Note that

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\Biggl.\Biggr|_{g_2=g_1+1,g_3=g_2+1,\ldots,g_{2r}=g_{2r-1}+1} A_{g_1,g_3,\ldots,g_{2r-1}} =
$$

$$
= \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} A_{s_1,s_2,\ldots,s_r}, \tag{94}
$$

where $A_{g_1,g_3,\ldots,g_{2r-1}}$, $A_{s_1,s_2,\ldots,s_r}$ are scalar values, $g_{2i-1}=s_i$; $i=1,2,\ldots,r$; $r=1,2,\ldots,[k/2]$, $\mathrm{A}_{k,r}$ is defined by (26):

$$
\mathrm{A}_{k,r}=\bigl\{(s_r,\ldots,s_1):\ s_r>s_{r-1}+1,\ldots,s_2>s_1+1,\ s_r,\ldots,s_1=1,\ldots,k-1\bigr\}.
$$

Using (79), (93), (94), and Theorem 5, we finally get

$$
\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} =
$$

$$
= J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} = J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \tag{95}
$$

w. p. 1, where (see (25))

$$
J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \stackrel{\text{def}}{=} \prod_{p=1}^{r}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}}\times
$$

$$
\times\int_t^T\psi_k(t_k)\ldots\int_t^{t_{s_r+3}}\psi_{s_r+2}(t_{s_r+2})\int_t^{t_{s_r+2}}\psi_{s_r}(t_{s_r+1})\psi_{s_r+1}(t_{s_r+1})\times
$$

$$
\times\int_t^{t_{s_r+1}}\psi_{s_r-1}(t_{s_r-1})\ldots\int_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times
$$

$$
\times\int_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots
$$

$$
\ldots\ d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}dt_{s_r+1}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}. \tag{96}
$$

Theorem 13 is proved.

Let us make a number of remarks about Theorem 13. An expansion similar to (44) was obtained in [87], where the author used the definition (394) of the Stratonovich stochastic integral, which differs from the definition we use in this article [2]. The proof from [87] is somewhat simpler than the proof proposed in this work. However, the results from [87] were obtained under the condition of convergence of trace series. The verification of this condition for the kernel (12) is a separate problem. In our proof, we essentially use the structure of the Fourier coefficients (45) corresponding to the kernel $K(t_1,\ldots,t_k)$ of the form (12). This circumstance actually made it possible to prove Theorem 13 using not the condition of finiteness of trace series, but using the condition of convergence to zero of explicit expressions for the remainders of the mentioned series. This leaves hope that it is possible to estimate the rate of convergence in Theorem 13 (see Theorems 19–22 below).

Note that under the conditions of Theorem 13 the sequential order of the series (also see (71), (78))

$$\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}$$

is not important.

We also note that the first and second conditions of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$ (see the proofs of Theorems 16–18 below). Moreover, the equality (42) is true for an arbitrary basis in $L_2([t,T])$ (see [15], Sect. 2.1.4 or [94]). Note that in the proofs of Theorems 6–12, 16–18, 23 the conditions of Theorem 13 are verified for various special cases of iterated Stratonovich stochastic integrals of multiplicities 2–6 with respect to the components of the multidimensional Wiener process.

It should be noted that (see (90))

$$(-1)^r \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}+$$

$$+(-1)^{r-1}\sum_{l_1=1}^{r} S_{l_1}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$+(-1)^{r-2}\sum_{\substack{l_1,l_2=1\\ l_1>l_2}}^{r} S_{l_1}S_{l_2}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}+$$

$$\ldots$$

$$+(-1)^{1}\sum_{\substack{l_1,l_2,\ldots,l_{r-1}=1\\ l_1>l_2>\ldots>l_{r-1}}}^{r} S_{l_1}S_{l_2}\ldots S_{l_{r-1}}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=$$

$$=\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}},\qquad (97)$$

where the meaning of the notations used in (90) is preserved.

For example, from (97) for the case $r=2$ we get

$$\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\,j_{g_2},\,j_{g_3}=\,j_{g_4}}-$$

$$-\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}-$$

$$-\frac{1}{2}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}=$$

$$=\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}-$$

$$-\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}.$$

As a result, Condition 3 of Theorem 13 can be replaced by a weaker condition

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\right)^2=0, \tag{98}$$

where $r=1,2,\ldots,[k/2]$.

However, Condition 3 of Theorem 13 itself contains a way of proving of the condition (98), which is partially realized in the proof of Theorems 16–18, 23 (see below).

In fact, when proving Theorem 18 (the case $r=3$ is proved in Theorem 23 for $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$), we proved the following equality

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\sum_{j_{g_3}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=$$

$$=\frac{1}{4}\mathbf{1}_{\{g_2=g_1+1\}}\mathbf{1}_{\{g_4=g_3+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)(j_{g_4}j_{g_3})\frown(\cdot),j_{g_1}=\ j_{g_2},j_{g_3}=\ j_{g_4}}.$$

On the other hand, iterative application of (78) gives

$$\sum_{j_{g_1}=0}^{\infty}\ldots\sum_{j_{g_{2r-1}}=0}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}},$$

where $r = 1, 2, \ldots, [k/2]$.

Taking into account the modification of Theorem 3 for the case of integration interval $[t, s]$ ($s \in (t, T]$) of iterated Ito stochastic integrals (see Theorem 1.11 in [15], [17] or Theorem 1.24 in [15]), we can formulate an analogue of Theorem 13 for the case of integration interval $[t, s]$ ($s \in (t, T]$) of iterated Stratonovich stochastic integrals of multiplicity k ($k \in \mathbb{N}$).

Denote

$$\bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}(s)\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}(s)\Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}}$$

and introduce the following notation

$$S_l \left\{ \bar{C}^{(p)}_{j_k \ldots j_q \ldots j_1}(s)\Bigg|_{q \ne g_1, g_2, \ldots, g_{2r-1}, g_{2r}} \right\} \stackrel{\mathsf{def}}{=} \frac{1}{2} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty} \sum_{j_{g_{2r-3}}=p+1}^{\infty} \ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty} \sum_{j_{g_{2l-3}}=p+1}^{\infty} \ldots \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_1}=p+1}^{\infty} C_{j_k \ldots j_1}(s)\Bigg|_{(j_{g_{2l}} j_{g_{2l-1}}) \frown (\cdot), j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}},$$

where $l = 1, 2, \ldots, r$,

$$C_{j_k \ldots j_1}(s)\Bigg|_{(j_{g_{2l}} j_{g_{2l-1}}) \frown (\cdot)}$$

is defined by analogy with (36),

$$C_{j_k \ldots j_1}(s) = \int_t^s \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k. \tag{99}$$

Theorem 15 [15], [47], [55], [62]. *Assume that the continuously differentiable functions* $\psi_l(\tau)$ ($l = 1, \ldots, k$) *and the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions* ($\phi_0(x) = 1/\sqrt{T-t}$) *in the space* $L_2([t, T])$ *are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2} \int_t^s \Phi_1(t_1) \Phi_2(t_1) dt_1 = \sum_{j_1=0}^{\infty} \int_t^s \Phi_2(t_2) \phi_{j_1}(t_2) \int_t^{t_2} \Phi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 \tag{100}$$

holds for all $s \in (t, T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t, T]$ *and the series on the right-hand side of* (100) *converges absolutely.*

2. *The estimates*

$$
\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad \left|\int\limits_\tau^s \phi_j(\theta)\Phi_2(\theta)d\theta\right| \le \frac{\Psi_2(s,\tau)}{j^{1/2+\alpha}},
$$

$$
\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^\tau \Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_3(s)}{p^\beta}
$$

hold for all s,τ such that $t<\tau<s<T$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$
\int\limits_t^s |\Psi_1(\tau)\Psi_2(s,\tau)|\,d\tau<\infty,\quad \int\limits_t^s |\Psi_3(\tau)|\,d\tau<\infty
$$

for all $s\in(t,T)$.

3. *The condition*

$$
\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0
$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (34)) and $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, where $r=1,2,\ldots,[k/2]$ and

$$
S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\mathsf{def}}{=}\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}(s)\Bigr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}
$$

for $d=0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$
J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^{*s}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{101}
$$

the following expansion

$$
J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}
$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (99), l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$, $s\in(t,T)$,*

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}
$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

The results presented below in this section show that Conditions 1 and 2 of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Note that Condition 1 of Theorem 13 is fulfilled for an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$ (see recent publications [15] (Sect. 2.1.4) or [94]).

In Sect. 2.1.2 of the monograpths [15]–[18], the following formula is proved

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}, \tag{102}$$

where

$$C_{j_1 j_1} = \int\limits_t^T \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$.

Moreover (see Sect. 2.1.2 of the monograpths [15]–[18]), the following estimate

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}\right| \le \frac{C}{p}, \tag{103}$$

holds under the above assumptions, where constant C does not depend on p.

The relations (102) and (103) have been modified for the Legendre polynomial system as follows (see Sect. 2.8, 2.13 of the monograpth [17])

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}(s), \tag{104}$$

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le \frac{C}{p}\left(\frac{1}{\left(1-z^2(s)\right)^{1/4}} + 1\right), \tag{105}$$

where $s \in (t,T)$ (s is fixed, the case $s = T$ corresponds to (102) and (103)), constant C does not depend p, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$,

$$C_{j_1 j_1}(s) = \int\limits_t^s \psi_2(t_2)\phi_{j_1}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 dt_2,$$

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}. \tag{106}$$

For the trigonometric case, the estimate (105) is replaced by [15], [17]

$$\left|\sum_{j_1=p+1}^{\infty} C_{j_1 j_1}(s)\right| \le \frac{C}{p}, \tag{107}$$

where $s\in[t,T]$, constant C does not depend on p.

Note the well known estimate for the Legendre polynomials

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{108}$$

where $P_j(y)$ is the Legendre polynomial, constant K does not depend on y and j.

We also note the following useful estimates for the case of Legendre polynomials ([15]-[18], Chapters 1, 2)

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{109}$$

$$\left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+1\right), \tag{110}$$

$$\left|\int\limits_v^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}}+\frac{1}{(1-(z(v))^2)^{1/4}}+1\right), \tag{111}$$

where $j\in\mathbb{N}$, $z(x),z(v)\in(-1,1)$, $x,v\in(t,T)$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t,T]$, constant C does not depend on j.

For the case of trigonometric functions we note the following obvious estimates

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{112}$$

$$\left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{113}$$

$$\left|\int\limits_v^x \psi(\tau)\phi_j(\tau)d\tau\right|<\frac{C}{j}, \tag{114}$$

where $j\in\mathbb{N}$, $x,v\in[t,T]$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t,T]$, constant C does not depend on j.

8. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 3. The Case $p_1=p_2=p_3\to\infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

In this section, we present a simple proof of Theorem 11 based on Theorem 13. In this case, the conditions of Theorem 11 will be weakened.

First, consider the following equalities

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_2(\tau)\phi_j(\tau)\int\limits_{t_1}^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau, \tag{115}$$

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_{t_1}^{t_2}\Phi_1(\theta)\phi_j(\theta)\int\limits_{\theta}^{t_2}\Phi_2(\tau)\phi_j(\tau)d\tau d\theta \tag{116}$$

that will be used further, where $t\le t_1<t_2\le T$, $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of funtions in $L_2([t,T])$. The equality (116) has been proved in [15] (Sect. 2.7.2). Using (116) and Fubini's Theorem, we get (115).

Theorem 16 [15], [47], [55], [62]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \tag{117}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

that converges in the mean-square sense is valid, where $i_1,i_2,i_3=0,1,\ldots,m$,

$$C_{j_3j_2j_1}=\int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. As follows from Sect. 7, Conditions 1 and 2 of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 13 for the iterated Stratonovich stochastic integral (117). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_1j_1}\right)^2=0, \tag{118}$$

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1}\right)^2=0, \tag{119}$$

$$\lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2=0. \tag{120}$$

We have

$$\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_1j_1}\right)^2=$$

$$=\sum_{j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{121}$$

$$=\sum_{j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2\le \tag{122}$$

$$\le\sum_{j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2= \tag{123}$$

$$=\int\limits_t^T\psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3\le \tag{124}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{125}$$

if $p\to\infty$, where constant K does not depend on p.

Note that the transition from (121) to (122) is based on the estimate (105) for the polynomial case and its analogue (107) for the trigonometric case, the transition from (123) to (124) is based on the Parseval equality, and the transition from (124) to (125) is also based on the estimate (105) and its analogue (107) for the trigonometric case.

By analogy with the previous case we have

$$\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1}\right)^2=$$

$$=\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_3}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3\right)^2=$$

$$
= \sum_{j_1=0}^{p} \left(\sum_{j_3=p+1}^{\infty} \int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2) \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) dt_3 dt_2 dt_1 \right)^2 = \tag{126}
$$

$$
= \sum_{j_1=0}^{p} \left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \sum_{j_3=p+1}^{\infty} \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2) \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) dt_3 dt_2 dt_1 \right)^2 \le \tag{127}
$$

$$
\le \sum_{j_1=0}^{\infty} \left(\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1) \sum_{j_3=p+1}^{\infty} \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2) \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) dt_3 dt_2 dt_1 \right)^2 =
$$

$$
= \int\limits_t^T \psi_1^2(t_1) \left(\sum_{j_3=p+1}^{\infty} \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2) \int\limits_{t_2}^T \psi_1(t_3)\phi_{j_3}(t_3) dt_3 dt_2 \right)^2 dt_1 \le \tag{128}
$$

$$
\le \frac{K}{p^2} \to 0 \tag{129}
$$

if $p \to \infty$, where constant K is independent of p.

The transition from (126) to (127) is based on analogues of the estimates (105), (107) for the value

$$
\left| \sum_{j_3=p+1}^{\infty} \int\limits_{t_1}^T \psi_2(t_2)\phi_{j_3}(t_2) \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_3}(t_3) dt_3 dt_2 \right|
$$

for the polynomial and trigonometric cases, the transition from (128) to (129) is also based on the mentioned analogues of the estimates (105), (107).

Further, we have

$$
\sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1} \right)^2 =
$$

$$
= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_3(t_3)\phi_{j_1}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 \right)^2 =
$$

$$
= \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} \int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3) dt_3 dt_2 \right)^2 = \tag{130}
$$

$$
= \sum_{j_2=0}^{p} \left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2) \sum_{j_1=p+1}^{\infty} \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^T \psi_3(t_3)\phi_{j_1}(t_3) dt_3 dt_2 \right)^2 \le \tag{131}
$$

$$\le \sum_{j_2=0}^{\infty}\left(\int\limits_t^T \psi_2(t_2)\phi_{j_2}(t_2)\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2\right)^2 =$$

(132) $$= \int\limits_t^T \psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2 dt_2.$$

The transition from (130) to (131) is based on the estimates (109), (110) and its obvious analogues (112), (113) for the trigonometric case. However, the estimates (109), (110) cannot be used to estimate the right-hand side of (132), since we get the divergent integral. For this reason, we will obtain a new estimate based on the relation [15]-[18]

$$\int\limits_t^x \psi(s)\phi_{j_1}(s)ds = \frac{\sqrt{T-t}\sqrt{2j_1+1}}{2}\int\limits_{-1}^{z(x)} P_{j_1}(y)\psi(u(y))dy =$$

$$= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}}\Biggl((P_{j_1+1}(z(x)) - P_{j_1-1}(z(x)))\psi(x) -$$

(133) $$-\frac{T-t}{2}\int\limits_{-1}^{z(x)}((P_{j_1+1}(y) - P_{j_1-1}(y))\psi'(u(y))dy\Biggr),$$

where $x \in (t,T)$, $j_1 \ge p+1$, $z(x)$ is defined by (106), $P_j(x)$ is the Legendre polynomial, ψ' is a derivative of the continuously differentiable function $\psi(s)$ with respect to the variable $u(y)$,

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}.$$

From (108) and the estimate $|P_j(y)| \le 1$, $y \in [-1,1]$ we obtain

(134) $$|P_j(y)| = |P_j(y)|^{\varepsilon}\cdot|P_j(y)|^{1-\varepsilon} \le |P_j(y)|^{1-\varepsilon} < \frac{C}{j^{1/2-\varepsilon/2}(1-y^2)^{1/4-\varepsilon/4}},$$

where $y \in (-1,1)$, $j \in \mathbb{N}$, and ε is an arbitrary small positive real number.

Combining (133) and (134), we have the following estimate

(135) $$\left|\int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau\right| < \frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right),$$

where $s \in (t,T)$, $z(s)$ is defined by (106), constant C does not depend on j_1.

Similarly to (135) we obtain

(136) $$\left|\int\limits_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau\right| < \frac{C}{(j_1)^{1-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right),$$

where $s \in (t,T)$, constant C is independent of j_1.

Combining (109) and (136), we have

$$\left|\int\limits_t^s \psi_1(\tau)\phi_{j_1}(\tau)d\tau \int\limits_s^T \psi_3(\tau)\phi_{j_1}(\tau)d\tau\right| <$$

$$< \frac{L}{(j_1)^{2-\varepsilon/2}}\left(\frac{1}{(1-z^2(s))^{1/4-\varepsilon/4}}+1\right)\left(\frac{1}{(1-z^2(s))^{1/4}}+1\right), \qquad (137)$$

where $s \in (t,T)$, $z(s)$ is defined by (106), constant L does not depend on j_1.

Observe that

$$\sum_{j_1=p+1}^{\infty}\frac{1}{(j_1)^{2-\varepsilon/2}} \le \int\limits_p^{\infty}\frac{dx}{x^{2-\varepsilon/2}} = \frac{1}{(1-\varepsilon/2)p^{1-\varepsilon/2}}. \qquad (138)$$

Applying (137) and (138) to estimate the right-hand side of (132) gives

$$\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \qquad (139)$$

if $p \to \infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

The estimation of the right-hand side of (132) for the trigonometric case is carried out using the estimates (112), (113). At that we obtain the estimate (139) with $\varepsilon = 0$. Theorem 16 is proved.

9. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 4. The Case $p_1 = \ldots = p_4 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\ldots$, $\psi_4(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 17 [15], [47], [55], [62]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_4(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \qquad (140)$$

the following expansion

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3, i_4 = 0, 1, \ldots, m$,

$$C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \times$$

$$\times dt_2 dt_3 dt_4$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As follows from Sect. 7, Conditions 1 and 2 of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 13 for the iterated Stratonovich stochastic integral (140). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_3,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_3 j_1 j_1} \right)^2 = 0, \tag{141}$$

$$\lim_{p\to\infty} \sum_{j_2,j_4=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_4 j_1 j_2 j_1} \right)^2 = 0, \tag{142}$$

$$\lim_{p\to\infty} \sum_{j_2,j_3=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_3 j_2 j_1} \right)^2 = 0, \tag{143}$$

$$\lim_{p\to\infty} \sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4 j_2 j_2 j_1} \right)^2 = 0, \tag{144}$$

$$\lim_{p\to\infty} \sum_{j_1,j_3=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_2 j_3 j_2 j_1} \right)^2 = 0, \tag{145}$$

$$\lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_1} \right)^2 = 0, \tag{146}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_2 j_1 j_2 j_1} \right)^2 = 0, \tag{147}$$

$$\lim_{p\to\infty} \left(\sum_{j_2=p+1}^{\infty} \sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1} \right)^2 = 0, \tag{148}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=0, \tag{149}$$

$$\lim_{p\to\infty}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigl|_{(j_1j_1)\frown(\cdot)}\right)^2=0, \tag{150}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Bigl|_{(j_3j_3)\frown(\cdot)}\right)^2=0, \tag{151}$$

$$\lim_{p\to\infty}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_2j_1}\Bigl|_{(j_2j_2)\frown(\cdot)}\right)^2=0, \tag{152}$$

where we use the notation (36) in (150)–(152).

Applying arguments similar to those we used in the proof of Theorem 16, we obtain for (141)

$$\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_3j_1j_1}\right)^2=\sum_{j_3,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{153}$$

$$=\sum_{j_3,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2\le \tag{154}$$

$$\le\sum_{j_3,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{155}$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_3<t_4\}}\psi_4^2(t_4)\psi_3^2(t_3)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\right)^2dt_3dt_4\le \tag{156}$$

$$\le\frac{K}{p^2}\ \to\ 0 \tag{157}$$

if $p\to\infty$, where constant K is independent of p.

Note that the transition from (153) to (154) is based on the estimate (105) for the polynomial case and its analogue for the trigonometric case, the transition from (155) to (156) is based on the Parseval equality, and the transition from (156) to (157) is also based on the estimate (105) and its analogue for the trigonometric case.

Further, we have for (142)

$$\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_4j_1j_2j_1}\right)^2=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2= \tag{158}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2= \tag{159}$$

$$=\sum_{j_2,j_4=0}^{p}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2\le$$

$$\le\sum_{j_2,j_4=0}^{\infty}\left(\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3dt_2dt_4\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_4\}}\psi_4^2(t_4)\psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\psi_3(t_3)\phi_{j_1}(t_3)dt_3\right)^2dt_2dt_4\le$$

$$\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{160}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The relation (160) was obtained by the same method as (157). Note that in obtaining (160) we used the estimates (111), (135) for the polynomial case and (112), (114) for the trigonometric case. We also used the integration order replacement in the iterated Riemann integrals (see (158), (159)).

Repeating the previous steps for (143) and (144), we get

$$\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_3j_2j_1}\right)^2=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_1}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_2,j_3=0}^{p}\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2=$$

$$=\sum_{j_2,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2\le$$

$$\le\sum_{j_2,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4dt_2dt_3\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_2<t_3\}}\psi_3^2(t_3)\psi_2^2(t_2)\left(\sum_{j_1=p+1}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_1}(t_4)dt_4\right)^2dt_2dt_3\le$$

$$\le\frac{K}{p^2}\ \to\ 0 \qquad (161)$$

if $p\to\infty$, where constant K does not depend on p;

$$\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_4j_2j_2j_1}\right)^2=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_2}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.$$

$$
\times \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\Biggr)^2 =
$$

$$
= \sum_{j_1,j_4=0}^{p} \Biggl(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\Biggr)^2 \le
$$

$$
\le \sum_{j_1,j_4=0}^{\infty} \Biggl(\int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1)\times
$$

$$
\times \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2dt_1dt_4\Biggr)^2 =
$$

$$
= \int\limits_{[t,T]^2} \mathbf{1}_{\{t_1<t_4\}}\psi_4^2(t_4)\psi_1^2(t_1)\left(\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2\right)^2 dt_1dt_4. \tag{162}
$$

Note that, by virtue of the additivity property of the integral, we have

$$
\sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_{t_2}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3dt_2 = \tag{163}
$$

$$
= \sum_{j_2=p+1}^{\infty} \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3-
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_{t}^{t_1} \psi_3(t_3)\phi_{j_2}(t_3) \int\limits_{t}^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_3-
$$

$$
- \sum_{j_2=p+1}^{\infty} \int\limits_{t_1}^{t_4} \psi_3(t_3)\phi_{j_2}(t_3)dt_3 \int\limits_{t}^{t_1} \psi_2(t_2)\phi_{j_2}(t_2)dt_2. \tag{164}
$$

However, all three series on the right-hand side of (164) have already been evaluated in (157) and (160). From (162) and (164) we finally obtain

$$
\sum_{j_1,j_4=0}^{p} \left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}} \ \to \ 0 \tag{165}
$$

if $p \to \infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p.

In complete analogy with (160), we have for (145)

$$
\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_2j_1}\right)^2=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_2}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.
$$

$$
\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.
$$

$$
\left.\times\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty}\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2=
$$

$$
=\sum_{j_1,j_3=0}^{p}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_3\right)^2\le
$$

$$
\le\sum_{j_1,j_3=0}^{\infty}\left(\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)\times\right.
$$

$$
\left.\times\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4dt_1dt_3\right)^2=
$$

$$
=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_3\}}\psi_3^2(t_3)\psi_1^2(t_1)\left(\sum_{j_2=p+1}^{\infty}\int\limits_{t_1}^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)dt_2\int\limits_{t_3}^T\psi_4(t_4)\phi_{j_2}(t_4)dt_4\right)^2dt_1dt_3\le
$$

$$
\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{166}
$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

We have for (146)

$$\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2=\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times\right.$$

$$\left.\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4\right)^2=$$

$$=\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2=$$

$$=\sum_{j_1,j_2=0}^{p}\left(\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2\le$$

$$\le\sum_{j_1,j_2=0}^{\infty}\left(\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^{T}\psi_2(t_2)\phi_{j_2}(t_2)\times\right.$$

$$\left.\times\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1\right)^2=$$

$$=\int\limits_{[t,T]^2}\mathbf{1}_{\{t_1<t_2\}}\psi_1^2(t_1)\psi_2^2(t_2)\left(\sum_{j_3=p+1}^{\infty}\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3\right)^2dt_2dt_1. \tag{167}$$

It is easy to see that the integral (see (167))

$$\int\limits_{t_2}^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_{t_3}^{T}\psi_4(t_4)\phi_{j_3}(t_4)dt_4dt_3$$

is similar to the integral from the formula (163) if in the last integral we substitute $t_4=T$. Therefore, by analogy with (165), we obtain

$$\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2\le\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{168}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

Now consider (147)–(149). We have for (147) (see **Step 2** in the proof of Theorem 13)

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2. \tag{169}$$

Consider (145) and (166). We have

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 = \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2\Biggl|_{j_1=j_3} \le$$

$$\le \sum_{j_1,j_3=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_3j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{170}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (169) and (170), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_2j_1j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0$$

if $p\to\infty$, where constant K_1 does not depend on p.

Similarly for (148) we have (see (144), (165))

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \left(\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le$$

$$\le (p+1)\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2, \tag{171}$$

$$\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 = \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2\Biggl|_{j_1=j_4} \le$$

$$\le \sum_{j_1,j_4=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_4j_2j_2j_1}\right)^2 \le \frac{K}{p^{2-\varepsilon}}, \tag{172}$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon = 0$ for the trigonometric case, constant K does not depend on p. Combining (171) and (172), we obtain

$$\left(\sum_{j_2=p+1}^{\infty}\sum_{j_1=p+1}^{\infty} C_{j_1j_2j_2j_1}\right)^2 \le \frac{(p+1)K}{p^{2-\varepsilon}} \le \frac{K_1}{p^{1-\varepsilon}} \ \to\ 0$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider (149). Using (78), we get

$$\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}=\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{\infty}C_{j_3j_3j_1j_1}-\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p}C_{j_3j_3j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)}-\sum_{j_3=p+1}^{\infty}\sum_{j_1=0}^{p}C_{j_3j_3j_1j_1},\tag{173}$$

where (see (36))

$$C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_3}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2dt_3dt_4.$$

From the estimate (103) for the polynomial and trigonometric cases we get

$$\left|\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggl|_{(j_1j_1)\frown(\cdot)}\right|\le\frac{C}{p},\tag{174}$$

where constant C is independent of p.

Further, we have (see (168))

$$\left(\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2\le(p+1)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2=$$

$$=(p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2\Biggl|_{j_1=j_2}\le$$

$$\le(p+1)\sum_{j_1,j_2=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1}\right)^2\le\frac{(p+1)K}{p^{2-\varepsilon}}\le\frac{K_1}{p^{1-\varepsilon}},\tag{175}$$

where constant K_1 does not depend on p.

Combining (173)–(175), we obtain

$$\left(\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\right)^2\le\frac{K_2}{p^{1-\varepsilon}}\ \to\ 0$$

if $p\to\infty$, where constant K_2 does not depend on p.

Let us prove (150)–(152). It is not difficult to see that the estimate (174) proves (150).

Using the integration order replacement, we have

$$\sum_{j_1=p+1}^{\infty}C_{j_3j_3j_1j_1}\Biggl|_{(j_3j_3)\frown(\cdot)}=$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \psi_4(t_4)\psi_3(t_4) \int_t^{t_4} \psi_2(t_2)\phi_{j_1}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \left(\psi_2(t_2) \int_{t_2}^T \psi_4(t_4)\psi_3(t_4) dt_4 \right) \phi_{j_1}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2, \tag{176}$$

$$\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1} \Big|_{(j_2 j_2) \curvearrowright (\cdot)} =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int_t^{t_4} \psi_3(t_3)\psi_2(t_3) \int_t^{t_3} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_3 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \int_{t_1}^{t_4} \psi_3(t_3)\psi_2(t_3) dt_3 dt_1 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) \left(\int_t^{t_4} - \int_t^{t_1} \right) \psi_3(t_3)\psi_2(t_3) dt_3 dt_1 dt_4 =$$

$$= \sum_{j_1=p+1}^{\infty} \int_t^T \left(\psi_4(t_4) \int_t^{t_4} \psi_3(t_3)\psi_2(t_3) dt_3 \right) \phi_{j_1}(t_4) \int_t^{t_4} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_4 - \tag{177}$$

$$- \sum_{j_1=p+1}^{\infty} \int_t^T \psi_4(t_4)\phi_{j_1}(t_4) \int_t^{t_4} \left(\psi_1(t_1) \int_t^{t_1} \psi_3(t_3)\psi_2(t_3) dt_3 \right) \phi_{j_1}(t_1) dt_1 dt_4. \tag{178}$$

Applying the estimate (103) (polynomial and trigonometric cases) to the right-hand sides of (176)–(178), we get

$$\left| \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_1 j_1} \Big|_{(j_3 j_3) \curvearrowright (\cdot)} \right| \le \frac{C}{p}, \tag{179}$$

$$\left| \sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_2 j_1} \Big|_{(j_2 j_2) \curvearrowright (\cdot)} \right| \le \frac{C}{p}, \tag{180}$$

where constant C is independent of p. The estimates (179), (180) prove (151), (152).

The relations (141)–(152) are proved. Theorem 17 is proved.

10. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case $p_1 = \ldots = p_5 \to \infty$ and Continuously Differentiable Weight Functions $\psi_1(\tau)$, $\ldots$, $\psi_5(\tau)$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 18 [15], [47], [55], [62]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots \int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{181}$$

the following expansion

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_5 = 0,1,\ldots,m,$

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. Note that in this proof we write k instead of 5 when this is true for an arbitrary k ($k\in\mathbb{N}$). As follows from Sect. 7, Conditions 1 and 2 of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 13 for the iterated Stratonovich stochastic integral (181). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2 = 0, \tag{182}$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2 = 0, \tag{183}$$

$$\lim_{p\to\infty}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2 = 0, \tag{184}$$

where $(\{g_1,g_2\},\{g_3,g_4\},\{q_1\})$ and $(\{g_1,g_2\},\{q_1,q_2,q_3\})$ are partitions of the set $\{1,2,\ldots,5\}$ that is $\{g_1,g_2,g_3,g_4,q_1\}=\{g_1,g_2,q_1,q_2,q_3\}=\{1,2,\ldots,5\}$; braces mean an unordered set, and parentheses mean an ordered set.

Let us find a representation for $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2>g_1+1}$ that will be convenient for further consideration.

Using the integration order replacement in the Riemann integrals, we obtain

$$\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^{t_{l+1}} h_l(t_l)dt_l\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_1(t_1)\int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots\int\limits_{t_{l-2}}^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\times$$

$$\times dt_{l-1}\ldots dt_2dt_1dt_{l+1}\ldots dt_k=$$

$$=\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k-$$

$$-\int\limits_t^T h_k(t_k)\ldots\int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1})\int\limits_t^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots$$

$$\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \tag{185}$$

where $1<l<k$ and $h_1(\tau),\ldots,h_k(\tau)$ are continuous functions on the interval $[t,T]$. By analogy with (185) we have for $l=k$

$$\int\limits_t^T h_l(t_l)\int\limits_t^{t_l} h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_l =$$

$$= \int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\int\limits_{t_{l-1}}^T h_l(t_l)dt_l dt_{l-1}\ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})dt_{l-1}\ldots dt_2 dt_1 -$$

$$-\int\limits_t^T h_1(t_1)\int\limits_{t_1}^T h_2(t_2)\ldots\int\limits_{t_{l-2}}^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)dt_{l-1}\ldots dt_2 dt_1 =$$

$$= \left(\int\limits_t^T h_l(t_l)dt_l\right)\int\limits_t^T h_{l-1}(t_{l-1})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1} -$$

$$-\int\limits_t^T h_{l-1}(t_{l-1})\left(\int\limits_t^{t_{l-1}} h_l(t_l)dt_l\right)\int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots\int\limits_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}. \tag{186}$$

The formulas (185), (186) will be used further.

Our further proof will not fundamentally depend on the weight functions $\psi_1(\tau),\ldots,\psi_k(\tau)$. Therefore, sometimes in subsequent consideration we assume for simplicity that $\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$.

Let us continue the proof. Applying (185) to $C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1}$ (more precisely to $h_s(t_s)$ $= \psi_s(t_s)\phi_{j_l}(t_s)$), we obtain for $l+1\le k$, $s-1\ge 1$, $l-1\ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_l j_{l-1}\ldots j_{s+1}j_l j_{s-1}\ldots j_1} = \tag{187}$$

$$= \sum_{j_l=p+1}^{\infty}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots\int\limits_t^{t_{s+2}}\phi_{j_{s+1}}(t_{s+1})\int\limits_t^{t_{s+1}}\phi_{j_l}(t_s)\int\limits_t^{t_s}\phi_{j_{s-1}}(t_{s-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{s-1}dt_s dt_{s+1}\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\phi_{j_l}(t_l)\int\limits_t^{t_l}\phi_{j_{l-1}}(t_{l-1})\ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \left(\int\limits_t^{t_{s+1}} \phi_{j_l}(t_s)dt_s \right) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_{s-1}dt_{s+1} \ldots dt_{l-1}dt_l dt_{l+1} \ldots dt_k -$$

$$- \sum_{j_l=p+1}^{\infty} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_{l+2}} \phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \phi_{j_l}(t_l) \int\limits_t^{t_l} \phi_{j_{l-1}}(t_{l-1}) \ldots$$

$$\ldots \int\limits_t^{t_{s+2}} \phi_{j_{s+1}}(t_{s+1}) \int\limits_t^{t_{s+1}} \phi_{j_{s-1}}(t_{s-1}) \left(\int\limits_t^{t_{s-1}} \phi_{j_l}(t_s)dt_s \right) \int\limits_t^{t_{s-1}} \phi_{j_{s-2}}(t_{s-2}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1 \ldots dt_{s-2}dt_{s-1}dt_{s+1} \ldots dt_{l-1}dt_l dt_{l+1} \ldots dt_k =$$

$$= \sum_{j_l=p+1}^{\infty} A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} - \sum_{j_l=p+1}^{\infty} B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}.$$

Now we apply the formula (185) to $A_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$, $B_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1}$ (more precisely to $h_l(t_l) = \psi_l(t_l)\phi_{j_l}(t_l)$). Then we have for $l+1 \le k$, $s-1 \ge 1$, $l-1 \ge s+1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_{l-1} \ldots j_{s+1} j_l j_{s-1} \ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} F_p^{(d)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) \times$$

$$\times \prod_{\substack{g=1 \\ g \ne l,s}}^{k} \psi_g(t_g)\phi_{j_g}(t_g) dt_1 \ldots dt_{s-1}dt_{s+1} \ldots dt_{l-1}dt_{l+1} \ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_{l+1} j_{l-1} \ldots j_{s+1} j_{s-1} \ldots j_1} = \sum_{d=1}^{4} C^{*(d)}_{j_k \ldots j_q \ldots j_1}\Bigg|_{q \ne l,s}, \tag{188}$$

where

$$F_p^{(1)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1 < \ldots < t_{s-1} < t_{s+1} < \ldots < t_{l-1} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{189}$$

$$F_p^{(2)}(t_1, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{190}$$

$$F_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s-1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l+1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau, \tag{191}$$

$$F_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_k) =$$

$$= -\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{s+1}} \psi_s(\tau)\phi_{j_l}(\tau)d\tau \int\limits_t^{t_{l-1}} \psi_l(\tau)\phi_{j_l}(\tau)d\tau. \tag{192}$$

By analogy with (188) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l}, \tag{193}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1} j_l j_{l-1}\ldots j_2 j_l} \quad (l+1\le k), \tag{194}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1} j_l j_{s-1}\ldots j_1} \quad (s-1\ge 1). \tag{195}$$

Then we have for (193)–(195) (see (185), (186))

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} G_p^{(d)}(t_2,\ldots,t_{k-1}) \prod_{g=2}^{k-1} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{k-1}, \tag{196}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1} j_l j_{l-1}\ldots j_2 j_l} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{2} E_p^{(d)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)\times$$

$$\times \prod_{\substack{g=2\\ g\ne l}}^{k} \psi_g(t_g)\phi_{j_g}(t_g)dt_2\ldots dt_{l-1}dt_{l+1}\ldots dt_k, \tag{197}$$

$$\sum_{j_l=p+1}^{\infty} C_{j_l j_{k-1}\ldots j_{s+1} j_l j_{s-1}\ldots j_1} = \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} D_p^{(d)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})\times$$

$$\times\prod_{\substack{g=1\\ g\ne s}}^{k-1}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{s-1}dt_{s+1}\ldots dt_{k-1}, \tag{198}$$

where

$$G_p^{(1)}(t_2,\ldots,t_{k-1})=\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$G_p^{(2)}(t_2,\ldots,t_{k-1})=-\mathbf{1}_{\{t_2<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(1)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$E_p^{(2)}(t_2,\ldots,t_{l-1},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_2<\ldots<t_{l-1}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_2}\psi_1(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(1)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(2)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(3)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau,$$

$$D_p^{(4)}(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{k-1})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{k-1}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-1}}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{s-1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau.$$

Now let us consider the value $C_{j_k\ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ g_2=g_1+1}$. To do this, we will make the following transformations

$$\int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_l(t_l) \int\limits_t^{t_l} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots \int\limits_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-2}dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-2}}\right) h_l(t_{l-1}) \left(\int\limits_t^{t_{l+1}} - \int\limits_t^{t_{l-1}}\right) h_l(t_l)dt_l dt_{l-1}dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l \int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times$$

$$\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l\right) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots$$

$$\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1})dt_{l-1}\right) dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_1(t_1)\times$$

$$\times \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k +$$

$$+ \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_1(t_1) \int\limits_{t_1}^{t_{l+1}} h_2(t_2)\ldots \int\limits_{t_{l-3}}^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l)dt_l dt_{l-1}\right) dt_{l-2}\ldots dt_2dt_1dt_{l+1}\ldots dt_k =$$

$$= \int\limits_t^T h_k(t_k)\ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l)dt_l \int\limits_t^{t_{l+1}} h_l(t_{l-1})dt_{l-1}\right) \int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2})\times$$

$$\times \int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_l) dt_l \right) \int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2}) \times$$

$$\times \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1}) dt_{l-1} \right) \int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k -$$

$$- \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \left(\int\limits_t^{t_{l+1}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l) dt_l dt_{l-1} \right) \times$$

$$\times \int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2}) \int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k +$$

$$+ \int\limits_t^T h_k(t_k) \ldots \int\limits_t^{t_{l+2}} h_{l+1}(t_{l+1}) \int\limits_t^{t_{l+1}} h_{l-2}(t_{l-2}) \left(\int\limits_t^{t_{l-2}} h_l(t_{l-1}) \int\limits_t^{t_{l-1}} h_l(t_l) dt_l dt_{l-1} \right) \times$$

$$\times \int\limits_t^{t_{l-2}} h_{l-3}(t_{l-3}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{l-3} dt_{l-2} dt_{l+1} \ldots dt_k, \tag{199}$$

where $l+1 \le k$, $l-2 \ge 1$, and $h_1(\tau), \ldots, h_k(\tau)$ are continuous functions on the interval $[t, T]$.

Applying (199) to $C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots \ldots j_1}$, we obtain for $l+1 \le k$, $l-2 \ge 1$

$$\sum_{j_l=p+1}^{\infty} C_{j_k \ldots j_{l+1} j_l j_l j_{l-2} \ldots \ldots j_1} =$$

$$= \int\limits_{[t,T]^{k-2}} \sum_{d=1}^{4} H_p^{(d)}(t_1, \ldots, t_{l-2}, t_{l+1}, \ldots, t_k) \times$$

$$\times \prod_{\substack{g=1 \\ g \ne l-1, l}}^{k} \psi_g(t_g) \phi_{j_g}(t_g) dt_1 \ldots dt_{l-2} dt_{l+1} \ldots dt_k =$$

$$= \sum_{d=1}^{4} C^{**(d)}_{j_k \ldots j_{l+1} j_{l-2} \ldots j_1} = \sum_{d=1}^{4} C^{**(d)}_{j_k \ldots j_q \ldots j_1} \Bigg|_{q \ne l-1, l}, \tag{200}$$

where

$$H_p^{(1)}(t_1, \ldots, t_{l-2}, t_{l+1}, \ldots, t_k) =$$

$$= \mathbf{1}_{\{t_1 < \ldots < t_{l-2} < t_{l+1} < \ldots < t_k\}} \sum_{j_l=p+1}^{\infty} \int\limits_t^{t_{l+1}} \psi_l(\tau) \phi_{j_l}(\tau) d\tau \int\limits_t^{t_{l+1}} \psi_{l-1}(\tau) \phi_{j_l}(\tau) d\tau, \tag{201}$$

$$H_p^{(2)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-2}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)d\tau, \tag{202}$$

$$H_p^{(3)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+1}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau, \tag{203}$$

$$H_p^{(4)}(t_1,\ldots,t_{l-2},t_{l+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+1}<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-2}}\psi_{l-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_l(\theta)\phi_{j_l}(\theta)d\theta d\tau. \tag{204}$$

By analogy with (200) we can consider the expressions

$$\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_l}, \tag{205}$$

$$\sum_{j_l=p+1}^{\infty}C_{j_lj_lj_{k-2}\ldots j_1}. \tag{206}$$

Then we have for (205), (206) (see (199) and its analogue for $t_{l+1}=T$)

$$\sum_{j_l=p+1}^{\infty}C_{j_k\ldots j_{l+1}j_lj_l}=\int\limits_{[t,T]^{k-2}}L_p(t_3,\ldots,t_k)\prod_{g=3}^{k}\psi_g(t_g)\phi_{j_g}(t_g)dt_3\ldots dt_k, \tag{207}$$

$$\sum_{j_l=p+1}^{\infty}C_{j_lj_lj_{k-2}\ldots j_1}=\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}M_p^{(d)}(t_1,\ldots,t_{k-2})\prod_{g=1}^{k-2}\psi_g(t_g)\phi_{j_g}(t_g)dt_1\ldots dt_{k-2}, \tag{208}$$

where

$$L_p(t_3,\ldots,t_k)=\mathbf{1}_{\{t_3<\ldots<t_k\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_3}\psi_2(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(1)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{T}\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{T}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(2)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_k(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)d\tau,$$

$$M_p^{(3)}(t_1,\ldots,t_{k-2})=$$

$$=-\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^T\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau,$$

$$M_p^{(4)}(t_1,\ldots,t_{k-2})=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{k-2}\}}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{k-2}}\psi_{k-1}(\tau)\phi_{j_l}(\tau)\int\limits_t^{\tau}\psi_k(\theta)\phi_{j_l}(\theta)d\theta d\tau.$$

It is important to note that $C^{*(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$, $C^{**(d)}_{j_k\ldots j_{l+1}j_{l-2}\ldots j_1}$ $(d=1,\ldots,4)$ are Fourier coefficients (see (188), (200)), that is, we can use Parseval's equality in the further proof.

Combining the equalities (188)–(192) (the case $g_2>g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 16, 17, we obtain for (188)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_2>g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C^{*(d)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\right)^2=$$

$$=\int\limits_{[t,T]^{k-2}}\left(\sum_{d=1}^{4}F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2}}^{k}\psi_q(t_q)\right)^2\times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\le$$

$$\leq 4\sum_{d=1}^{4}\int\limits_{[t,T]^{k-2}}\left(F_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_2-1},t_{g_2+1},\ldots,t_k)\prod_{\substack{q=1\\ q\neq g_1,g_2}}^{k}\psi_q(t_q)\right)^2\times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_2-1}dt_{g_2+1}\ldots dt_k\leq$$

$$\leq\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{209}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (193)–(195) are considered analogously.

Absolutely similarly (see (209)) combining the equalities (200)–(204) (the case $g_2=g_1+1$), using Parseval's equality and applying the estimates for integrals from basis functions that we used in the proof of Theorems 16, 17, we get for (200)

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},g_2=g_1+1}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{p}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\neq g_1,g_2}\right)^2\leq\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{\infty}\left(\sum_{d=1}^{4}C_{j_k\ldots j_q\ldots j_1}^{**(d)}\Biggr|_{q\neq g_1,g_2}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2}}^{\infty}\left(\int\limits_{[t,T]^{k-2}}\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\neq g_1,g_2}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\right)^2=$$

$$=\int\limits_{[t,T]^{k-2}}\left(\sum_{d=1}^{4}H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\neq g_1,g_2}}^{k}\psi_q(t_q)\right)^2dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots\ldots dt_k\leq$$

$$\leq 4\sum_{d=1}^{4}\int\limits_{[t,T]^{k-2}}\left(H_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+2},\ldots,t_k)\prod_{\substack{q=1\\ q\neq g_1,g_2}}^{k}\psi_q(t_q)\right)^2dt_1\ldots dt_{g_1-1}dt_{g_1+2}\ldots dt_k\leq$$

$$\leq\frac{K}{p^{2-\varepsilon}}\ \to\ 0 \tag{210}$$

if $p\to\infty$, where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p. The cases (205), (206) are considered analogously.

From (209), (210) and their analogues for the cases (193)–(195), (205), (206) we obtain

$$\sum_{j_{q_1},\ldots,j_{q_{k-2}}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2}}\right)^2\le\frac{K}{p^{2-\varepsilon}},\tag{211}$$

where constant K is independent of p. Thus the equality (182) is proved.

Let us prove the equality (183). Consider the following cases

1. $g_2>g_1+1,\ g_4=g_3+1,$ 2. $g_2=g_1+1,\ g_4>g_3+1,$

3. $g_2>g_1+1,\ g_4>g_3+1,$ 4. $g_2=g_1+1,\ g_4=g_3+1.$

The proof for Cases 1–3 will be similar. Consider, for example, Case 2. Using (77), we obtain

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=0}^{p}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2\le\tag{212}$$

$$\le(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2=$$

$$=(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2\Biggl|_{j_{g_3}=j_{g_4}}\le$$

$$\le(p+1)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3},j_{g_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},g_4>g_3+1,g_2=g_1+1}\right)^2.\tag{213}$$

It is easy to see that the expression (213) (without the multiplier $p+1$) is a particular case $(g_4>g_3+1,g_2=g_1+1)$ of the left-hand side of (211). Combining (211) and (213), we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4>g_3+1,g_2=g_1+1}\right)^2\le$$

$$\le\frac{(p+1)K}{p^{2-\varepsilon}}\le\frac{K_1}{p^{1-\varepsilon}}\ \to\ 0\tag{214}$$

if $p\to\infty$, where constant K_1 does not depend on p.

Consider Case 4 ($g_2=g_1+1,\ g_4=g_3+1$). We have (see (78))

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=
$$

$$
=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\left(\sum_{j_{g_3}=0}^{\infty}-\sum_{j_{g_3}=0}^{p}\right)C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=
$$

$$
=\sum_{j_{q_1}=0}^{p}\left(\frac{1}{2}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)}-\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2\le
$$

$$
\le\frac{1}{2}\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)}\right)^2+ \tag{215}
$$

$$
+2\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=0}^{p}\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2. \tag{216}
$$

An expression similar to (216) was estimated (see (212)–(214)). Let us estimate (215). We have

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown(\cdot)}\right)^2=
$$

$$
=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown 0}\right)^2\le
$$

$$
\le(T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_3}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},(j_{g_3}j_{g_3})\frown j_{g_3}}\right)^2, \tag{217}
$$

where the notations are the same as in the proof of Theorem 13.

The expression (217) without the multiplier $T-t$ is an expression of type (141)–(146) before passing to the limit $\lim\limits_{p\to\infty}$ (the only difference is the replacement of one of the weight functions $\psi_1(\tau),\ldots,\psi_4(\tau)$ in (141)–(146) by the product $\psi_{l+1}(\tau)\psi_l(\tau)$ ($l=1,\ldots,4$). Therefore, for Case 4 ($g_2=g_1+1,\ g_4=g_3+1$), we obtain the estimate

$$
\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_5\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1,g_2=g_1+1}\right)^2\le
$$

$$
\le\frac{K}{p^{1-\varepsilon}}, \tag{218}
$$

where constant K is independent of p.

The estimates (214), (218) prove (183). Let us prove (184). By analogy with (217) we have

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=$$

$$=\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2=$$

$$=(T-t)\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright 0,j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2\le$$

$$\le(T-t)\sum_{j_{q_1}=0}^{p}\sum_{j_{g_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_1}j_{g_1})\curvearrowright j_{g_1},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2. \quad (219)$$

Thus, we obtain the estimate (see (217) and the proof of Theorem 17)

$$\sum_{j_{q_1}=0}^{p}\left(\sum_{j_{g_3}=p+1}^{\infty} C_{j_5\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_2=g_1+1}\right)^2\le$$

$$\le\frac{K}{p^{2-\varepsilon}}, \quad (220)$$

where ε is an arbitrary small positive real number for the polynomial case and $\varepsilon=0$ for the trigonometric case, constant K does not depend on p.

The estimate (220) proves (184). Theorem 18 is proved.

11. Estimates for the Mean-Square Approximation Error of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicity k in Theorems 13, 15

In this section, we estimate the mean-square approximation error for iterated Stratonovich stochastic integrals of multiplicity k ($k\in\mathbb{N}$) in Theorems 13, 15.

Theorem 19 [15], [47], [55], [62]. *Suppose that every $\psi_l(\tau)$ ($l=1,\ldots,k$) is a continuously differentiable nonrandom function at the interval $[t,T]$. Furthermore, let $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then the following estimates*

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\le$$

$$\leq K_1\left(\frac{1}{p}+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right), \tag{221}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}(s)\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^2\right\}\leq$$

$$\leq K_2(s)\left(\frac{1}{p}+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \mathsf{M}\left\{\left(R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right)^2\right\}\right) \tag{222}$$

hold, where $s\in(t,T]$ *(*s *is fixed),* $i_1,\ldots,i_k=1,\ldots,m$,

$$R_{s,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}=R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\Biggr|_{T=s},$$

$R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ *is defined by* (90), $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *and* $J^*[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}$ *are iterated Stratonovich stochastic integrals* (43) *and* (101), $C_{j_k\ldots j_1}$ *and* $C_{j_k\ldots j_1}(s)$ *are Fourier coefficients* (35) *and* (99), *constants* K_1, $K_2(s)$ *are independent of* p; *another notations are the same as in Theorems* 3, 13, 15.

Proof. As follows from Sect. 7 and 8, Conditions 1 and 2 of Theorems 13, 15 are satisfied under the conditions of Theorem 19. Then from the proof of Theorem 13 it follows that the expression (95) before passing to limit $\underset{p\to\infty}{\text{l.i.m.}}$ has the form

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}+$$

$$+\sum_{r=1}^{[k/2]}\left(\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}+\right.$$

$$\left.+\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right), \tag{223}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the approximation for the iterated Ito stochastic integral (2), which is obtained using Theorem 4, i.e.

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \tag{224}$$

$I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}$ is the approximation obtained using (224) for the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ (see (96)).

Using (223) and Theorem 5, we have

$$\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} = J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}+$$

$$+\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p} - I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} =$$

$$= J^{*}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}}\frac{1}{2^r}\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p} - I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} R_{T,t}^{(p)r,g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{225}$$

w. p. 1, where we denote $J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}$ as $I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}$.

In [15] (Sect. 1.7.2, Remark 1.7) it is shown that under the conditions of Theorem 19 the following estimate

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}\right)^2\right\} \le \frac{C}{p}, \tag{226}$$

holds, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (2), $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ has the form (224), $i_1,\ldots,i_k = 0,1,\ldots,m$, constant C depends only on k and $T-t$.

Applying (226), we obtain the following estimates

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} \le \frac{C}{p}, \tag{227}$$

$$\mathsf{M}\left\{\left(I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)p}-I[\psi^{(k)}]_{T,t}^{(i_1\ldots i_{s_1-1}i_{s_1+2}\ldots i_{s_r-1}i_{s_r+2}\ldots i_k)}\right)^2\right\}\le$$

$$\le\frac{C}{p}, \tag{228}$$

where constant C does not depend on p.

From (225)–(228) and the elementary inequality

$$(a_1+a_2+\ldots+a_n)^2\le n\left(a_1^2+a_2^2+\ldots+a_n^2\right),\quad n\in\mathbb{N}$$

we obtain (221).

The estimate (222) is obtained similarly to the estimate (221) using Theorems 1.11, 1.24 in [15], Theorem 15 and the estimate [15] (Sect. 1.8.1, Remark 1.12)

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}\right)^2\right\}\le\frac{C}{p},$$

where

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{s,t}^{(i_1\ldots i_k)p}=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}(s)\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $s\in(t,T]$ (s is fixed), $C_{j_k\ldots j_1}(s)$ is the Fourier coefficient (99), $i_1,\ldots,i_k=0,1,\ldots,m$, constant C depends only on k and $s-t$; another notations are the same as in Theorem 4, 15. Theorem 19 is proved.

12. Rate of the Mean-Square Convergence of Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3–5 in Theorems 16–18

In this section, we consider the rate of convergence of approximations of iterated Stratonovich stochastic integrals in Theorems 16–18. It is easy to see that in Theorems 16–18 the second term in parentheses on the right-hand side of (221) is estimated. Combining these results with Theorem 19, we obtain the following theorems.

Theorem 20 [15], [47], [55], [62]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p}$$

is fulfilled, where $i_1,i_2,i_3=1,\ldots,m,$ *constant* C *is independent of* p,

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j.

Theorem 21 [15], [47], [55], [62]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\ \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}d\mathbf{f}_{t_4}^{(i_4)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

holds, where $i_1,i_2,i_3,i_4=1,\ldots,m,$ *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\times$$

$$\times dt_2dt_3dt_4;$$

another notations are the same as in Theorem 20.

Theorem 22 [15], [47], [55], [62]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_5}^{(i_5)}$$

the following estimate

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

is valid, where $i_1,\ldots,i_5=1,\ldots,m$, *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^{T}\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorem 20, 21.

13. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case $p_1=\ldots=p_6\to\infty$ and $\psi_1(\tau),\ \ldots,\ \psi_6(\tau)\equiv 1$ (The Cases of Legendre Polynomials and Trigonometric Functions)

Theorem 23 [15], [47], [55], [63]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \qquad (229)$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^{T}\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. As noted in Sect. 7, Conditions 1 and 2 of Theorem 13 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. Let us verify Condition 3 of Theorem 13 for the iterated Stratonovich stochastic integral (229). Thus, we have to check the following conditions

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}, j_{q_3}, j_{q_4}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}} \right)^2 = 0, \tag{230}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}} \right)^2 = 0, \tag{231}$$

$$\lim_{p\to\infty} \sum_{j_{q_1}, j_{q_2}=0}^{p} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{(j_{g_4} j_{g_3})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, g_4=g_3+1} \right)^2 = 0, \tag{232}$$

$$\lim_{p\to\infty} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} \sum_{j_{g_5}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}} \right)^2 = 0, \tag{233}$$

$$\lim_{p\to\infty} \left(\sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_3}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}, g_6=g_5+1} \right)^2 = 0, \tag{234}$$

$$\lim_{p\to\infty} \left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_6 \ldots j_1}\Bigg|_{(j_{g_4} j_{g_3})\curvearrowright(\cdot)(j_{g_6} j_{g_5})\curvearrowright(\cdot), j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}, g_4=g_3+1, g_6=g_5+1} \right)^2 = 0, \tag{235}$$

where the expressions

$$(\{g_1, g_2\}, \{g_3, g_4\}, \{g_5, g_6\}\}), \quad (\{g_1, g_2\}, \{g_3, g_4\}, \{q_1, q_2\}\}), \quad (\{g_1, g_2\}, \{q_1, q_2, q_3, q_4\})$$

are partitions of the set $\{1, 2, \ldots, 6\}$ that is $\{g_1, g_2, g_3, g_4, g_5, g_6\} = \{g_1, g_2, g_3, g_4, q_1, q_2\} = \{g_1, g_2, q_1, q_2, q_3, q_4\} = \{1, 2, \ldots, 6\}$; braces mean an unordered set, and parentheses mean an ordered set.

The equalities (230), (232) were proved earlier (see the proof of equalities (211), (217)). The relation (235) follows from the estimate (103) for the polynomial case and its analogue for the trigonometric case. It is easy to see that the equalities (231) and (234) are proved in complete analogy with the proof of (183), (217).

Thus, we have to prove the relation (233). The equality (233) is equivalent to the following equalities

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=0, \tag{236}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_3j_2j_3j_2j_1}=0, \tag{237}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_1j_2j_1}=0, \tag{238}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=0, \tag{239}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_2j_3j_3j_1}=0, \tag{240}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=0, \tag{241}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=0, \tag{242}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=0, \tag{243}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=0, \tag{244}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0, \tag{245}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_1j_3j_3j_2j_1}=0, \tag{246}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0, \tag{247}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0, \tag{248}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0, \tag{249}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0. \tag{250}$$

Consider in detail the case of Legendre polynomials (the case of trigonometric functions is considered in complete analogy).

First, we prove the following equality for the Fourier coefficients for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

$$C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+$$

$$+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}. \tag{251}$$

Using the integration order replacement, we have

$$C_{j_6j_5j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6=$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6+$$

$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$

$$=C_{j_6}C_{j_5j_4j_3j_2j_1}-$$

$$-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)dt_5dt_6\ C_{j_4j_3j_2j_1}+$$

$$+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} +$$

$$+\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\int\limits_{t_5}^T \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\ldots\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6 =$$

$$\ldots$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} - C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} -$$

$$-\int\limits_t^T \phi_{j_6}(t_6)\int\limits_{t_6}^T \phi_{j_5}(t_5)\ldots\int\limits_{t_2}^T \phi_{j_1}(t_1)dt_1\ldots dt_5dt_6 =$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} -$$

$$-C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} - C_{j_1j_2j_3j_4j_5j_6}. \tag{252}$$

The equality (252) completes the proof of the relation (251).
Let us consider (236). From (71) we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}. \tag{253}$$

Applying (251), we get

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_1j_2j_3} = 2\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1} - C_{j_2j_3}C_{j_1j_3j_2j_1} + C_{j_1j_2j_3}C_{j_3j_2j_1} -$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1} + C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr). \tag{254}$$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots, \tag{255}$$

where

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j$$

is the Legendre polynomial.

Note that

$$C_{j_2 j_1}=\int\limits_t^T \phi_{j_2}(\tau)\int\limits_t^{\tau}\phi_{j_1}(\theta)d\theta d\tau =$$

$$=\frac{T-t}{2}\begin{cases}1/\sqrt{(2j_1+1)(2j_1+3)} & \text{if } j_2=j_1+1,\ j_1=0,1,2,\ldots\\ -1/\sqrt{4j_1^2-1} & \text{if } j_2=j_1-1,\ j_1=1,2,\ldots\\ 1 & \text{if } j_1=j_2=0\\ 0 & \text{otherwise}\end{cases}, \tag{256}$$

$$C_{j_1}=\int\limits_t^T \phi_{j_1}(\tau)d\tau=\begin{cases}\sqrt{T-t} & \text{if } j_1=0\\ 0 & \text{if } j_1\ne 0\end{cases}. \tag{257}$$

Moreover, the generalized Parseval equality gives

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3}C_{j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T \phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_t^T \phi_{j_3}(t_3)\int\limits_{t_3}^{T}\phi_{j_2}(t_2)\int\limits_{t_2}^{T}\phi_{j_1}(t_1)dt_1dt_2dt_3\times$$

$$\times\int\limits_t^T \phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_3<t_2<t_1\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3\times$$

$$\times\int\limits_{[t,T]^3}\mathbf{1}_{\{t_1<t_2<t_3\}}\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3=$$

$$=\int\limits_{[t,T]^3}\mathbf{1}_{\{t_3<t_2<t_1\}}\mathbf{1}_{\{t_1<t_2<t_3\}}dt_1dt_2dt_3=0. \tag{258}$$

Using the above arguments and also (71), (253), and (254), we get

$$-\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_1j_3j_2j_1}=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1j_3j_2j_1}=$$

$$=\frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_2j_3}C_{j_1j_3j_2j_1}-$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1}+C_{j_2j_3j_1j_2j_3}C_{j_1}\Biggr)=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1}-C_{j_3j_1j_2j_3}C_{j_2j_1}\Biggr)=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}-\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_1j_2j_3}C_{j_2j_1}=$$

$$=\sqrt{T-t}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}+\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}. \tag{259}$$

By analogy with the proof of (147) (see the proof of Theorem 17) we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_10j_2j_1}=\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_10j_2j_1}=0, \tag{260}$$

where we used the following representation

$$C_{j_2j_10j_2j_1}=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}dt_3dt_2dt_4dt_5=$$

$$=\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)(t_4-t)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5+$$

$$+\frac{1}{\sqrt{T-t}}\int\limits_t^T\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5\stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\bar{C}_{j_2j_1j_2j_1}+\tilde{C}_{j_2j_1j_2j_1}.$$

Further, we have (see (256))

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3}C_{j_2j_1}=\lim_{p\to\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{00}C_{j_300j_3}+$$

$$+\sum_{j_1=1}^{p}C_{j_1-1,j_1}C_{j_3j_1,j_1-1,j_3}+\sum_{j_1=1}^{p-1}C_{j_1+1,j_1}C_{j_3j_1,j_1+1,j_3}+C_{1,0}C_{j_301j_3}\Biggr). \tag{261}$$

Observe that

$$|C_{j_1-1,j_1}|+|C_{j_1+1,j_1}|\le\frac{K}{j_1}\quad (j_1=1,\ldots,p), \tag{262}$$

$$|C_{j_300j_3}|+|C_{j_3j_1,j_1-1,j_3}|+|C_{j_3j_1,j_1+1,j_3}|+|C_{j_301j_3}|\le\frac{K_1}{j_3^2}\quad (j_3\ge p+1), \tag{263}$$

where constants K, K_1 do not depend on j_1, j_3.

The estimate (262) follow from (256). At the same time, the estimate (263) can be obtained using the following reasoning. First note that the integration order replacement gives

$$C_{j_3j_1j_2j_3}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_3}(t_1)dt_1\right)dt_2\left(\int\limits_{t_3}^{T}\phi_{j_3}(t_4)dt_4\right)dt_3. \tag{264}$$

Consider the well-known estimate for Legendre polynomials

$$|P_j(y)|<\frac{K}{\sqrt{j+1}(1-y^2)^{1/4}},\quad y\in(-1,1),\quad j\in\mathbb{N}, \tag{265}$$

where constant K does not depend on y and j.

The estimate (265) can be rewritten for the function $\phi_j(x)$ (see (255)) in the following form

$$|\phi_j(x)|<\sqrt{\frac{2j+1}{j+1}}\frac{K}{\sqrt{T-t}}\frac{1}{(1-z^2(x))^{1/4}}<\frac{K_1}{\sqrt{T-t}}\frac{1}{(1-z^2(x))^{1/4}}, \tag{266}$$

where $K_1=K\sqrt{2}$, $x\in(t,T)$, $j\in\mathbb{N}$,

$$z(x)=\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

Note analogues of the estimates (109), (110)

$$\left|\int\limits_t^x \phi_{j_1}(s)ds\right| < \frac{C}{j_1(1-(z(x))^2)^{1/4}}, \quad \left|\int\limits_x^T \phi_{j_1}(s)ds\right| < \frac{C}{j_1(1-(z(x))^2)^{1/4}}, \quad x \in (t,T), \tag{267}$$

where $j_1 > 0$, constant C does not depend on j_1.

Applying the estimates (266) and (267) to (264) gives the estimate (263). Using (261), (262), and (263), we obtain

$$\left|\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_2j_3}C_{j_2j_1}\right| \le K\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\left(1+\sum_{j_1=1}^{p}\frac{1}{j_1}\right) \le$$

$$\le K\int\limits_p^{\infty}\frac{dx}{x^2}\left(2+\int\limits_1^p\frac{dx}{x}\right) = \frac{K(2+lnp)}{p} \to 0 \tag{268}$$

if $p \to \infty$, where constant K is independent of p. Thus, the equality (236) is proved (see (259), (260), (268)).

The relation (237) is proved in complete analogy with the proof of equality (236). For (237) we have (see (251))

$$\lim_{p\to\infty}\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_2j_3j_1}\right) = 2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1} =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_1}C_{j_3j_2j_3j_2j_1} - C_{j_3j_1}C_{j_2j_3j_2j_1} + C_{j_2j_3j_1}C_{j_3j_2j_1} -$$

$$-C_{j_3j_2j_3j_1}C_{j_2j_1} + C_{j_2j_3j_2j_3j_1}C_{j_1}\Bigr) =$$

$$= 2\lim_{p\to\infty}\left(\sqrt{T-t}\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_20} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}\right) =$$

$$= -2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}.$$

To estimate the Fourier coefficient $C_{j_3j_2j_3j_1}$, we use the following (see the proof of (236) for more details)

$$C_{j_3j_2j_3j_1} = \int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4 =$$

$$= \int\limits_t^T \phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_3}\phi_{j_3}(t_2)dt_2dt_1dt_3dt_4 =$$

$$= \int_t^T \phi_{j_3}(t_4) \int_t^{t_4} \phi_{j_2}(t_3) \left(\int_t^{t_3} \phi_{j_3}(t_2)dt_2 \right) \int_t^{t_3} \phi_{j_1}(t_1)dt_1 dt_3 dt_4 -$$

$$- \int_t^T \phi_{j_3}(t_4) \int_t^{t_4} \phi_{j_2}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) \left(\int_t^{t_1} \phi_{j_3}(t_2)dt_2 \right) dt_1 dt_3 dt_4 =$$

$$= \int_t^T \phi_{j_2}(t_3) \left(\int_t^{t_3} \phi_{j_3}(t_2)dt_2 \right) \int_t^{t_3} \phi_{j_1}(t_1)dt_1 \left(\int_{t_3}^T \phi_{j_3}(t_4)dt_4 \right) dt_3 -$$

$$- \int_t^T \phi_{j_2}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) \left(\int_t^{t_1} \phi_{j_3}(t_2)dt_2 \right) dt_1 \left(\int_{t_3}^T \phi_{j_3}(t_4)dt_4 \right) dt_3.$$

Let us prove (238). From (71) we obtain

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_2 j_3 j_1 j_2 j_1} = - \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_2 j_3 j_1 j_2 j_1}. \tag{269}$$

Applying (251) and (269), we get (we replaced j_3 by j_4)

$$\sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4 j_1 j_2 j_1} + \sum_{j_1,j_2,j_4=0}^{p} C_{j_1 j_2 j_1 j_4 j_2 j_4} = 2 \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4 j_1 j_2 j_1} =$$

$$= \sum_{j_1,j_2,j_4=0}^{p} \Biggl(C_{j_4} C_{j_2 j_4 j_1 j_2 j_1} - C_{j_2 j_4} C_{j_4 j_1 j_2 j_1} + C_{j_4 j_2 j_4} C_{j_1 j_2 j_1} -$$

$$- C_{j_1 j_4 j_2 j_4} C_{j_2 j_1} + C_{j_2 j_1 j_4 j_2 j_4} C_{j_1} \Biggr) =$$

$$= 2 \sum_{j_1,j_2,j_4=0}^{p} \left(C_{j_2 j_1 j_4 j_2 j_4} C_{j_1} - C_{j_1 j_4 j_2 j_4} C_{j_2 j_1} \right) +$$

$$+ \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4} C_{j_1 j_2 j_1}. \tag{270}$$

Further, we have (see (71))

$$\lim_{p\to\infty} \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4} C_{j_1 j_2 j_1} = \lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_2 j_1} \right)^2 =$$

$$= \lim_{p\to\infty}\sum_{j_2=0}^{p}\left(\sum_{j_1=p+1}^{\infty}C_{j_1j_2j_1}\right)^2 = 0, \tag{271}$$

where we applied the equality (120).

Furthermore, by analogy with the proof of (236), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\Bigl(C_{j_2j_1j_4j_2j_4}C_{j_1} - C_{j_1j_4j_2j_4}C_{j_2j_1}\Bigr) = 0. \tag{272}$$

To estimate the Fourier coefficient $C_{j_1j_4j_2j_4}$ in (272), we use the following (see the proof of (236) for more details)

$$C_{j_1j_4j_2j_4} = \int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_3dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4 -$$

$$-\int\limits_t^T \phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_3)dt_3\right)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4.$$

The relations (269)–(272) complete the proof of equality (238).

Let us prove (239). Using (71), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1} = \sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}. \tag{273}$$

Applying (251) and (273), we obtain

$$2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1} =$$

$$= \sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_1}C_{j_2j_3j_3j_2j_1} - C_{j_2j_1}C_{j_3j_3j_2j_1} + \left(C_{j_3j_2j_1}\right)^2 -$$

$$-C_{j_3j_3j_2j_1}C_{j_2j_1} + C_{j_2j_3j_3j_2j_1}C_{j_1}\Bigr) =$$

$$
=2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}\Bigr)+
$$

$$
+\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2. \tag{274}
$$

In [15] (Sect. 1.7.2) the following estimate

$$
\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le
$$

$$
\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2}\le L_k\int\limits_p^{\infty}\frac{dx}{x^2}=\frac{L_k}{p} \tag{275}
$$

is proved for the polynomial and trigonometric cases, where $s=1,\ldots,k$, constant L_k depends on k and $T-t$.

Using the estimate (275), we get

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_2j_1}\right)^2=0. \tag{276}
$$

By analogy with the proof of (236), we have

$$
\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}\Bigr)=0, \tag{277}
$$

where we applied the equality (148). To estimate the Fourier coefficient $C_{j_3j_3j_2j_1}$ in (277), we used the following (see the proof of (236) for more details)

$$
C_{j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=
$$

$$
=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_3}(t_3)\int\limits_{t_3}^T\phi_{j_3}(t_4)dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\left(\int\limits_{t_2}^T\phi_{j_3}(t_3)dt_3\right)^2dt_2dt_1. \tag{278}
$$

Combining the equalities (273)–(277), we obtain (239).

Let us prove (240) (we replace j_2 by j_4 and j_3 by j_2 in (240)). As noted in Sect. 7, the sequential order of the series

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}$$

is not important. This follows directly from the formulas (78) and (71).

Applying the mentioned property and (71), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{279}$$

Observe that (see the above reasoning)

$$\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = \sum_{j_4=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{280}$$

Using (251) and (280), we obtain

$$\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\left(C_{j_1j_4j_4j_2j_2j_1}+C_{j_1j_2j_2j_4j_4j_1}\right) = 2\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1} - C_{j_4j_1}C_{j_4j_2j_2j_1} + C_{j_4j_4j_1}C_{j_2j_2j_1} -$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1} + C_{j_2j_2j_4j_4j_1}C_{j_1}\Bigr) =$$

$$= \sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\left(C_{j_1}C_{j_4j_4j_2j_2j_1} - C_{j_4j_1}C_{j_4j_2j_2j_1} - C_{j_2j_4j_4j_1}C_{j_2j_1} + C_{j_2j_2j_4j_4j_1}C_{j_1}\right) +$$

$$+\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_2j_1}\right)^2. \tag{281}$$

The equality

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\left(\sum_{j_2=p+1}^{\infty} C_{j_2j_2j_1}\right)^2 = 0 \tag{282}$$

follows from the relation (119).

By analogy with the proof of equality (236) we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Bigl(C_{j_1}C_{j_4j_4j_2j_2j_1} - C_{j_4j_1}C_{j_4j_2j_2j_1} -$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Biggr)=0, \tag{283}$$

where we applied the equality (149). To estimate the Fourier coefficient $C_{j_2j_4j_4j_1}$ in (283), we used the following (see the proof of (236) for more details)

$$C_{j_2j_4j_4j_1}=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_{t_1}^{t_4}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4+$$

$$+\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)^2dt_1dt_4-$$

$$-\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_2)dt_2\right)\int\limits_t^{t_4}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_4}(t_2)dt_2\right)dt_1dt_4.$$

The relation (240) follows from (279), (281)–(283).

Consider (241). Using the integration order replacement, we obtain

$$C_{j_3j_3j_2j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{4}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3. \tag{284}$$

Applying the estimates (267) to (284) gives the following estimate

$$|C_{j_3j_3j_2j_2j_1j_1}| \le \frac{K}{j_1^2 j_3^2} \quad (j_1, j_3 > 0,\ j_2 \ge 0), \tag{285}$$

where constant K does not depend on j_1, j_2, j_3.

Further, we get (see (78))

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1} = \sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1} =$$

$$= \frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)} - \sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1}, \tag{286}$$

where

$$C_{j_3j_3j_2j_2j_1j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)} =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\int\limits_t^{t_4}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_5}dt_4dt_2dt_5dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)(t_5-t)\int\limits_t^{t_5}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6+$$

$$+ \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_5dt_6 \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} C'_{j_3j_3j_1j_1} + C''_{j_3j_3j_1j_1}. \tag{287}$$

Let us substitute (287) into (286)

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1} = \frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C'_{j_3j_3j_1j_1}+$$

$$+\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C''_{j_3j_3j_1j_1} - \sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1}. \tag{288}$$

The relation (149) implies that

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C'_{j_3j_3j_1j_1}=0,\quad \lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C''_{j_3j_3j_1j_1}=0. \tag{289}
$$

From the estimate (285) we get

$$
\left|\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_2j_1j_1}\right| \le K(p+1)\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\le
$$

$$
\le K(p+1)\left(\int\limits_{p}^{\infty}\frac{dx}{x^2}\right)^2\le \frac{K(p+1)}{p^2}\ \to\ 0 \tag{290}
$$

if $p\to\infty$, where constant K is independent of p.

The relations (288)–(290) complete the proof of (241).

Let us prove (242). Using the integration order replacement, we get

$$
C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6dt_5dt_4dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)\left(\int\limits_{t}^{t_5}\phi_{j_3}(t_4)dt_4\right)dt_5dt_3-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_3}(t_5)\left(\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\right)dt_5dt_3. \tag{291}
$$

Applying (71) and (78), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=-\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$=\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Bigr|_{(j_3j_3)\frown(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Bigr|_{(j_3j_3)\frown(\cdot)}-\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}-$$

$$-\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}-\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}-$$

$$-\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}. \quad (292)$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Bigr|_{(j_3j_3)\frown(\cdot)}=0 \quad (293)$$

follows from the inequality similar to (175) (see the proof of Theorem 17), where we used the following representation

$$C_{j_2j_3j_3j_2j_1j_1}\Bigr|_{(j_3j_3)\frown(\cdot)}=$$

$$=\int_t^T\phi_{j_2}(t_6)\int_t^{t_6}\int_t^{t_4}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_6=$$

$$=\int_t^T\phi_{j_2}(t_6)\int_t^{t_6}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\int_{t_3}^{t_6}dt_4dt_3dt_6=$$

$$+\int_t^T\phi_{j_2}(t_6)(t_6-t)\int_t^{t_6}\phi_{j_2}(t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6+$$

$$+\int_t^T\phi_{j_2}(t_6)\int_t^{t_6}\phi_{j_2}(t_3)(t-t_3)\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_6\stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}C^{*}_{j_2j_2j_1j_1}+C^{**}_{j_2j_2j_1j_1}. \quad (294)$$

Applying the estimates (267) and (135) ($\varepsilon=1/2$) to (291) gives the following estimates

$$|C_{j_2j_3j_3j_2j_1j_1}| \le \frac{K}{j_1^2 j_2 j_3^{3/4}} \quad (j_1, j_2, j_3 > 0), \tag{295}$$

$$|C_{j_200j_2j_1j_1}| \le \frac{K}{j_1^2 j_2} \quad (j_1, j_2 > 0), \tag{296}$$

$$|C_{0j_3j_30j_1j_1}| \le \frac{K}{j_1^2 j_3} \quad (j_1, j_3 > 0), \tag{297}$$

$$|C_{0000j_1j_1}| \le \frac{K}{j_1^2} \quad (j_1 > 0). \tag{298}$$

Using the estimate (295), we have

$$\left|\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_2j_3j_3j_2j_1j_1}\right| \le K \sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_2=1}^{p}\frac{1}{j_2}\sum_{j_3=1}^{p}\frac{1}{j_3^{3/4}} \le$$

$$\le K \int\limits_p^{\infty}\frac{dx}{x^2}\left(1+\int\limits_1^p\frac{dx}{x}\right)\left(1+\int\limits_1^p\frac{dx}{x^{3/4}}\right) \le K_1\frac{1+lnp}{p^{3/4}} \to 0 \tag{299}$$

if $p \to \infty$, where constants K, K_1 do not depend on p.

Similarly we get (see (296)–(298))

$$\left|\sum_{j_1=p+1}^{\infty} C_{0000j_1j_1}\right| + \left|\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{0j_3j_30j_1j_1}\right| + \left|\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_200j_2j_1j_1}\right| \to 0 \tag{300}$$

if $p \to \infty$.

The relations (292), (293), (299), (300) prove (242).

Consider (243). Using the integration order replacement, we get

$$C_{j_3j_2j_3j_2j_1j_1} =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_2}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\left(\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\right)dt_5dt_3-$$

$$-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_2}(t_5)\left(\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\right)dt_5dt_3. \tag{301}$$

Applying (71), we obtain

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=$$

$$=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}. \tag{302}$$

Further proof of the equality (243) is based on the relations (301), (302) and is similar to the proof of the formula (242).

Let us prove (244). Applying the integration order replacement, we obtain

$$C_{j_3j_3j_2j_1j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3dt_2dt_1=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_1}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)dt_2dt_4-$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)^2 dt_2dt_4. \qquad (303)
$$

Using (71), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=
$$

$$
=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}. \qquad (304)
$$

Further proof of the equality (244) is based on the relations (303), (304) and is similar to the proof of the relations (242), (243).

Consider (245). Using the integration order replacement, we have

$$
C_{j_3j_3j_1j_2j_2j_1}=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_2}(t_3)\int\limits_{t_3}^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_2}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)dt_2dt_4-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_2}\phi_{j_2}(t_3)dt_3\right)dt_2dt_4. \qquad (305)
$$

Applying (71) and (78), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=-\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}. \tag{306}$$

The equality

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}=0 \tag{307}$$

follows from the inequality (175), where we proceed similarly to the proof of equality (293) (see (294)).

The relation

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0 \tag{308}$$

is proved on the basis of (305) and similarly with the proof of (242). The equalities (306)–(308) prove (245).

Let us prove (246). Using (71) and (78), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_1j_3j_3j_2j_1}=\sum_{j_3=p+1}^{\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_3j_2j_1}=$$

$$=\frac{1}{2}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1j_3j_3j_2j_1}. \tag{309}$$

Using the equality (147) we have

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_3j_3j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=0, \tag{310}$$

where we proceed similarly to the proof of equality (293) (see (294)).

Further, we will prove the following relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}C_{j_2j_1j_3j_3j_2j_1}=0 \tag{311}$$

using the equality (251). From (251) we have

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1} = \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_1j_3j_3j_2j_1} + C_{j_1j_2j_3j_3j_1j_2}\Biggr) =$$

$$= \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2}C_{j_1j_3j_3j_2j_1} - C_{j_1j_2}C_{j_3j_3j_2j_1} + C_{j_3j_1j_2}C_{j_3j_2j_1} -$$

$$-C_{j_3j_3j_1j_2}C_{j_2j_1} + C_{j_2j_3j_3j_1j_2}C_{j_1}\Biggr) =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1} - C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr) +$$

$$+\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}. \tag{312}$$

The generalized Parseval equality gives (by analogy with (258))

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1} = 0. \tag{313}$$

Let us prove the following equality

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1} - C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr) = 0. \tag{314}$$

The relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2}C_{j_1} = 0 \tag{315}$$

is proved by the same methods as in the proof of equality (236) and also using Theorem 17 and (78).

Further, we have (see (78))

$$\sum_{j_3=0}^{p} C_{j_3j_3j_1j_2} = \frac{1}{2}C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)} - \sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}. \tag{316}$$

Moreover,

$$C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)} = \int_t^T\int_t^{t_3}\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3 =$$

$$= \int_t^T\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_2}(t_1)dt_1\int_{t_2}^{T}dt_3dt_2 = \int_t^T(T-t_2)\phi_{j_1}(t_2)\int_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2 =$$

$$= \int\limits_t^T \phi_{j_2}(t_1) \int\limits_{t_1}^T (T-t_2)\phi_{j_1}(t_2)dt_2dt_1 = \int\limits_t^T \phi_{j_2}(t_2) \int\limits_{t_2}^T (T-t_1)\phi_{j_1}(t_1)dt_1dt_2 =$$

$$= \int\limits_{[t,T]^2} (T-t_1)\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2 \stackrel{\text{def}}{=} \tilde{C}_{j_2j_1}. \qquad (317)$$

Using (316), (317), and the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1} = \frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1}\tilde{C}_{j_2j_1} -$$

$$-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1} = -\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1}. \qquad (318)$$

We have (see (278))

$$C_{j_3j_3j_1j_2} = \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_1)\int\limits_{t_1}^T \phi_{j_1}(t_2)\left(\int\limits_{t_2}^T \phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \qquad (319)$$

By analogy with (268) and also using (319), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \qquad (320)$$

Combining (318) and (320), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \qquad (321)$$

The relation (314) follows from (315) and (321). From (312)–(314) we get (311). The equalities (309)–(311) complete the proof of (246).

For the proof of (247)–(250) we will use a new idea. More precisely, we will consider the sums of expressions (247)–(250) with the expressions already studied throughout this proof.

Let us begin from (247). Applying the integration order replacement, we obtain

$$C_{j_3j_1j_2j_3j_2j_1} + C_{j_3j_1j_2j_3j_1j_2} =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\int\limits_t^{t_5} \phi_{j_2}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6 =$$

$$=\int_t^T \phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_3}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int_{t_3}^{t_5}\phi_{j_2}(t_4)dt_4dt_3dt_5dt_6 =$$

$$=\int_t^T \phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\left(\int_t^{t_5}\phi_{j_2}(t_4)dt_4\right)\int_t^{t_5}\phi_{j_3}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int_t^T \phi_{j_3}(t_6)\int_t^{t_6}\phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_3}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)^2\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6 =$$

$$=\int_t^T \phi_{j_1}(t_5)\left(\int_t^{t_5}\phi_{j_2}(t_4)dt_4\right)\int_t^{t_5}\phi_{j_3}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5-$$

$$-\int_t^T \phi_{j_1}(t_5)\int_t^{t_5}\phi_{j_3}(t_3)\left(\int_t^{t_3}\phi_{j_2}(t_2)dt_2\right)^2\left(\int_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5. \qquad (322)$$

Using (71), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr). \qquad (323)$$

Further, by analogy with the proof of equality (242) and using (322), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=0. \qquad (324)$$

From (323) and (324) we get

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Bigl(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\Bigr)=0. \qquad (325)$$

Moreover (see (236)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_1j_2}=0. \qquad (326)$$

Combining (325) and (326), we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0.$$

The equality (247) is proved.

Consider (248). Using the integration order replacement, we have

$$C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-$$

$$-\int\limits_t^T\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5. \qquad (327)$$

Using (71), we obtain

$$-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right)=$$

$$=\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\right). \qquad (328)$$

By analogy with the proof of (242) and applying (327), we get

$$
\lim_{p\to\infty}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Biggr)=0. \qquad (329)
$$

From (328) and (329) we have

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}\Biggr)=0. \qquad (330)
$$

Moreover (see (237)),

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_1j_2}=0. \qquad (331)
$$

Combining (330) and (331), we finally obtain

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0.
$$

The equality (248) is proved.

Now consider (249). Using the integration order replacement, we obtain

$$
C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-
$$

$$
-\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5-
$$

(332)
$$
-\int\limits_t^T \phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)dt_5.
$$

Applying (71) and (78), we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=
$$

$$
=-\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)-
$$

(333)
$$
-\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}.
$$

The equality

(334)
$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}\Biggl|_{(j_2j_2)\curvearrowright(\cdot)}=0
$$

follows from the equality (147), where we proceed similarly to the proof of equality (293) (see (294)).

By analogy with the proof of (242) and applying (332), we get

(335)
$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0.
$$

From (333)–(335) we have

(336)
$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0.
$$

Moreover (see (238)),

(337)
$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_1j_2}=0.
$$

Combining (336) and (337), we finally obtain

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0.
$$

The equality (249) is proved.

Finally consider (250). Using the integration order replacement, we have

$$C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5-$$

(338)

$$-\int\limits_t^T\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3\left(\int\limits_{t_5}^{T}\phi_{j_2}(t_6)dt_6\right)dt_5.$$

Using (71) and (78), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)-$$

$$-\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=$$

$$=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}\right)+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)-$$

$$-\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}. \tag{339}$$

The equalities

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Bigg|_{(j_3j_3)\curvearrowright(\cdot)}\right)=0, \tag{340}$$

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}=$$

$$=\lim_{p\to\infty}\frac{1}{4}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)(j_3j_3)\curvearrowright(\cdot)}-$$

$$-\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Bigg|_{(j_1j_1)\curvearrowright(\cdot)}=0 \tag{341}$$

follows from the equalities (147), (148), where we used the same technique as in (294). When proving (341), we also applied (78) and (103).

By analogy with the proof of (242) and applying (338), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=0. \tag{342}$$

From (339)–(342) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\right)=0. \tag{343}$$

Furthermore (see (240)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}=0. \tag{344}$$

Combining (343) and (344), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0.$$

The equality (250) is proved. Theorem 23 is proved.

14. Generalization of Theorem 16. The Case p_1, p_2, $p_3 \to \infty$ and Continuously Differetiable Weight Functions (The Cases of Legendre Polynomials and Trigonometric Functions). Proof of Hypothesis 3 for the Case $k = 3$

This section is devoted to the following theorem.

Theorem 24 [15], [47], [55]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)}$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{345}$$

that converges in the mean-square sense is valid, where $i_1, i_2, i_3 = 0, 1, \ldots, m$,

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$.

Proof. Let us consider the case of Legendre polynomials (the trigonometric case is simpler and can be considered similarly). Applying (65), we obtain

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} = J'[K_{p_1 p_2 p_3}]_{T,t}^{(i_1 i_2 i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}} \sum_{j_3=0}^{p_3} \sum_{j_1=0}^{\min\{p_1,p_2\}} C_{j_3 j_1 j_1} J'[\phi_{j_3}]_{T,t}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}} \sum_{j_1=0}^{p_1} \sum_{j_3=0}^{\min\{p_2,p_3\}} C_{j_3 j_3 j_1} J'[\phi_{j_1}]_{T,t}^{(i_1)} +$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}} C_{j_1j_2j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)} \tag{346}$$

w. p. 1, where notations are the same as in (65).

Using Theorem 5 (see (27) for the case $k=3$), Theorem 3 (see (48)) as well as (83) (see the derivation of (83)) and (78), we get

$$J^*[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} = J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T \psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2 d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3 =$$

$$= J[\psi^{(3)}]_{T,t}^{(i_1i_2i_3)} + \frac{1}{2}J[\psi^{(3)}]_{T,t}^{1} + \frac{1}{2}J[\psi^{(3)}]_{T,t}^{2} =$$

$$= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Bigg|_{(j_2j_1)\curvearrowright(\cdot),\, j_1=j_2} J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\ \frac{1}{2}\sum_{j_1=0}^{p_1} C_{j_3j_2j_1}\Bigg|_{(j_3j_2)\curvearrowright(\cdot),\, j_2=j_3} J'[\phi_{j_1}]_{T,t}^{(i_1)} =$$

$$= \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ J'[K_{p_1p_2p_3}]_{T,t}^{(i_1i_2i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty} C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)} \tag{347}$$

w. p. 1.

Using (346), (347) and the elementary inequality

$$(a+b+c+d)^2 \le 4\left(a^2+b^2+c^2+d^2\right),$$

we obtain

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le
$$

$$
\le 4\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}-J'[K_{p_1 p_2 p_3}]_{T,t}^{(i_1 i_2 i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3 j_1 j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3 j_1 j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}\times
$$

$$
\times\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3 j_3 j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3 j_3 j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}+
$$

$$
+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathsf{M}\left\{\left(\sum_{j_2=0}^{p_2}\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1 j_2 j_1}J'[\phi_{j_2}]_{T,t}^{(i_2)}\right)^2\right\}=
$$

$$
=4A_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_2\ne 0\}}B_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_2=i_3\ne 0\}}C_{p_1p_2p_3}+4\cdot\mathbf{1}_{\{i_1=i_3\ne 0\}}D_{p_1p_2p_3}. \tag{348}
$$

Theorem 3 gives (see (48))

$$
\lim_{p_1,p_2,p_3\to\infty}A_{p_1p_2p_3}=0. \tag{349}
$$

Further, in complete analogy with (139) and using (71), we obtain

$$
D_{p_1p_2p_3}=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=0}^{\min\{p_1,p_3\}}C_{j_1 j_2 j_1}\right)^2=\sum_{j_2=0}^{p_2}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1 j_2 j_1}\right)^2\le
$$

$$
\le\sum_{j_2=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_3\}+1}^{\infty}C_{j_1 j_2 j_1}\right)^2\le\frac{K}{\left(\min\{p_1,p_3\}\right)^{2-\varepsilon}}\ \to\ 0 \tag{350}
$$

if $p_1,p_2,p_3\to\infty$, where ε is an arbitrary small positive real number, constant K is independent of p.

We have

$$B_{p_1p_2p_3}=\mathsf{M}\left\{\left(\left(\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)\right)^2\right\}\le$$

(351)
$$\le 2E_{p_3}+2F_{p_1p_2p_3},$$

where

$$E_{p_3}=\mathsf{M}\left\{\left(\underset{p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\},$$

$$F_{p_1p_2p_3}=\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}-\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{\min\{p_1,p_2\}}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}J'[\phi_{j_3}]_{T,t}^{(i_3)}\right)^2\right\}=$$

(352)
$$=\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2.$$

By analogy with (125) we get

$$\sum_{j_3=0}^{p_3}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2\le\sum_{j_3=0}^{\infty}\left(\sum_{j_1=\min\{p_1,p_2\}+1}^{\infty}C_{j_3j_1j_1}\right)^2\le$$

(353)
$$\le\frac{K}{\left(\min\{p_1,p_2\}\right)^2}\ \to\ 0$$

if $p_1,p_2,p_3\to\infty$, where constant K does not depend on p.

Moreover,

(354)
$$\lim_{p_3\to\infty}E_{p_3}=\lim_{p_1,p_2,p_3\to\infty}E_{p_3}=0.$$

Combining (351)–(354), we obtain

(355)
$$\lim_{p_1,p_2,p_3\to\infty}B_{p_1p_2p_3}=0.$$

Consider $C_{p_1p_2p_3}$. We have

$$C_{p_1p_2p_3}=\mathsf{M}\left\{\left(\left(\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)+\right.\right.$$

$$\left.\left.+\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)\right)^2\right\}\le$$

$$\le 2G_{p_1}+2H_{p_1p_2p_3}, \tag{356}$$

where

$$G_{p_1}=\mathsf{M}\left\{\left(\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\},$$

$$H_{p_1p_2p_3}=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}-\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{\min\{p_2,p_3\}}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}J'[\phi_{j_1}]_{T,t}^{(i_1)}\right)^2\right\}=$$

$$=\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2. \tag{357}$$

By analogy with (129) we get

$$\sum_{j_1=0}^{p_1}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le\sum_{j_1=0}^{\infty}\left(\sum_{j_3=\min\{p_2,p_3\}+1}^{\infty}C_{j_3j_3j_1}\right)^2\le$$

$$\le\frac{K}{\left(\min\{p_2,p_3\}\right)^2}\ \to\ 0 \tag{358}$$

if $p_1,p_2,p_3\to\infty$, where constant K does not depend on p.

Moreover,

$$\lim_{p_1\to\infty}G_{p_1}=\lim_{p_1,p_2,p_3\to\infty}G_{p_1}=0. \tag{359}$$

Combining (356)–(359), we obtain

$$\lim_{p_1,p_2,p_3\to\infty} C_{p_1p_2p_3} = 0. \tag{360}$$

The relations (348)–(350), (355), (360) complete the proof of Theorem 24. Theorem 24 is proved.

15. Hypotheses 1–3 from the Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_\tau$. Let $\mathbf{f}_\tau^{(i)p}$ $(p\in\mathbb{N})$ be some approximation of $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_\tau^{(i)p}$ converges to $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_\tau^{(i)}$ by $\mathbf{f}_\tau^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_\tau^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_\tau$?

The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [69], [70] (also see [71]-[79]), it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to the iterated Ito stochastic integrals and solutions of Ito SDEs.

The piecewise linear approximation as well as the regularization by convolution [69]-[79]) relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [80], [81]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{361}$$

where $\tau\in[t,T]$, $t\geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (361) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{362}$$

From (362) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{363}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{364}$$

where $p_1,\ldots,p_k\in\mathbb{N}$, $i_1,\ldots,i_k=0,1,\ldots,m$ and

$$d\mathbf{w}_{\tau}^{(i)p}=\begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for}\quad i=1,2,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases},\qquad p\in\mathbb{N}. \tag{365}$$

Let us substitute (363) into (364)

$$\int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^k \zeta_{j_l}^{(i_l)}, \tag{366}$$

where $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j (if $i\ne 0$),

$$C_{j_k\ldots j_1}=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [69], [70], [74] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [74] (see Definition 7.1 on Pages 480–481). At least the proof of an analogue of Theorem 7.2 (see [74], Page 497) for approximations of the Wiener process based on its series expansion (361) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (366) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [69], [70] (also see [74], Theorems 7.1, 7.2).

From the other hand, Theorems 3, 4, 6–12, 16–18, 23, 24 from this paper (see proofs of Theorems 3, 4, 6–12 in Chapters 1 and 2 of the monographs [15]-[18]) can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 (or of multiplicity k under the condition of convergence of trace series (Theorem 13)) based on the approximation (362) of the Wiener process in the form of its series expansion. At that, the Riemann–Stieltjes integrals (364) converge (according to Theorems 6–13, 16–18, 23, 24) to the mentioned Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (361), (362), and Theorems 6–12, 16–18, 23, 24) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$, $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [69], [70], [74]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)} = \mathbf{f}_{(k+1)\Delta}^{(i)} - \mathbf{f}_{k\Delta}^{(i)}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, 2, \ldots, N-1.$$

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T \int_0^s d\mathbf{b}_\Delta^{(i_1)}(\tau) d\mathbf{b}_\Delta^{(i_2)}(s), \quad i_1, i_2 = 1, \ldots, m.$$

We can write w. p. 1

$$\int_0^T \int_0^s d\mathbf{b}_\Delta^{(i_1)}(\tau) d\mathbf{b}_\Delta^{(i_2)}(s) = \int_0^T \int_0^s \frac{\partial \mathbf{b}_\Delta^{(i_1)}}{\partial \tau}(\tau) d\tau \frac{\partial \mathbf{b}_\Delta^{(i_2)}}{\partial s}(s) ds =$$

$$= \sum_{l=0}^{N-1} \int_{l\Delta}^{(l+1)\Delta} \left(\sum_{q=0}^{l-1} \int_{q\Delta}^{(q+1)\Delta} \frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta} d\tau + \int_{l\Delta}^{s} \frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta} d\tau \right) \frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta} ds =$$

$$= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta\mathbf{f}_{q\Delta}^{(i_1)} \Delta\mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{\Delta^2} \sum_{l=0}^{N-1} \Delta\mathbf{f}_{l\Delta}^{(i_1)} \Delta\mathbf{f}_{l\Delta}^{(i_2)} \int_{l\Delta}^{(l+1)\Delta} \int_{l\Delta}^{s} d\tau ds =$$

$$= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta\mathbf{f}_{q\Delta}^{(i_1)} \Delta\mathbf{f}_{l\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta\mathbf{f}_{l\Delta}^{(i_1)} \Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{367}$$

Using (367), it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}} \int_0^T \int_0^s d\mathbf{b}_\Delta^{(i_1)}(\tau) d\mathbf{b}_\Delta^{(i_2)}(s) = \int_0^T \int_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \frac{1}{2} \mathbf{1}_{\{i_1 = i_2\}} \int_0^T ds =$$

$$= \int_0^{*T} \int_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}, \tag{368}$$

where $\Delta \to 0$ if $N \to \infty$ ($N\Delta = T$).

Obviously, (368) agrees with Theorem 7.1 (see [74], Page 486).

The next example relates to the approximation (362) of the Wiener process based on its series expansion (361), where $t = 0$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0, T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T \int_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p}, \quad i_1, i_2 = 1, 2, \ldots, m, \tag{369}$$

where $d\mathbf{f}_\tau^{(i)p}$ is defined by the relation (363).

Let us substitute (363) into (369)

$$\int_0^T\int_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{370}$$

where

$$C_{j_2j_1}=\int_0^T\phi_{j_2}(s)\int_0^s\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient; another notations are the same as in (366).

As we noted above, approximations of the Wiener process that are similar to (362) were not considered in [69], [70] (also see Theorems 7.1, 7.2 in [74]). Furthermore, transferring of the results of Theorems 7.1 and 7.2 [74] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [15]-[18]. More precisely, using Theorems 6 and 7, we obtain from (370) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}}\int_0^T\int_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\int_0^{*T}\int_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{371}$$

From the other hand, by Theorems 3, 4 (see (17)) for the case $k=2$ we obtain from (370) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}}\int_0^T\int_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int_0^T\int_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{372}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int_0^T ds,$$

then from Theorem 5 ($k=2$) and (372) we obtain (371).

16. Wong–Zakai Type Theorems for Iterated Stratonovich Stochastic Integrals. The Case of Approximation of the Multidimensional Wiener Process Based on its Series Expansion Using Legendre Polynomials and Trigonometric Functions

As we mentioned above, there exists a lot of publications on the subject of Wong–Zakai approximation of stochastic integrals and SDEs [69]-[79]. However, these works did not consider the approximation of iterated stochastic integrals and systems of SDEs for the case of approximation of the multidimensional Wiener process based on its series expansions. Usually, as an approximation of the Wiener process in the theorems of the Wong–Zakai type, the authors [69]-[79] choose a piecewise linear approximation or an approximation based on the regularization by convolution.

The Wong–Zakai approximation is widely used to approximate stochastic integrals and SDEs. In particular, the Wong–Zakai approximation can be used to approximate the iterated Stratonovich stochastic integrals in the context of numerical integration of Ito SDEs in the framework of the approach based on the Taylor–Stratonovich expansion [2]-[62], [67], [68]. It should be noted that the authors of the works [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [67] (pp. 438-439), [68] (pp. 263-264) mention the Wong–Zakai approximation [69]-[71], [74] within the frames of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process. However, in these works there is no rigorous proof of convergence for approximations of the mentioned stochastic integrals of miltiplicity 3 and higher.

From the other hand, the theory constructed in Chapters 1 and 2 of the monographs [15]-[18] can be considered as the proof of the Wong–Zakai approximation for iterated Stratonovich stochastic integrals of multiplicities 1 to 6 based on the Wiener process series expansion using Legendre polynomials and trigonometric functions.

The subject of this section is to reformulate the results of Chapter 2 of the monograph [15] (also see [16]–[18]) in the form of theorems on convergence of iterated Riemann–Stiltjes integrals to iterated Stratonovich stochastic integrals.

Let us reformulate Theorems 1, 2, 7–13, 16–18, 23, 24 of this paper as theorems on the convergence of iterated Riemann–Stiltjes integrals (364) to the iterated Stratonovich stochastic integrals (3).

Theorem 25. *Suppose that the following conditions are fulfilled*:

1. *Every* $\psi_l(\tau)$ $(l=1,2)$ *is a continuously differentiable function at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, for the iterated Stratonovich stochastic integral of second multiplicity

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\quad (i_1,i_2=0,1,\ldots,m)$$

the following equality

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}d\mathbf{w}_{t_2}^{(i_2)p_2}\tag{373}$$

is valid, where here and further l.i.m. *is a limit in the mean-square sense.*

Theorem 26. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following formula

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \int\limits_t^{T}\int\limits_t^{t_3}\int\limits_t^{t_2} d{\bf f}_{t_1}^{(i_1)p_1}d{\bf f}_{t_2}^{(i_2)p_2}d{\bf f}_{t_3}^{(i_3)p_3}$$

is valid.

Theorem 27. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\int\limits_t^{*T}(t-t_3)^{l_3}\int\limits_t^{*t_3}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following equality

$$I_{l_1l_2l_3T,t}^{*(i_1i_2i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\int\limits_t^{T}(t-t_3)^{l_3}\int\limits_t^{t_3}(t-t_2)^{l_2}\int\limits_t^{t_2}(t-t_1)^{l_1}d{\bf f}_{t_1}^{(i_1)p_1}d{\bf f}_{t_2}^{(i_2)p_2}d{\bf f}_{t_3}^{(i_3)p_3}$$

holds for each of the following cases

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
2. $i_1=i_2\ne i_3$ *and* $l_1=l_2\ne l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
3. $i_1\ne i_2=i_3$ *and* $l_1\ne l_2=l_3$ *and* $l_1,l_2,l_3=0,1,2,\ldots$
4. $i_1,i_2,i_3=1,\ldots,m;\ l_1=l_2=l_3=l$ *and* $l=0,1,2,\ldots$

Theorem 28. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_l(\tau)$ $(l=1,2,3)$ are continuously differentiable functions at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following formula

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d{\bf f}_{t_1}^{(i_1)p}d{\bf f}_{t_2}^{(i_2)p}d{\bf f}_{t_3}^{(i_3)p}$$

holds for each of the following cases:

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$,
2. $i_1=i_2\ne i_3$ *and* $\psi_1(\tau)\equiv\psi_2(\tau)$,
3. $i_1\ne i_2=i_3$ *and* $\psi_2(\tau)\equiv\psi_3(\tau)$,
4. $i_1,i_2,i_3=1,\ldots,m$ *and* $\psi_1(\tau)\equiv\psi_2(\tau)\equiv\psi_3(\tau)$.

Theorem 29. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space $L_2([t,T])$. Furthermore, let the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the functions $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following formula

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)p}d\mathbf{f}_{t_2}^{(i_2)p}d\mathbf{f}_{t_3}^{(i_3)p}$$

is valid.

Theorem 30. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}\quad (i_1,i_2,i_3,i_4=0,1,\ldots,m)$$

the following equality

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}$$

holds, where $\mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes and $\mathbf{w}_{\tau}^{(0)}=\tau$.

Theorem 31. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)$ are twice continuously differentiable functions at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (i_1,\ldots,i_k=0,1,\ldots,m)$$

the following relation

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}=0$$

is valid, where

$$J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k},$$

$n\in\mathbb{N}$, and $\overline{\lim}$ means $\limsup$.

Theorem 32. *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)$ are twice continuously differentiable functions at the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (i_1,\ldots,i_k=0,1,\ldots,m)$$

the following equality

$$\lim_{p_k\to\infty}\overline{\lim_{p_{k-1}\to\infty}}\ldots\overline{\lim_{p_1\to\infty}}\,\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}=0$$

is valid, where

$$J^*[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k},$$

$n\in\mathbb{N}$, *and* $\overline{\lim}$ *means* $\limsup$.

Theorem 33. *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^{s}\Phi_1(t_1)\Phi_2(t_1)dt_1=\sum_{j_1=0}^{\infty}\int\limits_t^{s}\Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \qquad (374)$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (374) converges absolutely.

2. *The estimates*

$$\left|\int\limits_t^{s}\phi_j(\tau)\Phi_1(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\qquad \left|\int\limits_s^{T}\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^{s}\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s\in(t,T)$ and for some $\alpha,\beta>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and

$$\int\limits_t^{T}\Psi_1^2(\tau)d\tau<\infty,\qquad \int\limits_t^{T}\left|\Psi_2(\tau)\right|d\tau<\infty.$$

3. *The condition*

$$
\lim_{p\to\infty} \sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p} \left(S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1} \Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} \right)^2 = 0
$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (34)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$
S_{l_1} S_{l_2} \ldots S_{l_d} \left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1} \Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \right\} \stackrel{\mathsf{def}}{=} \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1} \Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}
$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$
J^*[\psi^{(k)}]^{(i_1\ldots i_k)}_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k},
$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, *the following formula*

$$
J^*[\psi^{(k)}]^{(i_1\ldots i_k)}_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)p}_{t_1} \ldots d\mathbf{w}^{(i_k)p}_{t_k}
$$

is valid.

Theorem 34. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$
J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} d\mathbf{w}^{(i_2)}_{t_2} d\mathbf{w}^{(i_3)}_{t_3},
$$

where $i_1, i_3, i_3 = 0, 1, \ldots, m$, *the following equality*

$$
J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)p}_{t_1} d\mathbf{w}^{(i_2)p}_{t_2} d\mathbf{w}^{(i_3)p}_{t_3}
$$

holds.

Theorem 35. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} d\mathbf{w}^{(i_2)}_{t_2} d\mathbf{w}^{(i_3)}_{t_3} d\mathbf{w}^{(i_4)}_{t_4},
$$

where $i_1,\ldots,i_4=0,1,\ldots,m$, *the following relation*

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^T\psi_4(t_4)\int\limits_t^{t_4}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}$$

is valid.

Theorem 36. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)},$$

where $i_1,\ldots,i_5=0,1,\ldots,m$, *the following equality*

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^T\psi_5(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p}\ldots d\mathbf{w}_{t_5}^{(i_5)p}$$

holds.

Theorem 37. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\int\limits_t^{*t_6}\int\limits_t^{*t_5}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)}d\mathbf{w}_{t_6}^{(i_6)}$$

where $i_1,\ldots,i_6=0,1,\ldots,m$, *the following formula*

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\int\limits_t^T\int\limits_t^{t_6}\int\limits_t^{t_5}\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)p}d\mathbf{w}_{t_2}^{(i_2)p}d\mathbf{w}_{t_3}^{(i_3)p}d\mathbf{w}_{t_4}^{(i_4)p}d\mathbf{w}_{t_5}^{(i_5)p}d\mathbf{w}_{t_6}^{(i_6)p}$$

is valid.

Theorem 38. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(s),\psi_2(s),\psi_3(s)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)},$$

where $i_1,i_2,i_3=0,1,\ldots,m$, *the following formula*

$$J^*[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \int\limits_t^T \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} d\mathbf{w}_{t_2}^{(i_2)p_2} d\mathbf{w}_{t_3}^{(i_3)p_3}$$

is valid.

Let us reformulate Hypotheses 1–3 in terms of the convergence of iterated Riemann–Stiltjes integrals to iterated Stratonovich stochastic integrals.

Hypothesis 4. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of kth multiplicity*

$$\int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1,\ldots,i_k = 0,1,\ldots,m)$$

the following formula

$$\int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \ldots \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)p} \ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

is valid, where l.i.m. *is a limit in the mean-square sense, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i = 1,\ldots,m)$ and $\mathbf{w}_\tau^{(0)} = \tau$.*

Hypothesis 5. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral (3) of kth multiplicity*

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1,\ldots,i_k = 0,1,\ldots,m)$$

the following relation

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \int\limits_t^{T} \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p} \ldots d\mathbf{w}_{t_k}^{(i_k)p}$$

holds, where l.i.m. *is a limit in the mean-square sense, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ are independent standard Wiener processes $(i = 1,2,\ldots,m)$ and $\mathbf{w}_\tau^{(0)} = \tau$.*

Hypothesis 6. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Moreover, every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is an enough smooth nonrandom function on $[t,T]$. Then, for the iterated Stratonovich stochastic integral (3) of kth multiplicity*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad(i_1,\ldots,i_k=0,1,\ldots,m)$$

the following equality

$$J^*[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}$$

holds, where l.i.m. *is a limit in the mean-square sense,* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i=1,2,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

17. Generalization of Theorem 13 for Complete Orthonormal Systems of Functions in $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$ such that the Condition (376) is Satisfied

First, note that (see the proof of Thorem 13 and (92))

$$\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ \ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\ \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ \ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}J'[\phi_{j_{q_1}}\ \ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\frac{1}{2^r}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)\times$$

$$\times\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}+$$

$$+\frac{1}{2^r}\prod_{s=1}^{r}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\quad\text{w. p. 1.}\tag{375}$$

Using (375) and the condition (98), we obtain (93). This means that we get (95). Thus the expansion (44) is proved.

Analyzing the proof of Theorems 13 and conditions of Theorem 5 as well as taking into account the above arguments, it is easy to see that the following theorem is true.

Theorem 39 [15], [55]. *Assume that the continuous functions $\psi_1(\tau),\ldots,\psi_k(\tau)$ at the interval $[t,T]$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of functions ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ are such that the following condition*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\ j_{g_2},\ldots,j_{g_{2r-1}}=\ j_{g_{2r}}}\Biggr)^2=0\tag{376}$$

is satisfied for all $r=1,2,\ldots,[k/2]$. Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

Further in this section, we generalize Theorems 13, 39 to the case of complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$ such that the condition (376) is satisfied.

Let $(\Omega,\mathrm{F},\mathsf{P})$ be a complete probability space and let $f(t,\omega)\stackrel{\mathsf{def}}{=}f_t:[0,T]\times\Omega\to\mathbb{R}$ be the standard Wiener process defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega,\mathrm{F},\mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s\subset\mathrm{F}_t\subset\mathrm{F}$ for $s<t$.
2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.
3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0$, $\Delta>0$ is independent with the events of σ-algebra F_t.

Let $\xi(\tau,\omega)\stackrel{\mathsf{def}}{=}\xi_\tau:[0,T]\times\Omega\to\mathbb{R}$ be some random process, which is measurable with respect to the pair of variables (τ,ω) and satisfies to the following condition

$$\int\limits_t^T|\xi_\tau|d\tau<\infty\quad\text{w. p. 1}\quad(t\ge 0).$$

Let $\tau_j^{(N)}$, $j=0,1,\ldots,N$ be a partition of the interval $[t,T]$, $t\ge 0$ such that

$$t=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad\max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty.\tag{377}$$

Further, for simplicity, we write τ_j instead of $\tau_j^{(N)}$.

Consider the definition of the Stratonovich stochastic integral, which differs from the definition given in [2] (recall that we use definition [2] above in this article).

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\frac{1}{\tau_{j+1}-\tau_j}\int\limits_{\tau_j}^{\tau_{j+1}}\xi_s ds\ \left(f_{\tau_{j+1}}-f_{\tau_j}\right)\stackrel{\sf def}{=}\int\limits_t^T\xi_\tau\circ df_\tau \tag{378}$$

is called [90] the Stratonovich stochastic integral of the process ξ_τ, $\tau\in[t,T]$, where τ_j, $j=0,1,\ldots,N$ is a partition of the interval $[t,T]$ satisfying the condition (377).

We also denote by

$$\int\limits_t^\tau\xi_s\circ df_s$$

the Stratonovich stochastic integral like (378) (if it exists) of $\xi_s\mathbf{1}_{\{s\in[t,\tau]\}}$ for $\tau\in[t,T]$, $t\ge 0$.

It is known [90] (Lemma A.2) that the following iterated Stratonovich stochastic integral

$$J^S[\psi^{(k)}]_{\tau,t}^{(i_1\ldots i_k)}=\int\limits_t^\tau\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{w}_{t_1}^{(i_1)}\ldots\circ d\mathbf{w}_{t_k}^{(i_k)} \tag{379}$$

exists for the case $i_1=\ldots=i_k\ne 0$, where $\tau\in[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=0,1,\ldots,m$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$, $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

In [91] (2021) an analogue of Theorem 5 (1997) is proved for the case $i_1=\ldots=i_k\ne 0$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Let us denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}\stackrel{\sf def}{=}\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}, \tag{380}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ $(l=2,3,\ldots,k)$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (383), $\sum\limits_{\emptyset}$ is supposed to be equal to zero; another notations are the same as in Theorem 5.

Further, by analogy with (56), (59) and using the version of (53) for the case of an arbitrary complete orthonormal system $\{\phi_j(x)\}_{j=0}^\infty$ in $L_2([t,T])$ (see [15] or [18], Sect. 1.11) instead of (53), we obtain the following generalization of (56) to the case of an arbitrary complete orthonormal system $\{\phi_j(x)\}_{j=0}^\infty$ in $L_2([t,T])$ and $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$

$$\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\times$$

$$\times\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}\quad\text{w. p. 1}, \tag{381}$$

where $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$, $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ are multiple Wiener stochastic integrals defined as in [85] (1951). Note that in [85] the case of a scalar Wiener process has been considered.

It should be noted that Theorem 1.16 [15] (Sect. 1.11) and Theorem 4 can be reformulated as follows (also see [42], Sect. 15)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1,} \tag{382}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system in $L_2([t,T])$, $\psi_1(\tau),\ \ldots,\ \psi_k(\tau)\in L_2([t,T])$, $J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is the multiple Wiener stochastic integral defined as in [85] (1951) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}; \tag{383}$$

another notations are the same as in Theorem 4.

Passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in (381) and using the equality (382), we get w. p. 1

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{r=1}^{[k/2]}\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{s=1}^{r}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}, \tag{384}$$

where $J'[\phi_{j_{q_1}}\ldots\phi_{j_{q_{k-2r}}}]_{T,t}^{(i_{q_1}\ldots i_{q_{k-2r}})}$ is the multiple Wiener stochastic integral defined as in [85] (1951) and $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is the iterated Ito stochastic integral (383).

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$. Then we have

$$\sum_{j=0}^{\infty}\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le$$

$$\le\frac{1}{2}\sum_{j=0}^{\infty}\left(\left(\int\limits_t^T\mathbf{1}_{\{\tau<s\}}\phi_j(\tau)\Phi_1(\tau)d\tau\right)^2+\left(\int\limits_t^T\mathbf{1}_{\{\tau>s\}}\phi_j(\tau)\Phi_2(\tau)d\tau\right)^2\right)=$$

$$=\frac{1}{2}\left(\int\limits_t^s\Phi_1^2(\tau)d\tau+\int\limits_s^T\Phi_2^2(\tau)d\tau\right)\le\frac{1}{2}\left(\|\Phi_1\|_{L_2([t,T])}^2+\|\Phi_2\|_{L_2([t,T])}^2\right)<\infty, \tag{385}$$

i.e. the series

$$\sum_{j=0}^{\infty}\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau$$

converges absolutely.

By interpreting the integrals in (72)–(75) as Lebesgue integrals, using Fubini's theorem in (72) and Lebesgue's Dominated Convergence Theorem in (74), we obtain (70) (see (75), (385)) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Using the equality (104) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\psi_2(\tau)\in L_2([t,T])$ as well as absolute convergence of the series on the right-hand side of (104) for this case (see [15], Sect. 2.1.4 or [94]), we obtain [92] (Sect. 3.5.2, Theorem 3.5.2) the generalization of (78) for the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Repeating the steps of the proof of Theorem 13 below the formula (79) using (380), (384) or steps of the proof of Theorem 39 using (380), (384), we obtain for complete orthonormal systems $\{\phi_j(x)\}_{j=0}^{\infty}$ ($\phi_0(x)=1/\sqrt{T-t}$) in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ ($l=2,3,\ldots,k$) (for which the condition (376) is satisfied) the following equality

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}=$$

$$=J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}}J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}=\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\tag{386}$$

w. p. 1, where notations in (386) are the same as in Theorem 5 and $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (380).

Thus the following two theorems are proved.

Theorem 40 [15], [46], [55]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ ($\phi_0(x)=1/\sqrt{T-t}$) *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau)\in L_2([t,T])$ ($l=2,3,\ldots,k$) *are such that the folowing condition*

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_q=0}^{p_q}\ldots\sum_{j_k=0}^{p_k}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\times$$

$$\times\Biggl(\sum_{j_{g_1}=0}^{\min\{p_{g_1},p_{g_2}\}}\ldots\sum_{j_{g_{2r-1}}=0}^{\min\{p_{g_{2r-1}},p_{g_{2r}}\}}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowleft(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowleft(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\Biggr)^2=0\tag{387}$$

is satisfied for all $r=1,2,\ldots,[k/2]$. *Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Ito stochastic integrals defined by* (380) *the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{388}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*),* $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Theorem 41 [15], [55]. *Assume that the complete orthonormal system* $\{\phi_j(x)\}_{j=0}^{\infty}$ $(\phi_0(x) = 1/\sqrt{T-t})$ *in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $\psi_l(\tau)\psi_{l-1}(\tau) \in L_2([t,T])$ $(l = 2,3,\ldots,k)$ *are such that the condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see* (34)*) and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d \in \{1,2,\ldots,r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0,1,2,\ldots,r-1$, *where* $r = 1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\text{def}}{=} \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Bigg|_{q\neq g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$.

Then, for the sum $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Ito stochastic integrals defined by* (380) *the following expansion*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \ne 0$), $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ *for* $i = 1, \ldots, m$ *and* $\mathbf{w}_\tau^{(0)} = \tau$.

Note that in Theorems 40, 41 (the case $k = 2$) the condition $\psi_1(\tau)\psi_2(\tau) \in L_2([t, T])$ can be omitted.

Using Theorem 5 together with Proposition 3.1 [91] and the proof of Lemma A.2 [90], we can write $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = J^S[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)}$ w. p. 1 and reformulate Theorems 40, 41 for $J^S[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)}$ ($J^S[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)}$ is defined by (379)).

Let us consider the special case $k = 2$ of Theorem 40 in more detail. In this case, the condition (387) takes the following form (compare with (102))

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \int_t^T \psi_1(t_1)\psi_2(t_1) dt_1. \tag{389}$$

As follows from [15] (Sect. 2.1.4), the equality (389) is valid for the case of an arbitrary complete orthonormal system of functions in $L_2([t, T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$.

From Proposition 3.1 [91] for the case $k = 2$ we obtain

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)} +$$

$$+\frac{1}{2} \int_t^T \psi_1(t_1)\psi_2(t_1) dt_1 \tag{390}$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$, $i = 1, \ldots, m$,

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(i)}$$

is defined by (378), (379) and

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(i)}$$

is the iterated Ito stochastic integral of the form (2) ($k = 2$).

On the other hand, it is not difficult to show that

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i)} \circ d\mathbf{w}_{t_2}^{(j)} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i)} d\mathbf{w}_{t_2}^{(j)} \tag{391}$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$, $i \ne j$ ($i, j = 1, \ldots, m$), another notations are the same as in (390).

Combining (390) and (391), we get (see (380))

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i_1)} \circ d\mathbf{w}_{t_2}^{(i_2)} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} +$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int_t^T \psi_1(t_1)\psi_2(t_1)dt_1 \stackrel{\text{def}}{=} \bar{J}^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} \tag{392}$$

w. p. 1, where $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, $i_1, i_2 = 1, \ldots, m.$

It is easy to see that the condition $\phi_0(x) = 1/\sqrt{T-t}$ can be omitted in Theorems 40, 41 for the case $k = 2$ (see the proof of Theorem 13).

Summing up the above arguments, we obtain the following generalization of Theorem 7 to the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$.

Theorem 42 [15]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral*

$$J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{f}_{t_1}^{(i_1)} \circ d\mathbf{f}_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)$$

the following expansion

$$J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \tag{393}$$

that converges in the mean-square sence is valid, where the notations are the same as in Theorems 6, 7 *and $J^S[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ is defined by* (379).

In this section, it is also appropriate to mention the so-called multiple Stratonovich stochastic integral [90] (also see [86]).

The mean-square limit (if it exists)

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1=0}^{N-1} \ldots \sum_{l_k=0}^{N-1} \frac{1}{\Delta\tau_{l_1} \ldots \Delta\tau_{l_k}} \int\limits_{[\tau_{l_1},\tau_{l_1+1}]\times\ldots\times[\tau_{l_k},\tau_{l_k+1}]} K(t_1,\ldots,t_k)dt_1\ldots dt_k \ \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \bar{J}^S[K]_{T,t}^{(i_1\ldots i_k)} \tag{394}$$

is called [90] the multiple Stratonovich stochastic integral of the function $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$, where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\Delta\tau_j = \tau_{j+1} - \tau_j$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$ satisfying the condition (377), $i_1,\ldots,i_k = 0,1,\ldots,m$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $\mathbf{f}_\tau^{(i)}$ $(i = 1,\ldots,m)$ are independent standard Wiener processes defined as above in this section.

Note that in [90] the case $i_1 = \ldots = i_k \neq 0$ was considered. We also denote by $\bar{J}^S[K]_{s,t}^{(i_1\ldots i_k)}$ the multiple Stratonovich stochastic integral (394) (if it exists) of the function $K(t_1,\ldots,t_k)\mathbf{1}_{\{(t_1,\ldots,t_k)\in[t,s]^k\}}$, where $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$, $s \in [t,T]$, $t \geq 0$.

Let the function $K(t_1,\ldots,t_k)$ be chosen as follows

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1\le\ldots\le t_k\\ \\ 0, & \text{otherwise}\end{cases}, \tag{395}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

We will denote the multiple Stratonovich stochastic integral (394) of the function (395) as follows $\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$. It is known [90] (Lemma A.2) that the Stratonovich stochastic integrals $J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ and $\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ exist for the case $i_1=\ldots=i_k\ne 0$. Moreover,

$$J^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\bar{J}^S[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}$$

for this case [90] (Lemma A.2).

Recall that an expansion similar to (44) was obtained in [87] for the multiple Stratonovich stochastic integral (394) under the condition of convergence of trace series.

Recently, another approach to the expansion of integral (394) has been proposed (assuming that the integral (394) exists), where multiple Fourier–Walsh and Fourier–Haar series ($k\in\mathbb{N}$) have been applied [96]. The convergence was proved with respect to the special subsequence ($p_1=\ldots=p_k=p=2^m$, $m\to\infty$ in a formula similar to (388) [96]).

18. Modification of Condition 3 of Theorem 13 Using Parseval's Equality

Let us make some remarks about the development of the approach based on Theorem 13 and describe the algorithm of the verification of Condition 3 of Theorem 13. First, consider the case $k=2n+1$, $n=3,4,\ldots$ (k is the multiplicity of the iterated Stratonovich stochastic integral (43)). Let Conditions 1 and 2 of Theorem 13 be satisfied. Consider the equality (97). The right-hand side of (97) has the form

$$\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-$$

$$-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}.$$

Iterated application of the formulas (185), (186), (199) separately to the values

$$\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Biggl.\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

and

$$\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl.\Biggr|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}$$

$(g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (34), $r = 1, 2, \ldots, [k/2]$, $2r < k$) gives the following representation (see (98))

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2\le$$

$$\le\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot),\,j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2=$$

$$=\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\,dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\right)^2, \tag{396}$$

where

$$R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=$$

$$=\sum_{d=1}^{4^r}\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)-$$

$$-\sum_{d=1}^{2^r}\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\in L_2([t,T]^{k-2r})$$

and

$$\int\limits_{[t,T]^{k-2r}} R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times$$

$$\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\,dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k$$

is the Fourier coefficient of

$$\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)=$$

$$=R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q).$$

Also note that some of the functions

$$\bar{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$$

and

$$\tilde{R}_p^{(d)}(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)$$

can be identically equal to zero.

Obviously, we could use another representation for the function

$$R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k) \tag{397}$$

based on the left-hand side of the equality (97) and (185), (186), (199) (see Sect. 7, 10 for details). In Sect. 10, we considered the function (397) in detail for the case $k\ge 5$, $r=1$.

Parseval's equality gives

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{\infty}\left(\int\limits_{[t,T]^{k-2r}}R_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\times\right.$$

$$\left.\times\prod_{\substack{q=1\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{k}\psi_q(t_q)\phi_{j_q}(t_q)\ dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k\right)^2=$$

$$=\int\limits_{[t,T]^{k-2r}}\left(\hat{R}_p(t_1,\ldots,t_{g_1-1},t_{g_1+1},\ldots,t_{g_{2r}-1},t_{g_{2r}+1},\ldots,t_k)\right)^2\times$$

$$\times dt_1\ldots dt_{g_1-1}dt_{g_1+1}\ldots dt_{g_{2r}-1}dt_{g_{2r}+1}\ldots dt_k=$$

$$=\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{398}$$

Combining (396) and (398), we obtain

$$\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2\le$$

$$\le\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}. \tag{399}$$

Assume that we have succeeded in proving the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0. \tag{400}$$

Applying (399) and (400), we get (compare with (98))

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}\right)^2=0. \tag{401}$$

As noted in Sect. 7, Condition 3 of Theorem 13 can be replaced by a weaker condition (98) (or (401)). Also Condition 3 of Theorem 13 can be replaced by (400). From (401) we obviously obtain

$$\lim_{p\to\infty}\sum_{j_{g_1}=0}^{p}\ldots\sum_{j_{g_{2r-1}}=0}^{p}C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\frown(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\frown(\cdot),j_{g_1}=\,j_{g_2},\ldots,j_{g_{2r-1}}=\,j_{g_{2r}}}. \tag{402}$$

According to (97), the equality (402) will be satisfied if

$$\lim_{p\to\infty}S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigg|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}=0, \tag{403}$$

where $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ as in (34), $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, $r=1,2,\ldots,[k/2]$,

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d=0$, where

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}},\quad S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}$$

are defined by (38), (39), $l=1,2,\ldots,r$ (see Sect. 7 for details).

Let us make some remarks about the function (397) for the case $k>5$, $r=2$. In this case, using the left-hand side of the equality (97) and (185), (186), (199), we represent the function (397) as the sum of several functions. In particular, among these functions will be the following functions

$$Q_p(t_1,\ldots,t_{s-1},t_{s+1},\ldots,t_{l-1},t_{l+1},\ldots,t_{q-1},t_{q+1},\ldots,t_{g-1},t_{g+1},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{s-1}<t_{s+1}<\ldots<t_{l-1}<t_{l+1}<\ldots<t_{q-1}<t_{q+1}<\ldots<t_{g-1}<t_{g+1}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{s+1}}\psi_s(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times$$

$$\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{q+1}}\psi_q(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{g-1}}\psi_g(\tau)\phi_{j_q}(\tau)d\tau, \tag{404}$$

$$\bar Q_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right), \tag{405}$$

$$\tilde Q_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau, \tag{406}$$

$$\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-1}<t_{l+2}<\ldots<t_{q-1}<t_{q+2}<\ldots<t_k\}}\times$$

$$\times\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times$$

$$\times\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right). \tag{407}$$

Note that the pairs (g_1,g_2), (g_3,g_4) for the functions (405) and (406) have the property: $g_2=g_1+1$, $g_4=g_3+1$, $g_3=g_2+1$. At the same time, the pairs (g_1,g_2), (g_3,g_4) for the function (404) have the following property: $g_2>g_1+1$, $g_4>g_3+1$, $g_3\ge g_2+1$. For the function (407), the pairs (g_1,g_2), (g_3,g_4) chosen as follows: $g_2>g_1+1$, $g_4>g_3+1$, $g_4=g_2+1$, $g_3=g_1+1$. Generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered. We consider the functions (404)–(407) only as an example.

Suppose that $s+1=l-1$, $l+1=q-1$, $q+1=g-1$ in (404). Let us show that (we consider the case of Legendre polynomials; the trigonometric case is simpler and can be considered similarly)

$$\lim_{p\to\infty}\left\|Q_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{408}$$

$$\lim_{p\to\infty}\left\|\bar{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{409}$$

$$\lim_{p\to\infty}\left\|\tilde{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0, \tag{410}$$

$$\lim_{p\to\infty}\left\|\hat{Q}_p\right\|^2_{L_2([t,T]^{k-4})}=0. \tag{411}$$

First consider the proof of (408). We have ($s+1=l-1$, $l+1=q-1$, $q+1=g-1$)

$$\left(Q_p(t_1,\ldots,t_{l-3},t_{l-1},t_{l+1},t_{l+3},t_{l+5},\ldots,t_k)\right)^2=$$

$$=\mathbf{1}_{\{t_1<\ldots<t_{l-3}<t_{l-1}<t_{l+1}<t_{l+3}<t_{l+5}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2. \tag{412}$$

Using the estimate (135), we obtain

$$\left|\int\limits_t^s \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{K}{j^{1-\varepsilon/2}(1-z^2(s))^{1/4-\varepsilon/4}}, \tag{413}$$

where $j \in \mathbb{N}$, $s \in (t,T)$, $z(s)$ is defined by (106), $\varepsilon \in (0,1)$, constant K does not depend on j, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, $\psi(\tau)$ is a continuously differentiable nonrandom function on $[t,T]$.

Applying (413) and (138) (we take ε instead of $\varepsilon/2$ in (138)), we get

$$\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l-1}}\psi_{l-2}(\tau)\phi_{j_l}(\tau)d\tau\int\limits_t^{t_{l-1}}\psi_l(\tau)\phi_{j_l}(\tau)d\tau\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+2}(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^{t_{l+3}}\psi_{l+4}(\tau)\phi_{j_q}(\tau)d\tau\right)^2 \le$$

$$\le \frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-1}))^{1-\varepsilon}(1-z^2(t_{l+3}))^{1-\varepsilon}}, \tag{414}$$

where $t_{l-1}, t_{l+3} \in (t,T)$, constant K_1 is independent of p. Combining (412) and (414), we have (408).

Let us prove (409). The following equality is proved in Sect. 12 [47] (also see Sect. 2.9 [15]) for the case of Legendre polynomials ($n > m$; $n, m \in \mathbb{N}$)

$$\sum_{j=m+1}^{n} C_{jj}(s) = \sum_{j=m+1}^{n}\int\limits_t^s \psi_2(\theta)\phi_j(\theta)\int\limits_t^{\theta}\psi_1(\tau)\phi_j(\tau)d\tau d\theta =$$

$$= \frac{T-t}{4}\int\limits_{-1}^{z(s)}\psi_1(u(x))\psi_2(u(x))\left(P_{n+1}(x)P_n(x) - P_{m+1}(x)P_m(x)\right)dx -$$

$$-\frac{(T-t)^2}{8}\sum_{j=m+1}^{n}\frac{1}{2j+1}\int\limits_{-1}^{z(s)}\left(P_{j+1}(y) - P_{j-1}(y)\right)\psi_1'(u(y))\times$$

$$\times\Biggl(\left(P_{j+1}(z(s)) - P_{j-1}(z(s))\right)\psi_2(s) - \left(P_{j+1}(y) - P_{j-1}(y)\right)\psi_2(u(y)) -$$

$$-\frac{T-t}{2}\int\limits_y^{z(s)}\left(P_{j+1}(x) - P_{j-1}(x)\right)\psi_2'(u(x))dx\Biggr)dy, \tag{415}$$

where $s \in (t,T)$,

$$C_{jj}(s) = \int\limits_t^s \psi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\psi_1(\theta)\phi_j(\theta)d\theta d\tau,$$

$$u(y)=\frac{T-t}{2}y+\frac{T+t}{2},\quad z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ_1', ψ_2' are derivatives of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ with respect to the variable $u(y)$.

Applying the estimate (134) in (415) and taking into account the boundedness of the functions $\psi_1(\tau)$, $\psi_2(\tau)$ and their derivatives, we obtain

$$\left|\sum_{j=m+1}^{n}C_{jj}(s)\right|\le C_1\left(\frac{1}{n^{1-\varepsilon}}+\frac{1}{m^{1-\varepsilon}}\right)\int\limits_{-1}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/2-\varepsilon/2}}+$$

$$+C_2\sum_{j=m+1}^{n}\frac{1}{j^{2-\varepsilon}}\left(\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/2-\varepsilon/2}}+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\int\limits_{-1}^{z(s)}\frac{dy}{\left(1-y^2\right)^{1/4-\varepsilon/4}}+\right.$$

$$\left.+\int\limits_{-1}^{z(s)}\frac{1}{\left(1-y^2\right)^{1/4-\varepsilon/4}}\int\limits_{y}^{z(s)}\frac{dx}{\left(1-x^2\right)^{1/4-\varepsilon/4}}dy\right),\tag{416}$$

where $s\in(t,T)$, constants C_1,C_2 do not depend on n and m.

From (416) we have

$$\left|\sum_{j=m+1}^{\infty}C_{jj}(s)\right|\le\frac{K_1}{m^{1-\varepsilon}}+K_2\sum_{j=m+1}^{\infty}\frac{1}{j^{2-\varepsilon}}\left(1+\frac{1}{\left(1-z^2(s)\right)^{1/4-\varepsilon/4}}\right),\tag{417}$$

where $s\in(t,T)$, constants K_1,K_2 do not depend on m.

Applying (138) (we take ε instead of $\varepsilon/2$ in (138)) in (417), we get

$$\left|\sum_{j=m+1}^{\infty}C_{jj}(s)\right|\le\frac{K}{m^{1-\varepsilon}\left(1-z^2(s)\right)^{1/4-\varepsilon/4}},\tag{418}$$

where $s\in(t,T)$, constant K is independent of m.

Using the estimate (418), we obtain (see (405))

$$\left(\bar{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2=\mathbf{1}_{\{t_1<\ldots<t_{l-2}<t_{l+3}<\ldots<t_k\}}\times$$

$$\times\left(\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.$$

$$\left.\times\sum_{j_q=p+1}^{\infty}\left(\int\limits_t^{t_{l-2}}\psi_{l+1}(\theta)\phi_{j_q}(\theta)\int\limits_t^{\theta}\psi_{l+2}(u)\phi_{j_q}(u)dud\theta\right)\right)^2\le$$

$$
\le \frac{K_1}{p^{4(1-\varepsilon)}(1-z^2(t_{l-2}))^{1-\varepsilon}}, \tag{419}
$$

where $t_{l-2} \in (t,T)$, constant K_1 is independent of p. The inequality (419) completes the proof of (409).

Let us prove (410). Using (115), we obtain the following equality for the cases of Legendre polynomials and trigonometric functions

$$
\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1=0}^{p} C_{j_1j_1}(s) = \sum_{j_1=p+1}^{\infty} C_{j_1j_1}(s), \tag{420}
$$

where $s \in (t,T)$ and

$$
C_{jj}(s) = \int\limits_t^s \psi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau} \psi_1(\theta)\phi_j(\theta)d\theta d\tau.
$$

Applying (420) in (406), we get

$$
\left(\tilde{Q}_p(t_1,\ldots,t_{l-2},t_{l+3},\ldots,t_k)\right)^2 \le
$$

$$
\le \left(\sum_{j_l=p+1}^{\infty}\sum_{j_q=p+1}^{\infty}\int\limits_t^{t_{l+3}} \psi_{l+1}(\tau)\phi_{j_q}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times\right.
$$

$$
\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 =
$$

$$
= \left(\frac{1}{2}\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\psi_{l+2}(\tau)d\tau-\right.
$$

$$
-\sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}}\psi_{l+1}(\tau)\phi_{j_q}(\tau)\sum_{j_l=p+1}^{\infty}\left(\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)\times
$$

$$
\left.\times\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)dud\tau\right)^2 =
$$

$$
= (a-b)^2 \le 2(|a|^2+|b|^2). \tag{421}
$$

Further, we have

$$|a| \le \frac{1}{2}\int\limits_t^{t_{l+3}} \left|\psi_{l+1}(\tau)\right| \left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right| \left|\psi_{l+2}(\tau)\right| d\tau, \tag{422}$$

$$|b| \le \sum_{j_q=0}^{p}\int\limits_t^{t_{l+3}} \left|\psi_{l+1}(\tau)\phi_{j_q}(\tau)\right| \left|\sum_{j_l=p+1}^{\infty}\int\limits_t^{\tau}\psi_{l-1}(\theta)\phi_{j_l}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right| \times$$

$$\times \left|\int\limits_t^{\tau}\psi_{l+2}(u)\phi_{j_q}(u)du\right| d\tau. \tag{423}$$

Combining (418) and (422), we obtain

$$|a| \le \frac{C}{p^{1-\varepsilon}}, \tag{424}$$

where constant C is independent of p.

Separating in (423) the term with the number $j_q = 0$ and then applying (266), (109), (418), we obtain

$$|b| \le \frac{K}{p^{1-\varepsilon}}\left(\int\limits_t^{t_{l+3}}\frac{d\tau}{\left(1-z^2(\tau)\right)^{1/2-\varepsilon/4}} + \sum_{j_q=1}^{p}\frac{1}{j_q}\int\limits_t^{t_{l+3}}\frac{d\tau}{\left(1-z^2(\tau)\right)^{3/4-\varepsilon/4}}\right) \le$$

$$\le \frac{K_1}{p^{1-\varepsilon}}\left(1 + \sum_{j_q=1}^{p}\frac{1}{j_q}\right) \le \frac{K_1}{p^{1-\varepsilon}}\left(2 + \int\limits_1^{p}\frac{dx}{x}\right) =$$

$$= \frac{K_1\left(2 + lnp\right)}{p^{1-\varepsilon}} \ \to\ 0 \tag{425}$$

if $p \to \infty$. The estimates (421), (424), (425) complete the proof of (410).

Finally, consider the proof of (411). Using the elementary inequality $|ab| \le (a^2+b^2)/2$ and Parseval's equality, we have

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2 \le$$

$$\le \left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left|\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right| \times\right.$$

$$\left.\times \left|\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right|\right)^2 \le$$

$$\le \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=p+1}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\le$$

$$\le \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{l+2}}\psi_{l+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)dud\theta\right)^2+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\sum_{j_{l+1}=0}^{\infty}\left(\int\limits_t^{t_{q+2}}\psi_{q+1}(\theta)\phi_{j_{l+1}}(\theta)\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)dud\theta\right)^2\right)^2\le$$

$$\le \frac{1}{4}\left(\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{l+2}}\psi_{l+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_l(u)\phi_{j_l}(u)du\right)^2 d\theta+\right.$$

$$\left.+\sum_{j_l=p+1}^{\infty}\int\limits_t^{t_{q+2}}\psi_{q+1}^2(\theta)\left(\int\limits_t^{\theta}\psi_q(u)\phi_{j_l}(u)du\right)^2 d\theta\right)^2. \qquad (426)$$

Note that

$$\sum_{j=p+1}^{\infty}\frac{1}{j^2}\le\int\limits_p^{\infty}\frac{dx}{x^2}=\frac{1}{p}. \qquad (427)$$

From (426) and (427), (109) we obtain

$$\left(\hat{Q}_p(t_1,\ldots,t_{l-1},t_{l+2},\ldots,t_{q-1},t_{q+2},\ldots,t_k)\right)^2\le$$
$$\le\frac{K}{p^2}\ \to\ 0$$

if $p\to\infty$, where constant K does not depend on p. Thus the equalities (408)–(411) are proved.

Recall that the function (397) (this function is defined using the left-hand side of the equality (97)) for the case $k>5,\ r=2$ is represented as the sum of several functions. Four of them, namely Q_p, $\bar{Q}_p$, $\tilde{Q}_p$, $\hat{Q}_p$ (these functions correspond to the particular case of choosing the pairs (g_1,g_2), (g_3,g_4); generally speaking, all possible pairs (g_1,g_2), (g_3,g_4) must be considered), have been studied above. Absolutely similarly, we can consider the remaining functions (for all possible pairs (g_1,g_2), (g_3,g_4)) whose sum is the function (397) for the case $k>5$, $r=2$. As a result, we will have

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})}=0\quad (k>5,\ r=2).$$

After that, we can go to the function (397) for the case $k > 5$, $r = 3$, $2r < k$ (this function is defined using the left-hand side of the equality (97)) and follow the same steps as above. This will lead us to the following equality

$$\lim_{p\to\infty}\left\|\hat{R}_p\right\|^2_{L_2([t,T]^{k-2r})} = 0 \quad (k > 5,\ r = 3,\ 2r < k).$$

Then we can move on to the next step and so on. As a result, we get the equality (400) ($r = 1, 2, \ldots, [k/2]$). Thus the condition (98) is satisfied for the case $k = 2n + 1$, $n = 3, 4, \ldots$ (recall that the condition (98) is weaker than Condition 3 of Theorem 13 and the condition (98) can be used in Theorem 13 instead of Condition 3).

For the case $k = 2n$, $n = 3, 4, \ldots$ we follow the above steps for $r = 1, 2, \ldots, [k/2] - 1$ ($2r \le k - 2$). For $2r = k$ we use the same technique as in the proof of the equalities (147)–(149). Recall that we used (71), (78) and Parseval's equality in the proof of (147)–(149).

The obvious disadvantage of the proposed algorithm is the drastic increase of complexity of the proof when moving from $r = 1$ to $r = 2$, $r = 2$ to $r = 3$ and so on.

The proofs of Theorems 17 and 18 contain a rather simple trick of passing from $r = 1$ to $r = 2$. Unfortunately, this procedure cannot be applied already at the transition from $r = 2$ to $r = 3$.

Note that the case $k = 6$, $r = 3$ was successfully considered in Theorem 23 under the following simplifying assumption: $\psi_1(\tau), \ldots, \psi_6(\tau) \equiv 1$.

Nevertheless, the results obtained in this paper are quite sufficient for practical needs (see Chapters 4 and 5 [15] for details).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.

[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.

[5] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Berlin, Springer, 1994, 292 pp.

[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.

[7] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227
Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233
Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[12] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385.

DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073.
Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[15] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184v46 [math.PR], 2023, 998 pp. [In English].

[16] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606.
Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[17] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[18] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs (Third Edition). Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), A.1-A.947.
Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228
Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[23] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7
Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[24] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[25] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[26] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[27] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[28] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1805.12527 [math.PR]. 2018, 29 pp. [In English].

[29] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 29 pp. [In English].
[30] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[31] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[32] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[33] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].
[34] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html
[35] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].
[36] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html
[37] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367.
Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html
Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)
[38] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
[39] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf
[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].
[41] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604
[42] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2023, 144 pp. [in English].
[43] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. [In English]. arXiv:2307.11006 [math.PR], 2023, 58 pp.
[44] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2023, 71 pp. [in English].
[45] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [in English].
[46] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2023, 158 pp. [in English].
[47] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2023, 223 pp. [in English].

[48] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 66 pp. [in English].

[49] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. arXiv:1801.01962 [math.PR]. 2023, 49 pp. [In English].

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR]. 2018, 20 pp. [In English].

[51] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2023, 58 pp. [in English].

[52] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2023, 80 pp. [In English].

[53] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR]. 2018, 44 pp. [In English].

[54] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 40 pp. [In English].

[55] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 148 pp. [in English].

[56] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 46 pp. [In English].

[57] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[58] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:2006.16040 [math.PR]. 2020, 33 pp. [In English].

[59] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1806.10705 [math.PR]. 2018, 29 pp. [In English].

[60] Kuznetsov D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing, Saarbrucken, 2012, 409 pp.
Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[61] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[62] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[63] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022), 135-194. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[64] Allen E. Approximation of triple stochastic integrals through region subdivision. Communicat. in Appl. Anal. Special Tribute Issue to Prof. V. Lakshmikantham. 17 (2013), 355-366.

[65] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[66] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[67] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl. 10, 4 (1992), 431-441.

[68] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations With Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[69] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat. 5, 36 (1965), 1560-1564.

[70] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci. 3 (1965), 213-229.

[71] Wong E., Zakai M. Riemann-Stieltjes approximations of stochastic integrals. Z. Warsch. verw. Gebiete. 12 (1969), 87-97.
[72] Ikeda N., Nakao S., and Yamato, Y. A class of approximations of Brownian motion. Publ. RIMS Kyoto Univ. 13 (1977), 285-300.
[73] Koneczny F. On the Wong-Zakai approximation of stochastic differential equations. J. Mult. Anal. 13 (1983), 605-611.
[74] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.
[75] Karatzas I., Shreve S.E. Brownian Motion and Stochastic Calculus. 2nd Edition. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, 1991, 470 pp.
[76] Mackevicius V., Zibaitis B. Gaussian approximations of Brownian motion in a stochastic integral. Lith. Math. J. 33 (1993), 393-406. https://doi.org/10.1007/BF00995993
[77] Twardowska K. Wong-Zakai approximations for stochastic differential equations. Acta Appl. Math. 43 (1996), 317-359. https://doi.org/10.1007/BF00047670
[78] Gyongy I., Michaletzky G. On Wong-Zakai approximations with δ-martingales. Royal Society of London Proceedings Series A. 460, 2041 (2004), 309-324.
[79] Gyongy I., Shmatkov A. Rate of convergence of Wong-Zakai approximations for stochastic partial differential equations. Appl. Math. Optim. 54 (2006), 315-341.
[80] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974, 696 pp.
[81] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.
[82] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 343 pp. [In English].
[83] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf
[84] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].
[85] Ito K. Multiple Wiener integral. Journal of the Mathematical Society of Japan. 3, 1 (1951), 157-169.
[86] Budhiraja A. Multiple stochastic integrals and Hilbert space valued traces with applications to asymptotic statistics and non-linear filtering. Ph. D. Thesis, The University of North Caroline at Chapel Hill, 1994, VII+132 pp.
[87] Rybakov K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), 81-115. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.5.html
[88] Johnson G.W., Kallianpur G. Homogeneous chaos, p-forms, scaling and the Feynman integral. Transactions of the American Mathematical Society. 340 (1993), 503-548.
[89] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2
[90] Bardina X., Jolis M. Weak convergence to the multiple Stratonovich integral. Stochastic Processes and their Applications, Elsevier, 90, 2 (2000), 277-300.
[91] Bardina X., Rovira C. On the strong convergence of multiple ordinary integrals to multiple Stratonovich integrals. Publicacions Matemàtiques, 65 (2021), 859-876. DOI: http://doi.org/10.5565/PUBLMAT6522114
[92] Pugachev V.S. Lectures on Fuctional Analysis. MAI, Moscow, 1996, 744 pp.
[93] Hairer M. On Malliavin's proof of Hörmander's theorem. Bulletin Des Sciences Mathématiques, 135, 6-7 (2011), 650-666.
[94] Rybakov K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821
[95] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional Wiener process based on Multiple Fourier-Legendre series. MATEC Web of Conferences, 362 (2022), article id: 01014, 10 pp. DOI: http://doi.org/10.1051/matecconf/202236201014

[96] Rybakov K.A. Features of the expansion of multiple stochastic Stratonovich integrals using Walsh and Haar functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2023), 137-150. Available at: http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.9.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Chapter 3.

Expansions of Specific Iterated Ito and Stratonovich Stochastic Integrals From the Taylor–Ito and Taylor–Stratonovich Expansions

MEAN-SQUARE APPROXIMATION OF ITERATED ITO AND STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITIES 1 TO 6 FROM THE TAYLOR–ITO AND TAYLOR–STRATONOVICH EXPANSIONS USING LEGENDRE POLYNOMIALS

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the practical material on expansions and mean-square approximations of specific iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 with respect to components of the multidimensional Wiener process on the base of the method of generalized multiple Fourier series. More precisely, we used the multiple Fourier–Legendre series converging in the sense of norm in the space $L_2([t,T]^k)$ $(k=1,\ldots,6)$ for approximation of iterated Ito and Stratonovich stochastic integrals. The considered iterated Ito and Stratonovich stochastic integrals are part of the stochastic Taylor expansions (Taylor–Ito and Taylor–Stratonovich expansions). Therefore, the results of the article can be useful for the construction of high-order strong numerical methods for Ito stochastic differential equations. Expansions of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 using Legendre polynomials are derived. The convergence with probability 1 of the mentioned method of generalized multiple Fourier series is proved for iterated Ito stochastic integrals of arbitrary multiplicity k $(k \in \mathbb{N})$ for the cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.

Keywords: Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Ito stochastic differential equation, Numerical solution, Mean-square approximation, Convergence with propability 1, Expansion.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[4]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$,

$$\int \ \text{and} \ \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively; $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[4]. At the same time $\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [5]-[36].

Effective solution of the problem of mean-square approximation for collections of iterated Ito and Stratonovich stochastic integrals (2) and (3) composes the subject of this article.

2. Theorems on Expansions of Iterated Ito and Stratonovich Stochastic Itegrals

Let us consider the effective approach to expansion of the iterated Ito stochastic integrals (2) [8] (2006), [9]-[35] (the so-called method of generalized multiple Fourier series). Sometimes these stochastic integrals are referred to in the literature as multiple stochastic integrals (see, for example, [2]).

The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral. Then, the indicated nonrandom function is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2).

Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function on $[t,T]$ (the case $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$ will be considered in Theorem 2).

Define the following function on the hypercube $[t,T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k) & \text{for} \ \ t_1 < \ldots < t_k \\ \\ 0 & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t,T], \quad k \ge 2, \tag{4}$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1, \ldots, t_k)$ is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t,T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

Theorem 1 [8] (2006), [9]-[35], [37]-[48]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$,*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (5), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$, which satisfies the condition (6).

It was shown that Theorem 1 is valid for convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [11]-[25] and for convergence with probablity 1 (w. p. 1) [22]-[25], [27], [42]. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorem 1 [8]-[25]. The generalization of Theorem 1 for complete orthonormal with weigth $r(t_1)\ldots r(t_k) \ge 0$ systems of functions in the space $L_2([t,T]^k)$ can be found in [21]-[25], [31], [37]. Another modification of Theorem 1 and Theorem 2 (see below) is connected with the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [22]-[24] (Chapter 7), [34], [38]-[40].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1,\ldots,6$ [8]-[35]

$$
J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}
$$

$$
J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{10}
$$

$$
\begin{gathered}
J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),
\end{gathered} \tag{11}
$$

$$
\begin{gathered}
J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} - \\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} + \\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
\end{gathered} \tag{12}
$$

$$
\begin{gathered}
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} - \\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
\end{gathered}
$$

$$
\begin{gathered}
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
\left.+\,\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right),
\end{gathered}
\tag{13}
$$

$$
\begin{gathered}
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
\end{gathered}
$$

$$
\begin{gathered}
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
\end{gathered}
$$

$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Bigg), \tag{14}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is an explicit formula (see (5)) for calculation of expansion coefficients of the iterated Ito stochastic integral with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square approximation error for iterated Ito stochastic integrals (see Theorem 3 below).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, we have new possibilities for approximation — we can use not only the trigonometric functions as in [2]-[4] but the Legendre polynomials.

4. As it turned out (see below), it is more convenient to work with Legendre polynomials for constructing approximations of iterated stochastic integrals. We can choose different numbers q (see Sect. 4) for approximations of different iterated Ito stochastic integrals. This is impossible for approximations based on the approach from [2]-[4]. Approximations based on Legendre polynomials are much simpler than approximations based on trigonometric functions (see (52), (53), (114), (118) below).

5. The approach from [2]-[4], [49]-[51] leads to iterated series (iterated application of the operation of limit transition) in contrast with multiple series from Theorem 1 (operation of limit transition is implemented only once) starting at least from the second or third multiplicity of iterated stochastic integrals. Multiple series are more convenient for approximation than the iterated ones, since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202–204), [49] (pp. 82-84), [50] (pp. 438-439), [51] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the mentioned approach [2]-[4], [49]-[51] based on the Karhunen–Loeve expansion of the Brownian bridge process [3] together with the Wong–Zakai approximation [54]-[56] (see discussion in Sect. 8 for detail).

6. In a number of works of the author [12]-[24], [26], [30] Theorem 1 has been adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6.

For further consideration, let us consider the generalization of formulas (9)–(14) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{15}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (15) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{16}$$

Below there are several examples of sums in the form (16)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$
$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$
$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (7) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{17}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (17) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (13).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [22] (Sect. 1.11), [25] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{18}$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [52]. Note that we use another notations [22] (Sect. 1.11), [25] (Sect. 15) in comparison with [52]. Moreover, the proof of an analogue of Theorem 2 from [52] is somewhat different from the proof given in [22] (Sect. 1.11), [25] (Sect. 15).

As noted above, in a number of works of the author [12]-[24], [26], [30] Theorem 1 has been adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6. Let us first present some old results as the following theorem.

Theorem 3 [12]-[24], [26], [30]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function on* $[t,T]$ *and* $\psi_1(\tau),\psi_3(\tau)$ *are twice continuously differentiable functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\quad(i_1,i_2=1,\ldots,m), \tag{19}$$

$$J^*[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=0,1,\ldots,m), \tag{20}$$

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{21}$$

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\quad(i_1,i_2,i_3,i_4=0,1,\ldots,m), \tag{22}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (3) *and* $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ *in* (20), (22); *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [22] (Sect. 2.10–2.16), [26] (Sect. 13–19), [30] (Sect. 5–11), [41] (Sect. 7–13). Let us formulate four theorems that were obtained using this approach.

Theorem 4 [22], [26], [30], [41]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$*,* $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{23}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{24}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (23) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (24)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 5 [22], [26], [30], [41]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \tag{25}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \tag{26}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{27}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (25), (26) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (27), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 4.

Theorem 6 [22], [26], [30], [41]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{28}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{29}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{30}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (28), (29) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (30), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 4, 5.

Theorem 7 [22], [26], [30], [41]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{31}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^{T} \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 4–6.

As we mentioned above, Theorems 1 and 2 allow us to accurately calculate the mean-square approximation error for iterated Ito stochastic integrals (see Theorem 8 below).

Assume that $J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}$ is the approximation of (2), which is the expression on the right-hand side of (18) before passing to the limit

$$J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$E_k^{p_1,\ldots,p_k} \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\},$$

$$E_k^{p} \stackrel{\text{def}}{=} E_k^{p_1,\ldots,p_k} \ \ \text{if} \ \ p_1=\ldots=p_k=p,$$

$$I_k \stackrel{\text{def}}{=} \|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k.$$

In [17]-[25], [31] it was shown that

$$E_k^{p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right) \tag{32}$$

if $i_1,\ldots,i_k = 1,\ldots,m$ and $0 < T-t < \infty$ or $i_1,\ldots,i_k = 0,1,\ldots,m$ and $0 < T-t < 1$.

Moreover, in [10]-[25] the following estimate is obtained

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\} \le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right)^n, \tag{33}$$

where $n \in \mathbb{N}$.

The value E_k^p can be calculated exactly.

Theorem 8 [22] (Sect. 1.12), [31] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $i_1,\ldots,i_k = 1,\ldots,m$. *Then*

$$E_k^p = I_k -$$

$$-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{34}$$

where $i_1,\ldots,i_k = 1,\ldots,m$; *expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\} = C_{j_k\ldots j_1}.$$

Then from Theorem 8 for pairwise different $i_1,\ldots,i_k$ and for $i_1 = \ldots = i_k$ we obtain

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2,$$

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right).$$

Consider some examples of the application of Theorem 8 ($i_1,\ldots,i_5 = 1,\ldots,m$)

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}C_{j_1j_2} \quad (i_1 = i_2),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \ne i_3),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_2j_3j_1}C_{j_3j_2j_1} \quad (i_1 \ne i_2 = i_3),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \ne i_2),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 \ne i_3, i_4;\ i_3 \ne i_4), \tag{35}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_3 \ne i_2, i_4;\ i_2 \ne i_4), \tag{36}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_4 \ne i_2, i_3;\ i_2 \ne i_3), \tag{37}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_3 \ne i_1, i_4;\ i_1 \ne i_4), \tag{38}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_4 \ne i_1, i_3;\ i_1 \ne i_3), \tag{39}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_3 = i_4 \ne i_1, i_2;\ i_1 \ne i_2), \tag{40}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 = i_3 \ne i_4), \tag{41}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_3 = i_4 \ne i_1), \tag{42}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 = i_4 \ne i_3), \tag{43}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_3 = i_4 \ne i_2), \tag{44}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_2 \ne i_3 = i_4), \tag{45}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_3 \ne i_2 = i_4), \tag{46}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_4 \ne i_2 = i_3), \tag{47}$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right) \quad (i_1 \ne i_2 = i_4 \ne i_3 = i_5 \ne i_1),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right) \quad (i_1 = i_2 = i_3 \ne i_4 = i_5),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1}\right) \quad (i_1 = i_3 = i_4 = i_5 \ne i_2).$$

3. Expansions and Approximations of Specific Iterated Ito and Stratonovich Stochastic Integrals of Multiplicities 1 to 6 Using Legendre Polynomials

In this section, we provide considerable practical material (based on Theorems 1–7) on expansions and approximations of iterated Ito and Stratonovich stochastic integrals of the following form

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},\tag{48}$$

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},\tag{49}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ l_1,\ldots,l_k=0,1,\ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots,\tag{50}$$

where $P_j(x)$ is the Legendre polynomial. It is well known that the polynomials $P_j(x)$ can be represented, for example, in the form

$$P_j(x)=\frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

Consider some well known properties of the polynomials $P_j(x)$

$$P_j(1)=1,\quad P_{j+1}(-1)=-P_j(-1),\quad j=0,1,2,\ldots,$$

$$\frac{dP_{j+1}(x)}{dx}-\frac{dP_{j-1}(x)}{dx}=(2j+1)P_j(x),$$

$$xP_j(x)=\frac{(j+1)P_{j+1}(x)+jP_{j-1}(x)}{2j+1},\quad j=1,2,\ldots,$$

$$\int\limits_{-1}^{1}x^kP_j(x)dx=0,\quad k=0,1,2,\ldots,j-1,$$

$$\int\limits_{-1}^{1}P_k(x)P_j(x)dx=\begin{cases}0 & \text{if}\ \ k\ne j\\ 2/(2j+1) & \text{if}\ \ k=j\end{cases},$$

$$P_n(x)P_m(x)=\sum_{k=0}^{m}K_{m,n,k}P_{n+m-2k}(x),$$

where

$$K_{m,n,k}=\frac{a_{m-k}a_ka_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1},\quad a_k=\frac{(2k-1)!!}{k!},\quad m\le n.$$

Using the above properties, system of functions (50) and Theorems 1–7, we obtain the following expansions of iterated Ito and Stratonovich stochastic integrals (48) and (49) based on multiple Fourier–Legendre series

$$I_{(0)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{51}$$

$$I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{52}$$

$$I_{(2)T,t}^{(i_1)}=\frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{53}$$

$$I_{(00)T,t}^{*(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{54}$$

$$I_{(00)T,t}^{(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(01)T,t}^{*(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{55}$$

$$I_{(10)T,t}^{*(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{56}$$

or

$$I_{(01)T,t}^{*(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{01}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$I_{(10)T,t}^{*(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{10}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2j_1}^{01}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2j_1}^{01},$$

$$C_{j_2j_1}^{10}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2j_1}^{10},$$

$$\bar{C}_{j_2j_1}^{01}=-\int\limits_{-1}^{1}(1+y)P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2j_1}^{10}=-\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}(1+x)P_{j_1}(x)dxdy;$$

$$I_{(10)T,t}^{(i_1i_2)}=I_{(10)T,t}^{*(i_1i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2\quad \text{w. p. 1},$$

$$I_{(01)T,t}^{(i_1i_2)}=I_{(01)T,t}^{*(i_1i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2\quad \text{w. p. 1},$$

$$I_{(01)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

$$I_{(10)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

or

$$I_{(01)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{01}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(10)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{10}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(000)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},\tag{57}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

(58)
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right) \quad \text{w. p. 1},$$

(59)
$$I_{(000)T,t}^{*(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\zeta_0^{(i_1)}\right)^3 \quad \text{w. p. 1},$$

where

(60)
$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

(61)
$$\bar{C}_{j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz;$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)} = I_{(000)T,t}^{*(i_1 i_2 i_3)} + \mathbf{1}_{\{i_1=i_2\neq 0\}}\frac{1}{2}I_{(1)T,t}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\neq 0\}}\frac{1}{2}\left((T-t)I_{(0)T,t}^{(i_1)} + I_{(1)T,t}^{(i_1)}\right) \quad \text{w. p. 1},$$

$$I_{(02)T,t}^{*(i_1 i_2)} = -\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)} - (T-t)I_{(01)T,t}^{*(i_1 i_2)} + \frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} +$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

(62)
$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr],$$

$$I_{(20)T,t}^{*(i_1 i_2)} = -\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1 i_2)} - (T-t)I_{(10)T,t}^{*(i_1 i_2)} + \frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} +$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

(63)
$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr],$$

$$I_{(11)T,t}^{*(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{*(i_1 i_2)} - \frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1 i_2)} + I_{(01)T,t}^{*(i_1 i_2)}\right) +$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)} + \sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{64}$$

or

$$I_{(02)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{02}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$I_{(20)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{20}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$I_{(11)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{11}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2 j_1}^{02} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2 j_1}^{02},$$

$$C_{j_2 j_1}^{20} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2 j_1}^{20},$$

$$C_{j_2 j_1}^{11} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2 j_1}^{11},$$

$$\bar{C}_{j_2 j_1}^{02} = \int_{-1}^{1} P_{j_2}(y)(y+1)^2\int_{-1}^{y} P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2 j_1}^{20} = \int_{-1}^{1} P_{j_2}(y)\int_{-1}^{y} P_{j_1}(x)(x+1)^2dxdy,$$

$$\bar{C}_{j_2 j_1}^{11} = \int_{-1}^{1} P_{j_2}(y)(y+1)\int_{-1}^{y} P_{j_1}(x)(x+1)dxdy;$$

$$I_{(11)T,t}^{*(i_1 i_1)} = \frac{1}{2}\left(I_{(1)T,t}^{(i_1)}\right)^2 \quad \text{w. p. } 1,$$

$$I_{(02)T,t}^{(i_1 i_2)} = I_{(02)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1}, \tag{65}$$

$$I_{(20)T,t}^{(i_1 i_2)} = I_{(20)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1}, \tag{66}$$

$$I_{(11)T,t}^{(i_1 i_2)} = I_{(11)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1}, \tag{67}$$

$$I_{(02)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - (T-t) I_{(01)T,t}^{(i_1 i_2)} + \frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} + \right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.$$

$$\left.\left. + \frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$- \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \tag{68}$$

$$I_{(20)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - (T-t) I_{(10)T,t}^{(i_1 i_2)} + \frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} + \right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{\infty}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.$$

$$\left.\left. + \frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$- \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \tag{69}$$

$$I_{(11)T,t}^{(i_1 i_2)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)} - \frac{T-t}{2}\left(I_{(10)T,t}^{(i_1 i_2)} + I_{(01)T,t}^{(i_1 i_2)}\right) +$$

$$+\frac{(T-t)^3}{8}\left[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)} + \sum_{i=0}^{\infty}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.\right.$$

$$\left.\left. + \frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$- \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \tag{70}$$

or

$$I_{(02)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{02} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(20)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{20} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(11)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{11} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(3)T,t}^{(i_1)} = -\frac{(T-t)^{7/2}}{4} \left(\zeta_0^{(i_1)} + \frac{3\sqrt{3}}{5} \zeta_1^{(i_1)} + \frac{1}{\sqrt{5}} \zeta_2^{(i_1)} + \frac{1}{5\sqrt{7}} \zeta_3^{(i_1)} \right), \tag{71}$$

$$I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$\begin{aligned} I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\ -\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\ +\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\ + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \end{aligned} \tag{72}$$

$$I_{(0000)T,t}^{(i_1 i_1 i_1 i_1)} = \frac{1}{24} (T-t)^2 \left(\left(\zeta_0^{(i_1)} \right)^4 - 6 \left(\zeta_0^{(i_1)} \right)^2 + 3 \right) \quad \text{w. p. 1},$$

$$I_{(0000)T,t}^{*(i_1 i_1 i_1 i_1)} = \frac{1}{24} (T-t)^2 \left(\zeta_0^{(i_1)} \right)^4 \quad \text{w. p. 1}, \tag{73}$$

where

$$C_{j_4 j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16} (T-t)^2 \bar{C}_{j_4 j_3 j_2 j_1}, \tag{74}$$

$$\bar{C}_{j_4 j_3 j_2 j_1} = \int_{-1}^{1} P_{j_4}(u) \int_{-1}^{u} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy dz du; \tag{75}$$

$$I_{(001)T,t}^{*(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{001} \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(010)T,t}^{*(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{010} \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(100)T,t}^{*(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{100} \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(001)T,t}^{(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{001} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{76}$$

$$I_{(010)T,t}^{(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{77}$$

$$I_{(100)T,t}^{(i_1i_2i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{78}$$

where

$$C_{j_3j_2j_1}^{001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{001},$$

$$C_{j_3j_2j_1}^{010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{010},$$

$$C_{j_3j_2j_1}^{100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{100},$$

$$\bar{C}_{j_3j_2j_1}^{100} = -\int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)(x+1)dxdydz,$$

$$\bar{C}_{j_3j_2j_1}^{010}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_3j_2j_1}^{001}=-\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz;$$

$$I_{(lll)T,t}^{(i_1i_1i_1)}=\frac{1}{6}\left(\left(I_{(l)T,t}^{(i_1)}\right)^3-3I_{(l)T,t}^{(i_1)}\Delta_{l(T,t)}\right)\quad \hbox{w. p. 1},$$

$$I_{(lll)T,t}^{*(i_1i_1i_1)}=\frac{1}{6}\left(I_{(l)T,t}^{(i_1)}\right)^3\quad \hbox{w. p. 1},$$

$$I_{(llll)T,t}^{(i_1i_1i_1i_1)}=\frac{1}{24}\left(\left(I_{(l)T,t}^{(i_1)}\right)^4-6\left(I_{(l)T,t}^{(i_1)}\right)^2\Delta_{(l)T,t}+3\left(\Delta_{(l)T,t}\right)^2\right)\quad \hbox{w. p. 1},$$

$$I_{(llll)T,t}^{*(i_1i_1i_1i_1)}=\frac{1}{24}\left(I_{(l)T,t}^{(i_1)}\right)^4\quad \hbox{w. p. 1},$$

where

$$I_{(l)T,t}^{(i_1)}=\sum_{j=0}^{l}C_j^l\zeta_j^{(i_1)}\quad \hbox{w. p. 1},$$

$$\Delta_{l(T,t)}=\int\limits_t^T(t-s)^{2l}ds,\quad C_j^l=\int\limits_t^T(t-s)^l\phi_j(s)ds;$$

$$I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\hbox{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^5\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{79}$$

$$I_{(00000)T,t}^{(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(00000)T,t}^{*(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\zeta_0^{(i_1)}\right)^5\quad \text{w. p. 1},$$

where

$$C_{j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar{C}_{j_5j_4j_3j_2j_1},$$

$$\bar{C}_{j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv;$$

$$I_{(0001)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0010)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0100)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(1000)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0001)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$
I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$
$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$
$$
+\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$
$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$
$$
+\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{1000} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$
$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$
$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$
$$
+\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

where

$$
C_{j_4 j_3 j_2 j_1}^{0001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32} (T-t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0001},
$$

$$
C_{j_3 j_2 j_1}^{0010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32} (T-t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0010},
$$

$$
C_{j_4 j_3 j_2 j_1}^{0100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32} (T-t)^3 \bar{C}_{j_3 j_2 j_1}^{0100},
$$

$$
C_{j_4 j_3 j_2 j_1}^{1000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32} (T-t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{1000},
$$

$$\bar{C}^{1000}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydzdu,$$

$$\bar{C}^{0100}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}^{0010}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}^{0001}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu;$$

$$I^{*(i_1i_2i_3i_4i_5i_6)}_{(000000)T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6},$$

$$\begin{gathered}
I^{(i_1i_2i_3i_4i_5i_6)}_{(000000)T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{6}\zeta^{(i_l)}_{j_l}-\\
-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-\mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-\\
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-\mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}-\\
-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\\
-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\\
-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\\
-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-\\
-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-\\
-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+\\
+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_5)}_{j_5}+\\
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}+\\
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}+
\end{gathered}$$

$$
\begin{gathered}
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-\\
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-\\
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-\\
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-\\
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-\\
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-\\
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-\\
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-\\
\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),
\end{gathered}
$$

$$
I_{(000000)T,t}^{(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\left(\zeta_0^{(i_1)}\right)^6-15\left(\zeta_0^{(i_1)}\right)^4+45\left(\zeta_0^{(i_1)}\right)^2-15\right)\quad \text{w. p. 1},
$$

$$
I_{(000000)T,t}^{*(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\zeta_0^{(i_1)}\right)^6\quad \text{w. p. 1},
$$

where

$$
C_{j_6j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3\bar{C}_{j_6j_5j_4j_3j_2j_1},
$$

$$\bar{C}_{j_6j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_6}(w)\int\limits_{-1}^{w}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudvdw.$$

It should be noted that instead of the expansion (57) we may to consider the following expansion, which is derived by direct calculation

$$I_{(000)T,t}^{*(i_1i_2i_3)}=-\frac{1}{T-t}\left(I_{(0)T,t}^{(i_3)}I_{(10)T,t}^{*(i_2i_1)}+I_{(0)T,t}^{(i_1)}I_{(10)T,t}^{*(i_2i_3)}\right)+\frac{1}{2}I_{(0)T,t}^{(i_3)}\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_2i_1)}\right)-$$

(80) $$-(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)}+\sqrt{3}\zeta_1^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right)+\frac{1}{4}D_{T,t}^{(i_1i_2i_3)}\right),$$

where

$$D_{T,t}^{(i_1i_2i_3)}=\sum_{\substack{i=1,\ j=0,\ k=i\\ 2i\ge k+i-j\ge -2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i+1,k+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,\ j=0,\ 1\le k\le i-1\\ 2k\ge k+i-j\ge -2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k+1,i+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,\ j=0,\ k=i+2\\ 2i+2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i+1,k-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,\ j=0,\ 1\le k\le i+1\\ 2k-2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k-1,i+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,\ j=0,\ k=i-2,k\ge 1\\ 2i-2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i-1,k+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,\ j=0,\ 1\le k\le i-3\\ 2k+2\ge k+i-j\ge 0;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k+1,i-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,\ j=0,\ k=i\\ 2i\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{i-1,k-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,\ j=0\ 1\le k\le i-1\\ 2k\ge k+i-j\ge 2;\ k+i-j\ -\text{even}}}^{\infty}N_{ijk}K_{k-1,i-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)},$$

where

$$N_{ijk}=\sqrt{\frac{1}{(2k+1)(2j+1)(2i+1)}}\ ,$$

$$K_{m,n,k}=\frac{a_{m-k}a_ka_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1},\quad a_k=\frac{(2k-1)!!}{k!},\quad m\le n.$$

However, as we will see further, the expansion (58) is more convenient for practical implementation then (80).

Also note the following relation between iterated Ito and Stratonovich stochastic integrals

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} = I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} I_{(10)T,t}^{*(i_3 i_4)} - \frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I_{(10)T,t}^{*(i_1 i_4)} - I_{(01)T,t}^{*(i_1 i_4)}\right) -$$

$$-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left((T-t)I_{(00)T,t}^{*(i_1 i_2)} + I_{(01)T,t}^{*(i_1 i_2)}\right) + \frac{1}{8}(T-t)^2\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}} \quad \text{w. p. 1.}$$

Let

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)q}, \quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}$$

be approximations of the iterated Ito and Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}, \quad I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$$

defined by (48), (49), i.e. we replace ∞ with q in the expansions of these stochastic integrals. For example, $I_{(00)T,t}^{*(i_1 i_2)q}$ be the approximation of the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1 i_2)}$ obtained from (54) by replacing ∞ with q, etc.

It is easy to prove that

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)} - I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \quad (i_1 \neq i_2). \tag{81}$$

Moreover, using Theorem 8, we obtain for $i_1 \neq i_2$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)} - I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)} - I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\} =$$

$$= \frac{(T-t)^4}{16}\left(\frac{5}{9} - 2\sum_{i=2}^{q}\frac{1}{4i^2-1} - \sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2} -\right.$$

$$\left. - \sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \tag{82}$$

For the case $i_1 = i_2$, using Theorem 8, we have

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_1)} - I_{(10)T,t}^{(i_1 i_1)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_1)} - I_{(01)T,t}^{(i_1 i_1)q}\right)^2\right\} =$$

$$= \frac{(T-t)^4}{16}\left(\frac{1}{9} - \sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2} - 2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{83}$$

In Tables 1–3 we have calculations according to the formulas (81)–(83) for various values of q. In the given tables ε means the right-hand sides of these formulas.

Let us consider (55), (56) for $i_1 = i_2$

Table 1. Confirmation of the formula (81)

$2\varepsilon/(T-t)^2$	0.1667	0.0238	0.0025	$2.4988\cdot 10^{-4}$	$2.4999\cdot 10^{-5}$
q	1	10	100	1000	10000

Table 2. Confirmation of the formula (82)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450\cdot 10^{-4}$	$6.2495\cdot 10^{-5}$
q	1	10	100	1000	10000

Table 3. Confirmation of the formula (83)

$16\varepsilon/(T-t)^4$	0.0070	$4.3551\cdot 10^{-5}$	$6.0076\cdot 10^{-8}$	$6.2251\cdot 10^{-11}$	$6.3178\cdot 10^{-14}$
q	1	10	100	1000	10000

$$I_{(01)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}-\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right), \tag{84}$$

$$I_{(10)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}+\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right). \tag{85}$$

From (84), (85), considering (51) and (52), we obtain

$$I_{(10)T,t}^{*(i_1 i_1)}+I_{(01)T,t}^{*(i_1 i_1)}=-\frac{(T-t)^2}{2}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}\right)=I_{(0)T,t}^{(i_1)}I_{(1)T,t}^{(i_1)}\quad \text{w. p. 1.} \tag{86}$$

Obtaining (86), we supposed that the formulas (55), (56) are valid w. p. 1. The complete proof of this fact will be given in Sect. 5, 6.

Applying the Ito formula and standard relations between iterated Ito and Stratonovich stochastic integrals, it is easy to get the equality (86).

Furthermore, using the Ito formula, we obtain

$$I_{(11)T,t}^{*(i_1 i_1)}=\frac{\left(I_{(1)T,t}^{(i_1)}\right)^2}{2}\quad \text{w. p. 1.} \tag{87}$$

In addition, applying the Ito formula, we have

$$I_{(20)T,t}^{(i_1 i_1)} + I_{(02)T,t}^{(i_1 i_1)} = I_{(0)T,t}^{(i_1)} I_{(2)T,t}^{(i_1)} - \frac{(T-t)^3}{3} \quad \text{w. p. 1.} \tag{88}$$

From (88), considering the formulas (65), (66), we get

$$I_{(20)T,t}^{*(i_1 i_1)} + I_{(02)T,t}^{*(i_1 i_1)} = I_{(0)T,t}^{(i_1)} I_{(2)T,t}^{(i_1)} \quad \text{w. p. 1.} \tag{89}$$

Let us check whether the formulas (87), (89) follow from (62)–(64), if we suppose $i_1 = i_2$ in the last ones. From (62)–(64) for $i_1 = i_2$ we obtain

$$I_{(20)T,t}^{*(i_1 i_1)} + I_{(02)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{2} I_{(00)T,t}^{*(i_1 i_1)} - (T-t)\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) +$$

$$+ \frac{(T-t)^3}{4}\left(\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2 + \frac{2}{3\sqrt{5}}\zeta_2^{(i_1)}\zeta_0^{(i_1)}\right), \tag{90}$$

$$I_{(11)T,t}^{*(i_1 i_1)} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{*(i_1 i_1)} - \frac{T-t}{2}\left(I_{(10)T,t}^{*(i_1 i_1)} + I_{(01)T,t}^{*(i_1 i_1)}\right) + \frac{(T-t)^3}{24}\left(\zeta_1^{(i_1)}\right)^2. \tag{91}$$

It is easy to see that from (90) and (91), considering (86) and (51)–(54), we actually obtain the equalities (87) and (89), and it indirectly confirm the correctness of the formulas (62)–(64).

Obtaining (87), (89), we supposed that the formulas (62)–(64) are valid w. p. 1. The complete proof of this fact will be given in Sect. 5, 6.

On the basis of the presented expansions of iterated stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to noticeable complication of formulas for mentioned expansions.

However, increasing of mentioned parameters leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated stochastic integrals that leads to a sharp decrease of member quantities in expansions of iterated stochastic integrals, which are required for achieving the acceptable accuracy of approximation. In this context, let us consider the approach to the approximation of iterated stochastic integrals, which provides a possibility to obtain the mean-square approximations of the required accuracy without using the complex expansions like (80).

Let us consider the following approximation of iterated Ito stochastic integral $I_{(000)T,t}^{(i_1 i_2 i_3)}$ using (58)

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{92}$$

where $C_{j_3 j_2 j_1}$ is defined by (60), (61).

In particular, from (92) for $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we obtain

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{93}$$

Furthermore, using Theorem 8 for $k=3$, we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=$$

$$=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^2\quad(i_1\ne i_2, i_1\ne i_3, i_2\ne i_3),\tag{94}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=$$

$$=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad(i_1\ne i_2=i_3),\tag{95}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=$$

$$=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad(i_1=i_3\ne i_2),\tag{96}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=$$

$$=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad(i_1=i_2\ne i_3).\tag{97}$$

From the other hand, from (32) for $k=3$ we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^2\right),\tag{98}$$

where $i_1,i_2,i_3=1,\ldots,m$.

We may act similarly with more complicated iterated stochastic integrals. For example, for approximation of the stochastic integral $I_{(0000)T,t}^{(i_1i_2i_3i_4)}$ we can write (see (72))

$$I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}=\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),\tag{99}$$

where $C_{j_4 j_3 j_2 j_1}$ is defined by (74), (75).

Moreover, according to (32) for $k = 4$, we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} \le 24\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2\right),
$$

where $i_1, i_2, i_3, i_4 = 1, \ldots, m$.

For pairwise different $i_1, i_2, i_3, i_4 = 1, \ldots, m$ from Theorem 8 we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2. \tag{100}
$$

Using Theorem 8, we can calculate exactly the left-hand side of (100) for any possible combinations of i_1, i_2, i_3, i_4. These relations were obtained in [22]-[24], [31]. For example,

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} =
$$

$$
= \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_4 j_3 j_2 j_1}\right)\right) \quad (i_1 = i_2 \ne i_3 = i_4),
$$

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} =
$$

$$
= \frac{(T-t)^4}{24} - \sum_{j_1,\ldots,j_4=0}^{q_2} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 = i_3 \ne i_4),
$$

where

$$
\sum_{(j_1,j_2)}
$$

means the sum with respect to permutations (j_1, j_2).

Assume that $q_1 = 6$. In Tables 4–10 we have the exact values of coefficients $\bar{C}_{j_3 j_2 j_1}$, $j_1, j_2, j_3 = 0, 1, \ldots, 6$. Note that in [43], [44] the database with 270,000 exactly calculated Fourier–Legendre coefficients was described.

Calculating the value (94) for $q_1 = 6$, $i_1 \ne i_2$, $i_1 \ne i_3$, $i_3 \ne i_2$, we obtain the following approximate equality

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} \approx 0.01956(T-t)^3.
$$

Let us choose, for example, $q_2 = 2$. In Tables 11–19 we have the exact values of coefficients $\bar{C}_{j_4 j_3 j_2 j_1}$ $(j_1, j_2, j_3, j_4 = 0, 1, 2)$. In the case of pairwise different i_1, i_2, i_3, i_4 we have from (100) the following approximate equality

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} \approx 0.0236084(T-t)^4. \tag{101}
$$

Let us consider the following four approximations of iterated Ito stochastic integrals (see (76)–(79))

$$I_{(001)T,t}^{(i_1 i_2 i_3)q_3} = \sum_{j_1,j_2,j_3=0}^{q_3} C_{j_3 j_2 j_1}^{001} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{102}$$

$$I_{(010)T,t}^{(i_1 i_2 i_3)q_3} = \sum_{j_1,j_2,j_3=0}^{q_3} C_{j_3 j_2 j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{103}$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)q_3} = \sum_{j_1,j_2,j_3=0}^{q_3} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{104}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_4} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4} C_{j_5 j_4 j_3 j_2 j_1} \Biggl(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} -$$

$$- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} -$$

$$- \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}} \mathbf{1}_{\{j_1=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} -$$

$$- \mathbf{1}_{\{i_2=i_5\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_5}^{(i_5)} -$$

$$- \mathbf{1}_{\{i_3=i_5\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} +$$

$$+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_5\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_4}^{(i_4)} +$$

$$+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_4=i_5\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_5}^{(i_5)} +$$

$$+ \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_5\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_4=i_5\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_2}^{(i_2)} +$$

$$+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_5\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_3}^{(i_3)} +$$

$$+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_3=i_5\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_4}^{(i_4)} +$$

$$+ \mathbf{1}_{\{i_1=i_5\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_2}^{(i_2)} +$$

$$+ \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_4=i_5\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{i_3=i_5\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} +$$

$$+ \mathbf{1}_{\{i_2=i_5\}} \mathbf{1}_{\{j_2=j_5\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \Biggr). \tag{105}$$

TABLE 4. Coefficients $\bar{C}_{0j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{4}{3}$	$\frac{-2}{3}$	$\frac{2}{15}$	0	0	0	0
$j_2=1$	0	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{4}{105}$	0	0	0
$j_2=2$	$\frac{-4}{15}$	$\frac{2}{15}$	$\frac{2}{105}$	$\frac{-2}{35}$	$\frac{2}{105}$	0	0
$j_2=3$	0	$\frac{-2}{35}$	$\frac{2}{35}$	$\frac{2}{315}$	$\frac{-2}{63}$	$\frac{8}{693}$	0
$j_2=4$	0	0	$\frac{-8}{315}$	$\frac{2}{63}$	$\frac{2}{693}$	$\frac{-2}{99}$	$\frac{10}{1287}$
$j_2=5$	0	0	0	$\frac{-10}{693}$	$\frac{2}{99}$	$\frac{2}{1287}$	$\frac{-2}{143}$
$j_2=6$	0	0	0	0	$\frac{-4}{429}$	$\frac{2}{143}$	$\frac{2}{2145}$

TABLE 5. Coefficients $\bar{C}_{1j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{2}{3}$	$\frac{-4}{15}$	0	$\frac{2}{105}$	0	0	0
$j_2=1$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2=2$	$\frac{-2}{15}$	$\frac{8}{105}$	0	$\frac{-2}{105}$	0	$\frac{4}{1155}$	0
$j_2=3$	$\frac{-2}{35}$	0	$\frac{8}{315}$	0	$\frac{-38}{3465}$	0	$\frac{20}{9009}$
$j_2=4$	0	$\frac{-4}{315}$	0	$\frac{46}{3465}$	0	$\frac{-64}{9009}$	0
$j_2=5$	0	0	$\frac{-4}{693}$	0	$\frac{74}{9009}$	0	$\frac{-32}{6435}$
$j_2=6$	0	0	0	$\frac{-10}{3003}$	0	$\frac{4}{715}$	0

TABLE 6. Coefficients $\bar{C}_{2j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j_2=1$	$\frac{2}{15}$	$\frac{-4}{105}$	0	$\frac{-2}{315}$	0	$\frac{8}{3465}$	0
$j_2=2$	$\frac{2}{105}$	0	0	0	$\frac{-2}{495}$	0	$\frac{4}{3003}$
$j_2=3$	$\frac{-2}{35}$	$\frac{8}{315}$	0	$\frac{-2}{3465}$	0	$\frac{-116}{45045}$	0
$j_2=4$	$\frac{-8}{315}$	0	$\frac{4}{495}$	0	$\frac{-2}{6435}$	0	$\frac{-16}{9009}$
$j_2=5$	0	$\frac{-4}{693}$	0	$\frac{38}{9009}$	0	$\frac{-8}{45045}$	0
$j_2=6$	0	0	$\frac{-8}{3003}$	0	$\frac{118}{45045}$	0	$\frac{-4}{36465}$

TABLE 7. Coefficients $\bar{C}_{3j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	$\frac{2}{105}$	0	$\frac{-4}{315}$	0	$\frac{2}{693}$	0
$j_2=1$	$\frac{4}{105}$	0	$\frac{-2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{10}{9009}$
$j_2=2$	$\frac{2}{35}$	$\frac{-2}{105}$	0	$\frac{4}{3465}$	0	$\frac{-74}{45045}$	0
$j_2=3$	$\frac{2}{315}$	0	$\frac{-2}{3465}$	0	$\frac{16}{45045}$	0	$\frac{-10}{9009}$
$j_2=4$	$\frac{-2}{63}$	$\frac{46}{3465}$	0	$\frac{-32}{45045}$	0	$\frac{2}{9009}$	0
$j_2=5$	$\frac{-10}{693}$	0	$\frac{38}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{122}{765765}$
$j_2=6$	0	$\frac{-10}{3003}$	0	$\frac{20}{9009}$	0	$\frac{-226}{765765}$	0

Table 8. Coefficients $\bar{C}_{4j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	$\frac{2}{315}$	0	$\frac{-4}{693}$	0	$\frac{2}{1287}$
$j_2=1$	0	$\frac{2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{-10}{9009}$	0
$j_2=2$	$\frac{2}{105}$	0	$\frac{-2}{495}$	0	$\frac{4}{6435}$	0	$\frac{-38}{45045}$
$j_2=3$	$\frac{2}{63}$	$\frac{-38}{3465}$	0	$\frac{16}{45045}$	0	$\frac{2}{9009}$	0
$j_2=4$	$\frac{2}{693}$	0	$\frac{-2}{6435}$	0	0	0	$\frac{2}{13923}$
$j_2=5$	$\frac{-2}{99}$	$\frac{74}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{-2}{153153}$	0
$j_2=6$	$\frac{-4}{429}$	0	$\frac{118}{45045}$	0	$\frac{-4}{13923}$	0	$\frac{-2}{188955}$

Table 9. Coefficients $\bar{C}_{5j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	$\frac{2}{693}$	0	$\frac{-4}{1287}$	0
$j_2=1$	0	0	$\frac{8}{3465}$	0	$\frac{-10}{9009}$	0	$\frac{-4}{6435}$
$j_2=2$	0	$\frac{4}{1155}$	0	$\frac{-74}{45045}$	0	$\frac{16}{45045}$	0
$j_2=3$	$\frac{8}{693}$	0	$\frac{-116}{45045}$	0	$\frac{2}{9009}$	0	$\frac{8}{58905}$
$j_2=4$	$\frac{2}{99}$	$\frac{-64}{9009}$	0	$\frac{2}{9009}$	0	$\frac{4}{153153}$	0
$j_2=5$	$\frac{2}{1287}$	0	$\frac{-8}{45045}$	0	$\frac{-2}{153153}$	0	$\frac{4}{415701}$
$j_2=6$	$\frac{-2}{143}$	$\frac{4}{715}$	0	$\frac{-226}{765765}$	0	$\frac{-8}{415701}$	0

Table 10. Coefficients $\bar{C}_{6j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$	$j_1=3$	$j_1=4$	$j_1=5$	$j_1=6$
$j_2=0$	0	0	0	0	$\frac{2}{1287}$	0	$\frac{-4}{2145}$
$j_2=1$	0	0	0	$\frac{10}{9009}$	0	$\frac{-4}{6435}$	0
$j_2=2$	0	0	$\frac{4}{3003}$	0	$\frac{-38}{45045}$	0	$\frac{8}{36465}$
$j_2=3$	0	$\frac{20}{9009}$	0	$\frac{-10}{9009}$	0	$\frac{8}{58905}$	0
$j_2=4$	$\frac{10}{1287}$	0	$\frac{-16}{9009}$	0	$\frac{2}{13923}$	0	$\frac{4}{188955}$
$j_2=5$	$\frac{2}{143}$	$\frac{-32}{6435}$	0	$\frac{122}{765765}$	0	$\frac{4}{415701}$	0
$j_2=6$	$\frac{2}{2145}$	0	$\frac{-4}{36465}$	0	$\frac{-2}{188955}$	0	0

Assume that $q_3 = 2$, $q_4 = 1$. In Tables 20–36 we have the exact values of Fourier–Legendre coefficients $\bar{C}^{001}_{j_3j_2j_1}$, $\bar{C}^{010}_{j_3j_2j_1}$, $\bar{C}^{100}_{j_3j_2j_1}$ $(j_1, j_2, j_3 = 0, 1, 2)$, $\bar{C}_{j_5j_4j_3j_2j_1}$ $(j_1, \ldots, j_5 = 0, 1)$.

In the case of pairwise different $i_1, \ldots, i_5$ from Tables 20–36 we have

$$\mathsf{M}\left\{\left(I^{(i_1i_2i_3)}_{(100)T,t} - I^{(i_1i_2i_3)q_3}_{(100)T,t}\right)^2\right\} =$$

$$= \frac{(T-t)^5}{60} - \sum_{j_1,j_2,j_3=0}^{2}\left(C^{100}_{j_3j_2j_1}\right)^2 \approx 0.00815429(T-t)^5,$$

TABLE 11. Coefficients $\bar{C}_{00j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{3}$	$\frac{-2}{5}$	$\frac{2}{15}$
$j_2=1$	$\frac{-2}{15}$	$\frac{2}{15}$	$\frac{-2}{21}$
$j_2=2$	$\frac{-2}{15}$	$\frac{2}{35}$	$\frac{2}{105}$

TABLE 12. Coefficients $\bar{C}_{10j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{5}$	$\frac{-2}{9}$	$\frac{2}{35}$
$j_2=1$	$\frac{-2}{45}$	$\frac{2}{35}$	$\frac{-2}{45}$
$j_2=2$	$\frac{-2}{21}$	$\frac{2}{45}$	$\frac{2}{315}$

TABLE 13. Coefficients $\bar{C}_{02j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{15}$	$\frac{2}{21}$	$\frac{-4}{105}$
$j_2=1$	$\frac{2}{35}$	$\frac{-4}{105}$	$\frac{2}{105}$
$j_2=2$	$\frac{4}{105}$	$\frac{-2}{105}$	0

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{010}\right)^2\approx 0.0173903(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{001}\right)^2\approx 0.0252801(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.$$

Note that from (32) for $k=5$ we obtain

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^2\right),$$

Table 14. Coefficients $\bar{C}_{01j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{45}$	$\frac{-2}{105}$	$\frac{2}{315}$
$j_2=2$	$\frac{-2}{35}$	$\frac{2}{63}$	$\frac{-2}{315}$

Table 15. Coefficients $\bar{C}_{11j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 16. Coefficients $\bar{C}_{20j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$j_2=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$j_2=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 17. Coefficients $\bar{C}_{21j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{21}$	$\frac{-2}{45}$	$\frac{2}{315}$
$j_2=1$	$\frac{2}{315}$	$\frac{2}{315}$	$\frac{-2}{225}$
$j_2=2$	$\frac{-2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 18. Coefficients $\bar{C}_{12j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{35}$	$\frac{2}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{2}{63}$	$\frac{-2}{105}$	$\frac{2}{225}$
$j_2=2$	$\frac{2}{105}$	$\frac{-2}{225}$	$\frac{-2}{3465}$

Table 19. Coefficients $\bar{C}_{22j_2j_1}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{2}{105}$	$\frac{-2}{315}$	0
$j_2=1$	$\frac{2}{315}$	0	$\frac{-2}{1155}$
$j_2=2$	0	$\frac{2}{3465}$	0

TABLE 20. Coefficients $\bar{C}_{0j_2j_1}^{001}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	-2	$\frac{14}{15}$	$\frac{-2}{15}$
$j_2=1$	$\frac{-2}{15}$	$\frac{-2}{15}$	$\frac{6}{35}$
$j_2=2$	$\frac{2}{5}$	$\frac{-22}{105}$	$\frac{-2}{105}$

TABLE 21. Coefficients $\bar{C}_{1j_2j_1}^{001}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-6}{5}$	$\frac{22}{45}$	$\frac{-2}{105}$
$j_2=1$	$\frac{-2}{9}$	$\frac{-2}{105}$	$\frac{26}{315}$
$j_2=2$	$\frac{22}{105}$	$\frac{-38}{315}$	$\frac{-2}{315}$

TABLE 22. Coefficients $\bar{C}_{2j_2j_1}^{001}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{5}$	$\frac{2}{21}$	$\frac{4}{105}$
$j_2=1$	$\frac{-22}{105}$	$\frac{4}{105}$	$\frac{2}{105}$
$j_2=2$	0	$\frac{-2}{105}$	0

TABLE 23. Coefficients $\bar{C}_{0j_2j_1}^{100}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{3}$	$\frac{2}{15}$	$\frac{2}{15}$
$j_2=1$	$\frac{-2}{15}$	$\frac{-2}{45}$	$\frac{2}{35}$
$j_2=2$	$\frac{2}{15}$	$\frac{-2}{35}$	$\frac{-4}{105}$

where $i_1,\ldots,i_5=1,\ldots,m$.

Moreover, from the inequality (32) we get the following useful estimates

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),
$$

TABLE 24. Coefficients $\bar{C}_{1j_2j_1}^{100}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{5}$	$\frac{2}{45}$	$\frac{2}{21}$
$j_2=1$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2=2$	$\frac{2}{35}$	$\frac{-2}{63}$	$\frac{-2}{105}$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(20))T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(1000)T,t}^{(i_1i_2i_3i_4)}-I_{(1000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0100)T,t}^{(i_1i_2i_3i_4)}-I_{(0100)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0010)T,t}^{(i_1i_2i_3i_4)}-I_{(0010)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(0001)T,t}^{(i_1i_2i_3i_4)}-I_{(0001)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)}-I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)q}\right)^2\right\}\le 720\left(\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6j_5j_4j_3j_2j_1}^2\right).$$

In addition, from Theorem 8 for $k=2$ we have

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{10}C_{j_1j_2}^{10}\quad (i_1=i_2),$$

Table 25. Coefficients $\bar{C}_{2j_2j_1}^{100}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j_2=1$	$\frac{-2}{21}$	$\frac{-2}{315}$	$\frac{2}{105}$
$j_2=2$	$\frac{-2}{105}$	$\frac{-2}{315}$	0

Table 26. Coefficients $\bar{C}_{0j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{3}$	$\frac{8}{15}$	0
$j_2=1$	$\frac{-4}{15}$	0	$\frac{8}{105}$
$j_2=2$	$\frac{4}{15}$	$\frac{-16}{105}$	0

Table 27. Coefficients $\bar{C}_{1j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{5}$	$\frac{4}{15}$	$\frac{4}{105}$
$j_2=1$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2=2$	$\frac{4}{35}$	$\frac{-8}{105}$	0

Table 28. Coefficients $\bar{C}_{2j_2j_1}^{010}$

	$j_1=0$	$j_1=1$	$j_1=2$
$j_2=0$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j_2=1$	$\frac{-4}{21}$	$\frac{4}{105}$	$\frac{4}{315}$
$j_2=2$	$\frac{-4}{105}$	0	0

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{01}C_{j_1j_2}^{01}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{20}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{20}C_{j_1j_2}^{20}\quad (i_1=i_2),$$

Table 29. Coefficients $\bar{C}_{000j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{15}$	$\frac{-8}{45}$
$j_2=1$	$\frac{-4}{45}$	$\frac{8}{105}$

Table 30. Coefficients $\bar{C}_{010j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{45}$	$\frac{-16}{315}$
$j_2=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 31. Coefficients $\bar{C}_{110j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{105}$	$\frac{-2}{45}$
$j_2=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 32. Coefficients $\bar{C}_{011j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{315}$	$\frac{-4}{315}$
$j_2=1$	0	$\frac{2}{945}$

$$\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{11}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{11}C_{j_1j_2}^{11}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(02)}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{02}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{02}C_{j_1j_2}^{02}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\quad (i_1\ne i_2).$$

Clearly, expansions for iterated Stratonovich stochastic integrals (see above) are simpler than expansions for iterated Ito stochastic integrals (see Theorems 1, 2, and (9)–(14)). However, the

Table 33. Coefficients $\bar{C}_{001j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	0	$\frac{4}{315}$
$j_2=1$	$\frac{8}{315}$	$\frac{-2}{105}$

Table 34. Coefficients $\bar{C}_{100j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{8}{45}$	$\frac{-4}{35}$
$j_2=1$	$\frac{-16}{315}$	$\frac{2}{45}$

Table 35. Coefficients $\bar{C}_{101j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{4}{315}$	0
$j_2=1$	$\frac{4}{315}$	$\frac{-8}{945}$

Table 36. Coefficients $\bar{C}_{111j_2j_1}$

	$j_1=0$	$j_1=1$
$j_2=0$	$\frac{2}{105}$	$\frac{-8}{945}$
$j_2=1$	$\frac{2}{945}$	0

calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Ito stochastic integrals. Below we consider how we can estimate or calculate exactly (for some particular cases) the mean-square approximation error for iterated Stratonovich stochastic integrals.

As we mentioned above, on the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated Stratonovich stochastic integrals ($T-t \ll 1$ since the length of integration interval $[t,T]$ for iterated Stratonovich stochastic integrals plays the role of integration step for the numerical methods for Ito SDEs, i.e. $T-t$ is already fairly small). This leads to a sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals, which are required for achieving the acceptable accuracy of approximation.

From (81) ($i_1 \ne i_2$) we obtain

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)} - I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1} \le$$

$$\le \frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx = -\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right| \le C_1\frac{(T-t)^2}{q}, \qquad (106)$$

where C_1 is a constant.

It is easy to notice that for a sufficiently small $T-t$ (recall that $T-t \ll 1$ since it is a step of integration for numerical schemes for Ito SDEs) there exists a constant C_2 such that

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\} \le C_2\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}, \tag{107}$$

where $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}$ is an approximation of the iterated Stratonovich stochastic integral $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$.

From (106) and (107) we finally obtain

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\} \le C\frac{(T-t)^2}{q}, \tag{108}$$

where constant C is independent of $T-t$.

The same idea can be found in [2] in the framework of the method of approximation of iterated Stratonovich stochastic integrals based on the trigonometric expansion of the Brownian bridge process [3]. Note that, in contrast to the estimate (108), the constant C in Theorems 4–6 does not depend on p.

We can get more information about the numbers q (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since for pairwise different $i_1,\ldots,i_k=1,\ldots,m$

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t} \quad \text{w. p. 1},$$

where $J[\psi^{(k)}]_{T,t}$, $J^*[\psi^{(k)}]_{T,t}$ are defined by (2) and (3) correspondingly, then for pairwise different $i_1,\ldots,i_6=1,\ldots,m$ from Theorem 8 we obtain

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{*(i_1i_2i_3)}-I_{(100)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(010))T,t}^{*(i_1i_2i_3)}-I_{(010)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{*(i_1i_2i_3)}-I_{(001)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)q}\right)^2\right\}=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(20)T,t}^{*(i_1i_2)}-I_{(20)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{*(i_1i_2)}-I_{(11)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{*(i_1i_2)}-I_{(02)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(1000)T,t}^{*(i_1i_2i_3i_4)}-I_{(1000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0100)T,t}^{*(i_1i_2i_3i_4)}-I_{(0100)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0010)T,t}^{*(i_1i_2i_3i_4)}-I_{(0010)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(0001)T,t}^{*(i_1i_2i_3i_4)}-I_{(0001)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}-I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)q}\right)^2\right\}=\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6j_5j_4j_3j_2j_1}^2.$$

4. Legendre Polynomials of Trigonometry?

This section is devoted to the comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals.

Using Theorems 1, 2, 8 and the complete orthonormal system of trigonometric functions in the space $L_2([t,T])$, we obtain for $i_1\ne i_2$ $(i_1,i_2=1,\ldots,m)$

(109)
$$I_{(00)T,t}^{(i_1i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\ +\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr),$$

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1i_2)}-I_{(00)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{3(T-t)^2}{2\pi^2}\Biggl(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\Biggr), \tag{110}$$

(111)
$$I_{(00)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\ +\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr),$$

where
$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j,

$$\phi_j(s)=\frac{1}{\sqrt{T-t}}\begin{cases}1 & \text{for } j=0\\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)) & \text{for } j=2r-1\\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)) & \text{for } j=2r\end{cases},$$

where $r=1,2,\ldots$; another notations are the same as in Theorems 1, 2.

The expansion (109) was first derived by Milstein G.N. in [3] on the base of the Karhunen–Loeve expansion of the Brownian bridge process.

However, this approach has an obvious drawback. Indeed, we have too complex formulas (in comparison with (52), (53)) for the following stochastic integrals with Gaussian distribution

$$I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\Biggl(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\Biggr), \tag{112}$$

$$I_{(2)T,t}^{(i_1)}=(T-t)^{5/2}\Biggl(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\Biggr), \tag{113}$$

where $i_1=1,\ldots,m$.

In [3] Milstein G.N. proposed the following mean-square approximations on the base of the expansions (109), (112)

$$I_{(1)T,t}^{(i_1)q}=-\frac{(T-t)^{3/2}}{2}\Biggl(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr), \tag{114}$$

$$I_{(00)T,t}^{(i_1 i_2)q} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.$$

$$\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right) + \frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr), \tag{115}$$

where

$$\xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)}, \quad \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}, \tag{116}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$ $(r = 1, \ldots, q;\ i = 1, \ldots, m)$ are independent standard Gaussian random variables.

Obviously, for the approximations (114) and (115) we obtain

$$\mathsf{M}\left\{\left(I_{(1)T,t}^{(i_1)} - I_{(1)T,t}^{(i_1)q}\right)^2\right\} = 0,$$

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)} - I_{(00)T,t}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{117}$$

This idea has been developed in [2]. For example, the approximation $I_{(2)T,t}^{(i_1)q}$, which corresponds to (114), (115), has the form [2]

$$I_{(2)T,t}^{(i_1)q} = (T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)} + \sqrt{\beta_q}\mu_q^{(i_1)}\right) - \frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{118}$$

$$\mathsf{M}\left\{\left(I_{(2)T,t}^{(i_1)} - I_{(2)T,t}^{(i_1)q}\right)^2\right\} = 0,$$

where $\xi_q^{(i)}$, α_q have the form (116) and

$$\mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)}, \quad \beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}, \tag{119}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r = 1, \ldots, q;\ i = 1, \ldots, m)$ are independent standard Gaussian random variables.

Nevetheless, the expansions (114), (118) are too complex for the numerical modeling of two Gaussian random variables $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$.

Further, we will see that the using of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will drastically complicate the approximation of the stochastic integral $I_{(000)T,t}^{(i_1 i_2 i_3)}$; $i_1, i_2, i_3 = 1, \ldots, m$. This is due to the fact that for this approach the number q is fixed for all stochastic integrals included into the considered collection [2]. However, it is clear that due to the smallness of $T-t$, the number q for $I_{(000)T,t}^{(i_1 i_2 i_3)}$ could be taken significantly less than in the formula (115) (see for comparison the case of Legendre polynomials). This feature is also valid for the formulas (114), (118).

To obtain the expansion for (3) on the base of the approach from [3] the truncated trigonometric expansions of components of the multidimensional Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple integrals (3) were obtained (see [2], [3], [49]-[51]).

At that, in [3] the case $\psi_1(s), \psi_2(s) \equiv 1$ and $i_1, i_2 = 0, 1, \ldots, m$ is considered. In [2], [49]-[51] the attempt to consider the case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$ and $i_1, i_2, i_3 = 0, 1, \ldots, m$ is realized.

Note that the mean-square convergence of $J^{*(i_1 i_2 i_3)q}_{(111)T,t}$ to $J^{*(i_1 i_2 i_3)}_{(111)T,t}$ if $q \to \infty$ was not proved rigorously in [2] (Sect. 5.8, pp. 202–204), [49] (pp. 82-84), [50] (pp. 438-439), [51] (pp. 263-264) within the frames of the Milstein approach [3] together with the Wong–Zakai approximation [54]-[56] (see discussion in Sect. 8 for detail).

Consider the approximation $I^{(i_1 i_2)q}_{(00)T,t}$ of the iterated stochastic integral $I^{(i_1 i_2)}_{(00)T,t}$ obtained from (54) by replacing ∞ with q

$$(120) \qquad I^{(i_1 i_2)q}_{(00)T,t} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right) \quad (i_1 \ne i_2).$$

Let us compare computational costs for the approximations (115), (120). It is not difficult to show that [5]-[24]

$$(121) \qquad \mathsf{M}\left\{\left(I^{(i_1 i_2)}_{(00)T,t} - I^{(i_1 i_2)q}_{(00)T,t}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right).$$

Let us compare (120) with (115) and (121) with (117). Consider minimal natural numbers $q_{\rm trig}$ and $q_{\rm pol}$, which satisfy to (see Table 37)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q_{\rm pol}}\frac{1}{4i^2-1}\right) \le (T-t)^3, \qquad \frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right) \le (T-t)^3.$$

Thus, we have

$$\frac{q_{\rm pol}}{q_{\rm trig}} \approx 1.67, \quad 2.22, \quad 2.43, \quad 2.36, \quad 2.41, \quad 2.43, \quad 2.45, \quad 2.45.$$

From the other hand, the formula (115) includes $(4q+4)m$ independent standard Gaussian random variables. At the same time the folmula (120) includes only $(2q+2)m$ independent standard Gaussian random variables. Moreover, the formula (120) is simpler than the formula (115). Thus, in this case we can talk about approximately equal computational costs for the formulas (115) and (120).

There is one important feature. As we mentioned above, further we will see that the using of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will drastically complicate the approximation of the stochastic integral $I^{(i_1 i_2 i_3)}_{(000)T,t}$; $i_1, i_2, i_3 = 1, \ldots, m$. This is due to the fact that the number q is fixed for all stochastic integrals, which included into the considered collection (the case of trigonometric functions). However, it is clear that due to the smallness of $T-t$, the number q for $I^{(i_1 i_2 i_3)}_{(000)T,t}$ could be chosen significantly less than in the formula (115). This feature is also valid for the formulas (114), (118). However, for the case of Legendre polynomials we can choose different numbers q for different stochastic integrals (see Sect. 3).

TABLE 37. Numbers $q_{\rm trig}$, $q_{\rm pol}$

$T-t$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}	2^{-11}	2^{-12}
$q_{\rm trig}$	3	4	7	14	27	53	105	209
$q^*_{\rm trig}$	6	11	20	40	79	157	312	624
$q_{\rm pol}$	5	9	17	33	65	129	257	513

From the other hand, if we will not use the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the mean-square error of approximation of the stochastic integral $I_{(00)T,t}^{(i_1i_2)}$ will be three times larger (see (110)). Moreover, in this case the stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$ (with Gaussian distribution) will be approximated worse.

Consider minimal natural numbers $q^*_{\rm trig}$, which satisfy the condition (see Table 37)

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q^*_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^3.$$

In this situation we can talk about the advantage of Ledendre polynomials ($q^*_{\rm trig}>q_{pol}$ and (111) is more complex than (120)).

Using Theorems 1, 2 for the system of trigonometric functions, we have ($i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3$) [8]-[24] (also see [5], [6])

$$I_{(000)T,t}^{(i_1i_2i_3)q}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$
$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+$$
$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$
$$+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$
$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+$$
$$+\frac{1}{8\pi^2r^2}\Bigl(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+$$
$$+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\Bigr)\Biggr)+D_{T,t}^{(i_1i_2i_3)q}\Biggr), \tag{122}$$

where

$$D_{T,t}^{(i_1i_2i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\Biggl(\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+$$
$$+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+$$

Table 38. Confirmation of the formula (123)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

$$
+\frac{1}{4\sqrt{2}\pi^2}\left(\sum_{r,m=1}^{q}\left(\frac{2}{rm}\left(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\right.\right.\right.
$$

$$
\left.+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{m(r+m)}\left(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.
$$

$$
\left.\left.-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)\right)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\left(\frac{1}{m(l-m)}\left(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.\right.
$$

$$
\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+
$$

$$
+\frac{1}{l(l-m)}\left(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-\right.
$$

$$
\left.\left.\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)\right)\right),
$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q;\ i=1,\ldots,m)$ are independent standard Gaussian random variables (see (116), (119)).

The mean-square error of approximation (122) $(i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3)$ has the following form [8]-[24] (also see [5], [6])

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}=(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

(123)
$$
\left.-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right).
$$

In Table 38 we can see the numerical confirmation of the formula (123) (ε is the right-hand side of (123)).

As we mentioned above, the Milstein expansion [3] (i.e. expansion based on the Karhunen–Loeve expansion of the Brownian bridge process) for iterated stochastic integrals leads to iterated application of the operation of limit transition. The analogue of (122) for iterated Stratonovich stochastic integrals has been derived in [2], [49]-[51] on the base of the Milstein expansion together with the Wong–Zakai approximation [54]-[56] (without rigorous proof). It means that the authors in [2] (Sect. 5.8, pp. 202–204), [49] (pp. 82-84), [50] (pp. 438-439), [51] (pp. 263-264) formally could not use the double sum with the upper limit q in the analogue of (122). From the other hand the correctness of (122) follows directly from Theorems 1, 2. Note that (122) has been obtained reasonably for the first time in [8]. The version of (122) without using the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ can be found in [5] (1997).

The mean-square error (123) has been obtained for the first time in [8] on the base of the simplified variant of Theorem 8 (the case of pairwise different $i_1,\ldots,i_k$).

As we noted above, the number q must be the same in (114), (115), (122). This is the main drawback of this approach, because really the number q in (122) can be chosen essentially smaller than in (115).

Note that in (122) we can replace $I_{(000)T,t}^{(i_1i_2i_3)q}$ on $I_{(000)T,t}^{*(i_1i_2i_3)q}$ and (122) then will be valid for any $i_1,i_2,i_3=1,\ldots,m$ (see Theorem 3).

Let us compare the efficiency of application of Legendre polynomials and trigonometric functions for approximation of the iterated stochastic integrals $I_{(00)T,t}^{(i_1i_2)}$, $I_{(000)T,t}^{(i_1i_2i_3)}$ $(i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3)$.

Consider the following conditions $(i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3)$

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le (T-t)^4, \tag{124}$$

$$(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}\frac{\left(C_{j_3j_2j_1}\right)^2}{(T-t)^3}\right)\le (T-t)^4, \tag{125}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p}\frac{1}{r^2}\right)\le (T-t)^4, \tag{126}$$

$$(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{p_1}\frac{1}{r^2}-\frac{55}{32\pi^4}\sum_{r=1}^{p_1}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{p_1}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \tag{127}$$

where $C_{j_3j_2j_1}$ is defined by (60).

In Tables 39 and 40 we can see minimal numbers q, q_1, p, p_1, which satisfy the conditions (124)–(127). As we mentioned above, the numbers q, q_1 are different. At that $q_1\ll q$ (the case of Legendre polynomials). Moreover, we cannot take different numbers p, p_1 for the case of trigonometric functions. Thus, we must choose $q=p$ in (114), (115), (122). This leads to huge computational costs (see very complex formula (122)). From the other hand, we can choose different numbers q in (114), (115), (122). At that we must exclude random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (114), (115), (122). At this situation we have

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p^*}\frac{1}{r^2}\right)\le (T-t)^4, \tag{128}$$

$$(T-t)^3\left(\frac{5}{36}-\frac{1}{2\pi^2}\sum_{r=1}^{p_1^*}\frac{1}{r^2}-\frac{79}{32\pi^4}\sum_{r=1}^{p_1^*}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{p_1^*}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \tag{129}$$

where the left-hand sides of (128), (129) correspond to (115), (122) but without $\xi_q^{(i)}$, $\mu_q^{(i)}$. In Table 40 we can see minimal numbers p^*, p_1^*, which satisfy the conditions (128), (129). Moreover,

$$\mathsf{M}\left\{\left(I_{(1)T,t}^{(i_1)}-I_{(1)T,t}^{(i_1)q}\right)^2\right\}=\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{130}$$

Table 39. Numbers q, q_1

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 40. Numbers p, p_1, p^*, p_1^*

$T-t$	0.08222	0.05020	0.02310	0.01956
p	8	21	96	133
p_1	1	1	3	4
p^*	23	61	286	398
p_1^*	1	2	4	5

Table 41. Confirmation of the formula (129)

$\varepsilon/(T-t)^3$	0.0629	0.0097	0.0010	$1.0129\cdot 10^{-4}$	$1.0132\cdot 10^{-5}$
q	1	10	100	1000	10000

where

$$I_{(1)T,t}^{(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right).$$

It is not difficult to see that numbers $q_{\rm trig}$ in Table 37 correspond to minimal numbers $q_{\rm trig}$, which satisfy the condition

$$\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^4.$$

From the other hand, the right-hand side of (52) includes only 2 random variables. In this situation we again can talk about the advantage of Ledendre polynomials.

In Table 41 we can see the numerical confirmation of (129) (ε is the left-hand side of (129)).

Let us compare computational costs for the approximation $I_{(10)T,t}^{*(i_1i_2)q}$ obtained from (56) by replacing ∞ with q (the case of Legendre olynomials) and for the approximation $I_{(10)T,t}^{*(i_1i_2)q}$ obtained by Theorem 3 (the case of trigonometric functions)

$$I_{(10)T,t}^{*(i_1i_2)q}=-(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right)-$$

TABLE 42. Confirmation of the formulas (132)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right)+$$

$$+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\right.$$

$$\left.+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\right)\Biggr). \tag{131}$$

For the formula (131) ($i_1\neq i_2$) from Theorem 8 we obtain [8]-[24]

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.$$

$$\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\neq l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\right). \tag{132}$$

In Table 42 we can see the numerical confirmation of (132) (ε is the right-hand side of (132)).

Let us compare the complexity of approximation based on the formula (56) with the complexity of approximation (131). The formula (131) includes the double sum

$$\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right).$$

Thus, the approximation (131) is more complex than the approximation based on the formula (56) even if we take identical numbers q in these approximations. As we noted above, the number q in (131) must be equal to the number q from the formula (115), so it is much larger than the number q from the approximation based on the formula (56). As s result, we have an obvious advantage of the Legendre polynomials in computational costs in the considered case. As we mentioned above, if we will not use the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the number q in (131) can be chosen smaller, but the mean-square error of approximation of the stochastic integral $I_{(00)T,t}^{(i_1i_2)}$ will be three times larger (see (110)). Moreover, in this case the stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$ (with Gaussian distribution) will be approximated worse. In this situation we can again talk about the advantage of Ledendre polynomials.

Summing up the results of this section, we obtain to the following conclusions.

(I) We can talk about the approximately equal computational costs for the formulas (115) and (120). This means that computational costs for the implementation of Milstein scheme (explicit one-step strong numerical method with the order of accuracy 1.0 for Ito SDEs) for the case of Legendre polynomials and for the case of trigonometric functions are approximately the same.

(II) If we will not use the random variables $\xi_q^{(i)}$ (see (115)), then the mean-square error of approximation of the stochastic integral $I_{(00)T,t}^{(i_1 i_2)}$ will be three times larger (see (110)). In this situation we can talk about the advantage of Ledendre polynomials within the frames of the Milstein scheme for Ito SDEs. Moreover, in this case the stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$ (with Gaussian distribution) will be approximated worse.

(III) If we talk about an explicit one-step strong Taylor–Ito scheme of the order of accuracy $\gamma = 1.5$ for Ito SDEs, then the numbers q, q_1 (see (92), (120)) are different. At that $q_1 \ll q$ (the case of Legendre polynomials). The number q must be the same in (114), (115), (122) (the case of trigonometric functions). This leads to huge computational costs (see very complex formula (122)). From the other hand, we can choose different numbers q in (114), (115), (122). At that we must exclude the random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (114), (115), (122). This leads to another problems which we discussed above (see Conclusion (II)).

(IV) In addition, the author of this article supposes that the effect described in Conclusion (III) will be more impressive when analyzing more complex families of iterated Ito and Stratonovich stochastic integrals (when $\gamma = 2.0,\ 2.5,\ 3.0,\ \ldots$). This supposition is based on the fact that the polynomial system of functions has the significant advantage (in comparison with the trigonometric system of functions) for approximation of iterated stochastic integrals for which not all weight functions are equal to 1.

5. Convergence With Probability 1 of Expansions of Iterated Stochastic Integrals of Multiplicities 1 and 2

Let us address now to the convergence with probability 1. Note that proving Theorem 1 [22] (Theorem 1.1, Sect. 1.1.3) or Theorem 2 [22] (Theorem 1.16, Sect. 1.11) we obtained the following representation

$$J[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr) + R_{T,t}^{p_1,\ldots,p_k}$$

w. p. 1, where

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int_t^T \ldots \int_t^{t_2} R_{p_1,\ldots,p_k}(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{133}$$

$$R_{p_1,\ldots,p_k}(t_1,\ldots,t_k) = K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l),$$

where permutations $(t_1,\ldots,t_k)$ when summing in (133) are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. Another notations are the same as in Theorems 1, 2.

Let us consider in detail the following expansion of iterated Ito stochastic integral

$$I_{(00)T,t}^{(i_1 i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right). \tag{134}$$

If $i_1=i_2$, then from (134) we obtain the following equality

$$I_{(00)T,t}^{(i_1 i_1)}=\frac{1}{2}(T-t)\left(\left(\zeta_0^{(i_1)}\right)^2-1\right),$$

which is correct w. p. 1 and can be obtained using the Ito formula.

Let us consider the case $i_1\ne i_2$. In this case

$$I_{(00)T,t}^{*(i_1 i_2)}=I_{(00)T,t}^{(i_1 i_2)}\quad \text{w. p. 1.}$$

First, note the well-known fact.

Lemma 1. *If for the sequence of random variables ξ_p and for some $\alpha>0$ the number series*

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^{\alpha}\right\}$$

converges, then the sequence ξ_p converges to zero w. p. 1.

In our specific case ($i_1\ne i_2$)

$$I_{(00)T,t}^{(i_1 i_2)}=I_{(00)T,t}^{(i_1 i_2)p}+\xi_p,\quad \xi_p=\frac{T-t}{2}\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right),$$

$$I_{(00)T,t}^{(i_1 i_2)p}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{p}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \tag{135}$$

Let

$$R_{T,t}^{p_1,p_2}\stackrel{\mathsf{def}}{=}R_{T,t}^{p},\quad R_{p_1p_2}(t_1,t_2)\stackrel{\mathsf{def}}{=}R_p(t_1,t_2)\quad \text{for}\quad p_1=p_2=p.$$

Then

$$\xi_p=R_{T,t}^p=\int_t^T\int_t^{t_2}R_p(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}+\int_t^T\int_t^{t_1}R_p(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)},$$

$$\mathsf{M}\{|\xi_p|^2\}=\int_t^T\int_t^{t_2}\left(R_p(t_1,t_2)\right)^2dt_1dt_2+\int_t^T\int_t^{t_1}\left(R_p(t_1,t_2)\right)^2dt_2dt_1=\int_{[t,T]^2}\left(R_p(t_1,t_2)\right)^2dt_1dt_2, \tag{136}$$

$$\mathsf{M}\left\{|\xi_p|^2\right\}=\frac{(T-t)^2}{2}\sum_{i=p+1}^{\infty}\frac{1}{4i^2-1}, \tag{137}$$

$$R_p(t_1,t_2)=K(t_1,t_2)-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2),$$

(138) $$\sum_{i=p+1}^{\infty}\frac{1}{4i^2-1}\le\int\limits_{p}^{\infty}\frac{1}{4x^2-1}dx=-\frac{1}{4}\ln\left|1-\frac{2}{2p+1}\right|\le\frac{C}{p},$$

where constant C does not depend on p.

Therefore, taking $\alpha=2$ in Lemma 1, we cannot prove the convergence of ξ_p to zero w. p. 1, since the series

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^2\right\}$$

will be majorized by the divergent Dirichlet series with the index 1. Let us take $\alpha=4$ and estimate the value $\mathsf{M}\left\{|\xi_p|^4\right\}$.

According to (33), we can write

(139) $$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le C_{n,k}\left(\int\limits_{[t,T]^k}R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^n,$$

where $C_{n.k}=(k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$

From (139) for $k=2$, $n=2$ and (136)–(138) we obtain

(140) $$\mathsf{M}\left\{|\xi_p|^4\right\}\le K\left(\int\limits_{[t,T]^2}R_p^2(t_1,t_2)dt_1dt_2\right)^2\le\frac{K_1}{p^2}$$

and

(141) $$\sum_{p=1}^{\infty}\mathsf{M}\left\{|\xi_p|^4\right\}\le K_1\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty,$$

where constants K, K_1 do not depend on p.

Since the series in (141) converges, then according to Lemma 1 we obtain that $\xi_p\to 0$ when $p\to\infty$ w. p. 1. Then

$$I_{(00)T,t}^{(i_1i_2)p}\to I_{(00)T,t}^{(i_1i_2)}\ \ \text{when}\ \ p\to\infty\ \ \text{w. p. 1.}$$

Let us consider the stochastic integrals $I_{(01)T,t}^{*(i_1i_2)}$, $I_{(10)T,t}^{*(i_1i_2)}$ whose expansions look as (55), (56).

Consider the case $i_1\ne i_2$. In this case

$$I_{(01)T,t}^{*(i_1i_2)}=I_{(01)T,t}^{(i_1i_2)},\quad I_{(10)T,t}^{*(i_1i_2)}=I_{(10)T,t}^{(i_1i_2)},\quad I_{(00)T,t}^{*(i_1i_2)}=I_{(00)T,t}^{(i_1i_2)}\quad\text{w. p. 1,}$$

and

$$I_{(01)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)p} - \frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)} + \right.$$

$$\left. + \sum_{i=0}^{p}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right) + \xi_p^{(01)},$$

$$I_{(10)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)p} - \frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} + \right.$$

$$\left. + \sum_{i=0}^{p}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right) + \xi_p^{(10)},$$

where

$$\xi_p^{(01)} = -\frac{(T-t)^2}{4}\left(\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) + \right.$$

$$\left. + \sum_{i=p+1}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),$$

$$\xi_p^{(10)} = -\frac{(T-t)^2}{4}\left(\sum_{i=p+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) + \right.$$

$$\left. + \sum_{i=p+1}^{\infty}\left(\frac{(i+1)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+2)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right).$$

Then

$$\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^2\right\} = \int\limits_{[t,T]^2}\left(R_p^{(01)}(t_1,t_2)\right)^2 dt_1 dt_2 = \frac{(T-t)^4}{16}\times$$

$$\times \sum_{i=p+1}^{\infty}\left(\frac{2}{4i^2-1} + \frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2} + \frac{1}{(2i-1)^2(2i+3)^2}\right) \le C\sum_{i=p+1}^{\infty}\frac{1}{i^2} \le \frac{K}{p}, \tag{142}$$

where constants C, K do not depend on p.

Analogously, we obtain

$$\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^2\right\} = \int\limits_{[t,T]^2}\left(R_p^{(10)}(t_1,t_2)\right)^2 dt_1 dt_2 \le \frac{K}{p}, \tag{143}$$

where constant K does not depend on p.

According (139) when $k=2$, $n=2$ and (142), (143), we obtain

$$
\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\} \le K\left(\int\limits_{[t,T]^2}\left(R_p^{(01)}(t_1,t_2)\right)^2 dt_1dt_2\right)^2 \le \frac{K_1}{p^2},
$$

$$
\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\} \le K\left(\int\limits_{[t,T]^2}\left(R_p^{(10)}(t_1,t_2)\right)^2 dt_1dt_2\right)^2 \le \frac{K_1}{p^2},
$$

and

$$
\sum_{p=1}^{\infty}\mathsf{M}\left\{\left|\xi_p^{(01)}\right|^4\right\} \le K_1\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \quad \sum_{p=1}^{\infty}\mathsf{M}\left\{\left|\xi_p^{(10)}\right|^4\right\} \le K_1\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty, \tag{144}
$$

where constant K_1 does not depend on p.

From (144) and Lemma 1 we obtain that $\xi_p^{(01)},\ \xi_p^{(10)} \to 0$ when $p\to\infty$ w. p. 1. Then

$$
I_{(01)T,t}^{(i_1i_2)p} \to I_{(01)T,t}^{(i_1i_2)}, \quad I_{(10)T,t}^{(i_1i_2)p} \to I_{(10)T,t}^{(i_1i_2)} \quad \text{when} \quad p\to\infty \quad \text{w. p. 1},
$$

where $i_1\ne i_2$.

Let us consider the case $i_1=i_2$

$$
I_{(01)T,t}^{(i_1i_1)} = -\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2 - 1 + \frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)} + \right.
$$

$$
\left. + \sum_{i=0}^{p}\left(\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)} - \frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right) + \mu_p^{(01)},
$$

$$
I_{(10)T,t}^{(i_1i_1)} = -\frac{(T-t)^2}{4}\left(\left(\zeta_0^{(i_1)}\right)^2 - 1 + \frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)} + \right.
$$

$$
\left. + \sum_{i=0}^{p}\left(-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)} + \frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right)\right) + \mu_p^{(10)},
$$

where

$$
\mu_p^{(01)} = -\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)} - \frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right),
$$

$$
\mu_p^{(10)} = -\frac{(T-t)^2}{4}\sum_{i=p+1}^{\infty}\left(-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)} + \frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right).
$$

Then

$$\mathsf{M}\left\{\left(\mu_p^{(01)}\right)^2\right\} = \mathsf{M}\left\{\left(\mu_p^{(10)}\right)^2\right\} = \frac{(T-t)^4}{16}\times$$

$$\times\left(\sum_{i=p+1}^{\infty}\frac{1}{(2i+1)(2i+5)(2i+3)^2} + \sum_{i=p+1}^{\infty}\frac{2}{(2i-1)^2(2i+3)^2} + \left(\sum_{i=p+1}^{\infty}\frac{1}{(2i-1)(2i+3)}\right)^2\right) \le \frac{K}{p^2}$$

and

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{\left|\mu_p^{(01)}\right|^2\right\} \le K\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty, \quad \sum_{p=1}^{\infty}\mathsf{M}\left\{\left|\mu_p^{(10)}\right|^2\right\} \le K\sum_{p=1}^{\infty}\frac{1}{p^2} < \infty, \tag{145}$$

where constant K does not depend on p.

According Lemma 1 and (145), we obtain that $\mu_p^{(01)},\ \mu_p^{(10)} \to 0$ when $p\to\infty$ w. p. 1. Then

$$I_{(01)T,t}^{(i_1i_1)p} \to I_{(01)T,t}^{(i_1i_1)}, \quad I_{(10)T,t}^{(i_1i_1)p} \to I_{(10)T,t}^{(i_1i_1)} \quad \text{when} \quad p\to\infty \quad \text{w. p. 1.}$$

Analogously, we obtain

$$I_{(02)T,t}^{(i_1i_2)p} \to I_{(02)T,t}^{(i_1i_2)}, \quad I_{(11)T,t}^{(i_1i_2)p} \to I_{(11)T,t}^{(i_1i_2)}, \quad I_{(20)T,t}^{(i_1i_2)p} \to I_{(20)T,t}^{(i_1i_2)} \quad \text{when} \quad p\to\infty \quad \text{w. p. 1,}$$

where $i_1, i_2 = 1,\ldots,m$. This result based on the following truncated expansions of the stochastic integrals $I_{(02)T,t}^{(i_1i_2)}$, $I_{(20)T,t}^{(i_1i_2)}$, $I_{(11)T,t}^{(i_1i_2)}$ (see (68)–(70))

$$I_{(02)T,t}^{(i_1i_2)p} = -\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1i_2)p} - (T-t)I_{(01)T,t}^{(i_1i_2)p} + \frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} + \right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{p}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.$$

$$\left.\left. + \frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,$$

$$I_{(20)T,t}^{(i_1i_2)p} = -\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1i_2)p} - (T-t)I_{(10)T,t}^{(i_1i_2)p} + \frac{(T-t)^3}{8}\left[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} + \right.$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{p}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.$$

$$\left.\left. + \frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,$$

$$I_{(11)T,t}^{(i_1 i_2)p} = -\frac{(T-t)^2}{4} I_{(00)T,t}^{(i_1 i_2)p} - \frac{T-t}{2}\left(I_{(10)T,t}^{(i_1 i_2)p} + I_{(01)T,t}^{(i_1 i_2)p}\right) +$$

$$+\frac{(T-t)^3}{8}\left[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)} + \sum_{i=0}^{p}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} + \right.\right.$$

$$\left.\left. + \frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\right] -$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3.$$

The expansions (51)–(53), (71) for the stochastic integrals $I_{(0)T,t}^{(i_1)}$, $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$, $I_{(3)T,t}^{(i_1)}$ are initially correct w. p. 1 (they include 1, 2, 3, and 4 members of expansion, correspondently).

Apparently, using the proposed scheme we can prove convergence w. p. 1 for other iterated stochastic integrals. In the next section, we consider the more general and effective approach.

6. Convergence With Probability 1 of Expansion of Iterated Ito Stochastic Integrals in Theorem 1 for the Case of Multiplicity k ($k \in \mathbb{N}$)

This section is written on the base of Sect. 6 from [31] and Sect. 9 from [25] (also see [22]-[24] (Chapter 1)). Remind that in a lot of author's publications [8]-[41] the convergence in Theorem 1 has been considered in different probabilistic senses. For example, the mean-square convergence [8] (2006) (also see [9]-[41]) and convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) [10] (2007) (also see [11]-[17], [20]-[25]) have been proved. On the examples of specific iterated Ito stochastic integrals of mutiplicities 1 and 2 the convergence with probability 1 has been considered in the previous section (also see [10] (2007), [11]-[17], [20]-[25], [27]). However, these examples are narrow particular cases of the iterated Ito stochastic integrals (2).

In this section, we formulate and prove the theorem [22]-[25], [31], [42] on convergence with probability 1 of the expansions of iterated Ito stochastic integrals from Theorem 1.

Let us remind the well-known fact from the mathematical analysis, which is connected to existence of iterated limits.

Proposition 1. *Let* $\left\{x_{n,m}\right\}_{n,m=1}^{\infty}$ *be a double sequence and let there exists the limit*

$$\lim_{n,m\to\infty} x_{n,m} = a < \infty.$$

Moreover, let there exist the limits

$$\lim_{n\to\infty} x_{n,m} < \infty \quad \text{for any} \quad m, \qquad \lim_{m\to\infty} x_{n,m} < \infty \quad \text{for any} \quad n.$$

Then there exist the iterated limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m}, \qquad \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m}$$

and moreover,

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = a.$$

Theorem 9 [22]-[25], [31], [42]. *Let $\psi_l(\tau)$ $(l=1,\ldots,k)$ are continuously differentiable nonrandom functions on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad if \quad p\to\infty$$

w. p. 1, where $J[\psi^{(k)}]_{T,t}^{p,\ldots,p}$ is the expression on the right-hand side of (7) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=p$, i.e. (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),$$

where $i_1,\ldots,i_k=1,\ldots,m$.

Proof. Let us consider the Parseval equality

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k=\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2, \tag{146}$$

where

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\ =\ \prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{147}$$

where $t_1,\ldots,t_k\in[t,T]$ for $k\ge 2$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, $\mathbf{1}_A$ denotes the indicator of the set A,

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{148}$$

is the Fourier coefficient.

Using (147), we obtain

$$C_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\psi_k(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_k.$$

Further, we denote

$$\lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\ \stackrel{\mathsf{def}}{=}\ \sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2.$$

If $p_1 = \ldots = p_k = p$, then we also write

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C^2_{j_k\ldots j_1} \stackrel{\sf def}{=} \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

From the other hand, for iterated limits we write

$$\lim_{p_1\to\infty}\ldots\lim_{p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \stackrel{\sf def}{=} \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1},$$

$$\lim_{p_1\to\infty}\lim_{p_2,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \stackrel{\sf def}{=} \sum_{j_1=0}^{\infty}\sum_{j_2,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}$$

and so on.

Let us consider the following lemma.

Lemma 2. *The following equalities are fulfilled*

$$\sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_k=0}^{\infty}\ldots\sum_{j_1=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_{q_1}=0}^{\infty}\ldots\sum_{j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} \qquad (149)$$

for any permutation $(q_1,\ldots,q_k)$ *such that* $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$.

Proof. Let us consider the value

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} \qquad (150)$$

for any permutation $(q_l,\ldots,q_k)$, where $l = 1,2,\ldots,k$, $\{q_1,\ldots,q_k\} = \{1,\ldots,k\}$.

Obviously, (150) is the non-decreasing sequence with respect to p. Moreover,

$$\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} \le \sum_{j_{q_1}=0}^{p}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} \le$$

$$\le \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} < \infty.$$

Then the following limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} = \sum_{j_{q_l},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1}$$

exists.

Let $p_l,\ldots,p_k$ simultaneously tend to infinity. Then $g,r\to\infty$, where $g=\min\{p_l,\ldots,p_k\}$ and $r=\max\{p_l,\ldots,p_k\}$. Moreover,

$$\sum_{j_{q_l}=0}^{g}\ldots\sum_{j_{q_k}=0}^{g}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2\le\sum_{j_{q_l}=0}^{r}\ldots\sum_{j_{q_k}=0}^{r}C_{j_k\ldots j_1}^2.$$

This means that the existence of the limit

$$\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2 \tag{151}$$

implies the existence of the limit

$$\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2 \tag{152}$$

and equality of the limits (151) and (152).

Taking into account the above reasoning, we have

$$\lim_{p,q\to\infty}\sum_{j_{q_l}=0}^{q}\sum_{j_{q_{l+1}}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\lim_{p\to\infty}\sum_{j_{q_l}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{p_l,\ldots,p_k\to\infty}\sum_{j_{q_l}=0}^{p_l}\ldots\sum_{j_{q_k}=0}^{p_k}C_{j_k\ldots j_1}^2. \tag{153}$$

Since the limit

$$\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2$$

exists (see the Parseval equality (146)), then from Proposition 1 we have

$$\sum_{j_{q_1}=0}^{\infty}\sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$=\lim_{q,p\to\infty}\sum_{j_{q_1}=0}^{q}\sum_{j_{q_2}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2. \tag{154}$$

Using (153) and Proposition 1, we get

$$\sum_{j_{q_2}=0}^{\infty}\sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty}C_{j_k\ldots j_1}^2=\lim_{q\to\infty}\lim_{p\to\infty}\sum_{j_{q_2}=0}^{q}\sum_{j_{q_3}=0}^{p}\ldots\sum_{j_{q_k}=0}^{p}C_{j_k\ldots j_1}^2=$$

$$= \lim_{q,p\to\infty} \sum_{j_{q_2}=0}^{q} \sum_{j_{q_3}=0}^{p} \ldots \sum_{j_{q_k}=0}^{p} C^2_{j_k\ldots j_1} = \sum_{j_{q_2},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{155}$$

Combining (155) and (154), we obtain

$$\sum_{j_{q_1}=0}^{\infty} \sum_{j_{q_2}=0}^{\infty} \sum_{j_{q_3},\ldots,j_{q_k}=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1,\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1}.$$

Repeating the previous steps, we complete the proof of Lemma 2.

Further, let us show that for $s = 1, \ldots, k$

$$\sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1}. \tag{156}$$

Using the arguments which we used when proving Lemma 2, we obtain

$$\lim_{n\to\infty} \sum_{j_1=0}^{n} \ldots \sum_{j_{s-1}=0}^{n} \sum_{j_s=0}^{p} \sum_{j_{s+1}=0}^{n} \ldots \sum_{j_k=0}^{n} C^2_{j_k\ldots j_1} =$$

$$= \sum_{j_s=0}^{p} \sum_{j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_s=0}^{p} \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} \tag{157}$$

for any permutation $(q_1, \ldots, q_{k-1})$ such that $\{q_1, \ldots, q_{k-1}\} = \{1, \ldots, s-1, s+1, \ldots, k\}$, where p is a fixed natural number.

Obviously, we have

$$\sum_{j_s=0}^{p} \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_s=0}^{p} \ldots \sum_{j_{q_{k-1}}=0}^{\infty} C^2_{j_k\ldots j_1} = \ldots =$$

$$= \sum_{j_{q_1}=0}^{\infty} \ldots \sum_{j_{q_{k-1}}=0}^{\infty} \sum_{j_s=0}^{p} C^2_{j_k\ldots j_1}. \tag{158}$$

Using (157), (158), and Lemma 2, we get

$$\sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=0}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} -$$

$$
-\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=0}^{p}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=0}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_s=0}^{p}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2.
$$

The equality (156) is proved.
Using the Parseval equality and Lemma 2, we obtain

$$
\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_1,\ldots,j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}+
$$

$$
+\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2=\ldots=
$$

$$
=\sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+
$$

$$
+\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2+\ldots+\sum_{j_1=0}^{p}\ldots\sum_{j_{k-1}=0}^{p}\sum_{j_k=p+1}^{\infty}C_{j_k\ldots j_1}^2\le
$$

$$\leq \sum_{j_1=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \sum_{j_1=0}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 +$$

$$+\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_4=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 + \ldots + \sum_{j_1=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\sum_{j_k=p+1}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{s=1}^{k}\left(\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2\right). \tag{159}$$

Note that deriving (159), we used the following

$$\sum_{j_1=0}^{p}\ldots\sum_{j_{s-1}=0}^{p}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \lim_{m_{s-1}\to\infty}\sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-1}=0}^{m_{s-1}}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 =$$

$$= \sum_{j_1=0}^{m_1}\ldots\sum_{j_{s-2}=0}^{m_{s-2}}\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2 \leq$$

$$\leq \ldots \leq$$

$$\leq \sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty} C_{j_k\ldots j_1}^2,$$

where $m_1,\ldots,m_{s-1} > p$.

Denote

$$C_{j_s\ldots j_1}(\tau) = \int\limits_t^{\tau}\phi_{j_s}(t_s)\psi_s(t_s)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s,$$

where $s = 1,\ldots,k-1$.

Let us remind the Dini Theorem, which we will use further.

Theorem (Dini). *Let the functional sequence $u_n(x)$ be non-decreasing at each point of the interval $[a,b]$. In addition, all the functions $u_n(x)$ of this sequence and the limit function $u(x)$ are continuous on the interval $[a,b]$. Then the convergence $u_n(x)$ to $u(x)$ is uniform on the interval $[a,b]$.*

For $s < k$ due to the Parseval equality, Dini Theorem and (156) we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 =$$

$$\stackrel{(156)}{=}\quad \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2 =$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =$$

$$\stackrel{\text{(Dini Th.)}}{=}\quad \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\sum_{j_{k-1}=0}^{\infty}\left(C_{j_{k-1}\ldots j_1}(t_k)\right)^2 dt_k =$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad \sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\psi_k^2(t_k)\int\limits_t^{t_k}\psi_{k-1}^2(t_{k-1})\left(C_{j_{k-2}\ldots j_1}(t_{k-1})\right)^2\times$$

$$\times dt_{k-1}dt_k \le$$

$$\le C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =$$

$$\stackrel{\text{(Dini Th.)}}{=}\quad C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\sum_{j_{k-2}=0}^{\infty}\left(C_{j_{k-2}\ldots j_1}(\tau)\right)^2 d\tau =$$

$$\stackrel{\text{(Parseval Eq.)}}{=}\quad C\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\int\limits_t^{\tau}\psi_{k-2}^2(\theta)\left(C_{j_{k-3}\ldots j_1}(\theta)\right)^2 d\theta d\tau \le$$

$$\le K\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_{k-3}=0}^{\infty}\int\limits_t^T\left(C_{j_{k-3}\ldots j_1}(\tau)\right)^2 d\tau \le$$

$$\le\ldots\le$$

$$\le C_k\sum_{j_s=p+1}^{\infty}\sum_{j_{s-1}=0}^{\infty}\ldots\sum_{j_1=0}^{\infty}\int\limits_t^T\left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

(160) $$\stackrel{\text{(Dini Th.)}}{=} \quad C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \sum_{j_1=0}^{\infty} \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau,$$

where constants C, K depend on $T-t$ and constant C_k depends on k and $T-t$.

Let us explane more precisely how we obtain (160). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$\sum_{j=0}^{\infty} \left(\int\limits_t^{\tau} \phi_j(s)g(s)ds\right)^2 = \sum_{j=0}^{\infty} \left(\int\limits_t^{T} \mathbf{1}_{\{s<\tau\}}\phi_j(s)g(s)ds\right)^2 =$$

(161) $$= \int\limits_t^T \left(\mathbf{1}_{\{s<\tau\}}\right)^2 g^2(s)ds = \int\limits_t^{\tau} g^2(s)ds.$$

The equality (161) has been applied repeatedly when we obtaining (160).

Using the replacement of the integrating order in Riemann integrals, we have

$$C_{j_s\ldots j_1}(\tau) = \int\limits_t^{\tau} \phi_{j_s}(t_s)\psi_s(t_s)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\psi_1(t_1)dt_1\ldots dt_s =$$

$$= \int\limits_t^{\tau} \phi_{j_1}(t_1)\psi_1(t_1) \int\limits_{t_1}^{\tau} \phi_{j_2}(t_2)\psi_2(t_2)\ldots \int\limits_{t_{s-1}}^{\tau} \phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_2dt_1 \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \tilde{C}_{j_s\ldots j_1}(\tau).$$

For $l=1,\ldots,s$ we will use the following notation

$$\tilde{C}_{j_s\ldots j_l}(\tau,\theta) = \int\limits_{\theta}^{\tau} \phi_{j_l}(t_l)\psi_l(t_l) \int\limits_{t_l}^{\tau} \phi_{j_{l+1}}(t_{l+1})\psi_{l+1}(t_{l+1})\ldots \int\limits_{t_{s-1}}^{\tau} \phi_{j_s}(t_s)\psi_s(t_s)dt_s\ldots dt_{l+1}dt_l.$$

Using the Parseval equality and Dini Theorem, from (160) we obtain

$$\sum_{j_1=0}^{\infty}\ldots \sum_{j_{s-1}=0}^{\infty} \sum_{j_s=p+1}^{\infty} \sum_{j_{s+1}=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C^2_{j_k\ldots j_1} \le$$

$$\le C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \sum_{j_1=0}^{\infty} \left(C_{j_s\ldots j_1}(\tau)\right)^2 d\tau =$$

$$
= C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \sum_{j_1=0}^{\infty} \left(\tilde{C}_{j_s\ldots j_1}(\tau)\right)^2 d\tau =
$$

$$
\stackrel{\text{(Parseval Eq.)}}{=} C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_2=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau = \tag{162}
$$

$$
\stackrel{\text{(Dini Th.)}}{=} C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=0}^{\infty} \left(\tilde{C}_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1 d\tau = \tag{163}
$$

$$
\stackrel{\text{(Parseval Eq.)}}{=} C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t_1}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2 dt_2 dt_1 d\tau \le
$$

$$
\le C_k \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \int\limits_{t}^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2 dt_2 dt_1 d\tau \le
$$

$$
\le C_k^{'} \sum_{j_s=p+1}^{\infty} \sum_{j_{s-1}=0}^{\infty} \ldots \sum_{j_3=0}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_2^2(t_2) \left(\tilde{C}_{j_s\ldots j_3}(\tau,t_2)\right)^2 dt_2 d\tau \le
$$

$$
\le \ldots \le
$$

$$
\le C_k^{''} \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_{s-1}^2(t_{s-1}) \left(\tilde{C}_{j_s}(\tau,t_{s-1})\right)^2 dt_{s-1} d\tau \le
$$

$$
\le \tilde{C}_k \sum_{j_s=p+1}^{\infty} \int\limits_t^T \int\limits_t^{\tau} \left(\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta)d\theta \right)^2 du d\tau, \tag{164}
$$

where constants $C_k^{'}$, $C_k^{''}$, $\tilde{C}_k$ depend on k and $T-t$.

Let us explane more precisely how we obtain (164). For any function $g(s) \in L_2([t,T])$ we have the following Parseval equality

$$
\sum_{j=0}^{\infty} \left(\int\limits_{\theta}^{\tau} \phi_j(s)g(s)ds \right)^2 = \sum_{j=0}^{\infty} \left(\int\limits_t^T \mathbf{1}_{\{\theta<s<\tau\}} \phi_j(s)g(s)ds \right)^2 =
$$

$$
=\int\limits_t^T\left(\mathbf{1}_{\{\theta<s<\tau\}}\right)^2 g^2(s)ds=\int\limits_\theta^\tau g^2(s)ds. \tag{165}
$$

The equality (165) has been applied repeatedly when we obtaining (164).

Let us explane more precisely the passing from (162) to (163) (the same steps have been used when we deriving (164)).

We have

$$
\int\limits_t^T\int\limits_t^\tau \psi_1^2(t_1)\sum_{j_2=0}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau-\sum_{j_2=0}^{n}\int\limits_t^T\int\limits_t^\tau \psi_1^2(t_1)\left(\tilde C_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau=
$$

$$
=\int\limits_t^T\int\limits_t^\tau \psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau,t_1)\right)^2 dt_1d\tau=
$$

$$
=\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2 dt_1\Delta\tau_j, \tag{166}
$$

where $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition (6).

Since the non-decreasing functional sequence $u_n(\tau_j,t_1)$ and its limit function $u(\tau_j,t_1)$ are continuous on the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1, where

$$
u_n(\tau_j,t_1)=\sum_{j_2=0}^{n}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2,
$$

$$
u(\tau_j,t_1)=\sum_{j_2=0}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2=\int\limits_{t_1}^{\tau_j}\psi_2^2(t_2)\left(\tilde C_{j_s\ldots j_3}(\tau_j,t_2)\right)^2 dt_2,
$$

then by Dini Theorem we have the uniform convergence of $u_n(\tau_j,t_1)$ to $u(\tau_j,t_1)$ at the interval $[t,\tau_j]\subseteq[t,T]$ with respect to t_1. As a result, we obtain

$$
\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2<\varepsilon,\quad t_1\in[t,\tau_j] \tag{167}
$$

for $n>N(\varepsilon)$ ($N(\varepsilon)$ exists for any $\varepsilon>0$ and it does not depend on t_1).

From (166) and (167) we obtain

$$
\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)\sum_{j_2=n+1}^{\infty}\left(\tilde C_{j_s\ldots j_2}(\tau_j,t_1)\right)^2 dt_1\Delta\tau_j\le\varepsilon\lim_{N\to\infty}\sum_{j=0}^{N-1}\int\limits_t^{\tau_j}\psi_1^2(t_1)dt_1\Delta\tau_j=
$$

$$
= \varepsilon \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) dt_1 d\tau. \tag{168}
$$

Using (168), we get

$$
\lim_{n\to\infty} \int\limits_t^T \int\limits_t^{\tau} \psi_1^2(t_1) \sum_{j_2=n+1}^{\infty} \left(\tilde{C}_{j_s\ldots j_2}(\tau, t_1)\right)^2 dt_1 d\tau = 0.
$$

This fact completes the proof of passing from (162) to (163).

Let us estimate the integral

$$
\int\limits_u^{\tau} \phi_{j_s}(\theta)\psi_s(\theta) d\theta \tag{169}
$$

from (164) for the case when $\{\phi_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

Note that the estimates for the integral

$$
\int\limits_t^{\tau} \phi_j(\theta)\psi(\theta) d\theta, \quad j \ge p+1, \tag{170}
$$

where $\psi(\theta)$ is a continuously differentiable function on the interval $[t, T]$, have been obtained in [26], [32]. The same estimates can also be found in early publications [13]-[17], [20], [21] and in the monographs [22]-[24].

Let us estimate the integral (169) using the approach from [26], [32].

First, consider the case of Legendre polynomials. Then $\phi_j(s)$ looks as follows

$$
\phi_j(\theta) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(\theta - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j \ge 0, \tag{171}
$$

where $P_j(x)$ $(j = 0, 1, 2\ldots)$ is the Legendre polynomial.

Further, we have

$$
\int\limits_v^x \phi_j(\theta)\psi(\theta) d\theta = \frac{\sqrt{T-t}\sqrt{2j+1}}{2} \int\limits_{z(v)}^{z(x)} P_j(y)\psi(u(y)) dy =
$$

$$
= \frac{\sqrt{T-t}}{2\sqrt{2j+1}} \Biggl((P_{j+1}(z(x)) - P_{j-1}(z(x)))\psi(x) - (P_{j+1}(z(v)) - P_{j-1}(z(v)))\psi(v) -
$$

$$
- \frac{T-t}{2} \int\limits_{z(v)}^{z(x)} ((P_{j+1}(y) - P_{j-1}(y))\psi'(u(y)) dy \Biggr), \tag{172}
$$

where $x, v \in (t, T)$, $j \ge p+1$, $u(y)$ and $z(x)$ are defined by the following relations

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(x) = \left(x - \frac{T+t}{2}\right)\frac{2}{T-t},$$

and ψ' is a derivative of the function $\psi(\theta)$ with respect to the variable $u(y)$.

Note that in (172) we used the following well known property of the Legendre polynomials

$$\frac{dP_{j+1}}{dx}(x) - \frac{dP_{j-1}}{dx}(x) = (2j+1)P_j(x), \quad j = 1, 2, \ldots$$

From (172) and the well known estimate for the Legendre polynomials

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j \in \mathbb{N}, \tag{173}$$

where constant K does not depend on y and j, it follows that

$$\left|\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right| < \frac{C}{j}\left(\frac{1}{(1-(z(x))^2)^{1/4}} + \frac{1}{(1-(z(v))^2)^{1/4}} + C_1\right), \tag{174}$$

where $j \in \mathbb{N}$, $z(x), z(v) \in (-1, 1)$, $x, v \in (t, T)$, constants C, C_1 do not depend on j.

From (174) we obtain

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2 < \frac{C_2}{j^2}\left(\frac{1}{(1-(z(x))^2)^{1/2}} + \frac{1}{(1-(z(v))^2)^{1/2}} + C_3\right), \tag{175}$$

where $j \in \mathbb{N}$, constants C_2, C_3 do not depend on j.

Let us apply (175) for estimating of the right-hand side of (164). We have

$$\int\limits_t^T \int\limits_t^\tau \left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 du d\tau \le$$

$$\le \frac{K_1}{j_s^2}\left(\int\limits_{-1}^1 \frac{dy}{(1-y^2)^{1/2}} + \int\limits_{-1}^1\int\limits_{-1}^x \frac{dy}{(1-y^2)^{1/2}}dx + K_2\right) \le$$

$$\le \frac{K_3}{j_s^2}, \tag{176}$$

where $j_s \in \mathbb{N}$, constants K_1, K_2, K_3 are independent of j_s.

Now, consider the trigonometric case. The complete orthonormal system of trigonometric functions in the space $L_2([t, T])$ has the following form

$$\phi_j(\theta)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & j=0\\ \sqrt{2}\sin\left(2\pi r(\theta-t)/(T-t)\right), & j=2r-1,\\ \sqrt{2}\cos\left(2\pi r(\theta-t)/(T-t)\right), & j=2r\end{cases} \tag{177}$$

where $r=1,2,\ldots$

Using the system of functions (177), we have

$$\int\limits_v^x \phi_{2r-1}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=-\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\cos\frac{2\pi r(x-t)}{T-t}-\psi(v)\cos\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{178}$$

$$\int\limits_v^x \phi_{2r}(\theta)\psi(\theta)d\theta=\sqrt{\frac{2}{T-t}}\int\limits_v^x \cos\frac{2\pi r(\theta-t)}{T-t}\psi(\theta)d\theta=$$

$$=\sqrt{\frac{T-t}{2}}\frac{1}{\pi r}\left(\psi(x)\sin\frac{2\pi r(x-t)}{T-t}-\psi(v)\sin\frac{2\pi r(v-t)}{T-t}-\right.$$

$$\left.-\int\limits_v^x \sin\frac{2\pi r(\theta-t)}{T-t}\psi'(\theta)d\theta\right), \tag{179}$$

where $\psi'(\theta)$ is a derivative of the function $\psi(\theta)$ with respect to the variable θ.

Combining (178) and (179), we obtain for the trigonometric case

$$\left(\int\limits_v^x \phi_j(\theta)\psi(\theta)d\theta\right)^2\le\frac{C_4}{j^2}, \tag{180}$$

where $j\in\mathbb{N}$, constant C_4 is independent of j.

From (180) we finally have

$$\int\limits_t^T\int\limits_t^\tau\left(\int\limits_u^\tau \phi_{j_s}(\theta)\psi_s(\theta)d\theta\right)^2 dud\tau\le\frac{K_4}{j_s^2}, \tag{181}$$

where $j_s\in\mathbb{N}$, constant K_4 is independent of j_s.

Combibing (164), (176), and (181), we obtain

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_{s-1}=0}^{\infty}\sum_{j_s=p+1}^{\infty}\sum_{j_{s+1}=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}^2\le$$

$$\le L_k\sum_{j_s=p+1}^{\infty}\frac{1}{j_s^2}\le\frac{L_k}{p}, \tag{182}$$

where constant L_k depends on k and $T-t$.

Obviously, the case $s=k$ can be considered absolutely analogously to the case $s<k$. Then from (159) and (182) we obtain

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\le\frac{G_k}{p}, \tag{183}$$

where constant G_k depends on k and $T-t$.

For the further consideration we will use the estimate (33). Using (183) and the estimate (33) for the case $p_1=\ldots=p_k=p$ and $n=2$, we get

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\}\le$$

$$\le C_{2,k}\left(\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2\right)^2\le\frac{H_{2,k}}{p^2}, \tag{184}$$

where

$$C_{n,k}=(k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

and $H_{2,k}=G_k^2C_{2,k}$.

Let us consider Lemma 1 and put

$$\xi_p=\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|$$

and $\alpha=4$.

Then from (184) we obtain

$$\sum_{p=1}^{\infty}\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^4\right\}\le$$

$$\le H_{2,k}\sum_{p=1}^{\infty}\frac{1}{p^2}<\infty. \tag{185}$$

Using Lemma 1, from (185) we have

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} \ \to \ J[\psi^{(k)}]_{T,t} \quad \text{if} \quad p\to\infty$$

w. p. 1, where (see Theorem 1)

$$J[\psi^{(k)}]_{T,t}^{p,\ldots,p} = \sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{186}$$

where $i_1,\ldots,i_k=1,\ldots,m$ in (186). Theorem 9 is proved.

Taking into account (32) and (183), we obtain the following inequality

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^2\right\} \le \frac{k! P_k (T-t)^k}{p}, \tag{187}$$

where constant P_k depends only on k.

The estimates (33) and (183) imply the following inequality

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\} \le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\ \frac{(P_k)^n\,(T-t)^{nk}}{p^n}, \tag{188}$$

where $n\in\mathbb{N}$ and constant P_k depends only on k.

Consider the question on the rate of convergence w. p. 1 in Theorem 9. Using the inequality (188), we obtain

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{2n}\right\}\right)^{1/2n} \le \frac{Q_{n,k}}{\sqrt{p}}, \tag{189}$$

where $n\in\mathbb{N}$ and

$$Q_{n,k} = k!\ (n(2n-1))^{(k-1)/2}\,((2n-1)!!)^{1/2n}\ \sqrt{P_k}\ (T-t)^{k/2}.$$

According to the Lyapunov inequality and (189), we have

$$\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right)^{n}\right\}\right)^{1/n} \le \frac{Q_{n,k}}{\sqrt{p}} \tag{190}$$

for all $n\in\mathbb{N}$. Following [53] (Lemma 2.1), we get

$$\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|=\frac{p^{1/2-\varepsilon}}{p^{1/2-\varepsilon}}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\le$$

$$\le\frac{1}{p^{1/2-\varepsilon}}\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)=\frac{\eta_{\varepsilon}}{p^{1/2-\varepsilon}} \tag{191}$$

w. p. 1, where

$$\eta_{\varepsilon}=\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)$$

and $\varepsilon>0$ is fixed.

For $q>1/\varepsilon$, $q\in\mathbb{N}$ we obtain [53] (see (190))

$$\mathsf{M}\left\{|\eta_{\varepsilon}|^q\right\}=\mathsf{M}\left\{\left(\sup_{p\in\mathbb{N}}\left(p^{1/2-\varepsilon}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|\right)\right)^q\right\}=$$

$$=\mathsf{M}\left\{\sup_{p\in\mathbb{N}}\left(p^{(1/2-\varepsilon)q}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right)\right\}\le$$

$$\le\mathsf{M}\left\{\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right\}=$$

$$=\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\mathsf{M}\left\{\left|J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p,\ldots,p}\right|^q\right\}\le$$

$$\le\sum_{p=1}^{\infty}p^{(1/2-\varepsilon)q}\frac{(Q_{q,k})^q}{p^{q/2}}=(Q_{q,k})^q\sum_{p=1}^{\infty}\frac{1}{p^{\varepsilon q}}<\infty. \tag{192}$$

From (191) we have that for all $\varepsilon>0$ there exists a random variable η_{ε} such that the inequality (191) is fulfilled w. p. 1 for all $p\in\mathbb{N}$. Moreover, from the Lyapunov inequality and (192) we obtain $\mathsf{M}\left\{|\eta_{\varepsilon}|^q\right\}<\infty$ for all $q\ge 1$.

7. About the Structure of Functions $K(t_1,\ldots,t_k)$ Used in Applications

The systems of iterated stochastic integrals (2), (3), (48), (49) are part of the stochastic Taylor–Ito and Taylor–Stratonovich expansions (classical [2], [3] and unified [8]-[17], [20]-[24]).

The function $K(t_1,\ldots,t_k)$ from Theorems 1, 2 for the family (48) looks as follows

$$K(t_1,\ldots,t_k)=(t-t_k)^{l_k}\ldots(t-t_1)^{l_1}\ \mathbf{1}_{\{t_1<\ldots<t_k\}},\quad t_1,\ldots,t_k\in[t,T], \tag{193}$$

where $\mathbf{1}_A$ is the indicator of the set A.

In particular, for the stochastic integrals

$$I_{(1)T,t}^{(i_1)},\ I_{(2)T,t}^{(i_1)},\ I_{(00)T,t}^{(i_1 i_2)},\ I_{(000)T,t}^{(i_1 i_2 i_3)},\ I_{(01)T,t}^{(i_1 i_2)},\ I_{(10)T,t}^{(i_1 i_2)},\ I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)},$$

$$I_{(20)T,t}^{(i_1 i_2)},\ I_{(11)T,t}^{(i_1 i_2)},\ I_{(02)T,t}^{(i_1 i_2)}\quad (i_1,\ldots,i_4=1,\ldots,m)$$

the functions $K(t_1,\ldots,t_k)$ (see (193)) correspondently look as follows

$$K_1(t_1)=t-t_1,\quad K_2(t_1)=(t-t_1)^2,\quad K_{00}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2\}}, \tag{194}$$

$$K_{000}(t_1,t_2,t_3)=\mathbf{1}_{\{t_1<t_2<t_3\}},\quad K_{01}(t_1,t_2)=(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{195}$$

$$K_{10}(t_1,t_2)=(t-t_1)\mathbf{1}_{\{t_1<t_2\}},\quad K_{0000}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}}, \tag{196}$$

$$K_{20}(t_1,t_2)=(t-t_1)^2\mathbf{1}_{\{t_1<t_2\}},\quad K_{11}(t_1,t_2)=(t-t_1)(t-t_2)\mathbf{1}_{\{t_1<t_2\}}, \tag{197}$$

$$K_{02}(t_1,t_2)=(t-t_2)^2\mathbf{1}_{\{t_1<t_2\}}, \tag{198}$$

where $t_1,\ldots,t_4\in[t,T]$.

It is obviously that the most simple expansion for the polynomial of a finite degree into the Fourier series using the complete orthonormal system of functions in the space $L_2([t,T])$ will be its Fourier–Legendre expansion (finite sum). The polynomial functions are included in the functions (194)–(198) as their components if $l_1^2+\ldots+l_k^2>0$. So, it is logical to expect that the most simple expansions for the functions (194)–(198) into multiple Fourier series will be their Fourier–Legendre expansions when $l_1^2+\ldots+l_k^2>0$.

Note that the given assumption is confirmed completely (compare the formulas (52), (56) with the formulas (114), (131) correspondently). So, the usage of Legendre polynomials in the considered scientific field is an obvious step forward.

8. Theorems 1–7 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [54], [55], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [54]-[56] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [57], [58]

$$\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{199}$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (199) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{200}$$

From (200) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{201}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{202}$$

where $i_1,\ldots,i_k=0,1,\ldots,m,\ \ p_1,\ldots,p_k\in\mathbb{N}$,

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{203}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (201).

Let us substitute (201) into (202)

$$\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{204}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [54]-[56] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [56] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (200) were not considered in [54], [55] (also see [56], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [56] for approximations of the Wiener process based on its series expansion (199) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (204) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [54], [55] (also see [56], Theorems 7.1, 7.2).

From the other hand, Theorems 1–7 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the the Riemann–Stieltjes integrals (202) and approximation (200) of the Wiener process. At that, the Riemann–Stieltjes integrals (202) converge (according to Theorems 1–7) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (199), (200), and Theorems 3–7) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [54]-[56]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{205}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T\int_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (205) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int_0^T\int_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int_0^T\int_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int_{l\Delta}^{(l+1)\Delta}\int_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{206}$$

Using (206) and the standard relation between Stratonovich and Ito stochastic integrals, it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int_0^T\int_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int_0^T\int_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int_0^T ds=$$

$$=\int_0^{*T}\int_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{207}$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (207) agrees with Theorem 7.1 (see [56], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (199) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T\int_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \tag{208}$$

where $d\mathbf{f}_\tau^{(i)p}$ is defined by the relation (201).

Let us substitute (201) into (208)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{209}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (204).

As we noted above, approximations of the Wiener process that are similar to (200) were not considered in [54], [55] (also see Theorems 7.1, 7.2 in [56]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [56] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [22]-[24]. More precisely, using Theorem 3 for the case $k = 2$, we obtain from (209) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{210}$$

From the other hand, by Theorem 1 (see (10)) for the case $k = 2$ we obtain from (209) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right) + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{211}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from (211) and the standard relation between Stratonovich and Ito stochastic integrals we obtain (210).

9. Exact Calculation of the Mean-Square Approximation Errors for Iterated Stratonovich Stochastic Integrals $I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{*(i_1)}$, $I_{(00)T,t}^{*(i_1 i_2)}$, $I_{(000)T,t}^{*(i_1 i_2 i_3)}$, $I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)}$

First, consider the question on the exact calculation of the mean-square approximation errors for the following iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)},\quad I_{(1)T,t}^{*(i_1)},\quad I_{(00)T,t}^{*(i_1 i_2)},\quad I_{(000)T,t}^{*(i_1 i_2 i_3)}\quad (i_1,i_2,i_3=1,\ldots,m) \tag{212}$$

defined by (49).

We assume that the stochastic integrals (212) are approximated using Theorems 1, 3 and the Legendre polynomial system. Since $I_{(0)T,t}^{(i_1)}=I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{(i_1)}=I_{(1)T,t}^{*(i_1)}$ w. p. 1 (see (48)), then we can use (51), (52) to approximate the stochastic integrals $I_{(0)T,t}^{*(i_1)}$, $I_{(1)T,t}^{*(i_1)}$. In this case, we will have zero mean-square approximation errors.

To approximate the iterated Stratonovich stochastic integral $I_{(00)T,t}^{*(i_1 i_2)}$ we can use the formula (see (54))

$$I_{(00)T,t}^{*(i_1 i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right). \tag{213}$$

The mean-square approximation error for (213) will be determined by the formula (81) $(i_1\ne i_2)$. For the case $i_1=i_2$ we can use the well known equality

$$I_{(00)T,t}^{*(i_1 i_1)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\right)^2\quad \text{w. p. 1.}$$

Consider now the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1 i_2 i_3)}$ of multiplicity 3 $(i_1,i_2,i_3=1,\ldots,m)$. For the case of pairwise different i_1,i_2,i_3 we have the following relation

$$I_{(000)T,t}^{*(i_1 i_2 i_3)}=I_{(000)T,t}^{(i_1 i_2 i_3)}\quad \text{w. p. 1.}$$

Thus, in this case we can use the formulas (93) and (94). For the case $i_1=i_2=i_3$, to approximate the stochastic integral $I_{(000)T,t}^{*(i_1 i_1 i_1)}$, we use the formula (59).

Thus, it remains to consider the following three cases

$$i_1=i_2\ne i_3, \tag{214}$$

$$i_1\ne i_2=i_3, \tag{215}$$

$$i_1=i_3\ne i_2. \tag{216}$$

Taking into account the standard relations between Ito and Stratonovich stochastic integrals and Theorem 1 (the case $k=3$) together with Theorem 3, we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + I_{(000)T,t}^{(i_1 i_2 i_3)q} + \mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \right.\right.$$

$$\left.\left. +\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}, \tag{217}$$

where the approximations $I_{(000)T,t}^{*(i_1 i_2 i_3)q}$, $I_{(000)T,t}^{(i_1 i_2 i_3)q}$ are defined by the relations (see (57), (58))

$$I_{(000)T,t}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \right.$$

$$\left. -\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{218}$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}. \tag{219}$$

Substituting (218) and (219) into (217) yields

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + \mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right) + \right.\right.$$

$$\left.\left. +\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right) - \mathbf{1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{q} C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \tag{220}$$

Consider the case (214). From (220) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{221}$$

According to the results of Sect. 3 in [31] (also see Sect. 1.2.2 in [22]-[24]), the quantity

$$I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}$$

includes only iterated Ito stochastic integrals of multiplicity 3. At the same time, the quantity

$$\frac{1}{2}\int_t^T\int_t^\tau ds d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}$$

contains only iterated Ito stochastic integrals of multiplicity 1. This means that from (221) we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int_t^T(\tau-t)d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{222}$$

We have

$$\mathsf{M}\left\{\left(\frac{1}{2}\int_t^T(\tau-t)d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} = \frac{1}{4}\int_t^T(\tau-t)^2 d\tau -$$

$$-\sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\int_t^T(\tau-t)\phi_{j_3}(\tau)d\tau + \sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_3 j_1 j_1}\right)^2, \tag{223}$$

where $\phi_{j_3}(\tau)$ is the Legendre polynomial defined by (50).

According to the properties of Legendre polynomials, we obtain

$$\int_t^T(\tau-t)\phi_{j_3}(\tau)d\tau = \frac{(T-t)^{3/2}}{2}\begin{cases}1, & j_3=0\\ 1/\sqrt{3}, & j_3=1.\\ 0, & j_3\ge 2\end{cases} \tag{224}$$

Combining (222)–(224) and (97), we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{(T-t)^3}{4} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_1 j_2}C_{j_3 j_2 j_1} -$$

$$-\frac{(T-t)^{3/2}}{2}\sum_{j_1=0}^{q}\left(C_{0 j_1 j_1} + \frac{1}{\sqrt{3}}C_{1 j_1 j_1}\right) + \sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_3 j_1 j_1}\right)^2, \tag{225}$$

where $i_1 = i_2 \ne i_3$.

Consider the case (215). From (220) we obtain

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =
$$

$$
= {\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + \frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d{\bf f}_s^{(i_1)}d\tau - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\} =
$$

$$
= {\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + \frac{1}{2}\int\limits_t^T (T-s)d{\bf f}_s^{(i_1)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\} =
$$

$$
= {\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} +
$$

$$
+{\sf M}\left\{\left(\frac{1}{2}\int\limits_t^T (T-s)d{\bf f}_s^{(i_1)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\} =
$$

$$
= {\sf M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} +
$$

$$
+\frac{1}{4}\int\limits_t^T (T-s)^2 ds - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\int\limits_t^T (T-s)\phi_{j_1}(s)ds + \sum_{j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_1}\right)^2, \tag{226}
$$

where $\phi_{j_1}(\tau)$ is the Legendre polynomial defined by (50).

Moreover,

$$
\int\limits_t^T (T-s)\phi_{j_1}(s)ds = \frac{(T-t)^{3/2}}{2}\begin{cases} 1, & j_1 = 0\\ -1/\sqrt{3}, & j_1 = 1.\\ 0, & j_1 \ge 2\end{cases} \tag{227}
$$

Combining (226)–(227) and (95), we get

$$
{\sf M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{(T-t)^3}{4} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q} C_{j_2 j_3 j_1}C_{j_3 j_2 j_1} -
$$

$$
-\frac{(T-t)^{3/2}}{2}\sum_{j_3=0}^{q}\left(C_{j_3 j_3 0} - \frac{1}{\sqrt{3}}C_{j_3 j_3 1}\right) + \sum_{j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_1}\right)^2, \tag{228}
$$

where $i_1 \ne i_2 = i_3$.

Consider the case (216). From (220) we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}-\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+\mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}=$$

(229)
$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}+\sum_{j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_2j_1}\right)^2.$$

Combining (229) and (96), we have

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}C_{j_1j_2j_3}+$$

(230)
$$+\sum_{j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_2j_1}\right)^2,$$

where $i_1=i_3\ne i_2$.

Thus, the exact calculaton of the mean-square approximation error for the iterated Stratonovich stochastic integral $I_{(000)T,t}^{*(i_1i_2i_3)}$ $(i_1,i_2,i_3=1,\ldots,m)$ is given by the formulas (94), (225), (228), and (230).

Consider now the iterated Stratonovich stochastic integral $I_{(0000)T,t}^{*(i_1i_2i_3i_4)}$ of multiplicity 4 $(i_1,i_2,i_3,i_4=1,\ldots,m)$. For $i_1=i_2=i_3=i_4$ we can use the formula (73). For the case of pairwise different i_1,i_2,i_3,i_4 we have the following relation

$$I_{(0000)T,t}^{*(i_1i_2i_3i_4)}=I_{(0000)T,t}^{(i_1i_2i_3i_4)}\quad \text{w. p. 1.}$$

Then in this case we can use the formulas (99) (for pairwise different i_1,i_2,i_3,i_4) and (100) to approximate the stochastic integral $I_{(0000)T,t}^{*(i_1i_1i_1i_1)}$.

Thus, it remains to consider the following 13 cases

(231) $$i_1=i_2\ne i_3,i_4;\ i_3\ne i_4,$$

(232) $$i_1=i_3\ne i_2,i_4;\ i_2\ne i_4,$$

(233) $$i_1=i_4\ne i_2,i_3;\ i_2\ne i_3,$$

(234) $$i_2=i_3\ne i_1,i_4;\ i_1\ne i_4,$$

(235) $$i_2=i_4\ne i_1,i_3;\ i_1\ne i_3,$$

$$i_3 = i_4 \neq i_1, i_2;\ i_1 \neq i_2, \tag{236}$$

$$i_1 = i_2 = i_3 \neq i_4, \tag{237}$$

$$i_2 = i_3 = i_4 \neq i_1, \tag{238}$$

$$i_1 = i_2 = i_4 \neq i_3, \tag{239}$$

$$i_1 = i_3 = i_4 \neq i_2, \tag{240}$$

$$i_1 = i_2 \neq i_3 = i_4, \tag{241}$$

$$i_1 = i_3 \neq i_2 = i_4, \tag{242}$$

$$i_1 = i_4 \neq i_2 = i_3. \tag{243}$$

By analogy with (220) and using the standard relation between Stratonovich and Ito stochastic integrals (49), (48) of multiplicity 4 as well as (99), we obtain

$$
\begin{aligned}
&\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \\
&= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} + \right.\right.\\
&+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 d\mathbf{w}_{t_4}^{(i_4)} + \frac{1}{2}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 + \\
&+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\int\limits_t^T\int\limits_t^{t_2} dt_1 dt_2 - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q} - \\
&-\mathbf{1}_{\{i_1=i_2\neq 0\}}\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_4\neq 0\}}\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_1 j_3 j_2 j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_4 j_2 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
&+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\sum_{j_3,j_1=0}^{q} C_{j_3 j_3 j_1 j_1} + \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\sum_{j_2,j_1=0}^{q} C_{j_2 j_1 j_2 j_1} + \\
&\left.\left.+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\sum_{j_2,j_1=0}^{q} C_{j_1 j_2 j_2 j_1}\right)^2\right\},
\end{aligned}
\tag{244}
$$

where $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}$ is defined by (99).

Consider the case (231). From (244) we get

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0000)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}=$$

(245)
$$=\mathsf{M}\left\{\left(I^{(i_1i_2i_3i_4)}_{(0000)T,t}-I^{(i_1i_2i_3i_4)q}_{(0000)T,t}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}^{(i_3)}_{t_3}d\mathbf{w}^{(i_4)}_{t_4}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\right)^2\right\}.$$

Note that

$$\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)d\mathbf{w}^{(i_3)}_{t_3}d\mathbf{w}^{(i_4)}_{t_4}+\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_4)d\mathbf{w}^{(i_4)}_{t_4}d\mathbf{w}^{(i_3)}_{t_3}\qquad(246)$$

w. p. 1, where $i_3\ne i_4$.

According to the results of Sect. 3 in [31] (also see Sect. 1.2.2 in [22]-[24]), the quantity

$$I^{(i_1i_2i_3i_4)}_{(0000)T,t}-I^{(i_1i_2i_3i_4)q}_{(0000)T,t}$$

includes only iterated Ito stochastic integrals of multiplicity 4. At the same time (see (246)), the quantity

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}^{(i_3)}_{t_3}d\mathbf{w}^{(i_4)}_{t_4}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{p}C_{j_4j_3j_1j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}$$

contains only iterated Ito stochastic integrals of multiplicity 2. This means that from (245) we get

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0000)T,t}-I^{*(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}=\mathsf{M}\left\{\left(I^{(i_1i_2i_3i_4)}_{(0000)T,t}-I^{(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}^{(i_3)}_{t_3}d\mathbf{w}^{(i_4)}_{t_4}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I^{(i_1i_2i_3i_4)}_{(0000)T,t}-I^{(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}+\frac{1}{4}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)^2dt_3dt_4+$$

$$+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t_3-t)dt_3dt_4=$$

$$=\mathsf{M}\left\{\left(I^{(i_1i_2i_3i_4)}_{(0000)T,t}-I^{(i_1i_2i_3i_4)q}_{(0000)T,t}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+$$

$$+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}C_{j_4j_3}^{10}, \tag{247}$$

where

$$C_{j_4j_3}^{10}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)(t-t_3)dt_3dt_4. \tag{248}$$

Using (35) and (247), we finally obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}C_{j_4j_3}^{10}, \tag{249}$$

where $i_1=i_2\ne i_3,i_4;\ i_3\ne i_4$.

Consider the cases (232), (233) by analogy with the case (231) using (36), (37). We have

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_4,j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_1j_2j_1}\right)^2,$$

where $i_1=i_3\ne i_2,i_4$ and $i_2\ne i_4$;

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_4)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_3,j_2=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\right)^2,$$

where $i_1=i_4\ne i_2,i_3$ and $i_2\ne i_3$.

Consider the case (234) by analogy with the case (231). We have

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)(t_4-t_1)dt_3dt_4.$$

Then applying (38), we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_3)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(C_{j_4j_1}^{10}-C_{j_4j_1}^{01}\right),$$

where $i_2=i_3\ne i_1,i_4$ and $i_1\ne i_4$; $C_{j_4j_1}^{10}$ is defined by (248) and

$$C_{j_4j_1}^{01}=\int\limits_t^T\phi_{j_4}(t_4)(t-t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4. \tag{250}$$

For the case (235) by analogy with the case (231) and using (39), we get

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_4)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_3,j_1=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_2j_3j_2j_1}\right)^2,$$

where $i_2=i_4\ne i_1,i_3$ and $i_1\ne i_3$.

Consider the case (236) by analogy with the case (231). Note that [22]-[24] (see Example 3.1 in Sect. 3.6)

$$
\int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} dt_3 = \int\limits_t^T (T-t_2) \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \quad \text{w. p. 1.} \tag{251}
$$

Using (251), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\} +
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T (T-t_2)\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} - \sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\} + \frac{(T-t)^4}{48} + \sum_{j_2,j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\right)^2 -
$$

$$
- \sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\int\limits_t^T (T-t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1 dt_2.
$$

Then applying (40), we get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} - I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{(T-t)^4}{16} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}\left(\sum_{(j_3,j_4)} C_{j_4 j_3 j_2 j_1}\right) +
$$

$$
+ \sum_{j_2,j_1=0}^{q}\left(\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\right)^2 - \sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q} C_{j_3 j_3 j_2 j_1}\left((T-t)C_{j_2 j_1} + C_{j_2 j_1}^{01}\right),
$$

where $i_3 = i_4 \ne i_1, i_2$ and $i_1 \ne i_2$; $C_{j_2 j_1}^{01}$ is defined by (250) and

$$
C_{j_2 j_1} = \int\limits_t^T \phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1 dt_2.
$$

Consider the case (237). From (244) we have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_1 i_1 i_4)} - I_{(0000)T,t}^{*(i_1 i_1 i_1 i_4)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_1 i_4)} + \frac{1}{2}\int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} dt_1 d\mathbf{w}_{t_3}^{(i_1)} d\mathbf{w}_{t_4}^{(i_4)} +\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2 d\mathbf{w}_{t_4}^{(i_4)} - I_{(0000)T,t}^{(i_1 i_1 i_1 i_4)q} - \sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q} C_{j_4 j_3 j_1 j_1}\zeta_{j_3}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$-\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_1)}\zeta_{j_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\Biggr)^2\Biggr\}. \tag{252}$$

Furthermore,

$$\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}dt_1d\mathbf{w}_{t_3}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}+\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_4)}=$$

$$=\int\limits_t^T\int\limits_t^{t_4}(t_1-t)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}+\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}=$$

$$=\int\limits_t^T(t_4-t)\int\limits_t^{t_4}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}\quad \text{w. p. 1.} \tag{253}$$

From (252) and (253) we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_1i_1i_4)}-I_{(0000)T,t}^{*(i_1i_1i_1i_4)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_1i_4)}-I_{(0000)T,t}^{(i_1i_1i_1i_4)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(t_4-t)\int\limits_t^{t_4}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_4)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2}+C_{j_4j_2j_1j_2}+C_{j_4j_2j_2j_1}\right)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_1i_4)}-I_{(0000)T,t}^{(i_1i_1i_1i_4)q}\right)^2\right\}+\frac{(T-t)^4}{16}+$$

$$+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2}+C_{j_4j_2j_1j_2}+C_{j_4j_2j_2j_1}\right)\right)^2-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2}+C_{j_4j_2j_1j_2}+C_{j_4j_2j_2j_1}\right)\int\limits_t^T(t_4-t)\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)dt_1dt_4. \tag{254}$$

Using (41) and (254), we finally get

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{5(T-t)^4}{48}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2,j_3)}C_{j_4j_3j_2j_1}\right)+$$

$$+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2}+C_{j_4j_2j_1j_2}+C_{j_4j_2j_2j_1}\right)\right)^2+$$

$$+\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_1j_2j_2}+C_{j_4j_2j_1j_2}+C_{j_4j_2j_2j_1}\right)C_{j_4j_2}^{01},$$

where $i_1=i_2=i_3\neq i_4$.

Consider the case (238). From (244) we have

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_2i_2)}-I_{(0000)T,t}^{*(i_1i_2i_2i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_2)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_2)}+\right.\right.$$

$$+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3-I_{(0000)T,t}^{(i_1i_2i_2i_2)q}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_2)}-$$

$$\left.\left.-\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_2)}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\}. \quad (255)$$

Moreover,

$$\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}dt_3=$$

$$=\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_4}(T-t_4)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}=$$

$$=\int\limits_t^T\int\limits_t^{t_4}(T-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}\quad \text{w. p. 1.} \quad (256)$$

From (255) and (256) we get

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_2i_2)}-I_{(0000)T,t}^{*(i_1i_2i_2i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_2)}-I_{(0000)T,t}^{(i_1i_2i_2i_2)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(T-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_2)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_2)}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_2)}-I_{(0000)T,t}^{(i_1i_2i_2i_2)q}\right)^2\right\}+\frac{(T-t)^4}{16}+$$

$$+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\right)^2-$$

$$
(257) \qquad -\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}(T-t_1)\phi_{j_1}(t_1)dt_1dt_4.
$$

Applying (42) and (257), we finally obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{5(T-t)^4}{48}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_2,j_3,j_4)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_1=0}^{q}\left(\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\right)^2-
$$

$$
-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}\left(C_{j_4j_2j_2j_1}+C_{j_2j_4j_2j_1}+C_{j_2j_2j_4j_1}\right)\left((T-t)C_{j_4j_1}+C_{j_4j_1}^{10}\right),
$$

where $i_2=i_3=i_4\neq i_1$.

For the cases (239), (240) by analogy with the case (238) and using (43), (44), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2,j_4)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_3=0}^{q}\left(\sum_{j_1=0}^{q}\left(C_{j_4j_3j_1j_1}+C_{j_1j_3j_4j_1}+C_{j_1j_3j_1j_4}\right)\right)^2+
$$

$$
+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}\left(C_{j_4j_3j_1j_1}+C_{j_1j_3j_4j_1}+C_{j_1j_3j_1j_4}\right)C_{j_4j_3}^{10},
$$

where $i_1=i_2=i_4\neq i_3$;

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3,j_4)}C_{j_4j_3j_2j_1}\right)+
$$

$$
+\sum_{j_4,j_2=0}^{q}\left(\sum_{j_1=0}^{q}\left(C_{j_4j_1j_2j_1}+C_{j_1j_4j_2j_1}+C_{j_1j_1j_2j_4}\right)\right)^2-
$$

$$
-\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q}\left(C_{j_4j_1j_2j_1}+C_{j_1j_4j_2j_1}+C_{j_1j_1j_2j_4}\right)\left((T-t)C_{j_2j_3}+C_{j_2j_3}^{01}\right),
$$

where $i_1=i_3=i_4\neq i_2$.

Let us consider the case (241). Using (244), we have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)}-I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}+\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3+\frac{(T-t)^2}{8}-I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)q}-
$$

$$
\left.\left.-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4 j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}+\sum_{j_3,j_1=0}^{q}C_{j_3 j_3 j_1 j_1}\right)^2\right\}=
$$

$$
=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)}-I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)q}+\right.\right.
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4 j_3 j_1 j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)+
$$

$$
+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3 j_3 j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)+
$$

$$
\left.\left.+\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3 j_3 j_1 j_1}\right)^2\right\}. \tag{258}
$$

Note that

$$
\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}+\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_4}(t_4)d\mathbf{w}_{t_4}^{(i_3)}d\mathbf{w}_{t_3}^{(i_3)}, \tag{259}
$$

$$
\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}+\int\limits_t^T\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_2}(t_2)d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_1}^{(i_1)} \tag{260}
$$

w. p. 1.

The relations (258)–(260) and (251) imply the following

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)}-I_{(0000)T,t}^{*(i_1 i_1 i_3 i_3)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)}-I_{(0000)T,t}^{(i_1 i_1 i_3 i_3)q}\right)^2\right\}+
$$

$$
+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4 j_3 j_1 j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+
$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}dt_3-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+$$

$$+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_3i_3)}-I_{(0000)T,t}^{(i_1i_1i_3i_3)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_3-t)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_3)}-\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T(T-t_2)\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_1)}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+$$

$$+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_1i_3i_3)}-I_{(0000)T,t}^{(i_1i_1i_3i_3)q}\right)^2\right\}+\frac{(T-t)^4}{48}+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(C_{j_3j_4}^{10}+C_{j_4j_3}^{10}\right)+$$

$$+\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}+$$

$$+\frac{(T-t)^4}{48}-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left((T-t)\left(C_{j_1j_2}+C_{j_2j_1}\right)+C_{j_1j_2}^{01}+C_{j_2j_1}^{01}\right)+$$

$$+\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}+$$

$$+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2. \tag{261}$$

Furthermore,

$$\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}-2\left(\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2+\left(\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2=$$

$$=\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}-\left(\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2, \tag{262}$$

$$\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\right)^2\right\}-\left(\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1j_1}\right)^2. \tag{263}$$

We have [32], p. 71 (also see [22], Sect. 2.3)

$$\mathsf{M}\left\{\left(\sum_{j_3,j_4=0}^{q}a_{j_4j_3}\zeta_{j_3}^{(i)}\zeta_{j_4}^{(i)}\right)^2\right\}=\left(\sum_{j_3=0}^{q}a_{j_3j_3}\right)^2+\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(a_{j_3j_4}+a_{j_4j_3}\right)^2+2\sum_{j_4=0}^{q}\left(a_{j_4j_4}\right)^2, \tag{264}$$

where $i=1,\ldots,m$ and $a_{j_4j_3}$ $(j_3,j_4=0,1,\ldots,q)$ are scalar nonrandom coefficients.

Applying (264), we obtain

$$\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}\right)^2\right\}=$$

(265)

$$=\left(\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2+\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q}C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_4j_1j_1}\right)^2.$$

From (262) and (265) we get

$$\mathsf{M}\left\{\left(\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_3=j_4\}}\right)\right)^2\right\}=$$

$$=\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q}C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_4j_1j_1}\right)^2. \tag{266}$$

By analogy with (266) we obtain

$$
\mathsf{M}\left\{\left(\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\mathbf{1}_{\{j_1=j_2\}}\right)\right)^2\right\}=
$$

$$
=\sum_{j_2=0}^{q}\sum_{j_1=0}^{j_2-1}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_1j_2}+\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2+2\sum_{j_2=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_2}\right)^2. \tag{267}
$$

Combining (45), (261), (266), and (267), we finally have

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{12}-
$$

$$
-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)}C_{j_4j_3j_2j_1}\right)\right)+\sum_{j_4,j_3=0}^{q}\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\left(C_{j_3j_4}^{10}+C_{j_4j_3}^{10}\right)+
$$

$$
+\sum_{j_4=0}^{q}\sum_{j_3=0}^{j_4-1}\left(\sum_{j_1=0}^{q}C_{j_3j_4j_1j_1}+\sum_{j_1=0}^{q}C_{j_4j_3j_1j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_4j_4j_1j_1}\right)^2-
$$

$$
-\sum_{j_2,j_1=0}^{q}\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\left((T-t)C_{j_1}C_{j_2}+C_{j_1j_2}^{01}+C_{j_2j_1}^{01}\right)+
$$

$$
+\sum_{j_2=0}^{q}\sum_{j_1=0}^{j_2-1}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_1j_2}+\sum_{j_3=0}^{q}C_{j_3j_3j_2j_1}\right)^2+2\sum_{j_2=0}^{q}\left(\sum_{j_3=0}^{q}C_{j_3j_3j_2j_2}\right)^2+
$$

$$
+\left(\frac{(T-t)^2}{8}-\sum_{j_3,j_1=0}^{q}C_{j_3j_3j_1j_1}\right)^2,
$$

where $i_1=i_2\neq i_3=i_4$ and

$$
C_j=\int\limits_t^T\phi_j(\tau)d\tau=\begin{cases}\sqrt{T-t}, & j=0\\ 0, & j\neq 0\end{cases}.
$$

Consider the case (242) by analogy with the case (241). Using (244), we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_1i_2)}-I_{(0000)T,t}^{*(i_1i_2i_1i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_1i_2)}-I_{(0000)T,t}^{(i_1i_2i_1i_2)q}-\right.\right.
$$

$$
-\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} + \sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\Biggr)^2\Biggr\} =
$$

$$
= \mathsf{M}\Biggl\{\Biggl(I_{(0000)T,t}^{(i_1i_2i_1i_2)} - I_{(0000)T,t}^{(i_1i_2i_1i_2)q} - \sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \mathbf{1}_{\{j_2=j_4\}}\right) -
$$

$$
-\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right) - \sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\Biggr)^2\Biggr\} =
$$

$$
= \mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_1i_2)} - I_{(0000)T,t}^{(i_1i_2i_1i_2)q}\right)^2\right\} +
$$

$$
+\mathsf{M}\left\{\left(\sum_{j_4,j_2=0}^{q}\sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)} - \mathbf{1}_{\{j_2=j_4\}}\right)\right)^2\right\} +
$$

$$
+\mathsf{M}\left\{\left(\sum_{j_3,j_1=0}^{q}\sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)} - \mathbf{1}_{\{j_1=j_3\}}\right)\right)^2\right\} +
$$

$$
+\left(\sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2. \tag{268}
$$

Applying (46) and (268), we finally get

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)} - I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4j_3j_2j_1}\right)\right)+
$$

$$
+\sum_{j_4=0}^{q}\sum_{j_2=0}^{j_4-1}\left(\sum_{j_1=0}^{q} C_{j_2j_1j_4j_1} + \sum_{j_1=0}^{q} C_{j_4j_1j_2j_1}\right)^2 + 2\sum_{j_4=0}^{q}\left(\sum_{j_1=0}^{q} C_{j_4j_1j_4j_1}\right)^2 +
$$

$$
+\sum_{j_3=0}^{q}\sum_{j_1=0}^{j_3-1}\left(\sum_{j_2=0}^{q} C_{j_2j_1j_2j_3} + \sum_{j_2=0}^{q} C_{j_2j_3j_2j_1}\right)^2 + 2\sum_{j_3=0}^{q}\left(\sum_{j_2=0}^{q} C_{j_2j_3j_2j_3}\right)^2 +
$$

$$
+\left(\sum_{j_2,j_1=0}^{q} C_{j_2j_1j_2j_1}\right)^2,
$$

where $i_1 = i_3 \neq i_2 = i_4$.

Consider the case (243) by analogy with the cases (241) and (242). Applying (244), we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_2i_1)}-I_{(0000)T,t}^{*(i_1i_2i_2i_1)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}-\right.\right.$$

$$\left.\left.-\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}+\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}+\right.\right.$$

$$+\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}dt_2d\mathbf{w}_{t_4}^{(i_1)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)-$$

$$\left.\left.-\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)-\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{t_4}(t_4-t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_4}^{(i_1)}-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)\right)^2\right\}+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2=$$

$$=\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_2i_1)}-I_{(0000)T,t}^{(i_1i_2i_2i_1)q}\right)^2\right\}+\frac{(T-t)^4}{48}-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}(t_4-t_1)\phi_{j_1}(t_1)dt_1dt_4+\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}(t_4-t_1)\phi_{j_4}(t_1)dt_1dt_4\right)+$$

$$+\mathsf{M}\left\{\left(\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}-\mathbf{1}_{\{j_1=j_4\}}\right)\right)^2\right\}+$$

$$+\mathsf{M}\left\{\left(\sum_{j_3,j_2=0}^{q}\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\left(\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_2=j_3\}}\right)\right)^2\right\}+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2. \tag{269}$$

Applying (47) and (269), we finally obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{16}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)}C_{j_4j_3j_2j_1}\right)\right)-$$

$$-\sum_{j_4,j_1=0}^{q}\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\left(C_{j_4j_1}^{10}+C_{j_1j_4}^{10}-C_{j_4j_1}^{01}-C_{j_1j_4}^{01}\right)+$$

$$+\sum_{j_4=0}^{q}\sum_{j_1=0}^{j_4-1}\left(\sum_{j_2=0}^{q}C_{j_1j_2j_2j_4}+\sum_{j_2=0}^{q}C_{j_4j_2j_2j_1}\right)^2+2\sum_{j_4=0}^{q}\left(\sum_{j_2=0}^{q}C_{j_4j_2j_2j_4}\right)^2+$$

$$+\sum_{j_3=0}^{q}\sum_{j_2=0}^{j_3-1}\left(\sum_{j_1=0}^{q}C_{j_1j_2j_3j_1}+\sum_{j_1=0}^{q}C_{j_1j_3j_2j_1}\right)^2+2\sum_{j_3=0}^{q}\left(\sum_{j_1=0}^{q}C_{j_1j_3j_3j_1}\right)^2+$$

$$+\left(\sum_{j_2,j_1=0}^{q}C_{j_1j_2j_2j_1}\right)^2,$$

where $i_1=i_4\neq i_2=i_3$.

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992, 632 pp.
[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.
[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
[5] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[6] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: SPbGTU, Saint-Petersburg, 1998, 204 pp. (ISBN 5-7422-0045-5)

[7] Kuznetsov D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 32, Issue 12, (2000), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[8] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[9] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[13] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[14] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[15] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[16] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[17] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[18] Kuznetsov D.F. Development and Application of the Fourier Method for the Numerical Solution of Ito Stochastic Differential Equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[19] Kuznetsov D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 1.5 and 2.0 Orders of Strong Convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[22] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2022, 912 pp.

[23] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[24] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[25] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[26] Kuznetsov D.F. Expansions of Iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [in English]. arXiv:1712.09516 [math.PR]. 2022, 203 pp.

[27] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 105 pp.

[28] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 76 pp.

[29] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 42 pp.

[30] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 126 pp.

[31] Kuznetsov D.F. Exact calculation of mean-square error in the method of approximation of iterated Ito stochastic integrals based on the generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 68 pp.

[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[33] Kuznetsov D.F. On Numerical Modeling of the Multidimentional Dynamic Systems Under Random Perturbations With the 2.5 Order of Strong Convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[34] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[35] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[36] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N.Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[37] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[38] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[39] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.

[40] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[41] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2018, 138 pp.

[42] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and

Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[43] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier-Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[44] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 342 pp.

[45] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[46] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[47] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[48] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[49] Kloeden P.E., Platen E., Schurz H. Numerical solution of SDE through computer experiments. Berlin: Springer, 1994, 292 pp.

[50] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl., 10, 4 (1992), 431-441.

[51] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[52] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[53] Kloeden P.E., Neuenkirch A. The pathwise convergence of approximation schemes for stochastic differential equations. LMS Journal of Computation and Mathematics. 10 (2007), 235-253.

[54] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[55] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[56] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[57] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974, 696 pp.

[58] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

COMPARATIVE ANALYSIS OF THE EFFICIENCY OF APPLICATION OF LEGENDRE POLYNOMIALS AND TRIGONOMETRIC FUNCTIONS TO THE NUMERICAL INTEGRATION OF ITO STOCHASTIC DIFFERENTIAL EQUATIONS

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations in the framework of the method of approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. On the example of iterated Ito stochastic integrals of multiplicities 1 to 3 from the Taylor–Ito expansion it is shown that expansions of stochastic integrals based on Legendre polynomials are essentially simpler and require significantly less computational costs compared to their analogues obtained using the trigonometric system of functions. The results of the article can be useful for construction of high-order strong numerical methods for Ito stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
Keywords: Iterated Stratonovich stochastic integral, Iterated Ito stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Mean-square approximation, Expansion.

1. Introduction

In a lot of author's publications [1]-[43] the mean-square approximation method for iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series is proposed and developed (see Theorems 1–8 below). Further, we will call this method as the method of generalized multiple Fourier series. Under the term "generalized multiple Fourier series" we understand the Fourier series constructed using various complete orthonormal systems of functions in the space $L_2([t,T])$, and not only using the trigonometric system of functions. Here $[t,T]$ is an interval of integration of iterated Ito or Stratonovich stochastic integrals.

It is well known the another approach to series expansion of stochastic processes using eigenfunctions of their covariance operators (the so-called Karhunen–Loeve expansion) [44]. If the stochastic process is the Brownian bridge process on the time interval $[t,T]$, then the eigenfunctions of its covariance operator will be trigonometric functions which form a complete orthonormal system of functions in the space $L_2([t,T])$ [45]. This means that the basis functions in the mentioned approach can only be trigonometric functions. In [45]-[49] the series expansion of the Brownian bridge process was used for the expansion and mean-square approximation of iterated Ito and Stratonovich stochastic integrals. Further, we will call this expansion as the Milstein expansion.

As mentioned above, in contrast to the Milstein expansion the method of generalized multiple Fourier series [1]-[43] (see Theorems 1, 2 below) allows to use different systems of basis functions. Thus, we can set the problem of choice the optimal system of basis functions within the framework of the method of generalized multiple Fourier series. Some ideas on the solution of the mentioned problem were given in a number of the author's works [4]-[13], [16]-[20].

For example, in [4]-[13], [16], [17] it was shown that expansions for simplest iterated (double) Stratonovich stochastic integrals based on the systems of Haar and Rademacher–Walsh functions are too complex and ineffective in practice. In these works, a very brief comparison of the efficiency of application of Legendre polynomials and trigonometric functions in the framework of the method of generalized multiple Fourier series was also carried out. The subject of this article is the development and refinement of the results obtained in [4]-[13], [16], [17] in this direction.

2. Milstein Approach

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space. Let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent. Consider the Brownian bridge process [45]

$$\mathbf{f}_t-\frac{t}{\Delta}\mathbf{f}_{\Delta},\quad t\in[0,\Delta]. \tag{1}$$

The componentwise expansion of the stochastic process (1) into converging in the mean-square sense trigonometric Fourier series (version of the so-called Karhunen–Loeve expansion) has the following form [45]

$$\mathbf{f}_t^{(i)}-\frac{t}{\Delta}\mathbf{f}_\Delta^{(i)}=\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{2}$$

where

$$a_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_\Delta^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,\quad b_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_\Delta^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds,$$

where $r=0,1,\ldots;\ i=1,\ldots,m$.

It is easy to demonstrate [45] that the random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\}=\mathsf{M}\left\{a_{i,r}b_{i,k}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}a_{i,k}\right\}=\mathsf{M}\left\{b_{i,r}b_{i,k}\right\}=0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\}=\mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}^2\right\}=\mathsf{M}\left\{b_{i,r}^2\right\}=\frac{\Delta}{2\pi^2r^2},$$

where $i,i_1,i_2=1,\ldots,m;\ r\neq k;\ i_1\neq i_2$; M denotes a mathematical expectation.

According to (2), we have

$$\mathbf{f}_t^{(i)}=\mathbf{f}_\Delta^{(i)}\frac{t}{\Delta}+\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{3}$$

where the series converges in the mean-square sense.

Denote

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{4}$$

$$J_{(\lambda_1\ldots\lambda_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{5}$$

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{6}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{7}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function on $[t,T]$; $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$; $i_1,\ldots,i_k=0,1,\ldots,m$; $\lambda_l=0$ for $i_l=0$ and $\lambda_l=1$ for $i_l=1,\ldots,m$ $(l=1,\ldots,k)$;

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively. In this paper we use the definition of the Stratonovich stochastic integral from [46], [47].

In [45] Milstein G.N. obtained the following expansion of $J^{(i_1 i_2)}_{(11)T,t}$ using the expansion (3)

(8)
$$J^{(i_1 i_2)}_{(11)T,t} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr),$$

where the series converges in the mean-square sense; $i_1 \ne i_2$; $i_1, i_2 = 1,\ldots,m$;

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j;

$$\phi_j(s) = \frac{1}{\sqrt{T-t}}\begin{cases} 1 & \text{for } j=0 \\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)) & \text{for } j=2r-1 \\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)) & \text{for } j=2r \end{cases}, \tag{9}$$

where $r = 1, 2, \ldots$ Moreover, [45]

$$J^{(i_1)}_{(1)T,t} = \sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{10}$$

where $i_1 = 1,\ldots,m$.

In principle for implementing the strong numerical method with the order 1.0 of accuracy (Milstein method [45]) for Ito stochastic differential equations it is sufficient to take the following approximations

$$J^{(i_1)q}_{(1)T,t} \stackrel{\textsf{def}}{=} J^{(i_1)}_{(1)T,t} = \sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{11}$$

(12)
$$J^{(i_1 i_2)q}_{(11)T,t} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr),$$

where $i_1 \ne i_2$; $i_1, i_2 = 1,\ldots,m$.

It is not difficult to show that

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{13}$$

However, this approach has an obvious drawback. Indeed, we have too complex formulas for the stochastic integrals with Gaussian distribution

$$J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{14}$$

$$J_{(001)T,t}^{(00i_1)}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}-\frac{1}{2\sqrt{2}\pi}\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right),$$

where the sense of notations from (12) is hold.

In [45] Milstein G.N. proposed the following mean-square approximations on the base of the expansions (8), (14)

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{15}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$

$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr), \tag{16}$$

where $i_1\ne i_2$ in (16), and

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}, \tag{17}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ are independent standard Gaussian random variables.

Obviously, for the approximations (15) and (16) we obtain [45]

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,$$

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1 i_2)}-J_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{18}$$

This idea has been developed in [46]-[48]. For example, the approximation $J_{(001)T,t}^{(00i_1)q}$, which corresponds to (15), (16) has the form [46]-[48]

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.$$

$$\left.-\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{19}$$

$$\mathsf{M}\left\{\left(J_{(001)T,t}^{(00i_1)}-J_{(001)T,t}^{(00i_1)q}\right)^2\right\}=0,$$

where $\xi_q^{(i)}$ and α_q have the form (17) and

$$\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},$$

$\phi_j(s)$ is defined by (9); $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q;\ i=1,\ldots,m)$ are independent standard Gaussian random variables.

Nevetheless, the expansions (15), (19) are too complex for approximation of two Gaussian random variables $J_{(01)T,t}^{(0i_1)}$, $J_{(001)T,t}^{(00i_1)}$.

Further, we will see that introduction of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of the stochastic integral $J_{(111)T,t}^{(i_1 i_2 i_3)}$; $i_1,i_2,i_3=1,\ldots,m$ within the framework of the Milshtein approach. This is due to the fact that the number q is fixed for all stochastic integrals included into the considered collection. However, it is clear that due to the smallness of $T-t$, the number q for $J_{(111)T,t}^{(i_1 i_2 i_3)}$ could be taken significantly less than the number q in the formula (16). This feature is also valid for the formulas (15), (19).

On the other hand, the following very simple formulas are well known

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{20}$$

$$J_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{21}$$

$$J_{(001)T,t}^{(00i_1)}=\frac{(T-t)^{5/2}}{6}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{22}$$

where $\zeta_0^{(i)}$, $\zeta_1^{(i)}$, $\zeta_2^{(i)}$; $i=1,\ldots,m$ are indepentent standard Gaussian random variables.

Looking ahead, we note that the formulas (20)-(22) are part of the method that will be discussed in the next section (see Theorems 1, 2 below).

To obtain the Milstein expansion for (7) the truncated expansions (3) of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (7) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple integrals (7) were obtained [45]-[50].

At that, in [45], [49] the case $\psi_1(s), \psi_2(s) \equiv 1$ and $i_1, i_2 = 0, 1, \ldots, m$ is considered. In [46]-[48], [50] the attempt to consider the case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$ and $i_1, i_2, i_3 = 0, 1, \ldots, m$ is implemented.

Note that generally speaking the mean-square convergence of the approximation

$$J^{*(i_1 i_2 i_3)q}_{(111)T,t}$$

(obtained by the Milstein approach) to the appropriate iterated Stratonovich stochastic integral

$$J^{*(i_1 i_2 i_3)}_{(111)T,t}$$

must be proved separately due to iterated application of passing to the limit in the Milstein approach [45]. However, in [46] (pp. 438-439), [47] (Sect. 5.8, pp. 202–204), [48] (pp. 82-84), [50] (pp. 263-264) the authors use the Wong–Zakai approximation [52]-[54] (without rigorous proof) within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [45] (see discussion in Sect. 11 for details).

3. Method of Generalized Multiple Fourier Series

Let us consider an another approach to the expansion of iterated Ito stochastic integrals [4]-[43] (method of generalized multiple Fourier series).

The idea of this method is as follows: the iterated Ito stochastic integral (6) of multiplicity k is represented as the multiple stochastic integral from the certain non-random discontinuous function of k variables defined on the hypercube $[t, T]^k$, where $[t, T]$ is the interval of integration of the iterated Ito stochastic integral. Then, the indicated non-random function is expanded in the hypercube $[t, T]^k$ into the generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t, T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (6) into the multiple series of products of standard Gaussian random variables. Coefficients of this series are coefficients of generalized multiple Fourier series for the mentioned non-random function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (6).

Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a non-random function from the space $L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k) & \text{for} \quad t_1 < \ldots < t_k \\ \\ 0 & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2, \tag{23}$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$.

The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{24}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{25}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{26}$$

Theorem 1 [4] (2006), [5]-[32]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on the interval $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{27}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (6),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$,*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \tag{28}$$

are independent standard Gaussian random variables for various i or j (if $i \neq 0$), $C_{j_k \ldots j_1}$ is the Fourier coefficient (25)*, $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition* (26)*.*

Note that the continuity condition of $\phi_j(x)$ can be weakened (see [4]-[21]). Moreover, Theorem 1 can be generalized to the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t, T])$ (see Theorem 2 below).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [4]-[32]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{29}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{30}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1, \ldots, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{31}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1, \ldots, p_4 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_4=0}^{p_4} C_{j_4 \ldots j_1} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} +$$

$$+\mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} +$$

$$+ \mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \Biggr), \tag{32}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),\qquad (33)$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
&\qquad+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Bigg),
\end{aligned}
\tag{34}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

As a result we obtain the following new possibilities and advantages compared with the method based on the Milstein approach [45].

There is the explicit formula (see (25)) for calculation of expansion coefficients of the iterated Ito stochastic integral (6) with any fixed multiplicity k.

We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integrals (6) (see Theorem 8 below).

Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, we have new possibilities for approximation — we can use not only trigonometric functions as in the Milstein approach [45] but Legendre polynomials. As it turned out (see below), it is more convenient to work with Legendre polynomials for constructing approximations of the iterated Ito stochastic integrals (6). We can choose different numbers q (see Sect. 7) for approximations of different iterated Ito stochastic integrals from the family (6). This is impossible for approximations based on the Milstein approach [45]. Approximations based on Legendre polynomials essentially simpler than approximations based on trigonometric functions (see (15), (19), (21), (22)).

As we mentioned before, the Milstein approach [45] based on the Karhunen–Loeve expansion of the Brownian bridge process leads to iterated series (in contrast with multiple series from Theorems 1–7) starting at least from the second or third multiplicity of iterated stochastic integrals. Multiple series are more convenient for approximation than the iterated ones, since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series.

However, in [46] (pp. 438-439), [47] (Sect. 5.8, pp. 202–204), [48] (pp. 82-84), [50] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1=p_2=p_3=q\to\infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [45] together with the Wong–Zakai approximation [52]-[54] (see discussion in Sect. 11 for details).

For further consideration, let us consider the generalization of formulas (29)–(34) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (6). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{35}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (35) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{36}$$

Below there are several examples of sums in the form (36)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$
$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$
$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (27) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{37}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (37) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (33).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [18] (Sect. 1.11), [21] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{38}$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [51]. Note that we use another notations [18] (Sect. 1.11), [21] (Sect. 15) in comparison with [51]. Moreover, the proof of an analogue of Theorem 2 from [51] is somewhat different from the proof given in [18] (Sect. 1.11), [21] (Sect. 15).

4. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6

In a number the author's works [8]-[20], [22], [27] Theorems 1, 2 have been adapted for the integrals (7) of multiplicities 2 to 4. Let us collect some old results in the following theorem.

Theorem 3 [8]-[20], [22], [27]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable function on $[t,T]$ and $\psi_1(\tau),\psi_3(\tau)$ are twice continuously differentiable functions on $[t,T]$. Then*

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\quad(i_1,i_2=1,\ldots,m), \tag{39}$$

$$J^*[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=0,1,\ldots,m), \tag{40}$$

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m), \tag{41}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \quad (i_1,\ldots,i_4 = 0,1,\ldots,m), \tag{42}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (7), *and* $\psi_l(\tau) \equiv 1$ $(l = 1,\ldots,4)$ *in* (40), (42); *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [18] (Sect. 2.10–2.16), [22] (Sect. 13–19), [23] (Sect. 7–13), [27] (Sect. 5–11), [42] (Sect. 4–9). Let us formulate four theorems that were proved using this approach.

Theorem 4 [18], [22], [23], [27], [42]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3 = 0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{43}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{44}$$

are fulfilled, where $i_1,i_2,i_3 = 0,1,\ldots,m$ *in* (43) *and* $i_1,i_2,i_3 = 1,\ldots,m$ *in* (44), *constant* C *is independent of* p,

$$C_{j_3 j_2 j_1} = \int\limits_t^{T} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$); *another notations are the same as in Theorems* 1, 2.

Theorem 5 [18], [22], [23], [27], [42]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

(45) $$J^{*}[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following relations

(46) $$J^{*}[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

(47) $$\mathsf{M}\left\{\left(J^{*}[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (45), (46) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (47), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=$$

$$=\int\limits_t^{T}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 4.

Theorem 6 [18], [22], [23], [27], [42]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

(48) $$J^{*}[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)}$$

the following relations

(49) $$J^{*}[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)},$$

(50) $$\mathsf{M}\left\{\left(J^{*}[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (48), (49) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (50), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 4, 5.

Theorem 7 [18], [22], [23], [27], [43]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{51}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6$$

another notations are the same as in Theorems 4–6.

Note that an analogue of Theorem 3 for the case of iterated Stratonovich stochastic integrals of multiplicity 1 follows from (29).

5. Exact Calculation of the Mean-Square Error in Theorems 1, 2

As we mentioned above, Theorems 1, 2 give new possibilities for exact calculation of the mean-square error of approximation of iterated Ito stochastic integrals (see Theorem 8 below).

Assume that $J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}$ is the approximation of (6), which is the expression before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ on the right-hand side of (38)

$$J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\stackrel{\rm def}{=}E_k^{p_1,\ldots,p_k},$$

$$E_k^{p_1,\ldots,p_k}\stackrel{\rm def}{=}E_k^p\ \ \hbox{if}\ \ p_1=\ldots=p_k=p,$$

$$\|K\|_{L_2([t,T]^k)}^2=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\rm def}{=}I_k.$$

In [13]-[21], [28] it was shown that

$$E_k^{p_1,\ldots,p_k}\le k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right) \tag{52}$$

if $i_1,\ldots,i_k=1,\ldots,m$ $(0<T-t<\infty)$ or $i_1,\ldots,i_k=0,1,\ldots,m$ $(0<T-t<1)$.
Moreover [6]-[21],

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le C_{n,k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n,$$

where constant $C_{n,k}$ depends only on n and k $(n\in\mathbb{N})$.
The value E_k^p can be calculated exactly.

Theorem 8 [18] (Sect. 1.12), [28] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. Then*

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{53}$$

where $i_1,\ldots,i_k=1,\ldots,m$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in Theorems 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Then from Theorem 8 for pairwise different $i_1,\ldots,i_k$ and for $i_1=\ldots=i_k$ we obtain

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2,$$

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right).$$

Consider some examples of application of Theorem 8 $(i_1,i_2,i_3=1,\ldots,m)$

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2),$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3), \tag{54}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3), \tag{55}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2). \tag{56}$$

6. Comparative Analysis of the Efficiency of Application of Legendre Polynomials and Trigonometric Functions for the Integral $J_{(11)T,t}^{(i_1i_2)}$

Using Theorems 1, 2 and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ it is shown [4]-[43] (also see [1]-[3]) that

$$J_{(11)T,t}^{(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{57}$$

where the series converges in the mean-square sense; $i_1,i_2=1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

Table 1. Numbers $q_{\rm trig}$, $q_{\rm trig}^*$, $q_{\rm pol}$

$T-t$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}	2^{-11}	2^{-12}
$q_{\rm trig}$	3	4	7	14	27	53	105	209
$q_{\rm trig}^*$	6	11	20	40	79	157	312	624
$q_{\rm pol}$	5	9	17	33	65	129	257	513

are independent standard Gaussian random variables for various i or j,

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right);\ j=0,1,2,\ldots, \tag{58}$$

where $P_j(x)$ is the Legendre polynomial.

The formula (57) can also be found in [1]-[3]. It is not difficult to show that [1]-[32]

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{59}$$

where

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right). \tag{60}$$

Let us compare (60) with (16) and (59) with (18). Consider minimal natural numbers $q_{\rm trig}$ and $q_{\rm pol}$, which satisfy to (see Table 1)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q_{\rm pol}}\frac{1}{4i^2-1}\right)\le (T-t)^3,$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^3.$$

Thus, we have

$$\frac{q_{\rm pol}}{q_{\rm trig}}\ \approx\ 1.67,\ \ 2.22,\ \ 2.43,\ \ 2.36,\ \ 2.41,\ \ 2.43,\ \ 2.45,\ \ 2.45.$$

The formula (16) includes $(4q+4)m$ independent standard Gaussian random variables. At the same time the folmula (60) includes only $(2q+2)m$ independent standard Gaussian random variables. Moreover, the formula (60) is simpler than the formula (16). Thus, in this case we can talk about approximately equal computational costs for the formulas (16) and (60).

There is one important feature. As we mentioned above, further we will see that introduction of random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ will sharply complicate the approximation of the iterated stochastic integral $J_{(111)T,t}^{(i_1i_2i_3)}$; $i_1,i_2,i_3=1,\ldots,m$. This is due to the fact that the number q is fixed for all stochastic integrals, which included into the considered collection. However, it is clear that due to the smallness of $T-t$, the number q for $J_{(111)T,t}^{(i_1i_2i_3)}$ could be chosen significantly less than in the formula (16). This feature is also valid for the formulas (15), (19). However, for the case of Legendre polynomials we can choose different numbers q for different iterated stochastic integrals.

From the other hand, if we will not introduce the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the mean-square error of approximation of the iterated stochastic integral $J_{(11)T,t}^{(i_1i_2)}$ will be three times larger (see (13)). Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0i_1)}$, $J_{(001)T,t}^{(00i_1)}$ (with Gaussian distribution) will be approximated worse.

Consider minimal natural numbers $q_{\rm trig}^*$, which satisfy to (see Table 1)

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}^*}\frac{1}{r^2}\right)\le (T-t)^3.$$

In this situation we can talk about the advantage of Legendre polynomials ($q_{\rm trig}^*>q_{pol}$ and (16) is more complex than (60)).

7. Comparative Analysis of the Efficiency of Application of Legendre Polynomials and Trigonometric Functions for the Integrals $J_{(1)T,t}^{(i_1)}$, $J_{(11)T,t}^{(i_1i_2)}$, $J_{(01)T,t}^{(0i_1)}$, $J_{(10)T,t}^{(i_10)}$, $J_{(111)T,t}^{(i_1i_2i_3)}$

It is well known [45]-[49] that for implementation of strong Taylor–Ito numerical methods with the order 1.5 of accuracy for Ito stochastic differential equations we need to approximate the following collection of iterated Ito stochastic integrals

$$J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1i_2)},\quad J_{(01)T,t}^{(0i_1)},\quad J_{(10)T,t}^{(i_10)},\quad J_{(111)T,t}^{(i_1i_2i_3)}.$$

Using Theorems 1, 2 for the system of trigonometric functions, we have [4]-[32] (also see [1]-[3])

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{61}$$

$$J_{(11)T,t}^{(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$
$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+$$
$$+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\Biggr), \tag{62}$$

$$J_{(01)T,t}^{(0i_1)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{63}$$

$$J_{(10)T,t}^{(i_1 0)q}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{64}$$

$$J_{(111)T,t}^{(i_1 i_2 i_3)q}=(T-t)^{3/2}\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$
$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+$$
$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$
$$\left.+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\right)+$$
$$+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+\right.$$
$$+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\right.$$
$$\left.\left.\left.+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\right)+D_{T,t}^{(i_1 i_2 i_3)q}\right), \tag{65}$$

where

$$D_{T,t}^{(i_1 i_2 i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\left(\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+\right.\right.$$
$$\left.\left.+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\right)+$$
$$+\frac{1}{4\sqrt{2}\pi^2}\left(\sum_{r,m=1}^{q}\left(\frac{2}{rm}\left(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\right.\right.\right.$$
$$\left.+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+$$
$$+\frac{1}{m(r+m)}\left(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.$$
$$\left.\left.-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)\right)+$$
$$+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\left(\frac{1}{m(l-m)}\left(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-\right.\right.$$
$$\left.-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\right)+$$
$$+\frac{1}{l(l-m)}\left(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-\right.$$

TABLE 2. Confirmation of the formula (67)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722 \cdot 10^{-4}$	$7.5973 \cdot 10^{-5}$	$7.5990 \cdot 10^{-6}$
q	1	10	100	1000	10000

$$-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)\Biggr)\Biggr),$$

where

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},$$

$$\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},$$

and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q;\ i=1,\ldots,m)$ are independent standard Gaussian random variables. Moreover, in (65) we suppose that $i_1\ne i_2$, $i_1\ne i_3$, $i_2\ne i_3$.

Mean-square errors for the approximations (62)–(65) are represented by the formulas

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=0,$$

$$\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=0,$$

$$\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{66}$$

$$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q}\right)^2\right\}=(T-t)^3\Biggl(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-$$

$$-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\Biggr), \tag{67}$$

where $i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3$.

In Table 2, we can see the numerical confirmation of the formula (67) (ε is a right-hand side of (67)).

Note that the formulas (61), (62) have been obtained for the first time in [45]. Using (61), (62), we can realize numerically the explicit one-step strong Taylor–Ito numerical method with the order 1.0 of accuracy (Milstein scheme [45]). The analogue of the formula (65) has been obtained for the first time in [46]-[48].

As we mentioned above, the Milstein approach (see Sect. 2) leads to iterated application of the operation of limit transition. The analogue of (65) has been derived in [46]-[48], [50] on the base of the Milstein approach [45]. It means that the authors of the works [46]-[48], [50] could not formally use the double sum with the upper limit q in the analogue of (65) in [46] (pp. 438-439), [47] (Sect. 5.8, pp. 202–204), [48] (pp. 82-84), [50] (pp. 263-264) on the base of the Wong–Zakai approximation [52]-[54] (see discussion in Sect. 11 for details). From the other hand, the correctness of (65) follows directly from Theorems 1, 2. Note that (65) has been obtained reasonably for the first time in [4]. The version of (65) but without using the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$ can be found in [1]-[3].

The formula (66) appears for the first time in [45]. The mean-square error (67) has been obtained for the first time in [4] on the base of the simplified variant of Theorem 8 (the case of pairwise different $i_1,\ldots,i_k$).

As we noted above, the number q must be the same in (62)–(65). This is the main drawback of this approach because really the number q in (65) can be chosen essentially smaller than in (62).

Note that in (65) we can replace $J_{(111)T,t}^{(i_1 i_2 i_3)q}$ with $J_{(111)T,t}^{*(i_1 i_2 i_3)q}$ and (65) will when be valid for any $i_1, i_2, i_3 = 0, 1, \ldots, m$ (see Theorem 3).

Consider approximations of the iterated Ito stochastic integrals

$$J_{(1)T,t}^{(i_1)},\quad J_{(11)T,t}^{(i_1 i_2)},\quad J_{(01)T,t}^{(0 i_1)},\quad J_{(10)T,t}^{(i_1 0)},\quad J_{(111)T,t}^{(i_1 i_2 i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

on the base of Theorems 1, 2 (the case of Legendre polynomials) [1]-[32]

$$J_{(1)T,t}^{(i_1)}=\sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{68}$$

$$J_{(11)T,t}^{(i_1 i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{69}$$

$$J_{(01)T,t}^{(0 i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{70}$$

$$J_{(10)T,t}^{(i_1 0)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{71}$$

$$J_{(111)T,t}^{(i_1 i_2 i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),\quad q_1\ll q, \tag{72}$$

$$J_{(111)T,t}^{(i_1 i_1 i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right),$$

$$C_{j_3 j_2 j_1}=\int\limits_t^T\phi_{j_3}(z)\int\limits_t^z\phi_{j_2}(y)\int\limits_t^y\phi_{j_1}(x)dxdydz=$$

$$
=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1},
$$

$$
\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{73}
$$

where $\phi_j(x)$ has the form (58) and $P_i(x)$ is the Legendre polynomial ($i=0,1,2,\ldots$).

Mean-square errors for the approximations (69), (72) are represented by the formulas (see Theorem 8 and (52)) [1]-[32]

$$
\mathsf{M}\left\{\left(J_{(11)T,t}^{(i_1i_2)}-J_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2), \tag{74}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2\quad (i_1\ne i_2, i_1\ne i_3, i_2\ne i_3), \tag{75}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2-
$$

$$
-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3), \tag{76}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2-
$$

$$
-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2), \tag{77}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2-
$$

$$
-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3), \tag{78}
$$

$$
\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1i_2i_3)}-J_{(111)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2\right)\quad (i_1,i_2,i_3=1,\ldots,m). \tag{79}
$$

Let us compare the efficiency of application of Legendre polynomials and trigonometric functions for the iterated stochastic integrals $J_{(11)T,t}^{(i_1 i_2)}$, $J_{(111)T,t}^{(i_1 i_2 i_3)}$.

Consider the following conditions $(i_1 \neq i_2,\ i_1 \neq i_3,\ i_2 \neq i_3)$

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le (T-t)^4, \tag{80}$$

$$(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}\frac{\left(C_{j_3j_2j_1}\right)^2}{(T-t)^3}\right)\le (T-t)^4, \tag{81}$$

$$\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p}\frac{1}{r^2}\right)\le (T-t)^4, \tag{82}$$

$$(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{p_1}\frac{1}{r^2}-\frac{55}{32\pi^4}\sum_{r=1}^{p_1}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\neq l}}^{p_1}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \tag{83}$$

where

$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_i(x)$ is the Legendre polynomial.

In Tables 3 and 4, we can see minimal numbers q, q_1, p, p_1, which satisfy the conditions (80)–(83). As we mentioned above, the numbers q, q_1 are different. At that $q_1 \ll q$ (the case of Legendre polynomials). As we saw in the previous sections, we cannot take different numbers p, p_1 for the case of trigonometric functions. Thus, we have to choose $q = p$ in (62)–(65). This leads to huge computational costs (see very complex formula (65)). From the other hand, we can choose different numbers q in (62)–(65). At that we must exclude the random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (62)–(65).

At this situation for the case $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we have

$$\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{p^*}\frac{1}{r^2}\right)\le (T-t)^4, \tag{84}$$

$$(T-t)^3\left(\frac{5}{36}-\frac{1}{2\pi^2}\sum_{r=1}^{p_1^*}\frac{1}{r^2}-\frac{79}{32\pi^4}\sum_{r=1}^{p_1^*}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\neq l}}^{p_1^*}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right)\le (T-t)^4, \tag{85}$$

where the left-hand sides of (84), (85) correspond to (16), (65) but without $\xi_q^{(i)}$, $\mu_q^{(i)}$. In Table 4, we can see minimal numbers p^*, p_1^*, which satisfy the conditions (84), (85).

TABLE 3. Numbers q, q_1

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

TABLE 4. Numbers p, p_1, p^*, p_1^*

$T-t$	0.08222	0.05020	0.02310	0.01956
p	8	21	96	133
p_1	1	1	3	4
p^*	23	61	286	398
p_1^*	1	2	4	5

TABLE 5. Confirmation of the formula (85)

$\varepsilon/(T-t)^3$	0.0629	0.0097	0.0010	$1.0129\cdot 10^{-4}$	$1.0132\cdot 10^{-5}$
q	1	10	100	1000	10000

Moreover,

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(0i_1)}-J_{(01)T,t}^{(0i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(J_{(10)T,t}^{(i_10)}-J_{(10)T,t}^{(i_10)q}\right)^2\right\}=$$

$$=\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)\neq 0, \tag{86}$$

where $J_{(01)T,t}^{(0i_1)q}$, $J_{(10)T,t}^{(i_10)q}$ are defined by the formulas (63), (64).

It is not difficult to see that the numbers $q_{\rm trig}$ in Table 1 correspond to minimal numbers $q_{\rm trig}$, which satisfy the condition

$$\frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\rm trig}}\frac{1}{r^2}\right)\le (T-t)^4.$$

From the other hand, the right-hand sides of (70), (71) include only two random variables. In this situation we can again talk about the advantage of Ledendre polynomials.

In Table 5, we can see the numerical confirmation of the formula (85) (ε is a left-hand side of the formula (85)).

8. Comparative Analysis of the Efficiency of Application of Legendre Polynomials and Trigonometric Functions for the Integral $J_{(011)T,t}^{*(0i_1i_2)}$

In this section, we compare computational costs for the iterated Stratonovich stochastic integral $J_{(011)T,t}^{*(0i_1i_2)}$ $(i_1,i_2=1,\ldots,m)$ within the frames of the method of generalized multiple Fourier series for the systems of Legendre polynomials and trigomomenric functions.

Using Theorem 3 for the case of trigonometric system of functions, we obtain [4]-[32] (also see [1]-[3])

$$
\begin{aligned}
J_{(011)T,t}^{*(0i_1i_2)q}=(T-t)^2\Biggl(&\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+\\
&+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right)+\\
&+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right)-\\
&-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right)+\\
&+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\\
&+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr).
\end{aligned}
\tag{87}
$$

For the case $i_1\ne i_2$ from Theorem 8 we get [4]-[32] (also see [1]-[3])

$$
\begin{aligned}
\mathsf{M}\left\{\left(J_{(011)T,t}^{*(0i_1i_2)}-J_{(011)T,t}^{*(0i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\Biggl(&\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\\
&-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\Biggr).
\end{aligned}
\tag{88}
$$

Analogues of the formulas (87), (88) for the case of Legendre polynomials will look as follows [4]-[32] (also see [1]-[3])

$$
\begin{aligned}
J_{(011)T,t}^{*(0i_1i_2)q}=\frac{T-t}{2}J_{(11)T,t}^{*(i_1i_2)q}+\frac{(T-t)^2}{4}\Biggl(&\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+\\
&+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr),
\end{aligned}
\tag{89}
$$

where

TABLE 6. Confirmation of the formula (88)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261 \cdot 10^{-4}$	$8.4429 \cdot 10^{-5}$	$8.4435 \cdot 10^{-6}$
q	1	10	100	1000	10000

TABLE 7. Confirmation of the formula (90)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450 \cdot 10^{-4}$	$6.2495 \cdot 10^{-5}$
q	1	10	100	1000	10000

$$J^{*(i_1 i_2)q}_{(11)T,t} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

$$\mathsf{M}\left\{\left(J^{*(0i_1i_2)}_{(011)T,t} - J^{*(0i_1i_2)q}_{(011)T,t}\right)^2\right\} =$$

$$= \frac{(T-t)^4}{16}\left(\frac{5}{9} - 2\sum_{i=2}^{q}\frac{1}{4i^2-1} - \sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2} - \right.$$

$$\left. - \sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right) \quad (i_1 \neq i_2). \tag{90}$$

In Tables 6 and 7, we can see the numerical confirmation of the formulas (88) and (90) (ε is the right-hand side of (88), (90)).

Let us compare the complexity of the formulas (87) and (89). The formula (87) includes the double sum

$$\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)} + \frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right).$$

Thus, the formula (87) is more complex than the formula (89) even if we take identical numbers q in these formulas. As we noted above, the number q in (87) must be equal to the number q from the formula (16), so it is much larger than the number q from the formula (89). As a result, we have obvious advantage of the formula (89) in computational costs. As we mentioned above, if we will not use the random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$, then the number q in (87) can be chosen smaller, but the mean-square error of approximation of the stochastic integral $J^{(i_1i_2)}_{(11)T,t}$ will be three times larger (see (13)). Moreover, in this case the stochastic integrals $J^{(0i_1)}_{(01)T,t}$, $J^{(i_10)}_{(10)T,t}$, $J^{(00i_1)}_{(001)T,t}$ (with Gaussian distribution) will be approximated worse. In this situation we can again talk about the advantage of Ledendre polynomials.

9. Conclusions

Summing up the results of previous sections, we can come to the following conclusions.

1. We can talk about approximately equal computational costs for the formulas (16) and (60). This means that computational costs for implementing the Milstein scheme (explicit one-step strong Taylor–Ito numerical method with the order $\gamma = 1.0$ of accuracy for Ito stochastic differential equations [45]) for the case of Legendre polynomials and for the case of trigonometric functions are approximately the same.

2. If we will not use the random variables $\xi_q^{(i)}$ (see (16)), then the mean-square error of approximation of the stochastic integral $J_{(11)T,t}^{(i_1 i_2)}$ will be three times larger (see (13)). In this situation, we can talk about the advantage of Ledendre polynomials in the Milstein method. Moreover, in this case the stochastic integrals $J_{(01)T,t}^{(0 i_1)}$, $J_{(10)T,t}^{(i_1 0)}$, $J_{(001)T,t}^{(00 i_1)}$ (with Gaussian distribution) will be approximated worse.

3. If we talk about the explicit one-step strong Taylor–Ito scheme with the order $\gamma = 1.5$ of accuracy for Ito stochastic differential equations, then the numbers q, q_1 (see (69), (72)) are different. At that $q_1 \ll q$ (the case of Legendre polynomials). The number q must be the same in (62)–(65) (the case of trigonometric functions). This leads to huge computational costs (see very complex formula (65)). From the other hand, we can take different numbers q in (62)–(65). At that we should exclude the random variables $\xi_q^{(i)}$, $\mu_q^{(i)}$ from (62)–(65). This leads to another problems, which we discussed above (see Conclusion 1).

4. In addition, the author supposes that effect described in Conclusion 3 will be more impressive when analyzing more complex sets of iterated Ito and Stratonovich stochastic integrals (when $\gamma = 2.0,\ 2.5,\ 3.0,\ \ldots$; here γ has the same meaning as in Conclusion 3). This supposition is based on the fact that the polynomial system of functions has the significant advantage (compared with the trigonometric system) for approximation of iterated stochastic integrals for which not all weight functions are equal to 1.

10. Further Development of Multiple Fourier–Legendre Series Approach to the Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals of Multiplicities 3 to 5

From Theorems 1, 2 for $k = 4$ and 5 we obtain

$$
\begin{aligned}
J_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1} \Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
&+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}} + \\
&+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}} + \\
&+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
\end{aligned}
\tag{91}
$$

$$J_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_1} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_1} C_{j_5 j_4 j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)} \Biggr), \tag{92}$$

where

$$J_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{(i_1 i_2 i_3 i_4)}, \quad J_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$$

are defined by the formula (4); $q_1 < q$; $\mathbf{1}_A$ is the indicator of the set A; $i_1, i_2, i_3, i_4, i_5 = 0, 1, \ldots, m$, and

$$C_{j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(u) \int\limits_t^u \phi_{j_3}(z) \int\limits_t^z \phi_{j_2}(y) \int\limits_t^y \phi_{j_1}(x) dx dy dz du =$$
$$= \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}\ \frac{(T-t)^2}{16}\ \bar{C}_{j_4 j_3 j_2 j_1},$$

$$C_{j_5 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(v) \int\limits_t^v \phi_{j_4}(u) \int\limits_t^u \phi_{j_3}(z) \int\limits_t^z \phi_{j_2}(y) \int\limits_t^y \phi_{j_1}(x) dx dy dz du dv =$$
$$= \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}\ \frac{(T-t)^{5/2}}{32}\ \bar{C}_{j_5 j_4 j_3 j_2 j_1},$$

$$\bar{C}_{j_4 j_3 j_2 j_1} = \int\limits_{-1}^1 P_{j_4}(u) \int\limits_{-1}^u P_{j_3}(z) \int\limits_{-1}^z P_{j_2}(y) \int\limits_{-1}^y P_{j_1}(x) dx dy dz du, \tag{93}$$

(94) $$\bar{C}_{j_5 j_4 j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv,$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (if $i\neq 0$),

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots$$

is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

Note that the Fourier–Legendre coefficients $\bar{C}_{j_4 j_3 j_2 j_1}$, $\bar{C}_{j_5 j_4 j_3 j_2 j_1}$, and $\bar{C}_{j_3 j_2 j_1}$ (see (73)) can be calculated exactly using DERIVE or MAPLE (computer algebra systems). Several tables with these coefficients can be found in [4]-[20], [24], [31]. The database with 270,000 of exactly calculated Fourier–Legendre coefficients is descibed in [34], [35]. Note that the mentioned Fourier–Legendre coefficients not depend on the integration step $T-t$ of numerical methods for Ito stochastic differential equations. So, $T-t$ can be not a constant in this approach.

From (52) $(0<T-t<1)$ we obtain

(95) $$\mathsf{M}\left\{\left(J_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{(i_1 i_2 i_3 i_4)}-J_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t}^{(i_1 i_2 i_3 i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4 j_3 j_2 j_1}^2\right),$$

(96) $$\mathsf{M}\left\{\left(J_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}^{(i_1 i_2 i_3 i_4 i_5)}-J_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_1}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_1}C_{j_5 j_4 j_3 j_2 j_1}^2\right).$$

Note that in practice the numbers q,q_1 in (72), (91), (92) can be selected not large. For example, for the case of pairwise different $i_1,i_2,i_3,i_4,i_5=1,\ldots,m$ we obtain

(97) $$\mathsf{M}\left\{\left(J_{(111)T,t}^{(i_1 i_2 i_3)}-J_{(111)T,t}^{(i_1 i_2 i_3)6}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{6}C_{j_3 j_2 j_1}^2\approx 0.01956000(T-t)^3,$$

(98) $$\mathsf{M}\left\{\left(J_{(1111)T,t}^{(i_1 i_2 i_3 i_4)}-J_{(1111)T,t}^{(i_1 i_2 i_3 i_4)2}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C_{j_4 j_3 j_2 j_1}^2\approx 0.02360840(T-t)^4,$$

(99) $$\mathsf{M}\left\{\left(J_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)}-J_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)1}\right)^2\right\}=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5 j_4 j_3 j_2 j_1}^2\approx 0.00759105(T-t)^5.$$

From Theorems 3–6 we have

$$J^{*(i_1 i_2 i_3)q}_{(\lambda_1\lambda_2,\lambda_3)T,t}=\sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3},$$

$$J^{*(i_1 i_2 i_3 i_4)q_1}_{(\lambda_1\lambda_2,\lambda_3\lambda_4)T,t}=\sum_{j_1,j_2,j_3,j_4=0}^{q_1} C_{j_4 j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4},$$

$$J^{*(i_1 i_2 i_3 i_4 i_5)q_2}_{(\lambda_1\lambda_2,\lambda_3\lambda_4\lambda_5)T,t}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_2} C_{j_5 j_4 j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5},$$

where

$$J^{*(i_1 i_2 i_3)}_{(\lambda_1\lambda_2\lambda_3)T,t},\ J^{*(i_1 i_2 i_3 i_4)}_{(\lambda_1\lambda_2\lambda_3\lambda_4)T,t},\ J^{*(i_1 i_2 i_3 i_4 i_5)}_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)T,t}$$

are defined by the formula (5).

The values

$$\mathsf{M}\left\{\left(J^{*(i_1 i_2 i_3)}_{(111)T,t}-J^{*(i_1 i_2 i_3)6}_{(111)T,t}\right)^2\right\},\quad \mathsf{M}\left\{\left(J^{*(i_1 i_2 i_3 i_4)}_{(1111)T,t}-J^{*(i_1 i_2 i_3 i_4)2}_{(1111)T,t}\right)^2\right\},$$

$$\mathsf{M}\left\{\left(J^{*(i_1 i_2 i_3 i_4 i_5)}_{(11111)T,t}-J^{*(i_1 i_2 i_3 i_4 i_5)1}_{(11111)T,t}\right)^2\right\}$$

are equal to the right-hand sides of (97)–(99) for the case of pairwise different $i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m$.

Note that the optimization of the mean-square approximation procedures for the itertaed Ito stochastic integrals (6) of multiplicities 1 to 5 is carried out in [55], [56].

11. Theorems 1–7 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s \in [0, T]$. Let $\mathbf{f}_s^{(i)p}$, $p \in \mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i = 1, \ldots, m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [52], [53], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito stochastic differential equations. The piecewise linear approximation as well as the regularization by convolution [52]-[54] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s \in [0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [57], [58]

$$\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty} \int\limits_t^\tau \phi_j(s)ds\ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}, \tag{100}$$

where $\tau \in [t,T]$, $t \geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (100) converges for any $\tau \in [t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p} - \mathbf{f}_t^{(i)p} = \sum_{j=0}^{p} \int\limits_t^\tau \phi_j(s)ds\ \zeta_j^{(i)}. \tag{101}$$

From (101) we obtain

$$d\mathbf{f}_\tau^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau)\zeta_j^{(i)} d\tau. \tag{102}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{103}$$

where $p_1, \ldots, p_k \in \mathbb{N}$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$d\mathbf{w}_\tau^{(i)p} = \begin{cases} d\mathbf{f}_\tau^{(i)p} & \text{for} \quad i = 1, \ldots, m \\ \\ d\tau & \text{for} \quad i = 0 \end{cases}, \tag{104}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (102).

Let us substitute (102) into (103)

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)p_1} \ldots d\mathbf{w}_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{105}$$

where

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_s^{(0)} = s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [52]-[54] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [54] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (101) were not considered in [52], [53] (also see [54], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [54] for approximations of the Wiener process based on its series expansion (100) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (105) to the iterated Stratonovich stochastic integral (7) does not follow from the results of the papers [52], [53] (also see [54], Theorems 7.1, 7.2).

From the other hand, Theorems 1–7 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (7) of multiplicities 1 to 6 based on the approximation (101) of the Wiener process. At that, the Riemann–Stieltjes integrals (103) converge (according to Theorems 1–7) to the appropriate Stratonovich stochastic integrals (7). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (100), (101), and Theorems 3–7) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [52]-[54]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{106}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (106) and additive property of the Riemann–Stieltjes integrals, we can write w. p. 1

$$\int_0^T\int_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int_0^T\int_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{107}$$

Using (107) it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int_0^T\int_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int_0^T\int_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int_0^T ds=$$

$$=\int_0^{*T}\int_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{108}$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (108) agrees with Theorem 7.1 (see [54], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (100) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T\int_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \tag{109}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (102).

Let us substitute (102) into (109)

$$\int_0^T\int_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{110}$$

where

$$C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient; another notations are the same as in (105).

As we noted above, approximations of the Wiener process that are similar to (101) were not considered in [52], [53] (also see Theorems 7.1, 7.2 in [54]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [54] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [18]-[20]. More precisely, using Theorem 3, we obtain from (110) the desired result

$$\text{l.i.m.}_{p\to\infty}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\text{l.i.m.}_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=\int\limits_0^{*T}\int\limits_0^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{111}$$

From the other hand, by Theorems 1, 2 (see (30)) for the case $k=2$ we obtain from (110) the following relation

$$\text{l.i.m.}_{p\to\infty}\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\text{l.i.m.}_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=$$

$$=\text{l.i.m.}_{p\to\infty}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=$$

$$=\int\limits_0^T\int\limits_0^s d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{112}$$

Since

$$\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,$$

then from (112) we obtain (111).

References

[1] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[2] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publ., 204 pp. (ISBN 5-7422-0045-5)

[3] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[4] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[5] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[6] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[7] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[8] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[9] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[10] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[11] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[12] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[13] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[14] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[15] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[18] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR]. 2022, 923 pp.

[19] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English].

Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[20] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[21] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[22] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 204 pp.

[23] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.

[24] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[25] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[26] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[27] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 129 pp.

[28] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic based on the multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 68 pp.

[29] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604

[30] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[31] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[32] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[33] Kuznetsov D.F. Four new forms of the Taylor–Ito and Taylor–Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. [In English]. arXiv:2001.10192 [math.PR], 2020, 90 pp.

[34] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[35] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[36] Kuznetsov D.F. Application of multiple Fourier–Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[37] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[38] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[39] Kuznetsov D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.3/article.1.6.html

[40] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878

[41] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[42] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[43] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

[44] Gihman I.I., Skorohod A.V. Introduction to the Theory of Stochastic Processes. [In Russian]. Nauka, Moscow, 1977, 568 pp.

[45] Milstein G.N. Numerical Integration of Stochastic Differential Equations. [In Russian]. Ural University Press, Sverdlovsk, 1988, 225 pp.

[46] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.

[47] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992, 632 pp.

[48] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[49] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[50] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[51] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[52] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat. 5, 36 (1965), 1560-1564.

[53] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci. 3 (1965), 213-229.

[54] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[55] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[56] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[57] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974, 696 pp.

[58] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. Ph.D. Thesis, California Inst. of Technology, 2006, 225 pp.

[59] Kuznetsov D.F., Kuznetsov, M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

DEVELOPMENT AND APPLICATION OF THE FOURIER METHOD TO THE MEAN-SQUARE APPROXIMATION OF ITERATED ITO AND STRATONOVICH STOCHASTIC INTEGRALS

DMITRIY F. KUZNETSOV

ABSTRACT. The article is devoted to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals in the context of the numerical integration of Ito stochastic differential equations. The expansion of iterated Ito stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) and expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 6 have been obtained. Considerable attention is paid to expansions based on multiple Fourier–Legendre series. The exact and approximate expressions for the mean-square error of approximation of iterated Ito stochastic integrals are derived. The results of the article will be useful for numerical integration of Ito stochastic differential equations with non-commutative noise.

CONTENTS

MATHEMATICS SUBJECT CLASSIFICATION: 60H05, 60H10, 42B05, 42C10.

KEYWORDS: ITERATED ITO STOCHASTIC INTEGRAL, ITERATED STRATONOVICH STOCHASTIC INTEGRAL, GENERALIZED MULTIPLE FOURIER SERIES, MULTIPLE FOURIER–LEGENDRE SERIES, MULTIPLE TRIGONOMETRIC FOURIER SERIES, GENERALIZED ITERATED FOURIER SERIES, ITO STOCHASTIC DIFFERENTIAL EQUATION, NUMERICAL INTEGRATION, MEAN-SQUARE APPROXIMATION, EXPANSION.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to (1). The nonrandom functions $\mathbf{a}: \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B: \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems of different physical nature that are affected by random perturbations. For example, Ito SDEs are used as mathematical models in stochastic mathematical finance, hydrology, seismology, geophysics, chemical kinetics, population dynamics, electrodynamics, medicine and other fields [2]-[5]. Also these equations arise in optimal stochastic control, signal filtering against the background of random noises, parameter estimation for stochastic systems as well as in stochastic stability and bifurcations analysis [2], [4].

One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[26]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau,\ \ i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\int \quad \text{and} \quad \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1,\ldots,k)$ and $i_1,\ldots,i_k = 0,1,\ldots,m$ in [2]-[5]. At the same time $\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l = 1,\ldots,k;\ q_1,\ldots,q_k = 0,1,2,\ldots)$ and $i_1,\ldots,i_k = 1,\ldots,m$ in [6]-[26].

Effective solution of the problem of combined mean-square approximation for collections of iterated Ito and Stratonovich stochastic integrals (2) and (3) composes the subject of this article (also see author's publications [9]-[54]).

We want to mention in short that there are two main criteria of the numerical methods convergence for Ito SDEs [2]-[5]: a strong or mean-square criterion and a weak criterion, where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution.

Using the strong numerical methods, we can build sample pathes of Ito SDEs numerically. That is why strong numerical methods are using when constructing new mathematical models on the basis of Ito SDEs. Moreover, these methods are the tool for the numerical solution of different mathematical problems connected with Ito SDEs (see above) [2]-[5].

Strong numerical methods for Ito SDEs require the combined mean-square approximation of collections of iterated Ito and Stratonovich stochastic integrals (2) and (3). The problem of effective jointly numerical modeling (with respect to the mean-square criterion of convergence) of iterated Ito and Stratonovich stochastic integrals (2) and (3) is complex from both theoretical and computational points of view [2]-[59].

The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)$. This case allows the investigation with using of the Ito formula [2]-[5].

Note that even for the mentioned coincidence ($i_1 = \ldots = i_k \neq 0$), but for different functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ the mentioned difficulties persist. As a result, relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (with respect to the mean-square criterion of approximation) using the system of standard Gaussian random variables.

Why the problem of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals is so complex?

Firstly, the mentioned stochastic integrals (in the case of fixed limits of integration) are the random variables, whose density functions are unknown in the general case. The exception is connected with the narrow particular case which is the simplest iterated Ito stochastic integral (2) with multiplicity 2 and $\psi_1(\tau), \psi_2(\tau) \equiv 1$, $i_1, i_2 = 1, \ldots, m$. Nevertheless, the knowledge of this density function not gives a simple way for approximation of iterated Ito stochastic integral (2) of multiplicity 2 [55].

Secondly, we need to approximate not only one stochastic integral, but several iterated stochastic integrals which are complexly dependent in a probabilistic meaning.

Often, the problem of combined mean-square approximation of iterated Ito and Stratonovich stochastic integrals occurs even in cases when the exact solution of Ito SDE is known. It means that even if we know the solution of Ito SDE, we cannot model it numerically without the combined numerical modeling of iterated Ito and Stratonovich stochastic integrals.

Note that for a number of special types of Ito SDEs the problem of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals can be simplified but cannot be solved. The equations with additive vector noise, with scalar additive noise, with scalar non-additive noise, with a small parameter are related to such types of equations [2]-[5].

For the mentioned types of equations, simplifications are connected to the fact that some members from stochastic Taylor expansions are equal to zero or we may neglect some members from these expansions due to the presence of a small parameter [2]-[5].

Seems that iterated stochastic integrals may be approximated by multiple integral sums of different types [3], [5], [56]. However, this approach implies the partitioning of the interval of integration $[t, T]$ for iterated stochastic integrals. The length $T - t$ of this interval is already fairly small (because it is a step of integration of numerical methods for Ito SDEs) and does not need to be partitioned. Computational experiments show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration $[t, T]$ is partitioned) leads to unacceptably high computational cost and accumulation of computation errors [9].

In [3] (also see [2], [4], [5], [57], [60]), Milstein G.N. proposed to expand (2) or (3) into the iterated series of products of standard Gaussian random variables by representing the Wiener process as a trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loeve expansion of the Brownian bridge process). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the multidimensional Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double and triple stochastic integrals (3) were obtained [2], [4], [57], [60] ($k = 1, 2, 3$), [3], [5] ($k = 1, 2$) for the case $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$.

It should be noted that the authors of the publications [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [57] (pp. 438-439), [60] (pp. 263-264) use the Wong–Zakai approximation [61]-[63] (without rigorous proof) within the frames of the mentioned approach [3] based on the approximation of the Wiener process in the form of its series expansion (see discussion in Sect. 7 for detail).

Note that in [58], [59] the truncated expansions of the Wiener processes based on the Haar functions [59] and trigonometric functions [58], [59] were applied for the expansion of double [58], [59] and triple [58] Ito stochastic integrals (2). The expansions from [58], [59] also lead to iterated application of the operation of limit transition as in the Milstein approach [3].

It is necessary to note that the Milstein approach [3] excelled at least in several times (or even in several orders) the methods of multiple integral sums [3], [5], [56] considering computational costs in the sense of their diminishing [3], [5], [9].

An alternative and more general strong approximation method was proposed for (3) in [47] (also see [6], [13]-[18], [21], [23]-[26]). In these papers $J^*[\psi^{(k)}]_{T,t}$ was represented as the multiple stochastic integral of the certain discontinuous nonrandom function of k ($k \in \mathbb{N}$) variables, and the function was then expanded into the generalized iterated Fourier series by complete system of continuously differentiable functions that are orthonormal in the space $L_2([t, T])$. As a result, the general iterated series expansion of products of standard Gaussian random variables was obtained in [47] (also see [6], [13]-[18], [21], [23]-[26]) for the iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbb{N}$). Hereinafter, this method is referred to as the method of generalized iterated Fourier series.

Consider the formulation of the method of generalized iterated Fourier series. Let us introduce the following function $K(t_1, \ldots, t_k)$ defined on the k-dimensional hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}}, \tag{4}$$

where $t_1, \ldots, t_k \in [t, T]$ ($k \ge 2$) and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$. Here $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ and $\mathbf{1}_A$ denotes the indicator of the set A.

Theorem 1 [47] (1997), [6], [13]-[18], [21], [23]-[26]. *Suppose that every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a continuously differentiable function at the interval $[t, T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then, the iterated Stratonovich stochastic integral (3) is expanded into the converging in the mean of degree $2n$ ($n \in \mathbb{N}$) iterated series*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{5}$$

which means the following

$$\lim_{p_1\to\infty}\overline{\lim_{p_2\to\infty}}\ldots\overline{\lim_{p_k\to\infty}}\ \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)^{2n}\right\}=0, \tag{6}$$

where $\overline{\lim}$ *means* $\limsup$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)} \tag{7}$$

are independent standard Gaussian random variables for various i or j (if $i\neq 0$) and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{8}$$

is the Fourier coefficient.

The proof of Theorem 1 is based on the following statement.

Lemma 1 [47] (also see [6], [13]-[18], [21], [23]-[26]). *Under the conditions of Theorem* 1 *the function $K^*(t_1,\ldots,t_k)$ is represented in any internal point of the hypercube $[t,T]^k$ by the generalized iterated Fourier series*

$$K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l),\quad (t_1,\ldots,t_k)\in(t,T)^k, \tag{9}$$

where

$$K^*(t_1,\ldots,t_k)=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\left(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\neq s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\right)$$

for $t_1,\ldots,t_k\in[t,T]$ $(k\geq 2)$ and $K^*(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, $\mathbf{1}_A$ is the indicator of the set A, the Fourier coefficient $C_{j_k\ldots j_1}$ has the form (8). *At that, the iterated series* (9) *converges at the boundary of the hypercube $[t,T]^k$ (not necessarily to the function $K^*(t_1,\ldots,t_k)$).*

In [6], [13]-[18], [21], [23]-[26], [47] it was shown that the method of generalized iterated Fourier series leads for $k=2$ and $\psi_1(\tau)$, $\psi_2(\tau)\equiv 1$ (the case of trigonometric system of functions) to the Milstein expansion of (3) [3].

As we noted above, the method of generalized iterated Fourier series as well as the method from [3] lead to iterated application of the operation of limit transition. So, the convergence problem of the following approximation

$$J^*[\psi^{(k)}]_{T,t}^p=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{10}$$

to $J^*[\psi^{(k)}]_{T,t}$ if $p \to \infty$ in the mean-square sense must be considered separately (see Sect. 2 and discussion in Sect. 7 for detail). The mentioned problem appears for triple stochastic integrals or even for some double stochastic integrals in the case, when $\psi_1(\tau),\ \psi_2(\tau) \not\equiv 1$ (see above).

2. Method of the Mean-Square Approximation of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

In the previous section we paid attention on the fact that the method from [3] and the method of generalized iterated Fourier series [6], [13]-[18], [21], [23]-[26], [47] lead to iterated application of the operation of limit transition. So these methods may not converge in the mean-square sense to the appropriate iterated stochastic integrals (3) for some methods of series summation (see (10)).

The difficulties noted above can be overcome by the another method. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k ($k \in \mathbb{N}$) is represented as the multiple stochastic integral from the nonrandom discontinuous function $K(t_1,\ldots,t_k)$ defined on the hypercube $[t,T]^k$ by the relation (4), where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral. Then, the function $K(t_1,\ldots,t_k)$ is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 2–4 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of the generalized multiple Fourier series for the function $K(t_1,\ldots,t_k)$, which can be calculated using the explicit formula regardless of multiplicity k of the iterated Ito stochastic integral. Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ (defined by (4)) belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{11}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{12}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{13}$$

Theorem 2 [9] (2006), [10], [21], [23]-[46]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \quad (14)$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$,*

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)} \quad (15)$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (12), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of $[t,T]$, which satisfies the condition* (13).

Proof. The proof of Theorem 2 is based on Lemmas 5.1–5.3 [23] (P. A.253–A.259), Lemmas 1.1–1.3 [24]-[26] or Lemmas 1–3 [34]. According to Lemma 5.1 [23], Lemma 1.1 [24]-[26] or Lemma 1 [34], we have

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$
= \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \Bigl(K(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \Bigr), \tag{16}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the expression, which is enclosed in parentheses.

It is easy to see that (16) can be written in the form [23]-[26], [34]

$$
J[\psi^{(k)}]_{T,t} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} K(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Since the integration of bounded function with respect to the set of measure zero for Riemann or Lebesgue integrals gives zero result, then the following formula is correct for these integrals

$$
\int\limits_{[t,T]^k} |G(t_1,\ldots,t_k)| dt_1 \ldots dt_k = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} |G(t_1,\ldots,t_k)| dt_1 \ldots dt_k, \tag{17}
$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $dt_1,\ldots,dt_k$. At the same time the indexes near upper limits of integration are changed correspondently and the function $|G(t_1,\ldots,t_k)|$ is supposed as integrated in the hypercube $[t,T]^k$.

According to Lemmas 5.1–5.3 [23] (P. A.253–A.259), Lemmas 1.1–1.3 [24]-[26] or Lemmas 1–3 [34], we get the following representation

$$
J[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \int\limits_t^T \ldots \int\limits_t^{t_2} \sum_{(t_1,\ldots,t_k)} \Bigl(\phi_{j_1}(t_1) \ldots \phi_{j_k}(t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \Bigr) +
$$

$$
+ R_{T,t}^{p_1,\ldots,p_k} =
$$

$$
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1}) \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} +
$$

$$
+ R_{T,t}^{p_1,\ldots,p_k} = \tag{18}
$$

$$
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \Biggl(\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} \phi_{j_1}(\tau_{l_1}) \ldots \phi_{j_k}(\tau_{l_k}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)} -
$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k}\quad\text{w. p. 1,}\tag{19}$$

where

$$R_{T,t}^{p_1,\ldots,p_k}=$$

$$=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\tag{20}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

Let us estimate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series.

By Lemma 5.2 [23] (pp. A.257–A.258), Lemma 1.2 [24]-[26] or Lemma 2 [34] we have (see (17))

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2dt_1\ldots dt_k\to 0\tag{21}$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral. Theorem 2 is proved.

In order to evaluate the significance of Theorem 2 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [9]-[21], [23]-[46]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{22}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{23}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{24}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{25}$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{26}
$$

$$
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{27}$$

where $\mathbf{1}_A$ is the indicator of the set A.

The convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) in Theorem 2 is proved in [11]-[18], [21], [23]-[26], [34]. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ also can be applied in Theorem 2 [11]-[18], [21], [23]-[26], [34]. The convergence w. p. 1 in Theorem 2 is proved in [24]-[26], [28], [33], [42] for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. The modifications of Theorem 2 were obtained in [23]-[26], [35] for complete orthonormal with weight $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in the space $L_2([t,T]^k)$ ($k\in\mathbb{N}$) as well as for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson measures and iterated stochastic integrals with respect to martingales). Application of Theorem 2 and Theorem 4 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [24]-[26] (Chapter 7) and in [43]-[46].

Note that the correctness of formulas (22)–(27) can be verified by the fact that if $i_1=\ldots=i_6=i=1,\ldots,m$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv\psi(\tau)$, then we can derive from (22)–(27) the well known equalities

$$J[\psi^{(1)}]_{T,t}=\frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t}=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}=\frac{1}{4!}\left(\delta_{T,t}^4-6\delta_{T,t}^2\Delta_{T,t}+3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}=\frac{1}{5!}\left(\delta_{T,t}^5-10\delta_{T,t}^3\Delta_{T,t}+15\delta_{T,t}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t} = \frac{1}{6!}\left(\delta^6_{T,t} - 15\delta^4_{T,t}\Delta_{T,t} + 45\delta^2_{T,t}\Delta^2_{T,t} - 15\Delta^3_{T,t}\right)$$

w. p. 1 [9]-[18], [21]-[26], where

$$\delta_{T,t} = \int\limits_t^T \psi(\tau) d\mathbf{f}^{(i)}_\tau, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(\tau) d\tau.$$

Note that the mentioned equalities can be independently obtained using the Ito formula and Hermite polynomials.

For further consideration, let us consider the generalization of formulas (22)–(27) for the case of an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{28}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (28) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{29}$$

Below there are several examples of sums in the form (29)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$
= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +
$$
$$
+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =
$$

$$
= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +
$$
$$
+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +
$$
$$
+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
$$

Now, we can formulate Theorem 2 (see (14)) using the alternative form.

Theorem 3 [12] (2009) (also see [13]-[18], [21], [23]-[26]. *Under the conditions of Theorem* 2 *the following expansion*

$$
J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \Biggr) \tag{30}
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 2.

In particular, from (30) for $k=5$ we obtain

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_5=0}^{p_5} C_{j_5\ldots j_1} \Biggl(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} -
$$

$$
- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \prod_{l=1}^{3} \zeta_{j_{q_l}}^{(i_{q_l})} +
$$

$$
+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}} \mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}} \mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}} \zeta_{j_{q_1}}^{(i_{q_1})} \Biggr).
$$

The last equality obviously agrees with (26).

3. A Generalization of Theorems 2, 3 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$

In this section, we will use the definition of the multiple Wiener stochastic integral from [64], [65] to generalize Theorems 2, 3 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$.

Consider the following step function on the hypercube $[t,T]^k$

$$\Phi_N(t_1,\ldots,t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{31}$$

where $a_{l_1\ldots l_k} \in \mathbb{R}$ and such that $a_{l_1\ldots l_k} = 0$ if $l_p = l_q$ for some $p \neq q$,

$$\mathbf{1}_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A \\ 0 & \text{otherwise} \end{cases},$$

$N \in \mathbb{N}$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (13):

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} \Delta\tau_j \to 0 \ \text{ if } \ N \to \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \tag{32}$$

Let us define the multiple Wiener stochastic integral for $\Phi_N(t_1,\ldots,t_k)$ [64], [65]

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} \stackrel{\mathsf{def}}{=} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{33}$$

where $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$, $i = 0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)} = \tau$.

It is known (see [65], Lemma 9.6.4) that for any $\Phi(t_1,\ldots,t_k) \in L_2([t,T]^k)$ there exists a sequence of step functions $\Phi_N(t_1,\ldots,t_k)$ of the form (31) such that

$$\lim_{N\to\infty} \int\limits_{[t,T]^k} \left(\Phi(t_1,\ldots,t_k) - \Phi_N(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k = 0. \tag{34}$$

We have

$$\Phi_N(t_1,\ldots,t_k) = \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k) =$$

$$= \sum_{(l_1,\ldots,l_k)} \sum_{\substack{l_1,\ldots,l_k=0 \\ l_1<l_2<\ldots<l_k}}^{N-1} a_{l_1\ldots l_k} \mathbf{1}_{[\tau_{l_1},\tau_{l_1+1})}(t_1)\ldots \mathbf{1}_{[\tau_{l_k},\tau_{l_k+1})}(t_k), \tag{35}$$

where permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$ (recall that $a_{l_1\ldots l_k}=0$ if $l_p=l_q$ for some $p\ne q$).

Using (35), we get

$$\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}= \tag{36}$$

$$=\sum_{(l_1,\ldots,l_k)}\sum_{\substack{l_1,\ldots,l_k=0\\ l_1<l_2<\ldots<l_k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}, \tag{37}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$ and permutations $(l_1,\ldots,l_k)$ when summing are performed only in the expression $l_1<l_2<\ldots<l_k$. At the same time the indices near upper limits of integration in the iterated stochastic integrals in (36) are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$ (see (36)). In addition, the multiple Wiener stochastic integral $J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}$ is defined by (33) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

Using (34), (37), Lemma 2, and (17) for Lebesgue integrals, we have

$$\mathsf{M}\left\{\left(J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}-J'[\Phi_M]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi_M(t_1,\ldots,t_k)\right)^2dt_1\ldots dt_k=$$

$$=C_k\left\|\Phi_N-\Phi_M\right\|_{L_2([t,T]^k)}^2\le$$

$$\le 2C_k\left(\left\|\Phi_N-\Phi\right\|_{L_2([t,T]^k)}^2+\left\|\Phi-\Phi_M\right\|_{L_2([t,T]^k)}^2\right)^2\to 0$$

if $N,M\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

Thus, there exists the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)}.$$

We will define the multiple Wiener stochastic integral for $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ by the formula [64], [65]

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} \stackrel{\sf def}{=} \underset{N\to\infty}{\text{l.i.m.}}\ J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_1,\ldots,l_k=0}^{N-1} a_{l_1\ldots l_k}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}, \tag{38}$$

where $\Phi_N(t_1,\ldots,t_k)$ is defined by (31), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$, $i=0,1,\ldots,m$, $\mathbf{w}_{\tau}^{(0)}=\tau$.

Let us prove the following equality

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1}, \tag{39}$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[\Phi]_{T,t}^{(i_1\ldots i_k)}$ is defined by (38) and

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Ito stochastic integral.

The equality (39) has already been proved for the case $\Phi(t_1,\ldots,t_k)=\Phi_N(t_1,\ldots,t_k)$ (see (37)). From (37) we have

$$J'[\Phi_N]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi_N(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1}. \tag{40}$$

Passing to the limit $\underset{N\to\infty}{\text{l.i.m.}}$ in the equality (40), we obtain

$$J'[\Phi]_{T,t}^{(i_1\ldots i_k)} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad \text{w. p. 1.} \tag{41}$$

Using Lemma 1.2 [24]-[26] or Lemma 2 [34] as well as (17) for Lebesgue integrals and (34), we get

$$\mathsf{M}\left\{\left(\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\left(\Phi_N(t_1,\ldots,t_k)-\Phi(t_1,\ldots,t_k)\right)^2 dt_1\ldots dt_k\ \to 0 \tag{42}$$

if $N\to\infty$, where constant C_k depends only on the multiplicity k of the multiple Wiener stochastic integral.

The relations (41) and (42) prove the equality (39). From (39) we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1,} \tag{43}$$

where $K=K(t_1,\ldots,t_k)$ is defined by (4).

Applying (43), we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=J'[K]_{T,t}^{(i_1\ldots i_k)}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}J'[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \tag{44}$$

w. p. 1, where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k)\stackrel{\text{def}}{=}K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l) \tag{45}$$

and

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{46}$$

is the Fourier coefficient corresponding to $K(t_1,\ldots,t_k)$.

Again applying (39), we have

(47)

$$J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition, the multiple Wiener stochastic integral $J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (38).

Using Lemma 1.2 [24]-[26] or Lemma 2 [34], (11) as well as (17) for Lebesgue integrals, we have

$$\mathsf{M}\left\{\left(J'[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\}\le$$

$$\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k=$$

(48)

$$=C_k\int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k\to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant C_k depends only on the multiplicity k of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$.

Thus, the following theorem is proved.

Theorem 4 [24] (Sect. 1.11), [34] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^r\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorems* 2, 3.

It should be noted that an analogue of Theorem 4 was considered in [66]. Note that we use another notations [24] (Sect. 1.11), [34] (Sect. 15) in comparison with [66]. Moreover, the proof of an analogue of Theorem 4 from [66] is somewhat different from the proof given in [24] (Sect. 1.11), [34] (Sect. 15).

4. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6 Based on Multiple Fourier–Legendre Series and Multiple Trigonometric Fourier Series

In a number of works of the author [14]-[18], [21], [23]-[27], [31] Theorems 2, 4 have been adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6. Let us collect some old results in the following statement.

Theorem 5 [14]-[18], [21], [23]-[27], [31]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable functions on $[t,T]$. Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m), \tag{49}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m), \tag{50}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{51}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \quad (i_1,\ldots,i_4=0,1,\ldots,m), \tag{52}$$

where $J^*[\psi^{(k)}]_{T,t}$ is defined by (3) *and $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ in* (50), (52); *another notations are the same as in Theorems* 2–4.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [24] (Sect. 2.10–2.16), [27] (Sect. 13–19), [31] (Sect. 5–11), [32] (Sect. 7–13), [54] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 6 [24], [27], [31], [32], [54]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{53}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{54}$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (53) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (54)*, constant* C *is independent of* p,

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*); another notations are the same as in Theorems* 2–4.

Theorem 7 [24], [27], [31], [32], [54]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\ldots,\ \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{55}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{56}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{57}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (55), (56) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (57)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 6.

Theorem 8 [24], [27], [31], [32], [54]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{58}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{59}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{60}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (58), (59) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (60), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^{T}\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 6, 7.

Theorem 9 [24], [27], [31], [32]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^{T}\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 6–8.

5. Exact and Approximate Expressions for the Mean-Square Error of Approximation of Iterated Ito Stochastic Integrals in Theorems 2, 4

Theorem 10 [19]-[21], [23]-[26], [33]. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuous functions in the space* $L_2([t,T])$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\},$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad(i_1,\ldots,i_k=1,\ldots,m),$$

(61)
$$J[\psi^{(k)}]_{T,t}^p=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right),$$

(62)
$$S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)},$$

the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form* (12),

(63)
$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j $(i=1,\ldots,m)$,

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$ *(at that, if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$*, then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$*); another notations are the same as in Theorem* 2.

Proof. Using Theorem 2 for the case $p_1=\ldots=p_k=p$ and $i_1,\ldots,i_k=1,\ldots,m$, we obtain

$$J[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p} \ldots \sum_{j_k=0}^{p} C_{j_k\ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} \right). \tag{64}$$

For $n > p$ we can write

$$J[\psi^{(k)}]_{T,t}^{n} = \left(\sum_{j_1=0}^{p} + \sum_{j_1=p+1}^{n} \right) \ldots \left(\sum_{j_k=0}^{p} + \sum_{j_k=p+1}^{n} \right) C_{j_k\ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} \right) =$$

$$= J[\psi^{(k)}]_{T,t}^{p} + \xi[\psi^{(k)}]_{T,t}^{p+1,n}. \tag{65}$$

Let us prove that due to the special structure of random variables $S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}$ (also see (23)–(27)) the following relations are correct

$$\mathsf{M}\left\{ \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} \right\} = 0, \tag{66}$$

$$\mathsf{M}\left\{ \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} \right) \left(\prod_{l=1}^{k} \zeta_{j_l'}^{(i_l)} - S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)} \right) \right\} = 0, \tag{67}$$

where

$$(j_1,\ldots,j_k) \in \mathrm{K}_p, \quad (j_1',\ldots,j_k') \in \mathrm{K}_n \backslash \mathrm{K}_p$$

and

$$\mathrm{K}_n = \left\{ (j_1,\ldots,j_k) :\ 0 \le j_1,\ldots,j_k \le n \right\},$$

$$\mathrm{K}_p = \left\{ (j_1,\ldots,j_k) :\ 0 \le j_1,\ldots,j_k \le p \right\}.$$

Let us prove (66). For the case $i_1,\ldots,i_k = 1,\ldots,m$ and $p_1 = \ldots = p_k = p$ from (18) and (19) we obtain

$$\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,\ldots,l_k=0 \\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k}) \Delta\mathbf{f}_{\tau_{l_1}}^{(i_1)} \ldots \Delta\mathbf{f}_{\tau_{l_k}}^{(i_k)} =$$

$$= \sum_{(j_1,\ldots,j_k)} \int_t^T \phi_{j_k}(t_k) \ldots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{w. p. 1}, \tag{68}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

From (68) due to the moment property of Ito stochastic integral we obtain (66).

Let us prove (67). From (68) we have

$$
0\le\left|{\sf M}\left\{\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-S_{j_1,\ldots,j_k}^{(i_1\ldots i_k)}\right)\left(\prod_{l=1}^{k}\zeta_{j_l'}^{(i_l)}-S_{j_1',\ldots,j_k'}^{(i_1\ldots i_k)}\right)\right\}\right|=
$$

$$
=\left|{\sf M}\left\{\sum_{(j_1,\ldots,j_k)}\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\times\right.\right.
$$

$$
\left.\left.\times\int\limits_t^T\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right\}\right|\le
$$

$$
\le\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k}(t_k)\phi_{j_k'}(t_k)dt_k\ldots\int\limits_t^T\phi_{j_1}(t_1)\phi_{j_1'}(t_1)dt_1=
$$

$$
=\sum_{(j_1',\ldots,j_k')}{\bf 1}_{\{j_1=j_1'\}}\ldots{\bf 1}_{\{j_k=j_k'\}}, \tag{69}
$$

where where ${\bf 1}_A$ is the indicator of the set A. From (69) we obtain (67).

Consider in detail the case $k=3$ in (69). We have

$$
\left|{\sf M}\left\{\sum_{(j_1,j_2,j_3)}\sum_{(j_1',j_2',j_3')}\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\times\right.\right.
$$

$$
\left.\left.\times\int\limits_t^T\phi_{j_3'}(t_3)\int\limits_t^{t_3}\phi_{j_2'}(t_2)\int\limits_t^{t_2}\phi_{j_1'}(t_1)d{\bf f}_{t_1}^{(i_1)}d{\bf f}_{t_2}^{(i_2)}d{\bf f}_{t_3}^{(i_3)}\right\}\right|=
$$

$$
=\left|\int\limits_t^T\phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_1'}(s)ds+\right.
$$

$$
+{\bf 1}_{\{i_1=i_2\}}\int\limits_t^T\phi_{j_3}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds+
$$

$$
+{\bf 1}_{\{i_2=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_1'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_2'}(s)ds+
$$

$$
+{\bf 1}_{\{i_1=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_1'}(s)ds+
$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_2}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_1}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_1'}(s)ds+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\int\limits_t^T\phi_{j_1}(s)\phi_{j_3'}(s)ds\int\limits_t^T\phi_{j_3}(s)\phi_{j_2'}(s)ds\int\limits_t^T\phi_{j_2}(s)\phi_{j_1'}(s)ds\Biggr|=$$

$$=\Bigl|\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}}+\mathbf{1}_{\{i_1=i_2\}}\cdot\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+$$

$$+\mathbf{1}_{\{i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}+\mathbf{1}_{\{i_1=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$

$$+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+\mathbf{1}_{\{i_1=i_2=i_3\}}\cdot\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}\Bigr|\le$$

$$\le\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_1=j_1'\}}+\mathbf{1}_{\{j_3=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}+$$

$$+\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}+\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+$$

$$+\mathbf{1}_{\{j_2=j_3'\}}\mathbf{1}_{\{j_1=j_2'\}}\mathbf{1}_{\{j_3=j_1'\}}+\mathbf{1}_{\{j_1=j_3'\}}\mathbf{1}_{\{j_3=j_2'\}}\mathbf{1}_{\{j_2=j_1'\}}=$$

$$=\sum_{(j_1',j_2',j_3')}\mathbf{1}_{\{j_1=j_1'\}}\mathbf{1}_{\{j_2=j_2'\}}\mathbf{1}_{\{j_3=j_3'\}},$$

where we used the relation

$$\int\limits_t^T\phi_g(s)\phi_q(s)ds=\mathbf{1}_{\{g=q\}},\quad g,q=0,1,2\ldots$$

From (66) and (67) we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{p}\xi[\psi^{(k)}]_{T,t}^{p+1,n}\right\}=0.$$

Due to (61), (64), and (65) we can write

$$\xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}^{n}-J[\psi^{(k)}]_{T,t}^{p},$$

$$\underset{n\to\infty}{\text{l.i.m.}}\ \xi[\psi^{(k)}]_{T,t}^{p+1,n}=J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\stackrel{\text{def}}{=}\xi[\psi^{(k)}]_{T,t}^{p+1}.$$

We have

$$0\le\left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1}J[\psi^{(k)}]_{T,t}^{p}\right\}\right|=$$

$$= \left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1} - \xi[\psi^{(k)}]_{T,t}^{p+1,n} + \xi[\psi^{(k)}]_{T,t}^{p+1,n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| =$$

$$\le \left|\mathsf{M}\left\{\left(\xi[\psi^{(k)}]_{T,t}^{p+1} - \xi[\psi^{(k)}]_{T,t}^{p+1,n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| + \left|\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1,n} J[\psi^{(k)}]_{T,t}^{p}\right\}\right| =$$

$$= \left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right) J[\psi^{(k)}]_{T,t}^{p}\right\}\right| \le$$

$$\le \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}}\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}} \le$$

$$\le \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}} \times$$

$$\times \left(\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p} - J[\psi^{(k)}]_{T,t}\right)^2\right\}} + \sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}}\right) \le$$

$$\le K\sqrt{\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{n}\right)^2\right\}} \to 0 \quad \text{if} \quad n \to \infty, \tag{70}$$

where K is a constant.

From (70) it follows that

$$\mathsf{M}\left\{\xi[\psi^{(k)}]_{T,t}^{p+1} J[\psi^{(k)}]_{T,t}^{p}\right\} = 0$$

or

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p}\right) J[\psi^{(k)}]_{T,t}^{p}\right\} = 0.$$

The last equality means that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t} J[\psi^{(k)}]_{T,t}^{p}\right\} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}. \tag{71}$$

Taking into account (71), we obtain

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\} +$$

$$+\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\} - 2\mathsf{M}\left\{J[\psi^{(k)}]_{T,t} J[\psi^{(k)}]_{T,t}^{p}\right\} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\} -$$

$$-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}=$$

$$=\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}. \tag{72}$$

Consider the value

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{p}\right\}.$$

From (61) and (68) we obtain

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}. \tag{73}$$

After substituting (73) into (72) we obtain (79). Theorem 10 is proved.

Let $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ be the expression before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ in (14). Denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\stackrel{\text{def}}{=}E_k^{p_1,\ldots,p_k},$$

$$E_k^{p_1,\ldots,p_k}\stackrel{\text{def}}{=}E_k^{p}\quad\text{if}\quad p_1=\ldots=p_k=p,$$

$$\|K\|_{L_2([t,T]^k)}^2=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\text{def}}{=}I_k.$$

In [21], [23]-[26], [33] it was shown that

$$E_k^{p_1,\ldots,p_k}\le k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right) \tag{74}$$

if

$$i_1,\ldots,i_k=1,\ldots,m\quad(0<T-t<\infty)$$

or

$$i_1,\ldots,i_k=0,1,\ldots,m\quad(0<T-t<1).$$

Moreover, in [11]-[18], [21], [23]-[26], [34] it was shown that

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\}\le C_{n,k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)^n,$$

where

$$C_{n,k}=(k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!! \quad (n\in\mathbb{N}).$$

Remark 1. *Note that*

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k=C_{j_k\ldots j_1}.$$

Then from Theorem 10 *for pairwise different* $i_1,\ldots,i_k$ *and for* $i_1=\ldots=i_k$ *we get*

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2, \tag{75}$$

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right).$$

Consider some examples of application of Theorem 10 $(i_1,\ldots,i_5=1,\ldots,m)$

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2),$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3), \tag{76}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3), \tag{77}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2), \tag{78}$$

$$E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}C_{j_4\ldots j_1}\right)\quad (i_1=i_2\ne i_3,i_4;\ i_3\ne i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_4 \neq i_2, i_3;\ i_2 \neq i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_2 = i_3 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1} \right) \quad (i_2 = i_3 = i_4 \neq i_1),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \right) \right) \quad (i_1 = i_2 \neq i_3 = i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right) \right) \quad (i_1 = i_4 \neq i_2 = i_3),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right) \right) \quad (i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} \left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1} \right) \right) \quad (i_1 = i_2 = i_3 \neq i_4 = i_5),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1} \right) \quad (i_1 = i_3 = i_4 = i_5 \neq i_2).$$

Consider a generalization of Theorem 10 to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ \ldots, \psi_k(\tau) \in L_2([t,T])$

Theorem 11 [24] (Sect. 1.12), [33] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$, $i_1,\ldots,i_k = 1,\ldots,m$. Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^p\right)^2\right\} = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k -$$

$$-\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \mathsf{M}\left\{J[\psi^{(k)}]_{T,t} \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{79}$$

where $i_1,\ldots,i_k = 1,\ldots,m$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$,

$$J[\psi^{(k)}]_{T,t}^{p} = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr);$$

another notations are the same as in Theorems 2–4.

6. Approximation of Specific Iterated Ito and Stratonovich Stochastic Integrals

In this section we provide considerable practical material (based on Theorems 2–9 and Legendre polynomials) concerning expansions and approximations of iterated Ito and Stratonovich stochastic integrals. The question about what kind of functions (polynomial or trigonometric) is more convenient for the mean-square approximation of iterated stochastic integrals is also considered.

Let us consider the following iterated Ito and Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)} = \int\limits_t^T (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{80}$$

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T} (t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{81}$$

where $i_1,\ldots,i_k = 1,\ldots,m,\ \ l_1,\ldots,l_k = 0,1,\ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{82}$$

where $P_j(x)$ is the Legendre polynomial.

Using the system of functions (82) as well as Theorems 2–9, we obtain the following expansions of iterated Ito and Stratonovich stochastic integrals (80), (81) [24]

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{83}$$

$$I_{(2)T,t}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{84}$$

(85)
$$I_{(00)T,t}^{*(i_1 i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

$$I_{(00)T,t}^{(i_1 i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(01)T,t}^{*(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr),$$

$$I_{(10)T,t}^{*(i_1 i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1 i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

(86)
$$+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr),$$

or

$$I_{(01)T,t}^{*(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{01}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$I_{(10)T,t}^{*(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{10}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2 j_1}^{01}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2 j_1}^{01},$$

$$C_{j_2 j_1}^{10}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^2\bar{C}_{j_2 j_1}^{10},$$

$$\bar{C}_{j_2 j_1}^{01}=-\int\limits_{-1}^{1}(1+y)P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2 j_1}^{10}=-\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}(1+x)P_{j_1}(x)dxdy;$$

$$I_{(10)T,t}^{(i_1 i_2)}=I_{(10)T,t}^{*(i_1 i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2,\quad I_{(01)T,t}^{(i_1 i_2)}=I_{(01)T,t}^{*(i_1 i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2\quad \text{w. p. 1},$$

$$I_{(01)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

$$I_{(10)T,t}^{(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{(i_1i_2)}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

or

$$I_{(01)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{01}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(10)T,t}^{(i_1i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^{10}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(000)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(000)T,t}^{(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

(87)
$$-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$I_{(000)T,t}^{(i_1i_1i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(000)T,t}^{*(i_1i_1i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\zeta_0^{(i_1)}\right)^3\quad \text{w. p. 1},$$

where

(88)
$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz; \tag{89}$$

$$I_{(000)T,t}^{(i_1i_2i_3)}=I_{(000)T,t}^{*(i_1i_2i_3)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\frac{1}{2}I_{(1)T,t}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\left((T-t)I_{(0)T,t}^{(i_1)}+I_{(1)T,t}^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(02)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(01)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{90}$$

$$I_{(20)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(10)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{91}$$

$$I_{(11)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{92}$$

or

$$I_{(02)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{02} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$I_{(20)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{20} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$I_{(11)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{11} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2 j_1}^{02} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{02},$$

$$C_{j_2 j_1}^{20} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{20},$$

$$C_{j_2 j_1}^{11} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{11},$$

$$\bar{C}_{j_2 j_1}^{02} = \int\limits_{-1}^{1} P_{j_2}(y)(y+1)^2 \int\limits_{-1}^{y} P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2 j_1}^{20} = \int\limits_{-1}^{1} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)(x+1)^2 dxdy,$$

$$\bar{C}_{j_2 j_1}^{11} = \int\limits_{-1}^{1} P_{j_2}(y)(y+1) \int\limits_{-1}^{y} P_{j_1}(x)(x+1)dxdy;$$

$$I_{(11)T,t}^{*(i_1 i_1)} = \frac{1}{2}\left(I_{(1)T,t}^{(i_1)}\right)^2 \quad \text{w. p. 1},$$

$$I_{(02)T,t}^{(i_1 i_2)} = I_{(02)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \quad I_{(20)T,t}^{(i_1 i_2)} = I_{(20)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$I_{(11)T,t}^{(i_1 i_2)} = I_{(11)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$
I_{(02)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-(T-t)I_{01_{T,t}}^{(i_1 i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+
$$

$$
+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+
$$

$$
+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,
$$

$$
I_{(20)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-(T-t)I_{(10)T,t}^{(i_1 i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+
$$

$$
+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+
$$

$$
+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,
$$

$$
I_{(11)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-\frac{T-t}{2}\left(I_{(10)T,t}^{(i_1 i_2)}+I_{(01)T,t}^{(i_1 i_2)}\right)+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+
$$

$$
+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+
$$

$$
+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,
$$

or

$$
I_{(02)T,t}^{(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{02}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(20)T,t}^{(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{20}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I_{(11)T,t}^{(i_1 i_2)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2 j_1}^{11}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$I_{(3)T,t}^{(i_1)}=-\frac{(T-t)^{7/2}}{4}\left(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\right),$$

$$I_{(0000)T,t}^{*(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0000)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
(93)
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$I_{(0000)T,t}^{(i_1i_1i_1i_1)}=\frac{1}{24}(T-t)^2\left(\left(\zeta_0^{(i_1)}\right)^4-6\left(\zeta_0^{(i_1)}\right)^2+3\right)\quad \text{w. p. 1},$$

$$I_{(0000)T,t}^{*(i_1i_1i_1i_1)}=\frac{1}{24}(T-t)^2\left(\zeta_0^{(i_1)}\right)^4\quad \text{w. p. 1},$$

where

(94)
$$C_{j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2\bar{C}_{j_4j_3j_2j_1},$$

(95)
$$\bar{C}_{j_4j_3j_2j_1}=\int_{-1}^{1}P_{j_4}(u)\int_{-1}^{u}P_{j_3}(z)\int_{-1}^{z}P_{j_2}(y)\int_{-1}^{y}P_{j_1}(x)dxdydzdu;$$

$$I_{(001)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^{001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(010)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^{010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(100)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{100} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$I_{(001)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{001} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{96}$$

$$I_{(010)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{97}$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{98}$$

where

$$C_{j_3 j_2 j_1}^{001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{001},$$

$$C_{j_3 j_2 j_1}^{010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{010},$$

$$C_{j_3 j_2 j_1}^{100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{100},$$

$$\bar{C}_{j_3 j_2 j_1}^{100} = -\int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x)(x+1) dx dy dz,$$

$$\bar{C}_{j_3 j_2 j_1}^{010} = -\int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y)(y+1) \int_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}^{001}_{j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz;$$

$$I_{(lll)T,t}^{(i_1i_1i_1)}=\frac{1}{6}\left(\left(I_{(l)T,t}^{(i_1)}\right)^3-3I_{(l)T,t}^{(i_1)}\Delta_{l(T,t)}\right)\quad \text{w. p. 1},$$

$$I_{(lll)T,t}^{*(iii)}=\frac{1}{6}\left(I_{(l)T,t}^{(i_1)}\right)^3\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{(i_1i_1i_1i_1)}=\frac{1}{24}\left(\left(I_{(l)T,t}^{(i_1)}\right)^4-6\left(I_{(l)T,t}^{(i_1)}\right)^2\Delta_{(l)T,t}+3\left(\Delta_{(l)T,t}\right)^2\right)\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{*(i_1i_1i_1i_1)}=\frac{1}{24}\left(I_{(l)T,t}^{(i_1)}\right)^4\quad \text{w. p. 1},$$

where

$$I_{(l)T,t}^{(i_1)}=\sum_{j=0}^{l}C_j^l\zeta_j^{(i_1)}\quad \text{w. p. 1},$$

$$\Delta_{l(T,t)}=\int\limits_t^T(t-s)^{2l}ds,\quad C_j^l=\int\limits_t^T(t-s)^l\phi_j(s)ds;$$

$$I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\right.$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{99}$$

$$I_{(00000)T,t}^{(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(00000)T,t}^{*(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\zeta_0^{(i_1)}\right)^5\quad \text{w. p. 1},$$

where

$$C_{j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar{C}_{j_5j_4j_3j_2j_1},$$

$$\bar{C}_{j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv;$$

$$I_{(0001)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0010)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0100)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(1000)T,t}^{*(i_1i_2i_3)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{(0001)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{0001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$
\begin{aligned}
I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010} \Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
&+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} + \\
&+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
\end{aligned}
$$

$$
\begin{aligned}
I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
&+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} + \\
&+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
\end{aligned}
$$

$$
\begin{aligned}
I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{1000} \Biggl(&\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} - \\
&-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} + \\
&+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} + \\
&+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
\end{aligned}
$$

where

$$
C_{j_4 j_3 j_2 j_1}^{0001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0001},
$$

$$
C_{j_3 j_2 j_1}^{0010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0010},
$$

$$
C_{j_4 j_3 j_2 j_1}^{0100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3 \bar{C}_{j_3 j_2 j_1}^{0100},
$$

$$C_{j_4j_3j_2j_1}^{1000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar C_{j_4j_3j_2j_1}^{1000},$$

$$\bar C_{j_4j_3j_2j_1}^{1000}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydzdu,$$

$$\bar C_{j_4j_3j_2j_1}^{0100}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar C_{j_4j_3j_2j_1}^{0010}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar C_{j_4j_3j_2j_1}^{0001}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu;$$

$$I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)},$$

$$I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$
$$-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$
$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$
+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+
$$
$$
+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),
$$

$$
I_{(000000)T,t}^{(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\left(\zeta_0^{(i_1)}\right)^6-15\left(\zeta_0^{(i_1)}\right)^4+45\left(\zeta_0^{(i_1)}\right)^2-15\right)\quad \text{w. p. 1},
$$

$$
I_{(000000)T,t}^{*(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\zeta_0^{(i_1)}\right)^6\quad \text{w. p. 1},
$$

where

$$C_{j_6j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3\bar{C}_{j_6j_5j_4j_3j_2j_1},$$

$$\bar{C}_{j_6j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_6}(w)\int\limits_{-1}^{w}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudvdw.$$

Consider the approximation $I_{(00)T,t}^{*(i_1i_2)q}$ of the iterated stochastic integral $I_{(00)T,t}^{*(i_1i_2)}$ obtained from (85) by replacing ∞ on q.

It is easy to prove that [47] (1997)

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad(i_1\neq i_2).$$

Further, using Theorems 10, 11, we obtain for $i_1\neq i_2$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right).$$

For the case $i_1=i_2$ using Theorems 10, 11, we have

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_1)}-I_{(10)T,t}^{(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_1)}-I_{(01)T,t}^{(i_1i_1)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right).\qquad(100)$$

On the basis of the presented expansions of iterated stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to a noticeable complication of formulas for the mentioned expansions.

However, increasing of the mentioned parameters leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated stochastic integrals. As a result, this feature leads to a sharp decrease of member quantities in expansions of iterated stochastic integrals, which are required for achieving the acceptable accuracy of approximation. In the context of it, let us consider the approach to approximation of iterated stochastic integrals, which provides a possibility to obtain the mean-square approximations of the required accuracy without the using of complex expansions.

Let us analyze the following approximation of triple stochastic integral on the base of (87)

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \qquad (101)$$

where $C_{j_3 j_2 j_1}$ is defined by (88), (89).

In particular, from (101) for $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we obtain

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}. \qquad (102)$$

Using (74), (75), (76)–(78), we get

$$\mathsf{M}\left\{ \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1} \right)^2 \right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 \quad (i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3), \qquad (103)$$

$$\mathsf{M}\left\{ \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1} \right)^2 \right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3), \qquad (104)$$

$$\mathsf{M}\left\{ \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1} \right)^2 \right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \quad (i_1 = i_3 \neq i_2), \qquad (105)$$

$$\mathsf{M}\left\{ \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1} \right)^2 \right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3), \qquad (106)$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}\le 6\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3 j_2 j_1}^2\right)\quad (i_1,i_2,i_3=1,\ldots,m). \tag{107}$$

We can act similarly with more complicated iterated stochastic integrals. For example, for the approximation of stochastic integral $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}$ we can write (see (93))

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}=\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4 j_3 j_2 j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

where $C_{j_4 j_3 j_2 j_1}$ is defined by (94), (95). Moreover, according to (74)

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}-I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\}\le 24\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4 j_3 j_2 j_1}^2\right)\quad (i_1,i_2,i_3,i_4=1,\ldots,m).$$

For pairwise different $i_1,i_2,i_3,i_4=1,\ldots,m$ from (75) we obtain

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}-I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4 j_3 j_2 j_1}^2. \tag{108}$$

Using Theorems 10, 11, we can calculate exactly the left-hand side of (108) for any possible combinations of i_1,i_2,i_3,i_4. These relations were obtained in [23]-[26], [33].

In Tables 1–3, we have some examples of exact values of the Fourier–Legendre coefficients (here and further in this article the Fourier–Legendre coefficients have been calculated exactly using Derive (computer algebra system)). Note that in [49], [50] we used the database with 270,000 exactly calculated Fourier–Legendre coefficients. These coefficients [49], [50] were calculated using the Python programming language.

Assume that $q_1=6$. Calculating the value of expression (103) for $q_1=6$, $i_1\ne i_2$, $i_1\ne i_3$, $i_3\ne i_2$, we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}\approx 0.01956(T-t)^3.$$

Table 1. Coefficients $\bar{C}_{3j_2j_1}$.

$j_2 \backslash j_1$	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

Table 2. Coefficients $\bar{C}_{21j_2j_1}$.

$j_2 \backslash j_1$	0	1	2
0	$\frac{2}{21}$	$-\frac{2}{45}$	$\frac{2}{315}$
1	$\frac{2}{315}$	$\frac{2}{315}$	$-\frac{2}{225}$
2	$-\frac{2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 3. Coefficients $\bar{C}_{101j_2j_1}$.

$j_2 \backslash j_1$	0	1
0	$\frac{4}{315}$	0
1	$\frac{4}{315}$	$-\frac{8}{945}$

Let us choose, for example, $q_2 = 2$. In the case of pairwise different i_1, i_2, i_3, i_4 we have from (108) the following approximate equality

(109) $$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)} - I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\} \approx 0.0236084(T-t)^4.$$

Consider the approximations

$$I_{(001)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(010)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(100)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}$$

based on the expansions (96)–(99).

Assume that $q_3 = 2$, $q_4 = 1$. In the case of pairwise different $i_1, \ldots, i_5$ we obtain

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)} - I_{(100)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{60} - \sum_{j_1,j_2,j_3=0}^{2} \left(C_{j_3j_2j_1}^{100}\right)^2 \approx 0.00815429(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)} - I_{(010)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{20} - \sum_{j_1,j_2,j_3=0}^{2} \left(C_{j_3j_2j_1}^{010}\right)^2 \approx 0.0173903(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{001}\right)^2\approx 0.0252801(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}=$$

$$=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.$$

Note that from (74) we have

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(0000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^2\right),$$

where $i_1,\ldots,i_5=1,\ldots,m$.

Let us consider the expansions of Ito stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$ based on the Milstein approach from [3], which was mentioned in Sect. 1 (also see [2], [57])

(110) $$I_{(1)T,t}^{(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),$$

$$I_{(2)T,t}^{(i_1)q}=(T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.$$

(111) $$\left.-\frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),$$

where $\zeta_j^{(i)}$ is defined by the formula (15), $\phi_j(\tau)$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$, and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,\ i=1,\ldots,m)$ are independent standard Gaussian random variables, $i_1=1,\ldots,m$,

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},$$

$$\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}.$$

It is obvious that (110), (6) significantly more complicated compared to (83), (84).

Another example of obvious advantage of the Legendre polynomials over the trigonometric functions (in the framework of the considered problem) is the truncated expansion of iterated Stratonovich stochastic integral $I_{(10)T,t}^{*(i_1 i_2)}$ obtained by Theorem 5, in which instead of the double Fourier–Legendre series (see (85), (86)) is taken the double trigonometric Fourier series

$$
\begin{aligned}
I_{(10)T,t}^{*(i_1 i_2)q} = -(T-t)^2 \Biggl(& \frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)} + \\
& + \frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)} - 2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right) + \\
& + \frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)} + \frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)} - 2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right) - \\
& - \frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)} + \frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right) + \\
& + \sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \\
& + \frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} + \zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr), \qquad (112)
\end{aligned}
$$

where the meaning of the notations included in (110), (6) is saved.

7. Theorems 2–9 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [61], [62], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [61]-[63] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [67], [68]

$$
\mathbf{f}_\tau^{(i)} - \mathbf{f}_t^{(i)} = \sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \qquad (113)
$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (113) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{114}$$

From (114) we obtain

$$d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{115}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{116}$$

where $p_1,\ldots,p_k\in\mathbb{N},\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$d\mathbf{w}_\tau^{(i)p}=\begin{cases}d\mathbf{f}_\tau^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{117}$$

and $d\mathbf{f}_\tau^{(i)p}$ in defined by the relation (115).

Let us substitute (115) into (116)

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{118}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [61]-[63] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [63] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (114) were not considered in [61], [62] (also see [63], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [63] for approximations of the Wiener process based on its series expansion (113) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (118) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [61], [62] (also see [63], Theorems 7.1, 7.2).

Nevetheless, the authors of the publications [2] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [57] (pp. 438-439), [60] (pp. 263-264) use (without rigorous proof) the Wong–Zakai approximation [61]-[63] based on the series expansion of the Brownian bridge process [3].

From the other hand, Theorems 2–9 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the approximation (114) of the Wiener process. At that, the Riemann–Stieltjes integrals (116) converge (according to Theorems 2–9) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (113), (114), and Theorems 5–9) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [61]-[63]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{119}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (119) and additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{120}$$

Using (120) and standard relation between Stratonovich and Ito stochastic integrals, it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_0^T ds=$$

$$=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_{s}^{(i_2)}, \tag{121}$$

where $\Delta\to 0$ if $N\to\infty$ $(N\Delta=T)$.

Obviously, (121) agrees with Theorem 7.1 (see [63], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (113) for $t=0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p},\quad i_1,i_2=1,\ldots,m, \tag{122}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (115).

Let us substitute (115) into (122)

$$\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_{s}^{(i_2)p}=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{123}$$

where

$$C_{j_2j_1}=\int\limits_0^T\phi_{j_2}(s)\int\limits_0^s\phi_{j_1}(\tau)d\tau ds$$

is the Fourier coefficient; another notations are the same as in (118).

As we noted above, approximations of the Wiener process that are similar to (114) were not considered in [61], [62] (also see Theorems 7.1, 7.2 in [63]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [63] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [24]-[26]. More precisely, using Theorem 5 for the case $k = 2$, we obtain from (123) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)}. \tag{124}$$

From the other hand, by Theorems 2, 4 (see (23)) for the case $k = 2$ we obtain from (123) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right) + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{125}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from (125) and standard relation between Stratonovich and Ito stochastic integrals we obtain (124).

References

[1] Gichman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
[6] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: SPbGTU Publishing House, 1998, 204 pp. (ISBN 5-7422-0045-5)
[7] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
[8] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf
[9] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)
[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)
[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)
[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)
[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)
[14] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html
[15] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)
[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)
[17] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)
[18] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html
[19] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096
[20] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056
[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[22] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N. Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[23] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[24] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2022, 923 pp.

[25] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[26] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[27] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 204 pp.

[28] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[29] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[30] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[31] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 129 pp.

[32] Kuznetsov, D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.

[33] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.

[34] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[35] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[36] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604

[37] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[38] Kuznetsov D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: https://doi.org/10.1134/S0965542520030100

[39] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[41] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Pringsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[42] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[43] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[44] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[45] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.

[46] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[47] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[48] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[49] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[50] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[51] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.

[52] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[53] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[54] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[55] Wiktorsson M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487.

[56] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[57] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[58] Averina T.A., Prigarin S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[59] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[60] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[61] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[62] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[63] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[64] Itô, K. Multiple Wiener integral. Journal of the Mathematical Society of Japan, 3, 1 (1951), 157-169.

[65] Kuo, H.-H. Introduction to Stochastic Integration. Universitext (UTX), Springer. N. Y., 2006, 289 pp.

[66] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes". ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[67] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974, 696 pp.

[68] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[69] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

[70] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional Wiener process based on Multiple Fourier-Legendre series. [In English]. MATEC Web of Conferences, Vol. 362 (2022), article id: 01014, 10 pp. DOI: http://doi.org/10.1051/matecconf/202236201014

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS FROM THE TAYLOR–STRATONOVICH EXPANSION BASED ON MULTIPLE TRIGONOMETRIC FOURIER SERIES. COMPARISON WITH THE MILSTEIN EXPANSION

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to comparison of the Milstein expansion of iterated Stratonovich stochastic integrals with the method of expansion of iterated stochastic integrals based on generalized multiple Fourier series. We consider the practical material connected with the expansions of iterated Stratonovich stochastic integrals from the Taylor–Stratonovich expansion based on multiple trigonometric Fourier series. The comparison of effectiveness of the Fourier–Legendre series as well as the trigonomertic Fourier series for expansions of iterated Stratonovich stochastic integrals is considered.

Contents

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

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$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a}: \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B: \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[7]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a nonrandom function on $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$; $i_1, \ldots, i_k = 0, 1, \ldots, m$;

$$\int \ \text{and} \ \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [3]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ for the classical Taylor–Ito and Taylor–Stratonovich expansions [2]-[7] and $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ for the unified Taylor–Ito and Taylor–Stratonovich expansions [8]-[24].

2. Milstein Expansion and Method of Generatized Multiple Fourier Series

Milstein G.N. proposed [2] (1988) an approach to the expansion of iterated stochastic integrals based on the trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karhunen–Loeve expansion).

Let us consider the Brownian bridge process [2]

$$\mathbf{f}_t - \frac{t}{\Delta}\mathbf{f}_\Delta, \quad t \in [0, \Delta], \quad \Delta > 0, \tag{4}$$

where $\mathbf{f}_t$ is a standard m-dimensional Wiener process with independent components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$.

Consider the componentwise Karhunen–Loeve expansion of the process (4) [2]

$$\mathbf{f}_t^{(i)}-\frac{t}{\Delta}\mathbf{f}_{\Delta}^{(i)}=\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right) \tag{5}$$

converging in the mean-square sense, where

$$a_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,$$

$$b_{i,r}=\frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{f}_s^{(i)}-\frac{s}{\Delta}\mathbf{f}_{\Delta}^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds,$$

where $r=0,1,\ldots;\ i=1,\ldots,m.$

It is easy to demonstrate [2] that the random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\}=\mathsf{M}\left\{a_{i,r}b_{i,k}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}a_{i,k}\right\}=\mathsf{M}\left\{b_{i,r}b_{i,k}\right\}=0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\}=\mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\}=0,\quad \mathsf{M}\left\{a_{i,r}^2\right\}=\mathsf{M}\left\{b_{i,r}^2\right\}=\frac{\Delta}{2\pi^2r^2},$$

where $i,i_1,i_2=1,\ldots,m;\ r\neq k;\ i_1\neq i_2.$

According to (5), we have

$$\mathbf{f}_t^{(i)}=\mathbf{f}_{\Delta}^{(i)}\frac{t}{\Delta}+\frac{1}{2}a_{i,0}+\sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta}+b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{6}$$

where the series converges in the mean-square sense.

Note that the trigonometric functions are the eigenfunctions of the covariance operator of the Brownian bridge process. That is why the basis functions are the trigonometric functions in the considered approach.

In [2] Milstein G.N. proposed to expand (2) or (3) (for the case $k=2$ and $\psi_1(s),\psi_2(s)\equiv 1$) into iterated series of products of standard Gaussian random variables by representing the Wiener process as the series (6). To obtain the Milstein expansion of (2) or (3), the truncated expansions (6) of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (2) or (3) valid for an arbitrary multiplicity k. For this reason, only expansions of simplest single, double, and triple integrals (2), (3) were obtained (see [2]-[7]).

At that, in [2], [7] the case $\psi_1(s),\psi_2(s)\equiv 1$ and $i_1,i_2=0,1,\ldots,m$ is considered. In [3]-[6] the attempt to consider the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$ and $i_1,i_2,i_3=0,1,\ldots,m$ is realized.

It should be noted that the authors of the works [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) use the Wong–Zakai approximation [38]-[40] (without rigorous proof) within the frames of the Milstein approach [2] based on the series expansion of the Brownian bridge process. See discussion in Sect. 7 of this paper for details.

Let us consider an another approach to the expansion of iterated stochastic integrals [10]-[37], which is reffered to as the method of generalized multiple Fourier series.

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ \\ 0,\ \text{otherwise}\end{cases}=\prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{7}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ denotes the indicator of the set A.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$.

The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{8}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{9}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{10}$$

Theorem 1 [10] (2006), [11]-[37]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{11}$$

where

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1\},$$

$$\mathrm{L}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \tag{12}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i \neq 0$*),* $C_{j_k \ldots j_1}$ *is the Fourier coefficient* (9)*,* $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$*,* $\{\tau_j\}_{j=0}^N$ *is the partition of* $[t, T]$*, which satisfies the condition* (10)*.*

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [10]-[37]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{13}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \Biggr), \tag{14}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1, \ldots, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{15}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1, \ldots, p_4 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_4=0}^{p_4} C_{j_4 \ldots j_1} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -$$

$$- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$

$$- \mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$

$$- \mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$

$$+ \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \mathbf{1}_{\{i_3 = i_4 \neq 0\}} \mathbf{1}_{\{j_3 = j_4\}} +$$

$$+ \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \mathbf{1}_{\{i_2 = i_4 \neq 0\}} \mathbf{1}_{\{j_2 = j_4\}} +$$

$$+ \mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \Biggr), \tag{16}$$

$$
\begin{gathered}
J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr),
\end{gathered}
\tag{17}
$$

$$
\begin{gathered}
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
\end{gathered}
$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
&+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
&+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
&\qquad+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
&\qquad-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr),
\end{aligned}
\tag{18}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

For further consideration, let us consider the generalization of formulas (13)–(18) for the case of an arbitrary multiplicity k $(k\in\mathbb{N})$ of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{19}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (19) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{20}$$

Below there are several examples of sums in the form (20)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (11) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),\qquad(21)$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (21) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (17).

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [22] (Sect. 1.11), [29] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(22)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [41]. Note that we use another notations [22] (Sect. 1.11), [29] (Sect. 15) in comparison with [41]. Moreover, the proof of an analogue of Theorem 2 from [41] is somewhat different from the proof given in [22] (Sect. 1.11), [29] (Sect. 15).

3. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 6

In a number of works of the author [15]-[24], [30] Theorems 1, 2 have been adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6. Let us first present some old results as the following theorem.

Theorem 3 [15]-[24], [30]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function on* $[t,T]$ *and* $\psi_1(\tau), \psi_3(\tau)$ *are twice continuously differentiable functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m), \tag{23}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m), \tag{24}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{25}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \quad (i_1,i_2,i_3,i_4=0,1,\ldots,m), \tag{26}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (3) *and* $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ *in* (24), (26); *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [22] (Sect. 2.10–2.16), [30] (Sect. 13–19), [33] (Sect. 7–13), [34] (Sect. 5–11), [54] (Sect. 4–9), [55]. Let us formulate four theorems that were obtained using this approach.

Theorem 4 [22], [30], [33], [34], [54]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{27}$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{28}
$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (27) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (28)*, constant* C *is independent of* p*,*

$$
C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*); another notations are the same as in Theorems* 1, 2*.*

Theorem 5 [22], [30], [33], [34], [54]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{29}
$$

the following relations

$$
J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{30}
$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{31}
$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (29), (30) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (31)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$
C_{j_4j_3j_2j_1}=
$$

$$
=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;
$$

another notations are the same as in Theorem 4.

Theorem 6 [22], [30], [33], [34], [54]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{32}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{33}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{34}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (32), (33) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (34), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^{T}\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 4, 5.

Theorem 7 [22], [30], [33], [34], [55]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{35}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^{T}\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 4–6.

4. Exact Calculation of the Mean-Square Error in Theorems 1, 2

Theorems 1 and 2 allow us to accurately calculate the mean-square approximation error for iterated Ito stochastic integrals (see Theorem 8 below).

Assume that $J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}$ is the approximation of (2), which is the expression on the right-hand side of (22) before passing to the limit

$$J[\psi^{(k)}]_{T,t}^{p_1\ldots p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$E_k^{p_1,\ldots,p_k} \stackrel{\rm def}{=} {\sf M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\},$$

$$E_k^{p_1,\ldots,p_k} \stackrel{\rm def}{=} E_k^p \quad \text{if} \quad p_1=\ldots=p_k=p,$$

$$I_k \stackrel{\rm def}{=} \|K\|^2_{L_2([t,T]^k)} = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k.$$

In [10]-[24], [29] it was shown that

$$E_k^{p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right) \tag{36}$$

if $i_1,\ldots,i_k=1,\ldots,m$ and $0<T-t<\infty$ or $i_1,\ldots,i_k=0,1,\ldots,m$ and $0<T-t<1$.

Moreover, in [12]-[24], [29] the following estimate

$${\sf M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\} \le$$

$$\le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right)^n \tag{37}$$

is obtained, where $n\in\mathbb{N}$.

The value E_k^p can be calculated exactly.

Theorem 8 [22] (Sect. 1.12), [35] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. Then*

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{38}$$

where $i_1,\ldots,i_k=1,\ldots,m$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in Theorems 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\} = C_{j_k\ldots j_1}.$$

Then from Theorem 8 for pairwise different $i_1,\ldots,i_k$ and for $i_1=\ldots=i_k$ we obtain

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2,$$

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right).$$

Consider some examples of the application of Theorem 8 ($i_1,i_2,i_3=1,\ldots,m$)

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\neq i_3),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\neq i_2=i_3),$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\neq i_2),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 \neq i_3, i_4;\ i_3 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_3 \neq i_2, i_4;\ i_2 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_4 \neq i_2, i_3;\ i_2 \neq i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_3 \neq i_1, i_4;\ i_1 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_4 \neq i_1, i_3;\ i_1 \neq i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_3 = i_4 \neq i_1, i_2;\ i_1 \neq i_2),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 = i_3 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_3 = i_4 \neq i_1),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_2 = i_4 \neq i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_3 = i_4 \neq i_2),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_2 \neq i_3 = i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_3 \neq i_2 = i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right)\right) \quad (i_1 = i_4 \neq i_2 = i_3),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right) \quad (i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1).$$

5. Some Technical Problems of the Milstein Approach

Let us denote

$$I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)T,t} = \int\limits_t^{*T} (t-t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{39}$$

where $i_1,\ldots,i_k = 1,\ldots,m$; $l_1,\ldots,l_k = 0,1,\ldots$.

Consider the Milstein expansions for the simplest iterated Stratonovich stochastic integrals (39)

$$I^{*(i_1)}_{(0)T,t} = \sqrt{T-t}\zeta_0^{(i_1)}, \tag{40}$$

$$I^{*(i_1)}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{41}$$

$$I^{*(i_1 i_2)}_{(00)T,t} = \frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.$$

$$\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)\Biggr), \tag{42}$$

$$I^{*(i_1)}_{(2)T,t} = (T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\zeta_{2r}^{(i_1)} - \frac{1}{\sqrt{2}\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right), \tag{43}$$

where $i_1, i_2 = 1,\ldots,m$;

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j, and

$$\phi_j(s) = \frac{1}{\sqrt{T-t}}\begin{cases} 1 & \text{when } j=0 \\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)) & \text{when } j=2r-1 \\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)) & \text{when } j=2r \end{cases}, \tag{44}$$

where $r=1,2,\ldots$

Obviously, that $I_{(1)T,t}^{*(i_1)}$, $I_{(2)T,t}^{*(i_1)}$ have Gaussian distribution and the expansions (41), (43) are too complex for such simple stochastic integrals as $I_{(1)T,t}^{*(i_1)}$, $I_{(2)T,t}^{*(i_1)}$.

Milstein G.N. proposed [2] the following mean-square approximations on the base of the expansions (41), (42)

$$I_{(1)T,t}^{*(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{45}$$

$$I_{(00)T,t}^{*(i_1i_2)q}=\frac{1}{2}(T-t)\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.$$

$$\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\right), \tag{46}$$

where

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}, \tag{47}$$

where $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ are independent standard Gaussian random variables.

The approximation $I_{(2)T,t}^{*(i_1)q}$, which corresponds to (45), (46) has the form [3]

$$I_{(2)T,t}^{*(i_1)q}=(T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.$$

$$\left.-\frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{48}$$

where $\xi_q^{(i)}$, α_q has the form (47) and

$$\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)},\quad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4},$$

$\phi_j(s)$ is defined by (44); $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ are independent standard Gaussian random variables; $i=1,\ldots,m$.

Nevetheless, the expansions (45), (48) are too complex for the approximation of two Gaussian random variables $I_{(1)T,t}^{*(i_1)}$, $I_{(2)T,t}^{*(i_1)}$.

Using Theorems 1–3 and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, we obtain for $i_1,i_2=1,\ldots,m$ [10]-[37]

(49) $$I^{*(i_1)}_{(0)T,t}=\sqrt{T-t}\zeta_0^{(i_1)},$$

(50) $$I^{*(i_1)}_{(1)T,t}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

(51) $$I^{*(i_1)}_{(2)T,t}=\frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

(52) $$I^{*(i_1 i_2)}_{(00)T,t}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j, where

(53) $$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right);\ j=0,1,2,\ldots,$$

where $P_j(x)$ is the Legendre polynomial.

It is not difficult to see that the expansions (50), (51) are much simpler than the expansions (45), (48).

Obviously that the Milstein approach [2] leads to iterated series (iterated application of the operation of limit transitions) in contradiction to multiple series (the operation of limit transition is implemented only once) from Theorems 1–7.

For the case of simplest stochastic integral $I^{*(i_1 i_2)}_{(00)T,t}$ of second multiplicity this problem was avoided as we saw earlier. However, the situation is not the same for the simplest iterated stochastic integral $I^{*(i_1 i_2 i_3)}_{(000)T,t}$ of third multiplicity.

Let us denote

$$J^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)T,t}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\lambda_l=1$ if $i_l=1,\ldots,m$ and $\lambda_l=0$ if $i_l=0$; $l=1,\ldots,k$ ($\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$).

Consider the expansion of iterated Stratonovich stochastic integral of third multiplicity obtained in [3]-[6] by the Milstein approach

$$J^{*(i_1 i_2 i_3)}_{(111)\Delta,0}=\frac{1}{\Delta}J^{*(i_1)}_{(1)\Delta,0}J^{*(0 i_2 i_3)}_{(011)\Delta,0}+\frac{1}{2}a_{i_1,0}J^{*(i_2 i_3)}_{(11)\Delta,0}+\frac{1}{2\pi}b_{i_1}J^{(i_2)}_{(1)\Delta,0}J^{*(i_3)}_{(1)\Delta,0}-$$

(54) $$-\Delta J^{*(i_2)}_{(1)\Delta,0}B_{i_1 i_3}+\Delta J^{*(i_3)}_{(1)\Delta,0}\left(\frac{1}{2}A_{i_1 i_2}-C_{i_2 i_1}\right)+\Delta^{3/2}D_{i_1 i_2 i_3},$$

where

$$J_{(011)\Delta,0}^{*(0i_2i_3)}=\frac{1}{6}J_{(1)\Delta,0}^{*(i_2)}J_{(1)\Delta,0}^{*(i_3)}-\frac{1}{\pi}\Delta J_{(1)\Delta,0}^{*(i_3)}b_{i_2}+$$

$$+\Delta^2 B_{i_2i_3}-\frac{1}{4}\Delta a_{i_3,0}J_{(1)\Delta,0}^{*(i_2)}+\frac{1}{2\pi}\Delta b_{i_3}J_{(1)\Delta,0}^{*(i_2)}+\Delta^2 C_{i_2i_3}+\frac{1}{2}\Delta^2 A_{i_2i_3},$$

$$A_{i_2i_3}=\frac{\pi}{\Delta}\sum_{r=1}^{\infty}r\left(a_{i_2,r}b_{i_3,r}-b_{i_2,r}a_{i_3,r}\right),$$

$$C_{i_2i_3}=-\frac{1}{\Delta}\sum_{l=1}^{\infty}\sum_{r=1(r\neq l)}^{\infty}\frac{r}{r^2-l^2}\left(ra_{i_2,r}a_{i_3,l}+lb_{i_2,r}b_{i_3,l}\right),$$

$$B_{i_2i_3}=\frac{1}{2\Delta}\sum_{r=1}^{\infty}\left(a_{i_2,r}a_{i_3,r}+b_{i_2,r}b_{i_3,r}\right),\ b_i=\sum_{r=1}^{\infty}\frac{1}{r}b_{i,r},$$

$$D_{i_1i_2i_3}=-\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}l\Biggl(a_{i_2,l}\left(a_{i_3,l+r}b_{i_1,r}-a_{i_1,r}b_{i_3,l+r}\right)+$$

$$+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r+l}+b_{i_1,r}b_{i_3,l+r}\right)\Biggr)+$$

$$+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{\infty}\sum_{r=1}^{l-1}l\Biggl(a_{i_2,l}\left(a_{i_1,r}b_{i_3,l-r}+a_{i_3,l-r}b_{i_1,r}\right)-$$

$$-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\Biggr)+$$

$$+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{\infty}\sum_{r=l+1}^{\infty}l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+$$

$$+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\Biggr).$$

From the form of expansion (54) and expansion of the stochastic integral $J_{(011)\Delta,0}^{*(0i_2i_3)}$ we can conclude that they include iterated (double) series. Moreover, for approximation of the considered stochastic integral $J_{(111)\Delta,0}^{*(i_1i_2i_3)}$ in the works [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) it is proposed to put upper limits of summation by equal q (on the base of the Wong–Zakai approximation [38]-[40], but without rigorous proof; also see discussion in Sect. 7).

For example, the value $D_{i_1i_2i_3}$ is approximated in [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) by the double sums of the form

$$D_{i_1i_2i_3}^{(q)}=-\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{q}l\Biggl(a_{i_2,l}\left(a_{i_3,l+r}b_{i_1,r}-a_{i_1,r}b_{i_3,l+r}\right)+$$

$$+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r+l}+b_{i_1,r}b_{i_3,l+r}\right)\Biggr)+$$

$$+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=1}^{l-1}l\Biggl(a_{i_2,l}\left(a_{i_1,r}b_{i_3,l-r}+a_{i_3,l-r}b_{i_1,r}\right)-$$

$$-b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r}-b_{i_1,r}b_{i_3,l-r}\right)\Biggr)+$$

$$+\frac{\pi}{2\Delta^{3/2}}\sum_{l=1}^{q}\sum_{r=l+1}^{2q}l\Biggl(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r}-a_{i_1,r}b_{i_3,r-l}\right)+$$

$$+b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l}+b_{i_1,r}b_{i_3,r-l}\right)\Biggr).$$

Obviously, we can avoid this problem (iterated application of the operation of limit transition) using the method based on Theorems 1–7.

If we prove that the terms of the expansion (54) coincide with the terms of its analogue obtained using Theorems 1–3 (this fact is proved in [10]-[24] for the simplest stochastic integrals $I_{(1)T,t}^{*(i_1)}$, $I_{(00)T,t}^{*(i_1i_2)}$ of first and second multiplicity), then we can replace the iterated (double) series in (54) by the multiple ones, as in Theorems 1–3 (as was made formally in [3]-[6]). However, it requires a separate argumentation.

6. Approximation of Specific Iterated Stochastic Integrals of Multiplicities 1 to 3 Using Theorem 3 and Trigonometric System of Functions

In [10]-[24] on the base of Theorems 1–3 the author of this paper obtained the following expansions of the iterated Stratonovich stochastic integrals (39) (independently from the papers [2]-[7] excepting the method in which additional random variables $\xi_q^{(i)}$ and $\mu_q^{(i)}$) are introduced)

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{55}$$

$$I_{(1)T,t}^{*(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{56}$$

$$I_{(00)T,t}^{*(i_1i_2)q}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.$$

$$\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr), \tag{57}$$

$$
I_{(000)T,t}^{*(i_1i_2i_3)q}=(T-t)^{3/2}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+
$$

$$
+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2r^2}\Bigl(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+
$$

$$
+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\Bigr)\Biggr)+D_{T,t}^{(i_1i_2i_3)q}\Biggr), \tag{58}
$$

where

$$
D_{T,t}^{(i_1i_2i_3)q}=\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\Biggl(\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+
$$

$$
+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{4\sqrt{2}\pi^2}\Biggl(\sum_{r,m=1}^{q}\Biggl(\frac{2}{rm}\Bigl(-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+
$$

$$
+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr)+
$$

$$
+\frac{1}{m(r+m)}\Bigl(-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr)\Biggr)+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl(\frac{1}{m(l-m)}\Bigl(\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr)+
$$

$$
+\frac{1}{l(l-m)}\Bigl(-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Bigr)\Biggr)\Biggr),
$$

$$I_{(10)T,t}^{*(i_1 i_2)q} = -(T-t)^2 \Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)} +$$

$$+ \frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_2)}\zeta_0^{(i_1)} - 2\mu_q^{(i_1)}\zeta_0^{(i_2)}\Biggr) +$$

$$+ \frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)} + \frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)} - 2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\Biggr) -$$

$$- \frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)} + \frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\Biggr) +$$

$$+ \sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) + \frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} + \zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\Biggr)\Biggr), \tag{59}$$

$$I_{(01)T,t}^{*(i_1 i_2)q} = (T-t)^2 \Biggl(-\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)} - 2\xi_q^{(i_2)}\zeta_0^{(i_1)}\right) +$$

$$+ \frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)} - 2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr) -$$

$$- \frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - 2\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}\right) - \frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)} - 2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr) +$$

$$+ \frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)} + \zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr) -$$

$$- \sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right) - \frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} + \zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr), \tag{60}$$

$$I_{(2)T,t}^{*(i_1)q} = (T-t)^{5/2}\Biggl(\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)} + \sqrt{\beta_q}\mu_q^{(i_1)}\Biggr) -$$

$$- \frac{1}{\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr), \tag{61}$$

where

$$\xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)}, \quad \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}, \quad \mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},$$

$$\beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4}, \quad \zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)},$$

where $\phi_j(s)$ has the form (44); $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ are independent standard Gaussian random variables; $i_1,i_2,i_3=1,\ldots,m$.

Note that from (59), (60) it follows that

$$\sum_{j=0}^{\infty} C_{jj}^{10}=\sum_{j=0}^{\infty} C_{jj}^{01}=-\frac{(T-t)^2}{4}, \tag{62}$$

where

$$C_{jj}^{10}=\int\limits_t^T \phi_j(x)\int\limits_t^x \phi_j(y)(t-y)dydx,$$

$$C_{jj}^{01}=\int\limits_t^T \phi_j(x)(t-x)\int\limits_t^x \phi_j(y)dydx.$$

The formulas (62) are particular cases of the more general relation, which we applied for the proof of Theorem 3 for the case $k=2$ (see [15]-[24]).

Let us consider the mean-square errors of approximations (57)–(60). From the relations (57)–(60) when $i_1\ne i_2$, $i_2\ne i_3$, $i_1\ne i_3$ we obtain by direct calculation

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1i_2)}-I_{(00)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{63}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=(T-t)^3\left(\frac{1}{4\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\right.$$

$$\left.+\frac{55}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{4\pi^4}\left(\sum_{\substack{r,l=1\\ r\ne l}}^{\infty}-\sum_{\substack{r,l=1\\ r\ne l}}^{q}\right)\frac{5l^4+4r^4-3l^2r^2}{r^2l^2(r^2-l^2)^2}\right), \tag{64}$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=(T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\right.$$

$$\left.+\frac{5}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}-\sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\right), \tag{65}$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=(T-t)^4\left(\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\right.$$

$$\left.+\frac{5}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}-\sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2+k^2}{l^2(l^2-k^2)^2}\right). \tag{66}$$

It is easy to demonstrate that the relations (64), (65), and (66) can be represented using Theorem 8 in the following form

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=(T-t)^3\left(\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.$$

(67)
$$\left.-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right),$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)}-I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.$$

(68)
$$\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\neq l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\right),$$

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)}-I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left(\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.$$

(69)
$$\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\neq l}}^{q}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}\right).$$

Comparing (67)–(69) and (64)–(66), we obtain

(70)
$$\sum_{\substack{k,l=1\\ k\neq l}}^{\infty}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}=\sum_{\substack{k,l=1\\ k\neq l}}^{\infty}\frac{l^2+k^2}{l^2\left(l^2-k^2\right)^2}=\frac{\pi^4}{48},$$

(71)
$$\sum_{\substack{r,l=1\\ r\neq l}}^{\infty}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}=\frac{9\pi^4}{80}.$$

Let us consider approximations of the stochastic integrals $I_{(10)T,t}^{*(i_1 i_1)}$, $I_{(01)T,t}^{*(i_1 i_1)}$ and conditions for selecting the number q using the trigonometric system of functions

$$I_{(10)T,t}^{*(i_1 i_1)q}=-(T-t)^2\left(\frac{1}{6}\left(\zeta_0^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-\right.$$

$$-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)-$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr),
$$

$$
I_{(01)T,t}^{*(i_1i_1)q}=(T-t)^2\Biggl(-\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-
$$

$$
-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{\pi^2r^2}\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_1)}+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right)+
$$

$$
+\frac{1}{8\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}\left(3\left(\zeta_{2r-1}^{(i_1)}\right)^2+\left(\zeta_{2r}^{(i_1)}\right)^2\right)\Biggr).
$$

Then, we obtain

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_1)}-I_{(01)T,t}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_1)}-I_{(10)T,t}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{2}{\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{\pi^4}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^2+
$$

(72)
$$
+\frac{1}{\pi^4}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{\infty}-\sum_{\substack{k,l=1\\ k\neq l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr).
$$

Using (70), we write the relation (72) in the following form

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_1)}-I_{(01)T,t}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_1)}-I_{(10)T,t}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl(\frac{17}{240}-\frac{1}{3\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\frac{2}{\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}+
$$

(73)
$$
+\frac{1}{\pi^4}\left(\sum_{r=1}^{q}\frac{1}{r^2}\right)^2-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\ k\neq l}}^{q}\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr).
$$

Table 1. Confirmation of the formula (67)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722 \cdot 10^{-4}$	$7.5973 \cdot 10^{-5}$	$7.5990 \cdot 10^{-6}$
q	1	10	100	1000	10000

Table 2. Confirmation of the formulas (68), (69)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261 \cdot 10^{-4}$	$8.4429 \cdot 10^{-5}$	$8.4435 \cdot 10^{-6}$
q	1	10	100	1000	10000

Table 3. Confirmation of the formula (73)

$4\varepsilon/(T-t)^4$	0.0268	0.0034	$3.3955 \cdot 10^{-4}$	$3.3804 \cdot 10^{-5}$	$3.3778 \cdot 10^{-6}$
q	1	10	100	1000	10000

Table 4. Confirmation of the formula (70)

ε_q	2.0294	0.3241	0.0330	0.0033	$3.2902 \cdot 10^{-4}$
q	1	10	100	1000	10000

In Tables 1–3, we confirm numerically the formulas (67)–(69), (73) for various values q. In Tables 1–3, the number ε means the right-hand sides of the mentioned formulas.

The formulas (70), (71) appear to be interesting. Let us confirm numerically their correctness in Tables 4 and 5 (the number ε_q is the absolute deviation of multiple partial sums with the upper limit of summation q for the series (70), (71) from the right-hand sides of the formulas (70), (71); convergence of multiple series is regarded here when $p_1 = p_2 = q \to \infty$, which is acceptable according to Theorems 1, 2).

Using the trigonometric system of functions, let us consider the approximations of iterated stochastic integrals of the following form

$$J^{*(i_1\ldots i_k)}_{(\lambda_1\ldots\lambda_k)T,t} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k},$$

where $\lambda_l = 1$ if $i_l = 1,\ldots,m$ and $\lambda_l = 0$ if $i_l = 0$; $l = 1,\ldots,k$ ($\mathbf{w}^{(i)}_\tau = \mathbf{f}^{(i)}_\tau$ for $i = 1,\ldots,m$ and $\mathbf{w}^{(0)}_\tau = \tau$).

It is easy to see that the approximations

$$J^{*(i_1 i_2)q}_{(\lambda_1\lambda_2)T,t}, \quad J^{*(i_1 i_2 i_3)q}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

of the stochastic integrals

$$J^{*(i_1 i_2)}_{(\lambda_1\lambda_2)T,t}, \quad J^{*(i_1 i_2 i_3)}_{(\lambda_1\lambda_2\lambda_3)T,t}$$

are defined by the right-hand sides of the formulas (57), (58), where it is necessary to take

TABLE 5. Confirmation of the formula (71)

ε_q	10.9585	1.8836	0.1968	0.0197	0.0020
q	1	10	100	1000	10000

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{74}$$

and $i_1,i_2,i_3=0,1,\ldots,m$.

Since

$$\int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(0)}=\begin{cases}\sqrt{T-t} & \text{if } j=0\\ \\ 0 & \text{if } j\neq 0\end{cases},$$

then it is easy to get from (57) and (58), considering that in these equalities $\zeta_j^{(i)}$ has the form (74) and $i_1,i_2,i_3=0,1,\ldots,m$, the following family of formulas

$$J_{(10)T,t}^{(i_1 0)q}=\frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),$$

$$J_{(01)T,t}^{(0 i_2)q}=\frac{1}{2}(T-t)^{3/2}\left(\zeta_0^{(i_2)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_2)}+\sqrt{\alpha_q}\xi_q^{(i_2)}\right)\right),$$

$$J_{(001)T,t}^{(00 i_3)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_3)}+\sqrt{\beta_q}\mu_q^{(i_3)}\right)-\right.$$
$$\left.-\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_3)}+\sqrt{\alpha_q}\xi_q^{(i_3)}\right)\right),$$

$$J_{(010)T,t}^{(0 i_2 0)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_2)}-\frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_2)}+\sqrt{\beta_q}\mu_q^{(i_2)}\right)\right),$$

$$J_{(100)T,t}^{(i_1 00)q}=(T-t)^{5/2}\left(\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)+\right.$$
$$\left.+\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),$$

$$J_{(011)T,t}^{*(0i_2i_3)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_3)}\zeta_0^{(i_2)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_3)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_3)}\Biggr)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\right)\Biggr)-$$

$$-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\zeta_{2r}^{(i_2)}\zeta_{2l}^{(i_3)}+\frac{l}{r}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_2)}\zeta_{2r-1}^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\right)+$$

(75)
$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),$$

$$J_{(110)T,t}^{*(i_1i_20)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)}+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right)\Biggr)+$$

$$+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{r^2-l^2}\Biggl(\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr)+$$

$$+\sum_{r=1}^{q}\Biggl(\frac{1}{4\pi r}\left(\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_1)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+$$

$$+\frac{1}{8\pi^2r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\Biggr)\Biggr),$$

$$J_{(101)T,t}^{*(i_10i_3)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl(\frac{1}{\pi r}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$
+\frac{1}{\pi^2 r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}\right)\Biggr)-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_3)}-
$$

$$
-\sum_{r=1}^{q}\frac{1}{4\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\right)\Biggr).
$$

7. Theorems 1–7 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [38], [39], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [38]-[40] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [42], [43]

$$
\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{76}
$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (76) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$
\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^{\tau}\phi_j(s)ds\ \zeta_j^{(i)}. \tag{77}
$$

From (77) we obtain

$$
d\mathbf{f}_\tau^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{78}
$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{79}$$

where $p_1,\ldots,p_k\in\mathbb{N},\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$d\mathbf{w}_{\tau}^{(i)p}=\begin{cases} d\mathbf{f}_{\tau}^{(i)p} & \text{for}\quad i=1,\ldots,m\\ \\ d\tau & \text{for}\quad i=0\end{cases}, \tag{80}$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (78).

Let us substitute (78) into (79)

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{81}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [38]-[40] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [40] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (77) were not considered in [38], [39] (also see [40], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [40] for approximations of the Wiener process based on its series expansion (76) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (81) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [38], [39] (also see [40], Theorems 7.1, 7.2).

From the other hand, Theorems 1–7 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the approximation (77) of the Wiener process. At that, the iterated Riemann–Stieltjes integrals (79) converge (according to Theorems 1–7) to the appropriate iterated Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (76), (77), and Theorems 3–7) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2,\ \psi_1(s),\ \psi_2(s)\equiv 1;$ $i_1,i_2=1,\ldots,m.$

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [38]-[40]).

Let $\mathbf{b}_{\Delta}^{(i)}(t),\ t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t,\ t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)},\ i=1,\ldots,m,$ i.e.

$$
\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},
$$

where

$$
\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.
$$

Note that w. p. 1

$$
\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{82}
$$

Consider the following iterated Riemann–Stieltjes integral

$$
\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.
$$

Using (82) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

$$
\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=
$$

$$
=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=
$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=
$$

$$
=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{83}
$$

Using (83), it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau) d\mathbf{b}_{\Delta}^{(i_2)}(s) = \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_0^T ds =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)}, \tag{84}$$

where $\Delta \to 0$ if $N \to \infty$ ($N\Delta = T$).

Obviously, (84) agrees with Theorem 7.1 (see [40], p. 486).

The next example relates to the approximation (77) of the Wiener process based on its series expansion (76) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0, T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{85}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (78).

Let us substitute (78) into (85)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{86}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (81).

As we noted above, approximations of the Wiener process that are similar to (77) were not considered in [38], [39] (also see Theorems 7.1, 7.2 in [40]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [40] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [22]-[24]. More precisely, using Theorem 3, we obtain from (86) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)}. \tag{87}$$

From the other hand, by Theorems 1, 2 (see (14)) for the case $k = 2$ we obtain from (86) the following relation

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)p} d\mathbf{f}_s^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right) + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1} =$$

$$= \int\limits_0^T \int\limits_0^s d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} + \mathbf{1}_{\{i_1=i_2\}} \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \tag{88}$$

Since

$$\sum_{j_1=0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^{\infty} \left(\int\limits_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left(\int\limits_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int\limits_0^T ds,$$

then from (88) we obtain (87).

References

[1] Gihman I.I., Skorochod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982. 354 pp.

[2] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[5] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.

[6] Platen, E., Bruti-Liberati, N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010. 868 pp.

[7] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. J. Math. Sci. (N. Y.), 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. J. Math. Sci. (N. Y.), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[15] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[18] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[22] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184 [math.PR]. 2022, 923 pp. [In English].

[23] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[24] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[25] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[26] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[27] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[28] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[29] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [in English].

[30] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 204 pp. [In English].

[31] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2019, 106 pp. [In English].

[32] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series, arXiv:1807.02190 [math.PR]. 2018, 44 pp. [In English].

[33] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [in English].

[34] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 129 pp. [In English].

[35] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 68 pp.

[36] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].

[37] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[38] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[39] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[40] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[41] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[42] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[43] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[44] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 343 pp. [In English].

[45] Kuznetsov D.F. Application of multiple Fourier–Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at:
http://www.sde-kuznetsov.spb.ru/20e.pdf

[46] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[47] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[48] Kuznetsov D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.3/article.1.6.html

[49] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].

[50] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[51] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[52] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[53] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[54] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[55] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: `sde_kuznetsov@inbox.ru`

NEW SIMPLE METHOD OF EXPANSION OF ITERATED ITO STOCHASTIC INTEGRALS OF MULTIPLICITY 2 BASED ON EXPANSION OF THE BROWNIAN MOTION USING LEGENDRE POLYNOMIALS AND TRIGONOMETRIC FUNCTIONS

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the expansion of iterated Ito stochastic integrals of second multiplicity based on expansion of the Brownian motion (standard Wiener process) using complete orthonormal systems of functions in the space $L_2([t,T])$. The cases of Legendre polynomials and trigonometric functions are considered in details. We obtained a new representation of the Levy stochastic area based on the Legendre polynomials. This representation was first derived in the author's work [1] (1997). In this article, we obtain the mentioned representation by a simpler method compared to [1] (1997). Also, we get the polynomial representation of the Levy stochastic area using the method of expansion of iterated Ito stochastic integrals based on generalized multple Fourier series. The polynomial representation of the Levy stochastic area has more simple form in comparison with the classical trigonometric representation of the Levy stochastic area. The convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) as well as the convergence with probability 1 for approximations of the Levy stochastic area are proved. The results of the article can be applied to the numerical solution of Ito stochastic differential equations as well as to the numerical approximation of mild solution for non-commutative semilinear stochastic partial differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.
Keywords: Iterated Ito stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Levy stochastic area, Mean square-convergence, Milstein method, Ito stochastic differential equation, Approximation, Expansion.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{w}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{w}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [2]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

One of the effective approaches to the numerical integration of Ito stochastic differential equations is an approach based on the Taylor–Ito expansion [3]-[5]. The most important feature of the Taylor–Ito expansion is a presence in this expansion of the so-called iterated Ito stochastic integrals, which play the key role for solving the problem of numerical integration of Ito stochastic differential equations and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \quad (i_1, \ldots, i_k = 0, 1, \ldots, m), \tag{2}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes, and $\mathbf{w}_\tau^{(0)} = \tau$.

In this article, we pay a special attention to the case $k = 2$, $i_1, i_2 = 1, \ldots, m$, $\psi_1(\tau), \psi_2(\tau) \equiv 1$. This case corresponds to the so-called Milstein method [4], [5] for the numerical integration of Ito stochastic differential equations. It is well known that the Milstein method has the order 1.0 of strong convergence under the specific conditions [4], [5].

The Milstein method has the following form [4], [5]

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} I_{\tau_{p+1},\tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1,i_2=1}^m G_{i_1} B_{i_2} \hat{I}_{\tau_{p+1},\tau_p}^{(i_1 i_2)},$$

where $\Delta = T/N$ $(N > 1)$ is a constant (for simplicity) step of integration, $\tau_p = p\Delta$ $(p = 0, 1, \ldots, N)$,

$$G_i = \sum_{j=1}^n B_{ji}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}_j} \quad (i = 1, \ldots, m),$$

B_i is the ith column of the matrix function B and B_{ij} is the ijth element of the matrix function B, $\mathbf{a}_i$ is the ith element of the vector function $\mathbf{a}$, and $\mathbf{x}_i$ is the ith element of the column $\mathbf{x}$, the columns B_{i_1}, $\mathbf{a}$, $G_{i_1} B_{i_2}$ are calculated in the point $(\mathbf{y}_p, p)$,

$$I_{\tau_{p+1},\tau_p}^{(i_1)}=\int\limits_{\tau_p}^{\tau_{p+1}} d\mathbf{w}_{\tau}^{(i_1)},$$

$\hat{I}_{\tau_{p+1},\tau_p}^{(i_1 i_2)}$ is an approximation of the following iterated Ito stochastic integral

$$I_{\tau_{p+1},\tau_p}^{(i_1 i_2)}=\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(i_1)}d\mathbf{w}_{s}^{(i_2)}.$$

The Levy stochastic area $A_{T,t}^{(i_1 i_2)}$ is defined as follows [6]

$$A_{T,t}^{(i_1 i_2)}=\frac{1}{2}\left(I_{T,t}^{(i_1 i_2)}-I_{T,t}^{(i_2 i_1)}\right).$$

It is clear that

$$I_{T,t}^{(i_1 i_2)}=\frac{1}{2}I_{T,t}^{(i_1)}I_{T,t}^{(i_2)}+A_{T,t}^{(i_1 i_2)}\quad \text{w. p. 1,} \tag{3}$$

where w. p. 1 means with probability 1, $i_1\ne i_2$.

The relation (3) implies that the problem of numerical simulation of the iterated Ito stochastic integral $I_{T,t}^{(i_1 i_2)}$ is equivalent to the problem of numerical simulation of the Levy stochastic area.

There are some methods for representation of the Levy stochastic area (see, for example, [3]–[5]). In this article, we consider a new representation of the Levy stochastic area based on the Legendre polynomials. This representation is simpler than its existing analogue based on the Karhunen–Loeve expansion of the Brownian bridge process [4] (also see [3]).

2. Method of Expansion of Iterated Ito stochastic integrals Based on Generalized Multiple Fourier Series

Consider the iterated Ito stochastic integrals (2) and define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ & \\ 0, & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2,$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here we suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{4}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{5}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\qquad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\qquad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{6}$$

Theorem 1 [7] (2006), [8]-[52]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continous functions in $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k=0,1,\ldots,m$, $C_{j_k\ldots j_1}$ is the Fourier coefficient* (5),

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{8}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of $[t,T]$, which satisfies the condition (6).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [7]-[52]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{9}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{10}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{11}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{12}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{13}$$

where $\mathbf{1}_A$ is the indicator of the set A.

It was shown that Theorem 1 is valid for convergence w. p. 1 [12]-[14], [32], [44] (the cases of Legendre polynomials and trigonometric functions) and for convergence in the mean of degree $2n$ ($n\in\mathbb{N}$) [12]-[14], [16]-[21], [32]. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ can also be applied in Theorem 1 [7]-[21]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [11]-[14], [42]. Recently, Theorem 1 and Theorem 2 (see below) has been applied to the expansion and mean-square approximation of iterated stochastic integrals with respect to the infinite-dmensional Q-Wiener process [12]-[14] (Chapter 7), [28], [29], [45]-[47]. These results can be directly applied to construction of high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise [12]-[14] (Chapter 7), [29], [47].

Note that we obtain the following useful possibilities of the approach based on Theorem 1.

1. There is the explicit formula (5) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square approximation error of iterated Ito stochastic integral (2) [9]-[14], [22], [33] (also see Theorem 3 below).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [3]-[5] but Legendre polynomials.

4. As it turned out [7]-[43] it is more convenient to work with Legendre polynomials for approximation of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials are much simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [12]-[14], [26], [30].

5. The approach to expansion of iterated Ito and Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process [4] (also see [3], [5]) as well as the approach from [53] lead to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1 and Theorem 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$; $i_1,i_2,i_3=1,\ldots,m$) of iterated stochastic integrals. Multiple series from Theorems 1, 2 (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series. However, the authors of the works [3] (Sect. 5.8, pp. 202–204), [54] (pp. 82-84), [55] (pp. 438-439), [56] (pp. 263-264) use the Wong–Zakai approximation [57]-[59] (without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals [4] (1988) based on the series expansion of the Brownian bridge process (version of the so-called Karhunen-Loeve expansion). See discussions in [12] (Sect. 2.16, 6.2), [13] (Sect. 2.6.2, 6.2), [14] (Sect. 2.6.2, 6.2), [32] (Sect. 11), [34] (Sect. 8), [36] (Sect. 6) for detail.

Note that the correctness of the formulas (9)–(13) can be verified by the fact that if $i_1 = \ldots = i_5 = i = 1, \ldots, m$ and $\psi_1(s), \ldots, \psi_5(s) \equiv \psi(s)$ in (9)–(13), then we can obtain from (9)–(13) the following equalities

$$J[\psi^{(1)}]_{T,t} = \frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t} = \frac{1}{2!}\left(\delta^2_{T,t} - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t} = \frac{1}{3!}\left(\delta^3_{T,t} - 3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t} = \frac{1}{4!}\left(\delta^4_{T,t} - 6\delta^2_{T,t}\Delta_{T,t} + 3\Delta^2_{T,t}\right),$$

$$J[\psi^{(5)}]_{T,t} = \frac{1}{5!}\left(\delta^5_{T,t} - 10\delta^3_{T,t}\Delta_{T,t} + 15\delta_{T,t}\Delta^2_{T,t}\right)$$

w. p. 1, where

$$\delta_{T,t} = \int\limits_t^T \psi(s)d\mathbf{w}_s^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(s)ds.$$

The above formulas can be independently obtained using the Ito formula and Hermite polynomials. Note that the cases $k = 2, 3$ and $p_1 = p_2 = p_3 = p$ are considered in detail in [8]-[21], [32].

Consider the generalization of formulas (9)–(13) for the case of an arbitrary multiplicity k of the stochastic integral $J[\psi^{(k)}]_{T,t}$ as well as for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{14}$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (14) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}\}, \{q_1, \ldots, q_{k-2r}\}) \\ \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}}} a_{g_1 g_2, \ldots, g_{2r-1} g_{2r}, q_1 \ldots q_{k-2r}}. \tag{15}$$

Below there are several examples of sums in the form (15)

$$\sum_{\substack{(\{g_1, g_2\}) \\ \{g_1, g_2\} = \{1, 2\}}} a_{g_1 g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1, g_2\}, \{g_3, g_4\}\}) \\ \{g_1, g_2, g_3, g_4\} = \{1, 2, 3, 4\}}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} = a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135}+$$

$$+a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4}+$$

$$+a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can generalize Theorem 1.

Theorem 2 [12] (Sect. 1.11), [32] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (16)$$

that converges in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x*; another notations are the same as in Theorem* 1.

In particular from (16) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (13).

It should be noted that an analogue of Theorem 2 for multiple Ito stochastic integrals was considered in [60]. Note that we use another notations in comparison with [60]. Moreover, the proof of an analogue of Theorem 2 from [60] is somewhat different from the proof given in [12] (Sect. 1.11), [32] (Sect. 15).

Let us denote

$$E_k^{p_1,\ldots,p_k} \stackrel{\rm def}{=} {\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\},\quad E_k^p \stackrel{\rm def}{=} E_k^{p_1,\ldots,p_k}\Bigg|_{p_1=\ldots=p_k=p},$$

$$I_k \stackrel{\rm def}{=} \|K\|^2_{L_2([t,T]^k)} = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k,$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is the expression on the right-hand side of (16) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\hbox{l.i.m.}}$, i.e.

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^r {\bf 1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}{\bf 1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr).$$

In [12]-[14], [32] it was shown that

$$E_k^{p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right)$$

if $i_1,\ldots,i_k=1,\ldots,m$ and $0<T-t<\infty$ or $i_1,\ldots,i_k=0,1,\ldots,m$ and $0<T-t<1$.

Moreover, in [12]-[14], [32] the following estimate is obtained

$$E_k^{p_1,\ldots,p_k} \le (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\times$$

$$\times\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right)^n, \tag{17}$$

where $n\in\mathbb{N}$.

The value E_k^p can be calculated exactly.

Theorem 3 [12] (Sect. 1.12), [33] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then*

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}{\sf M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right\}, \tag{18}$$

where $i_1,\ldots,i_k=1,\ldots,m$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Then from Theorem 3 for pairwise different $i_1,\ldots,i_k$ and for $i_1=\ldots=i_k$ we obtain

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2,$$

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right).$$

3. New Representation of the Levy Stochastic Area Based on the Legendre Polynomials

Let us consider (10) for the case $i_1\ne i_2$, $\psi_1(s),\ \psi_2(s)\equiv 1$. At that we suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then

$$I_{T,t}^{(i_1 i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{19}$$

where

$$I_{T,t}^{(i_1 i_2)}=\int\limits_t^T\int\limits_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)}\quad (i_1,i_2=1,\ldots,m), \tag{20}$$

$\zeta_j^{(i)}$ are independent standard Gaussian random variables (for various i or j), which have the following form

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)},$$

where

$$\phi_i(s)=\sqrt{\frac{2i+1}{T-t}}P_i\left(\left(s-t-\frac{T-t}{2}\right)\frac{2}{T-t}\right),\quad i=0,1,2,\ldots, \tag{21}$$

and $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial.

Note that the representation (19) was first obtained in the author's works [1] (1997), [61] (1998).

From (19) we obtain

$$\frac{T-t}{2}\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right)=\frac{1}{2}\left(I_{T,t}^{(i_1 i_2)}-I_{T,t}^{(i_2 i_1)}\right).$$

Then, a new representation of the Levy stochastic area based on the Legendre polynomials has the following form

$$A_{T,t}^{(i_1 i_2)}=\frac{T-t}{2}\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_{i}^{(i_2)}-\zeta_{i}^{(i_1)}\zeta_{i-1}^{(i_2)}\right). \tag{22}$$

4. The Classical Representation of the Levy Stochastic Area

Let us consider (10) for the case $i_1 \neq i_2,\ \ \psi_1(s),\ \psi_2(s)\equiv 1$. At that we suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is the complete orthonormal system of trigonometric functions in $L_2([t,T])$. Then

$$I_{T,t}^{(i_1 i_2)}=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr)\Biggr), \tag{23}$$

where we use the same notations as in (19), but $\phi_j(s)$ has the following form

$$\phi_j(s)=\frac{1}{\sqrt{T-t}}\begin{cases}1, & \text{if } j=0\\ \sqrt{2}\sin(2\pi r(s-t)/(T-t)), & \text{if } j=2r-1\ ,\quad r=1,\ 2,\ldots\\ \sqrt{2}\cos(2\pi r(s-t)/(T-t)), & \text{if } j=2r\end{cases} \tag{24}$$

From (23) we obtain

$$\frac{T-t}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)=$$

$$=\frac{1}{2}\left(I_{T,t}^{(i_1 i_2)}-I_{T,t}^{(i_2 i_1)}\right).$$

Then, the representation of the Levy stochastic area based on the trigonometric functions has the following form

$$\hat{A}_{T,t}^{(i_1 i_2)}=\frac{T-t}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right). \tag{25}$$

As we mentioned above, Milstein G.N. proposed [4] the method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on the trigonometric Fourier expansion of the following Brownian bridge process

$$\mathbf{w}_t - \frac{t}{\Delta}\mathbf{w}_{\Delta}, \quad t \in [0, \Delta], \quad \Delta > 0,$$

where $\mathbf{w}_t$ is a standard multidimensional Wiener process with independent components $\mathbf{w}_t^{(i)}$, $i = 1, \ldots, m$.

The trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karunen–Loeve expansion) has the form [4]

$$\mathbf{w}_t^{(i)} - \frac{t}{\Delta}\mathbf{w}_{\Delta}^{(i)} = \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{26}$$

where

$$a_{i,r} = \frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{w}_s^{(i)} - \frac{s}{\Delta}\mathbf{w}_{\Delta}^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,$$

$$b_{i,r} = \frac{2}{\Delta}\int\limits_0^{\Delta}\left(\mathbf{w}_s^{(i)} - \frac{s}{\Delta}\mathbf{w}_{\Delta}^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds,$$

$r = 0, 1, \ldots,\ \ i = 1, \ldots, m.$

It is easy to demonstrate [4] that the random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\} = \mathsf{M}\left\{a_{i,r}b_{i,k}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i,r}a_{i,k}\right\} = \mathsf{M}\left\{b_{i,r}b_{i,k}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\} = \mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i,r}^2\right\} = \mathsf{M}\left\{b_{i,r}^2\right\} = \frac{\Delta}{2\pi^2 r^2},$$

where $i,\ i_1, i_2 = 1, \ldots, m,\ \ r \ne k,\ \ i_1 \ne i_2$.

According to (26), we have

$$\mathbf{w}_t^{(i)} = \mathbf{w}_{\Delta}^{(i)}\frac{t}{\Delta} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \tag{27}$$

where the series converges in the mean-square sense.

The expansion (23) has been obtained in [4] using (27).

5. New Simple Method for Obtainment of Representation of the Levy Stochastic Area

It is well known that the idea of representing of the Wiener process as a functional series with random coefficients using the complete orthonormal system of trigonometric functions in $L_2([0,T])$ goes back to the works of Wiener [62] (1924) and Levy [63] (1951). The specified series was used in [62] and [63] for construction of the Brownian motion process (Wiener process). A little later, Ito and McKean in [64] (1965) used for this purpose the complete orthonormal system of Haar functions in $L_2([0,T])$.

Let $\mathbf{w}_\tau$, $\tau\in[0,T]$ be an m-dimestional standard Wiener process with independent components $\mathbf{w}_\tau^{(i)}$ $(i=1,\ldots,m)$. We have

$$\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}=\int\limits_t^s d\mathbf{w}_\tau^{(i)}=\int\limits_t^T \mathbf{1}_{\{\tau<s\}}d\mathbf{w}_\tau^{(i)},$$

where

$$\mathbf{1}_{\{\tau<s\}}=\begin{cases}1, & \tau<s\\ \\ 0, & \text{otherwise}\end{cases},\quad \tau,s\in[t,T],\quad 0\le t<T.$$

Consider the Fourier expansion of $\mathbf{1}_{\{\tau<s\}}$ at the interval $[t,T]$ (see, for example, [65])

$$\mathbf{1}_{\{\tau<s\}}=\sum_{j=0}^{\infty}\int\limits_t^T \mathbf{1}_{\{\tau<s\}}\phi_j(\tau)d\tau\cdot\phi_j(\tau)=\sum_{j=0}^{\infty}\int\limits_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau), \tag{28}$$

where $\{\phi_j(\tau)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$ and the series on the right-hand side of (28) converges in the mean-square sence, i.e.

$$\int\limits_t^T\left(\mathbf{1}_{\{\tau<s\}}-\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau)\right)^2 d\tau\to 0\quad\text{if}\quad q\to\infty.$$

Let $\left(\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}\right)^{(q)}$ be the mean-square approximation of the process $\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}$, which has the following form

$$\left(\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}\right)^{(q)}=\int\limits_t^T\left(\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau)\right)d\mathbf{w}_\tau^{(i)}=\sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\cdot\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}. \tag{29}$$

Moreover,

$$\mathsf{M}\left\{\left(\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}-\left(\mathbf{w}_s^{(i)}-\mathbf{w}_t^{(i)}\right)^{(q)}\right)^2\right\}=$$

$$= \mathsf{M}\left\{\left(\int\limits_t^T \left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau)\right)d\mathbf{w}_\tau^{(i)}\right)^2\right\} =$$

$$= \int\limits_t^T \left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^{q}\int\limits_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau)\right)^2 d\tau \to 0 \quad \text{if } q\to\infty. \tag{30}$$

In [53] it was proposed to use the expansion similar to (29) for construction of expansion of the iterated Ito stochastic integral (20) of multiplicity 2. At that, to obtain the mentioned expansion of (20), the truncated expansions (29) of components of the Wiener process $\mathbf{w}_s$ have been iteratively substituted in the single integrals [53]. This procedure leads to the calculation of coefficients of the double Fourier series, which is a time-consuming task for not too complex problem of expansion of the iterated Ito stochastic integral (20).

In contrast to [53] we subsitute the truncated expansion (29) only one time and only into the innermost integral in (20). This procedure leads to the simple calculation of the coefficients

$$\int\limits_t^s \phi_j(\tau)d\tau \quad (j=0,1,2,\ldots)$$

of the usual (not double) Fourier series.

Moreover, we use the Legendre polynomials for construction of the expansion of (20). For the first time the Legendre polynomials have been applied in the framework of the mentioned problem in the author's papers [1] (1997), [61] (1998), [66] (2000), [67] (2001) (also see [7]-[52], [68], [69]). At the same time in the papers of other author's these polynomials have not been considered as the basis functions for construction of expansions of iterated Ito and Stratonovich stochastic integrals.

Theorem 4 [12]-[14], [68], [69]. *Let $\phi_j(\tau)$ $(j=0,1,\ldots)$ be an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Let*

$$\int\limits_t^T \left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)} = \sum_{j=0}^{q}\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i_1)}\int\limits_t^T\int\limits_t^s \phi_j(\tau)d\tau d\mathbf{w}_s^{(i_2)} \tag{31}$$

be the approximation of the iterated Ito stochastic integral

$$\int\limits_t^T\int\limits_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)} \quad (i_1\ne i_2),$$

where $i_1, i_2 = 1,\ldots,m$. Then

$$\int\limits_t^T\int\limits_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)} = \underset{q\to\infty}{\text{l.i.m.}}\ \int\limits_t^T \left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)} =$$

$$= \underset{q\to\infty}{\text{l.i.m.}}\ \sum_{j=0}^{q}\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i_1)}\int\limits_t^T\int\limits_t^s \phi_j(\tau)d\tau d\mathbf{w}_s^{(i_2)},$$

where $i_1 \neq i_2$ $(i_1, i_2 = 1, \ldots, m)$.

Proof. Using standard properties of the Ito stochastic integral as well as (30) and the property of orthonormality of the functions $\phi_j(\tau)$ $(j = 0, 1, \ldots)$ at the interval $[t, T]$, we obtain

$$
\mathsf{M}\left\{\left(\int_t^T\int_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)} - \int_t^T\left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)}\right)^2\right\} =
$$

$$
= \int_t^T \mathsf{M}\left\{\left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)} - \left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)}\right)^{(q)}\right)^2\right\} ds =
$$

$$
= \int_t^T\int_t^T\left(\mathbf{1}_{\{\tau<s\}} - \sum_{j=0}^{q}\int_t^s \phi_j(\tau)d\tau\cdot\phi_j(\tau)\right)^2 d\tau ds =
$$

$$
= \int_t^T\left((s-t) - \sum_{j=0}^{q}\left(\int_t^s \phi_j(\tau)d\tau\right)^2\right) ds. \tag{32}
$$

Applying the continuity of the functions $u_q(s)$ (see below), the nondecreasing property of the functional sequence

$$
u_q(s) = \sum_{j=0}^{q}\left(\int_t^s \phi_j(\tau)d\tau\right)^2,
$$

and the continuity of the limit function $u(s) = s - t$ according to Dini's Theorem, we have the uniform convergence $u_q(s)$ to $u(s)$ at the interval $[t, T]$.

Then from this fact as well as from (32) we obtain

$$
\int_t^T\int_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)} = \underset{q\to\infty}{\text{l.i.m.}} \int_t^T\left(\mathbf{w}_s^{(i_1)} - \mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)}. \tag{33}
$$

Theorem 4 is proved.

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be the complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$, which has the form (21). Then

$$
\int_t^s \phi_j(\tau)d\tau = \frac{T-t}{2}\left(\frac{\phi_{j+1}(s)}{\sqrt{(2j+1)(2j+3)}} - \frac{\phi_{j-1}(s)}{\sqrt{4j^2-1}}\right) \quad \text{for} \quad j \geq 1. \tag{34}
$$

Denote (see Theorem 1)

$$
\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)} \quad (i = 1, \ldots, m).
$$

From (31) and (34) we get

$$\int\limits_t^T \left(\mathbf{w}_s^{(i_1)}-\mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)} = \frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\int\limits_t^T (s-t)\mathbf{w}_s^{(i_2)}+$$

$$+\frac{T-t}{2}\sum_{j=1}^{q}\zeta_j^{(i_1)}\left(\frac{1}{\sqrt{(2j+1)(2j+3)}}\zeta_{j+1}^{(i_2)}-\frac{1}{\sqrt{4j^2-1}}\zeta_{j-1}^{(i_2)}\right)=$$

$$=\frac{T-t}{2}\zeta_0^{(i_1)}\left(\zeta_0^{(i_2)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_2)}\right)+$$

$$+\frac{T-t}{2}\sum_{j=1}^{q}\zeta_j^{(i_1)}\left(\frac{1}{\sqrt{(2j+1)(2j+3)}}\zeta_{j+1}^{(i_2)}-\frac{1}{\sqrt{4j^2-1}}\zeta_{j-1}^{(i_2)}\right)=$$

$$=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{j=1}^{q}\frac{1}{\sqrt{4j^2-1}}\left(\zeta_{j-1}^{(i_1)}\zeta_j^{(i_2)}-\zeta_j^{(i_1)}\zeta_{j-1}^{(i_2)}\right)\right)+$$

$$+\frac{T-t}{2}\zeta_q^{(i_1)}\zeta_{q+1}^{(i_2)}\frac{1}{\sqrt{(2q+1)(2q+3)}}. \tag{35}$$

Then from (33) and (35) we obtain

$$\int\limits_t^T\int\limits_t^s d\mathbf{w}_\tau^{(i_1)}d\mathbf{w}_s^{(i_2)} = \underset{q\to\infty}{\text{l.i.m.}}\int\limits_t^T \left(\mathbf{w}_s^{(i_1)}-\mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)} =$$

$$=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{j=1}^{\infty}\frac{1}{\sqrt{4j^2-1}}\left(\zeta_{j-1}^{(i_1)}\zeta_j^{(i_2)}-\zeta_j^{(i_1)}\zeta_{j-1}^{(i_2)}\right)\right). \tag{36}$$

From (36) it follows that the equality (22) is fulfilled. It is not difficult to see that the relation (25) can also be obtained using the approach from this section.

Let $\{\phi_j(\tau)\}_{j=0}^{\infty}$ be the complete orthonormal system of trigonomertic functions in the space $L_2([t,T])$, which has the form (24).

We have

$$\int\limits_t^s \phi_j(\tau)d\tau = \frac{T-t}{2\pi r}\begin{cases}\phi_{2r-1}(s), & j=2r\\ \\ \sqrt{2}\phi_0(s)-\phi_{2r}(s), & j=2r-1\end{cases}, \tag{37}$$

where $j\ge 1$ and $r=1,2,\ldots$.

From (31) and (37) we obtain

$$\int\limits_t^T \left(\mathbf{w}_s^{(i_1)}-\mathbf{w}_t^{(i_1)}\right)^{(q)} d\mathbf{w}_s^{(i_2)}=\frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\int\limits_t^T (s-t)\mathbf{w}_s^{(i_2)}+$$

$$+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right)=$$

$$=\frac{1}{\sqrt{T-t}}\zeta_0^{(i_1)}\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_2)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right)+$$

$$+\frac{T-t}{2}\sum_{r=1}^{q}\frac{1}{\pi r}\left(\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\sqrt{2}\zeta_0^{(i_2)}\zeta_{2r-1}^{(i_1)}\right)=$$

$$=\frac{1}{2}(T-t)\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Biggl(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr)\Biggr)-$$

$$-\frac{T-t}{\pi\sqrt{2}}\zeta_0^{(i_1)}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}. \tag{38}$$

From (38) and (33) we obviously get (23).

6. Convergence in the Mean of Degree $2n$ and With Probability 1

Let us denote

$$A_{T,t}^{(i_1i_2)q}=\frac{T-t}{2}\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right), \tag{39}$$

$$\hat{A}_{T,t}^{(i_1i_2)q}=\frac{T-t}{2\pi}\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right). \tag{40}$$

Then, from (3) we get

$$I_{T,t}^{(i_1i_2)q}=\frac{T-t}{2}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+A_{T,t}^{(i_1i_2)q}, \tag{41}$$

$$I_{T,t}^{(i_1i_2)q}=\frac{T-t}{2}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\hat{A}_{T,t}^{(i_1i_2)q}. \tag{42}$$

It is not difficult to demonstrate [4] that from (23) we can get an another representation for the Levy stochastic area

$$
\frac{T-t}{2\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+\right.\right.
$$

$$
\left.\left.+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_{0}^{(i_2)}-\zeta_{0}^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right)+\sqrt{2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^{1/2}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\right),
$$

where

$$
\xi_q^{(i)}=\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^{-1/2}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)},
$$

and $\zeta_0^{(i)},\ \zeta_{2r}^{(i)},\ \zeta_{2r-1}^{(i)},\ \xi_q^{(i)}$ $(r=1,\ldots,q,\ i=1,\ldots,m)$ are independent standard Gaussian random variables.

From (39) and (40) we obtain

$$
\mathsf{M}\left\{\left(A_{T,t}^{(i_1i_2)}-A_{T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)=
$$

$$
=\frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le\frac{(T-t)^2}{2}\int\limits_{q}^{\infty}\frac{1}{4x^2-1}dx=
$$

$$
=-\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{(T-t)^2}{q}, \tag{43}
$$

$$
\mathsf{M}\left\{\left(\hat A_{T,t}^{(i_1i_2)}-\hat A_{T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)=
$$

$$
=\frac{3(T-t)^2}{2\pi^2}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\le\frac{3(T-t)^2}{2\pi^2}\int\limits_{q}^{\infty}\frac{dx}{x^2}=
$$

$$
=\frac{3(T-t)^2}{2\pi^2 q}\le C_2\frac{(T-t)^2}{q}, \tag{44}
$$

where constants $C_1,\ C_2$ does not depend on q.

For the case $k=2,\ i_1\ne i_2$, and $\psi_1(s),\ \psi_2(s)\equiv 1$ from (17) we obtain

$$
\mathsf{M}\left\{\left(I_{T,t}^{(i_1i_2)}-I_{T,t}^{(i_1i_2)q}\right)^{2n}\right\}\le C_{n,2}\left(\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\right)^n\to 0\quad\text{if } q\to\infty, \tag{45}
$$

$$\mathsf{M}\left\{\left(I_{T,t}^{(i_1i_2)}-I_{T,t}^{(i_1i_2)q}\right)^{2n}\right\}\le C_{n,2}\left(\frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)\right)^n\to 0\quad \text{if } q\to\infty, \tag{46}$$

where $C_{n,k}=(k!)^{2n}\bigl(n(2n-1)\bigr)^{n\cdot(k-1)}(2n-1)!!$, $I_{T,t}^{(i_1i_2)q}$ has the form (41) in the inequality (45), and $I_{T,t}^{(i_1i_2)q}$ has the form (42) in the inequality (46),

From (43)–(46) we get

$$\mathsf{M}\left\{\left(A_{T,t}^{(i_1i_2)}-A_{T,t}^{(i_1i_2)q}\right)^{2n}\right\}\to 0\quad \text{if}\quad q\to\infty,$$

$$\mathsf{M}\left\{\left(\hat{A}_{T,t}^{(i_1i_2)}-\hat{A}_{T,t}^{(i_1i_2)q}\right)^{2n}\right\}\to 0\quad \text{if}\quad q\to\infty.$$

Let us address now to the convergence w. p. 1 for $A_{T,t}^{(i_1i_2)q}$. First, note the well known fact.

Lemma 1. *If for the sequence of random variables ξ_q and for some $\alpha>0$ the number series*

$$\sum_{q=1}^{\infty}\mathsf{M}\left\{|\xi_q|^{\alpha}\right\}$$

converges, then the sequence ξ_q converges to zero w. p. 1.

From (43) and (45) $(n=2)$ we obtain

$$\mathsf{M}\left\{\left(I_{T,t}^{(i_1i_2)}-I_{T,t}^{(i_1i_2)q}\right)^{4}\right\}=\mathsf{M}\left\{\left(A_{T,t}^{(i_1i_2)}-A_{T,t}^{(i_1i_2)q}\right)^{4}\right\}\le\frac{K}{q^2},$$

where constant K does not depend on q.

Since the series

$$\sum_{q=1}^{\infty}\frac{K}{q^2}$$

converges, then according to Lemma 1 we obtain that $A_{T,t}^{(i_1i_2)}-A_{T,t}^{(i_1i_2)q}\to 0$ if $q\to\infty$ w. p. 1. Then $A_{T,t}^{(i_1i_2)q}\to A_{T,t}^{(i_1i_2)}$ if $q\to\infty$ w. p. 1.

In addition, using (44) and (46) $(n=2)$, we get $\hat{A}_{T,t}^{(i_1i_2)q}\to\hat{A}_{T,t}^{(i_1i_2)}$ if $q\to\infty$ w. p. 1.

References

[1] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[2] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.

[4] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.

[6] Watanabe S. Levy's stochastic area formula and Brownian motion on compact Lie groups. In: Ikeda N., Watanabe S., Fukushima M., Kunita H. (Eds.) Ito's Stochastic Calculus and Probability Theory. Springer, Tokyo, 1996, pp. 401–412.

[7] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[8] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[9] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[12] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184 [math.PR]. 2022, 869 pp. [In English].

[13] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[14] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[19] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[20] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[21] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.

DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[22] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[23] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[24] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 28 pp. [In English].

[25] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[26] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[27] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[28] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[29] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3, (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[30] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English]

[31] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77.
DOI: http://doi.org/10.13108/2019-11-4-49
Available at: http://matem.anrb.ru/en/article?art_id=604

[32] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [In English].

[33] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 67 pp. [In English].

[34] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 105 pp. [In English].

[35] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 104 pp. [In English].

[36] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. arXiv:1712.09516 [math.PR]. 2022, 174 pp. [In English].

[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 64 pp. [In English].

[38] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. arXiv:1801.01962 [math.PR]. 2018, 40 pp. [In English].

[39] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 46 pp. [In English].

[40] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 78 pp. [In English].

[41] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR]. 2018, 42 pp. [In English].

[42] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 37 pp. [In English].
[43] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2018, 94 pp. [In English].
[44] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html
[45] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].
[46] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453.
Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf
[47] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.3/article.1.6.html
[48] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878
[49] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100
[50] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html
[51] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2
[52] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 342 pp. [In English].
[53] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].
[54] Kloeden P.E., Platen E., Schurz H. Numerical solution of SDE through computer experiments. Berlin: Springer, 1994, 292 pp.
[55] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl.. 10, 4 (1992), 431-441.
[56] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.
[57] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[58] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[59] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[60] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[61] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at:

http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[62] Wiener N. Un problème de probabilités dénombrables. Bulletin de la Société Mathématique de France. 52 (1924), 569-578.

[63] Lévy P. Wiener's random function and other Laplacian random functions. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. 1951, 171-187.

[64] Ito K., McKean H. Diffusion processes and their sample paths. Springer-Verlag, Berlin-Heidelberg-New York, 1965, 395 p.

[65] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Institute of Technology, 2006, 225 p.

[66] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[67] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[68] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4, (2019), 32-52. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[69] Kuznetsov D.F. New representation of the Levy stochastic area based on Legendre polynomials. [In English]. arXiv:1807.00409v1 [math.PR]. 2018, 10 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Chapter 4.

Application to the High-Order Strong Numerical Methods for Ito SDEs

FOUR NEW FORMS OF THE TAYLOR–ITO AND TAYLOR–STRATONOVICH EXPANSIONS AND ITS APPLICATION TO THE HIGH-ORDER STRONG NUMERICAL METHODS FOR ITO STOCHASTIC DIFFERENTIAL EQUATIONS

DMITRIY F. KUZNETSOV

Abstract. The problem of the Taylor–Ito and Taylor–Stratonovich expansions of the Ito stochastic processes in a neighborhood of a fixed moment of time is considered. The classical forms of the Taylor–Ito and Taylor–Stratonovich expansions are transformed to the four new representations, which includes the minimal sets of different types of iterated Ito and Stratonovich stochastic integrals. Therefore, these representations (the so-called unified Taylor–Ito and Taylor–Stratonovich expansions) are more convenient for constructing of high-order strong numerical methods for Ito stochastic differential equations. Explicit one-step strong numerical schemes with the orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 based on the unified Taylor–Ito and Taylor–Stratonovich expansions are derived. Effective mean-square approximations of iterated Ito and Stratonovich stochastic integrals from these numerical schemes are constructed on the base of the multiple Fourier–Legendre series with multiplicities 1 to 6.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05.

Keywords: Taylor–Ito expansion, Taylor–Stratonovich expansion, Unified Taylor–Ito expansion, Unified Taylor–Stratonovich expansion, Ito stochastic differential equation, High-order strong numerical method for Ito SDEs, Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Mean-square approximation, Expansion.

1. Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-subfields of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau,\quad \mathbf{x}_0=\mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to Ito SDE (1). The nonrandom functions $\mathbf{a}:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $B:\mathbb{R}^n\times[0,T]\to\mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\}<\infty$ (M denotes a mathematical expectation). Also we assume that $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ are independent when $t>0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems of different physical origin that are affected by random perturbations. For example, Ito SDEs are used as mathematical models in stochastic mathematical finance, hydrology, seismology, geophysics, chemical kinetics, population dynamics, electrodynamics, medicine and other fields [2]-[5]. Numerical integration of Ito SDEs based on the strong convergence criterion of approximations [2] is widely used for the numerical simulation of sample trajectories of solutions to Ito SDEs (which is required for constructing new mathematical models on the basis of such equations and for the numerical solution of different mathematical problems connected with Ito SDEs). Among these problems, we note the following: filtering of signals under influence of random noises in various statements (linear Kalman–Bucy filtering, nonlinear optimal filtering, filtering of continuous time Markov chains with a finite space of states, etc.), optimal stochastic control (including incomplete data control), testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis [2], [3].

Exact solutions of Ito SDEs are known in rather rare cases. It is for this reason that it becomes necessary to construct numerical procedures for solving these equations.

In this paper, a promising approach [2]-[5] to the numerical integration of Ito SDEs based on the stochastic analogues of the Taylor formula (Taylor–Ito and Taylor–Stratonovich expansions) [6]-[9] is used. This approach uses a finite discretization of the time variable and the numerical simulation of the solution to the Ito SDE at discrete instants of time using the stochastic analogues of the Taylor formula mentioned above. A number of works (e.g., [2]-[5]) describe numerical schemes with the strong orders of convergence $1.5, 2.0,$ 2.5, and 3.0 for the Ito SDEs; however, they do not contain efficient procedures of the mean-square approximation of iterated stochastic integrals involved in these schemes for the case of non-commutative noise.

In this paper we consider the unified Taylor–Ito and Taylor–Stratonovich expansions [8], [9] which makes it possible (in contrast with its classical analogues [6], [7]) to use the minimal sets of iterated Ito and Stratonovich stochastic integrals; this is a simplifying factor for the numerical methods implementation. We prove the unified Taylor–Ito expansion [8] with using the slightly different approach (which is taken from [9]) in comparison with the approach from [8]. Moreover, we obtain another (second) version of the unified Taylor–Ito expansion [10], [11]. In addition, we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions [9]). Futhermore, in this paper we study methods [12]-[63] of numerical simulation for iterated Ito and Stratonovich stochastic integrals of multiplicities $1, 2, 3, 4, 5, 6, \ldots$ used in the strong numerical methods for Ito SDEs [2]-[5], [46]-[50], [56]-[61]. To approximate the iterated Ito and Stratonovich stochastic integarls appearing in the numerical schemes with the strong orders of convergence $1.0, 1.5,$ $2.0, 2.5, 3.0$ etc., the method of generalized multiple Fourier series and especially method of multiple Fourier–Legendre series are studied in [12]-[63]. It is important to note that the method of generalized multiple Fourier series [12]-[63] does not lead to the partitioning of the integration interval of the iterated Ito and Stratonovich stochastic integrals under consideration; this interval is the integration step of the numerical methods used to solve Ito SDEs; therefore, it is already fairly small and does not need to be partitioned. Computational experiments [46] show that the application of numerical simulation for iterated stochastic integrals (in which the interval of integration is partitioned) leads to unacceptably high computational cost and accumulation of computation errors. Also note that the Legendre polynomials have essential advantage over the trigonomentric functions (see [23], [39]) in the framework of the method of generalized multiple Fourier series [12]-[63] for the mean-square approximation of iterated Ito and Stratonovich stochastic integrals.

The rest of the article is organized as follows. In the introduction (below) we consider a brief review of the literature on the problem of construction of the Taylor–Ito and Taylor–Stratonovich expansions for the solutions of Ito SDEs. Sect. 2 is devoted to the integration order replacement technique for iterated Ito stochastic integrals. In Sect. 3, we consider the classical Taylor–Ito expansion while Sect. 4 and Sect. 5 are devoted to the first and second forms of the so-called unified Taylor–Ito expansion correspondingly. The classical Taylor–Stratonovich expansion is considered in Sect. 6. The first and second forms of the unified Taylor–Stratonovich expansion are derived in Sect. 7 and Sect. 8. In Sect. 9, we give a comparative analysis of the unified Taylor–Ito and Taylor–Stratonovich expansions with the classical Taylor–Ito and Taylor–Stratonovich expansions. Application of the first form of the unified Taylor–Ito expansion to the high-order strong numerical methods for Ito SDEs is considered in Sect. 10. In Sect. 11, we construct the high-order strong numerical methods for Ito SDEs on the base of the first form of the unified Taylor–Stratonovich expansion. Sect. 12 is devoted to the effective method of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. In Sect. 13, we discuss the connection between Theorems 8, 10–13 (see Sect. 12) and Wong–Zakai approximation.

Let us consider the following iterated Ito and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{s,t}=\int\limits_t^s\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{s,t}=\int\limits_t^{*s}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $0\le t<s\le T$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function at the interval $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau,\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$\int\ \ \text{and}\ \ \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively. In this paper we use the definition of the Stratonovich stochastic integral from [2].

It sould be noted that one of the main problems when constructing the high-order strong numerical methods for Ito SDEs on the base of the Taylor–Ito and Taylor–Stratonovich expansions is the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. Obviously, in the absence of procedures for the numerical simulation of stochastic integrals, the mentioned numerical methods are unrealizable in practice. For this reason, in Sect. 12 we give a brief overview to the effective method of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals (2) and (3) of arbitrary multiplicity k $(k\in\mathbb{N})$, which is proposed and developed by the author of this article in a number of publications [12]-[63]. This method is based on the generalized multiple Fourier series converging in the mean-square sense. The extensive practical material on expansions and mean-square approximations of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions is given in Sect. 12. In the mentioned section, the main focus is on approximations based on multiple Fourier–Legendre series. Such approximations is more effective in comparison with the trigonometric approximations [23], [39] at least for the numerical methods with the strong order 1.5 of convergence and higher [23], [39].

Let us give a brief review of the literature on the problem of construction of the Taylor–Ito and Taylor–Stratonovich expansions for the solutions of Ito SDEs. A few variants of a stochastic analog of the Taylor formula have been obtained in [2]-[7] for the stochastic processes in the form $R(\mathbf{x}_s,s)$, where $\mathbf{x}_s$ is a solution of the Ito SDE (1) and $R:\mathbb{R}^n\times[0,T]\to\mathbb{R}^1$ is a nonrandom sufficiently smooth function.

The first result in this direction called the Ito–Taylor expansion has been obtained in [6], [7]. This result gives an expansion of the process $R(\mathbf{x}_s,s)$ into a series such that every term (if $k>0$) contains an iterated Ito stochastic integral

$$\int\limits_t^s\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{4}$$

as a factor, where $0 \le t < s \le T$, $i_1, \ldots, i_k = 0, 1, \ldots, m$. Obviously that the iterated Ito stochastic integral (4) is a particular case of (2) for $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$.

In [7], another expansion of the stochastic process $R(\mathbf{x}_s, s)$ in a series has been derived. Instead of the Ito integrals, the iterated Stratonovich stochastic integrals

$$\int\limits_t^{*s} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{5}$$

were used; the corresponding expansion was called the Stratonovich–Taylor expansion. In the formula (5), the indices $i_1, \ldots, i_k$ take values $0, 1, \ldots, m$.

In [8] the Ito–Taylor expansion from [6], [7] is reduced to the interesting and unexpected form (called the unified Taylor–Ito expansion) with the help of special transformations [66] (also see [33], [67], [68]). Every term of this expansion (if $k > 0$) contains an iterated Ito stochastic integral of the form

$$\int\limits_t^{s} (s - t_k)^{l_k} \ldots \int\limits_t^{t_2} (s - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{6}$$

where $0 \le t < s \le T$, $l_1, \ldots, l_k = 0, 1, 2, \ldots$ and $i_1, \ldots, i_k = 1, \ldots, m$.

It is worth to mention another form of the unified Taylor–Ito expansion [10], [11], [67] (also see [46]-[50], [54]-[61]). Terms of the latter expansion contain iterated Ito stochastic integrals of the form

$$\int\limits_t^{s} (t - t_k)^{l_k} \ldots \int\limits_t^{t_2} (t - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{7}$$

where $l_1, \ldots, l_k = 0, 1, 2, \ldots$ and $i_1, \ldots, i_k = 1, \ldots, m$.

In this paper, we derive two new forms of the Taylor–Ito expansions (the so-called unified Taylor–Ito expansions [10], [11], [67] (also see [46]-[50], [54]-[61])) using an approach which is taken from [9]. Obviously that some of iterated Ito stochastic integrals of the form (4) or (5) are connected by linear relations, while this is not the case for integrals of the form (6), (7). In this sense, the total quantity of stochastic integrals of the form (6) or (7) is minimal. Futhermore, in this article we construct two new forms of the Taylor–Stratonovich expansion (the so-called unified Taylor–Stratonovich expansions [9]) such that every term (if $k > 0$) contains as a multiplier an iterated Stratonovich stochastic integral of one of two types

$$\int\limits_t^{*s} (t - t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{8}$$

$$\int\limits_t^{*s} (s - t_k)^{l_k} \ldots \int\limits_t^{*t_2} (s - t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{9}$$

where $l_1, \ldots, l_k = 0, 1, 2, \ldots$, $i_1, \ldots, i_k = 1, \ldots, m$, and $k = 1, 2, \ldots$

It is not difficult to see that for the sets of iterated Stratonovich stochastic integrals (8) and (9) the property of minimality (see above) also holds as for the sets of iterated Ito stochastic integrals (6), (7).

As we noted above, the main problem in implementation of high-order strong numerical methods for Ito SDEs is the mean-square approximation of iterated stochastic integrals (4)–(9). Obviously, these stochastic integrals are particular cases of the stochastic integrals (2), (3).

Taking into account the results of [12]-[65] and the minimality of the sets of stochastic integrals of the forms (6)–(9), we conclude that the unified Taylor–Ito and Taylor–Stratonovich expansions based on the iterated stochastic integrals (6)–(9) may be useful for constructing of high-order strong numerical methods with the orders of convergence 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, ... for Ito SDEs.

2. Integration Order Replacement Technique for Iterated Ito Stochastic Integrals

Let $f_\tau, \tau \in [0,T]$ be a scalar standard Wiener process that is F_τ-measurable for every $\tau \in [0,T]$. We introduce a class $\mathfrak{M}_2([t,T])$ $(t \ge 0)$ of random functions $\xi(\tau,\omega) \stackrel{\sf def}{=} \xi_\tau : [t,T] \times \Omega \to \mathbb{R}^1$ having the following properties: these functions are measurable with respect to the pair (τ,ω) of variables, F_τ-measurable for every $\tau \in [t,T]$, and satisfy the conditions

$$\int\limits_t^T \mathsf{M}\left\{\xi_\tau^2\right\} d\tau < \infty$$

and $\mathsf{M}\left\{\xi_\tau^2\right\} < \infty$ for any $\tau \in [t,T]$.

On the class $\mathfrak{M}_2([t,T])$ $(t \ge 0)$, we introduce the Hilbert norm

$$\|\xi\|_{2,T,t} = \left(\int\limits_t^T \mathsf{M}\left\{\xi_\tau^2\right\} d\tau\right)^{1/2}.$$

Let $\left\{\tau_j^{(N)}\right\}_{j=1}^N$ be a partition of the interval $[t,T]$ such that

$$t = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \Delta_N = \max_{0\le j\le N-1} \left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0 \quad \text{as} \quad N \to \infty. \tag{10}$$

Let $\xi^{(N)}(\tau,\omega)$ be a sequence of step functions from the space $\mathfrak{M}_2([t,T])$ defined as follows

$$\xi^{(N)}(\tau,\omega) = \xi_j(\omega) \quad \text{w. p. 1} \quad \text{for} \quad \tau \in \left[\tau_j^{(N)}, \tau_{j+1}^{(N)}\right), \quad j = 0,1,\ldots,N-1,$$

where here and further w. p. 1 means with probability 1.

It is known [1] that for any function $\xi_\tau \in \mathfrak{M}_2([t,T])$ there exists a sequence $\xi^{(N)}(\tau,\omega) \in \mathfrak{M}_2([t,T])$, which converges to the function ξ_τ in the sence of norm $\|\cdot\|_{2,T,t}$.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}(\tau_j^{(N)},\omega)\left(f(\tau_{j+1}^{(N)},\omega) - f(\tau_j^{(N)},\omega)\right) \stackrel{\sf def}{=} \int\limits_t^T \xi_\tau df_\tau \tag{11}$$

is called [1] the Ito stochastic integral of a function $\xi_\tau \in \mathfrak{M}_2([t,T])$. Here $\xi^{(N)}(\tau,\omega)$ is an arbitrary sequence of step functions from the class $\mathfrak{M}_2([t,T])$ converging to the function $\xi(\tau,\omega)$ in the sense of norm $\|\cdot\|_{2,T,t}$, i.e.

$$\lim_{N\to\infty} \int\limits_t^T \mathsf{M}\left\{\left|\xi^{(N)}(\tau,\omega) - \xi(\tau,\omega)\right|^2\right\} d\tau = 0. \tag{12}$$

We introduce the class $\mathfrak{Q}_m([t,T])$ $(t \ge 0)$ of Ito processes η_τ, $\tau \in [t,T]$ of the form

$$\eta_\tau = \eta_t + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s df_s, \tag{13}$$

where $(a_\tau)^m, (b_\tau)^m \in \mathfrak{M}_2([t,T])$ and

$$\lim_{s\to\tau} \mathsf{M}\left\{|b_s - b_\tau|^4\right\} = 0$$

for all $\tau \in [t,T]$.

Let $C^{2,1}(\mathbb{R}^1 \times [t,T])$ $(t \geq 0)$ be the space of functions $F(x,\tau) : \mathbb{R}^1 \times [t,T] \to \mathbb{R}^1$ with the following property: these functions are twice differentiable in x and have one derivative in τ. Moreover, all these derivatives are bounded.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j^{(N)}} + \eta_{\tau_{j+1}^{(N)}}\right), \tau_j^{(N)}\right)\left(f_{\tau_{j+1}^{(N)}} - f_{\tau_j^{(N)}}\right) \stackrel{\text{def}}{=} \int\limits_t^{*T} F(\eta_\tau, \tau) df_\tau \tag{14}$$

is called [69] the Stratonovich stochastic integral of the process $F(\eta_\tau, \tau)$, $\tau \in [t,T]$ $(t \geq 0)$, where $F(x,\tau) \in C^{2,1}(\mathbb{R}^1 \times [t,T])$. We apply in the formula (14) the same notations as in the formula (11).

It is known [69] (also see [2]) that under proper conditions, the following relation holds

$$\int\limits_t^{*T} F(\eta_\tau, \tau) df_\tau = \int\limits_t^{T} F(\eta_\tau, \tau) df_\tau + \frac{1}{2}\int\limits_t^{T} \frac{\partial F}{\partial x}(\eta_\tau, \tau) b_\tau d\tau \quad \text{w. p. 1.} \tag{15}$$

If the Wiener processes in the formulas (13) and (14) are independent, then

$$\int\limits_t^{*T} F(\eta_\tau, \tau) df_\tau = \int\limits_t^{T} F(\eta_\tau, \tau) df_\tau \quad \text{w. p. 1.} \tag{16}$$

Note that a possible variant of conditions providing the correctness of the formulas (15) and (16) consists of the following conditions

$$\eta_\tau \in \mathfrak{Q}_4([t,T]), \quad F(\eta_\tau, \tau) \in \mathfrak{M}_2([t,T]), \quad \text{and} \quad F(x,\tau) \in C^{2,1}(\mathbb{R}^1 \times [t,T]).$$

Note that if $F(x,\tau) = F_1(x)F_2(\tau)$, then the smoothness condition $F(x,\tau) \in C^{2,1}(\mathbb{R}^1 \times [t,T])$ can be weakened. Namely, it suffices to replace the condition with respect to τ by continuity with respect to this variable.

A theorem allowing the change of order of integration in the iterated Ito stochastic integrals has been proved in [66]-[68] (also see [33]). In what follows, we apply this theorem; let us cite its exact formulation and the notations.

It is well known that the Ito stochastic integral exists in the mean-square sense (see (11)), if the stochastic process $\xi(\tau,\omega) \in \mathfrak{M}_2([0,T])$, that is, perhaps this process does not satisfy the property of the mean-square continuity on the interval $[0,T]$. Let us formulate the theorem on integration order replacement for the special class of iterated Ito stochastic integrals. At the same time, the condition of the mean-square continuity of integrand in the innermost stochastic integral will be significant.

Let $\mathfrak{S}_2([t,T])$ $(t \geq 0)$ be the class of functions $\xi : [t,T] \times \Omega \to \mathbb{R}^1$, which satisfy the conditions:

1. $\xi_\tau \in \mathfrak{M}_2([t,T])$.

2. ξ_τ is the mean-square continuous stochastic process at the interval $[t,T]$.

Let us introduce the following class of iterated Ito stochastic integrals

$$J[\phi,\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_1(t_1)\ldots \int\limits_t^{t_{k-1}} \psi_k(t_k) \int\limits_t^{t_k} \phi_\tau dw_\tau^{(k+1)} dw_{t_k}^{(k)} \ldots dw_{t_1}^{(1)},$$

where $\phi_\tau \in \mathfrak{S}_2([t,T])$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$, here and further $w_\tau^{(l)} = f_\tau$ or $w_\tau^{(l)} = \tau$ for $\tau \in [t,T]$ $(l=1,\ldots,k+1)$, $(\psi_1,\ldots,\psi_k) \stackrel{\rm def}{=} \psi^{(k)}$, $\psi^{(1)} \stackrel{\rm def}{=} \psi_1$.

In [69] Stratonovich introduced the definition of the so-called combined stochastic integral for the specific class of integrated processes. Taking this definition as a foundation, let us consider the following construction of stochastic integral

$$\underset{N\to\infty}{\rm l.i.m.} \sum_{j=0}^{N-1} \phi_{\tau_j} \left(f_{\tau_{j+1}} - f_{\tau_j}\right) \theta_{\tau_{j+1}} \stackrel{\rm def}{=} \int\limits_t^T \phi_\tau df_\tau \theta_\tau, \tag{17}$$

where $\phi_\tau,\ \theta_\tau \in \mathfrak{S}_2([t,T])$, $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies the condition (10) (for simplicity we write here and sometimes further τ_j instead of $\tau_j^{(N)}$).

Further, we will use integrals of the type (17) ($\phi_\tau \in \mathfrak{S}_2([t,T])$ and θ_τ from a little bit narrower class of stochastic processes than $\mathfrak{S}_2([t,T])$) for formulation the theorem on integration order replacement for iterated Ito stochastic integrals $J[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$.

Note that under the appropriate conditions the following properties of stochastic integrals defined by the formula (17) can be proved

$$\int\limits_t^T \phi_\tau df_\tau g(\tau) = \int\limits_t^T \phi_\tau g(\tau) df_\tau \quad \text{w. p. 1},$$

where $g(\tau)$ is a continuous nonrandom function at the interval $[t,T]$,

$$\int\limits_t^T \left(\alpha\phi_\tau + \beta\psi_\tau\right) df_\tau \theta_\tau = \alpha \int\limits_t^T \phi_\tau df_\tau \theta_\tau + \beta \int\limits_t^T \psi_\tau df_\tau \theta_\tau \quad \text{w. p. 1},$$

$$\int\limits_t^T \phi_\tau df_\tau \left(\alpha\theta_\tau + \beta\psi_\tau\right) = \alpha \int\limits_t^T \phi_\tau df_\tau \theta_\tau + \beta \int\limits_t^T \phi_\tau df_\tau \psi_\tau \quad \text{w. p. 1},$$

where $\alpha,\ \beta \in \mathbb{R}^1$. At that, we suppose that the stochastic processes ϕ_τ, θ_τ, and ψ_τ are such that all integrals (included in the mentioned properties) exist.

Let us define the stochastic integrals $\hat{I}[\psi^{(k)}]_{T,s}$, $k\ge 1$ of the form

$$\hat{I}[\psi^{(k)}]_{T,s} = \int\limits_s^T \psi_k(t_k) dw_{t_k}^{(k)} \int\limits_{t_k}^T \psi_{k-1}(t_{k-1}) dw_{t_{k-1}}^{(k-1)} \ldots \int\limits_{t_2}^T \psi_1(t_1) dw_{t_1}^{(1)}$$

in accordance with the definition (17) by the following recurrence relation

$$\hat{I}[\psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_k(\tau_l)\left(w_{\tau_{l+1}}^{(k)}-w_{\tau_l}^{(k)}\right)\hat{I}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \tag{18}$$

where $k\ge 1,\ \ \hat{I}[\psi^{(0)}]_{T,s}\stackrel{\rm def}{=}1$, and $[s,T]\subseteq[t,T]$.

Then, we will define the iterated stochastic integral $\hat{J}[\phi,\psi^{(k)}]_{T,t},\ k\ge 1$

$$\hat{J}[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T \phi_s dw_s^{(k+1)}\hat{I}[\psi^{(k)}]_{T,s}$$

similarly in accordance with the definition (17)

$$\hat{J}[\phi,\psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l}\left(w_{\tau_{l+1}}^{(k+1)}-w_{\tau_l}^{(k+1)}\right)\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}.$$

Let us formulate the theorem on integration order replacement for iterated Ito stochastic integrals.

Theorem 1 [66]-[68] (also see [33]). *Suppose that $\phi_\tau\in\mathfrak{S}_2([t,T])$ and every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the stochastic integral $\hat{J}[\phi,\psi^{(k)}]_{T,t}$ $(k\ge 1)$ exists and*

$$J[\phi,\psi^{(k)}]_{T,t}=\hat{J}[\phi,\psi^{(k)}]_{T,t}\quad \text{w. p. } 1.$$

Let us consider some propositions related to Theorem 1.

Proposition 1 [66]-[68] (also see [33]). *Let the conditions of Theorem 1 are fulfilled and $h(\tau)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$\int\limits_t^T \phi_\tau dw_\tau^{(k+1)}h(\tau)\hat{I}[\psi^{(k)}]_{T,\tau}=\int\limits_t^T \phi_\tau h(\tau)dw_\tau^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau}\quad \text{w. p. } 1, \tag{19}$$

and the integrals on the left-hand side of (19) as well as on the right-hand side of (19) exist.

Proposition 2 [66]-[68] (also see [33]). *Under the conditions of Theorem 1 the following equality is satisfied*

$$\int\limits_t^T h(t_1)\int\limits_t^{t_1}\phi_\tau dw_\tau^{(k+2)}dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}=$$

$$=\int\limits_t^T \phi_\tau dw_\tau^{(k+2)}\int\limits_\tau^T h(t_1)dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}\quad \text{w. p. } 1. \tag{20}$$

Moreover, the stochastic integrals in (20) exist.

Using the integration order replacement technique for iterated Ito stochastic integrals (Theorem 1), we can obtain different equalities for iterated Ito stochastic integrals. At that, the mentioned technique is essentially simpler (for the specific class of Ito processes which are the iterated Ito stochastic integrals) in application than the Ito formula. Let us consider two examples on application of the integration order replacement technique for iterated Ito stochastic integrals.

Example 1. *Using Theorem* 1 *and Proposition* 1, *we obtain*

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}df_{t_2}dt_3=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}\int\limits_{t_2}^T dt_3=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}(T-t_2)=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T (T-t_2)df_{t_2}=$$

$$=\int\limits_t^T (T-t_2)\int\limits_t^{t_2} df_{t_1}df_{t_2}\quad \text{w. p. 1.}$$

Example 2. *Using Theorem* 1 *and Proposition* 1, *we obtain*

$$\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2df_{t_3}dt_4=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}\int\limits_{t_3}^T dt_4=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}(T-t_3)=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T (T-t_3)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2df_{t_3}=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}\int\limits_{t_1}^{t_3} dt_2\right)df_{t_3}=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}(t_3-t_1)\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} (t_3-t_1)df_{t_1}\right)df_{t_3}=\int\limits_t^T (T-t_2)\int\limits_t^{t_2} (t_2-t_1)df_{t_1}df_{t_2}\quad \text{w. p. 1.}$$

Let us apply Theorem 1 to deriving of one propetry for Ito stochastic integrals.

Lemma 1. *Let* $h(\tau), g(\tau), G(\tau) : [t,s]\to\mathbb{R}^1$ *be continuous nonrandom functions at the interval* $[t,s]$ *and let* $G(\tau)$ *be a antiderivative of the function* $g(\tau)$. *Furthermore, let* $\xi_\tau\in\mathfrak{S}_2([t,s])$. *Then*

$$\int\limits_t^s g(\tau)\int\limits_t^\tau h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau=\int\limits_t^s (G(s)-G(\theta))h(\theta)\int\limits_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} \tag{21}$$

w. p. 1, *where* $i,j=1,2$ *and* $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ *are independent standard Wiener processes that are* F_τ*–measurable for all* $\tau\in[t,s]$.

Proof. Applying Theorem 1 twice and Proposition 1, we get the following relations

$$\int_t^s g(\tau)\int_t^\tau h(\theta)\int_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau = \int_t^s \xi_u d\mathbf{f}_u^{(i)}\int_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int_\theta^s g(\tau)d\tau =$$

$$= G(s)\int_t^s \xi_u d\mathbf{f}_u^{(i)}\int_u^s h(\theta)d\mathbf{f}_\theta^{(j)} - \int_t^s \xi_u d\mathbf{f}_u^{(i)}\int_u^s G(\theta)h(\theta)d\mathbf{f}_\theta^{(j)} =$$

$$= G(s)\int_t^s h(\theta)\int_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} - \int_t^s G(\theta)h(\theta)\int_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} =$$

$$= \int_t^s (G(s)-G(\theta))h(\theta)\int_t^\theta \xi_u d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} \quad \text{w. p. 1.} \tag{22}$$

The proof of Lemma 1 is completed. Let us consider an analogue of Lemma 1 for Stratonovich stochastic integrals.

Lemma 2 [9]. *Let $h(\tau), g(\tau), G(\tau) : [t,s] \to \mathbb{R}^1$ be continuous nonrandom functions at the interval $[t,s]$ and let $G(\tau)$ be a antiderivative of the function $g(\tau)$. Let $\xi_\tau^{(l)} \in \mathfrak{Q}_4([t,s])$ and*

$$\xi_\tau^{(l)} = \int_t^\tau a_u du + \int_t^\tau b_u d\mathbf{f}_u^{(l)}, \quad l = 1,2.$$

Then

$$\int_t^s g(\tau)\int_t^{*\tau} h(\theta)\int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau = \int_t^{*s} (G(s)-G(\theta))h(\theta)\int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} \tag{23}$$

w. p. 1, where $i,j,l = 1,2$ and $\mathbf{f}_\tau^{(1)}$, $\mathbf{f}_\tau^{(2)}$ are independent standard Wiener processes that are F_τ–measurable for all $\tau \in [t,s]$.

Proof. Under the conditions of Lemma 2, we can apply equalities (15) and (16) with $F(x,\theta) \equiv xh(\theta)$,

$$\eta_\theta = \int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)},$$

since the function $xh(\theta)$ is sufficiently smooth and the following obvious inclusions hold: $\eta_\theta \in \mathfrak{Q}_4([t,s])$ and $\eta_\theta h(\theta) \in \mathfrak{M}_2([t,s])$. Thus, we have the equalities

$$\int_t^{*\tau} h(\theta)\int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} = \int_t^\tau h(\theta)\int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}}\int_t^\tau h(\theta)\xi_\theta^{(l)}d\theta, \tag{24}$$

$$\int_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)} = \int_t^\theta \xi_u^{(l)}d\mathbf{f}_u^{(i)} + \frac{1}{2}\mathbf{1}_{\{l=i\}}\int_t^\theta b_u du \tag{25}$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of a set A. Substituting formulas (24) and (25) into the left-hand side of equality (23) and applying Theorem 1 twice and Proposition 1, we get the following relations

$$\int\limits_t^s g(\tau)\int\limits_t^{*\tau} h(\theta)\int\limits_t^{*\theta} \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}d\tau =$$

$$=\int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau +$$

$$+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\int\limits_\theta^s g(\tau)d\tau+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta\int\limits_\theta^s g(\tau)d\tau =$$

$$=G(s)\left(\int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta+\right.$$

$$\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)d\mathbf{f}_\theta^{(j)}\right)-$$

$$-\left(\int\limits_t^s \xi_u^{(l)}d\mathbf{f}_u^{(i)}\int\limits_u^s G(\theta)h(\theta)d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)}d\theta+\right.$$

$$\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s b_u du\int\limits_u^s h(\theta)G(\theta)d\mathbf{f}_\theta^{(j)}\right)=$$

$$=G(s)\left(\int\limits_t^s h(\theta)\int\limits_t^\theta \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s h(\theta)\xi_\theta^{(l)}d\theta+\right.$$

$$\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s h(\theta)\int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right)-$$

$$-\left(\int\limits_t^s G(\theta)h(\theta)\int\limits_t^\theta \xi_u^{(l)}d\mathbf{f}_u^{(i)}d\mathbf{f}_\theta^{(j)}+\frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^s G(\theta)h(\theta)\xi_\theta^{(l)}d\theta+\right.$$

$$\left.+\frac{1}{2}\mathbf{1}_{\{l=i\}}\int\limits_t^s h(\theta)G(\theta)\int\limits_t^\theta b_u du d\mathbf{f}_\theta^{(j)}\right) \tag{26}$$

w. p. 1. Applying successively the formulas (24), (25) together with the formula (24) in which $h(\theta)$ replaced by $G(\theta)h(\theta)$ as well as the relation (26), we obtain the equality (23).

3. The Taylor–Ito Expansion

In this section, we cite the Taylor-Ito expansion [7] and introduce some necessary notations. At that, we will use the original notation introduced by the author of this paper.

Let $\mathfrak{L}$ be the class of functions $R(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^1$ with the following property: these functions are twice continuously differentiable in $\mathbf{x}$ and have one continuous derivative in t. We consider the following operators on the space $\mathfrak{L}$

$$LR(\mathbf{x},t)=\frac{\partial R}{\partial t}(\mathbf{x},t)+\sum_{i=1}^{n}a^{(i)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x},t)+$$

$$+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2 R}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}}(\mathbf{x},t), \tag{27}$$

$$G_0^{(i)}R(\mathbf{x},t)=\sum_{j=1}^{n}B^{(ji)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x},t),\quad i=1,\ldots,m. \tag{28}$$

By the Ito formula, we have the equality

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\int\limits_t^s LR(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^{m}\int\limits_t^s G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)} \tag{29}$$

w. p. 1, where $0\le t<s\le T$. In the formula (29) it is assumed that the functions $\mathbf{a}(\mathbf{x},t)$, $B(\mathbf{x},t)$, and $R(\mathbf{x},t)$ satisfy the following condition: $LR(\mathbf{x}_\tau,\tau)$, $G_0^{(i)}R(\mathbf{x}_\tau,\tau)\in\mathfrak{M}_2([0,T])$ for $i=1,\ldots,m$.

Introduce the following notation

$${}^{(k)}A=\Biggl\Vert A^{(i_1\ldots i_k)}\Biggr\Vert_{i_1=1,\ldots,i_k=1}^{m_1\ \ldots\ m_k},\quad m_1,\ldots,m_k\ge 1, \tag{30}$$

$${}^{(k+l)}A\stackrel{l}{\cdot}{}^{(l)}B^{(k)}=\begin{cases}\Biggl\Vert \sum\limits_{i_1=1}^{m_1}\ldots\sum\limits_{i_l=1}^{m_l}A^{(i_1\ldots i_{k+l})}B^{(i_1\ldots i_l)}\Biggr\Vert_{i_{l+1}=1,\ldots,i_{l+k}=1}^{m_{l+1}\ \ldots\ m_{l+k}} & \text{for}\quad k\ge 1\\ \sum\limits_{i_1=1}^{m_1}\ldots\sum\limits_{i_l=1}^{m_l}A^{(i_1\ldots i_l)}B^{(i_1\ldots i_l)} & \text{for}\quad k=0\end{cases},$$

$$\Biggl\Vert A_{k+1}D_k^{(i_k)}A_k\ldots A_2D_1^{(i_1)}A_1R(\mathbf{x},t)\Biggr\Vert_{i_1=1,\ldots,i_k=1}^{m_1\ \ldots\ m_k}={}^{(k)}A_{k+1}D_kA_k\ldots A_2D_1A_1R(\mathbf{x},t), \tag{31}$$

where A_p and $D_q^{(i_q)}$ are operators defined on the space $\mathfrak{L}$ for $p=1,\ldots,k+1$, $q=1,\ldots,k$, and $i_q=1,\ldots,m_q$. It is assumed that the left-hand side of (31) exists. The symbol $\stackrel{0}{\cdot}$ is treated as the usual multiplication. If $m_l=0$ in (30) for some $l\in\{1,\ldots,k\}$, then the right-hand side of (30) is treated as

$$\Bigl\Vert A^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_k)}\Bigr\Vert_{i_1=1,\ldots,i_{l-1}=1,i_{l+1}=1,\ldots,i_k=1}^{m_1\ \ldots\ m_{l-1}\ \ m_{l+1}\ \ldots\ m_k},$$

(shortly, ${}^{(k-1)}A$).

We also introduce the following notation

$$\Bigl\Vert Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\Bigr\Vert_{i_1=\lambda_1,\ldots,i_l=\lambda_l}^{m\lambda_1\ \ldots\ m\lambda_l}\ \stackrel{\sf def}{=}\ {}^{(p_l)}Q_{\lambda_l}\ldots Q_{\lambda_1}R(\mathbf{x},t),$$

$${}^{(p_k)}J_{(\lambda_k\ldots\lambda_1)s,t}=\Bigl\Vert J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}\Bigr\Vert_{i_1=\lambda_1,\ldots,i_k=\lambda_k}^{m\lambda_1\ \ldots\ m\lambda_k},$$

$$M_k=\Bigl\{(\lambda_k,\ldots,\lambda_1):\lambda_l=1\ \text{or}\ \lambda_l=0;\ l=1,\ldots,k\Bigr\},\quad k\ge 1,$$

$$J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}=\int\limits_t^s\ldots\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(i_k)}\ldots d\mathbf{w}_{t_k}^{(i_1)},\quad k\ge 1,$$

where $\lambda_l=1$ or $\lambda_l=0$, $Q_{\lambda_l}^{(i_l)}=L$ and $i_l=0$ for $\lambda_l=0$, $Q_{\lambda_l}^{(i_l)}=G_0^{(i_l)}$ and $i_l=1,\ldots,m$ for $\lambda_l=1$,

$$p_l=\sum_{j=1}^{l}\lambda_j\quad\text{for}\quad l=1,\ldots,r+1,\quad r\in\mathbb{N},$$

$\mathbf{w}_\tau^{(i)}$ $(i=1,\ldots,m)$ are F_τ-measurable for all $\tau\in[0,T]$ independent standard Wiener processes and $\mathbf{w}_\tau^{(0)}=\tau$.

Applying the formula (29) to the process $R(\mathbf{x}_s,s)$ repeatedly, we obtain the following Taylor–Ito expansion [7]

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}{}^{(p_k)}Q_{\lambda_k}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_k}{\cdot}{}^{(p_k)}J_{(\lambda_k\ldots\lambda_1)s,t}+(D_{r+1})_{s,t}\tag{32}$$

w. p. 1, where

$$(D_{r+1})_{s,t}=\sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}}\int\limits_t^s\ldots\left(\int\limits_t^{t_2}{}^{(p_{r+1})}Q_{\lambda_{r+1}}\ldots Q_{\lambda_1}R(\mathbf{x}_{t_1},t_1)\stackrel{\lambda_{r+1}}{\cdot}d\mathbf{w}_{t_1}\right)\ldots\stackrel{\lambda_1}{\cdot}d\mathbf{w}_{t_{r+1}}.\tag{33}$$

It is assumed that the right-hand sides of (32), (33) exist.

A possible variant of the conditions under which the right-hand sides of (32), (33) exist is as follows

(i) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\in\mathfrak{L}$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r}M_g$;

(ii) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_\tau,\tau)\in\mathfrak{M}_2([0,T])$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r+1}M_g$.

Let us write the expansion (32) in the another form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\ \sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}\ \sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}+$$

$$+\left(D_{r+1}\right)_{s,t}\quad \text{w. p. 1.}$$

Denote

$$G_{rk}=\Biggl\{(\lambda_k,\ldots,\lambda_1):\ r+1\le 2k-\lambda_1-\ldots-\lambda_k\le 2r\Biggr\},$$

$$E_{qk}=\Biggl\{(\lambda_k,\ldots,\lambda_1):\ 2k-\lambda_1-\ldots-\lambda_k=q\Biggr\},$$

where $\lambda_l=1$ or $\lambda_l=0$ $(l=1,\ldots,k)$.

The Taylor–Ito expansion ordered according to the order of smallness (in the mean-square sense when $s\downarrow t$) of its terms has the form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q,k=1}^{r}\ \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}}\ \sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}+$$

$$+\left(H_{r+1}\right)_{s,t}\quad \text{w. p. 1,} \tag{34}$$

where

$$\left(H_{r+1}\right)_{s,t}=\sum_{k=1}^{r}\ \sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}}\ \sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}Q_{\lambda_k}^{(i_k)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{(i_k\ldots i_1)}+$$

$$+\left(D_{r+1}\right)_{s,t}.$$

4. The First Form of the Unified Taylor–Ito Expansion

In this section, we transform the right-hand side of (32) with the help of Theorem 1 and Lemma 1 to a representation including iterated Ito stochastic integrals of the form (7).

Denote

$$I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}=\int\limits_t^s (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad \text{for}\quad k\ge 1 \tag{35}$$

and

$$I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}=1\quad \text{for}\quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m$. Moreover, let

$$^{(k)}I_{l_1\ldots l_{k}s,t}=\left\|I_{l_1\ldots l_{k}s,t}^{(i_1\ldots i_k)}\right\|_{i_1,\ldots,i_k=1}^{m},$$

$$G_p^{(i)}\stackrel{\sf def}{=}\frac{1}{p}\left(G_{p-1}^{(i)}L-LG_{p-1}^{(i)}\right),\quad p=1,2,\ldots,\quad i=1,\ldots,m, \tag{36}$$

where L and $G_0^{(i)}$, $i=1,\ldots,m$, are determined by the equalities (27), (28). Denote

$$A_q\stackrel{\sf def}{=}\left\{(k,j,l_1,\ldots,l_k):\ k+j+\sum_{p=1}^{k}l_p=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\},$$

$$\left\|G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}G_{l_1}\ldots G_{l_k}L^jR(\mathbf{x},t),$$

$$L^jR(\mathbf{x},t)\stackrel{\sf def}{=}\begin{cases}\underbrace{L\ldots L}_{j}R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases}.$$

Theorem 2. *Let conditions* (i), (ii) *be satisfied. Then for any $s,t\in[0,T]$ such that $s>t$ and for any positive integer r, the following expansion takes place w. p.* 1

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in{\rm A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_{k}s,t}^{(i_1\ldots i_k)}+$$

$$+\left(D_{r+1}\right)_{s,t}, \tag{37}$$

where $\left(D_{r+1}\right)_{s,t}$ has the form (33).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q}{}^{(p_q)}Q_{\lambda_q}\ldots Q_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_q}{\cdot}{}^{(p_q)}J_{(\lambda_q\ldots\lambda_1)s,t}=$$

$$=\sum_{(k,j,l_1,\ldots,l_k)\in{\rm A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_{k}s,t}^{(i_1\ldots i_k)} \tag{38}$$

w. p. 1. The equality (38) is valid for $q=1$. Assume that (38) is valid for some $q>1$. In this case, using the induction hypothesis, we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}} {}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{p_{q+1}}{\cdot} {}^{(p_{q+1})}J_{(\lambda_1\ldots\lambda_{q+1})s,t} =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^s \sum_{(\lambda_q,\ldots,\lambda_1)\in M_q} \left({}^{(p_{q+1})}Q_{\lambda_1}\ldots Q_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{p_q}{\cdot} {}^{(p_q)}J_{(\lambda_1\ldots\lambda_q)\theta,t}\right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^s \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \frac{(\theta-t)^j}{j!}\times$$

$$\times\left({}^{(k+\lambda_{q+1})}G_{l_1}\ldots G_{l_k}L^jQ_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{k}{\cdot} {}^{(k)}I_{l_1\ldots l_{k\,s,t}}\right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \left({}^{(k)}G_{l_1}\ldots G_{l_k}L^{j+1}R(\mathbf{x}_t,t) \overset{k}{\cdot} \int\limits_t^s \frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta + \right.$$

$$\left. + \left({}^{(k+1)}G_{l_1}\ldots G_{l_k}L^jG_0R(\mathbf{x}_t,t) \overset{k}{\cdot} \int\limits_t^s \frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}\right) \overset{1}{\cdot} d\mathbf{f}_\theta\right) \tag{39}$$

w. p. 1.

Using Lemma 1, we obtain

$$\int\limits_t^s \frac{(\theta-t)^j}{j!}{}^{(k)}I_{l_1\ldots l_{k\,\theta,t}}d\theta =$$

$$= \frac{1}{(j+1)!}\begin{cases}(s-t)^{j+1} & \text{for } k=0\\ \\ (s-t)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k\,s,t}} - (-1)^{j+1}\cdot{}^{(k)}I_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0\end{cases} \tag{40}$$

w. p. 1. In addition (see (35)), we get

$$\int\limits_t^s \frac{(\theta-t)^j}{j!} I^{(i_1\ldots i_k)}_{l_1\ldots l_{k\,\theta,t}} d\mathbf{f}^{(i_{k+1})}_\theta = \frac{(-1)^j}{j!} I^{(i_1\ldots i_k i_{k+1})}_{l_1\ldots l_k j_{s,t}} \tag{41}$$

in the notations just introduced. Substitute the relations (40) and (41) into the formula (39). Grouping summands of the obtained expression with equal lower indices at iterated Ito stochastic integrals and using (36) and the equality

$$G^{(i)}_pR(\mathbf{x},t) = \frac{1}{p!}\sum_{q=0}^p(-1)^qC^q_pL^qG^{(i)}_0L^{p-q}R(\mathbf{x},t), \quad \text{where} \quad C^q_p = \frac{p!}{q!(p-q)!}, \tag{42}$$

(this equality follows from (36)), we note that the obtained expression is equal to

$$\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}} \frac{(s-t)^j}{j!} {}^{(k)}G_{l_1}\ldots G_{l_k}L^j\{\eta_t\} \stackrel{k}{\cdot} {}^{(k)}I_{l_1\ldots l_k s,t}$$

w. p. 1. Summing the equalities (38) for $q=1,2,\ldots,r$ and applying the formula (32), we obtain the expression (37). The proof is completed.

Let us order terms of the expansion (37) according to their smallness orders as $s\downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}+$$

$$+\left(H_{r+1}\right)_{s,t}\quad \text{w. p. 1,} \tag{43}$$

where

$$\left(H_{r+1}\right)_{s,t}=\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}+\left(D_{r+1}\right)_{s,t},$$

$$\mathrm{D}_q=\left\{(k,j,l_1,\ldots,l_k):k+2\left(j+\sum_{p=1}^{k}l_p\right)=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\}, \tag{44}$$

(45)

$$\mathrm{U}_r=\left\{(k,j,l_1,\ldots,l_k):k+j+\sum_{p=1}^{k}l_p\le r,\ k+2\left(j+\sum_{p=1}^{k}l_p\right)\ge r+1;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\},$$

and $\left(D_{r+1}\right)_{s,t}$ has the form (33). Note that the remainder term $\left(H_{r+1}\right)_{s,t}$ in (43) has a higher order of smallness in the mean-square sense as $s\downarrow t$ than the terms of the main part of expansion (43).

5. The Second Form of the Unified Taylor–Ito Expansion

Consider iterated Ito stochastic integrals of the form

$$J_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}=\int\limits_t^s(s-t_k)^{l_k}\ldots\int\limits_t^{t_2}(s-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad \text{for}\quad k\ge 1$$

and

$$J_{l_1\ldots l_k s,t}^{(i_1\ldots i_k)}=1\quad \text{for}\quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m$.

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$I_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}=\sum_{j_1=0}^{l_1}\ldots\sum_{j_k=0}^{l_k}\prod_{g=1}^{k}C_{l_g}^{j_g}(t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\ J_{j_1\ldots j_k\,s,t}^{(i_1\ldots i_k)}\quad \text{w. p. 1}, \tag{46}$$

where

$$C_l^k=\frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Ito expansion of the process $\eta_s = R(\mathbf{x}_s, s)$ can be constructed either using the iterated stochastic integrals $I_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}$ similarly to the previous section or using the iterated stochastic integrals $J_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}$. This is the main subject of this section.

Denote

$$\left\|J_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\text{def}}{=}{}^{(k)}J_{l_1\ldots l_k\,s,t},$$

$$\left\|L^jG_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\text{def}}{=}{}^{(k)}L^jG_{l_1}\ldots G_{l_k}R(\mathbf{x},t).$$

Theorem 3. *Let conditions* (i), (ii) *be satisfied. Then for any $s,t\in[0,T]$ such that $s>t$ and for any positive integer r, the following expansion is valid w. p.* 1

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}R(\mathbf{x}_t,t)\ J_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}+$$

$$+\left(D_{r+1}\right)_{s,t}, \tag{47}$$

where $\left(D_{r+1}\right)_{s,t}$ has the form (33).

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}L^jG_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}R(\mathbf{x}_t,t)\ J_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}=$$

$$\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}\quad\text{w. p. 1} \tag{48}$$

for $q=1,2,\ldots,r$. To check (48), substitute the expression (46) into the right-hand side of (48) and then use the formulas (36), (42).

Let us rank terms of the expansion (47) according to their orders of smallness in the mean-square sense as $s\downarrow t$

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t, t)\ J_{l_1\ldots l_k\, s,t}^{(i_1\ldots i_k)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1},$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} L^j G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} R(\mathbf{x}_t, t)\ J_{l_1\ldots l_k\, s,t}^{(i_1\ldots i_k)} + \left(D_{r+1}\right)_{s,t}.$$

The term $\left(D_{r+1}\right)_{s,t}$ has the form (33); the terms D_q and U_r have the forms (44) and (45), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Ito expansion (32) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s, s)$ as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$ has been proved in [2] (Proposition 5.9.2). Since expansions (37) and (47) are obtained from the Taylor–Ito expansion (32) without any additional conditions, the truncated expansions (37) and (47) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of [2] (Proposition 5.9.2) converge to the process $R(\mathbf{x}_s, s)$ w. p. 1 as $r \to \infty$ for all $s, t \in [0, T]$ such that $s > t$ and $T < \infty$.

6. The Taylor–Stratonovich Expansion

In this section, we cite the Taylor–Stratonovich expansion [7] and introduce some necessary notations. At that, we will use the original notations introduced by the author of this paper.

Assume that $LR(\mathbf{x}_\tau, \tau)$, $G_0^{(i)} R(\mathbf{x}_\tau, \tau) \in \mathfrak{M}_2([0, T])$ for $i = 1, \ldots, m$, and consider the Ito formula in the form (29).

In addition, suppose that the function $G_0^{(i)} R(\mathbf{x}, t)$ $(i = 1, \ldots, m)$ is such that the formulas (15) and (16) can be applied. In this case, the relations (15) and (16) imply that

$$\int\limits_t^s G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} = \int\limits_t^{*s} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} - \frac{1}{2} \int\limits_t^s G_0^{(i)} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\tau \tag{49}$$

w. p. 1 for $i = 1, \ldots, m$.

Using the relation (49), let us write the formula (29) in the following form

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \int\limits_t^s \bar{L} R(\mathbf{x}_\tau, \tau) d\tau + \sum_{i=1}^{m} \int\limits_t^{*s} G_0^{(i)} R(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau^{(i)} \quad \text{w. p. 1}, \tag{50}$$

where

$$\bar{L} R(\mathbf{x}, t) = LR(\mathbf{x}, t) - \frac{1}{2} \sum_{i=1}^{m} G_0^{(i)} G_0^{(i)} R(\mathbf{x}, t). \tag{51}$$

Introduce the following notation

$$\left\|D_{\lambda_l}^{(i_l)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\right\|_{i_1=\lambda_1,\ldots,i_l=\lambda_l}^{m\lambda_1\ \ldots\ m\lambda_l}\ \stackrel{\mathsf{def}}{=}\ {}^{(p_l)}D_{\lambda_l}\ldots D_{\lambda_1}R(\mathbf{x},t),$$

$${}^{(p_k)}J^*_{(\lambda_k\ldots\lambda_1)s,t}=\left\|J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}\right\|_{i_1=\lambda_1,\ldots,i_k=\lambda_k}^{m\lambda_1\ \ldots\ m\lambda_k},$$

$$M_k=\Bigl\{(\lambda_k,\ldots,\lambda_1):\lambda_l=1\ \text{or}\ \lambda_l=0;\ l=1,\ldots,k\Bigr\},\quad k\ge 1,$$

$$J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}=\int\limits_t^{*s}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_k)}\ldots d\mathbf{w}_{t_k}^{(i_1)},\quad k\ge 1,$$

where $\lambda_l=1$ or $\lambda_l=0$, $D_{\lambda_l}^{(i_l)}=\bar{L}$ and $i_l=0$ for $\lambda_l=0$, $D_{\lambda_l}^{(i_l)}=G_0^{(i_l)}$ and $i_l=1,\ldots,m$ for $\lambda_l=1$,

$$p_l=\sum_{j=1}^{l}\lambda_j\quad\text{for}\quad l=1,\ldots,r+1,\quad r\in\mathbb{N},$$

$\mathbf{w}_\tau^{(i)}$ $(i=1,\ldots,m)$ are F_τ-measurable for all $\tau\in[0,T]$ independent standard Wiener processes and $\mathbf{w}_\tau^{(0)}=\tau$.

Applying the formula (50) to the process $R(\mathbf{x}_s,s)$ repeatedly, we obtain the following Taylor–Stratonovich expansion [7]

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}{}^{(p_k)}D_{\lambda_k}\ldots D_{\lambda_1}R(\mathbf{x}_t,t)\stackrel{p_k}{\cdot}{}^{(p_k)}J^*_{(\lambda_k\ldots\lambda_1)s,t}+(D_{r+1})_{s,t}\tag{52}$$

w. p. 1, where

(53)

$$(D_{r+1})_{s,t}=\sum_{(\lambda_{r+1},\ldots,\lambda_1)\in M_{r+1}}\int\limits_t^{*s}\ldots\left(\int\limits_t^{*t_2}{}^{(p_{r+1})}D_{\lambda_{r+1}}\ldots D_{\lambda_1}R(\mathbf{x}_{t_1},t_1)\stackrel{\lambda_{r+1}}{\cdot}d\mathbf{w}_{t_1}\right)\ldots\stackrel{\lambda_1}{\cdot}d\mathbf{w}_{t_{r+1}}.$$

It is assumed that the right-hand sides of (52), (53) exist.

A possible variant of the conditions under which the right-hand sides of (52), (53) exist is as follows

(i*) $Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\in\mathfrak{L}$ for all $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r}M_g$;

(ii*)

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)-Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{y},t)\right|\le K|\mathbf{x}-\mathbf{y}|,\tag{54}$$

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)\right|\le K(1+|\mathbf{x}|),\tag{55}$$

and

$$\left|Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},t)-Q_{\lambda_l}^{(i_l)}\ldots Q_{\lambda_1}^{(i_1)}R(\mathbf{x},s)\right|\le K|t-s|^{\nu}(1+|\mathbf{x}|)$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, $t,s\in[0,T]$, $(\lambda_l,\ldots,\lambda_1)\in\bigcup\limits_{g=1}^{r+1}M_g$ and for some $\nu>0$, where $K<\infty$ is a constant, $Q_{\lambda_l}^{(i_l)}=L$ and $i_l=0$ for $\lambda_l=0$, $Q_{\lambda_l}^{(i_l)}=G_0^{(i_l)}$ and $i_l=1,\ldots,m$ for $\lambda_l=1$;

(iii*) the functions $\mathbf{a}(\mathbf{x},t)$ and $B(\mathbf{x},t)$ are measurable with respect to all of the variables and satisfy the conditions (54) and (55);

(iv*) $\mathbf{x}_0$ is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^8\right\}<\infty$.

Let us write the expansion (52) in the another form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in M_k}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_k}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)}+$$

$$+\left(D_{r+1}\right)_{s,t}\quad \text{w. p. 1.}$$

Denote

$$G_{rk}=\Biggl\{(\lambda_k,\ldots,\lambda_1):\ r+1\le 2k-\lambda_1-\ldots-\lambda_k\le 2r\Biggr\},$$

$$E_{qk}=\Biggl\{(\lambda_k,\ldots,\lambda_1):\ 2k-\lambda_1-\ldots-\lambda_k=q\Biggr\},$$

where $\lambda_l=1$ or $\lambda_l=0$ $(l=1,\ldots,k)$.

The Taylor–Stratonovich expansion ordered according to the order of smallness (in the mean-square sense when $s\downarrow t$) of its terms has the form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q,k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_k}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)}+$$

$$+\left(H_{r+1}\right)_{s,t}\quad \text{w. p. 1,} \tag{56}$$

where

$$\left(H_{r+1}\right)_{s,t}=\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_k}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\ J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)}+$$

$$+\left(D_{r+1}\right)_{s,t}.$$

7. The First Form of the Unified Taylor–Stratonovich Expansion

In this section, we transform the right-hand side of (52) with the help of Theorem 1 and Lemma 2 to a representation including iterated Stratonovich stochastic integrals of the form (8).

Denote

$$I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*s} (t-t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \quad \text{for} \quad k \ge 1 \tag{57}$$

and

$$I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} = 1 \quad \text{for} \quad k = 0,$$

where $i_1, \ldots, i_k = 1, \ldots, m$. Moreover, let

$$^{(k)}I_{l_1\ldots l_k s,t}^{*} = \left\|I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)}\right\|_{i_1,\ldots,i_k=1}^{m},$$

$$\bar{G}_p^{(i)} \stackrel{\mathsf{def}}{=} \frac{1}{p}\left(\bar{G}_{p-1}^{(i)}\bar{L} - \bar{L}\bar{G}_{p-1}^{(i)}\right), \quad p = 1, 2, \ldots, \quad i = 1, \ldots, m, \tag{58}$$

where $\bar{G}_0^{(i)} \stackrel{\mathsf{def}}{=} G_0^{(i)}$, $i = 1, \ldots, m$. The operators $\bar{L}$ and $G_0^{(i)}$, $i = 1, \ldots, m$, are determined by the equalities (27), (28), and (51). Denote

$$A_q \stackrel{\mathsf{def}}{=} \left\{(k, j, l_1, \ldots, l_k) : \ k + j + \sum_{p=1}^{k} l_p = q; \ k, j, l_1, \ldots, l_k = 0, 1, \ldots\right\},$$

$$\left\|\bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}, t)\right\|_{i_1,\ldots,i_k=1}^{m} \stackrel{\mathsf{def}}{=} {}^{(k)}\bar{G}_{l_1} \ldots \bar{G}_{l_k} \bar{L}^j R(\mathbf{x}, t),$$

$$\bar{L}^j R(\mathbf{x}, t) \stackrel{\mathsf{def}}{=} \begin{cases} \underbrace{\bar{L} \ldots \bar{L}}_{j} R(\mathbf{x}, t) & \text{for } j \ge 1 \\ \\ R(\mathbf{x}, t) & \text{for } j = 0 \end{cases}.$$

Theorem 4. *Let conditions* (i*)-(iv*) *be satisfied. Then for any* $s, t \in [0, T]$ *such that* $s > t$ *and for any positive integer* r, *the following expansion takes place w. p.* 1

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) \ I_{l_1\ldots l_k s,t}^{*(i_1\ldots i_k)} +$$

$$+ \left(D_{r+1}\right)_{s,t}, \tag{59}$$

where $(D_{r+1})_{s,t}$ *has the form* (53).

Proof. We claim that

$$\sum_{(\lambda_q,\ldots,\lambda_1)\in M_q} {}^{(p_q)}D_{\lambda_q}\ldots D_{\lambda_1}R(\mathbf{x}_t,t) \overset{p_q}{\cdot} {}^{(p_q)}J^*_{(\lambda_q\ldots\lambda_1)s,t} =$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^j R(\mathbf{x}_t,t)\ I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}} \tag{60}$$

w. p. 1. The equality (60) is valid for $q=1$. Assume that (60) is valid for some $q>1$. In this case, using the induction hypothesis, we obtain

$$\sum_{(\lambda_{q+1},\ldots,\lambda_1)\in M_{q+1}} {}^{(p_{q+1})}D_{\lambda_1}\ldots D_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{p_{q+1}}{\cdot} {}^{(p_{q+1})}J^*_{(\lambda_1\ldots\lambda_{q+1})s,t} =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^{*s} \sum_{(\lambda_q,\ldots,\lambda_1)\in M_q} \left({}^{(p_{q+1})}D_{\lambda_1}\ldots D_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{p_q}{\cdot} {}^{(p_q)}J^*_{(\lambda_1\ldots\lambda_q)\theta,t}\right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{\lambda_{q+1}\in\{1,\ 0\}} \int\limits_t^{*s} \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \frac{(\theta-t)^j}{j!}\times$$

$$\times\left({}^{(k+\lambda_{q+1})}\bar{G}_{l_1}\ldots\bar{G}_{l_k}\bar{L}^j D_{\lambda_{q+1}}R(\mathbf{x}_t,t) \overset{k}{\cdot} {}^{(k)}I^*_{l_1\ldots l_{k\,s,t}}\right) \overset{\lambda_{q+1}}{\cdot} d\mathbf{w}_\theta =$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in A_q} \Biggl({}^{(k)}\bar{G}_{l_1}\ldots\bar{G}_{l_k}\bar{L}^{j+1}R(\mathbf{x}_t,t) \overset{k}{\cdot} \int\limits_t^s \frac{(\theta-t)^j}{j!}\,{}^{(k)}I^*_{l_1\ldots l_{k\,\theta,t}}d\theta +$$

$$+ \left({}^{(k+1)}\bar{G}_{l_1}\ldots\bar{G}_{l_k}\bar{L}^j\bar{G}_0 R(\mathbf{x}_t,t) \overset{k}{\cdot} \int\limits_t^{*s} \frac{(\theta-t)^j}{j!}\,{}^{(k)}I^*_{l_1\ldots l_{k\,\theta,t}}\right) \overset{1}{\cdot} d\mathbf{f}_\theta\Biggr) \tag{61}$$

w. p. 1.

Using Lemma 1, we obtain

$$\int\limits_t^s \frac{(\theta-t)^j}{j!}\,{}^{(k)}I^*_{l_1\ldots l_{k\,\theta,t}}d\theta =$$

$$= \frac{1}{(j+1)!}\begin{cases}(s-t)^{j+1} & \text{for } k=0\\ & \\ (s-t)^{j+1}\cdot{}^{(k)}I^*_{l_1\ldots l_{k\,s,t}} - (-1)^{j+1}\cdot{}^{(k)}I^*_{l_1\ldots l_{k-1}\ l_k+j+1_{s,t}} & \text{for } k>0\end{cases} \tag{62}$$

w. p. 1. In addition (see (57)), we get

$$\int\limits_t^{*s} \frac{(\theta-t)^j}{j!} I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,\theta,t}} d\mathbf{f}^{(i_{k+1})}_\theta = \frac{(-1)^j}{j!} I^{*(i_1\ldots i_k i_{k+1})}_{l_1\ldots l_k j_{s,t}} \tag{63}$$

in the notations just introduced. Substitute the relations (62) and (63) into the formula (61). Grouping summands of the obtained expression with equal lower indices at iterated Stratonovich stochastic integrals and using (58) and the equality

$$\bar{G}^{(i)}_p R(\mathbf{x},t) = \frac{1}{p!}\sum_{q=0}^{p}(-1)^q C_p^q \bar{L}^q \bar{G}^{(i)}_0 \bar{L}^{p-q} R(\mathbf{x},t), \quad \text{where} \quad C_p^q = \frac{p!}{q!(p-q)!}, \tag{64}$$

(this equality follows from (58)), we note that the obtained expression is equal to

$$\sum_{(k,j,l_1,\ldots,l_k)\in A_{q+1}} \frac{(s-t)^j}{j!} {}^{(k)}\bar{G}_{l_1}\ldots \bar{G}_{l_k}\bar{L}^j\{\eta_t\} \stackrel{k}{\cdot} {}^{(k)}I^{*}_{l_1\ldots l_{k\,s,t}}$$

w. p. 1. Summing the equalities (60) for $q=1,2,\ldots,r$ and applying the formula (52), we obtain the expression (59). The proof is completed.

Let us order terms of the expansion (59) according to their smallness orders as $s\downarrow t$ in the mean-square sense

$$R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t) + \sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}^{(i_1)}_{l_1}\ldots \bar{G}^{(i_k)}_{l_k}\bar{L}^j R(\mathbf{x}_t,t)\ I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \tag{65}$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}^{(i_1)}_{l_1}\ldots \bar{G}^{(i_k)}_{l_k}\bar{L}^j R(\mathbf{x}_t,t)\ I^{*(i_1\ldots i_k)}_{l_1\ldots l_{k\,s,t}} + \left(D_{r+1}\right)_{s,t},$$

$$\mathrm{D}_q = \left\{(k,j,l_1,\ldots,l_k): k+2\left(j+\sum_{p=1}^{k} l_p\right) = q;\ k,j,l_1,\ldots,l_k = 0,1,\ldots\right\}, \tag{66}$$

(67)

$$\mathrm{U}_r = \left\{(k,j,l_1,\ldots,l_k): k+j+\sum_{p=1}^{k} l_p \le r,\ k+2\left(j+\sum_{p=1}^{k} l_p\right) \ge r+1;\ k,j,l_1,\ldots,l_k = 0,1,\ldots\right\},$$

and $\left(D_{r+1}\right)_{s,t}$ has the form (53). Note that the remainder term $\left(H_{r+1}\right)_{s,t}$ in (65) has a higher order of smallness in the mean-square sense as $s\downarrow t$ than the terms of the main part of expansion (65).

8. The Second Form of the Unified Taylor–Stratonovich Expansion

Consider iterated Stratonovich stochastic integrals of the form

$$J^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}=\int\limits_t^{*s}(s-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(s-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\quad\text{for}\quad k\ge 1$$

and

$$J^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}=1\quad\text{for}\quad k=0,$$

where $i_1,\ldots,i_k=1,\ldots,m$.

The additive property of stochastic integrals and the Newton binomial formula imply the following equality

$$I^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}=\sum_{j_1=0}^{l_1}\ldots\sum_{j_k=0}^{l_k}\prod_{g=1}^{k}C_{l_g}^{j_g}(t-s)^{l_1+\ldots+l_k-j_1-\ldots-j_k}\;J^{*(i_1\ldots i_k)}_{j_1\ldots j_k\,s,t}\quad\text{w. p. 1,}\tag{68}$$

where

$$C_l^k=\frac{l!}{k!(l-k)!}$$

is the binomial coefficient. Thus, the Taylor–Stratonovich expansion of the process $\eta_s=R(\mathbf{x}_s,s)$, $s\in[0,T]$ can be constructed either using the iterated stochastic integrals $I^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}$ similarly to the previous section or using the iterated stochastic integrals $J^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}$. This is the main subject of this section.

Denote

$$\left\|J^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}J^{*}_{l_1\ldots l_k\,s,t},$$

$$\left\|\bar{L}^j\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}R(\mathbf{x},t)\right\|_{i_1,\ldots,i_k=1}^{m}\stackrel{\sf def}{=}{}^{(k)}\bar{L}^j\bar{G}_{l_1}\ldots\bar{G}_{l_k}R(\mathbf{x},t).$$

Theorem 5. *Let conditions* (i*)-(iv*) *be satisfied. Then for any $s,t\in[0,T]$ such that $s>t$ and for any positive integer r, the following expansion is valid w. p.* 1

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar{L}^j\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}R(\mathbf{x}_t,t)\;J^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}+$$

$$+\left(D_{r+1}\right)_{s,t},\tag{69}$$

where $\left(D_{r+1}\right)_{s,t}$ has the form (53).

Proof. To prove the theorem, we check the equalities

$$\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t)\ J_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)} =$$

$$\sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{A}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t,t)\ I_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)} \quad \text{w. p. 1} \tag{70}$$

for $q = 1, 2, \ldots, r$. To check (70), substitute the expression (68) into the right-hand side of (70) and then use the formulas (58), (64).

Let us rank terms of the expansion (69) according to their orders of smallness in the mean-square sense as $s \downarrow t$

$$R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t)\ J_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)} +$$

$$+ \left(H_{r+1}\right)_{s,t} \quad \text{w. p. 1},$$

where

$$\left(H_{r+1}\right)_{s,t} = \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{U}_r} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t,t)\ J_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)} + \left(D_{r+1}\right)_{s,t}.$$

The term $\left(D_{r+1}\right)_{s,t}$ has the form (53); the terms D_q and U_r have the forms (66) and (67), respectively. Finally, we note that the convergence w. p. 1 of the truncated Taylor–Stratonovich expansion (52) (without the remainder term $\left(D_{r+1}\right)_{s,t}$) to the process $R(\mathbf{x}_s,s)$ as $r \to \infty$ for all $s, t \in [0,T]$ such that $s > t$ and $T < \infty$ has been proved in [2] (Proposition 5.10.2). Since the expansions (59) and (69) are obtained from the Taylor–Stratonovich expansion (52) without any additional conditions, the truncated expansions (59) and (69) (without the reminder term $\left(D_{r+1}\right)_{s,t}$) under the conditions of [2] (Proposition 5.10.2) converge to the process $R(\mathbf{x}_s,s)$ w. p. 1 as $r \to \infty$ for all $s, t \in [0,T]$ such that $s > t$ and $T < \infty$.

9. Comparison of the Unified Taylor–Ito and Taylor–Stratonovich Expansions With the Classical Taylor–Ito and Taylor–Stratonovich Expansions

Note that the truncated unified Taylor–Ito and Taylor–Stratonovich expansions contain the less number of various iterated Ito and Stratonovich stochastic integrals (moreover, their major part will have less multiplicity) in comparison with the classical Taylor–Ito and Taylor–Stratonovich expansions [7].

It is easy to notice that the stochastic integrals from the families (4), (5) are connected by linear relations. However, the stochastic integrals from the families (6), (7) cannot be connected by linear relations. This holds for the stochastic integrals from the families (8), (9). Therefore, we will call the families (6)–(9) as the *stochastic bases.*

Let us call the numbers $\mathrm{rank}_{\mathrm{A}}(r)$ and $\mathrm{rank}_{\mathrm{D}}(r)$ of various iterated Ito and Stratonovich stochastic integrals which are included in the families (6)–(9) as the *ranks of stochastic bases* when summation in the stochastic expansions is performed using the sets A_q $(q=1,\ldots,r)$ and D_q $(q=1,\ldots,r)$ correspondently. Here r is a fixed natural number.

At the beginning, let us analyze several examples related to the Taylor–Ito expansions (obviously, the same conclusions will hold for the Taylor–Stratonovich expansions).

Assume that summation in the unified Taylor–Ito expansions is performed using the sets D_q $(q=1,\ldots,r)$. It is easy to see that the truncated unified Taylor–Ito expansion (43), where summation is performed with respect to the sets D_q when $r=3$ includes 4 ($\mathrm{rank}_{\mathrm{D}}(3)=4$) various iterated Ito stochastic integrals

$$I_{0_{s,t}}^{(i_1)},\quad I_{00_{s,t}}^{(i_1i_2)},\quad I_{1_{s,t}}^{(i_1)},\quad I_{000_{s,t}}^{(i_1i_2i_3)}.$$

The same truncated classical Taylor–Ito expansion (34) [2] contains 5 various iterated Ito stochastic integrals

$$J_{(1)s,t}^{(i_1)},\quad J_{(11)s,t}^{(i_1i_2)},\quad J_{(10)s,t}^{(i_10)},\quad J_{(01)s,t}^{(0i_1)},\quad J_{(111)s,t}^{(i_1i_2i_3)}.$$

For $r=4$ we have 7 ($\mathrm{rank}_{\mathrm{D}}(4)=7$) integrals

$$I_{0_{s,t}}^{(i_1)},\quad I_{00_{s,t}}^{(i_1i_2)},\quad I_{1_{s,t}}^{(i_1)},\quad I_{000_{s,t}}^{(i_1i_2i_3)},\quad I_{01_{s,t}}^{(i_1i_2)},\quad I_{10_{s,t}}^{(i_1i_2)},\quad I_{0000_{s,t}}^{(i_1i_2i_3i_4)}$$

against 9 stochastic integrals

$$J_{(1)s,t}^{(i_1)},\quad J_{(11)s,t}^{(i_1i_2)},\quad J_{(10)s,t}^{(i_10)},\quad J_{(01)s,t}^{(0i_1)},\quad J_{(111)s,t}^{(i_1i_2i_3)},\quad J_{(101)s,t}^{(i_10i_3)},\quad J_{(110)s,t}^{(i_1i_20)},\quad J_{(011)s,t}^{(0i_1i_2)},\quad J_{(1111)s,t}^{(i_1i_2i_3i_4)}.$$

For $r=5$ ($\mathrm{rank}_{\mathrm{D}}(5)=12$) we get 12 integrals against 17 integrals and for $r=6$ and $r=7$ we have 20 against 29 and 33 against 50 correspondently.

We will obtain the same results when compare the unified Taylor–Stratonovich expansions [9], [46]-[50], [54]-[61] with their classical analogues [2], [7] (see previous sections).

Note that summation according to the sets D_q is usually used while constructing strong numerical methods (built according to the mean-square criterion of convergence) for Ito SDEs [2], [4], [46]-[50], [56]-[61]. Summation according to the sets A_q is usually used when building weak numerical methods (built in accordance with the weak criterion of convergence) for Ito SDEs [2], [4]. For example, $\mathrm{rank}_{\mathrm{A}}(4)=15$, while the total number of various iterated Ito stochastic integrals (included in the classical Taylor–Ito expansion [2] when $r=4$) equals to 26.

Let us show that [48]-[50], [56]-[61]

$$\mathrm{rank}_{\mathrm{A}}(r)=2^r-1.$$

Let $(l_1,\ldots,l_k)$ be an ordered set such that $l_1,\ldots,\,l_k=0,1,\ldots$ and $k=1,2,\ldots$ Consider $S(k)\stackrel{\mathsf{def}}{=} l_1+\ldots+l_k=p$ (p is a fixed natural number or zero). Let $N(k,p)$ be a number of all ordered combinations $(l_1,\ldots,l_k)$ such that $l_1,\ldots,l_k=0,1,\ldots,\ k=1,2,\ldots,$ and $S(k)=p$. First let us show that

$$N(k,p)=C_{p+k-1}^{k-1},$$

where

$$C_n^m=\frac{n!}{m!(n-m)!}$$

is a binomial coefficient.

It is not difficult to see that

$$N(1,p)=1=C_{p+1-1}^{1-1},$$

$$N(2,p)=p+1=C_{p+2-1}^{2-1},$$

$$N(3,p)=\frac{(p+1)(p+2)}{2}=C_{p+3-1}^{3-1}.$$

Moreover,

$$N(k+1,p)=\sum_{l=0}^{p}N(k,l)=\sum_{l=0}^{p}C_{l+k-1}^{k-1}=C_{p+k}^{k},$$

where we used the induction assumption and the well known property of binomial coefficients.
Then

$$\mathrm{rank}_A(r)=$$

$$=N(1,0)+(N(1,1)+N(2,0))+(N(1,2)+N(2,1)+N(3,0))+\ldots$$

$$\ldots+(N(1,r-1)+N(2,r-2)+\ldots+N(r,0))=$$

$$=C_0^0+(C_1^0+C_1^1)+(C_2^0+C_2^1+C_2^2)+\ldots$$

$$\ldots+(C_{r-1}^0+C_{r-1}^1+C_{r-1}^2+\ldots+C_{r-1}^{r-1})=$$

$$=2^0+2^1+2^2+\ldots+2^{r-1}=2^r-1.$$

Let $n_{\mathrm{M}}(r)$ be the total number of various iterated stochastic integrals included in the classical Taylor–Ito expansion (32) [2], where summation is performed with respect to the set

$$\bigcup_{k=1}^{r}\mathrm{M}_k.$$

If we exclude from the consideration the integrals which are equal to $(s-t)^j/j!$, then

$$n_M(r)=$$

$$=(2^1-1)+(2^2-1)+(2^3-1)+\ldots+(2^r-1)=$$

$$=2(1+2+2^2+\ldots+2^{r-1})-r=2(2^r-1)-r.$$

It means that

$$\lim_{r\to\infty}\frac{n_M(r)}{\mathrm{rank}_A(r)}=2.$$

In Table 1 we can see the numbers

$$\mathrm{rank}_{\mathrm{A}}(r),\quad n_{\mathrm{M}}(r),\quad f(r)=n_{\mathrm{M}}(r)/\mathrm{rank}_{\mathrm{A}}(r)$$

for various values r.
Let us show that [48]-[50], [56]-[61]

Таблица 1. Numbers $\text{rank}_{\text{A}}(r),\ n_{\text{M}}(r),\ f(r)=n_{\text{M}}(r)/\text{rank}_{\text{A}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{A}}(r)$	1	3	7	15	31	63	127	255	511	1023
$n_{\text{M}}(r)$	1	4	11	26	57	120	247	502	1013	2036
$f(r)$	1	1.3333	1.5714	1.7333	1.8387	1.9048	1.9449	1.9686	1.9824	1.9902

(71)
$$
\text{rank}_{\text{D}}(r)=\begin{cases}\displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{(r-1)/2+[s/2]} C_l^s & \text{for}\quad r=1,\ 3,\ 5,\ldots\\[2ex] \displaystyle\sum_{s=0}^{r-1}\ \sum_{l=s}^{r/2-1+[(s+1)/2]} C_l^s & \text{for}\quad r=2,\ 4,\ 6,\ldots\end{cases},
$$

where $[x]$ is an integer part of a number x, and C_n^m is a binomial coefficient.

For proving (71) we write the condition

$$
k+2(j+S(k))\le r,
$$

where $S(k)\stackrel{\text{def}}{=}l_1+\ldots+l_k$ $(k,j,l_1,\ldots,l_k=0,1,\ldots)$ in the form $j+S(k)\le(r-k)/2$, and perform the consideration of all possible combinations with respect to $k=1,\ldots,r$. Moreover, we take into account the above reasoning.

Let us calculate the number $n_E(r)$ of all different iterated Ito stochastic integrals from the classical Taylor–Ito expansion (34) [2] if the summation in this expansion is performed with respect to the set

$$
\bigcup_{q,k=1}^{r} E_{qk}.
$$

The summation condition can be written in this case in the form: $0\le p+2q\le r$, where q is a total number of integrations with respect to time while p is a total number of integrations with respect to the Wiener processes in the selected iterated stochastic integral from the Taylor–Ito expansion (34) [2]. At that, the multiplicity of the mentioned stochastic integral equals to $p+q$ and it is not more than r. Let us write the above condition $(0\le p+2q\le r)$ in the form: $0\le q\le(r-p)/2$ $\Leftrightarrow 0\le q\le[(r-p)/2]$, where $[x]$ means an integer part of a real number x. Then, performing the consideration of all possible combinations with respect to $p=1,\ldots,r$ and using the combinatorial reasoning, we obtain the formula

(72)
$$
n_E(r)=\sum_{s=1}^{r}\ \sum_{l=0}^{[(r-s)/2]} C^s_{[(r-s)/2]+s-l},
$$

where $[x]$ means an integer part of a real number x.

In Table 2 we can see the numbers

$$
\text{rank}_{\text{D}}(r),\quad n_{\text{E}}(r),\quad g(r)=n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)
$$

for various values r.

Таблица 2. Numbers $\text{rank}_{\text{D}}(r)$, $n_{\text{E}}(r)$, $g(r)=n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{D}}(r)$	1	2	4	7	12	20	33	54	88	143
$n_{\text{E}}(r)$	1	2	5	9	17	29	50	83	138	261
$g(r)$	1	1	1.2500	1.2857	1.4167	1.4500	1.5152	1.5370	1.5682	1.8252

10. Application of First Form of the Unified Taylor–Ito Expansion to the High-Order Strong Numerical Methods for Ito SDEs

Let us write (43) for all $s,t\in[0,T]$ such that $s>t$ in the following form

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\text{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)}+$$

$$+\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t)+\left(\bar{H}_{r+1}\right)_{s,t}\quad \text{w. p. 1,} \tag{73}$$

where

$$\left(\bar{H}_{r+1}\right)_{s,t}=\left(H_{r+1}\right)_{s,t}-\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t).$$

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0,T]$ such that

$$0=\tau_0<\tau_1<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}|\tau_{j+1}-\tau_j|.$$

From (73) for $s=\tau_{p+1}$, $t=\tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for Ito SDE (1), which is based on the first form of the unified Taylor–Ito expansion

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\text{D}_q}\frac{(\tau_{p+1}-\tau_p)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^j\,\mathbf{y}_p\ \hat{I}_{l_1\ldots l_{k\,\tau_{p+1},\tau_p}}^{(i_1\ldots i_k)}+$$

$$+\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}\,\mathbf{y}_p, \tag{74}$$

where $\hat{I}_{l_1\ldots l_{k\,\tau_{p+1},\tau_p}}^{(i_1\ldots i_k)}$ is an approximation of the iterated Ito stochastic integral $I_{l_1\ldots l_{k\,\tau_{p+1},\tau_p}}^{(i_1\ldots i_k)}$ defined as

$$I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)}=\int\limits_t^s(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}.$$

Note that we understand the equality (74) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $T = \tau_N$, $p = 0, 1, \ldots, N$.

It is known [2] that under the appropriate conditions the numerical scheme (74) has strong order of convergence $r/2$ $(r \in \mathbb{N})$.

Let $B_j(\mathbf{x}, t)$ is the j-th column of the matrix function $B(\mathbf{x}, t)$.

Below we consider particular cases of the numerical scheme (74) for $r = 2, 3, 4, 5$, and 6, i.e. explicit one-step strong numerical schemes for Ito SDE (1) with orders $1.0, 1.5, 2.0, 2.5$, and 3.0 of convergence. At that, for simplicity we will write $\mathbf{a}$, $L\mathbf{a}$, B_i, $G_0^{(i)} B_j$ etc. instead of $\mathbf{a}(\mathbf{y}_p, \tau_p)$, $L\mathbf{a}(\mathbf{y}_p, \tau_p)$, $B_i(\mathbf{y}_p, \tau_p)$, $G_0^{(i)} B_j(\mathbf{y}_p, \tau_p)$ etc. correspondingly. Moreover, the operators L and $G_0^{(i)}$, $i = 1, \ldots, m$, are determined by the equalities (27), (28) as before.

Scheme with strong order 1.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} B_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)}. \tag{75}$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} B_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} +$$

$$+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \right) - L B_{i_1} \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \right] +$$

$$+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)} G_0^{(i_2)} B_{i_1} \hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)} +$$

$$+ \frac{\Delta^2}{2} L\mathbf{a}. \tag{76}$$

Scheme with strong order 2.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} B_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2 i_1)} +$$

$$+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \right) - L B_{i_1} \hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)} \right] +$$

$$+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)} G_0^{(i_2)} B_{i_1} \hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3 i_2 i_1)} + \frac{\Delta^2}{2} L\mathbf{a} +$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}. \tag{77}
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-LB_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+
$$

$$
+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+
$$

$$
+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+
$$

$$
+\frac{1}{2}LLB_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\Bigr)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+
$$

$$
+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat{I}_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-
$$

$$-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I^{(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I^{(i_5i_4i_3i_2i_1)}_{00000_{\tau_{p+1},\tau_p}}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{78}$$

Scheme with strong order 3.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I^{(i_1)}_{0_{\tau_{p+1},\tau_p}}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat I^{(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I^{(i_1)}_{0_{\tau_{p+1},\tau_p}}+\hat I^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right)-LB_{i_1}\hat I^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I^{(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}LB_{i_1}\left(\hat I^{(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat I^{(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)-LG_0^{(i_2)}B_{i_1}\hat I^{(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat I^{(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\Delta\hat I^{(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I^{(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \tag{79}$$

where

$$\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I^{(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat I^{(i_1)}_{1_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat I^{(i_1)}_{0_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}LLB_{i_1}\hat I^{(i_1)}_{2_{\tau_{p+1},\tau_p}}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat I^{(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat I^{(i_1)}_{1_{\tau_{p+1},\tau_p}}\Bigr)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat I^{(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}-\hat I^{(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat I^{(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}-\hat I^{(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-$$

$$-LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{00000_{\tau_{p+1},\tau_p}}^{(i_5i_4i_3i_2i_1)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a},$$

and

$$\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{\Delta^2}{2}\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+$$

$$+\frac{1}{2}LLG_0^{(i_2)}B_{i_1}\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}$$

$$+G_0^{(i_2)}LG_0^{(i_1)}\mathbf{a}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right)+$$

$$+LG_0^{(i_2)}LB_{i_1}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+$$

$$+G_0^{(i_2)}LLB_{i_1}\left(\frac{1}{2}\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{1}{2}\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-$$

$$-LG_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\Biggl[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\hat I_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}LG_0^{(i_2)}B_{i_1}\left(\hat I_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat I_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)-$$

$$-LG_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+$$

$$+G_0^{(i_4)}LG_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat I_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat I_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}LB_{i_1}\left(\hat I_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat I_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I_{000000_{\tau_{p+1},\tau_p}}^{(i_6i_5i_4i_3i_2i_1)}.$$

It is well known [2] that under the standard conditions the numerical schemes (75)–(79) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Ito stochastic integrals from the numerical schemes (75)–(79) [2], [57]-[61]

$$
\mathsf{M}\left\{\left(I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}-\hat{I}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^{r+1},
$$

where $r/2$ are strong orders of convergence for the numerical schemes (75)–(79), i.e. $r/2=1.0, 1.5, 2.0, 2.5$, and 3.0. Moreover, constant C does not depends on Δ.

As we mentioned above, the numerical schemes (75)–(79) are unrealizable in practice without procedures for the numerical simulation of iterated Ito stochastic integrals from (73). In Sect. 12, we give a brief overview to the effective method of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of arbitrary multiplicity k ($k\in\mathbb{N}$).

11. Application of First Form of the Unified Taylor–Stratonovich Expansion to the High-Order Strong Numerical Methods for Ito SDEs

Let us write (65) for all $s,t\in[0,T]$ such that $s>t$ in the following from

$$
R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^jR(\mathbf{x}_t,t)\ I_{l_1\ldots l_k\,s,t}^{*(i_1\ldots i_k)}+
$$

$$
+\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t)+\left(\bar{H}_{r+1}\right)_{s,t}\quad \text{w. p. 1}, \tag{80}
$$

where

$$
\left(\bar{H}_{r+1}\right)_{s,t}=\left(H_{r+1}\right)_{s,t}-\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(s-t)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}R(\mathbf{x}_t,t).
$$

Consider the partition $\{\tau_p\}_{p=0}^N$ of the interval $[0,T]$ such that

$$
0=\tau_0<\tau_1<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\left|\tau_{j+1}-\tau_j\right|.
$$

From (80) for $s=\tau_{p+1}$, $t=\tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for Ito SDE (1), which is based on the first form of the unified Taylor–Stratonovich expansion

$$
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(\tau_{p+1}-\tau_p)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}\bar{G}_{l_1}^{(i_1)}\ldots\bar{G}_{l_k}^{(i_k)}\bar{L}^j\,\mathbf{y}_p\ \hat{I}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}+
$$

$$
+\mathbf{1}_{\{r=2d-1,d\in N\}}\frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}\mathbf{y}_p, \tag{81}
$$

where $\hat I^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}$ is an approximation of the iterated Stratonovich stochastic integral $I^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}$ defined as

$$I^{*(i_1\ldots i_k)}_{l_1\ldots l_k\,s,t}=\int\limits_t^{*s}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}.$$

Note that we understand the equality (81) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p=p\Delta$, $\Delta=T/N$, $T=\tau_N$, $p=0,1,\ldots,N$.

It is known [2] that under the appropriate conditions the numerical scheme (81) has strong order of convergence $r/2$ ($r\in\mathbb{N}$).

Denote

$$\bar{\mathbf{a}}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)-\frac{1}{2}\sum_{j=1}^m G_0^{(j)}B_j(\mathbf{x},t),$$

where $B_j(\mathbf{x},t)$ is the j-th column of the matrix function $B(\mathbf{x},t)$.

Below we consider particular cases of the numerical scheme (81) for $r=2,3,4,5$, and 6, i.e. explicit one-step strong numerical schemes for Ito SDE (1) with orders $1.0, 1.5, 2.0, 2.5$, and 3.0 of convergence. At that for simplicity we will write $\bar{\mathbf{a}}$, $\bar L\bar{\mathbf{a}}$, $L\mathbf{a}$, B_i, $G_0^{(i)}B_j$ etc. instead of $\bar{\mathbf{a}}(\mathbf{y}_p,\tau_p)$, $\bar L\bar{\mathbf{a}}(\mathbf{y}_p,\tau_p)$, $L\mathbf{a}(\mathbf{y}_p,\tau_p)$, $B_i(\mathbf{y}_p,\tau_p)$, $G_0^{(i)}B_j(\mathbf{y}_p,\tau_p)$ etc. correspondingly. Moreover, the operators $\bar L$ and $G_0^{(i)}$, $i=1,\ldots,m$, are determined by the equalities (27), (28), and (51) as before.

Scheme with strong order 1.0

(82)
$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^m B_{i_1}\hat I^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^m G_0^{(i_2)}B_{i_1}\hat I^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}.$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^m B_{i_1}\hat I^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^m G_0^{(i_2)}B_{i_1}\hat I^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}+$$

$$+\sum_{i_1=1}^m\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat I^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\hat I^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right)-\bar L B_{i_1}\hat I^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right]+$$

(83)
$$+\sum_{i_1,i_2,i_3=1}^m G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat I^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}L\mathbf{a}.$$

Scheme with strong order 2.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^m B_{i_1}\hat I^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^m G_0^{(i_2)}B_{i_1}\hat I^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}. \tag{84}$$

Scheme with strong order 2.5

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}+\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)-\bar{L}B_{i_1}\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}-\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}+\Delta\hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4i_3i_2i_1)}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}+\frac{\Delta^2}{2}\hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\Bigl(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)}+\Delta\hat{I}_{1\tau_{p+1},\tau_p}^{*(i_1)}\Bigr)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}-\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3i_2i_1)}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}\Bigg]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{00000_{\tau_{p+1},\tau_p}}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{85}$$

Scheme with strong order 3.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right)-\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Bigg[G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)-\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)\Bigg]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \tag{86}$$

where

$$\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Bigg[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{0_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{2_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_1)}_{1_{\tau_{p+1},\tau_p}}\right)\Bigg]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_3i_2i_1)}_{000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_3i_2i_1)}_{001_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{00000_{\tau_{p+1},\tau_p}}+$$

$$+\frac{\Delta^3}{6}\bar{L}\bar{L}\bar{\mathbf{a}},$$

and

$$\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}$$

$$+G_0^{(i_2)}\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)\right)+$$

$$+\bar{L}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_2)}\bar{L}\bar{L}B_{i_1}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\frac{1}{2}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\Biggl[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}\bar{L}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_4)}\bar{L}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}\right)+$$

$$
+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\bar{L}B_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}B_{i_1}\hat{I}^{*(i_6i_5i_4i_3i_2i_1)}_{000000_{\tau_{p+1},\tau_p}}.
$$

It is well known [2] that under the standard conditions the numerical schemes (82)–(86) have strong orders of convergence 1.0, 1.5, 2.0, 2.5, and 3.0 correspondingly. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from the numerical schemes (82)–(86) [2], [57]-[61]

$$
\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{l_1\ldots l_k{}_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_1\ldots i_k)}_{l_1\ldots l_k{}_{\tau_{p+1},\tau_p}}\right)^2\right\}\le C\Delta^{r+1},
$$

where $r/2$ are strong orders of convergence for the numerical schemes (82)–(86), i.e. $r/2=1.0,1.5,$ $2.0,2.5,$ and 3.0. Moreover, constant C does not depends on Δ.

As we mentioned above, the numerical schemes (82)–(86) are unrealizable in practice without procedures for the numerical simulation of iterated Stratonovich stochastic integrals from (80). In the next section, we give a brief overview to the effective method of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of arbitrary multiplicity k $(k\in\mathbb{N})$.

12. Method of the Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series

It should be noted that there is an approach to the mean-square approximation of iterated stochastic integrals based on multiple integral sums (see, for example, [4]). This method implies the partitioning of the integration interval of the iterated stochastic integral under consideration; this interval is the integration step of the numerical methods used to solve Ito SDEs; therefore, it is already fairly small and does not need to be partitioned. Computational experiments [46] show that the application of the method [4] to stochastic integrals with multiplicities $k\ge 2$ leads to unacceptably high computational cost and accumulation of computation errors. Another well-known method is based on the Karhunen–Loeve expansion of the Brownian bridge process [4]. This method has no the mentioned drawback (also see [2], [3]) but leads to iterated application of the operation of limit transition. So, the mentioned method may not converge in the mean-square sense to appropriate iterated stochastic integrals for some methods of series summation (see discussion in Sect. 13 for details).

The difficulties noted above can be overcome with a different and more effective method proposed and developed by the author in [46] (also see [12]-[43], [47]-[63]). The idea of this method is as follows: the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ of the form (2) with multiplicity k is represented as a multiple stochastic integral from the nonrandom discontinuous function $K(t_1,\ldots,t_k)$ of k variables (see (87) below) defined on the hypercube $[t,T]^k$, where here and further $[t,T]$ is an interval of integration of the iterated Ito stochastic integral. Then, the function $K(t_1,\ldots,t_k)$ is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 6, 7 below) to the mean-square convergening expansion of the iterated Ito stochastic integral into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the function $K(t_1,\ldots,t_k)$, which can be calculated using the explicit formula regardless of multiplicity k of the iterated Ito stochastic integral. Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\quad=\quad\prod_{l=1}^{k}\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}},\tag{87}$$

where $t_1,\ldots,t_k\in[t,T]$ $(k\geq 2)$, $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau):[t,T]\to\mathbb{R}$ are continuous nonrandom functions (the case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered below in this section). Here $\mathbf{1}_A$ denotes the indicator of the set A.

The function $K(t_1,\ldots,t_k)$ of the form (87) is piecewise continuous in the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,\tag{88}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k\tag{89}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\leq j\leq N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j.\tag{90}$$

Theorem 6 [46] (2006) (also see [12]-[43], [47]-[62]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -\right.$$

$$\left.-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right),\tag{91}$$

where

$$\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \left\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\right\},$$

$$\mathrm{L}_k = \left\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g \ne l_r\ (g\ne r);\ g,r=1,\ldots,k\right\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{92}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (89)*,* $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$*,* $\{\tau_j\}_{j=0}^N$ *is a partition of* $[t,T]$*, which satisfies the condition* (90)*.*

In order to evaluate the significance of Theorem 6 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [46] (2006) (also see [12]-[43], [47]-[62])

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{93}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{94}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{95}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^4 \zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{96}$$

$$
J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{97}
$$

$$
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-
$$

$$
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+
$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$

$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{98}$$

where $\mathbf{1}_A$ is the indicator of the set A.

It was shown in [12], [48]-[61] that Theorem 6 is valid for convergence in the mean of degree $2n$ ($n\in\mathbb{N}$). The convergence w. p. 1 in Theorem 6 is proved in [58]-[61] for the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorem 6 [12], [48]-[61]. The generalization of Theorem 6 for complete orthonormal with weigth $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in the space $L_2([t,T]^k)$ can be found in [24], [57]-[61]. Another modification of Theorem 6 is connected with the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [31], [32], [43]. The latters play the key role for implemetation of the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise [70]-[73].

For further consideration, let us consider the generalization of formulas (93)–(98) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{99}
$$

where

$$
\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (99) is a partition and consider the sum with respect to all possible partitions

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \tag{100}
$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}\in\mathbb{R}^1$.

Below there are several examples of sums in the form (100)

$$
\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2,g_3g_4}=a_{12,34}+a_{13,24}+a_{23,14},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=
$$

$$
=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=
$$

$$
=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+
$$
$$
+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =
$$

$$
= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4}+
$$
$$
+a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2}+
$$
$$
+a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
$$

Now we can write (91) as

$$
J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{101}
$$

where $\prod\limits_{\emptyset} \stackrel{\sf def}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\sf def}{=} 0$, $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 6.

In particular, from (101) for $k=5$ we obtain

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -
$$
$$
- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +
$$
$$
+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).
$$

The last equality obviously agrees with (97).

Let us consider the generalization of Theorem 6 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 7 [12] (Sect. 15), [58] (Sect. 1.11), [62]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$
J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{102}
$$

converging in the mean-square sense is valid, where $\prod\limits_{\emptyset} \stackrel{\sf def}{=} 1,\ \sum\limits_{\emptyset} \stackrel{\sf def}{=} 0,$ $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 6.

It should be noted that an analogue of Theorem 7 was considered in [74]. Note that we use another notations in comparison with [74]. Moreover, the proof of an analogue of Theorem 7 from [74] is different from the proof given in [12] (Sect. 15), [58] (Sect. 1.11), [62].

In a number of works of the author [16]-[22], [30], [34], [40], [42], [51]-[61], [63] Theorems 6, 7 have been adapted for the iterated Stratonovich stochastic integrals (3). Let us collect some of these results (old results) in the following statement.

Theorem 8 [16]-[22], [30], [34], [40], [42], [51]-[61]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function on* $[t,T]$ *and* $\psi_1(\tau),\psi_3(\tau)$ *are two times continuously differentiable functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m), \tag{103}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m), \tag{104}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{105}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \quad (i_1,\ldots,i_4=0,1,\ldots,m), \tag{106}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (3) *and* $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ *in* (104), (106); *another notations are the same as in Theorem* 6.

Note that the formula (103) is generalized to the case of continuous functions $\psi_1(\tau),\psi_2(\tau)$ in [58] (Sect. 2.1.4).

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [16] (Sect. 13–19), [21] (Sect. 5–11), [22] (Sect. 7–13), [58] (Sect. 2.10–2.16), [61] (Sect. 2.10–2.16), [63], [64]. Let us introduce some notations and formulate the main theorem of the approach noted above.

Consider the Fourier coefficient

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

corresponding to the function (87), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Bigg|_{(j_lj_l)\frown(\cdot)} \stackrel{\sf def}{=}$$

$$\stackrel{\mathsf{def}}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k = \tag{107}$$

$$= \sqrt{T-t}\int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k =$$

$$= \sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1},$$

i.e. $\sqrt{T-t}\hat{C}_{j_k\ldots j_{l+1}0j_{l-2}\ldots j_1}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$, $\psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see the relation (99)).

Denote

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}.$$

Introduce the following notation

$$S_l\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\mathsf{def}}{=} \frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} \sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots \sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots \sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty} C_{j_k\ldots j_1}\Biggr|_{(j_{g_{2l}}j_{g_{2l-1}})\frown(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}.$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value

$$\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \tag{108}$$

as follows: S_l multiplies (108) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

with

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{109}$$

Note that we write

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}},\quad\ldots$$

Since (109) is again the Fourier coefficient, then the action of superposition S_lS_m on (109) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^3}\prod_{s=1}^{3}\mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_3S_1\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},$$

$$S_2\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2}\mathbf{1}_{\{g_4=g_3+1\}}\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.$$

Theorem 9 [16] (Sect. 13), [21] (Sect. 5), [22] (Sect. 7), [58] (Sect. 2.10), [63] (Sect. 4). *Assume that the continuously differentiable functions* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *and the complete orthonormal system*

$\{\phi_j(x)\}_{j=0}^{\infty}$ *of continuous functions* ($\phi_0(x) = 1/\sqrt{T-t}$) *in the space* $L_2([t,T])$ *are such that the following conditions are satisfied*:

1. *The equality*

$$\frac{1}{2}\int\limits_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty}\int\limits_t^s \Phi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\Phi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2 \tag{110}$$

holds for all $s \in (t,T]$, *where the nonrandom functions* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable on* $[t,T]$ *and the series on the right-hand side of* (110) *converges absolutely.*

2. *The estimates*

$$\left|\int\limits_t^s \phi_j(\tau)\Phi_1(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \qquad \left|\int\limits_s^T \phi_j(\tau)\Phi_2(\tau)d\tau\right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s \in (t,T)$ *and for some* $\alpha,\beta > 0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$, $j,p \in \mathbb{N}$, *and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau < \infty, \qquad \int\limits_t^T |\Psi_2(\tau)|\, d\tau < \infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (*see* (99)) *and* $l_1,l_2,\ldots,l_d$ *such that* $l_1,l_2,\ldots,l_d \in \{1,2,\ldots,r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0,1,2,\ldots,r-1$, *where* $r = 1,2,\ldots,[k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\sf def}{=} \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1}\Bigl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{111}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{112}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *for* $i=1,\ldots,m$ *and* $\mathbf{w}_\tau^{(0)}=\tau.$

Note that (110) is fulfilled for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\Phi_1(\tau),\Phi_2(\tau)\in L_2([t,T])$ (see [58], Sect. 2.1.4 or [77]).

Consider the following four theorems, which were proved in [16], [21], [22], [58], [61], [63], [64].

Theorem 10 [16], [21], [22], [58], [63]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\psi_2(\tau),$ $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad(i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},\qquad(113)$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p}\qquad(114)$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (113) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (114)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*); another notations are the same as in Theorem* 6*.*

Theorem 11 [16], [21], [22], [58], [63]. *Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,$ $\psi_4(\tau)$ be continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{115}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{116}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{117}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ in (115), (116) *and $i_1,\ldots,i_4=1,\ldots,m$ in* (117), *constant C does not depend on p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t,T])$,*

$$C_{j_4j_3j_2j_1}=\int\limits_t^{T}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 10.

Theorem 12 [16], [21], [22], [58], [63]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,$ $\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{118}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{119}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{120}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (118), (119) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (120), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorem 10, 11.

Theorem 13 [16], [21], [22], [58], [64]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{121}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 10–12.

On the base of Theorems 8–13 in [22], [55]-[61] the folloing hypothesis was formulated.

Hypothesis 1 [22], [55]-[61]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of* k*th multiplicity*

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (i_1,\ldots,i_k=0,1,\ldots,m) \tag{122}$$

the following converging in the mean-square sense expansion

$$J^*[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{123}$$

holds, where the Fourier coefficient $C_{j_k\ldots j_1}$ *has the form*

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

l.i.m. *is a limit in the mean-square sense,*

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(if* $i\ne 0$*),* $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ *are independent standard Wiener processes* $(i=1,\ldots,m)$ *and* $\mathbf{w}_\tau^{(0)}=\tau$.

The hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p=\sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{124}$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\}=0.$$

The following theorem shows how to calculate exactly the mean-square approximation error for iterated Ito stochastic integrals in Theorems 6, 7.

Theorem 14 [14] (Sect. 6). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^\infty$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{125}$$

where $J[\psi^{(k)}]_{T,t}$ *is the iterated Ito stochastic integral* (2),

$$J[\psi^{(k)}]_{T,t}^p=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \tag{126}$$

$i_1,\ldots,i_k=1,\ldots,m$, *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$*. At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$*, then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$*; another notations are the same as in Theorems* 6, 7.

Denote

$$E_k^p \stackrel{\sf def}{=} {\sf M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^p\right)^2\right\}.$$

Note that

$${\sf M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d{\bf f}_{t_1}^{(i_1)}\ldots d{\bf f}_{t_k}^{(i_k)}\right\}=$$

$$=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k=C_{j_k\ldots j_1}.$$

Then from Theorem 14 for pairwise different $i_1,\ldots,i_k$ and for $i_1=\ldots=i_k$ we obtain

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2, \tag{127}$$

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right)$$

correspondingly.

Consider some examples of application of Theorem 14 ($i_1,\ldots,i_5=1,\ldots,m$)

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}\quad(i_1=i_2), \tag{128}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad(i_1=i_2\ne i_3), \tag{129}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad(i_1\ne i_2=i_3), \tag{130}$$

$$E_3^p=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad(i_1=i_3\ne i_2),$$

$$E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}C_{j_4\ldots j_1}\right)\quad(i_1=i_2\ne i_3,i_4;\ i_3\ne i_4),$$

$$E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)}C_{j_4\ldots j_1}\right)\quad(i_1=i_3\ne i_2,i_4;\ i_2\ne i_4),$$

$$E_4^p=I_4-\sum_{j_1,\ldots,j_4=0}^{p}C_{j_4\ldots j_1}\left(\sum_{(j_1,j_4)}C_{j_4\ldots j_1}\right)\quad(i_1=i_4\ne i_2,i_3;\ i_2\ne i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_3 \neq i_1, i_4;\ i_1 \neq i_4),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2 = i_4 \neq i_1, i_3;\ i_1 \neq i_3),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_3 = i_4 \neq i_1, i_2;\ i_1 \neq i_2),$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_1 = i_3 = i_4 \neq i_2),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right) \quad (i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right) \quad (i_1 = i_2 = i_3 \neq i_4 = i_5),$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1}\right) \quad (i_1 = i_3 = i_4 = i_5 \neq i_2).$$

Let $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ be the expression on the right-hand side of (102) before passing to the limit. Denote

$$I_k \stackrel{\text{def}}{=} \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k.$$

In [12], [14], [57]-[61] it was shown that

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right), \tag{131}$$

where $i_1,\ldots,i_k = 1,\ldots,m$ and $0 < T-t < \infty$ or $i_1,\ldots,i_k = 0,1,\ldots,m$ and $0 < T-t < 1$.

Moreover [12], [48]-[61],

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^{2n}\right\} \le C_{n,k}\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right)^n,$$

where $C_{n,k} = (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!,\ \ n \in \mathbb{N}$.

Below we provide practical material (based on Theorems 6–8, 10–13) concerning expansions and approximations of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 (the case of Legendre polynomial system). The question about what kind of functions (polynomial or trigonometric) is more convenient for the mean-square approximation of iterated stochastic integrals is also considered.

Let us introduce more convenient (for further) notations for the iterated Ito and Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{132}$$

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{133}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ l_1,\ldots,l_k=0,1,\ldots$

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{134}$$

where $P_j(x)$ is the Legendre polynomial.

Using the system of functions (134) and Theorems 6–8, 10–13, we obtain the following expansions of iterated Ito and Stratonovich stochastic integrals (132), (133) [13], [15], [36]-[39], [46]-[63] (also see early publications [44] (2000), [45] (2001), [65] (1997), [67] (1998))

$$I_{(0)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{135}$$

$$I_{(2)T,t}^{(i_1)}=\frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{136}$$

$$I_{(00)T,t}^{*(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{137}$$

$$I_{(00)T,t}^{(i_1i_2)}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(01)T,t}^{*(i_1i_2)}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right),$$

$$I_{(10)T,t}^{*(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{*(i_1 i_2)} - \frac{(T-t)^2}{4} \left(\frac{1}{\sqrt{3}} \zeta_0^{(i_2)} \zeta_1^{(i_1)} + \right.$$

(138)
$$\left. + \sum_{i=0}^{\infty} \left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right),$$

or

$$I_{(01)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{01} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$I_{(10)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{10} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2 j_1}^{01} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{01},$$

$$C_{j_2 j_1}^{10} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{10},$$

$$\bar{C}_{j_2 j_1}^{01} = -\int_{-1}^{1} (1+y) P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy,$$

$$\bar{C}_{j_2 j_1}^{10} = -\int_{-1}^{1} P_{j_2}(y) \int_{-1}^{y} (1+x) P_{j_1}(x) dx dy;$$

$$I_{(10)T,t}^{(i_1 i_2)} = I_{(10)T,t}^{*(i_1 i_2)} + \frac{1}{4} \mathbf{1}_{\{i_1=i_2\}} (T-t)^2 \quad \text{w. p. } 1,$$

$$I_{(01)T,t}^{(i_1 i_2)} = I_{(01)T,t}^{*(i_1 i_2)} + \frac{1}{4} \mathbf{1}_{\{i_1=i_2\}} (T-t)^2 \quad \text{w. p. } 1,$$

$$I_{(01)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)} - \frac{(T-t)^2}{4} \left(\frac{1}{\sqrt{3}} \zeta_0^{(i_1)} \zeta_1^{(i_2)} + \right.$$

$$\left. + \sum_{i=0}^{\infty} \left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)} \right) \right),$$

$$I_{(10)T,t}^{(i_1 i_2)} = -\frac{T-t}{2} I_{(00)T,t}^{(i_1 i_2)} - \frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} +$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

or

$$I_{(01)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{01}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(10)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{10}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr),$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

(139)
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$I_{(000)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\Biggl(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\Biggr) \quad \text{w. p. 1},$$

$$I_{(000)T,t}^{*(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\zeta_0^{(i_1)}\right)^3 \quad \text{w. p. 1},$$

where

(140)
$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1},$$

(141)
$$\bar{C}_{j_3 j_2 j_1} = \int_{-1}^{1} P_{j_3}(z)\int_{-1}^{z} P_{j_2}(y)\int_{-1}^{y} P_{j_1}(x)dxdydz;$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)} = I_{(000)T,t}^{*(i_1 i_2 i_3)} + \mathbf{1}_{\{i_1=i_2\neq 0\}}\frac{1}{2}I_{(1)T,t}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\frac{1}{2}\left((T-t)I_{(0)T,t}^{(i_1)}+I_{(1)T,t}^{(i_1)}\right)\quad \text{w. p. 1},$$

$$I_{(02)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(01)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{142}$$

$$I_{(20)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-(T-t)I_{(10)T,t}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{143}$$

$$I_{(11)T,t}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{*(i_1i_2)}-\frac{(T-t)}{2}\left(I_{(10)T,t}^{*(i_1i_2)}+I_{(01)T,t}^{*(i_1i_2)}\right)+$$

$$+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \tag{144}$$

or

$$I_{(02)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{02} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$I_{(20)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{20} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

$$I_{(11)T,t}^{*(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{11} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where

$$C_{j_2 j_1}^{02} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{02},$$

$$C_{j_2 j_1}^{20} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{20},$$

$$C_{j_2 j_1}^{11} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3 \bar{C}_{j_2 j_1}^{11},$$

$$\bar{C}_{j_2 j_1}^{02} = \int_{-1}^{1} P_{j_2}(y)(y+1)^2 \int_{-1}^{y} P_{j_1}(x) dx dy,$$

$$\bar{C}_{j_2 j_1}^{20} = \int_{-1}^{1} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x)(x+1)^2 dx dy,$$

$$\bar{C}_{j_2 j_1}^{11} = \int_{-1}^{1} P_{j_2}(y)(y+1) \int_{-1}^{y} P_{j_1}(x)(x+1) dx dy,$$

$$I_{(11)T,t}^{*(i_1 i_1)} = \frac{1}{2}\left(I_{(1)T,t}^{(i_1)}\right)^2 \quad \text{w. p. 1},$$

$$I_{(02)T,t}^{(i_1 i_2)} = I_{(02)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$I_{(20)T,t}^{(i_1 i_2)} = I_{(20)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$I_{(11)T,t}^{(i_1 i_2)} = I_{(11)T,t}^{*(i_1 i_2)} - \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3 \quad \text{w. p. 1},$$

$$I_{(02)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-(T-t)I_{01_{T,t}}^{(i_1 i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,$$

$$I_{(20)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-(T-t)I_{(10)T,t}^{(i_1 i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,$$

$$I_{(11)T,t}^{(i_1 i_2)}=-\frac{(T-t)^2}{4}I_{(00)T,t}^{(i_1 i_2)}-\frac{T-t}{2}\left(I_{(10)T,t}^{(i_1 i_2)}+I_{(01)T,t}^{(i_1 i_2)}\right)+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr]-$$

$$-\frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3,$$

or

$$I_{(02)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{02} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(20)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{20} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(11)T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^{11} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right),$$

$$I_{(3)T,t}^{(i_1)} = -\frac{(T-t)^{7/2}}{4} \left(\zeta_0^{(i_1)} + \frac{3\sqrt{3}}{5} \zeta_1^{(i_1)} + \frac{1}{\sqrt{5}} \zeta_2^{(i_1)} + \frac{1}{5\sqrt{7}} \zeta_3^{(i_1)} \right),$$

$$I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$\begin{aligned} I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \Biggl(& \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \\ & -\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\ & -\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\ & -\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\ & +\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \\ & +\mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\ & +\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \end{aligned} \tag{145}$$

$$I_{(0000)T,t}^{(i_1 i_1 i_1 i_1)} = \frac{1}{24}(T-t)^2 \left(\left(\zeta_0^{(i_1)} \right)^4 - 6 \left(\zeta_0^{(i_1)} \right)^2 + 3 \right) \quad \text{w. p. 1},$$

$$I_{(0000)T,t}^{*(i_1 i_1 i_1 i_1)} = \frac{1}{24}(T-t)^2 \left(\zeta_0^{(i_1)} \right)^4 \quad \text{w. p. 1},$$

where

$$C_{j_4 j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^2 \bar{C}_{j_4 j_3 j_2 j_1}, \tag{146}$$

$$\bar{C}_{j_4 j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du; \tag{147}$$

$$I^{*(i_1 i_2 i_3)}_{(001)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{001}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3},$$

$$I^{*(i_1 i_2 i_3)}_{(010)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{010}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3},$$

$$I^{*(i_1 i_2 i_3)}_{(100)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{100}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3},$$

$$I^{(i_1 i_2 i_3)}_{(001)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{001}_{j_3 j_2 j_1} \Biggl(\zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \Biggr), \tag{148}$$

$$I^{(i_1 i_2 i_3)}_{(010)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{010}_{j_3 j_2 j_1} \Biggl(\zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \Biggr), \tag{149}$$

$$I^{(i_1 i_2 i_3)}_{(100)T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C^{100}_{j_3 j_2 j_1} \Biggl(\zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \Biggr), \tag{150}$$

where

$$C_{j_3j_2j_1}^{001}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3j_2j_1}^{001},$$

$$C_{j_3j_2j_1}^{010}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3j_2j_1}^{010},$$

$$C_{j_3j_2j_1}^{100}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar C_{j_3j_2j_1}^{100},$$

$$\bar C_{j_3j_2j_1}^{100}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydz,$$

$$\bar C_{j_3j_2j_1}^{010}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

$$\bar C_{j_3j_2j_1}^{001}=-\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz;$$

$$I_{(lll)T,t}^{(i_1i_1i_1)}=\frac{1}{6}\left(\left(I_{(l)T,t}^{(i_1)}\right)^3-3I_{(l)T,t}^{(i_1)}\Delta_{l(T,t)}\right)\quad \text{w. p. 1},$$

$$I_{(lll)T,t}^{*(i_1i_1i_1)}=\frac{1}{6}\left(I_{(l)T,t}^{(i_1)}\right)^3\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{(i_1i_1i_1i_1)}=\frac{1}{24}\left(\left(I_{(l)T,t}^{(i_1)}\right)^4-6\left(I_{(l)T,t}^{(i_1)}\right)^2\Delta_{(l)T,t}+3\left(\Delta_{(l)T,t}\right)^2\right)\quad \text{w. p. 1},$$

$$I_{(llll)T,t}^{*(i_1i_1i_1i_1)}=\frac{1}{24}\left(I_{(l)T,t}^{(i_1)}\right)^4\quad \text{w. p. 1},$$

where

$$I_{(l)T,t}^{(i_1)}=\sum_{j=0}^{l}C_j^l\zeta_j^{(i_1)}\quad \text{w. p. 1},$$

$$\Delta_{l(T,t)}=\int\limits_t^T(t-s)^{2l}ds,\quad C_j^l=\int\limits_t^T(t-s)^l\phi_j(s)ds;$$

$$
I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},
$$

$$
I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1} \Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +
$$

$$
+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{151}
$$

$$
I_{(00000)T,t}^{(i_1 i_1 i_1 i_1 i_1)} = \frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5 - 10\left(\zeta_0^{(i_1)}\right)^3 + 15\zeta_0^{(i_1)}\right) \quad \text{w. p. 1},
$$

$$
I_{(00000)T,t}^{*(i_1 i_1 i_1 i_1 i_1)} = \frac{1}{120}(T-t)^{5/2}\left(\zeta_0^{(i_1)}\right)^5 \quad \text{w. p. 1},
$$

where

$$
C_{j_5 j_4 j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar{C}_{j_5 j_4 j_3 j_2 j_1},
$$

$$
\bar{C}_{j_5 j_4 j_3 j_2 j_1} = \int\limits_{-1}^{1} P_{j_5}(v)\int\limits_{-1}^{v} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdudv;
$$

$$I_{(0001)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0001} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$I_{(0010)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$I_{(0100)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$I_{(1000)T,t}^{*(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{1000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0001} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$
$$+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} +$$
$$+ \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +$$
$$+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr),$$

$$I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +$$
$$+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} +$$
$$+ \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +$$
$$+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr),$$

$$I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -$$
$$- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right),
$$

$$
I_{(1000)T,t}^{(i_1i_2i_3i_4)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^{1000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right),
$$

where

$$
C_{j_4j_3j_2j_1}^{0001}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0001},
$$

$$
C_{j_3j_2j_1}^{0010}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0010},
$$

$$
C_{j_4j_3j_2j_1}^{0100}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_3j_2j_1}^{0100},
$$

$$
C_{j_4j_3j_2j_1}^{1000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{1000},
$$

$$
\bar{C}_{j_4j_3j_2j_1}^{1000}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydzdu,
$$

$$
\bar{C}_{j_4j_3j_2j_1}^{0100}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,
$$

$$\bar{C}^{0010}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}^{0001}_{j_4j_3j_2j_1}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu;$$

$$I^{*(i_1i_2i_3i_4i_5i_6)}_{(000000)T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6},$$

$$I^{(i_1i_2i_3i_4i_5i_6)}_{(000000)T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p}C_{j_6j_5j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{6}\zeta^{(i_l)}_{j_l}-$$
$$-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-\mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-$$
$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}-\mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}-$$
$$-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-$$
$$-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-$$
$$-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-$$
$$-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-$$
$$-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}-$$
$$-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_5)}_{j_5}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+$$
$$+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_6)}_{j_6}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_5)}_{j_5}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_4)}_{j_4}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}+$$
$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_4)}_{j_4}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_3)}_{j_3}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_5)}_{j_5}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_4)}_{j_4}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta^{(i_3)}_{j_3}\zeta^{(i_5)}_{j_5}+$$
$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta^{(i_4)}_{j_4}\zeta^{(i_5)}_{j_5}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta^{(i_1)}_{j_1}\zeta^{(i_4)}_{j_4}+$$

$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-
$$
$$
-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-
$$
$$
\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),
$$

$$
I_{(000000)T,t}^{(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\left(\zeta_0^{(i_1)}\right)^6-15\left(\zeta_0^{(i_1)}\right)^4+45\left(\zeta_0^{(i_1)}\right)^2-15\right)\quad \text{w. p. 1},
$$

$$
I_{(000000)T,t}^{*(i_1i_1i_1i_1i_1i_1)}=\frac{1}{720}(T-t)^3\left(\zeta_0^{(i_1)}\right)^6\quad \text{w. p. 1},
$$

where

$$
C_{j_6j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3\bar{C}_{j_6j_5j_4j_3j_2j_1},
$$

$$
\bar{C}_{j_6j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_6}(w)\int\limits_{-1}^{w}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudvdw.
$$

Let us analyze the approximation $I_{(00)T,t}^{*(i_1 i_2)q}$ of the iterated stochastic integral $I_{(00)T,t}^{*(i_1 i_2)}$ obtained from (137) by replacing ∞ on q.

It is easy to prove that

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)}-I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2). \tag{152}$$

Moreover, using Theorem 14, we obtain for $i_1\ne i_2$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1 i_2)}-I_{(10)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1 i_2)}-I_{(01)T,t}^{*(i_1 i_2)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right).$$

For the case $i_1=i_2$, using Theorem 14, we have

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_1)}-I_{(10)T,t}^{(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_1)}-I_{(01)T,t}^{(i_1 i_1)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{153}$$

On the basis of the presented expansions of iterated stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to noticeable complication of formulas for the mentioned expansions.

However, increasing of the mentioned parameters leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities in expansions of iterated stochastic integrals, which are required for achieving the acceptable accuracy of approximation. In the context of it let us consider the approach to approximation of iterated stochastic integrals, which provides a possibility to obtain the mean-square approximations of the required accuracy without using the complex expansions.

Let us analyze the following approximation of triple stochastic integral using (139)

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{154}$$

where $C_{j_3 j_2 j_1}$ is defined by (140), (141).

In particular, from (154) when $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we obtain

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}. \tag{155}$$

Using (127), (129)–(131), we get

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 \quad (i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3), \tag{156}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3), \tag{157}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \quad (i_1 = i_3 \neq i_2), \tag{158}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} =$$

$$= \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3), \tag{159}$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2\right) \quad (i_1, i_2, i_3 = 1, \ldots, m). \tag{160}$$

We may act similarly with more complicated iterated stochastic integrals. For example, for the approximation of stochastic integral $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}$ we may write (see (145))

$$
I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}=\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

where $C_{j_4j_3j_2j_1}$ is defined by (146), (147). Moreover, according to (131)

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}\le 24\left(\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}^2\right)\quad (i_1,i_2,i_3,i_4=1,\ldots,m).
$$

For pairwise different $i_1,i_2,i_3,i_4=1,\ldots,m$ from (127) we obtain

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4j_3j_2j_1}^2. \tag{161}
$$

Using Theorem 14, we can calculate exactly the left-hand side of (161) for any possible combinations of i_1,i_2,i_3,i_4. These relations were obtained in [14], [57]-[61].

In Tables 3–5, we have some examples of exact values of the Fourier–Legendre coefficients (here and further the Fourier–Legendre coefficients have been calculated exactly using DERIVE (computer algebra system)). Note that in [75], [76] the database with 270,000 exactly calculated Fourier–Legendre coefficients was described. This database was used in the software package, which is written in the Python programming language for the implementation of the numerical schemes (75)-(79), (82)-(86).

Assume that $q_1=6$. Calculating the value of expression (156) for $q_1=6$, $i_1\ne i_2$, $i_1\ne i_3$, $i_3\ne i_2$, we obtain

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q_1}\right)^2\right\}\approx 0.01956(T-t)^3.
$$

Let us choose, for example, $q_2=2$. In the case of pairwise different i_1,i_2,i_3,i_4 we have from (161) the following approximate equality

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)q_2}\right)^2\right\}\approx 0.0236084(T-t)^4. \tag{162}
$$

Let us analyze the approximations

Table 3. Coefficients $\bar{C}_{3j_2j_1}$.

$j_2 \backslash j_1$	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

Table 4. Coefficients $\bar{C}_{21j_2j_1}$.

$j_2 \backslash j_1$	0	1	2
0	$\frac{2}{21}$	$-\frac{2}{45}$	$\frac{2}{315}$
1	$\frac{2}{315}$	$\frac{2}{315}$	$-\frac{2}{225}$
2	$-\frac{2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 5. Coefficients $\bar{C}_{101j_2j_1}$.

$j_2 \backslash j_1$	0	1
0	$\frac{4}{315}$	0
1	$\frac{4}{315}$	$-\frac{8}{945}$

$$I_{(001)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(010)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(100)T,t}^{(i_1i_2i_3)q_3}, \quad I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}$$

based on the expansions (148)–(151).

Assume that $q_3 = 2$, $q_4 = 1$. In the case of pairwise different $i_1, \ldots, i_5$ we obtain

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)} - I_{(100)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{60} - \sum_{j_1,j_2,j_3=0}^{2} \left(C_{j_3j_2j_1}^{100}\right)^2 \approx 0.00815429(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)} - I_{(010)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{20} - \sum_{j_1,j_2,j_3=0}^{2} \left(C_{j_3j_2j_1}^{010}\right)^2 \approx 0.0173903(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)} - I_{(001)T,t}^{(i_1i_2i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{10} - \sum_{j_1,j_2,j_3=0}^{2} \left(C_{j_3j_2j_1}^{001}\right)^2 \approx 0.0252801(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)} - I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\} =$$

$$
=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.
$$

Note that using (131), we can write

$$
\mathsf{M}\left\{\left(I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}\right)^2\right\}\le 120\left(\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^2\right),
$$

where $i_1,\ldots,i_5=1,\ldots,m$.

Moreover, from (131) we obtain the following useful estimates

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(100)T,t}^{(i_1i_2i_3)}-I_{(100)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(010)T,t}^{(i_1i_2i_3)}-I_{(010)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(001)T,t}^{(i_1i_2i_3)}-I_{(001)T,t}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(20))T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1i_2)}-I_{(02)T,t}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(1000)T,t}^{(i_1i_2i_3i_4)}-I_{(1000)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(0100)T,t}^{(i_1i_2i_3i_4)}-I_{(0100)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(0010)T,t}^{(i_1i_2i_3i_4)}-I_{(0010)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(0001)T,t}^{(i_1i_2i_3i_4)}-I_{(0001)T,t}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right),
$$

$$
\mathsf{M}\left\{\left(I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)}-I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)q}\right)^2\right\}\le 720\left(\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6j_5j_4j_3j_2j_1}^2\right).
$$

In addition, from (128) we get

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{10}C_{j_1j_2}^{10}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(10)T,t}^{(i_1i_2)}-I_{(10)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{01}C_{j_1j_2}^{01}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(01)T,t}^{(i_1i_2)}-I_{(01)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{20}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{20}C_{j_1j_2}^{20}\quad (i_1=i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(20)T,t}^{(i_1i_2)}-I_{(20)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\quad (i_1\ne i_2),
$$

$$
\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1i_2)}-I_{(11)T,t}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{11}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{11}C_{j_1j_2}^{11}\quad (i_1=i_2),
$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)}-I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{11}\right)^2 \quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(02)}^{(i_1 i_2)}-I_{(02)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{02}C_{j_1 j_2}^{02} \quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{(02)T,t}^{(i_1 i_2)}-I_{(02)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2 \quad (i_1\ne i_2),$$

Clearly, expansions for iterated Stratonovich stochastic integrals (see Theorems 8, 10-12) are simpler than expansions for iterated Ito stochastic integrals (see Theorems 6, 7, and (93)–(98)). However, the calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Ito stochastic integrals (see Theorem 14 and (131)). Below we consider how we can estimate or calculate exactly (for some particular cases) the mean-square approximation error for iterated Stratonovich stochastic integrals (the development of these results is contained in Chapter 5 [58] (also see Theorems 10–13 from this paper).

As we mentioned above, on the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ in the mean-square sense for iterated Stratonovich stochastic integrals ($T-t\ll 1$ because the length of integration interval $[t,T]$ of the iterated Stratonovich stochastic integrals plays the role of integration step for the numerical methods for Ito SDEs, so $T-t$ is already fairly small). This leads to a sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals, which are required for achieving the acceptable accuracy of approximation.

From (152) ($i_1\ne i_2$) we obtain

$$\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)}-I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le$$

$$\le\frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx=-\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{(T-t)^2}{q}, \tag{163}$$

where C_1 is a constant.

Since $T-t\ll 1$, then it is easy to notice that there exists such a constant C_2 that

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C_2\mathsf{M}\left\{\left(I_{(00)T,t}^{*(i_1 i_2)}-I_{(00)T,t}^{*(i_1 i_2)q}\right)^2\right\}, \tag{164}$$

where $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}$ is an approximation of the iterated Stratonovich stochastic integral $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$.

From (163) and (164) we finally obtain

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C\frac{(T-t)^2}{q}, \tag{165}$$

where constant C does not depends on $T-t$. Note that, in contrast to the estimate (165), the constant C in Theorems 10–12 does not depend on q.

The same idea can be found in [2] in the framework of the method based on the trigonometric expansion of the Brownian bridge process.

We can get more information about the numbers q (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since for pairwise different $i_1,\ldots,i_k=1,\ldots,m$

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}\quad \text{w. p. 1},$$

where $J[\psi^{(k)}]_{T,t},\ J^*[\psi^{(k)}]_{T,t}$ are defined by (2) and (3) correspondingly, then for pairwise different $i_1,\ldots,i_6=1,\ldots,m$ we can write (see (127))

$$\mathsf{M}\left\{\left(I_{(01)T,t}^{*(i_1i_2)}-I_{(01)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(10)T,t}^{*(i_1i_2)}-I_{(10)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1i_2i_3i_4)}-I_{(0000)T,t}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(100)T,t}^{*(i_1i_2i_3)}-I_{(100)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(010))T,t}^{*(i_1i_2i_3)}-I_{(010)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(001)T,t}^{*(i_1i_2i_3)}-I_{(001)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)}-I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)q}\right)^2\right\}=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{(20)T,t}^{*(i_1i_2)}-I_{(20)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{(11)T,t}^{*(i_1i_2)}-I_{(11)T,t}^{*(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2,$$

$$
\mathsf{M}\left\{\left(I_{(02)T,t}^{*(i_1 i_2)}-I_{(02)T,t}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I_{(1000)T,t}^{*(i_1 i_2 i_3 i_4)}-I_{(1000)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\}=\frac{(T-t)^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{1000}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I_{(0100)T,t}^{*(i_1 i_2 i_3 i_4)}-I_{(0100)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\}=\frac{(T-t)^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0100}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I_{(0010)T,t}^{*(i_1 i_2 i_3 i_4)}-I_{(0010)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\}=\frac{(T-t)^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0010}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I_{(0001)T,t}^{*(i_1 i_2 i_3 i_4)}-I_{(0001)T,t}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\}=\frac{(T-t)^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0001}\right)^2,
$$

$$
\mathsf{M}\left\{\left(I_{(000000)T,t}^{*(i_1 i_2 i_3 i_4 i_5 i_6)}-I_{(000000)T,t}^{*(i_1 i_2 i_3 i_4 i_5 i_6)q}\right)^2\right\}=\frac{(T-t)^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6 j_5 j_4 j_3 j_2 j_1}^2.
$$

For example [46] (also see [26]-[29], [38], [47]-[61]),

$$
\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)6}\right)^2\right\}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{6}C_{j_3 j_2 j_1}^2\approx 0.01956000(T-t)^3,
$$

$$
\mathsf{M}\left\{\left(I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)}-I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)2}\right)^2\right\}=\frac{(T-t)^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C_{j_4 j_3 j_2 j_1}^2\approx 0.02360840(T-t)^4,
$$

$$
\mathsf{M}\left\{\left(I_{(100)T,t}^{*(i_1 i_2 i_3)}-I_{(100)T,t}^{*(i_1 i_2 i_3)2}\right)^2\right\}=\frac{(T-t)^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{100}\right)^2\approx 0.00815429(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(010)T,t}^{*(i_1 i_2 i_3)}-I_{(010)T,t}^{*(i_1 i_2 i_3)2}\right)^2\right\}=\frac{(T-t)^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{010}\right)^2\approx 0.0173903(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(001)T,t}^{*(i_1 i_2 i_3)}-I_{(001)T,t}^{*(i_1 i_2 i_3)2}\right)^2\right\}=\frac{(T-t)^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{001}\right)^2\approx 0.0252801(T-t)^5,
$$

$$
\mathsf{M}\left\{\left(I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)}-I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)1}\right)^2\right\}=
$$

$$
=\frac{(T-t)^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1} C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105(T-t)^5.
$$

Let us consider expansions of the Ito stochastic integrals $I_{(1)T,t}^{(i_1)}$, $I_{(2)T,t}^{(i_1)}$ based on the approach from [4] (also see [2])

$$
I_{(1)T,t}^{(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{166}
$$

$$
I_{(2)T,t}^{(i_1)q}=(T-t)^{5/2}\left(\frac{1}{3}\zeta_0^{(i_1)}+\frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)-\right.
$$

$$
\left.-\frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right), \tag{167}
$$

where $\zeta_j^{(i)}$ is defined by the formula (92), $\phi_j(\tau)$ is a complete orthonormal system of trigonometric functions in the space $L_2([t,T])$, and $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$ $(r=1,\ldots,q,\ \ i=1,\ldots,m)$ are independent standard Gaussian random variables, $i_1=1,\ldots,m$,

$$
\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)},\qquad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},
$$

$$
\mu_q^{(i)}=\frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\ \zeta_{2r}^{(i)},\qquad \beta_q=\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}.
$$

It is obvious that (166), (12) significantly more complicated compared to (135), (136).

Another example of obvious advantage of the Legendre polynomials over the trigonometric functions (in the framework of the considered problem) is the truncated expansion of the iterated Stratonovich stochastic integral $I_{(10)T,t}^{*(i_1i_2)}$ obtained by Theorem 8, in which instead of the double Fourier–Legendre series (see (137), (138)) is taken the double trigonometric Fourier series

$$
I_{(10)T,t}^{*(i_1i_2)q}=-(T-t)^2\left(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_2)}\zeta_0^{(i_1)}+\right.
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_2)}\zeta_0^{(i_1)}-2\mu_q^{(i_1)}\zeta_0^{(i_2)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left(-\frac{1}{\pi r}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\right)\right)-
$$

$$
-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{1}{r^2-l^2}\left(\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{l}{r}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right)+
$$

$$
+\sum_{r=1}^{q}\left(\frac{1}{4\pi r}\left(\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right)+\right.
$$

$$
\left.+\frac{1}{8\pi^2 r^2}\left(3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_2)}\zeta_{2r}^{(i_1)}\right)\right)\Biggr), \tag{168}
$$

where the meaning of the notations included in (166), (12) is preserved.

A deep comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito SDEs is given in [23], [39].

13. Theorems 6–8, 10–13 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimensional Wiener process $\mathbf{f}_s$, $s\in[0,T]$. Let $\mathbf{f}_s^{(i)p}$, $p\in\mathbb{N}$ be some approximation of $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. Suppose that $\mathbf{f}_s^{(i)p}$ converges to $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ if $p\to\infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $\mathbf{f}_s^{(i)}$ by $\mathbf{f}_s^{(i)p}$, $i=1,\ldots,m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$ of the multidimentional Wiener process $\mathbf{f}_s$? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [78], [79], it was shown that under the special conditions and for some types of approximations of the Wiener process the answere is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [78]-[80] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $\mathbf{f}_s$, $s\in[0,T]$ be an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$, $i=1,\ldots,m$. It is well known that the following representation takes place [81], [82]

$$
\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}=\sum_{j=0}^{\infty}\int\limits_t^\tau\phi_j(s)ds\ \zeta_j^{(i)},\quad \zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}, \tag{169}
$$

where $\tau\in[t,T]$, $t\ge 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j. Moreover, the series (169) converges for any $\tau\in[t,T]$ in the mean-square sense.

Let $\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}$ be the mean-square approximation of the process $\mathbf{f}_\tau^{(i)}-\mathbf{f}_t^{(i)}$, which has the following form

$$
\mathbf{f}_\tau^{(i)p}-\mathbf{f}_t^{(i)p}=\sum_{j=0}^{p}\int\limits_t^\tau\phi_j(s)ds\ \zeta_j^{(i)}. \tag{170}
$$

From (170) we obtain

$$d\mathbf{f}_{\tau}^{(i)p}=\sum_{j=0}^{p}\phi_j(\tau)\zeta_j^{(i)}d\tau. \tag{171}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}, \tag{172}$$

where $p_1,\ldots,p_k\in\mathbb{N},\ \ i_1,\ldots,i_k=0,1,\ldots,m,$

$$d\mathbf{w}_{\tau}^{(i)p}=\begin{cases}d\mathbf{f}_{\tau}^{(i)p}\ \ \ \text{for}\ \ \ i=1,\ldots,m\\ \\ d\tau\ \ \ \ \ \ \ \text{for}\ \ \ i=0\end{cases}, \tag{173}$$

and $d\mathbf{f}_{\tau}^{(i)p}$ in defined by the relation (171).

Let us substitute (171) into (172)

$$\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)p_1}\ldots d\mathbf{w}_{t_k}^{(i_k)p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{174}$$

where

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_s^{(i)}=\mathbf{f}_s^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_s^{(0)}=s$,

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k$$

is the Fourier coefficient.

To best of our knowledge [78]-[80] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [80] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (170) were not considered in [78], [79] (also see [80], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [80] for approximations of the Wiener process based on its series expansion (169) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (174) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [78], [79] (also see [80], Theorems 7.1, 7.2).

From the other hand, Theorems 8, 10–13 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6 based on the approximation (170) of the Wiener process. At that, the Riemann–Stieltjes integrals

(172) converge (according to Theorems 8, 10–13) to the appropriate Stratonovich stochastic integrals (3). Recall that $\{\phi_j(x)\}_{j=0}^{\infty}$ (see (169), (170), and Theorems 8, 10–13) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.

To illustrate the above reasoning, consider two examples for the case $k=2$, $\psi_1(s)$, $\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [78]-[80]).

Let $\mathbf{b}_{\Delta}^{(i)}(t)$, $t\in[0,T]$ be the piecewise linear approximation of the ith component $\mathbf{f}_t^{(i)}$ of the multidimensional standard Wiener process $\mathbf{f}_t$, $t\in[0,T]$ with independent components $\mathbf{f}_t^{(i)}$, $i=1,\ldots,m$, i.e.

$$\mathbf{b}_{\Delta}^{(i)}(t)=\mathbf{f}_{k\Delta}^{(i)}+\frac{t-k\Delta}{\Delta}\Delta\mathbf{f}_{k\Delta}^{(i)},$$

where

$$\Delta\mathbf{f}_{k\Delta}^{(i)}=\mathbf{f}_{(k+1)\Delta}^{(i)}-\mathbf{f}_{k\Delta}^{(i)},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1.$$

Note that w. p. 1

$$\frac{d\mathbf{b}_{\Delta}^{(i)}}{dt}(t)=\frac{\Delta\mathbf{f}_{k\Delta}^{(i)}}{\Delta},\quad t\in[k\Delta,(k+1)\Delta),\quad k=0,1,\ldots,N-1. \tag{175}$$

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s),\quad i_1,i_2=1,\ldots,m.$$

Using (175) and the additive property of Riemann–Stieltjes integrals, we can write w. p. 1

$$\int\limits_0^T\int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau)d\mathbf{b}_{\Delta}^{(i_2)}(s)=\int\limits_0^T\int\limits_0^s \frac{d\mathbf{b}_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau\frac{d\mathbf{b}_{\Delta}^{(i_2)}}{ds}(s)ds=$$

$$=\sum_{l=0}^{N-1}\int\limits_{l\Delta}^{(l+1)\Delta}\left(\sum_{q=0}^{l-1}\int\limits_{q\Delta}^{(q+1)\Delta}\frac{\Delta\mathbf{f}_{q\Delta}^{(i_1)}}{\Delta}d\tau+\int\limits_{l\Delta}^{s}\frac{\Delta\mathbf{f}_{l\Delta}^{(i_1)}}{\Delta}d\tau\right)\frac{\Delta\mathbf{f}_{l\Delta}^{(i_2)}}{\Delta}ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{\Delta^2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}\int\limits_{l\Delta}^{(l+1)\Delta}\int\limits_{l\Delta}^{s}d\tau ds=$$

$$=\sum_{l=0}^{N-1}\sum_{q=0}^{l-1}\Delta\mathbf{f}_{q\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}+\frac{1}{2}\sum_{l=0}^{N-1}\Delta\mathbf{f}_{l\Delta}^{(i_1)}\Delta\mathbf{f}_{l\Delta}^{(i_2)}. \tag{176}$$

Using (176) and the standard relation between Stratonovich and Ito stochastic integrals, it is not difficult to show that

$$\underset{N\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{b}_{\Delta}^{(i_1)}(\tau) d\mathbf{b}_{\Delta}^{(i_2)}(s) = \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} \int\limits_0^T ds =$$

$$= \int\limits_0^{*T} \int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)}, \tag{177}$$

where $\Delta \to 0$ if $N \to \infty$ $(N\Delta = T)$.

Obviously, (177) agrees with Theorem 7.1 (see [80], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (169) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0,T])$.

Consider the following iterated Riemann–Stieltjes integral

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m, \tag{178}$$

where $d\mathbf{f}_{\tau}^{(i)p}$ is defined by the relation (171).

Let us substitute (171) into (178)

$$\int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \tag{179}$$

where

$$C_{j_2 j_1} = \int\limits_0^T \phi_{j_2}(s) \int\limits_0^s \phi_{j_1}(\tau) d\tau ds$$

is the Fourier coefficient; another notations are the same as in (174).

As we noted above, approximations of the Wiener process that are similar to (170) were not considered in [78], [79] (also see Theorems 7.1, 7.2 in [80]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [80] to the case under consideration is not obvious.

However, the authors of the works [2] (Sect. 5.8, pp. 202–204), [3] (pp. 82-84), [83] (pp. 438-439), [84] (pp. 263-264) use the Wong–Zakai approximation [78]-[80] (without rigorous proof) within the frames of the approach [4] based on the series expansion of the Brownian bridge process.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [58]-[61]. More precisely, using Theorem 8 from this paper we obtain from (179) the desired result

$$\underset{p\to\infty}{\text{l.i.m.}} \int\limits_0^T \int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p} d\mathbf{f}_{s}^{(i_2)p} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} =$$

$$
=\int\limits_0^{*T}\int\limits_0^{*s} d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_s^{(i_2)}. \tag{180}
$$

From the other hand, by Theorem 6 (see (94)) for the case $k=2$ we obtain from (179) the following relation

$$
\underset{p\to\infty}{\text{l.i.m.}}\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)p}d\mathbf{f}_s^{(i_2)p}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}=
$$

$$
=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right)+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}=
$$

$$
=\int\limits_0^T\int\limits_0^s d\mathbf{f}_{\tau}^{(i_1)}d\mathbf{f}_s^{(i_2)}+\mathbf{1}_{\{i_1=i_2\}}\sum_{j_1=0}^{\infty}C_{j_1j_1}. \tag{181}
$$

Since

$$
\sum_{j_1=0}^{\infty}C_{j_1j_1}=\frac{1}{2}\sum_{j_1=0}^{\infty}\left(\int\limits_0^T\phi_j(\tau)d\tau\right)^2=\frac{1}{2}\left(\int\limits_0^T\phi_0(\tau)d\tau\right)^2=\frac{1}{2}\int\limits_0^T ds,
$$

then from (181) and the standard relation between Stratonovich and Ito stochastic integrals we obtain (180).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.
[3] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.
[4] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.
[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.
[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.
[7] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.
[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N.Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635
[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N.Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[10] Kulchitski O.Yu., Kuznetsov D.F. Expansion of Ito processes into a Taylor–Ito series in a neighborhood of a fixed time moment, Dep. VINITI, No. 2637-93, 1993, 25 pp.
[11] Kuznetsov D.F. Methods of numerical simulation of stochastic differential Ito equations solutions in problems of mechanics. Ph. D., Saint-Petersburg, 1996. 260 pp.
[12] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2023, 144 pp. [in English].

[13] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2023, 58 pp. [in English].

[14] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2023, 71 pp. [in English].

[15] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [in English].

[16] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2023, 222 pp. [in English].

[17] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2023, 80 pp. [in English].

[18] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicities 1 to 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 66 pp. [in English].

[19] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combained approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 46 pp. [In English].

[20] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR]. 2018, 20 pp. [In English].

[21] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2023, 148 pp. [in English].

[22] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2023, 158 pp. [in English].

[23] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].

[24] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 40 pp. [In English].

[25] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 29 pp. [In English].

[26] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1806.10705 [math.PR]. 2018, 29 pp. [In English].

[27] Kuznetsov D.F, Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Ito expansion. arXiv:1805.12527 [math.PR]. 2018, 29 pp. [In English].

[28] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv: 1802.04844 [math.PR]. 2018, 37 pp. [in English].

[29] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR], 2018, 44 pp. [in English].

[30] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method arXiv:1801.01962 [math.PR]. 2023, 49 pp. [in English].

[31] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[32] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[33] Kuznetsov D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. arXiv:1801.04634 [math.PR], 2018, 27 pp. [In English].

[34] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor-Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. arXiv:1801.08862 [math.PR], 2018, 36 p. [In English].

[35] Kuznetsov D.F. New simple method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on expansion of the Brownian motion using Legendre polynomials and trigonometric functions. arXiv:1807.00409 [math.PR], 2019, 23 pp. [In English].

[36] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[37] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056
[38] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[39] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116
[40] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604
[41] Kuznetsov D.F. Explicit one-step numerical method with the order of strong convergence 2.5 for Ito stochastic differential equations with multidimensional nonadditive noise, based on the Taylor-Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100
[42] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[43] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html
[44] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
[45] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf
[46] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)
[47] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)
[48] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)
[49] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)
[50] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)
[51] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html
[52] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)
[53] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011,

284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[54] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[55] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2017), A.1-A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[56] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[57] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[58] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. arXiv:2003.14184v45 [math.PR], 2023, 996 pp. [In English].

[59] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[60] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[61] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs (Third Edition). [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2023), A.1-A.947. Available at:
http://diffjournal.spbu.ru/EN/numbers/2023.1/article.1.10.html

[62] Kuznetsov D.F. A new proof of the expansion of iterated Ito stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials. arXiv:2307.11006 [math.PR], 2023, 58 pp. [In English].

[63] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[64] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 4 (2022), 135-194. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.4/article.1.9.html

[65] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[66] Kuznetsov D.F. Theorems about integration order replacement in iterated stochastic integrals. Dep. VINITI. 3607-V97, 1997, 31 pp.

[67] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[68] Kuznetsov D.F. Integration order replacement in iterated stochastic integrals with respect to martingale. Preprint. Saint-Petersburg: SPbGTU Publ., 1999 , 11 pp. Available at: http://www.sde-kuznetsov.spb.ru/99c.pdf

[69] Stratonovich R.L. Conditional Markov Processes and Their Application to the Theory of Optimal Control [in Russian], Moscow University Press, Moscow, 1966, 320 pp.

[70] Jentzen A. and Röckner M. A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.

[71] Becker S., Jentzen A. and Kloeden P.E. An exponential Wagner-Platen type scheme for SPDEs. SIAM J. Numer. Anal. 54, 4 (2016), 2389-2426.

[72] Jentzen A. and Röckner M. Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. J. Differ. Eq. 252, 1 (2012), 114-136.

[73] Da Prato G., Jentzen A. and Röckner M. A mild Itô formula for SPDEs. arXiv:1009.3526 [math.PR] (2012), 39 pp.

[74] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differential Equations and Control Processes, 3 (2021), 109-140. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[75] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes"ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[76] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. arXiv:2009.14011 [math.PR], 2020, 342 pp. [In English].

[77] Rybakov, K.A. On traces of linear operators with symmetrized Volterra–type kernels. Symmetry, 15, 1821 (2023), 1-18. DOI: http://doi.org/10.3390/sym15101821

[78] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[79] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[80] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[81] Liptser R.Sh., Shirjaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[82] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[83] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl. 10, 4 (1992), 431-441.

[84] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

TO NUMERICAL MODELING WITH STRONG ORDERS 1.0, 1.5, AND 2.0 OF CONVERGENCE FOR MULTIDIMENSIONAL DYNAMICAL SYSTEMS WITH RANDOM DISTURBANCES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to explicit one-step numerical methods with strong orders 1.0, 1.5, and 2.0 of convergence for Ito stochastic differential equations with multidimensional and non-commutative noise. For numerical modeling of iterated Ito stochastic integrals with multiplicities 1 to 4 we use the method of multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k = 1,2,3,4$. The article is addressed to engineers who use numerical modeling in stochastic control and for solving the nonlinear filtering problem.

Contents

1. Introduction

The Ito stochastic differential equations (SDEs) are known to be adequate mathematical models of the dynamical systems of various physical nature subjected to random perturbations [2]-[5]. On the assumption of strong convergence criterion [2], the need for numerical integration of Ito SDEs arises

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at solving the different mathematical problems. Among them we mention the following problems: stochastic optimal control (also with incomplete data) [2], [6], signal filtering in random noise in various formulations [2], [6], estimating the parameters of stochastic systems [2], [3]. It is common knowledge that one of the promising approaches to the numerical integration of Ito SDEs is the approach based on the stochastic analogues of the Taylor formula, the so-called Taylor–Ito and Taylor–Stratonovich expansions [2], [3], [7]-[12]. This approach makes use of finite discretization of the time variable and implies numerical modeling of the solution of Ito SDE at the discrete time instants using the stochastic analogues of the Taylor formula obtained by iterative application of the Ito formula.

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Ito SDE in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0,\omega), \quad \omega \in \Omega. \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the Ito SDE (1). The nonrandom functions $\mathbf{a}: \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B: \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the Ito SDE (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

The most important feature of stochastic analogues of the Taylor formula [2], [3], [7]-[12] for solutions of the Ito SDE (1) consists in the presence of iterated Ito and Stratonovich stochastic integrals. These stochastic integrals are complicated functionals from the components of the multidimensional Wiener process. In one of the most general forms of notation of the present paper, the aforementioned iterated Ito and Stratonovich stochastic integrals are given, respectively, by

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$,

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively, $i_1,\ldots,i_k = 0,1,\ldots,m$ (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Consequently, the systems of stochastic integrals like (2), (3) play an important part in solving the problem of numerical integration of the Ito SDEs (1). In terms of the mean-square convergence criterion, the problem of efficient joint numerical modeling of the totalities of stochastic integrals of the kind (2), (3) (the case of a multidimensional Wiener process) is not only important, but also sufficiently complex in both the theoretical and computational terms. We note that the aforementioned problem does not arise at using the Euler method for the Ito SDEs (1) [2], [7]. However, despite its simplicity, the Euler method under the standard conditions [2], [7] for coefficients of the Ito SDE (1) has the mean-square convergence order 0.5 [2], [7], and its accuracy is insufficient to solve a number of practical problems. This fact motivates one to construct numerical methods for the Ito SDEs (1) having higher orders of strong convergence.

It may seem at the first glance that the stochastic integrals from the families (2), (3) can be approximated by the multiple integral sums. However, this leads to partitioning of the interval of integration $[t, T]$ of the iterated stochastic integrals. The mentioned interval is already a small value because it represents a step of integration in the numerical methods for Ito SDEs. As the numerical experiments show [13], the above partitioning gives rise to an unacceptably high computing costs.

A number of publications are devoted to methods of numerical modeling of families of stochastic integrals like (2), (3), which do not use partitioning of the aforementioned interval of integration $[t, T]$ and converge in the mean-square sense. It was suggested in [7] to use converging in the mean-square sense trigonometric Fourier expansions of the Wiener processes, which underlie the iterated stochastic integral. By this method, the mean-square approximations of the simplest integrals like (2) of multiplicities 1 and 2 ($k = 2$; $\psi_1(s)$, $\psi_2(s) \equiv 1$; $i_1, i_2 = 0, 1, \ldots, m$) were obtained in [7]. These approximations were used in [7] to construct a numerical method for the Ito SDE (1), which under certain conditions [7] has the order 1.0 of the mean-square convergence and is known as the Milstein method.

A more general method of the mean-square approximation of the stochastic integrals like (3), which based on the generalized iterated Fourier series was proposed in [14], [15]. It enables one to use the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$. In virtue of its characteristics, the method from [7] admits the application of only trigonometric basis functions.

In [2], [3], [16], [17] an attempt was made to extend the method from [7] to the stochastic integrals like (3) for $k = 3$; $\psi_1(s), \ldots, \psi_3(s) \equiv 1$; $i_1, \ldots, i_3 = 0, 1, \ldots, m$.

We note that the methods [2], [3], [16], [17] ($k = 3$) and [14] ($k \geq 3$) lead to iterated application of the operation of limit transition. As a result, these methods allow us to represent the integrals (3) as iterated series of products of standard Gaussian random variables (the operation of passing to the limit is carried out iteratively). This fact is essential and imposes some constraints related with the method of summation of the aforementioned series [2], [3], [14], [16], [17] if we consider the stochastic integrals like (2), (3) of multiplicities 3 and higher (we mean here at least triple integration over the Wiener processes). Additionally, the aforementioned methods in virtue of their features prevent precise calculation of the mean-square error of approximation with the exception of the simplest iterated stochastic integrals of multiplicity 2. This means that at the stage of realization of the numerical methods for Ito SDEs, possibly, one will need to allow for the redundant terms of the expansions of iterated stochastic integrals, which increases the computing costs and reduces efficiency of the numerical methods.

We notice [2], [13] that to construct numerical methods for the Ito SDE (1) having orders 1.5 and 2.0 of strong convergence one has to approximate (proceeding from the mean-square convergence criterion) the stochastic integrals not only of multiplicities 1 and 2, but also 3 and 4 from the families (2), (3). Some publications [2], [7], [8] contain the aforementioned numerical schemes with orders 1.5 and 2.0 of strong convergence but without the contained in them efficient procedures of the mean-square approximation of iterated stochastic integrals for the case of a multidimensional Wiener process, which corresponds to $i_1, \ldots, i_4 = 1, \ldots, m$ in (2), (3). Part of publications (see, for example, [2], [8]) contain representations of the stochastic integrals of multiplicities 3 and 4 like (2), (3) only for the simplest case $\psi_1(s), \ldots, \psi_4(s) \equiv 1$, $i_1 = \ldots = i_4$ (representations based on the Hermit polynomials). Some publications [8] use other simplifying assumptions about the Ito SDE (1). For example, assumptions are made about additivity of the stochastic perturbation or its smallness, which corresponds, respectively, to $B(\mathbf{x},t) \equiv C(t)$ or $B(\mathbf{x},t) \equiv \varepsilon D(\mathbf{x},t)$. Here, $\varepsilon > 0$ is a fixed small number and $C : [0,T] \to \mathbb{R}^{n\times m}$, $D : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$.

In the case at hand, the problem of efficient joint numerical modeling of the iterated stochastic integrals from the families (2), (3) becomes simpler due to the absence of some terms in the expressions of the numerical methods or the possibility of disregarding some of the aforementioned terms. Also, one may encounter approximation method [18] for iterated stochastic integrals of multiplicity 3 from the familiy (2) for $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s) \equiv 1$ $(i_1, i_2, i_3 = 1, \ldots, m)$ based on partitioning of the interval of integration $[t, T]$ of the iterated stochastic integrals and using multiple integral sums whose disadvantages were mentioned above.

The present paper is devoted to the development of efficient procedures for joint numerical modeling of the iterated stochastic integrals from the families (2), (3) in accordance with the mean-square criterion of convergence. At that we do not use any essential simplifying assumptions, that is, the Wiener process involved in the Ito SDE (1) is assumed to be the multidimensional one which corresponds to the condition $i_1, \ldots, i_k = 0, 1, \ldots, m$ in (2), (3). In addition, it is assumed that the stochastic perturbation is nonadditive (the simplifying assumptions about the function $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ involved in (1) are not introduced). Additionally, the functions $\psi_1(s), \ldots, \psi_k(s)$ in (2), (3) are, generally speaking, assumed to be different. Moreover, the assumption of commutativity [2], [3] of the stochastic perturbation is also not introduced.

More precisely, in this paper we consider the method of the mean-square approximation of iterated Ito stochastic integrals from the family (2), which is based on the generalized multiple (not iterated) Fourier series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ $(k \in \mathbb{N})$ [13] (2006), [19]-[55]. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202–204), [3] (pp. 82-84), [16] (pp. 438-439), [17] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the approach based on the Karhunen–Loeve expansion of the Brownian bridge process [7] together with the Wong–Zakai approximation [56]-[58]. See discussions in [32] (Sect. 2.18, 6.2), [33], [34] (Sect. 2.6.2, 6.2) for details.

2. Numerical Schemes With the Orders 1.0, 1.5, and 2.0 of Strong Convergence

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of the segment $[0,T]$ with the partition rank Δ_N such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T.$$

Denote by $\mathbf{y}_{\tau_j} \stackrel{\sf def}{=} \mathbf{y}_j;\ j = 0, 1, \ldots, N$ the discrete approximation of the process $\mathbf{x}_t$, $t \in [0,T]$ (solution of the Ito SDE (1) corresponding to the maximal step of discretization Δ_N.

Definition 1 [2]. *We will say that the discrete approximation* (*numerical method*) $\mathbf{y}_j;\ j = 0, 1, \ldots, N$ *corresponding to the maximal step of discretization* Δ_N *converges strongly with the order* $\gamma > 0$ *at the time instant* T *to the process* $\mathbf{x}_t$, $t \in [0,T]$ *if there exist a constant* $C > 0$ *independent of* Δ_N *and a number* $\delta > 0$ *such that*

$$\mathsf{M}\left\{|\mathbf{x}_T - \mathbf{y}_T|\right\} \le C(\Delta_N)^{\gamma} \qquad (4)$$

for all $\Delta_N \in (0, \delta)$.

We note that the authors of some publications [7], [8] prefer to consider the mean-square convergence instead of the strong convergence.

Definition 2 [7], [8]. *We will say that the numerical method* $\mathbf{y}_j;\ j=0,1,\ldots,N$ *converges in the mean-square sense with the order* $\gamma>0$ *to the process* $\mathbf{x}_t,\ t\in[0,T]$ *if there exist a constant* $C>0$ *independent of* Δ_N, j *and a number* $\delta>0$ *such that*

$$\left(\mathsf{M}\left\{|\mathbf{x}_j-\mathbf{y}_j|^2\right\}\right)^{1/2}\le C(\Delta_N)^{\gamma}$$

for all $\Delta_N\in(0,\delta)$.

Here, $\mathbf{x}_{\tau_j}\stackrel{\sf def}{=}\mathbf{x}_j;\ j=0,1,\ldots,N$.

We notice that sometimes the condition (4) in Definition 1 is replaced by the condition [2]

$$\mathsf{M}\left\{|\mathbf{x}_j-\mathbf{y}_j|\right\}\le C(\Delta_N)^{\gamma}\quad (j=0,1,\ldots,N)$$

At that, the constant C is independent of Δ_N and j.

Strong convergence follows, obviously, from the mean-square convergence in virtue of the Lyapunov inequality. In what follows, we rely on Definition 1 of strong convergence.

Consider the following explicit one-step numerical method

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i=1}^m B_i\hat I^{(i)}_{(0)\tau_{p+1},\tau_p}+\Delta\mathbf{a}+\sum_{i,j=1}^m G_jB_i\hat I^{(ji)}_{(00)\tau_{p+1},\tau_p}+$$
$$+\sum_{i=1}^m\left(G_i\mathbf{a}\left(\Delta\hat I^{(i)}_{(0)\tau_{p+1},\tau_p}+\hat I^{(i)}_{(1)\tau_{p+1},\tau_p}\right)-LB_i\hat I^{(i)}_{(1)\tau_{p+1},\tau_p}\right)+$$
$$+\sum_{i,j,l=1}^m G_lG_jB_i\hat I^{(lji)}_{(000)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}L\mathbf{a}\tag{5}$$

corresponding to the constant discretization step $\Delta=T/N$ ($\tau_p=p\Delta;\ p=0,1,\ldots,N;\ N>1$), where $\hat I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}$ denotes approximation of the iterated Ito stochastic integral

$$I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}=\int\limits_t^s(t-\tau_k)^{l_k}\ldots\int\limits_t^{\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}^{(i_1)}_{\tau_1}\ldots d\mathbf{f}^{(i_k)}_{\tau_k},\tag{6}$$

and

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^n\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^m\sum_{l,i=1}^n B_{lj}(\mathbf{x},t)B_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_i},$$

$$G_i=\sum_{j=1}^n B_{ji}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_j}\quad(i=1,\ldots,m),$$

$l_1,\ldots,l_k=0,1,2\ldots;\ i_1,\ldots,i_k=1,\ldots,m;\ k=1,2,\ldots;$ B_i and B_{ij} are, respectively, the ith column and ijth element of the matrix function B; $\mathbf{a}_i$ and $\mathbf{x}_i$ are, respectively, the ith components of the vector function $\mathbf{a}$ and column $\mathbf{x}$; the columns

$$B_i,\quad \mathbf{a},\quad G_jB_i,\quad G_i\mathbf{a},\quad LB_i,\quad G_lG_jB_i,\quad L\mathbf{a}$$

are calculated at the point $(\mathbf{y}_p, p)$.

The numerical scheme (5) can be found, for example, in a somewhat different form in [2], [7], [8]. The difference here lies in that the author of this work used in (5) the relation

$$\Delta I^{(i)}_{(0)\tau_{p+1},\tau_p} + I^{(i)}_{(1)\tau_{p+1},\tau_p} = \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{\tau} d\mathbf{f}^{(i)}_s d\tau \tag{7}$$

which follows with probability 1 from the Ito formula and enables one to reduce by one the number of iterated Ito stochastic integrals to be approximated. This is due to the fact that the Ito stochastic integral on the right-hand side of (7) is expressed as a linear combination of the Ito stochastic integrals

$$I^{(i)}_{(0)\tau_{p+1},\tau_p} \quad \text{and} \quad I^{(i)}_{(1)\tau_{p+1},\tau_p},$$

whose approximations are already included in the right-hand side of (5).

It is common knowledge that under certain conditions [2] the discrete approximation (numerical method) (5) has the order 1.5 of strong convergence. Among the aforementioned conditions we note only the condition for approximations of the iterated Ito stochastic integrals involved in (5)

$$\mathsf{M}\left\{\left(I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p} - \hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\right)^2\right\} \le C\Delta^r, \tag{8}$$

where $r = 4$ and the constant C is independent of Δ, because the present paper deals mostly with the approximation of the aforementioned stochastic integrals.

Conditions somewhat different from [2] are given in [8]. Under them the numerical method (5) has the order 1.5 of the mean-square convergence.

Note that the Milstein method [7] (method with the order 1.0 of strong convergence) corresponds to the first line in (5).

Consider the explicit one-step numerical method with the order 2.0 of strong convergence given by

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i=1}^{m} B_i \hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \Delta\mathbf{a} + \sum_{i,j=1}^{m} G_j B_i \hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p} +$$

$$+ \sum_{i=1}^{m}\left(G_i\mathbf{a}\left(\Delta\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}\right) - LB_i\hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}\right) +$$

$$+ \sum_{i,j,l=1}^{m} G_l G_j B_i \hat{I}^{(lji)}_{(000)\tau_{p+1},\tau_p} + \frac{\Delta^2}{2} L\mathbf{a} +$$

$$+ \sum_{i,j=1}^{m}\Biggl(G_0^{(j)} LB_i\left(\hat{I}^{(ji)}_{(10)\tau_{p+1},\tau_p} - \hat{I}^{(ji)}_{(01)\tau_{p+1},\tau_p}\right) - LG_jB_i\hat{I}^{(ji)}_{(10)\tau_{p+1},\tau_p} +$$

$$+ G_jG_i\mathbf{a}\left(\hat{I}^{(ji)}_{(01)\tau_{p+1},\tau_p} + \Delta\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}\right)\Biggr) +$$

$$+ \sum_{i,j,l,r=1}^{m} G_r G_l G_j B_i \hat{I}^{(rlji)}_{(0000)\tau_{p+1},\tau_p}, \tag{9}$$

where notation corresponds to (5).

The numerical scheme (9) can be found in another representation in [2], [8]. In this case the distinctions are due to the fact that along with (7) the author used in (9) the equalities

$$I_{(01)\tau_{p+1},\tau_p}^{(ji)} + \Delta I_{(00)\tau_{p+1},\tau_p}^{(ji)} = \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{\theta} \int\limits_{\tau_p}^{\tau} d\mathbf{f}_s^{(j)} d\mathbf{f}_\tau^{(i)} d\theta \qquad (10)$$

$$I_{(10)\tau_{p+1},\tau_p}^{(ji)} - I_{(01)\tau_{p+1},\tau_p}^{(ji)} = \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{\theta} \int\limits_{\tau_p}^{\tau} d\mathbf{f}_s^{(j)} d\tau d\mathbf{f}_\theta^{(i)}, \qquad (11)$$

which follow with probability 1 from the Ito formula and enable one to reduce by one more unit the number of iterated Ito stochastic integrals to be approximated. This is due to the fact that the Ito stochastic integrals on the right-hand sides of (10) and (11) are expressed as linear combinations of the Ito stochastic integrals

$$I_{(01)\tau_{p+1},\tau_p}^{(ji)}, \quad I_{(10)\tau_{p+1},\tau_p}^{(ji)}, \quad I_{(00)\tau_{p+1},\tau_p}^{(ji)},$$

whose approximations are already included in the right-hand side of (9).

We notice that under certain conditions [2] the numerical method (9) has the order 2.0 of strong convergence. Among the aforementioned conditions we mark only the condition (8) for $r = 5$ intended for approximations of the iterated Ito stochastic integrals included in (9).

Some modifications of the numerical methods (5) and (9) were constructed in [2], [8]. Among which there are finite-difference methods of the Runge–Kutta type as well as the implicit and two-step methods (also see [13], [19]-[22], [30]-[34]). In all aforementioned methods, however, a need arises for efficient joint mean-square approximation of the iterated Ito stochastic integrals of multiplicities 1 to 4. The collection of these integrals is the same as in the numerical methods (5) and (9).

3. Expansion of Iterated Ito Stochastic Integrals of Multiplicity k ($k \in \mathbb{N}$) Based on Generalized Multiple Fourier Series

An efficient mean-square approximation method for the iterated Ito stochastic integrals like (2) was proposed and developed by the author of this article in [13], [19]-[55] (see Theorems 1, 2 below). This method based on the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$, $k \in \mathbb{N}$. At that the method [13], [19]-[55] allows to use different complete orthonormal systems of functions in the space $L_2([t,T]^k)$, $k \in \mathbb{N}$. In this article, we use the system of Legendre polynomials, which has a series of advantages over the system of trigonometric functions in the framework of the considered problem [43], [44]. Moreover, in this method the passage to the limit is carried out only once, which leads to a correct choice of the lengths of sequences of the standard Gaussian random variables required to approximate the iterated Ito stochastic integrals.

Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a nonrandom function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\ & \\ 0 & \text{otherwise} \end{cases}, \quad t_1,\ldots,t_k \in [t,T], \quad k \geq 2, \qquad (12)$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \left\| K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)} = 0,$$

where

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{13}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2},$$

and the Parceval equality

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1} \tag{14}$$

takes place.

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{15}$$

Theorem 1 [13] (2006), [19]-[55]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{16}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$
\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k, \quad \mathrm{H}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1\},
$$

$$
\mathrm{L}_k = \{(l_1, \ldots, l_k) : \ l_1, \ldots, l_k = 0, \ 1, \ldots, N-1; \ l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},
$$

l.i.m. *is a limit in the mean-square sense*, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$
\zeta_j^{(i)} = \int\limits_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \tag{17}
$$

are independent standard Gaussian random variables for various i *or* j (*if* $i \neq 0$), $C_{j_k \ldots j_1}$ *is the Fourier coefficient* (13), $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t, T]$, *which satisfies the condition* (15).

It was shown in [20]-[27], [30]-[34] that Theorem 1 is valid for convergence in the mean of degree $2n$ ($n \in \mathbb{N}$). Moreover, the convergence with probability 1 in Theorem 1 is proved in [32]-[34], [63]. In addition, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t, T])$ also can be applied in Theorem 1 [13], [19]-[27], [30]-[34]. The modification of Theorem 1 for complete orthonormal with weigth $r(x) \geq 0$ systems of functions in the space $L_2([t, T])$ can be found in [31], [32]-[34], [40]. Application of Theorem 1 and Theorem 2 (see below) to the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process contains in [32]-[34] (Chapter 7), [46], [55], [64], [66].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 5$ [13] (2006), [19]-[55] (the cases $k = 6, 7$ and $k > 7$ ($k \in \mathbb{N}$) can also be found in these papers)

$$
J[\psi^{(1)}]_{T,t} = \underset{p_1 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \tag{18}
$$

$$
J[\psi^{(2)}]_{T,t} = \underset{p_1, p_2 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \right), \tag{19}
$$

$$
J[\psi^{(3)}]_{T,t} = \underset{p_1, p_2, p_3 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} -
$$

$$
- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{20}
$$

$$
J[\psi^{(4)}]_{T,t} = \underset{p_1, \ldots, p_4 \to \infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_4=0}^{p_4} C_{j_4 \ldots j_1} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -
$$

$$
- \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1 = i_3 \neq 0\}} \mathbf{1}_{\{j_1 = j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1 = i_4 \neq 0\}} \mathbf{1}_{\{j_1 = j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2 = i_3 \neq 0\}} \mathbf{1}_{\{j_2 = j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$
$$
\left.+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\right), \tag{21}
$$

$$
J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\right.
$$
$$
-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$
$$
-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$
$$
+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$
$$
+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$
$$
+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$
$$
\left.+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{22}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

For further consideration, let us consider the generalization of formulas (18)–(22) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{23}
$$

where

$$
\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (23) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{24}$$

Below there are several examples of sums in the form (24)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (16) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{25}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (25) for $k=5$ we obtain

$$
J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -
$$

$$
- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +
$$

$$
+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\ne 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\ne 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).
$$

The last equality obviously agrees with (22).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [32] (Sect. 1.11), [39] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$
J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{26}
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [68]. Note that we use another notations [32] (Sect. 1.11), [39] (Sect. 15) in comparison with [68]. Moreover, the proof of an analogue of Theorem 2 from [68] is somewhat different from the proof given in [32] (Sect. 1.11), [39] (Sect. 15).

4. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6

As it turned out [23]-[27], [30]-[34], [47], [71], [72] Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3). At that the expansions of the integrals (3) turn out to be much simpler than the expansions of the iterated Ito stochastic integrals (2). Let us first present some old results as the following theorem.

Theorem 3 [23]-[27], [30]-[34], [47]. *Assume that the following conditions are fulfilled*:

1. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is the complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

2. *The function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$, *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$ (*in* (27) *and* (29)).

Then, the iterated Stratonovich stochastic integrals (3) *of multiplicities* 2–4 *are expanded into the mean-square converging multiple series*

$$J^*[\psi^{(2)}]_{T,t} = \underset{q_1,q_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{27}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{q_1,q_2,q_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{28}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{29}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{30}$$

where we assume that $i_1, i_2, i_3 = 1, \ldots, m$ *in* (27)–(29) *and* $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (30)*. Additionally, we assume in* (28) *and* (30) *that* $\psi_1(\tau), \ldots, \psi_4(\tau) \equiv 1$*. Another notations are the same as in Theorems* 1, 2*.*

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [32] (Sect. 2.10–2.16), [37] (Sect. 5–11), [47] (Sect. 13–19), [49] (Sect. 7–13), [71] (Sect. 4–9), [72]. Let us formulate four theorems that were obtained using this approach.

Theorem 4 [32], [37], [47], [49], [71]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$*. Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{31}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{32}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (31) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (32)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 5 [32], [37], [47], [49], [71]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{33}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{34}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{35}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (33), (34) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (35), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 4.

Theorem 6 [32], [37], [47], [49], [71]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{36}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}, \tag{37}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{38}$$

are fulfilled, where $i_1,\ldots,i_5 = 0,1,\ldots,m$ *in* (36), (37) *and* $i_1,\ldots,i_5 = 1,\ldots,m$ *in* (38), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^{T} \psi_5(t_5)\phi_{j_5}(t_5)\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_5;$$

another notations are the same as in Theorems 4, 5*.*

Theorem 7 [32], [37], [47], [49], [72]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{39}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^{T} \phi_{j_6}(t_6)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_6;$$

another notations are the same as in Theorems 4–6*.*

5. Legendre Polynomial-Based Approximation of the Iterated Ito and Stratonovich Stochastic Integrals Used in the Applications

We notice that the collection of iterated Ito stochastic integrals used in the numerical methods (5), (9) is given by

$$I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1 i_2)},\quad I_{(000)T,t}^{(i_1 i_2 i_3)},\quad I_{(01)T,t}^{(i_1 i_2)},\quad I_{(10)T,t}^{(i_1 i_2)},\quad I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)},\tag{40}$$

where $i_1,\ldots,i_4=1,\ldots,m$.

The functions $K(t_1,\ldots,t_k)$ like (12) for the collection (40) are given, respectively, by

$$K_0(t_1)\equiv 1,\quad K_1(t_1)=t-t_1,\quad K_{00}(t_1,t_2)=\mathbf{1}_{\{t_1<t_2\}},$$

$$K_{000}(t_1,t_2,t_3)=\mathbf{1}_{\{t_1<t_2<t_3\}},\quad K_{01}(t_1,t_2)=(t-t_2)\mathbf{1}_{\{t_1<t_2\}},$$

$$K_{10}(t_1,t_2)=(t-t_1)\mathbf{1}_{\{t_1<t_2\}},\quad K_{0000}(t_1,\ldots,t_4)=\mathbf{1}_{\{t_1<t_2<t_3<t_4\}},$$

where $t_1,\ldots,t_4\in[t,T]$ and $\mathbf{1}_A$ is the indicator of the set A.

For a finite-degree polynomial, the simplest (having a finite number of terms) expansion into Fourier series by the complete orthonormal system of functions in the space $L_2([t,T])$ is the Fourier–Legendre series expansion. The polynomial functions are included in the functions $K_1(t_1)$, $K_{01}(t_1,t_2)$, $K_{10}(t_1,t_2)$ as their components. Therefore, it is logical to expect that the simplest expansions of these functions into multiple Fourier series are their Fourier–Legendre expansions.

The following example illustrates rather well the noticed feature.

Consider the approximation $I_{(1)T,t}^{(i_1)q}$ of the stochastic integral $I_{(1)T,t}^{(i_1)}$ based on the expansion of the Brownian bridge process into the trigonometric Fourier series with random coefficients [7]

$$I_{(1)T,t}^{(i_1)q}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right),\tag{41}$$

where

$$\xi_q^{(i_1)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)},\quad \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2},$$

where $\zeta_0^{(i_1)}$, $\zeta_{2r-1}^{(i_1)}$, $\xi_q^{(i_1)}$; $r=1,\ldots,q$; $i_1=1,\ldots,m$ are independent standard Gaussian random variables.

On the other hand, it is possible to obtain the following equality

$$I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),\tag{42}$$

which is valid with probability 1 and based on the expansion of the function $t-t_1$ into the Fourier–Legendre series at the interval $[t,T]$ (this expansion has just two terms).

The above example demonstrates the advantage of the Legendre polynomials over the trigonometric functions in the context of the issue under consideration. More detailed comparison can be found in [32]-[34], [43], [44].

We notice that, as was established in [13], [19]-[27], [30]-[34], in the Fourier method (Theorem 1) it is also possible to use the Haar and Rademacher–Walsh functions (also see Theorem 2). However, in [13], [19]-[27], [30]-[34] it was shown that the expansions of the iterated Ito stochastic integrals (2) of multiplicities 1 and 2 obtained with the use of Theorem 1 and systems of Haar and Rademacher–Walsh functions are overcomplicated as compared with their analogues obtained on the basis of the Legendre polynomials. In this connection, practical application of such expansions is hindered.

Consider approximations of the remaining stochastic integrals from the family (40) obtained using Theorems 1, 2 and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. First, we consider approximations of stochastic integrals of multiplicities 1 and 2

$$I_{(0)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{43}$$

$$I_{(00)T,t}^{(i_1i_2)q}=I_{(00)T,t}^{*(i_1i_2)q}-\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}(T-t),$$

$$I_{(00)T,t}^{*(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{44}$$

$$I_{(10)T,t}^{(i_1i_2)q}=I_{(10)T,t}^{*(i_1i_2)q}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2,\quad I_{(01)T,t}^{(i_1i_2)q}=I_{(01)T,t}^{*(i_1i_2)q}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2, \tag{45}$$

$$I_{(01)T,t}^{*(i_1i_2)q}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)q}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr), \tag{46}$$

$$I_{(10)T,t}^{*(i_1i_2)q}=-\frac{T-t}{2}I_{(00)T,t}^{*(i_1i_2)q}-\frac{(T-t)^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

$$+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\Biggr), \tag{47}$$

where here and below

$$I_{(l_1\ldots l_k)s,t}^{*(i_1\ldots i_k)q}\quad\text{and}\quad I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)q}$$

are the approximations of the iterated Stratonovich and Ito stochastic integrals like

$$I_{(l_1\ldots l_k)s,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*s}(t-\tau_k)^{l_k}\ldots\int\limits_t^{*\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}_{\tau_1}^{(i_1)}\ldots d\mathbf{f}_{\tau_k}^{(i_k)} \tag{48}$$

and, correspondingly, like (6); $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various i or j; $j=0,1,\ldots,p+2$; $i=1,\ldots,m$.

Calculate the mean-square errors of approximations (44)–(47). A precise formula for pairwise different $i_1,\ldots,i_k=1,\ldots,m$ was established in [13], [31]-[34], [38]

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2, \tag{49}$$

where in virtue of the Parseval equality (14) the right-hand side of (49) tends to zero for $q\to\infty$; $J[\psi^{(k)}]_{T,t}$ has the form (2), and $J[\psi^{(k)}]_{T,t}^q$ is the approximation of $J[\psi^{(k)}]_{T,t}$ defined as the prelimit expression in (26) for $p_1=\ldots=p_k=q$ (also see the prelimit expressions in (18)–(22)); the sense of the rest notations is the same as in Theorems 1, 2.

The following formula [13], [31]-[34], [38] takes place

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}-J[\psi^{(2)}]_{T,t}^q\right)^2\right\}=$$

$$=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{q}C_{j_1j_2}C_{j_2j_1}\quad(i_1=i_2), \tag{50}$$

where notations are the same as in (49).

The value $\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}$ can be calculated exactly.

Theorem 8 [32] (Sect. 1.12), [38] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{51}$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Using (49) and (50), we get

$$\textsf{M}\left\{\left(I_{(00)T,t}^{(i_1 i_2)}-I_{(00)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2), \tag{52}$$

$$\textsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_2)}-I_{(10)T,t}^{(i_1 i_2)q}\right)^2\right\}=\textsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_2)}-I_{(01)T,t}^{(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\times$$

$$\times\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right) \tag{53}$$

for $i_1\ne i_2$ and

$$\textsf{M}\left\{\left(I_{(10)T,t}^{(i_1 i_1)}-I_{(10)T,t}^{(i_1 i_1)q}\right)^2\right\}=\textsf{M}\left\{\left(I_{(01)T,t}^{(i_1 i_1)}-I_{(01)T,t}^{(i_1 i_1)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{54}$$

Let us consider the numerical modeling of the iterated Ito stochastic integral of multiplicity 3 $I_{(000)T,t}^{(i_1 i_2 i_3)}$. Using Theorems 1, 2 for the case $k=3$ (see (20)), we obtain

$$I_{(000)T,t}^{(i_1 i_2 i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{55}$$

where $i_1,i_2,i_3=1,\ldots,m$ and

$$C_{j_3 j_2 j_1}=\int\limits_t^T\phi_{j_3}(z)\int\limits_t^z\phi_{j_2}(y)\int\limits_t^y\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3 j_2 j_1}, \tag{56}$$

$$\bar{C}_{j_3 j_2 j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{57}$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial.

For the case $i_1 = i_2 = i_3$, one can use the well known equality which follows from the Ito formula and is valid with probability 1 [2]

$$I_{(000)T,t}^{(i_1i_1i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right). \tag{58}$$

The procedure of numerical modeling of the iterated Ito stochastic integral $I_{(000)T,t}^{(i_1i_2i_3)}$ may follow (55)–(58). The Fourier–Legendre coefficients $\bar{C}_{j_3j_2j_1}$ of the form (57) being precisely calculable for the given number q by PYTHON, DERIVE or MAPLE. The mean-square error of approximation is checked by (49) for $k = 3$ as well as by the formulas established in [31]-[34], [38]

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t} - J[\psi^{(3)}]_{T,t}^q\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3-$$

$$-\sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \ne i_3), \tag{59}$$

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t} - J[\psi^{(3)}]_{T,t}^q\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3-$$

$$-\sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_2j_3j_1}C_{j_3j_2j_1} \quad (i_1 \ne i_2 = i_3), \tag{60}$$

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t} - J[\psi^{(3)}]_{T,t}^q\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3-$$

$$-\sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \ne i_2). \tag{61}$$

The following estimate [31]-[34], [38] can also be applied for the case $k = 3$

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^q\right)^2\right\} \le$$

$$\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}^2\right) \tag{62}$$

where $i_1,\ldots,i_k = 1,\ldots,m$ and $0 < T-t < \infty$ or $i_1,\ldots,i_k = 0,1,\ldots,m$ and $0 < T-t < 1$.

In particular, for the pairwise different $i_1, i_2, i_3 = 1,\ldots,m$ and $q = 6$ we get from (49)

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)6}\right)^2\right\}\approx 0.01956(T-t)^3. \tag{63}$$

Taking into consideration that $T-t$ is the integration step of numerical methods for the Ito SDE (1) and $T-t$ is a sufficiently small number, we get that already for $q=6$ the mean-square error of approximation of the stochastic integral $I_{000_{T,t}}^{(i_1i_2i_3)}$ is sufficiently small as well (see (63)).

Consider now the iterated Ito stochastic integral $I_{(0000)T,t}^{(i_1i_2i_3i_4)}$ of multiplicity 4. Using Theorems 1, 2, we get the representation

$$\begin{gathered}I_{(0000)T,t}^{(i_1i_2i_3i_4)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),\end{gathered} \tag{64}$$

where $i_1,i_2,i_3,i_4=1,\ldots,m$ and

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(u)\int\limits_t^u\phi_{j_3}(z)\int\limits_t^z\phi_{j_2}(y)\int\limits_t^y\phi_{j_1}(x)dxdydzdu=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}\Delta^2\bar{C}_{j_4j_3j_2j_1},$$

$$\bar{C}_{j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial.

For precise calculation of the Fourier–Legendre coefficients $C_{j_4j_3j_2j_1}$ we can use the previous recommendations and check the mean-square error of approximation of the iterated Ito stochastic integral $I_{(0000)T,t}^{(i_1i_2i_3i_4)}$, for example, using the estimate (62) for $k=4$.

In particular, for pairwise different $i_1,\ldots,i_4=1,\ldots,m$ we get from (49) with regard for smallness of $T-t$ already for $q=2$ a sufficiently good accuracy of the mean-square approximation

$$\mathsf{M}\left\{\left(I_{(0000)T,t}^{(i_1i_2i_3i_4)}-I_{(0000)T,t}^{(i_1i_2i_3i_4)2}\right)^2\right\}\approx 0.0236084(T-t)^4. \tag{65}$$

We notice that at deriving (63) and (65) the coefficients $\bar{C}_{j_3j_2j_1}$ and $\bar{C}_{j_4j_3j_2j_1}$ were precisely calculated using the DERIVE package.

Note that the formulas (27)–(30) are simpler than (19)–(21). However, calculation of the mean-square approximation error for the iterated Stratonovich stochastic integrals (3) turned out more complex than for the iterated Ito stochastic integrals (2) [32]-[34], [42], [52].

6. Algorithms of Numerical Modeling With the Orders 1.5 and 2.0 of Strong Convergence

We formulate in algorithmic form the above formulas and recommendations for the numerical method of the order 1.5 of strong convergence. We assume that the necessary Fourier–Legendre coefficients $\bar{C}_{j_3j_2j_1}$, $\bar{C}_{j_4j_3j_2j_1}$ are already calculated. In particular, several tables of the precisely calculated Fourier–Legendre coefficients $\bar{C}_{j_3j_2j_1}$, $\bar{C}_{j_4j_3j_2j_1}$ were presented in [13], [31]-[34]. These coefficients were calculated by DERIVE. It should be noted that in [61], [62] the database with 270,000 precisely calculated Fourier–Legendre coefficients is presented. In [61], [62] we used the PYTHON programming language.

Algorithm. 1.

Step 1. Given are the initial parameters of the problem such as the interval of integration $[0, T]$, step of integration Δ (for example, constant $\Delta = T/N$, $N \geq 1$, although a variable step of integration is admissible), initial condition $\mathbf{y}_0$, and constant C involved in the condition (8).

Step 2. Assume that $p = 0$.

Step 3. Selection of the minimal natural numbers q and q_1 $(q \ll q_1)$ ensuring the necessary accuracy of approximation of the stochastic integrals

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}, \quad I_{(000)\tau_{p+1},\tau_k}^{(i_1i_2i_3)} \quad (\tau_p = p\Delta)$$

and satisfying the conditions

$$\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)} - I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q_1}\right)^2\right\} = \frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q_1}\frac{1}{4i^2-1}\right) \leq C\Delta^4, \tag{66}$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)} - I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\} \leq 6\left(\frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2\right) \leq C\Delta^4. \tag{67}$$

Remark 1. *If it is required to check the mean-square approximation error of the iterated Ito stochastic integral* $I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}$ *using the precise formulas* (49), (59)–(61), *rather than the estimate* (67) (*see* (62)), *then instead of the condition* (67) *one has to take the following conditions*

$$E_{p,q,\Delta}^{(i_1i_2i_3)} = \frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 \leq C\Delta^4 \quad (i_1 \neq i_2,\ i_1 \neq i_3,\ i_2 \neq i_3),$$

$$E_{p,q,\Delta}^{(i_1i_2i_3)} = \frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_2j_3j_1}C_{j_3j_2j_1} \leq C\Delta^4 \quad (i_1 \neq i_2 = i_3),$$

$$E_{p,q,\Delta}^{(i_1 i_2 i_3)} = \frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \le C\Delta^4 \quad (i_1 = i_3 \ne i_2),$$

$$E_{p,q,\Delta}^{(i_1 i_2 i_3)} = \frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \le C\Delta^4 \quad (i_1 = i_2 \ne i_3),$$

where

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\} \stackrel{\rm def}{=} E_{p,q,\Delta}^{(i_1 i_2 i_3)}.$$

Step 4. *Modeling of the sequence of independent standard Gaussian random variables* $\zeta_l^{(i)}$ $(l = 0, 1, \ldots, q_1;\ i = 1, \ldots, m)$.

Step 5. *Modeling of the iterated Ito stochastic integrals*

$$I_{(0)\tau_{p+1},\tau_p}^{(i_1)}, \quad I_{(1)\tau_{p+1},\tau_p}^{(i_1)}, \quad I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}, \quad I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}$$

using the formulas

$$I_{(0)\tau_{k+1},\tau_k}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)\tau_{p+1},\tau_p}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)q_1} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q_1}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

where $i_1, i_2, i_3 = 1, \ldots, m$.

Remark 2. *In the case of* $i_1 = i_2 = i_3$, *it is advisable to model the stochastic integral* $I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}$ *using the formula* (58), *where one has to assume that* $T - t = \Delta$.

Step 6. *Calculate* $\mathbf{y}_{p+1}$ *from* (5).

Step 7. *If* $p < N-1$, *then assume that* $p = p+1$ *and go to Step* 4; *otherwise, go to Step* 8.

Step 8. *End.*

We briefly note how to modify the algorithm to enable numerical modeling with the order 2.0 of strong convergence.

At Step 3 one has to take the following three iterated Ito stochastic integrals

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)},\quad I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)},\quad I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)},$$

whose approximations obey (45)–(47), (64) and add to the considered stochastic integrals. Moreover, we replace $C\Delta^4$ by $C\Delta^5$ in (66), (67). At that, one can use the estimate (62) for $k=4$ and the formulas (53), (54) to check the accuracy of modeling of the aforementioned integrals. As the result, we get the following conditions

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)q_2}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)q_2}\right)^2\right\}=\frac{\Delta^4}{16}\times$$

$$\times\left(\frac{5}{9}-2\sum_{i=2}^{q_2}\frac{1}{4i^2-1}-\sum_{i=1}^{q_2}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q_2}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\le C\Delta^5$$

for $i_1\ne i_2$ and

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_1)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_1)q_3}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_1)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_1)q_3}\right)^2\right\}=$$

$$=\frac{\Delta^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q_3}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q_3}\frac{1}{(2i-1)^2(2i+3)^2}\right)\le C\Delta^5$$

for $i_1=i_2$;

$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)q_4}\right)^2\right\}\le 24\left(\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q_4}C_{j_4 j_3 j_2 j_1}^2\right)\le C\Delta^5, \tag{68}$$

where $i_1,i_2,i_3,i_4=1,\ldots,m$; $q_2,q_3,q_4<q<q_1$.

Carry out Step 5 with allowance of the stochastic integrals

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)},\ I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)},\ I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)},$$

and calculate $\mathbf{y}_{p+1}$ at Step 6 according to (9).

It should be noted that instead of the estimate (68) we can use the precise relations for the value

$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)q_4}\right)^2\right\},$$

which were obtained in [31]-[34], [38] for all possible combinations of $i_1,i_2,i_3,i_4=1,\ldots,m$. Note that the optimization of the mentioned procedure is considered in [69].

7. Conclusions

The present paper provided efficient procedures for the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 4 based on multiple Fourier–Legendre series. These results can be used for implementation of the numerical methods with the orders 1.0, 1.5, and 2.0 of strong convergence for Ito stochastic differential equations with multidimensional non-commutative noise. The results of the article can be applied for numerical solution of the problems of optimal stochastic control and signal filtering in random noise in different formulations. The development of the approaches from this work can be found in [13], [19]-[55], [61]-[67], [69], [70].

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.
[3] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.
[4] Arato M. Linear Stochastic Systems With Constant Coefficients. A Statistical Approach. Springer, Berlin, Heidelberg, N.Y., 1982, 289 pp.
[5] Shiriaev A.N. Foundations of Financial Mathematics. Vol. 2, Fazis, Moscow, 1998, 544 pp.
[6] Liptser R.Sh., Shiriaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. Nauka, Moscow, 1974, 696 pp.
[7] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.
[8] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
[9] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.
[10] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.
[11] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.) 99, 2 (2000), 1130-1140. DOI: https://doi.org/10.1007/BF02673635
[12] Kuznetsov, D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[13] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)
[14] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html
[15] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)
[16] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.
[17] Platen, E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.
[18] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.
[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[23] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[24] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[25] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[26] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[27] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[28] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[29] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[30] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[31] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[32] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR]. 2022, 923 pp.

[33] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[34] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[35] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2019, 106 pp.

[36] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.
[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 129 pp.
[38] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.
[39] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.
[40] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.
[41] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.
[42] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[43] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116
[44] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.
[45] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html
[46] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html
[47] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [in English]. arXiv:1712.09516 [math.PR]. 2022, 204 pp.
[48] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.
[49] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.
[50] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor–Stratovovich expansion. [in English]. arXiv: 1806.10705 [math.PR]. 2018, 29 pp.
[51] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [in English]. arXiv:1807.02190 [math.PR], 2018, 44 pp.
[52] Kuznetsov D.F. Explicit one-step strong numerical methods of order 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich Expansions. [in English]. arXiv: 1802.04844 [math.PR]. 2018, 37 pp.
[53] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [in English]. arXiv:1801.01564 [math.PR]. 2018, 65 pp.
[54] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [in English]. arXiv:1712.08991 [math.PR]. 2017, 57 pp.
[55] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.
[56] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[57] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[58] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[59] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80
Available at: http://www.sde-kuznetsov.spb.ru/00a.pdf
[60] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf
[61] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html
[62] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.
[63] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html
[64] Kuznetsov, D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.3/article.1.6.html
[65] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878
[66] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf
[67] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100
[68] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[69] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series. [In English]. arXiv:2010.13564 [math.PR], 2020, 63 pp.
[70] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2
[71] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at:
http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html
[72] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html
[73] Kuznetsov D.F. The three-step strong numerical methods of the orders of accuracy 1.0 and 1.5 for Ito stochastic differential equations. [In English]. Journal of Automation and Information Sciences (Begell House), 2002, 34 (Issue 12), 14 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v34.i12.30
Available at: http://www.sde-kuznetsov.spb.ru/02a.pdf
[74] Kuznetsov D.F. Finite-difference strong numerical methods of order 1.5 and 2.0 for stochastic differential Ito equations with nonadditive multidimensional noise. [In English]. Journal of Automation and Information Sciences

(Begell House), 2001, 33 (Issue 5-8), 13 pp. DOI: http://doi.org/10.1615/JAutomatInfScien.v33.i5-8.180
Available at: http://www.sde-kuznetsov.spb.ru/01c.pdf

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

NUMERICAL SIMULATION OF 2.5-SET OF ITERATED ITO STOCHASTIC INTEGRALS OF MULTIPLICITIES 1 TO 5 FROM THE TAYLOR–ITO EXPANSION

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the construction of effective procedures of the mean-square approximation of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor–Ito expansion based on multiple Fourier–Legendre series. The results of the article can be applied to the implementation of numerical methods with orders 1.5, 2.0, and 2.5 of strong convergence for Ito stochastic differential equations with multidimensional non-commutative noise.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.

Keywords: Explicit one-step strong numerical method, Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Taylor–Ito expansion, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Ito stochastic differential equation, Numerical integration, Mean-square convergence, Expansion, Approximation.

1. Introduction

This paper is a continuation of the author's research [1], [2] on numerical methods with rather high orders of strong convergence for Ito stochastic differential equations (SDEs). The definition of strong convergence will be given below. The development of such numerical methods is topical due to a wide range of applications for Ito SDEs [3]-[9]. In particular, these equations arise in optimal stochastic control, signal filtering against the background of random noises, parameter estimation for stochastic systems as well as in stochastic stability and bifurcations analysis [3]-[9]. Also the Ito SDEs represent adequate mathematical models for dynamic systems of different physical origin that are affected by random perturbations. They are used as mathematical models in stochastic mathematical finance, hydrology and seismology, geophysics, chemical kinetics and population dynamics, electrodynamics, medicine and other fields (see [3]-[9]). On the other hand, new numerical methods with rather high orders of strong convergence are needed for the Ito SDEs because one of the elementary numerical methods — the Euler scheme has insufficient accuracy for a series of practical problems under standard assumptions; the details can be found in [3].

This paper follows a promising approach to the numerical integration of Ito SDEs [3], [7]-[9] that is based on the stochastic analogs of the Taylor formula (the so-called Taylor–Ito and Taylor–Stratonovich expansions [3], [7]-[14]) for the solutions of Ito SDEs. This approach includes the finite partitioning of the time variable and also the numerical solution of an Ito SDE at discrete moments of time using the stochastic analogs of the Taylor formula. The numerical methods with orders 1.5, 2.0, and 2.5 of strong convergence will be considered in the article.

This paper employs the so-called unified Taylor–Ito expansion [12]-[21] with a minimum set of iterated Ito stochastic integrals, which is a simplifying factor at the implementation stage of numerical methods. The iterated Ito stochastic integrals figuring in the numerical schemes with a strong convergence of orders 1.5, 2.0, and 2.5 are approximated using the method of generalized multiple Fourier series, which was considered in a series of papers of the author [14]-[57]. As was noted in [1], [2], this method does not require the splitting of an integration interval $[t, T]$ of the iterated Ito stochastic integrals; recall that its length $T - t$ gives the integration step of the numerical methods for Ito SDEs and hence is a sufficiently small value. In accordance with experimental results [14], the splitting of the interval $[t, T]$ leads to an inadmissibly high computational cost. As a rule, this splitting is used in the approximation methods of iterated Ito stochastic integrals based on integral sums [7], [8], [58].

As mentioned in [1], [2], in a number of publications [3], [7], [8], numerical schemes with strong convergence of high orders (1.5, 2.0, and 2.5) for the Ito SDEs have been proposed. However, these methods do not contain efficient mean-square approximation procedures for the iterated Ito stochastic integrals in the case of multidimensional non-commutative noises. Generally, the authors [3], [7], [8] introduced some simplifying assumptions on the additivity, commutativity or smallness of the noises, which results in a considerable simplification of the numerical modeling problem of iterated Ito stochastic integrals. Like [1], [2], this paper will partially eliminate this drawback.

Note that the properties of numerical schemes for the Ito SDEs (including the ones with a strong convergence of orders 1.5, 2.0, and 2.5) were well studied in [3], [7], [8]. In particular, their stability was thoroughly analyzed. The goal of this paper is to develop efficient numerical modeling procedures for the iterated Ito stochastic integrals of multiplicities 1 to 5 including the exact calculation and efficient estimation of the mean-square approximation errors of these stochastic integrals.

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0, T]\}$ be a nondecreasing right-continous family of σ-subfields of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0, T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito SDE in the integral form

(1) $$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega).$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [59]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

Consider the following iterated Ito stochastic integrals

(2) $$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a non-random function on $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$.

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [3], [7], [8], [10], [11] and $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [12]-[28].

Effective solution of the problem of combined mean-square approximation of the iterated Ito stochastic integrals (2) of multiplicities 1 to 5 composes the subject of this article.

2. Explicit One-Step Strong Numerical Schemes With Convergence Orders 1.5, 2.0, and 2.5 Based on the Unified Taylor–Ito Expansion

Introduce the definition of a strong convergence of a numerical method for the Ito SDEs.

Consider the partition $\{\tau_p\}_{p=0}^N$ of the interval $[0, T]$ with the maximum step of discretization Δ_N such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T.$$

Denote by $\mathbf{y}_{\tau_p} \stackrel{\mathsf{def}}{=} \mathbf{y}_p$, $p = 0, 1, \ldots, N$ the discrete approximation of the process $\mathbf{x}_t$, $t \in [0, T]$ (the solution of the Ito SDE (1)) that corresponds to the maximum discretization step Δ_N.

Definition 1 [3]. *The discrete approximation (numerical method) $\mathbf{y}_j$, $j = 0, 1, \ldots, N$ that corresponds to the maximum discretization step Δ_N is said to be strongly converging with an order $\gamma > 0$ to the process $\mathbf{x}_t$, $t \in [0, T]$ if there exist a constant $C > 0$ that is independent of Δ_N and j $(j = 0, 1, \ldots, N)$ and also a value $\delta > 0$ such that*

(3) $$\mathsf{M}\{|\mathbf{x}_j - \mathbf{y}_j|\} \le C(\Delta_N)^\gamma \quad (j = 0, 1, \ldots, N)$$

for all $\Delta_N \in (0, \delta)$.

In a series of publications [7], [8], the authors considered the mean-square convergence instead of the strong convergence, which corresponds to the replacement of the condition (3) with

(4) $$\left(\mathsf{M}\{|\mathbf{x}_j - \mathbf{y}_j|^2\}\right)^{1/2} \le C(\Delta_N)^\gamma \quad (j = 0, 1, \ldots, N).$$

In (3) and (4): $\mathbf{x}_{\tau_j} \stackrel{\mathsf{def}}{=} \mathbf{x}_j$, $j = 0, 1, \ldots, N$.

Obviously, by virtue of the Lyapunov inequality [3] the mean-square convergence implies the strong convergence. As it has appeared, a rather nontrivial question is which iterated stochastic integrals (Ito or Stratonovich) are preferable for the numerical integration of the Ito SDEs with a correct estimation of the mean-square approximation error. By their external view the approximations of the iterated Stratonovich stochastic integrals are simpler than the corresponding approximations of the iterated Ito stochastic integrals; see the details in Sections 3, 5 below. However, the estimation procedure of the mean-square approximation error turns out to be much easier for the iterated Ito stochastic integrals, which motivates the use of these integrals.

Consider the explicit one-step numerical scheme for the Ito SDEs that is based on the unified Taylor–Ito expansion [14]-[18]

$$
\begin{aligned}
\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_{i_2}B_{i_1}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_2i_1)}+\\
+\sum_{i_1=1}^{m}\left(G_{i_1}\mathbf{a}\left(\Delta\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)+\\
+\sum_{i_1,i_2,i_3=1}^{m}G_{i_3}G_{i_2}B_{i_1}\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+\\
+\mathbf{v}_{p+1,p}+\mathbf{r}_{p+1,p},
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\mathbf{v}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl(G_{i_2}LB_{i_1}\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_2i_1)}-\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_2i_1)}\right)-LG_{i_2}B_{i_1}\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_2i_1)}+\\
+G_{i_2}G_{i_1}\mathbf{a}\left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_2i_1)}+\Delta\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_2i_1)}\right)\Biggr)+\\
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_4i_3i_2i_1)},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl(G_{i_1}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}+\frac{\Delta^2}{2}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}\right)+\\
+\frac{1}{2}LLB_{i_1}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}-LG_{i_1}\mathbf{a}\left(\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)\Biggr)+\\
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl(G_{i_3}LG_{i_2}B_{i_1}\left(\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_3i_2i_1)}-\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_3i_2i_1)}\right)+\\
+G_{i_3}G_{i_2}LB_{i_1}\left(\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_3i_2i_1)}-\hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_3i_2i_1)}\right)+
\end{aligned}
$$

$$+G_{i_3}G_{i_2}G_{i_1}\mathbf{a}\left(\Delta\hat{I}^{(i_3i_2i_1)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{(i_3i_2i_1)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$\left.-LG_{i_3}G_{i_2}B_{i_1}\hat{I}^{(i_3i_2i_1)}_{(100)\tau_{p+1},\tau_p}\right)+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{(i_5i_4i_3i_2i_1)}_{(00000)\tau_{p+1},\tau_p}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a},$$

where $\Delta=T/N$ $(N>1)$ is a constant (for simplicity) integration step, $\tau_p=p\Delta$ $(p=0,1,\ldots,N)$, $\hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}$ denotes an approximation of the iterated Ito stochastic integral of multiplicity k

$$I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}=\int\limits_t^s(t-\tau_k)^{l_k}\ldots\int\limits_t^{\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}^{(i_1)}_{\tau_1}\ldots d\mathbf{f}^{(i_k)}_{\tau_k}, \tag{6}$$

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B_{lj}(\mathbf{x},t)B_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_i},$$

$$G_i=\sum_{j=1}^{n}B_{ji}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_j},\quad i=1,\ldots,m,$$

$l_1,\ldots,l_k=0,1,2,$ $i_1,\ldots,i_k=1,\ldots,m,$ $k=1,2,\ldots,5,$ B_i and B_{ij} are the ith column and the ijth element of the matrix function B, $\mathbf{a}_i$ is the ith element of the vector function $\mathbf{a}$, $\mathbf{x}_i$ is the ith element of the column $\mathbf{x}$, the functions

$$B_{i_1},\ \mathbf{a},\ G_{i_2}B_{i_1},\ G_{i_1}\mathbf{a},\ LB_{i_1},\ G_{i_3}G_{i_2}B_{i_1},\ L\mathbf{a},\ LL\mathbf{a},\ G_{i_2}LB_{i_1},$$

$$LG_{i_2}B_{i_1},\ G_{i_2}G_{i_1}\mathbf{a},\ G_{i_4}G_{i_3}G_{i_2}B_{i_1},\ G_{i_1}L\mathbf{a},\ LLB_{i_1},\ LG_{i_1}\mathbf{a},\ G_{i_3}LG_{i_2}B_{i_1},\ G_{i_3}G_{i_2}LB_{i_1},$$

$$G_{i_3}G_{i_2}G_{i_1}\mathbf{a},\ LG_{i_3}G_{i_2}B_{i_1},\ G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}$$

are calculated at the point $(\mathbf{y}_p,p)$.

Under the standard conditions [3], [14] the numerical scheme (5) has order 2.5 of strong convergence. The major emphasis below will be placed on the approximation of the iterated Ito stochastic integrals appearing in (5). Therefore, among the standard conditions, note the approximation condition of these integrals [3], [14], which has the form

$$\mathsf{M}\left\{\left(I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}-\hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\right)^2\right\}\le C\Delta^6, \tag{7}$$

where constant C is independent of Δ.

Note that if we exclude $\mathbf{v}_{p+1,p}+\mathbf{r}_{p+1,p}$ from the right-hand side of (5), then we have an explicit one-step strong numerical scheme of order 1.5 [3], [14]-[18]. The right-hand side of (5) but without the value $\mathbf{r}_{p+1,p}$ define an explicit one-step strong numerical scheme of order 2.0 [3], [14]-[18].

Using the numerical scheme (5) or its modification based on the Taylor–Ito expansion [11], the implicit or multistep analogs of (5) can be constructed; see [3]. The set of the iterated Ito stochastic integrals to be approximated for implementing these modifications is the same as for the numerical scheme (5) itself. Interestingly, the truncated unified Taylor–Ito expansion — the foundation of the numerical scheme (5) — contains 12 different iterated Ito stochastic integrals of the form (6), which cannot be interconnected by linear relations [14]-[18]. The analogous Taylor–Ito expansion [3] contains 17 different iterated Ito stochastic integrals, part of which are interconnected by linear relations and part of which have a higher multiplicity than the iterated Ito stochastic integrals (6). This fact well explains the use of the numerical scheme (5).

One of the main problems arising in the implementation of the numerical scheme (5) is the joint numerical modeling of the iterated Ito stochastic integrals figuring in (5). In the next section, we will consider an efficient numerical modeling method for the iterated Ito stochastic integrals and also demonstrate which stochastic integrals (Ito or Stratonovich) are preferable for numerical modeling with a correct estimation of the mean-square approximation error.

3. Method of Numerical Modeling for Iterated Ito Stochastic Integrals Based on Genegalized Multiple Fourier Series

An efficient numerical modeling method for the iterated Ito stochastic integrals based on generalized multiple Fourier series was considered in [14] (2006); also see [15]-[57]. This method rests on an important result presented below (Theorems 1, 2).

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{8}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{9}$$

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{10}$$

Theorem 1 [14] (2006), [15]-[34], [39]-[49], [51]-[57]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}\ =\ \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\ -$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{11}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{12}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (9), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (10).

The convergence in the mean of degree $2n$ $(n\in\mathbb{N})$ [19]-[21], [23]-[28] as well as the convergence with probability 1 [19]-[21], [41], [43] of approximations from Theorem 1 (also see Theorem 2 below) are proved. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorems 1 [14]-[28]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [18]-[21], [52]

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [14]-[34], [39]-[49], [51]-[57]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{13}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{14}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{15}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+ \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{16}$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +$$

$$+ \mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{17}$$

where $\mathbf{1}_A$ is the indicator of the set A.

We will consider the case $i_1,\ldots,i_5 = 1,\ldots,m$. This case corresponds to the numerical method (5).

For further consideration, let us consider the generalization of formulas (13)–(17) for the case of an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(18)\qquad (\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}),$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (18) is a partition and consider the sum with respect to all possible partitions

$$(19)\qquad \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}.$$

Below there are several examples of sums in the form (19)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$

$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can write (11) as

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{20}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (20) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (17).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [19] (Sect. 1.11), [41] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{21}$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [62]. Note that we use another notations [19] (Sect. 1.11), [41] (Sect. 15) in comparison with [62]. Moreover, the proof of an analogue of Theorem 2 from [62] is somewhat different from the proof given in [19] (Sect. 1.11), [41] (Sect. 15).

4. Calculation of the Mean-Square Approximation Error in the Method of Generalized Multiple Fourier Seires

Note that for the integrals $J[\psi^{(k)}]_{T,t}$ defined by (2) the mean-square approximation error can be exactly calculated and efficiently estimated.

Let $J[\psi^{(k)}]_{T,t}^q$ be the expression on the right-hand side of (21) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=q$, i.e.

$$J[\psi^{(k)}]_{T,t}^q=\sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{22}$$

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}\stackrel{\text{def}}{=}E_k^q,$$

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\text{def}}{=}I_k.$$

In [17]-[21], [41], [42] it was shown that

$$E_k^q\le k!\left(I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2\right) \tag{23}$$

for the following two cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$,
2. $i_1,\ldots,i_k=0,1,\ldots,m$ and $T-t\in(0,1)$.

The value E_k^q can be calculated exactly.

Theorem 3 [19] (Sect. 1.12), [42] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then*

$$E_k^q=I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{24}$$

where $i_1, \ldots, i_k = 1, \ldots, m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1, \ldots, j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1, \ldots, i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\} = C_{j_k\ldots j_1}.$$

Therefore, for the special case of pairwise different numbers $i_1, \ldots, i_k$ as well as for the case $i_1 = \ldots = i_k$ from Theorem 3 it follows that [18]-[21], [29], [42]

$$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C^2_{j_k\ldots j_1}, \tag{25}$$

$$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

is a sum with respect to all possible permutations $(j_1, \ldots, j_k)$.

Consider some examples [18]-[21], [29], [42] of application of Theorem 3 $(i_1, i_2, i_3 = 1, \ldots, m)$

$$E_2^q = I_2 - \sum_{j_1,j_2=0}^{q} C^2_{j_2j_1} - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}C_{j_1j_2} \quad (i_1 = i_2), \tag{26}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \neq i_3), \tag{27}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_2j_3j_1}C_{j_3j_2j_1} \quad (i_1 \neq i_2 = i_3), \tag{28}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \neq i_2). \tag{29}$$

The values E_4^q and E_5^q were calculated exaclty for all possible combinations of $i_1, \ldots, i_5 = 1, \ldots, m$ in [18]-[21], [42].

5. Expansions of Iterated Stratonovich Stochastic Integrals Based on Multiple Fourier–Legendre Series and Multiple Trigonometric Fourier Seires

In contrast to the iterated Ito stochastic integrals, the iterated Stratonovich stochastic integrals have simpler expansions than (11) and (21), but the calculation (or estimation) of the mean-square approximation errors for the latter is a much more difficult problem than for the former. Study this issue in detail.

Introduce the following iterated Stratonovich stochastic integrals

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{30}$$

where the notations are the same as in the formula (2) (in this paper, we use the definition of the Stratonovich stochastic integral from [3]).

Consider a slightly modified and extended theoretical result that adapts Theorems 1, 2 for the iterated Stratonovich stochastic integrals (30) of multiplicities 2 to 4 (some old results).

Theorem 4 [15]-[21], [26]-[28], [32], [40], [43]-[47], [53]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t,T])$. In addition, assume that $\psi_2(s)$ is a continuously differentiable function on the interval $[t,T]$ and $\psi_1(s)$, $\psi_3(s)$ are twice continuously differentiable functions on the interval $[t,T]$. Then*

$$J^*[\psi^{(k)}]_{T,t} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}, \tag{31}$$

where $k = 2, 3, 4$. At that $\psi_1(s), \ldots, \psi_k(s) \equiv 1$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in (31) for $k = 4$, while $i_1, \ldots, i_k = 1, \ldots, m$ in (31) for $k = 2, 3$; the other notations are the same as in Theorems 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [19] (Sect. 2.10–2.16), [44] (Sect. 7–13), [45] (Sect. 13–19), [53] (Sect. 5–11), [57] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 5 [19], [44], [45], [53], [57]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{32}$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le\frac{C}{p} \tag{33}
$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (32) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (33), *constant* C *is independent of* p,

$$
C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3
$$

and

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 6 [19], [44], [45], [53], [57]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$
J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{34}
$$

the following relations

$$
J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{35}
$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{36}
$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (34), (35) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (36), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$
C_{j_4j_3j_2j_1}=
$$

$$
=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;
$$

another notations are the same as in Theorem 5.

Theorem 7 [19], [44], [45], [53], [57]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{37}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{38}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{39}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (37), (38) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (39), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 5, 6.

Theorem 8 [19], [44], [45], [53]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{40}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 5–7.

Clearly, the expansion (31) is simpler than the expansions (11), (21). However, the calculation of the mean-square approximation error for the expansion (31) turns out to be much more difficult than for the expansions (11), (21). We will demonstrate this fact below.

The cases $k=1,2$ are actually not interesting: for $k=1$, the Ito and Stratonovich stochastic integrals of a smooth nonrandom function equal each other with probability 1 (w. p. 1); for $k=2$, the Ito stochastic integrals appearing in the numerical scheme (5) differ w. p. 1 from the corresponding Stratonovich stochastic integrals by constant values by virtue of the standard relations between the Ito and Stratonovich stochastic integrals [3]. Consider the triple Stratonovich stochastic integral defined by

$$I_{(000)T,t}^{*(i_1i_2i_3)}=\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m).$$

In view of the standard relations between the Ito and Stratonovich stochastic integrals [3] and also Theorems 1, 2, and 4 ($k=3$), we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}+\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\Biggl\{\Biggl(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}+I_{(000)T,t}^{(i_1i_2i_3)q}+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}+\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-I_{(000)T,t}^{*(i_1i_2i_3)q}\Biggr)^2\Biggr\},\tag{41}$$

$$I_{(000)T,t}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),\tag{42}$$

$$I_{(000)T,t}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},\tag{43}$$

where $I_{(000)T,t}^{(i_1 i_2 i_3)q}$ is the approximation defined by (22) (also see (15)) for $k=3$ and $I_{(000)T,t}^{*(i_1 i_2 i_3)q}$ is the approximation based on Theorem 4 for $k=3$.

Substituting (42) and (43) into (41) yields

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}=$$

$$=\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q}+\mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)+\right.\right.$$

$$\left.\left.+\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)-\mathbf{1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{q}C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}\le$$

$$\le 4\left(\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)}-I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\}+\mathbf{1}_{\{i_1=i_2\}}F_q^{(i_3)}+\right.$$

$$\left.+\mathbf{1}_{\{i_2=i_3\}}G_q^{(i_1)}+\mathbf{1}_{\{i_1=i_3\}}H_q^{(i_2)}\right),\tag{44}$$

where

$$F_q^{(i_3)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$G_q^{(i_1)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\},$$

$$H_q^{(i_2)}=\mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q}C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}.$$

For the case of Legendre polynomials or trigonometric functions we have the equalities (for details, see the proof of Theorem 4 for $k=3$ in [15]-[21], [26]-[28], [45])

$$\lim_{q\to\infty}F_q^{(i_3)}=0,\quad \lim_{q\to\infty}G_q^{(i_1)}=0,\quad \lim_{q\to\infty}H_q^{(i_2)}=0.$$

However, in accordance with (44) the value

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)}-I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\}$$

with a finite q can be estimated by the sum of

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}, \tag{45}$$

and three additional terms of a rather complex structure. The value (45) can be calculated exactly using Theorem 3 or estimated using (23) for the case $k=3$.

As is easily observed, this peculiarity will also apply to the iterated Stratonovich stochastic integrals of multiplicities 4 and 5, with the only difference that the number of additional terms like $F_q^{(i_3)}$, $G_q^{(i_1)}$, and $H_q^{(i_2)}$ will be considerably higher and their structure will be more complicated. Therefore, the payment for a relatively simple approximation of the iterated Stratonovich stochastic integrals (Theorems 4–8) in comparison with the iterated Ito stochastic integrals (Theorems 1, 2) is a much more difficult calculation or estimation procedure of their mean-square approximation errors (see Chapter 5 in [19] for detail). This well explains why the main emphasis of the paper is on the approximation of the iterated Ito stochastic integrals figuring in the numerical scheme (5). Their approximation involves Theorems 1, 2 for $k=1,\ldots,5$ and also a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. As was established in [19]-[21], [31], [39], the Legendre polynomials have a series of advantages over the trigonometric functions for the approximation of iterated stochastic integrals using Theorems 1, 2.

6. Approximation of Iterated Ito Stochastic Integrals Based on Multiple Fourier–Legendre Series

Consider the approximations of the iterated Ito stochastic integrals that appear in the numerical scheme (5) using Theorems 1, 2 and the complete orthonormal system of Legendre polynomials in the space $L_2([\tau_p,\tau_{p+1}])$ ($\tau_p=p\Delta$, $N\Delta=T$, $p=0,1,\ldots,N$) [14] (also see [15]-[51], [54]-[56])

$$I_{(0)\tau_{p+1},\tau_p}^{(i_1)}=\sqrt{\Delta}\zeta_0^{(i_1)}, \tag{46}$$

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q}=\frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{47}$$

$$I_{(1)\tau_{p+1},\tau_p}^{(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{48}$$

$$I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{49}$$

$$I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$\left.+\,\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right), \tag{50}$$

$$I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)q}=-\frac{\Delta}{2}I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q}-\frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+\right.$$
$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{51}$$

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)q}=-\frac{\Delta}{2}I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q}-\frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+\right.$$
$$\left.+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right) \tag{52}$$

or

$$I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)q}=\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{01}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right),$$

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)q}=\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^{10}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right);$$

$$I_{(2)\tau_{p+1},\tau_p}^{(i_1)}=\frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{53}$$

$$I_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^{001}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\right.$$
$$\left.-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{54}$$

$$I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3 j_2 j_1}^{010}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{55}$$

$$I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3 j_2 j_1}^{100}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{56}$$

$$I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)q}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5 j_4 j_3 j_2 j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{j_2=j_5\neq 0\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\neq 0\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{57}$$

where

$$C_{j_3 j_2 j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar{C}_{j_3 j_2 j_1},$$

$$C_{j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdu=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}\Delta^2\bar{C}_{j_4j_3j_2j_1},$$

$$C_{j_2j_1}^{01}=\int\limits_{\tau_p}^{\tau_{p+1}}(\tau_p-y)\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdy=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}\Delta^2\bar{C}_{j_2j_1}^{01},$$

$$C_{j_2j_1}^{10}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}(\tau_p-x)\phi_{j_1}(x)dxdy=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}\Delta^2\bar{C}_{j_2j_1}^{10},$$

$$C_{j_3j_2j_1}^{001}=\int\limits_{\tau_p}^{\tau_{p+1}}(\tau_p-z)\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{001},$$

$$C_{j_3j_2j_1}^{010}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}(\tau_p-y)\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{010},$$

$$C_{j_3j_2j_1}^{100}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}(\tau_p-x)\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{100},$$

$$C_{j_5j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_5}(v)\int\limits_{\tau_p}^{v}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdudv=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}\Delta^{5/2}\bar{C}_{j_5j_4j_3j_2j_1},$$

where

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_2j_1}^{01} = -\int\limits_{-1}^{1} (1+y) P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_2j_1}^{10} = -\int\limits_{-1}^{1} P_{j_2}(y) \int\limits_{-1}^{y} (1+x) P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_3j_2j_1}^{100} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)(x+1)dxdydz,$$

$$\bar{C}_{j_3j_2j_1}^{010} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y)(y+1) \int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_3j_2j_1}^{001} = -\int\limits_{-1}^{1} P_{j_3}(z)(z+1) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

$$\bar{C}_{j_5j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_5}(v) \int\limits_{-1}^{v} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)dxdydzdudv,$$

where $P_i(x)$ $(i = 0, 1, 2, \ldots)$ is the Legendre polynomial and

$$\phi_i(x) = \sqrt{\frac{2i+1}{\Delta}} P_i\left(\left(x - \tau_p - \frac{\Delta}{2}\right)\frac{2}{\Delta}\right), \quad i = 0, 1, 2, \ldots$$

Let us consider the exact relations and some estimates for the mean-square approximation errors of iterated Ito stochastic integrals.

Using Theorem 3, we get [16]-[28], [43], [51]

$$\text{(58)} \qquad \mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)} - I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\} = \frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \quad (i_1 \ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)} - I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\} = \mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)} - I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\} =$$

$$=\frac{\Delta^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\quad (i_1\ne i_2), \tag{59}$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1i_1)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1i_1)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1i_1)q}\right)^2\right\}=$$

$$=\frac{\Delta^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{60}$$

Applying (25)–(29), we obtain

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2\quad (i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3).$$

At the same time using the estimate (23) for $i_1,\ldots,i_5=1,\ldots,m$, we have

$$\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2\right), \tag{61}$$

$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right), \tag{62}$$

$$\mathsf{M}\left\{\left(I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{100}\right)^2\right), \tag{63}$$

$$\mathsf{M}\left\{\left(I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{010}\right)^2\right), \tag{64}$$

$$\mathsf{M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{001}\right)^2\right), \tag{65}$$

$$\mathsf{M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)q}\right)^2\right\}\le 120\left(\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5 i_4 i_3 i_2 j_1}^2\right). \tag{66}$$

As was emphasized in [1], [2], [19]-[21], the Fourier–Legendre coefficients $\bar{C}_{j_3 j_2 j_1}$ and $\bar{C}_{j_4 j_3 j_2 j_1}$ (as well as the Fourier–Legendre coefficients $\bar{C}_{j_3 j_2 j_1}^{001}$, $\bar{C}_{j_3 j_2 j_1}^{010}$, $\bar{C}_{j_3 j_2 j_1}^{100}$, $\bar{C}_{j_5 j_4 j_3 j_2 j_1}$) can be exactly calculated using symbolic transformation packages like Derive. The exact values of these Fourier–Legendre coefficients calculated in Derive were presented in tabular form in the monographs [14]-[28]. Note that the mendioned Fourier–Legendre coefficients do not depend on the integration step $\tau_{p+1}-\tau_p$ of the numerical method, which can be variable.

Recently, the database with 270,000 exactly calculated Fourier–Legendre coefficients was described [54]. This database was used in the software package, which is written in the Python programming language for the implementation of explicit one-step strong numerical methods with orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Ito SDEs with non-commutative noise. The optimization of the mean-square approximation procedures for iterated Ito stochastic integrals from these numerical schemes can be found in [56].

Generally speaking, the minimum values q that guarantee the fulfillment of the condition (7) for each of approximations (see above) are different and abruptly decreasing with the growth of orders of smallness with respect to Δ of approximations of iterated Ito stochastic integrals.

For pairwise different $i_1,\ldots,i_5=1,\ldots,m$ Theorem 3 gives

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)6}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_1,j_2,j_3=0}^{6}C_{j_3 j_2 j_1}^2\approx 0.01956000\Delta^3, \tag{67}$$

$$\mathsf{M}\left\{\left(I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)2}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{100}\right)^2\approx 0.00815429\Delta^5, \tag{68}$$

$$\mathsf{M}\left\{\left(I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)2}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{010}\right)^2\approx 0.01739030\Delta^5, \tag{69}$$

$$\mathsf{M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)2}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{001}\right)^2\approx 0.02528010\Delta^5, \tag{70}$$

$$\text{(71)}\qquad \mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)2}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C_{j_4j_3j_2j_1}^2\approx 0.02360840\Delta^4,$$

$$\text{(72)}\qquad \mathsf{M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5)1}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5j_4j_3j_2j_1}^2\approx 0.00759105\Delta^5.$$

Recall that the value Δ acts as the integration step of the numerical method (5) for the Ito SDE (1), thereby being rather small. Hence, even for $q=6,2,$ and 1 the mean-square approximation errors (67)–(72) of the iterated Ito stochastic integrals of multiplicities 3 to 5 are sufficiently small. Note that in [3], [7], [8] the iterated stochastic integrals were approximated using the trigonometric Fourier expansion of the multidimensional Brownian bridge process and the mean-square approximation error of the iterated stochastic integrals were estimated by the value

$$\frac{C_1\Delta^2}{q},$$

where C_1 is a constant and Δ, q have the same meaning as in (47). Clearly, such an approach is rougher than the one involving Theorem 3.

Note that the number q must be the same for all approximations of iterated stochastic integrals from the considered collection in the approach from [3], [7], [8] while the numbers q can be chosen different for different stochastic integrals from the considered collection in the method based on Theorems 1–3.

On the basis of the presented expansions (see above) of iterated Ito stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to increasing of orders of smallness with respect to Δ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities (the numbers q) in expansions of iterated Ito stochastic integrals, which are required for achieving the acceptable accuracy of approximation.

7. Numerical Algorithm With the Order 2.5 of Strong Convergence

In this section, we will write the formulas and recommendations on the numerical method (5) with the order 2.5 of strong convergence as an algorithm.

Let the Fourier–Legendre coefficients

$$\bar{C}_{j_3j_2j_1},\quad \bar{C}_{j_4j_3j_2j_1},\quad \bar{C}_{j_3j_2j_1}^{001},\quad \bar{C}_{j_3j_2j_1}^{010},\quad \bar{C}_{j_3j_2j_1}^{100},\quad \bar{C}_{j_5j_4j_3j_2j_1}$$

be precalculated [54].

Algorithm 1.

Step 1. Specify the initial parameters of the problem: the integration interval $[0,T]$, the integration step Δ (e.g., the constant one $\Delta=T/N$, where $N>1$; a variable step is also admissible), the initial condition $\mathbf{y}_0$, and the constant C appearing in the condition (7).

Step 2. Let $p=0$.

Step 3. Choose the minimum values q under which the right-hand sides of (58)–(66) are not exceeding the right-hand side of the inequality (7).

Step 4. Generate a sequence of independent standard Gaussian random variables $\zeta_l^{(i)}$ $(l = 0, 1, \ldots, q+2;\ i = 1, \ldots, m)$. Here the number q is a maximum from the numbers q chosen at Step 3.

Step 5. Model the iterated Ito stochastic integrals

$$I_{(0)\tau_{p+1},\tau_p}^{(i_1)},\ I_{(1)\tau_{p+1},\tau_p}^{(i_1)},\ I_{(2)\tau_{p+1},\tau_p}^{(i_1)},\ I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)},\ I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)},\ I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)},\ I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)},$$

$$I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)},\ I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)},\ I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)},\ I_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)},\ I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)}$$

using the formulas (46)–(57) with the values q chosen at Step 3.

Step 6. Find $\mathbf{y}_{p+1}$ by the formula (5).

Step 7. If $p < N-1$, then assign $p = p+1$ and go back to Step 4. Otherwise proceed to Step 8.

Step 8. End of the Algorithm 1.

8. Conclusions

In this paper, the efficient mean-square approximation procedures for the iterated Ito stochastic integrals of multiplicities 1 to 5 that are based on the multiple Fourier–Legendre series have been developed. These results can be used for the implementation of the numerical method (5) with the strong order 2.5 of convergence for the Ito SDEs with multidimensional non-commutative noises.

References

[1] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[2] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 28 pp. [In English].

[3] Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. Berlin: Springer, 1992, 632 pp.

[4] Arato M. Linear stochastic systems with constant coefficients. A statistical approach. Berlin–Heidelberg–N.Y.: Springer, 1982, 289 pp.

[5] Shiriaev A.N. Foundations of Financial Mathematics. Vol. 2, Moscow: Fazis, 1998, 544 pp. [In Russian]

[6] Liptser R.Sh., Shiriaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. Moscow: Nauka, 1974, 696 pp. [In Russian]

[7] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk: Ural University Press, 1988, 225 pp. [In Russian]

[8] Milstein G.N., Tretyakov M.V. Stochastic numerics for mathematical physics. Berlin: Springer, 2004, 616 pp.

[9] Kloeden P.E., Platen E., Schurz H. Numerical solution of SDE through computer experiments. Berlin: Springer, 1994, 292 pp.

[10] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.

[11] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor Expansions. Math. Nachr. 151 (1991), 33-50.

[12] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[13] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 90 pp. [In English].

[14] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[15] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[16] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2017, no. 1, 385 pp. (A.1–A.385). DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[19] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 923 pp. [In English].

[20] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/collection.html

[21] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[23] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[25] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[26] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[27] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[28] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[29] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[30] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[31] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[32] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[33] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[34] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[35] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[36] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[37] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[38] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[39] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English]

[40] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[41] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [In English].

[42] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on the generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 68 pp. [In English].

[43] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [In English].

[44] Kuznetsov D.F. The hypotheses on expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [In English].

[45] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 203 pp. [In English].

[46] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 65 pp. [In English].

[47] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. arXiv:1801.01962 [math.PR]. 2018, 49 pp. [In English].

[48] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 57 pp. [In English].

[49] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv:1802.04844 [math.PR]. 2018, 31 pp. [In English].

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 77 pp. [In English].

[51] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR]. 2018, 42 pp. [In English].

[52] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501v4 [math.PR]. 2018, 40 pp. [In English].

[53] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 126 pp. [In English].

[54] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 343 pp. [In English].

[55] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[56] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].

[57] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[58] Allen, E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.

[59] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev: Naukova Dumka, 1982, 612 pp. [In Russian]

[60] Kuznetsov, D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596.
DOI: http://doi.org/10.1023/A:1026138522239

[61] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[62] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

NUMERICAL SIMULATION OF 2.5-SET OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITIES 1 TO 5 FROM THE TAYLOR–STRATONOVICH EXPANSION

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to construction of effective procedures of the mean-square approximation for iterated Stratonovich stochastic integrals of multiplicities 1 to 5. We apply the method of generalized multiple Fourier series for approximation of iterated stochastic integrals. More precisely, we use multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k \in \mathbb{N}$. Considered iterated Stratonovich stochastic integrals are part of the Taylor–Stratonovich expansion. That is why the results of the article can be applied to implementation of numerical methods with the orders 1.0, 1.5, 2.0 and 2.5 of strong convergence for Ito stochastic differential equations with multidimensional non-commutative noise.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
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1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0, T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0, T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying to the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[17]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous nonrandom functions on $[t, T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau, \;\; i_1, \ldots, i_k = 0, 1, \ldots, m,$

$$\int \text{ and } \int^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in the classical Taylor–Ito and Taylor–Stratonovich expansions [2]-[7]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k;\; q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in the unified Taylor–Ito and Taylor–Stratonovich expansions [8]-[17].

Effective solution of the problem of combined mean-square approximation of collections of the iterated Ito and Stratonovich stochastic integrals (2), (3) of multiplicities 1 to 5 and beyond composes the subject of the article.

We want to mention in short that there are two main criteria of numerical methods convergence for Ito SDEs [2]-[4]: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution.

Using the strong numerical methods, we can build sample pathes of Ito SDEs numerically. These methods require the combined mean-square approximation of collections of the iterated Ito and Stratonovich stochastic integrals (2) and (3).

The strong numerical methods are using when constructing new mathematical models on the basis of Ito SDEs, when solving the filtering problem of signal under the influence of random disturbance in various arrangements, when solving the problem of stochastic optimal control, when solving the problem of testing procedures of evaluating parameters of stochastic systems etc. [2]-[5].

The problem of effective jointly numerical modeling (in accordance to the mean-square convergence criterion) of the iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[5], [10]-[66].

The only exception is connected with the narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)$. This case allows the investigation with using of the Ito formula [2]-[4].

Note that even for the mentioned coincidence ($i_1 = \ldots = i_k \neq 0$), but for different functions $\psi_1(s), \ldots, \psi_k(s)$ the mentioned difficulties persist, and relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be represented effectively in a finite form (for the mean-square approximation) using the system of standard Gaussian random variables.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated stochastic integrals can be simplified but cannot be solved. The equations with additive vector noise, with additive scalar or non-additive scalar noise, with a small parameter are related to such types of equations [2]-[4]. For the mentioned types of equations, simplifications are connected with the fact that either some coefficient functions from stochastic analogues of the Taylor formula (Taylor–Ito and Taylor–Stratonovich expansions) identically equal to zero, or scalar noise has an essential effect, or due to the presence of a small parameter we can neglect some members from stochastic analogues of the Taylor formula, which include difficult for approximation iterated stochastic integrals [2]-[4]. In this article, we consider Ito SDEs with multidimentional and non-additive noise. The conditions of commutativity of the noise [2] are also not used.

Seems that iterated stochastic integrals can be approximated by multiple integral sums of different types [3], [4], [59]. However, this approach implies partitioning of the interval of integration $[t, T]$ of iterated stochastic integrals (the length $T - t$ of this interval is a small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant calculating costs [10].

In [3] (also see [2], [4]) Milstein G.N. proposed to expand (2) or (3) into iterated series of products of standard Gaussian random variables by representing the Wiener process as a trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion for the Brownian bridge process). For example, to obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals (2) and (3) were presented in [2] (the integrals (3) for $k = 1, 2, 3$) and in [3], [4] (the integrals (2) for $k = 1, 2$) for the simplest case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$. Moreover, the Milstein approach [3] leads to iterated application of the operation of limit transition (see above).

It should be noted that the authors of the works [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [60] (pp. 438-439), [61] (pp. 263-264) use the Wong–Zakai approximation [62]-[64] (without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals [3] (1988) based on the series expansion of the Brownian bridge process (version of the so-called Karhunen-Loeve expansion). See discussions in [15] (Sect. 2.18, 6.2), [17] (Sect. 2.6.2, 6.2) [39] (Sect. 11), [41] (Sect. 8), [42] (Sect. 11), [43] (Sect. 6), [44] (Sect. 6) for detail.

Note that in [65] the method of expansion of iterated (double) Ito stochastic integrals (2) ($k = 2$; $\psi_1(s), \psi_2(s) \equiv 1$; $i_1, i_2 = 1, \ldots, m$) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered. The restrictions of the method [65] are also connected with iterated application of the operation of limit transition (as in the Milstein approach [3] (1988)) at least starting from the third multiplicity of iterated stochastic integrals.

It is necessary to note that the Milstein approach [3] excelled in several times or even in several orders the methods based on multiple integral sums [3], [4], [59] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [33], [34] (also see [11]-[17], [21]-[24], [47]), where $J^*[\psi^{(k)}]_{T,t}$ was represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables, and the function was then expressed as the iterated generalized Fourier series in complete systems of continuous functions that are orthonormal in the space $L_2([t,T])$. In [33], [34] (also see [11]-[17], [21]-[24], [47]) the cases of Legendre polynomials and trigonometric functions are considered in detail. As a result, the general iterated series expansion of (3) in terms of products of standard Gaussian random variables was obtained in [33], [34] (also see [11]-[17], [21]-[24], [47]) for an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series.

It was shown in [33], [34] (also see [11]-[17], [21]-[24], [47]) that the method of generalized iterated Fourier series leads to the Milstein expansion [3] of (3) in the case of trigonometric functions and to a substantially simpler expansion of (3) in the case of Legendre polynomials.

Note that the method of generalized iterated Fourier series as well as the Milstein approach [3] lead to iterated application of the operation of limit transition. As mentioned above, this problem appears for iterated (triple) stochastic integrals ($i_1, i_2, i_3 = 1, \ldots, m$) or even for some iterated (double) stochastic integrals in the case, when $\psi_1(s)$, $\psi_2(s) \not\equiv 1$ ($i_1, i_2 = 1, \ldots, m$) [10] (also see [11]-[32], [37]-[46], [48]-[50]). The mentioned problem (iterated application of the operation of limit transition) not appears in the efficient method, which is considered for (2) in Theorems 1, 2 (see below) [10]-[32], [37]-[46], [48]-[58].

The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated nonrandom function is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square convergening expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2). Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is an explicit formula (see (8) below) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) [12]-[17], [25], [40].

3. Since the used multiple Fourier series is a generalized in the sense that it is constructed using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [2]-[4], but Legendre polynomials.

4. As it turned out [10]-[32], [37]-[46], [48]-[57] it is more convenient to work with Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals (2). Approximations

based on the Legendre polynomials are essentially simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [15]-[17], [29], [37].

5. An approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see [65]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 1, 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$; $i_1, i_2, i_3 = 1, \ldots, m$) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, the authors of the works [2] (Sect. 5.8, pp. 202–204), [5] (pp. 82-84), [60] (pp. 438-439), [61] (pp. 263-264) use the condition $p_1 = p_2 = p_3 = p \to \infty$ together with the Wong–Zakai approximation [62]-[64] (but without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals [3] (1988) based on the series expansion of the Brownian bridge process. See discussions in [15] (Sect. 2.18, 6.2), [17] (Sect. 2.6.2, 6.2), [39] (Sect. 11), [41] (Sect. 8), [42] (Sect. 11), [43] (Sect. 6), [44] (Sect. 6) for detail.

As it turned out, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3) at least for multiplicities 1 to 6 [11]-[17], [22]-[24], [30], [33], [34], [38], [42]-[45], [47], [50], [58]. Expansions of these iterated Stratonovich stochastic integrals turned out much simpler (see Theorems 4–10 below), than the appropriate expansions of the iterated Ito stochastic integrals (2) from Theorems 1, 2.

2. Explicit One-Step Strong Numerical Schemes With Orders 2.0 and 2.5 for Ito SDEs Based on the Unified Taylor–Stratonovich expansion

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \le j \le N-1} \Delta\tau_j, \quad \Delta\tau_j = \tau_{j+1} - \tau_j.$$

Let $\mathbf{y}_{\tau_j} \stackrel{\sf def}{=} \mathbf{y}_j$, $j = 0, 1, \ldots, N$ be a time discrete approximation of the process $\mathbf{x}_t$, $t \in [0, T]$, which is a solution of the Ito SDE (1).

Definiton 1 [2]. *We will say that a time discrete approximation $\mathbf{y}_j$ $(j = 0, 1, \ldots, N)$ corresponding to the maximal step of discretization Δ_N, converges strongly with order $\gamma > 0$ at time moment T to the process $\mathbf{x}_t$, $t \in [0, T]$, if there exists a constant $C > 0$, which does not depend on Δ_N, and a $\delta > 0$ such that*

$$\mathsf{M}\{|\mathbf{x}_T - \mathbf{y}_T|\} \le C(\Delta_N)^\gamma$$

for each $\Delta_N \in (0, \delta)$.

Consider the explicit one-step strong numerical scheme with order 2.5 for Ito SDEs based on the so-called unified Taylor–Stratonovich expansion [9]-[21], [57]

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_{i_2} B_{i_1} \hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_2 i_1)} +$$

$$+\sum_{i_1=1}^{m}\left(G_{i_1}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)-\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{(000)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\left(G_{i_2}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{(10)\tau_{p+1},\tau_p}-\hat{I}^{*(i_2i_1)}_{(01)\tau_{p+1},\tau_p}\right)-\bar{L}G_{i_2}B_{i_1}\hat{I}^{*(i_2i_1)}_{(10)\tau_{p+1},\tau_p}+\right.$$

$$\left.+G_{i_2}G_{i_1}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{(01)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_2i_1)}_{(00)\tau_{p+1},\tau_p}\right)\right)+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{(0000)\tau_{p+1},\tau_p}+\frac{\Delta^3}{6}LL\mathbf{a}+$$

$$+\sum_{i_1=1}^{m}\left(G_{i_1}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}\right)+\right.$$

$$\left.+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}-LG_{i_1}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)\right)+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\left(G_{i_3}\bar{L}G_{i_2}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{(100)\tau_{p+1},\tau_p}-\hat{I}^{*(i_3i_2i_1)}_{(010)\tau_{p+1},\tau_p}\right)+\right.$$

$$+G_{i_3}G_{i_2}\bar{L}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{(010)\tau_{p+1},\tau_p}-\hat{I}^{*(i_3i_2i_1)}_{(001)\tau_{p+1},\tau_p}\right)+$$

$$+G_{i_3}G_{i_2}G_{i_1}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_3i_2i_1)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{*(i_3i_2i_1)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$\left.-\bar{L}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{(100)\tau_{p+1},\tau_p}\right)+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{(00000)\tau_{p+1},\tau_p}, \tag{4}$$

where $\Delta=T/N$ $(N>1)$ is a constant (for simplicity) step of integration, $\tau_p=p\Delta$ $(p=0,1,\ldots,N)$, $\hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}$ is an approximation of the iterated Stratonovich stochastic integral

$$I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}=\int\limits_t^{*s}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}^{(i_1)}_{t_1}\ldots d\mathbf{f}^{(i_k)}_{t_k}, \tag{5}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ l_1,\ldots,l_k=0,1,2,\ \ k=1,2,\ldots,5,$

$$\bar{\mathbf{a}}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)-\frac{1}{2}\sum_{j=1}^{m}G_jB_j(\mathbf{x},t),$$

$$\bar{L} = L - \frac{1}{2}\sum_{j=1}^{m} G_j G_j,$$

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}_i} + \frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n} B_{lj}(\mathbf{x},t)B_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}_l \partial \mathbf{x}_i},$$

$$G_i = \sum_{j=1}^{n} B_{ji}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}_j}\ , \quad i = 1,\ldots,m,$$

B_i and B_{ij} are the ith column and the ijth element of the matrix function B, $\mathbf{a}_i$ is the ith element of the vector function $\mathbf{a}$, $\mathbf{x}_i$ is the ith element of the column $\mathbf{x}$, the functions

$$B_{i_1},\ \bar{\mathbf{a}},\ G_{i_2}B_{i_1},\ G_{i_1}\bar{\mathbf{a}},\ \bar{L}B_{i_1},\ G_{i_3}G_{i_2}B_{i_1},\ \bar{L}\bar{\mathbf{a}},\ LL\mathbf{a},\ G_{i_2}\bar{L}B_{i_1},$$

$$\bar{L}G_{i_2}B_{i_1},\ G_{i_2}G_{i_1}\bar{\mathbf{a}},\ G_{i_4}G_{i_3}G_{i_2}B_{i_1},\ G_{i_1}\bar{L}\bar{\mathbf{a}},\ \bar{L}\bar{L}B_{i_1},\ \bar{L}G_{i_1}\bar{\mathbf{a}},\ G_{i_3}\bar{L}G_{i_2}B_{i_1},\ G_{i_3}G_{i_2}\bar{L}B_{i_1},$$

$$G_{i_3}G_{i_2}G_{i_1}\bar{\mathbf{a}},\ \bar{L}G_{i_3}G_{i_2}B_{i_1},\ G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}$$

are calculated at the point $(\mathbf{y}_p, p)$.

It is well known that under the standard conditions [2], [10] the numerical scheme (4) has strong order of convergence 2.5. The major emphasis below will be placed on the approximation of the iterated Stratonovich stochastic integrals appearing in (4). Therefore, among the standard conditions, we note the following approximation condition for these stochastic integrals [2], [10]

$$\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p} - \hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\right)^2\right\} \le C\Delta^6, \tag{6}$$

where constant C is independent of Δ.

Note that if we exclude from (4) the terms starting from the term $\Delta^3 LL\mathbf{a}/6$, then we have the explicit one-step strong numerical scheme with order 2.0 [2], [10], [13]-[21].

Using the numerical scheme (4) or its modifications based on the classical Taylor–Stratonovich expansion [7], the implicit or multistep analogues of (4) can be constructed [2], [10], [13]-[21]. The set of the iterated Stratonovich stochastic integrals to be approximated for implementing these modifications is the same as for the numerical scheme (4) itself. Interestingly, the truncated unified Taylor–Stratonovich expansion [9] (the foundation of the numerical scheme (4)) contains only 12 different types of the iterated Stratonovich stochastic integrals (5), which cannot be interconnected by linear relations [10], [13]-[21]. The analogues classical Taylor–Stratonovich expansion [2], [7] contains 17 different types of iterated Stratonovich stochastic integrals, part of which are interconnected by linear relations and part of which have a higher multiplicity than the iterated Stratonovich stochastic integrals (5). This fact well explains the use of the numerical scheme (4).

One of the main problems arising in the implementation of the numerical scheme (4) is the joint numerical modeling of the iterated Stratonovich stochastic integrals figuring in (4).

3. Expansions of Iterated Ito Stochastic Integrals (Method of Genegalized Multiple Fourier Series)

An efficient numerical modeling method for iterated Ito stochastic integrals based on generalized multiple Fourier series was considered in [10] (also see [11]-[32], [37]-[58]).

This method rests on important results presented below (Theorems 1, 2).

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ & \\ 0 & \text{otherwise}\end{cases},\quad t_1,\ldots,t_k\in[t,T],\quad k\ge 2, \tag{7}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{8}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{9}$$

Theorem 1 [10] (2006), [11]-[32], [37]-[58]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{10}$$

where $J[\psi^{(k)}]_{T,t}$ *is defined by* (2),

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k = 0,1,\ldots,m,$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{11}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i\neq 0$), $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (8), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (9).

It was shown in [19]-[24] that Theorem 1 is valid for convergence in the mean of degree $2n$ $(n\in\mathbb{N})$. The convergence with probability 1 in Theorem 1 is proved in [15]-[17], [39], [53] for the cases of Legendre polynomials and trigonometric functions. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ can also be applied in Theorem 1 [10]-[24]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [14], [15]-[17], [39], [49]. Application of Theorem 1 and Theorem 2 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [15]-[17] (Chapter 7) and in [31], [32], [54], [56].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [10]-[32], [37]-[58]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{12}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{13}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{14}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{15}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$
$$+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{16}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Note that we will consider the case $i_1,\ldots,i_5=1,\ldots,m$. This case corresponds to the numerical scheme (4).

For further consideration, let us consider the generalization of formulas (12)–(16) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2).

In order to do this, let us introduce some notations. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{17}
$$

where

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (17) is a partition and consider the sum with respect to all possible partitions

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2, \ldots, g_{2r-1} g_{2r}, q_1 \ldots q_{k-2r}}. \tag{18}
$$

Below there are several examples of sums in the form (18)

$$
\sum_{\substack{(\{g_1,g_2\}) \\ \{g_1,g_2\}=\{1,2\}}} a_{g_1 g_2} = a_{12},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\}) \\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\}) \\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1 g_2, q_1 q_2} =
$$

$$
= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},
$$

$$
\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\}) \\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1 g_2, q_1 q_2 q_3} =
$$

$$
= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +
$$
$$
+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},
$$

$$
\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\}) \\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1 g_2, g_3 g_4, q_1} =
$$

$$
= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +
$$
$$
+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +
$$
$$
+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
$$

Now we can write (10) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{19}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (19) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (16).

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [15] (Sect. 1.11), [39] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{20}$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [66]. Note that we use another notations [15] (Sect. 1.11), [39] (Sect. 15) in comparison with [66]. Moreover, the proof of an analogue of Theorem 2 from [66] is somewhat different from the proof given in [15] (Sect. 1.11), [39] (Sect. 15).

4. Calculation of the Mean-Square Approximation Error in the Method of Generalized Multiple Fourier Seires

Note that for the integrals $J[\psi^{(k)}]_{T,t}$ defined by (2) the mean-square approximation error can be exactly calculated and efficiently estimated.

Let $J[\psi^{(k)}]_{T,t}^q$ be the expression on the right-hand side of (20) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=q$, i.e.

$$J[\psi^{(k)}]_{T,t}^q=\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr). \tag{21}$$

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}\stackrel{\text{def}}{=}E_k^q,$$

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\text{def}}{=}I_k.$$

In [13]-[17], [39], [40] it was shown that

$$E_k^q\le k!\Biggl(I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2\Biggr) \tag{22}$$

for the following two cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$,
2. $i_1,\ldots,i_k=0,1,\ldots,m$ and $T-t\in(0,1)$.

The value E_k^q can be calculated exactly.

Theorem 3 [15] (Sect. 1.12), [40] (Sect. 6). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then*

$$E_k^q=I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{23}$$

where $i_1,\ldots,i_k=1,\ldots,m$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$*. At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$*, then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$*; another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Therefore, for the case of pairwise different numbers $i_1,\ldots,i_k$ as well as for the case $i_1=\ldots=i_k$ from Theorem 3 it follows that [14], [15]-[17], [25], [40]

$$E_k^q=I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2, \tag{24}$$

$$E_k^q=I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

is a sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

Consider some examples [14], [15]-[17], [25], [40] of application of Theorem 3 $(i_1,i_2,i_3=1,\ldots,m)$

$$E_2^q=I_2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{q}C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2), \tag{25}$$

$$E_3^q=I_3-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3), \tag{26}$$

$$E_3^q=I_3-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3), \tag{27}$$

$$E_3^q=I_3-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2). \tag{28}$$

The values E_4^q and E_5^q were calculated exaclty for all possible combinations of $i_1,\ldots,i_5=1,\ldots,m$ in [14], [15]-[17], [40].

5. Expansions of Iterated Stratonovich Stochastic Integrals Based on Multiple Fourier–Legendre Series and Multiple Trigonometric Fourier Seires

In contrast to the iterated Ito stochastic integrals (2), the iterated Stratonovich stochastic integrals (3) have simpler expansions (see Theorems 4–10 below) than (10) but the calculation (or estimation) of mean-square approximation errors for the latter is a more difficult problem than for the former. We will study this issue in details below.

As we mentioned above, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3) at least for multiplicities 1 to 6. Expansions of these iterated Stratonovich stochastic integrals turned out much simpler, than the appropriate expansions of the iterated Ito stochastic integrals (2) from Theorems 1, 2. Let us formulate some old results on expansions of the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 4.

Theorem 4 [11]-[17], [22]-[24], [30], [33], [34], [38], [43], [45], [47]. *Assume that the following conditions are fulfulled*:

1. *The function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the function* $\psi_1(\tau)$ *is twice continuously differentiable at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, the iterated Stratonovich stochastic integral of second multiplicity

$$\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

is expanded into the following series

$$\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

converging in the mean-square sense, where

$$C_{j_2j_1}=\int\limits_t^{T}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2;$$

another notations are the same as in Theorems 1, 2.

Theorem 5 [11]-[17], [22]-[24], [38], [43], [44]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space* $L_2([t,T])$. *Moreover, the function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$.

Then, for the iterated Stratonovich stochastic integral of third multiplicity

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{29}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 1, 2.

Theorem 6 [11]-[17], [22]-[24], [38], [43], [50]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of multiplicity* 4

$$\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following expansion

$$\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

converging in the mean-square sense is valid, where $i_1,i_2,i_3,i_4=0,1,\ldots,m$,

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorems 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [15] (Sect. 2.10–2.16), [42] (Sect. 7–13), [43] (Sect. 13–19), [50] (Sect. 5–11), [58] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 7 [15], [42], [43], [50], [58]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{30}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{31}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (30) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (31), *constant* C *is independent of* p,

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \ne 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 8 [15], [42], [43], [50], [58]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{32}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{33}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{34}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (32), (33) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (34), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$
C_{j_4j_3j_2j_1}=
$$

$$
=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;
$$

another notations are the same as in Theorem 7.

Theorem 9 [15], [42], [43], [50], [58]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{35}
$$

the following relations

$$
J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{36}
$$

$$
\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,\ldots,j_5=0}^{p}C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{37}
$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (35), (36) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (37), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$
C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;
$$

another notations are the same as in Theorems 7, 8.

Theorem 10 [15], [42], [43], [50], [58]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$
J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{38}
$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 7–9.

6. Approximation of Iterated Stratonovich Stochastic Integrals Based on Multiple Fourier–Legendre Series

As was mentioned above, one of the main problems arising in the implementation of the numerical scheme (4) is the joint numerical modeling of the iterated Stratonovich stochastic integrals figuring in (4). Let us consider efficient numerical modeling formulas for the iterated Stratonovich stochastic integrals based on Theorems 4–9.

Using Theorems 1, 2 ($k=1$), Theorems 4–9, and multiple Fourier–Legendre series, we obtain the following approximations of iterated Stratonovich stochastic integrals from (4) [10]-[50]

$$I_{(0)\tau_{p+1},\tau_p}^{*(i_1)}=\sqrt{\Delta}\zeta_0^{(i_1)}, \tag{39}$$

$$I_{(1)\tau_{p+1},\tau_p}^{*(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{40}$$

$$I_{(2)\tau_{p+1},\tau_p}^{*(i_1)}=\frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{41}$$

$$I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)q}=\frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{42}$$

$$I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)q}=-\frac{\Delta}{2}I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)q}-\frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{43}$$

$$I^{*(i_1 i_2)q}_{(10)\tau_{p+1},\tau_p}=-\frac{\Delta}{2}I^{*(i_1 i_2)q}_{(00)\tau_{p+1},\tau_p}-\frac{\Delta^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

(44) $$+\sum_{i=0}^{q}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr)$$

or

(45) $$I^{*(i_1 i_2)q}_{(01)\tau_{p+1},\tau_p}=\sum_{j_1,j_2=0}^{q}C^{01}_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

(46) $$I^{*(i_1 i_2)q}_{(10)\tau_{p+1},\tau_p}=\sum_{j_1,j_2=0}^{p}C^{10}_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)};$$

(47) $$I^{*(i_1 i_2 i_3)q}_{(000)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

(48) $$I^{*(i_1 i_2 i_3)q}_{(100)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3=0}^{q}C^{100}_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

(49) $$I^{*(i_1 i_2 i_3)q}_{(010)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3=0}^{q}C^{010}_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

(50) $$I^{*(i_1 i_2 i_3)q}_{(001)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3=0}^{q}C^{001}_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

(51) $$I^{*(i_1 i_2 i_3 i_4)q}_{(0000)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

(52) $$I^{*(i_1 i_2 i_3 i_4 i_5)q}_{(00000)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5 j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

where the Fourier–Legendre coefficients have the form

$$C^{01}_{j_2 j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}(\tau_p-y)\phi_{j_3}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdy=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}\Delta^2\bar{C}^{01}_{j_2 j_1},$$

$$C_{j_2j_1}^{10}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(y)\int\limits_{\tau_p}^{y}(\tau_p-x)\phi_{j_1}(x)dxdy=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}\Delta^2\bar C_{j_2j_1}^{10},$$

$$C_{j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar C_{j_3j_2j_1}, \tag{53}$$

$$C_{j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdu=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}\Delta^2\bar C_{j_4j_3j_2j_1}, \tag{54}$$

$$C_{j_3j_2j_1}^{001}=\int\limits_{\tau_p}^{\tau_{p+1}}(\tau_p-z)\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar C_{j_3j_2j_1}^{001}, \tag{55}$$

$$C_{j_3j_2j_1}^{010}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}(\tau_p-y)\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar C_{j_3j_2j_1}^{010}, \tag{56}$$

$$C_{j_3j_2j_1}^{100}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}(\tau_p-x)\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar C_{j_3j_2j_1}^{100}, \tag{57}$$

$$C_{j_5j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_5}(v)\int\limits_{\tau_p}^{v}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdudv=$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}\Delta^{5/2}\bar{C}_{j_5j_4j_3j_2j_1}, \tag{58}$$

where

$$\bar{C}^{01}_{j_2j_1} = -\int\limits_{-1}^{1}(1+y)P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdy,$$

$$\bar{C}^{10}_{j_2j_1} = -\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}(1+x)P_{j_1}(x)dxdy,$$

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{59}$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{60}$$

$$\bar{C}^{100}_{j_3j_2j_1} = -\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydz, \tag{61}$$

$$\bar{C}^{010}_{j_3j_2j_1} = -\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{62}$$

$$\bar{C}^{001}_{j_3j_2j_1} = -\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{63}$$

$$\bar{C}_{j_5j_4j_3j_2j_1} = \int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv, \tag{64}$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial and

$$\phi_i(x) = \sqrt{\frac{2i+1}{\Delta}}P_i\left(\left(x-\tau_p-\frac{\Delta}{2}\right)\frac{2}{\Delta}\right), \quad i=0,1,2,\ldots$$

The Fourier–Legendre coefficients

$$\bar{C}^{01}_{j_2j_1},\ \bar{C}^{10}_{j_2j_1},\ \bar{C}_{j_3j_2j_1},\ \bar{C}_{j_4j_3j_2j_1},\ \bar{C}^{001}_{j_3j_2j_1},\ \bar{C}^{010}_{j_3j_2j_1},\ \bar{C}^{100}_{j_3j_2j_1},\ \bar{C}_{j_5j_4j_3j_2j_1} \tag{65}$$

can be calculated exactly before start of the numerical method (4). The above calculation can be done with Python, Derive or Maple. In [10], [12]-[24], [41] several tables with these coefficients can be found. Moreover, in [51], [52] the database with 270,000 exactly calculated Fourier–Legendre coefficients (including (65)) was described. This database was used in the software package, which is written in the Python programming language for the implementation of high-order strong numerical schemes for Ito SDEs with non-commutative noise [51], [52]. Note that the mentioned Fourier–Legendre coefficients do not depend on the step of integration $\tau_{p+1}-\tau_p$ of the numerical scheme, which can be not a constant in a general case.

On the basis of the presented expansions (see (39)–(52)) of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to increasing of smallness orders with respect to Δ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities (the numbers q) in expansions of iterated Stratonovich stochastic integrals, which are required for achieving the acceptable accuracy of approximation. Generally speaking, the minimal values q that guarantee the condition (6) for each approximation (39)–(52) are various and abruptly decreasing with the growth of smallness orders with respect to Δ in the mean-square sense for iterated stochastic integrals.

Consider in detail the question on calculation and estimation of the mean-square approximation error for the iterated Stratonovich stochastic integrals (5) (see [15], Chapter 5 for details).

Let us consider the following iterated Ito stochastic integrals

$$I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ l_1,\ldots,l_k=0,1,2,\ \ k=1,2,\ldots,5.$

According to the standard relations between iterated Ito and Stratonovich stochastic integrals, we obtain w. p. 1 (with probability 1)

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}=I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)}-\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\Delta,$$

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}=I_{(10)\tau_{p+1},\tau_p}^{*(i_1i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}\Delta^2,$$

$$I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}=I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}\Delta^2.$$

Moreover, the mean-square approximation error for the iterated Ito stochastic integral

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}\quad (i_1\ne i_2)$$

equals to the mean-square approximation error for the iterated Stratonovich stochastic integral (see [15], Sect. 5.1 for details)

$$I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)}\quad (i_1\ne i_2).$$

From Theorem 3 we obtain [10]-[32], [37]-[50]

$$
\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2), \tag{66}
$$

$$
\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=
$$

$$
=\frac{\Delta^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\right.
$$

$$
\left.-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\quad (i_1\ne i_2). \tag{67}
$$

The case $i_1=i_2$ is considered in [15], Sect. 5.1.

Let us estimate the mean-square approximation error for the iterated Stratonovich stochastic integrals (5) of multiplicities $k\ge 3$. From (66) $(i_1\ne i_2)$ we get

$$
\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)}-I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le
$$

$$
\le\frac{\Delta^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx=-\frac{\Delta^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{\Delta^2}{q}, \tag{68}
$$

where constant C_1 does not depend on Δ.

As was mentioned above, the value Δ plays the role of integration step in the numerical procedures for Ito SDEs. Then this value is a sufficiently small. Keeping in mind this circumstance, it is easy to notice that there exists such a constant C_2 that

$$
\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C_2\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)}-I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)q}\right)^2\right\}, \tag{69}
$$

where $I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}$ is the approximation of the iterated Stratonovich stochastic integral (5) for $k\ge 3$.

From (68) and (69) we finally have

$$
\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}-I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\}\le K\frac{\Delta^2}{q}, \tag{70}
$$

where constant K does not depend on Δ.

The same idea can be found in [2] for the case of trigonometric functions. Note that, in contrast to the estimate (70), the constant C in Theorems 7–9 does not depend on q.

Essentially more information about numbers q can be obtained by another approach. We have

$$I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}=I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\quad \text{w. p. 1}$$

for pairwise different $i_1,\ldots,i_k=1,\ldots,m$.

Then, for pairwise different $i_1,\ldots,i_5=1,\ldots,m$ from ([24]) we obtain

$$\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}-I^{*(i_1i_2)q}_{(01)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C^{01}_{j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}-I^{*(i_1i_2)q}_{(10)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C^{10}_{j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)q}_{(000)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C^2_{j_3j_2j_1},$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{(0000)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3i_4)q}_{(0000)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C^2_{j_4j_3j_2j_1},$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)q}_{(100)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{100}_{j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)q}_{(010)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{010}_{j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)q}_{(001)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C^{001}_{j_3j_2j_1}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3i_4i_5)q}_{(00000)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C^2_{j_5i_4i_3i_2j_1}.$$

For example [10]-[24],

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)6}_{(000)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{6}C^2_{j_3j_2j_1}\approx 0.01956000\Delta^3,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)2}_{(100)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C^{100}_{j_3j_2j_1}\right)^2\approx 0.00815429\Delta^5,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}-I^{*(i_1i_2i_3)2}_{(010)\tau_{p+1},\tau_p}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C^{010}_{j_3j_2j_1}\right)^2\approx 0.01739030\Delta^5,$$

$$\mathsf{M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)}-I_{(001)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)2}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3 j_2 j_1}^{001}\right)^2\approx 0.02528010\Delta^5,$$

$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)2}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C_{j_4 j_3 j_2 j_1}^2\approx 0.02360840\Delta^4,$$

$$\mathsf{M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4 i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4 i_5)1}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5 i_4 i_3 i_2 j_1}^2\approx 0.00759105\Delta^5.$$

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. [In Russian]. Kiev: Naukova Dumka, 1982, 612 pp.

[2] Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. Berlin: Springer, 1992, 632 pp.

[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk: Ural University Press, 1988, 225 pp.

[4] Milstein G.N., Tretyakov M.V. Stochastic numerics for mathematical physics. Berlin: Springer, 2004, 616 pp.

[5] Kloeden P.E., Platen E., Schurz H. Numerical solution of SDE through computer experiments. Berlin: Springer, 1994, 292 pp.

[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.

[7] Kloeden P.E., Platen E. The Stratonovich and Ito–Taylor Expansions. Math. Nachr. 151 (1991), 33-50.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[12] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), 1, A.1–A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[15] Kuznetsov, D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR]. 2022, 923 pp.

[16] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[17] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at:
http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at:
http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[22] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at:
http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[23] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at:
http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[24] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[25] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[26] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[27] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [In English]. arXiv:1802.00888 [math.PR]. 2018, 28 pp.

[28] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[29] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[30] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[31] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[32] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[33] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[34] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[35] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[36] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[37] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[38] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[39] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [in English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[40] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [in English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.

[41] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. [in English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[42] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.

[43] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [in English]. arXiv:1712.09516 [math.PR]. 2022, 203 pp.

[44] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [in English]. arXiv:1801.01564 [math.PR]. 2018, 65 pp.

[45] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [in English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.

[46] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [in English]. arXiv:1712.08991 [math.PR]. 2022, 57 pp.

[47] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [in English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[48] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [in English]. arXiv:1807.02190 [math.PR]. 2018, 42 pp.

[49] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [in English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [in English]. arXiv:1802.00643 [math.PR]. 2022, 126 pp.

[51] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[52] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[53] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[54] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.3/article.1.6.html

[55] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878

[56] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[57] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

[58] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[59] Allen E. Approximation of Triple Stochastic Integrals Through Region Subdivision. Communicat. in Appl. Anal. Special Tribute Issue to Prof. V. Lakshmikantham. 17 (2013), 355-366.

[60] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.

[61] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.

[62] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[63] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[64] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[65] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. [In Russian]. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[66] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

EXPLICIT ONE-STEP STRONG NUMERICAL METHODS OF ORDERS 2.0 AND 2.5 FOR ITO STOCHASTIC DIFFERENTIAL EQUATIONS BASED ON THE UNIFIED TAYLOR–ITO AND TAYLOR–STRATONOVICH EXPANSIONS

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the construction of explicit one-step strong numerical methods with the orders 2.0 and 2.5 of convergence for Ito stochastic differential equations with multidimensional non-commutative noise. We consider numerical methods based on the unified Taylor–Ito and Taylor–Stratonovich expansions. For the numerical modeling of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 5 we apply the method of multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k=1,\ldots,5$. The article is addressed to engineers who use numerical modeling in stochastic control and for solving the non-linear filtering problem. The article will be interesting to scientists who working in the field of numerical integration of stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.

Keywords: Explicit one-step strong numerical method, Ito stochastic differential equation, Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Strong convergence, Mean-square convergence, Approximation, Expansion.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-subfields of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as the Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[4] that Ito stochastic differential equations are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito stochastic differential equations is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[10]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito stochastic differential equations and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1,\ldots,m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\int \text{ and } \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1,\ldots,k)$ and $i_1,\ldots,i_k = 0,1,\ldots,m$ in [2]-[7]. At the same time $\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l = 1,\ldots,k;\ q_1,\ldots,q_k = 0,1,2,\ldots)$ and $i_1,\ldots,i_k = 1,\ldots,m$ in [8]-[10].

We want to mention in short that there are two main criteria of numerical methods convergence for Ito stochastic differential equations: a strong or mean-square criterion and a weak criterion, where the subject of approximation is not the solution of Ito stochastic differential equation, simply stated, but the distribution of Ito stochastic differential equation solution [2]. Both of the above criteria are independent, that is, generally speaking, the fulfillment of a strong criterion does not imply the fulfillment of a weak criterion, and vice versa. Each of two convergence criteria is oriented on solution of specific classes of mathematical problems connected with stochastic differential equations.

Using the strong numerical methods, we may build sample pathes of Ito stochastic differential equation numerically. These methods require the combined mean-square approximation of collections of iterated Ito and Stratonovich stochastic integrals. Effective solution of this problem composes one of the subjects of this article.

The strong numerical methods are used for constructing new mathematical models on the basis of Ito stochastic differential equations and also for solving some mathematical problems connected with Ito stochastic differential equations. Among this problems we mention the following: signal filtering under the influence of random disturbances in various statements, stochastic optimal control, testing estimation procedures of parameters of stochastic systems [2].

2. Explicit One-Step Strong Numerical Schemes of Orders 2.0 and 2.5 Based on the Unified Taylor–Ito expansion

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$ such that

$$0=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j,\quad \Delta\tau_j=\tau_{j+1}-\tau_j.$$

Let $\mathbf{y}_{\tau_j}\stackrel{\sf def}{=}\mathbf{y}_j,\ j=0,1,\ldots,N$ be a time discrete approximation of the process $\mathbf{x}_t,\ t\in[0,T]$, which is a solution of the Ito stochastic differential equation (1).

Definiton 1 [2]. *We will say that a time discrete approximation* $\mathbf{y}_j,\ j=0,1,\ldots,N$, *corresponding to the maximal step of discretization* Δ_N, *converges strongly with order* $\gamma>0$ *at time moment* T *to the process* $\mathbf{x}_t,\ t\in[0,T]$, *if there exists a constant* $C>0$, *which does not depend on* Δ_N, *and a* $\delta>0$ *such that*

$$\mathsf{M}\{|\mathbf{x}_T-\mathbf{y}_T|\}\le C(\Delta_N)^{\gamma}$$

for each $\Delta_N\in(0,\delta)$.

Consider the explicit one-step strong numerical scheme of order 2.5 based on the so-called unified Taylor–Ito expansion [11], [14]-[22]

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^m B_{i_1}\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^m G_{i_2}B_{i_1}\hat I_{(00)\tau_{p+1},\tau_p}^{(i_2 i_1)}+$$

$$+\sum_{i_1=1}^m\left(G_{i_1}\mathbf{a}\left(\Delta\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)+$$

$$+\sum_{i_1,i_2,i_3=1}^m G_{i_3}G_{i_2}B_{i_1}\hat I_{(000)\tau_{p+1},\tau_p}^{(i_3 i_2 i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^m\left(G_{i_2}LB_{i_1}\left(\hat I_{(10)\tau_{p+1},\tau_p}^{(i_2 i_1)}-\hat I_{(01)\tau_{p+1},\tau_p}^{(i_2 i_1)}\right)-LG_{i_2}B_{i_1}\hat I_{(10)\tau_{p+1},\tau_p}^{(i_2 i_1)}+\right.$$

$$\left.+G_{i_2}G_{i_1}\mathbf{a}\left(\hat I_{(01)\tau_{p+1},\tau_p}^{(i_2 i_1)}+\Delta\hat I_{(00)\tau_{p+1},\tau_p}^{(i_2 i_1)}\right)\right)+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{(i_4 i_3 i_2 i_1)}_{(0000)\tau_{p+1},\tau_p}+\frac{\Delta^3}{6}LL\mathbf{a}+$$

$$+\sum_{i_1=1}^{m}\Biggl(G_{i_1}L\mathbf{a}\left(\frac{1}{2}\hat{I}^{(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{(i_1)}_{(1)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{(i_1)}_{(0)\tau_{p+1},\tau_p}\right)+$$

$$+\frac{1}{2}LLB_{i_1}\hat{I}^{(i_1)}_{(2)\tau_{p+1},\tau_p}-LG_{i_1}\mathbf{a}\left(\hat{I}^{(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{(i_1)}_{(1)\tau_{p+1},\tau_p}\right)\Biggr)+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl(G_{i_3}LG_{i_2}B_{i_1}\left(\hat{I}^{(i_3 i_2 i_1)}_{(100)\tau_{p+1},\tau_p}-\hat{I}^{(i_3 i_2 i_1)}_{(010)\tau_{p+1},\tau_p}\right)+$$

$$+G_{i_3}G_{i_2}LB_{i_1}\left(\hat{I}^{(i_3 i_2 i_1)}_{(010)\tau_{p+1},\tau_p}-\hat{I}^{(i_3 i_2 i_1)}_{(001)\tau_{p+1},\tau_p}\right)+$$

$$+G_{i_3}G_{i_2}G_{i_1}\mathbf{a}\left(\Delta\hat{I}^{(i_3 i_2 i_1)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{(i_3 i_2 i_1)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$-LG_{i_3}G_{i_2}B_{i_1}\hat{I}^{(i_3 i_2 i_1)}_{(100)\tau_{p+1},\tau_p}\Biggr)+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{(i_5 i_4 i_3 i_2 i_1)}_{(00000)\tau_{p+1},\tau_p}, \tag{4}$$

where $\Delta = T/N$ $(N > 1)$ is a constant (for simplicity) integration step, $\tau_p = p\Delta$ $(p = 0, 1, \ldots, N)$, $\hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}$ denotes an approximation of the iterated Ito stochastic integral of multiplicity k

$$I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}=\int\limits_t^s (t-\tau_k)^{l_k}\ldots\int\limits_t^{\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}^{(i_1)}_{\tau_1}\ldots d\mathbf{f}^{(i_k)}_{\tau_k}, \tag{5}$$

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B_{lj}(\mathbf{x},t)B_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_i},$$

$$G_i=\sum_{j=1}^{n}B_{ji}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_j},\quad i=1,\ldots,m,$$

$l_1,\ldots,l_k = 0, 1, 2,$ $i_1,\ldots,i_k = 1,\ldots,m,$ $k = 1, 2,\ldots,5,$ B_i and B_{ij} are the ith column and the ijth component of the matrix function B, $\mathbf{a}_i$ is the ith component of the vector function $\mathbf{a}$, $\mathbf{x}_i$ is the ith component of the column $\mathbf{x}$, the functions

$$B_{i_1},\ \mathbf{a},\ G_{i_2}B_{i_1},\ G_{i_1}\mathbf{a},\ LB_{i_1},\ G_{i_3}G_{i_2}B_{i_1},\ L\mathbf{a},\ LL\mathbf{a},\ G_{i_2}LB_{i_1},$$

$$LG_{i_2}B_{i_1},\ G_{i_2}G_{i_1}\mathbf{a},\ G_{i_4}G_{i_3}G_{i_2}B_{i_1},\ G_{i_1}L\mathbf{a},\ LLB_{i_1},\ LG_{i_1}\mathbf{a},\ G_{i_3}LG_{i_2}B_{i_1},\ G_{i_3}G_{i_2}LB_{i_1},$$

$$G_{i_3}G_{i_2}G_{i_1}\mathbf{a},\ LG_{i_3}G_{i_2}B_{i_1},\ G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}$$

are calculated at the point $(\mathbf{y}_p, p)$.

Under the standard conditions [2], [11] the numerical scheme (4) has strong order 2.5 of convergence. The major emphasis below will be placed on the approximation of the iterated Ito stochastic integrals appearing in (4). Therefore, among the mentioned conditions, we note only the approximation condition for iterated Ito stochastic integrals [2], [11], which has the form

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}-\hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^6, \tag{6}$$

where constant C is independent of Δ.

Note that if we exclude from (4) the terms starting from the term $\Delta^3 LL\mathbf{a}/6$, then we will have the explicit one-step strong numerical scheme of order 2.0 [2], [11], [14]-[22].

Using the numerical scheme (4) or its modifications based on the Taylor–Ito expansion [7], the implicit or multistep analogues of (4) can be constructed [2], [11], [14]-[22]. The set of the iterated Ito stochastic integrals to be approximated for implementing these modifications is the same as for the numerical scheme (4) itself. Interestingly, the truncated unified Taylor–Ito expansion (the foundation of the numerical scheme (4)) contains 12 different types of iterated Ito stochastic integrals of the form (5), which cannot be interconnected by linear relations [11], [14]-[22]. The analogous Taylor–Ito expansion [2], [7] contains 17 different types of iterated Ito stochastic integrals, part of which are interconnected by linear relations and part of which have a higher multiplicity than the iterated Ito stochastic integrals (5). This fact well explains the use of the numerical scheme (4).

One of the main problems arising in the implementation of the numerical scheme (4) is the joint numerical modeling of the iterated Ito stochastic integrals figuring in (4). In the subsequent sections, we will consider an efficient numerical modeling method for the iterated Ito stochastic integrals and also demonstrate which stochastic integrals (Ito or Stratonovich) are preferable for numerical modeling with a correct estimation of the mean-square approximation error.

3. Method of Numerical Modeling for Iterated Ito Stochastic Integrals Based on Genegalized Multiple Fourier Series. Direct Approach

An efficient numerical modeling method for the iterated Ito stochastic integrals (2) based on generalized multiple Fourier series was proposed in [11] (2006); also see [12]-[57]. This method rests on an important result presented below.

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{7}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{8}$$

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{9}$$

Theorem 1 [11]-[33], [38]-[47], [49]-[57]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{10}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{11}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (8), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval $[t,T]$, which satisfies the condition* (9).

The convergence in the mean of degree $2n$ $(n\in\mathbb{N})$ [16]-[25] as well as the convergence with probability 1 [16]-[18], [40], [42] are proved for the approximations from Theorem 1.

Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ can also be applied in Theorem 1 [11]-[25]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [15]-[18], [50].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [11]-[33], [38]-[47], [49]-[57]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{12}$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{13}$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{14}$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{15}$$

$$J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{16}$$

where $\mathbf{1}_A$ is the indicator of the set A.

We will consider the case $i_1,\ldots,i_5=1,\ldots,m$. Obviously, this case corresponds to the numerical method (4).

For further consideration, let us consider the generalization of formulas (12)–(16) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{17}$$

where

$$\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (17) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{18}$$

Below there are several examples of sums in the form (18)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$
$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$
$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (10) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times\right.$$

$$\left.\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \tag{19}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (19) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} - \right.$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$\left. + \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\right).$$

The last equality obviously agrees with (16).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [16] (Sect. 1.11), [40] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$
J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{20}
$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [59]. Note that we use another notations [16] (Sect. 1.11), [40] (Sect. 15) in comparison with [59]. Moreover, the proof of an analogue of Theorem 2 from [59] is somewhat different from the proof given in [16] (Sect. 1.11), [40] (Sect. 15).

Note that, for the integrals $J[\psi^{(k)}]_{T,t}$ defined by (2), the mean-square approximation error can be exactly calculated and efficiently estimated.

Let $J[\psi^{(k)}]_{T,t}^q$ be the expression on the right-hand side of (20) before passing to the limit for the case $p_1=\ldots=p_k=q$, i.e.

$$
J[\psi^{(k)}]_{T,t}^q=\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times
$$

$$
\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{21}
$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}\stackrel{\text{def}}{=}E_k^q,
$$

$$
\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\text{def}}{=}I_k.
$$

In [14]-[18], [40], [41] it was shown that

$$
E_k^q\le k!\left(I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2\right) \tag{22}
$$

for the following two cases:

1. $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$,
2. $i_1,\ldots,i_k=0,1,\ldots,m$ and $T-t\in(0,1)$.

The value E_k^q can be calculated exactly.

Theorem 3 [16] (Sect. 1.12), [41] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{23}$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Therefore, for the case of pairwise different numbers $i_1,\ldots,i_k$ as well as for the case $i_1=\ldots=i_k$ from Theorem 3 it follows that [15]-[18], [26], [41]

$$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}^2, \tag{24}$$

$$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

is a sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

Consider some examples [15]-[18], [26], [41] of application of Theorem 3 ($i_1,i_2,i_3=1,\ldots,m$)

$$E_2^q = I_2 - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}^2 - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2), \tag{25}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\neq i_3), \tag{26}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\neq i_2=i_3), \tag{27}$$

$$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1=i_3\ne i_2). \tag{28}$$

The values E_4^q and E_5^q were calculated exaclty for all possible combinations of $i_1,\ldots,i_5=1,\ldots,m$ in [15]-[18], [41].

4. Approximation of Specific Iterated Ito Stochastic Integrals Based on Multiple Fourier–Legendre Series

Consider approximations of the iterated Ito stochastic integrals that appear in the numerical scheme (4) using Theorems 1, 2 for the case of complete orthonormal system of Legendre polynomials in the space $L_2([\tau_p,\tau_{p+1}])$ ($\tau_p=p\Delta$, $N\Delta=T$, $p=0,1,\ldots,N$) [11] (also see [12]-[49], [52]-[54])

$$I_{(0)\tau_{p+1},\tau_p}^{(i_1)} = \sqrt{\Delta}\zeta_0^{(i_1)}, \tag{29}$$

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)q} = \frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{30}$$

$$I_{(1)\tau_{p+1},\tau_p}^{(i_1)} = -\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{31}$$

$$I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{32}$$

$$I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{33}$$

$$I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)q} = -\frac{\Delta}{2} I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)q} - \frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)} + \right.$$

$$\left. + \sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{34}$$

$$I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)q} = -\frac{\Delta}{2} I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)q} - \frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} + \right.$$

$$\left. + \sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{35}$$

$$I_{(2)\tau_{p+1},\tau_p}^{(i_1)} = \frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{36}$$

$$I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{001}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \right.$$

$$\left. -\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{37}$$

$$I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{010}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \right.$$

$$\left. -\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{38}$$

$$I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{100}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \right.$$

$$\left. -\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{39}$$

$$I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)q} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5 j_4 j_3 j_2 j_1}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} - \right.$$

$$-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}+$$

$$\left.+\mathbf{1}_{\{j_2=j_5\neq 0\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\neq 0\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{40}$$

where

$$C_{j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar{C}_{j_3j_2j_1}, \tag{41}$$

$$C_{j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdu=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}\Delta^{2}\bar{C}_{j_4j_3j_2j_1}, \tag{42}$$

$$C_{j_3j_2j_1}^{001}=\int\limits_{\tau_p}^{\tau_{p+1}}(\tau_p-z)\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{001}, \tag{43}$$

$$C_{j_3j_2j_1}^{010}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}(\tau_p-y)\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{010}, \tag{44}$$

$$C_{j_3j_2j_1}^{100}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}(\tau_p-x)\phi_{j_1}(x)dxdydz=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{100}, \tag{45}$$

$$C_{j_5j_4j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_5}(v)\int\limits_{\tau_p}^{v}\phi_{j_4}(u)\int\limits_{\tau_p}^{u}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydzdudv=$$

$$=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}\Delta^{5/2}\bar{C}_{j_5j_4j_3j_2j_1}, \tag{46}$$

where

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{47}$$

$$\bar{C}_{j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{48}$$

$$\bar{C}_{j_3j_2j_1}^{100}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydz, \tag{49}$$

$$\bar{C}_{j_3j_2j_1}^{010}=-\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{50}$$

$$\bar{C}_{j_3j_2j_1}^{001}=-\int\limits_{-1}^{1}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz, \tag{51}$$

$$\bar{C}_{j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv, \tag{52}$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial and

$$\phi_i(x)=\sqrt{\frac{2i+1}{\Delta}}P_i\left(\left(x-\tau_p-\frac{\Delta}{2}\right)\frac{2}{\Delta}\right),\quad i=0,1,2,\ldots$$

Let us consider the exact relations and some estimates for the mean-square errors of approximations of iterated Ito stochastic integrals.

Using Theorem 3, we obtain [13]-[25], [42], [49]

(53) $$\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}=$$

(54) $$=\frac{\Delta^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_1)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_1)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_1)q}\right)^2\right\}=$$

(55) $$=\frac{\Delta^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right).$$

Applying (24) and (25)–(28), we get

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2\quad (i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_2j_3j_1}C_{j_3j_2j_1}\quad (i_1\ne i_2=i_3),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}C_{j_1j_2j_3}\quad (i_1=i_3\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad (i_1=i_2\ne i_3).$$

At the same time using the estimate (22) for $i_1,\ldots,i_5=1,\ldots,m$, we have

$$\mathsf{M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}-I_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}-I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),$$

(56)
$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^{2}\right),$$

(57)
$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^{2}\right),$$

(58)
$$\mathsf{M}\left\{\left(I_{(100)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(100)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),$$

(59)
$$\mathsf{M}\left\{\left(I_{(010)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(010)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),$$

(60)
$$\mathsf{M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-I_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),$$

(61)
$$\mathsf{M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5)q}\right)^2\right\}\le 120\left(\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^{2}\right).$$

The Fourier–Legendre coefficients

$$\bar{C}_{j_3j_2j_1},\quad \bar{C}_{j_4j_3j_2j_1},\quad \bar{C}_{j_3j_2j_1}^{001},\quad \bar{C}_{j_3j_2j_1}^{010},\quad \bar{C}_{j_3j_2j_1}^{100},\quad \bar{C}_{j_5j_4j_3j_2j_1}$$

can be calculated exactly using computer algebra systems like Derive. The exact values of these Fourier–Legendre coefficients were presented in tabular form in the monographs [11]-[25]. Note that the mendioned Fourier–Legendre coefficients do not depend on the integration step $\tau_{p+1}-\tau_p$ of the numerical method, which can be variable.

Recently, the database with 270,000 exactly calculated Fourier–Legendre coefficients was described [52], [56]. This database was used in the software package, which is written in the Python programming language for the implementation of explicit one-step strong numerical schemes with orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Ito stochastic differential equations [52], [56]. The optimization of the mean-square approximation procedures for iterated Ito stochastic integrals from these numerical schemes can be found in [54].

Note that in [2]-[4] (also see [58]) the iterated stochastic integrals were approximated using the trigonometric Fourier expansion of the multidimensional Brownian bridge process. It is important to pay attention that the number q must be the same for all approximations of iterated stochastic integrals from the considered collection in the approcah from [2]-[4]. At the same time the numbers q can be selected individually for different stochastic integrals from the considred collection in the method based on Theorems 1–3 (see Sect. 5.3, 6.2 from [16] for details).

On the basis of the presented expansions (29)–(40) of iterated Ito stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to increasing of smallness orders with respect to Δ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities (the numbers q) in expansions of iterated Ito stochastic integrals, which are required for achieving the acceptable accuracy of approximation. Generally speaking, the minimum values q that guarantee the fulfillment of the condition (6) for each approximation (see (29)–(40)) are different and abruptly decreasing with the growth of smallness order (with respect to Δ) of the approximations of iterated stochastic integrals.

The detailed comparison of the method from [2]-[4] with the method based on Theorems 1–3 can be found in [16]-[18] (Chapters 2, 5, 6), [30], [38].

5. Approximation of Iterated Ito and Stratonovich Stochastic Integrals. Combined Approach

In contrast to the iterated Ito stochastic integrals (2), the iterated Stratonovich stochastic integrals (3) have simpler expansions (see Theorems 4–10 below) than (20). However, the calculation (or estimation) of the mean-square approximation error for the latter is a more difficult problem than for the former. Below in this section, we will study this issue in detail.

As it turned out, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3) at least for multiplicities 1 to 6. Expansions of these iterated Stratonovich stochastic integrals turned out much simpler than the appropriate expansions of the iterated Ito stochastic integrals (2) from Theorems 1, 2. Applying this feature and standard relations between iterated Ito and Stratonovich stochastic integrals, we will get simpler expansions for the iterated Ito stochastic integrals (2) than the expansions from the previous section. However, as was mentioned above, the estimation of the mean-square approximation error for the expansions from this section is a nontrivial problem.

Let us first present some old results on expansion of the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 4 (Theorems 4–6 below).

Theorem 4 [12]-[18], [23]-[25], [31], [34], [35], [39], [44], [46], [48]. *Assume that the following conditions are fulfilled*:

1. *The function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the function $\psi_1(\tau)$ is twice continuously differentiable at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$.*

Then, the iterated Stratonovich stochastic integral of the second multiplicity

$$\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

is expanded into the double series

$$\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

converging in the mean-square sense, where

$$C_{j_2j_1}=\int\limits_t^T\psi_2(s_2)\phi_{j_2}(s_2)\int\limits_t^{s_2}\psi_1(s_1)\phi_{j_1}(s_1)ds_1ds_2;$$

another notations are the same as in Theorems 1, 2.

Theorem 5 [12]-[18], [23]-[25], [39], [44], [45]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space* $L_2([t,T])$*. Furthermore, the function* $\psi_2(\tau)$ *is continuously differentiable at the interval* $[t,T]$ *and the functions* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable at the interval* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad(i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\tag{62}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 1, 2.

Theorem 6 [12]-[18], [23]-[25], [39], [44], [51]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of multiplicity* 4

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}$$

the following expansion

$$I_{T,t}^{*(i_1i_2i_3i_4)}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

converging in the mean-square sense is valid, where $i_1,i_2,i_3,i_4=0,1,\ldots,m,$

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4,$$

$\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ *are independent standard Wiener processes and* $\mathbf{w}_\tau^{(0)} = \tau$.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [16] (Sect. 2.10–2.16), [43] (Sect. 7–13), [44] (Sect. 13–19), [51] (Sect. 5–11), [55] (Sect. 4–9), [57]. Let us formulate four theorems that were obtained using this approach.

Theorem 7 [16], [43], [44], [51], [55]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{63}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{64}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (63) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (64), *constant* C *is independent of* p,

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j (*in the case when* $i \neq 0$); *another notations are the same as in Theorems* 1, 2.

Theorem 8 [16], [43], [44], [51], [55]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$. *Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t, T]$. *Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \tag{65}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{66}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{67}$$

are fulfilled, where $i_1,\ldots,i_4 = 0,1,\ldots,m$ *in* (65), (66) *and* $i_1,\ldots,i_4 = 1,\ldots,m$ *in* (67), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 7.

Theorem 9 [16], [43], [44], [51], [55]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5)\ldots\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{68}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{69}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{70}$$

are fulfilled, where $i_1,\ldots,i_5 = 0,1,\ldots,m$ *in* (68), (69) *and* $i_1,\ldots,i_5 = 1,\ldots,m$ *in* (70), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 7, 8.

Theorem 10 [16], [43], [44], [51], [57]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{71}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 7–9.

Let us denote

$$I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{72}$$

where $i_1,\ldots,i_k=1,\ldots,m,\ \ l_1,\ldots,l_k=0,\ 1,\ldots.$

Below we will consider the iterated Stratonovich stochastic integrals (72) as well as the iterated Ito stochastic integrals $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$ defined by (5).

According to the standard relations between iterated Ito and Stratonovich stochastic integrals as well as according to Theorems 5, 7, we obtain

$$I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}=I_{(000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}I_{(1)\tau_{p+1},\tau_p}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\left(\Delta I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)\quad \text{w. p. 1}, \tag{73}$$

where

$$I^{*(i_1 i_2 i_3)}_{(000)\tau_{p+1},\tau_p} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{74}$$

where $C_{j_3j_2j_1}$ is defined by (41), (47).

From (73), (74) and (29), (31) we obtain the following approximation

$$I^{(i_1 i_2 i_3)q}_{(000)\tau_{p+1},\tau_p} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3} - \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}\Delta^{3/2}\left(\zeta^{(i_3)}_0 + \frac{1}{\sqrt{3}}\zeta^{(i_3)}_1\right) -$$

$$- \frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}\Delta^{3/2}\left(\zeta^{(i_1)}_0 - \frac{1}{\sqrt{3}}\zeta^{(i_1)}_1\right). \tag{75}$$

For the case $i_1=i_2=i_3$ it is comfortable to use the folowing well known relation

$$I^{(i_1 i_1 i_1)}_{(000)\tau_{p+1},\tau_p} = \frac{1}{6}\Delta^{3/2}\left(\left(\zeta^{(i_1)}_0\right)^3 - 3\zeta^{(i_1)}_0\right) \quad \text{w. p. 1.} \tag{76}$$

Let us consider the iterated Ito stochastic integrals

$$I^{(i_3 i_2 i_1)}_{(100)\tau_{p+1},\tau_p}, \quad I^{(i_3 i_2 i_1)}_{(010)\tau_{p+1},\tau_p}, \quad I^{(i_3 i_2 i_1)}_{(001)\tau_{p+1},\tau_p}.$$

According to the standard relations between iterated Ito and Stratonovich stochastic integrals as well as according to Theorems 5, 7, we obtain

$$I^{(i_1 i_2 i_3)}_{(001)\tau_{p+1},\tau_p} = I^{*(i_1 i_2 i_3)}_{(001)\tau_{p+1},\tau_p} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}I^{(i_3)}_{(2)\tau_{p+1},\tau_p} +$$

$$+ \frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}\left(\Delta^2 I^{(i_1)}_{(0)\tau_{p+1},\tau_p} - I^{(i_1)}_{(2)\tau_{p+1},\tau_p}\right) \quad \text{w. p. 1,} \tag{77}$$

$$I^{(i_1 i_2 i_3)}_{(010)\tau_{p+1},\tau_p} = I^{*(i_1 i_2 i_3)}_{(010)\tau_{p+1},\tau_p} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}I^{(i_3)}_{(2)\tau_{p+1},\tau_p} +$$

$$+ \frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}\left(\Delta^2 I^{(i_1}_{(0)\tau_{p+1},\tau_p} - I^{(i_1)}_{(2)\tau_{p+1},\tau_p}\right) \quad \text{w. p. 1,} \tag{78}$$

$$I^{(i_1 i_2 i_3)}_{(100)\tau_{p+1},\tau_p} = I^{*(i_1 i_2 i_3)}_{(100)\tau_{p+1},\tau_p} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}I^{(i_3)}_{(2)\tau_{p+1},\tau_p} -$$

$$- \frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I^{(i_1)}_{(2)\tau_{p+1},\tau_p} + \Delta I^{(i_1)}_{(1)\tau_{p+1},\tau_p}\right) \quad \text{w. p. 1,} \tag{79}$$

where

$$I^{*(i_1 i_2 i_3)}_{(001)\tau_{p+1},\tau_p} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{q} C^{001}_{j_3 j_2 j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3},$$

$$I^{*(i_1 i_2 i_3)}_{(010)\tau_{p+1},\tau_p} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{q} C^{010}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3},$$

$$I^{*(i_1 i_2 i_3)}_{(100)\tau_{p+1},\tau_p} = \underset{q\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{q} C^{100}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3},$$

where $C^{001}_{j_3 j_2 j_1}$, $C^{010}_{j_3 j_2 j_1}$, $C^{100}_{j_3 j_2 j_1}$ are defined by (43)-(45) and (49)-(51). From (77)–(79) and (29), (31), (36) we obtain the following approximations

$$I^{(i_1 i_2 i_3)q}_{(001)\tau_{p+1},\tau_p} = \sum_{j_1,j_2,j_3=0}^{q} C^{001}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} + \frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}\Delta^{5/2}\left(\zeta^{(i_3)}_0 + \frac{\sqrt{3}}{2}\zeta^{(i_3)}_1 + \frac{1}{2\sqrt{5}}\zeta^{(i_3)}_2\right) +$$

$$+\frac{1}{12}\mathbf{1}_{\{i_2=i_3\}}\Delta^{5/2}\left(2\zeta^{(i_1)}_0 - \frac{\sqrt{3}}{2}\zeta^{(i_1)}_1 - \frac{1}{2\sqrt{5}}\zeta^{(i_1)}_2\right), \tag{80}$$

$$I^{(i_1 i_2 i_3)q}_{(010)\tau_{p+1},\tau_p} = \sum_{j_1,j_2,j_3=0}^{q} C^{010}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} + \frac{1}{12}\mathbf{1}_{\{i_1=i_2\}}\Delta^{5/2}\left(\zeta^{(i_3)}_0 + \frac{\sqrt{3}}{2}\zeta^{(i_3)}_1 + \frac{1}{2\sqrt{5}}\zeta^{(i_3)}_2\right) +$$

$$+\frac{1}{12}\mathbf{1}_{\{i_2=i_3\}}\Delta^{5/2}\left(2\zeta^{(i_1)}_0 - \frac{\sqrt{3}}{2}\zeta^{(i_1)}_1 - \frac{1}{2\sqrt{5}}\zeta^{(i_1)}_2\right), \tag{81}$$

$$I^{(i_1 i_2 i_3)q}_{(100)\tau_{p+1},\tau_p} = \sum_{j_1,j_2,j_3=0}^{q} C^{100}_{j_3 j_2 j_1} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} + \frac{1}{12}\mathbf{1}_{\{i_1=i_2\}}\Delta^{5/2}\left(\zeta^{(i_3)}_0 + \frac{\sqrt{3}}{2}\zeta^{(i_3)}_1 + \frac{1}{2\sqrt{5}}\zeta^{(i_3)}_2\right) +$$

$$+\frac{1}{12}\mathbf{1}_{\{i_2=i_3\}}\Delta^{5/2}\left(\zeta^{(i_1)}_0 - \frac{1}{\sqrt{5}}\zeta^{(i_1)}_2\right). \tag{82}$$

Let us consider the iterated Ito stochastic integral of multiplicity 4. According to the standard relations between iterated Ito and Stratonovich stochastic integrals as well as according to Theorems 6, 8, we get

$$I^{(i_1 i_2 i_3 i_4)}_{(0000)\tau_{p+1},\tau_p} = I^{*(i_1 i_2 i_3 i_4)}_{(0000)\tau_{p+1},\tau_p} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} I^{(i_3 i_4)}_{(10)\tau_{p+1},\tau_p} -$$

$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I^{(i_1 i_4)}_{(10)\tau_{p+1},\tau_p} - I^{(i_1 i_4)}_{(01)\tau_{p+1},\tau_p}\right) - \frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left(\Delta I^{(i_1 i_2)}_{(00)\tau_{p+1},\tau_p} + I^{(i_1 i_2)}_{(01)\tau_{p+1},\tau_p}\right) -$$

$$-\frac{1}{8}\Delta^2 \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}} \quad \text{w. p. 1}, \tag{83}$$

$$I^{*(i_1i_2i_3i_4)}_{(0000)\tau_{p+1},\tau_p}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I^{(i_1i_2i_3i_4)q}_{(0000)\tau_{p+1},\tau_p}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}I^{(i_3i_4)q}_{(10)\tau_{p+1},\tau_p}-$$

$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I^{(i_1i_4)q}_{(10)\tau_{p+1},\tau_p}-I^{(i_1i_4)q}_{(01)\tau_{p+1},\tau_p}\right)-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left(\Delta I^{(i_1i_2)q}_{(00)\tau_{p+1},\tau_p}+I^{(i_1i_2)q}_{(01)\tau_{p+1},\tau_p}\right)-$$

$$-\frac{1}{8}\Delta^2\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}},$$

where

$$I^{(i_1i_2)q}_{(00)\tau_{p+1},\tau_p},\quad I^{(i_1i_2)q}_{(01)\tau_{p+1},\tau_p},\quad I^{(i_1i_2)q}_{(10)\tau_{p+1},\tau_p}$$

are determined by the relations (30), (34), (35) and $C_{j_4j_3j_2j_1}$ is defined by (42), (48).

For the case $i_1=i_2=i_3=i_4$ it is comfortable to use the folowing well known relation

$$I^{(i_1i_1i_1i_1)}_{(0000)\tau_{p+1},\tau_p}=\frac{1}{24}\Delta^2\left(\left(\zeta_0^{(i_1)}\right)^4-6\left(\zeta_0^{(i_1)}\right)^2+3\right)\quad \text{w. p. 1.}$$

Let us consider the iterated Ito stochastic integral of fifth multiplicity using Theorems 6, 9

$$I^{(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}=I^{*(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}I^{(i_3i_4i_5)}_{(100)\tau_{p+1},\tau_p}-$$

$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I^{(i_1i_4i_5)}_{(100)\tau_{p+1},\tau_p}-I^{(i_1i_4i_5)}_{(010)\tau_{p+1},\tau_p}\right)-\frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left(I^{(i_1i_2i_5)}_{(010)\tau_{p+1},\tau_p}-I^{(i_1i_2i_5)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$-\frac{1}{2}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta I^{(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}+I^{(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}\right)-\frac{1}{8}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}}I^{(i_5)}_{(2)\tau_{p+1},\tau_p}-$$

$$-\frac{1}{8}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta^2 I^{(i_1)}_{(0)\tau_{p+1},\tau_p}+2\Delta I^{(i_1)}_{(1)\tau_{p+1},\tau_p}+I^{(i_1)}_{(2)\tau_{p+1},\tau_p}\right)+$$

(84)
$$-\frac{1}{8}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta I^{(i_3)}_{(1)\tau_{p+1},\tau_p}+I^{(i_3)}_{(2)\tau_{p+1},\tau_p}\right)\quad \text{w. p. 1,}$$

$$I^{*(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}=\underset{q\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)q} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5 j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} + \frac{1}{2}\mathbf{1}_{\{i_1=i_2\}} I_{(100)\tau_{p+1},\tau_p}^{(i_3 i_4 i_5)q} -$$

$$-\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\left(I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_4 i_5)q} - I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_4 i_5)q}\right) - \frac{1}{2}\mathbf{1}_{\{i_3=i_4\}}\left(I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_5)q} - I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_5)q}\right) -$$

$$-\frac{1}{2}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q} + I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}\right) - \frac{1}{8}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_3=i_4\}} I_{(2)\tau_{p+1},\tau_p}^{(i_5)} -$$

$$-\frac{1}{8}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta^2 I_{(0)\tau_{p+1},\tau_p}^{(i_1)} + 2\Delta I_{(1)\tau_{p+1},\tau_p}^{(i_1)} + I_{(2)\tau_{p+1},\tau_p}^{(i_1)}\right) +$$

$$-\frac{1}{8}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i_4=i_5\}}\left(\Delta I_{(1)\tau_{p+1},\tau_p}^{(i_3)} + I_{(2)\tau_{p+1},\tau_p}^{(i_3)}\right),$$

where

$$I_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}, \quad I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}, \quad I_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}, \quad I_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)q}, \quad I_{(0)\tau_{p+1},\tau_p}^{(i_1)}, \quad I_{(1)\tau_{p+1},\tau_p}^{(i_1)}, \quad I_{(2)\tau_{p+1},\tau_p}^{(i_1)}$$

are determined by (75), (80)–(82), (29), (31), (36) and $C_{j_5 j_4 j_3 j_2 j_1}$ is defined by (46), (52).

For the case $i_1 = \ldots = i_5$ it is comfortable to use the folowing well known relation

$$I_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_1 i_1 i_1 i_1)} = \frac{1}{120}\Delta^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5 - 10\left(\zeta_0^{(i_1)}\right)^3\Delta + 15\zeta_0^{(i_1)}\Delta^2\right) \quad \text{w. p. 1.}$$

Clearly, the expansions from Theorems 4–10 are simpler than the expansions from Theorems 1, 2. However, the calculation of the mean-square approximation error for the expansions from Theorems 4–10 turns out to be much more difficult than for the expansions from Theorems 1, 2. We will demonstrate this fact below.

The case $k = 1$ is actually not interesting. For $k = 1$, the Ito and Stratonovich stochastic integrals of a smooth non-random function are equal each other w. p. 1. Moreover, for $k = 2$

$$I_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} = I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \quad (i_1 \neq i_2) \quad \text{w. p. 1.}$$

Consider the triple Stratonovich stochastic integral defined by

$$I_{(000)T,t}^{*(i_1 i_2 i_3)} = \int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m).$$

In view of the standard relations between Ito and Stratonovich stochastic integrals and also Theorems 1, 2, 5, 7, we obtain

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} + \mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\Biggl\{\Biggl(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + I_{(000)T,t}^{(i_1 i_2 i_3)q} +$$

$$+ \mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} + \mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\Biggr)^2\Biggr\}, \qquad (85)$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (86)$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \qquad (87)$$

where $I_{(000)T,t}^{(i_1 i_2 i_3)q}$ is the approximation defined by the formula (21) (also see (14)) for the case $k = 3$ and $I_{(000)T,t}^{*(i_1 i_2 i_3)q}$ is the approximation based on Theorems 5, 7 (see (98) below).

Substituting (86) and (87) into (85) yields

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1 i_2 i_3)} - I_{(000)T,t}^{*(i_1 i_2 i_3)q}\right)^2\right\} =$$

$$= \mathsf{M}\Biggl\{\Biggl(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q} + \mathbf{1}_{\{i_1=i_2\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)} - \sum_{j_1,j_3=0}^{q} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right) +$$

$$+\mathbf{1}_{\{i_2=i_3\}}\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_{s}^{(i_1)} d\tau - \sum_{j_1,j_3=0}^{q} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)}\right) - \mathbf{1}_{\{i_1=i_3\}}\sum_{j_1,j_2=0}^{q} C_{j_1 j_2 j_1}\zeta_{j_2}^{(i_2)}\Biggr)^2\Biggr\} \le$$

$$\le 4\Biggl(\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)q}\right)^2\right\} + \mathbf{1}_{\{i_1=i_2\}}F_q^{(i_3)} +$$

$$+\mathbf{1}_{\{i_2=i_3\}}G_q^{(i_1)} + \mathbf{1}_{\{i_1=i_3\}}H_q^{(i_2)}\Biggr), \qquad (88)$$

where

$$F_q^{(i_3)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}-\sum_{j_1,j_3=0}^{q}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\},$$

$$G_q^{(i_1)}=\mathsf{M}\left\{\left(\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_{s}^{(i_1)}d\tau-\sum_{j_1,j_3=0}^{q}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\},$$

$$H_q^{(i_2)}=\mathsf{M}\left\{\left(\sum_{j_1,j_2=0}^{q}C_{j_1j_2j_1}\zeta_{j_2}^{(i_2)}\right)^2\right\}.$$

For the cases of Legendre polynomials and trigonometric functions, we have the equalities [12]-[18], [23]-[25], [39], [44], [45]

$$\lim_{q\to\infty}F_q^{(i_3)}=0,\quad \lim_{q\to\infty}G_q^{(i_1)}=0,\quad \lim_{q\to\infty}H_q^{(i_2)}=0.$$

However, in accordance with (88) the value

$$\mathsf{M}\left\{\left(I_{(000)T,t}^{*(i_1i_2i_3)}-I_{(000)T,t}^{*(i_1i_2i_3)q}\right)^2\right\}$$

with a finite q can be estimated by the sum of

$$4\mathsf{M}\left\{\left(I_{(000)T,t}^{(i_1i_2i_3)}-I_{(000)T,t}^{(i_1i_2i_3)q}\right)^2\right\}, \tag{89}$$

and three additional terms of a rather complex structure. The value (89) can be calculated exactly using Theorem 3 or estimated using (22) for the case $k=3$.

As is easily observed, this peculiarity will also apply to the iterated Stratonovich stochastic integrals of multiplicities 4 and 5 with the only difference that the number of additional terms like $F_q^{(i_3)}$, $G_q^{(i_1)}$, and $H_q^{(i_2)}$ will be considerably higher and their structure will be more complicated. Therefore, the payment for relatively simple expansions of the iterated Stratonovich stochastic integrals (Theorems 4–10) in comparison with the iterated Ito stochastic integrals (Theorems 1, 2) is a much more difficult calculation or estimation procedure of their mean-square approximation errors.

6. Explicit One-Step Strong Numerical Schemes of Orders 2.0 and 2.5 Based on the Unified Taylor–Stratonovich expansion

Consider the explicit one-step strong numerical scheme of order 2.5 based on the so-called unified Taylor–Stratonovich expansion [11], [14]-[22]

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_{i_2}B_{i_1}\hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_2i_1)}+$$

$$
+\sum_{i_1=1}^{m}\Biggl(G_{i_1}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)-\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\Biggr)+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{(000)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+
$$

$$
+\sum_{i_1,i_2=1}^{m}\Biggl(G_{i_2}\bar{L}B_{i_1}\left(\hat{I}^{*(i_2i_1)}_{(10)\tau_{p+1},\tau_p}-\hat{I}^{*(i_2i_1)}_{(01)\tau_{p+1},\tau_p}\right)-\bar{L}G_{i_2}B_{i_1}\hat{I}^{*(i_2i_1)}_{(10)\tau_{p+1},\tau_p}+
$$

$$
+G_{i_2}G_{i_1}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{(01)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_2i_1)}_{(00)\tau_{p+1},\tau_p}\right)\Biggr)+
$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{(0000)\tau_{p+1},\tau_p}+\frac{\Delta^3}{6}LL\mathbf{a}+
$$

$$
+\sum_{i_1=1}^{m}\Biggl(G_{i_1}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}\right)+
$$

$$
+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}-LG_{i_1}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)\Biggr)+
$$

$$
+\sum_{i_1,i_2,i_3=1}^{m}\Biggl(G_{i_3}\bar{L}G_{i_2}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{(100)\tau_{p+1},\tau_p}-\hat{I}^{*(i_3i_2i_1)}_{(010)\tau_{p+1},\tau_p}\right)+
$$

$$
+G_{i_3}G_{i_2}\bar{L}B_{i_1}\left(\hat{I}^{*(i_3i_2i_1)}_{(010)\tau_{p+1},\tau_p}-\hat{I}^{*(i_3i_2i_1)}_{(001)\tau_{p+1},\tau_p}\right)+
$$

$$
+G_{i_3}G_{i_2}G_{i_1}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_3i_2i_1)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{*(i_3i_2i_1)}_{(001)\tau_{p+1},\tau_p}\right)-
$$

$$
-\bar{L}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_3i_2i_1)}_{(100)\tau_{p+1},\tau_p}\Biggr)+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{(00000)\tau_{p+1},\tau_p}, \tag{90}
$$

where $\Delta=T/N$ $(N>1)$ is a constant (for simplicity) step of integration, $\tau_p=p\Delta$ $(p=0,1,\ldots,N)$, $\hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t}$ is an approximation of the iterated Stratonovich stochastic integral (72),

$$
\bar{\mathbf{a}}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)-\frac{1}{2}\sum_{j=1}^{m}G_jB_j(\mathbf{x},t),
$$

$$
\bar{L}=L-\frac{1}{2}\sum_{j=1}^{m}G_0^{(j)}G_0^{(j)}=\frac{\partial}{\partial t}+\sum_{j=1}^{n}\bar{\mathbf{a}}^{(j)}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}^{(j)}},
$$

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B_{lj}(\mathbf{x},t)B_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}_l\partial \mathbf{x}_i},$$

$$G_i=\sum_{j=1}^{n}B_{ji}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}_j},\quad i=1,\ldots,m,$$

$l_1,\ldots,l_k=0,1,2,\ i_1,\ldots,i_k=1,\ldots,m,\ k=1,2,\ldots,5,$ B_i and B_{ij} are the ith column and the ijth component of the matrix function B, $\mathbf{a}_i$ is the ith component of the vector function $\mathbf{a}$, $\mathbf{x}_i$ is the ith component of the column $\mathbf{x}$, the functions

$$B_{i_1},\ \bar{\mathbf{a}},\ G_{i_2}B_{i_1},\ G_{i_1}\bar{\mathbf{a}},\ \bar{L}B_{i_1},\ G_{i_3}G_{i_2}B_{i_1},\ \bar{L}\bar{\mathbf{a}},\ LL\mathbf{a},\ G_{i_2}\bar{L}B_{i_1},$$

$$\bar{L}G_{i_2}B_{i_1},\ G_{i_2}G_{i_1}\bar{\mathbf{a}},\ G_{i_4}G_{i_3}G_{i_2}B_{i_1},\ G_{i_1}\bar{L}\bar{\mathbf{a}},\ \bar{L}\bar{L}B_{i_1},\ \bar{L}G_{i_1}\bar{\mathbf{a}},\ G_{i_3}\bar{L}G_{i_2}B_{i_1},\ G_{i_3}G_{i_2}\bar{L}B_{i_1},$$

$$G_{i_3}G_{i_2}G_{i_1}\bar{\mathbf{a}},\ \bar{L}G_{i_3}G_{i_2}B_{i_1},\ G_{i_5}G_{i_4}G_{i_3}G_{i_2}B_{i_1}$$

are calculated at the point $(\mathbf{y}_p,p)$.

Under the standard conditions [2], [11] the numerical scheme (90) has strong order 2.5 of convergence. The major emphasis below will be placed on the approximation of the iterated Stratonovich stochastic integrals appearing in (90). Therefore, among the mentioned standard conditions, we note the approximation condition for these stochastic integrals [2], [11], which has the form

$$\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\right)^2\right\}\le C\Delta^6, \tag{91}$$

where constant C is independent of Δ.

Note that if we exclude from (90) the terms starting from the term $\Delta^3 LL\mathbf{a}/6$, then we will have the explicit one-step strong numerical scheme of order 2.0 [2], [11], [14]-[22].

Using the numerical scheme (90) or its modifications based on the Taylor–Stratonovich expansion [7], the implicit or multistep analogues of (90) can be constructed [2], [11], [14]-[22]. The set of the iterated Stratonovich stochastic integrals to be approximated for implementing these modifications is the same as for the numerical scheme (90) itself. Interestingly, the truncated unified Taylor–Stratonovich expansion (the foundation of the numerical scheme (90)) contains 12 different types of the iterated Stratonovich stochastic integrals (72), which cannot be interconnected by linear relations [11], [14]-[22]. The analogous Taylor–Stratonovich expansion [2], [7] contains 17 different types of iterated Stratonovich stochastic integrals, part of which are interconnected by linear relations and part of which have a higher multiplicity than the iterated Stratonovich stochastic integrals (72). This fact well explains the use of the numerical scheme (90).

One of the main problems arising in the implementation of the numerical scheme (90) is the joint numerical modeling of the iterated Stratonovich stochastic integrals figuring in (90). Let us consider an efficient numerical modeling method for the iterated Stratonovich stochastic integrals based on Theorems 4–10.

Using Theorems 4–9 and multiple Fourier–Legendre series, we obtain the following approximations of the iterated Stratonovich stochastic integrals from (90) [11]-[51]

$$I^{*(i_1)}_{(0)\tau_{p+1},\tau_p}=\sqrt{\Delta}\zeta_0^{(i_1)}, \tag{92}$$

$$I_{(1)\tau_{p+1},\tau_p}^{*(i_1)} = -\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{93}$$

$$I_{(2)\tau_{p+1},\tau_p}^{*(i_1)} = \frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{94}$$

$$I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)q} = \frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{95}$$

$$I_{(01)\tau_{p+1},\tau_p}^{*(i_1 i_2)q} = -\frac{\Delta}{2}I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)q} - \frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)} + \right.$$

$$\left. + \sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{96}$$

$$I_{(10)\tau_{p+1},\tau_p}^{*(i_1 i_2)q} = -\frac{\Delta}{2}I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)q} - \frac{\Delta^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} + \right.$$

$$\left. + \sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right), \tag{97}$$

$$I_{(000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{98}$$

$$I_{(100)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{99}$$

$$I_{(010)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{100}$$

$$I_{(001)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{101}$$

$$I_{(0000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{102}$$

$$I_{(00000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4i_5)q} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}, \qquad (103)$$

where the Fourier–Legendre coefficients

$$C_{j_3j_2j_1},\quad C_{j_3j_2j_1}^{100},\quad C_{j_3j_2j_1}^{010},\quad C_{j_3j_2j_1}^{001},\quad C_{j_4j_3j_2j_1},\quad C_{j_5j_4j_3j_2j_1}$$

are determined by (41)–(46), (47)–(52).

On the basis of the presented expansions (see (92)–(103)) of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to increasing of smallness orders with respect to Δ in the mean-square sense for iterated stochastic integrals. This leads to a sharp decrease of member quantities (the numbers q) in expansions of iterated Stratonovich stochastic integrals, which are required for achieving the acceptable accuracy of approximation. Generally speaking, the minimum values q that guarantee the fulfillment of the condition (91) for each approximation (92)–(103) are different and abruptly decreasing with the growth of smallness order (with respect to Δ) of the approximations of iterated stochastic integrals.

From Theorem 3 for the case $i_1 \ne i_2$ we obtain

$$\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)} - I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\} = \frac{\Delta^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1} \le$$

$$\le \frac{\Delta^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx = -\frac{\Delta^2}{8}\ln\left|1-\frac{2}{2q+1}\right| \le C_1\frac{\Delta^2}{q}, \qquad (104)$$

where constant C_1 does not depend on Δ.

As was mentioned above, the value Δ plays the role of integration step in the numerical procedures for Ito stochastic differential equations. Thus this value is a sufficiently small. Keeping in mind this circumstance, it is easy to notice that there exists such a constant C_2 that

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)} - I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\} \le C_2\mathsf{M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)} - I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\}, \qquad (105)$$

where $I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}$ is an approximation of the iterated Stratonovich stochastic integral (72).

From (104) and (105) we finally have

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)} - I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\} \le K\frac{\Delta^2}{q}, \qquad (106)$$

where constant K does not depend on Δ.

The same idea can be found in [2] for the case of trigonometric functions. Note that, in contrast to the estimate (106), the constant C in Theorems 7–9 does not depend on q.

We can get significantly more information about numbers q using a different approach. Applying the standard relation between iterated Ito and Stratonovich stochastic integrals, we have

$$
I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}=I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\quad \text{w. p. 1}
$$

for pairwise different $i_1,\ldots,i_k=1,\ldots,m$.

Then for $i_1\ne i_2$ the following mean-square errors

$$
{\sf M}\left\{\left(I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)}-I_{(00)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\},\quad {\sf M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{*(i_1i_2)}-I_{(10)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\},
$$

$$
{\sf M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)}-I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\}
$$

are defined by (53), (54).

Moreover, for pairwise different $i_1,\ldots,i_5=1,\ldots,m$ from (24) we obtain

$$
{\sf M}\left\{\left(I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)}-I_{(01)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\}=\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2,
$$

$$
{\sf M}\left\{\left(I_{(10)\tau_{p+1},\tau_p}^{*(i_1i_2)}-I_{(10)\tau_{p+1},\tau_p}^{*(i_1i_2)q}\right)^2\right\}=\frac{\Delta^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2,
$$

$$
{\sf M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2,
$$

$$
{\sf M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4)q}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2,
$$

$$
{\sf M}\left\{\left(I_{(100)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(100)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2,
$$

$$
{\sf M}\left\{\left(I_{(010)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(010)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2,
$$

$$
{\sf M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(001)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)q}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2,
$$

$$
{\sf M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4i_5)q}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^2.
$$

For example [11]-[25],

$$\mathsf{M}\left\{\left(I_{(000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)6}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{6}C_{j_3j_2j_1}^2\approx 0.01956000\Delta^3,$$

$$\mathsf{M}\left\{\left(I_{(100)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(100)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)2}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{100}\right)^2\approx 0.00815429\Delta^5,$$

$$\mathsf{M}\left\{\left(I_{(010)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(010)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)2}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{010}\right)^2\approx 0.01739030\Delta^5,$$

$$\mathsf{M}\left\{\left(I_{(001)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)}-I_{(001)\tau_{p+1},\tau_p}^{*(i_1i_2i_3)2}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C_{j_3j_2j_1}^{001}\right)^2\approx 0.02528010\Delta^5,$$

$$\mathsf{M}\left\{\left(I_{(0000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4)}-I_{(0000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4)2}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C_{j_4j_3j_2j_1}^2\approx 0.02360840\Delta^4,$$

$$\mathsf{M}\left\{\left(I_{(00000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4i_5)}-I_{(00000)\tau_{p+1},\tau_p}^{*(i_1i_2i_3i_4i_5)1}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C_{j_5i_4i_3i_2j_1}^2\approx 0.00759105\Delta^5.$$

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp. [In Russian]
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.
[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp. [In Russian]
[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.
[5] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Berlin, Springer, 1994, 292 pp.
[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.
[7] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor Expansions. Math. Nachr. 151 (1991), 33-50.
[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99: 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635
[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118: 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239
[10] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 90 pp. [In English].
[11] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[12] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[13] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions, Approximations, Formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2017, no. 1, 385 pp. (A.1–A.385). DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 923 pp. [In English].

[17] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/collection.html

[18] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: St.-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[23] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[24] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[25] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[26] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[27] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[28] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 28 pp. [in English].

[29] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[30] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[31] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[32] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724 [math.GM], 2019, 41 pp. [In English].

[33] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes. no. 3, 2019, P. 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[34] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at:
http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[35] Kuznetsov D.F. Problems of the Numerical Analysis of Ito Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at:
http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0045-5)

[36] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[37] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[38] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English]

[39] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals, based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[40] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [in English].

[41] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on the generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 68 pp. [in English].

[42] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [in English].

[43] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [in English].

[44] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 204 pp. [in English].

[45] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 65 pp. [in English].

[46] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. arXiv:1801.01962 [math.PR]. 2018, 49 pp. [in Russian].

[47] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 57 pp. [in English].
[48] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 77 pp. [In English].
[49] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR]. 2018, 44 pp. [In English].
[50] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 40 pp. [In English].
[51] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 129 pp. [In English].
[52] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 343 pp. [In English].
[53] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf
[54] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].
[55] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html
[56] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html
[57] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html
[58] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].
[59] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

STRONG NUMERICAL METHODS OF ORDERS 2.0, 2.5, AND 3.0 FOR ITO STOCHASTIC DIFFERENTIAL EQUATIONS BASED ON THE UNIFIED STOCHASTIC TAYLOR EXPANSIONS AND MULTIPLE FOURIER–LEGENDRE SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the construction of explicit one-step strong numerical methods with the orders of convergence 2.0, 2,5, and 3.0 for Ito stochastic differential equations with multidimensional non-commutative noise. We consider the numerical methods based on the unified Taylor–Ito and Taylor–Stratonovich expansions. For numerical modeling of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 we apply the method of multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$, $k=1,\ldots,6$. The article is addressed to engineers who use numerical modeling in stochastic control and for solving the non-linear filtering problem. The article can be interesting for mathematicians who working in the field of high-order strong numerical methods for Ito stochastic differential equations.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
Keywords: Explicit one-step strong numerical method, Unified Taylor–Ito expansion, Unified Taylor–Stratonovich expansion, Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Method of generalized multiple Fourier series, Multiple Fourier–Legendre series, Strong convergence, Mean-square convergence, Approximation, Expansion.

1. Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int\limits_0^t \Sigma(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \ \ \mathbf{x}_0 = \mathbf{x}(0, \omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $\Sigma : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as the Ito stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\{|\mathbf{x}_0|^2\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems of different physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[10]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\int \ \text{and} \ \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l = 1, \ldots, k)$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ in [2]-[4], [6], [7]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ $(l = 1, \ldots, k;\ q_1, \ldots, q_k = 0, 1, 2, \ldots)$ and $i_1, \ldots, i_k = 1, \ldots, m$ in [8]-[10].

Effective solution of the problem of combined mean-square approximation of the iterated Ito and Stratonovich stochastic integrals (2) and (3) of multiplicities 1 to 6 composes one of the subjects of this article.

We want to mention in short that there are two main criteria of numerical methods convergence for Ito SDEs [2]-[4]: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution.

Using strong numerical methods, we may build sample pathes of Ito SDEs numerically. These methods require combined mean-square approximation of the iterated Ito and Stratonovich stochastic integrals (2) and (3).

Also numerical integration of Ito SDEs based on the strong convergence criterion of approximation is widely used for the numerical solution of different mathematical problems connected with Ito SDEs. Among these problems, we note the following: signal filtering under influence of random noises in various statements, optimal stochastic control, testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis [2]-[4].

The problem of effective jointly numerical modeling (in accordance to the mean-square convergence criterion) of the iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[66].

The only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)$. This case allows the investigation with using of the Ito formula [2]-[4].

Note that even for the mentioned coincidence ($i_1 = \ldots = i_k \neq 0$), but for different functions $\psi_1(s), \ldots, \psi_k(s)$ the mentioned difficulties persist, and relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be represented effectively in a finite form (for the mean-square approximation) using the system of standard Gaussian random variables.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated stochastic integrals can be simplified but cannot be solved. The equations with additive scalar noise, with additive vector noise, with non-additive scalar noise, with a small parameter are related to such types of equations [2]-[4]. For the mentioned types of equations, simplifications are connected with the fact that either some coefficient functions from stochastic analogues of the Taylor formula identically equal to zero, or scalar noise has a strong effect, or due to the presence of a small parameter we may neglect some members from the stochastic analogues of the Taylor formula, which include difficult for approximation iterated stochastic integrals [2]-[4], [13]. In this article, we consider Ito SDEs with multidimensional, non-additive and non-commutative noise.

Seems that iterated stochastic integrals may be approximated by multiple integral sums of different types [3], [4], [14]. However, this approach implies partitioning of the interval of integration $[t, T]$ of iterated stochastic integrals (the length $T - t$ of this interval is a small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant calculating costs [5].

In [3] (also see [2], [4], [12], [13]), Milstein proposed to expand (2) or (3) into iterated series in terms of products of standard Gaussian random variables by representing the Wiener process as a trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion of the Brownian bridge process). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process $\mathbf{f}_s$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple stochastic integrals of the form (3) were presented in [2], [11]-[13] ($k = 1, 2, 3$) and in [3], [4] ($k = 1, 2$) for the simplest case $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$; $i_1, i_2, i_3 = 0, 1, \ldots, m$.

Moreover, in [2] (Sect. 5.8, pp. 202–204), [11] (pp. 82-84), [12] (pp. 438-439), [13] (pp. 263-264) the authors use (without rigorous proof) the Wong–Zakai approximation [67]-[69] within the frames of the Milstein approach [3] based on the Karhunen–Loeve expansion of the Brownian bridge process (see discussions in [54] (Sect. 2.18, 6.2), [55], [56] (Sect. 2.6.2, 6.2), [44]).

Note that in [66] the method of expansion of the iterated Ito stochastic integrals (2) ($k = 2$; $\psi_1(s), \psi_2(s) \equiv 1$; $i_1, i_2 = 1, \ldots, m$) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered.

It is necessary to note that the Milstein approach [3] excelled in several times or even in several orders the methods of multiple integral sums [3], [4], [14] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [15]-[17] (also see [24]-[30]), where $J^*[\psi^{(k)}]_{T,t}$ was represented as a multiple stochastic integral from the certain discontinuous non-random function of k variables, and the function was then expressed as an iterated generalized Fourier series in a complete systems of continuous functions that are orthonormal in $L_2([t,T])$. In [15]-[17] (also see [24]-[30]) the cases of Legendre polynomials and trigonometric functions are considered in details. As a result, a general iterated series expansion of (3) in terms of products of standard Gaussian random variables was obtained in [15]-[17] (also see [24]-[30]) for an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of generalized iterated Fourier series.

It was shown in [15], [16] (also see [18]-[28]) that the method of generalized iterated Fourier series leads to the Milstein expansion [3] of (3) in the case of trigonometric functions (at least for $k=2$; $\psi_1(s),\psi_2(s)\equiv 1$; $i_1,i_2=1,\ldots,m$) and to a substantially simpler expansion of (3) in the case of Legendre polynomials.

Note that the method of generalized iterated Fourier series as well as the Milstein approach [3] lead to iterated application of the operation of limit transition. This problem appears for triple stochastic integrals ($i_1,i_2,i_3=1,\ldots,m$) or even for some double stochastic integrals in the case, when $\psi_1(\tau)$, $\psi_2(\tau)\not\equiv 1$ ($i_1,i_2=1,\ldots,m$) [15], [16] (also see [18]-[28]).

The mentioned problem (iterated application of the operation of limit transition) not appears in the method, which is considered for (2) in Theorems 1, 2 (see below) [5], [18]-[28], [31]-[56]. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity k is represented as the multiple stochastic integral from the certain discontinuous non-random function of k variables defined on the hypercube $[t,T]^k$, where $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated non-random function of k variables is expanded in the hypercube into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square converging expansion of the iterated Ito stochastic integral (2) into the multiple series in terms of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned non-random function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2). Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (8)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k.

2. We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) (see [19], [20], [31]-[34], [36], [37], [39], [54]-[56]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we may use not only the trigonometric functions as in [2]-[4] but the Legendre polynomials.

4. As it turned out (see [5], [15], [16], [18]-[37], [40], [42], [45], [46], [50]-[52], [54]-[56]), it is more convenient to work with Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of Legendre polynomials in the framework of the mentioned problem are considered in [34], [52], [54]-[56].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see [66]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 1, 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$; $i_1,i_2,i_3=1,\ldots,m$) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1,\ldots,p_k$). For

example, when $p_1=\ldots=p_k=p\to\infty$. For iterated series, the condition $p_1=\ldots=p_k=p\to\infty$ obviously does not guarantee the convergence of this series.

However, the authors of the works [2] (Sect. 5.8, pp. 202–204), [11] (pp. 82-84), [12] (pp. 438-439), [13] (pp. 263-264) unreasonably use the condition $p_1=p_2=p_3=p\to\infty$ within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process together with the Wong–Zakai approximation [67]-[69].

2. Explicit One-Step Strong Numerical Schemes of Orders 2.0, 2.5, and 3.0 Based on the Unified Taylor–Ito expansion

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,\bar T]$ such that

$$0=\tau_0<\ldots<\tau_N=\bar T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j,\quad \Delta\tau_j=\tau_{j+1}-\tau_j.$$

Let $\mathbf{y}_{\tau_j}\stackrel{\sf def}{=}\mathbf{y}_j;\ j=0,1,\ldots,N$ be a time discrete approximation of the process $\mathbf{x}_s$, $s\in[0,\bar T]$, which is a solution of the Ito SDE (1).

Definition 1 [2]. *We will say that a time discrete approximation* $\mathbf{y}_j;\ j=0,1,\ldots,N$, *corresponding to the maximal step of discretization* Δ_N, *converges strongly with order* $\gamma>0$ *at time moment* $\bar T$ *to the process* $\mathbf{x}_s$, $s\in[0,\bar T]$ *if there exists a constant* $C>0$, *which does not depend on* Δ_N, *and a* $\delta>0$ *such that*

$${\sf M}\{|\mathbf{x}_{\bar T}-\mathbf{y}_{\bar T}|\}\le C(\Delta_N)^{\gamma}$$

for each $\Delta_N\in(0,\delta)$.

Consider the following explicit one-step strong numerical scheme of order 3.0 based on the so-called unified Taylor–Ito expansion [5], [10], [19]-[24], [54]-[56]

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^m\Sigma_{i_1}\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^m G_0^{(i_2)}\Sigma_{i_1}\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+$$

$$+\sum_{i_1=1}^m\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}+\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)-L\Sigma_{i_1}\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^m G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^m\left[G_0^{(i_2)}L\Sigma_{i_1}\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-LG_0^{(i_2)}\Sigma_{i_1}\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\right.$$

$$\left.+G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^m G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat I_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\mathbf{u}_{p+1,p}+\mathbf{v}_{p+1,p},\tag{4}$$

$$\mathbf{u}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}+\frac{\Delta^2}{2}\hat I_{0_{\tau_{p+1},\tau_p}}^{(i_1)}\right)+$$

$$+\frac{1}{2}LL\Sigma_{i_1}\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\left(\hat I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}+\Delta\hat I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_3)}LG_0^{(i_2)}\Sigma_{i_1}\left(\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}L\Sigma_{i_1}\left(\hat I_{010_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}-\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)+$$

$$+G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{000_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}+\hat I_{001_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\right)-$$

$$-LG_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat I_{100_{\tau_{p+1},\tau_p}}^{(i_3i_2i_1)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat I_{00000_{\tau_{p+1},\tau_p}}^{(i_5i_4i_3i_2i_1)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a},$$

$$\mathbf{v}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{\Delta^2}{2}\hat I_{00_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+$$

$$+\frac{1}{2}LLG_0^{(i_2)}\Sigma_{i_1}\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+$$

$$+G_0^{(i_2)}LG_0^{(i_1)}\mathbf{a}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\Delta\left(\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{01_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\right)+$$

$$+LG_0^{(i_2)}L\Sigma_{i_1}\left(\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)+$$

$$+G_0^{(i_2)}LL\Sigma_{i_1}\left(\frac{1}{2}\hat I_{02_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\frac{1}{2}\hat I_{20_{\tau_{p+1},\tau_p}}^{(i_2i_1)}-\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)-$$

$$-LG_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{10_{\tau_{p+1},\tau_p}}^{(i_2i_1)}+\hat I_{11_{\tau_{p+1},\tau_p}}^{(i_2i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\Biggl[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{0000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+\hat I_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}LG_0^{(i_2)}\Sigma_{i_1}\left(\hat I_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat I_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)-$$

$$-LG_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}+$$

$$+G_0^{(i_4)}LG_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\left(\hat{I}_{1000_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0100_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)+$$

$$\left.+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}L\Sigma_{i_1}\left(\hat{I}_{0010_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}-\hat{I}_{0001_{\tau_{p+1},\tau_p}}^{(i_4i_3i_2i_1)}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}_{000000_{\tau_{p+1},\tau_p}}^{(i_6i_5i_4i_3i_2i_1)},$$

where $\Delta=\bar{T}/N$ $(N>1)$ is a constant (for simplicity) step of integration, $\tau_p=p\Delta$ $(p=0,1,\ldots,N)$, $\hat{I}_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}$ is an approximation of the iterated Ito stochastic integral

$$I_{l_1\ldots l_k\,s,t}^{(i_1\ldots i_k)}=\int\limits_t^s(t-\tau_k)^{l_k}\ldots\int\limits_t^{\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}_{\tau_1}^{(i_1)}\ldots d\mathbf{f}_{\tau_k}^{(i_k)}, \tag{5}$$

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}\Sigma_{lj}(\mathbf{x},t)\Sigma_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_i},$$

$$G_0^{(i)}=\sum_{j=1}^{n}\Sigma_{ji}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_j},\quad i=1,\ldots,m,$$

$l_1,\ldots,l_k=0,1,2\ldots,$ $i_1,\ldots,i_k=1,\ldots,m,$ $k=1,2,\ldots,$ Σ_i is the ith column of the matrix function Σ and Σ_{ij} is the ijth component of the matrix function Σ, $\mathbf{a}_i$ is the ith component of the vector function $\mathbf{a}$ and $\mathbf{x}_i$ is the ith component of the column $\mathbf{x}$, the columns

$$\Sigma_{i_1},\ \mathbf{a},\ G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_1)}\mathbf{a},\ L\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ L\mathbf{a},\ G_0^{(i_2)}L\Sigma_{i_1},\ LG_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_2)}G_0^{(i_1)}\mathbf{a},$$

$$G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_1)}L\mathbf{a},\ LL\Sigma_{i_1},\ LG_0^{(i_1)}\mathbf{a},\ G_0^{(i_3)}LG_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}L\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a},$$

$$LG_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ LL\mathbf{a},\ G_0^{(i_2)}G_0^{(i_1)}L\mathbf{a},\ LLG_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_2)}LG_0^{(i_1)}\mathbf{a},\ LG_0^{(i_2)}L\Sigma_{i_1},$$

$$G_0^{(i_2)}LL\Sigma_{i_1},\ LG_0^{(i_2)}G_0^{(i_1)}\mathbf{a},\ G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\mathbf{a},\ G_0^{(i_4)}G_0^{(i_3)}LG_0^{(i_2)}\Sigma_{i_1},\ LG_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},$$

$$G_0^{(i_4)}LG_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}L\Sigma_{i_1},\ G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}$$

are calculated at the point $(\mathbf{y}_p,p)$.

It is well known [2] that under the standard conditions the numerical scheme (4) has strong order of convergence 3.0. Among these conditions we consider only the condition for approximations of iterated Ito stochastic integrals from the numerical scheme (4) [2], [5]

$$\mathsf{M}\left\{\left(I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}-\hat{I}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^7, \tag{6}$$

where $\hat{I}^{(i_1\ldots i_k)}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}$ is an approximation of $I^{(i_1\ldots i_k)}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}$, constant C does not depend on Δ.

Note that if we exclude $\mathbf{u}_{p+1,p}+\mathbf{v}_{p+1,p}$ from the right-hand side of (4), then we will have the explicit one-step strong numerical scheme of order 2.0. The right-hand side of (4) but without the value $\mathbf{v}_{p+1,p}$ define the explicit one-step strong numerical scheme of order 2.5.

Note that the truncated unified Taylor–Ito expansion [5], [8], [10], [16], [18]-[28], [54]-[56] contains the less number of various types of iterated Ito stochastic integrals (moreover, their major part will have less multiplicities) in comparison with the classical Taylor–Ito expansion [2], [7].

Furthermore, some iterated Ito stochastic integrals from the Taylor–Ito expansion [2], [7] are connected by linear relations. However, the iterated stochastic integrals from the unified Taylor–Ito expansion [5], [8], [10], [16], [18]-[28], [54]-[56] cannot be connected by linear relations. Therefore, we call these families of stochastic integrals from the unified Taylor–Ito expansion as the stochastic bases [5], [10], [54]-[56]. Note that (4) contains 20 different types of iterated Ito stochastic integrals. At the same time, the analogue of (4) based on the classical Taylor–Ito expansion [2], [7] contains 29 different types of iterated Ito stochastic integrals.

3. Approximation of iterated Ito Stochastic Integrals Based on Multiple Fourier–Legendre Series

Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a non-random function from the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for}\ \ t_1<\ldots<t_k\\ \\ 0 & \text{otherwise}\end{cases},\qquad t_1,\ldots,t_k\in[t,T],\qquad k\ge 2, \tag{7}$$

and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0,$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k, \tag{8}$$

$$\|f\|=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{9}$$

Theorem 1 [5] (2006), [18]-[28], [31]-[56]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \qquad (10)$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k = \mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k = \{(l_1,\ldots,l_k):\ l_1,\ldots,l_k = 0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense, $i_1,\ldots,i_k = 0,1,\ldots,m,$*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \qquad (11)$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (8), $\Delta\mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$, which satisfies the condition (9).

The convergence in the mean of degree $2n$ $(n\in\mathbb{N})$ [18]-[20], [22]-[28], [54]-[56] as well as the convergence with probability 1 [38], [40], [54]-[56] of approximations from Theorem 1 are proved. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t,T])$ can also be applied in Theorem 1 [5], [18]-[28], [54]-[56]. The modification of Theorem 1 for complete orthonormal with weigth $r(x)\ge 0$ systems of functions in the space $L_2([t,T])$ can be found in [20], [53]-[56]. Application of Theorem 1 and Theorem 2 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [54]-[56] (Chapter 7) and in [36], [37], [58], [62]-[64].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [5], [18]-[28], [31]-[56]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \qquad (12)$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \qquad (13)$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (14)
$$

$$
\begin{aligned}
J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
J[\psi^{(5)}]_{T,t}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
\end{aligned}
$$

$$
\begin{gathered}
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-\\
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-
\end{gathered}
$$

$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{17}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Note that we will consider the case $i_1,\ldots,i_6=1,\ldots,m$. This case corresponds to the numerical method (4).

For further consideration, let us consider the generalization of formulas (12)–(17) for the case of an arbitrary multiplicity k ($k\in\mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{18}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (18) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{19}$$

Below there are several examples of sums in the form (19)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} + a_{12,54,3} + a_{15,24,3} + a_{14,25,3} +$$
$$+ a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (10) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{20}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem **1**.

In particular, from (20) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (16).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [54] (Sect. 1.11), [38] (Sect. 15). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(21)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorem* 1.

It should be noted that an analogue of Theorem 2 was considered in [70]. Note that we use another notations [54] (Sect. 1.11), [38] (Sect. 15) in comparison with [70]. Moreover, the proof of an analogue of Theorem 2 from [70] is somewhat different from the proof given in [54] (Sect. 1.11), [38] (Sect. 15).

Let us consider the exact calculation and effective estimation of the mean-square error of approximation $J[\psi^{(k)}]_{T,t}^q$. Here $J[\psi^{(k)}]_{T,t}^q$ is the expression on the right-hand side of (21) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1=\ldots=p_k=q$

$$J[\psi^{(k)}]_{T,t}^q=\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^q\right)^2\right\}\stackrel{\rm def}{=}E_k^q,\qquad\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\stackrel{\rm def}{=}I_k.$$

In [19], [20], [31], [39], [54]-[56] it was shown that

$$E_k^q\le k!\left(I_k-\sum_{j_1,\ldots,j_k=0}^{q}C_{j_k\ldots j_1}^2\right),\qquad(22)$$

where $i_1,\ldots,i_k=1,\ldots,m$ $(T-t\in(0,+\infty))$ or $i_1,\ldots,i_k=0,1,\ldots,m$ $(T-t\in(0,1))$.

The value E_k^q can be calculated exactly.

Theorem 3 [54] (Sect. 1.12), [39] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

(23) $$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1} \mathsf{M}\left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \right\},$$

where $i_1,\ldots,i_k = 1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{ J[\psi^{(k)}]_{T,t} \int\limits_t^T \phi_{j_k}(t_k)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)} \right\} = C_{j_k\ldots j_1}.$$

Then from Theorem 3 for pairwise different $i_1,\ldots,i_k$ and for $i_1 = \ldots = i_k$ we obtain [20], [31], [39], [54]-[56]

(24) $$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C^2_{j_k\ldots j_1},$$

$$E_k^q = I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1}\right),$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

is a sum with respect to all possible permutations $(j_1,\ldots,j_k)$.

Consider some examples [20], [31], [39], [54]-[56] of application of Theorem 3 ($i_1,\ldots,i_5 = 1,\ldots,m$)

(25) $$E_2^q = I_2 - \sum_{j_1,j_2=0}^{q} C^2_{j_2j_1} - \sum_{j_1,j_2=0}^{q} C_{j_2j_1}C_{j_1j_2} \quad (i_1 = i_2),$$

(26) $$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \neq i_3),$$

(27) $$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_2j_3j_1}C_{j_3j_2j_1} \quad (i_1 \neq i_2 = i_3),$$

(28) $$E_3^q = I_3 - \sum_{j_3,j_2,j_1=0}^{q} C^2_{j_3j_2j_1} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \neq i_2),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\right) \quad (i_1=i_2\ne i_3,i_4;\ i_3\ne i_4),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1=i_3\ne i_2,i_4;\ i_2\ne i_4),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1}\right) \quad (i_1=i_2=i_3\ne i_4),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1}\right) \quad (i_2=i_3=i_4\ne i_1),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1=i_2\ne i_3=i_4),$$

$$E_4^q = I_4 - \sum_{j_1,\ldots,j_4=0}^{q} C_{j_4\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1}\right)\right) \quad (i_1=i_3\ne i_2=i_4),$$

$$E_5^q = I_5 - \sum_{j_1,\ldots,j_5=0}^{q} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right) \quad (i_1\ne i_2=i_4\ne i_3=i_5\ne i_1),$$

$$E_5^q = I_5 - \sum_{j_1,\ldots,j_5=0}^{q} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right) \quad (i_1=i_2=i_3\ne i_4=i_5),$$

$$E_5^q = I_5 - \sum_{j_1,\ldots,j_5=0}^{q} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1}\right) \quad (i_1=i_3=i_4=i_5\ne i_2).$$

The values E_4^q and E_5^q were calculated exaclty for all possible combinations of $i_1,\ldots,i_5=1,\ldots,m$ in [19], [20], [39], [54]-[56].

Let us consider the approximations of iterated Ito stochastic integrals from (4) using (12)–(17) and complete orthonormal system of Legendre polynomials in the space $L_2([\tau_p,\tau_{p+1}])$ ($\tau_p=p\Delta$, $N\Delta=\bar{T}$, $p=0,1,\ldots,N$) [20] (also see [15]-[19], [21]-[56])

$$I_{0_{\tau_{p+1},\tau_p}}^{(i_1)} = \sqrt{\Delta}\zeta_0^{(i_1)},$$

$$I_{00_{\tau_{p+1},\tau_p}}^{(i_1i_2)q} = \frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{29}$$

$$I_{1_{\tau_{p+1},\tau_p}}^{(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$I_{0000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}=-\frac{\Delta}{2}I_{00_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}-\frac{\Delta^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$
$$+\sum_{i=0}^{q}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

$$I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}=-\frac{\Delta}{2}I_{00_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}-\frac{\Delta^2}{4}\Biggl(\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$
$$+\sum_{i=0}^{q}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr),$$

$$I_{2_{\tau_{p+1},\tau_p}}^{(i_1)}=\frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

$$I_{001_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^{001}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$
I_{010_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)} \Biggr),
$$

$$
I_{100_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)} \Biggr),
$$

$$
I_{00000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4 i_5)q} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5 j_4 j_3 j_2 j_1} \Biggl(\prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)} +
$$

$$
+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)} +
$$

$$
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)} +
$$

$$
+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)} +
$$

$$
+\mathbf{1}_{\{j_2=j_5\neq 0\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\neq 0\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)} \Biggr),
$$

$$
I_{02_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} = -\frac{\Delta^2}{4} I_{00_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} - \Delta I_{01_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} + \frac{\Delta^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} +
$$

$$
+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{q}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +
$$

$$
+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)} \Biggr)\Biggr] - \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}\Delta^3, \tag{30}
$$

$$I_{20_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} = -\frac{\Delta^2}{4} I_{00_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} - \Delta I_{10_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} + \frac{\Delta^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} +$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=0}^{q}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - (i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - (i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr] - \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}\Delta^3, \tag{31}$$

$$I_{11_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} = -\frac{\Delta^2}{4} I_{00_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} - \frac{\Delta}{2}\left(I_{10_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q} + I_{01_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right) + \frac{\Delta^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)} +$$

$$+\sum_{i=0}^{q}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)} +$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)} - \zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr] - \frac{1}{24}\mathbf{1}_{\{i_1=i_2\}}\Delta^3, \tag{32}$$

$$I_{0001_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}^{0001}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$I_{0010_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}^{0010}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

$$
I_{0100_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{1000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}^{1000} \Biggl(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}} \Biggr),
$$

$$
I_{000000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3 i_4 i_5 i_6)q} = \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q} C_{j_6 j_5 j_4 j_3 j_2 j_1} \Biggl(\prod_{l=1}^{6} \zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{j_1=j_6\}}\mathbf{1}_{\{i_1=i_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_2=j_6\}}\mathbf{1}_{\{i_2=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_4=j_6\}}\mathbf{1}_{\{i_4=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_5=j_6\}}\mathbf{1}_{\{i_5=i_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)} +
$$

$$
+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)} + \mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)} +
$$

$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-$$

$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$

$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$

$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$

$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$

$$\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right),$$

where

$$C_{j_3j_2j_1}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_3}(z)\int\limits_{\tau_p}^{z}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdydz=$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\Delta^{3/2}\bar{C}_{j_3j_2j_1},
$$

$$
C_{j_4j_3j_2j_1} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_4}(u)\int\limits_{\tau_p}^{u} \phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydzdu =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}\Delta^{2}\bar{C}_{j_4j_3j_2j_1},
$$

$$
C_{j_3j_2j_1}^{001} = \int\limits_{\tau_p}^{\tau_{p+1}} (\tau_p - z)\phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydz =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{001},
$$

$$
C_{j_3j_2j_1}^{010} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_3}(z)\int\limits_{\tau_p}^{z} (\tau_p - y)\phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydz =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{010},
$$

$$
C_{j_3j_2j_1}^{100} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} (\tau_p - x)\phi_{j_1}(x)dxdydz =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}\Delta^{5/2}\bar{C}_{j_3j_2j_1}^{100},
$$

$$
C_{j_5j_4j_3j_2j_1} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_5}(v)\int\limits_{\tau_p}^{v} \phi_{j_4}(u)\int\limits_{\tau_p}^{u} \phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydzdudv =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}\Delta^{5/2}\bar{C}_{j_5j_4j_3j_2j_1},
$$

$$
C_{j_4j_3j_2j_1}^{0001} = \int\limits_{\tau_p}^{\tau_{p+1}} (\tau_p - u)\phi_{j_4}(u)\int\limits_{\tau_p}^{u} \phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydzdu =
$$

$$
= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}\Delta^{3}\bar{C}_{j_4j_3j_2j_1}^{0001},
$$

$$
C_{j_3j_2j_1}^{0010} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_4}(u)\int\limits_{\tau_p}^{u} (\tau_p - z)\phi_{j_3}(z)\int\limits_{\tau_p}^{z} \phi_{j_2}(y)\int\limits_{\tau_p}^{y} \phi_{j_1}(x)dxdydzdu =
$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}\Delta^3 \bar{C}_{j_4j_3j_2j_1}^{0010},$$

$$C_{j_4j_3j_2j_1}^{0100} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_4}(u) \int\limits_{\tau_p}^{u} \phi_{j_3}(z) \int\limits_{\tau_p}^{z} (\tau_p - y)\phi_{j_2}(y) \int\limits_{\tau_p}^{y} \phi_{j_1}(x) dx dy dz du =$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}\Delta^3 \bar{C}_{j_3j_2j_1}^{0100},$$

$$C_{j_4j_3j_2j_1}^{1000} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_4}(u) \int\limits_{\tau_p}^{u} \phi_{j_3}(z) \int\limits_{\tau_p}^{z} \phi_{j_2}(y) \int\limits_{\tau_p}^{y} (\tau_p - x)\phi_{j_1}(x) dx dy dz du =$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}\Delta^3 \bar{C}_{j_4j_3j_2j_1}^{1000},$$

$$C_{j_6j_5j_4j_3j_2j_1} = \int\limits_{\tau_p}^{\tau_{p+1}} \phi_{j_6}(w) \int\limits_{\tau_p}^{w} \phi_{j_5}(v) \int\limits_{\tau_p}^{v} \phi_{j_4}(u) \int\limits_{\tau_p}^{u} \phi_{j_3}(z) \int\limits_{\tau_p}^{z} \phi_{j_2}(y) \int\limits_{\tau_p}^{y} \phi_{j_1}(x) dx dy dz du dv dw =$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}\Delta^3 \bar{C}_{j_6j_5j_4j_3j_2j_1},$$

where

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}_{j_3j_2j_1}^{100} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x)(x+1) dx dy dz,$$

$$\bar{C}_{j_3j_2j_1}^{010} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y)(y+1) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}_{j_3j_2j_1}^{001} = -\int\limits_{-1}^{1} P_{j_3}(z)(z+1) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

$$\bar{C}_{j_5j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_5}(v) \int\limits_{-1}^{v} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du dv,$$

$$\bar{C}_{j_4j_3j_2j_1}^{1000}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0100}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0010}=-\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)(z+1)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_4j_3j_2j_1}^{0001}=-\int\limits_{-1}^{1}P_{j_4}(u)(u+1)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,$$

$$\bar{C}_{j_6j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_6}(w)\int\limits_{-1}^{w}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudvdw,$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial and

$$\phi_i(x)=\sqrt{\frac{2i+1}{\Delta}}P_i\left(\left(x-\tau_p-\frac{\Delta}{2}\right)\frac{2}{\Delta}\right),\quad i=0,1,2,\ldots$$

Let us consider the exact relations and some estimates for the mean-square errors of approximations of iterated Ito stochastic integrals.

Using Theorem 3, we get [18]-[28], [39] (also see [5], [15], [16], [29]-[37], [54]-[56])

$$\mathsf{M}\left\{\left(I_{00_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{00_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}=\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\quad(i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}=$$

$$=\frac{\Delta^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\quad(i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_1)}-I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_1)}-I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_1)q}\right)^2\right\}=$$

$$=\frac{\Delta^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right).$$

Applying (24), (25)–(28), we obtain

$$\mathsf{M}\left\{\left(I_{20_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{20_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{20}C_{j_1 j_2}^{20}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{20_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{20_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{30}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{11_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{11_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{11}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{11}C_{j_1 j_2}^{11}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{11_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{11_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{18}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{11}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{02_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{02_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2-\sum_{j_1,j_2=0}^{q}C_{j_2 j_1}^{02}C_{j_1 j_2}^{02}\quad (i_1=i_2),$$

$$\mathsf{M}\left\{\left(I_{02_{\tau_{p+1},\tau_p}}^{(i_1 i_2)}-I_{02_{\tau_{p+1},\tau_p}}^{(i_1 i_2)q}\right)^2\right\}=\frac{\Delta^6}{6}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2\quad (i_1\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)}-I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_2 j_1}^2\quad (i_1\ne i_2,\ i_1\ne i_3,\ i_2\ne i_3),$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)}-I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_2 j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_2 j_3 j_1}C_{j_3 j_2 j_1}\quad (i_1\ne i_2=i_3),$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)}-I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_2 j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_2 j_1}C_{j_1 j_2 j_3}\quad (i_1=i_3\ne i_2),$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)}-I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_2 j_1}^2-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3 j_1 j_2}C_{j_3 j_2 j_1}\quad (i_1=i_2\ne i_3),$$

where

$$C_{j_2 j_1}^{20}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_2}(y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)(\tau_p-x)^2 dxdy=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}\Delta^3\bar{C}_{j_2 j_1}^{20},$$

$$C_{j_2 j_1}^{02}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_2}(y)(\tau_p-y)^2\int\limits_{\tau_p}^{y}\phi_{j_1}(x)dxdy=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}\Delta^3\bar{C}_{j_2 j_1}^{02},$$

$$C_{j_2j_1}^{11}=\int\limits_{\tau_p}^{\tau_{p+1}}\phi_{j_2}(y)(\tau_p-y)\int\limits_{\tau_p}^{y}\phi_{j_1}(x)(\tau_p-x)dxdy=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}\Delta^3\bar C_{j_2j_1}^{11},$$

$$\bar C_{j_2j_1}^{20}=\int\limits_{-1}^{1}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)^2dxdy,$$

$$\bar C_{j_2j_1}^{02}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)^2\int\limits_{-1}^{y}P_{j_1}(x)dxdy,$$

$$\bar C_{j_2j_1}^{11}=\int\limits_{-1}^{1}P_{j_2}(y)(y+1)\int\limits_{-1}^{y}P_{j_1}(x)(x+1)dxdy,$$

where $P_i(x)$ $(i=0,1,2,\ldots)$ is the Legendre polynomial and

$$\phi_i(x)=\sqrt{\frac{2i+1}{\Delta}}P_i\left(\left(x-\tau_p-\frac{\Delta}{2}\right)\frac{2}{\Delta}\right),\quad i=0,1,2,\ldots$$

At the same time using the estimate (22) for $i_1,\ldots,i_6=1,\ldots,m$, we get

$$\mathsf{M}\left\{\left(I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{01_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{10_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^4}{12}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{10}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)}-I_{000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{q}C_{j_3j_2j_1}^2\right),$$

$$\mathsf{M}\left\{\left(I_{0000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)}-I_{0000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^2\right),$$

$$\mathsf{M}\left\{\left(I_{100_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)}-I_{100_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{010_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)}-I_{010_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{001_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)}-I_{001_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3)q}\right)^2\right\}\le 6\left(\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3j_2j_1}^{001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{00000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4i_5)}-I_{00000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4i_5)q}\right)^2\right\}\le 120\left(\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q}C_{j_5i_4i_3i_2j_1}^2\right),$$

$$\mathsf{M}\left\{\left(I_{20_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{20_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^6}{30}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{20}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{11_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{11_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^6}{18}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{11}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{02_{\tau_{p+1},\tau_p}}^{(i_1i_2)}-I_{02_{\tau_{p+1},\tau_p}}^{(i_1i_2)q}\right)^2\right\}\le 2\left(\frac{\Delta^6}{6}-\sum_{j_2,j_1=0}^{q}\left(C_{j_2j_1}^{02}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{1000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)}-I_{1000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^6}{360}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{0100_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)}-I_{0100_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^6}{120}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{0010_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)}-I_{0010_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^6}{60}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{0001_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)}-I_{0001_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4)q}\right)^2\right\}\le 24\left(\frac{\Delta^6}{36}-\sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right),$$

$$\mathsf{M}\left\{\left(I_{000000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4i_5i_6)}-I_{000000_{\tau_{p+1},\tau_p}}^{(i_1i_2i_3i_4i_5i_6)q}\right)^2\right\}\le 720\left(\frac{\Delta^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_6j_5j_4j_3j_2j_1}^2\right).$$

The Fourier–Legendre coefficients

$$\bar{C}_{j_3j_2j_1},\ \bar{C}_{j_4j_3j_2j_1},\ \bar{C}_{j_3j_2j_1}^{001},\ \bar{C}_{j_3j_2j_1}^{010},\ \bar{C}_{j_3j_2j_1}^{100},\ \bar{C}_{j_5j_4j_3j_2j_1},\ \bar{C}_{j_4j_3j_2j_1}^{0001},$$

$$\bar{C}_{j_4j_3j_2j_1}^{0010},\ \bar{C}_{j_4j_3j_2j_1}^{0100},\ \bar{C}_{j_4j_3j_2j_1}^{1000},\ \bar{C}_{j_6j_5j_4j_3j_2j_1}$$

can be calculated exactly before start of the numerical method (4) using DERIVE or MAPLE (computer algebra systems). In [5], [18]-[28], [40], [54]-[56] several tables with these coefficients can be found. Note that the mentioned Fourier–Legendre coefficients are independent of the integration step $\tau_{p+1}-\tau_p$ of the numerical scheme, which can be not a constant in a general case.

Note that in [57], [60] the database with 270,000 exactly calculated Fourier–Legendre coefficients was described. This database was used in the software package [57], [60], which is written in the Python programming language for the implementation of explicit one-step numerical schemes with strong orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Ito SDEs. The optimization of the mean-square approximation procedures for iterated Ito stochastic integrals from these numerical schemes can be found in [59].

On the basis of the presented approximations of iterated Ito stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $\tau_{p+1}-\tau_p$ ($\tau_{p+1}-\tau_p \ll 1$) in the mean-square sense for iterated Ito stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Ito stochastic integrals (see the number q in Theorem 3), which are required for achieving the acceptable accuracy of approximation.

4. Explicit One-Step Strong Numerical Schemes of Orders 2.0, 2.5, and 3.0 Based on the Unified Taylor–Stratonovich expansion

Consider the following explicit one-step strong numerical scheme of order 3.0 based on the so-called unified Taylor–Stratonovich expansion [9] (also see [5], [18]-[28], [54]-[56])

$$
\begin{aligned}
\mathbf{y}_{p+1} = \mathbf{y}_p &+ \sum_{i_1=1}^{m} \Sigma_{i_1} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} + \\
&+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} + \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} \right) - \bar{L} \Sigma_{i_1} \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} \right] + \\
&+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_3)} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} + \frac{\Delta^2}{2} \bar{L} \bar{\mathbf{a}} + \\
&+ \sum_{i_1,i_2=1}^{m} \left[G_0^{(i_2)} \bar{L} \Sigma_{i_1} \left(\hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} - \hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} \right) - \bar{L} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{10_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} + \right. \\
&\left. + G_0^{(i_2)} G_0^{(i_1)} \bar{\mathbf{a}} \left(\hat{I}_{01_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} + \Delta \hat{I}_{00_{\tau_{p+1},\tau_p}}^{*(i_2 i_1)} \right) \right] + \\
&+ \sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} \Sigma_{i_1} \hat{I}_{0000_{\tau_{p+1},\tau_p}}^{*(i_4 i_3 i_2 i_1)} + \mathbf{q}_{p+1,p} + \mathbf{r}_{p+1,p},
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\mathbf{q}_{p+1,p} = &\sum_{i_1=1}^{m} \left[G_0^{(i_1)} \bar{L} \bar{\mathbf{a}} \left(\frac{1}{2} \hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} + \frac{\Delta^2}{2} \hat{I}_{0_{\tau_{p+1},\tau_p}}^{*(i_1)} \right) + \right. \\
&\left. + \frac{1}{2} \bar{L} \bar{L} \Sigma_{i_1} \hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)} - \bar{L} G_0^{(i_1)} \bar{\mathbf{a}} \left(\hat{I}_{2_{\tau_{p+1},\tau_p}}^{*(i_1)} + \Delta \hat{I}_{1_{\tau_{p+1},\tau_p}}^{*(i_1)} \right) \right] + \\
&+ \sum_{i_1,i_2,i_3=1}^{m} \left[G_0^{(i_3)} \bar{L} G_0^{(i_2)} \Sigma_{i_1} \left(\hat{I}_{100_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} - \hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} \right) + \right. \\
&+ G_0^{(i_3)} G_0^{(i_2)} \bar{L} \Sigma_{i_1} \left(\hat{I}_{010_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} - \hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} \right) + \\
&+ G_0^{(i_3)} G_0^{(i_2)} G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{000_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} + \hat{I}_{001_{\tau_{p+1},\tau_p}}^{*(i_3 i_2 i_1)} \right) -
\end{aligned}
$$

$$-\bar{L}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}^{*(i_3i_2i_1)}_{100_{\tau_{p+1},\tau_p}}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}^{*(i_5i_4i_3i_2i_1)}_{00000_{\tau_{p+1},\tau_p}}+$$

$$+\frac{\Delta^3}{6}\bar{L}\bar{L}\bar{\mathbf{a}},$$

$$\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_2)}G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}+\frac{\Delta^2}{2}\hat{I}^{*(i_2i_1)}_{00_{\tau_{p+1},\tau_p}}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}G_0^{(i_2)}\Sigma_{i_1}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}$$

$$+G_0^{(i_2)}\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\Delta\left(\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{01_{\tau_{p+1},\tau_p}}\right)\right)+$$

$$+\bar{L}G_0^{(i_2)}\bar{L}\Sigma_{i_1}\left(\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_2)}\bar{L}\bar{L}\Sigma_{i_1}\left(\frac{1}{2}\hat{I}^{*(i_2i_1)}_{02_{\tau_{p+1},\tau_p}}+\frac{1}{2}\hat{I}^{*(i_2i_1)}_{20_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_2i_1)}_{10_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_2i_1)}_{11_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\Biggl[G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_4i_3i_2i_1)}_{0000_{\tau_{p+1},\tau_p}}+\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}\bar{L}G_0^{(i_2)}\Sigma_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}\right)-$$

$$-\bar{L}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}+$$

$$+G_0^{(i_4)}\bar{L}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{1000_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0100_{\tau_{p+1},\tau_p}}\right)+$$

$$+G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\bar{L}\Sigma_{i_1}\left(\hat{I}^{*(i_4i_3i_2i_1)}_{0010_{\tau_{p+1},\tau_p}}-\hat{I}^{*(i_4i_3i_2i_1)}_{0001_{\tau_{p+1},\tau_p}}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}\hat{I}^{*(i_6i_5i_4i_3i_2i_1)}_{000000_{\tau_{p+1},\tau_p}},$$

where $\Delta=\bar{T}/N$ $(N>1)$ is a constant (for simplicity) step of integration, $\tau_p=p\Delta$ $(p=0,1,\ldots,N)$, $\hat{I}_{l_1\ldots l_k\,s,t}^{*(i_1\ldots i_k)}$ is an approximation of the iterated Stratonovich stochastic integral

$$
I_{l_1\ldots l_k\,s,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*s}(t-\tau_k)^{l_k}\ldots\int\limits_t^{*\tau_2}(t-\tau_1)^{l_1}d\mathbf{f}_{\tau_1}^{(i_1)}\ldots d\mathbf{f}_{\tau_k}^{(i_k)}, \tag{34}
$$

$$
\bar{\mathbf{a}}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)-\frac{1}{2}\sum_{j=1}^{m}G_0^{(j)}\Sigma_j(\mathbf{x},t),
$$

$$
\bar{L}=L-\frac{1}{2}\sum_{j=1}^{m}G_0^{(j)}G_0^{(j)}=\frac{\partial}{\partial t}+\sum_{j=1}^{n}\bar{\mathbf{a}}^{(j)}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}^{(j)}},
$$

$$
L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mathbf{a}_i(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_i}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}\Sigma_{lj}(\mathbf{x},t)\Sigma_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_i},
$$

$$
G_0^{(i)}=\sum_{j=1}^{n}\Sigma_{ji}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}_j},\quad i=1,\ldots,m,
$$

$l_1,\ldots,l_k=0,1,2\ldots,$ $i_1,\ldots,i_k=1,\ldots,m,$ $k=1,2,\ldots,$ Σ_i is the ith column of the matrix function Σ and Σ_{ij} is the ijth component of the matrix function Σ, $\mathbf{a}_i$ is the ith component of the vector function $\mathbf{a}$ and $\mathbf{x}_i$ is the ith component of the column $\mathbf{x}$, the columns

$$
\Sigma_{i_1},\ \bar{\mathbf{a}},\ G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_1)}\bar{\mathbf{a}},\ \bar{L}\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ \bar{L}\bar{\mathbf{a}},\ G_0^{(i_2)}\bar{L}\Sigma_{i_1},\ \bar{L}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}},
$$

$$
G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_1)}\bar{L}\bar{\mathbf{a}},\ \bar{L}\bar{L}\Sigma_{i_1},\ \bar{L}G_0^{(i_1)}\bar{\mathbf{a}},\ G_0^{(i_3)}\bar{L}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}\bar{L}\Sigma_{i_1},\ G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}},
$$

$$
\bar{L}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ \bar{L}\bar{L}\bar{\mathbf{a}},\ G_0^{(i_2)}G_0^{(i_1)}\bar{L}\bar{\mathbf{a}},\ \bar{L}\bar{L}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_2)}\bar{L}G_0^{(i_1)}\bar{\mathbf{a}},\ \bar{L}G_0^{(i_2)}\bar{L}\Sigma_{i_1},
$$

$$
G_0^{(i_2)}\bar{L}\bar{L}\Sigma_{i_1},\ \bar{L}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}},\ G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}G_0^{(i_1)}\bar{\mathbf{a}},\ G_0^{(i_4)}G_0^{(i_3)}\bar{L}G_0^{(i_2)}\Sigma_{i_1},\ \bar{L}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},
$$

$$
G_0^{(i_4)}\bar{L}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1},\ G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\bar{L}\Sigma_{i_1},\ G_0^{(i_6)}G_0^{(i_5)}G_0^{(i_4)}G_0^{(i_3)}G_0^{(i_2)}\Sigma_{i_1}
$$

are calculated at the point $(\mathbf{y}_p,p)$.

It is well known [2] that under the standard conditions the numerical scheme (33) has strong order of convergence 3.0. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from the numerical scheme (33) [2], [5]

$$
\mathsf{M}\left\{\left(I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}-\hat{I}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^7,
$$

where $\hat{I}_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}$ is an approximation of $I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}$, constant C does not depend on Δ.

Note that if we exclude $\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}$ from the right-hand side of (33), then we will have the explicit one-step strong numerical scheme of order 2.0. The right-hand side of (33) but without the value $\mathbf{r}_{p+1,p}$ and with replacing the value $\Delta^3\bar{L}\bar{L}\bar{\mathbf{a}}/6$ by the value $\Delta^3 LL\mathbf{a}/6$ define the explicit one-step strong numerical scheme of order 2.5.

Note that the truncated unified Taylor–Stratonovich expansion [9] (also see [5], [18]-[28], [54]-[56]) contains the less number of various types of iterated Stratonovich stochastic integrals (moreover, their major part will have less multiplicities) in comparison with the classical Taylor–Stratonovich expansion [2], [7].

Furthermore, some iterated stochastic integrals from the Taylor–Stratonovich expansion [2], [7] are connected by linear relations. However, the iterated stochastic integrals from the unified Taylor–Stratonovich expansion [9] (also see [5], [18]-[28], [54]-[56]) cannot be connected by linear relations. Therefore, we call these families of stochastic integrals as the stochastic bases [5], [18]-[28], [54]-[56]. Note that (33) contains 20 different types of iterated Stratonovich stochastic integrals. At the same time, the analogue of (33) based on the classical Taylor–Stratonovich expansion [2], [7] contains 29 different types of iterated stochastic integrals.

5. Fourier–Legendre Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6

As noted above, in a number of works of the author Theorems 1, 2 have been adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 (the case of multiplicity 1 is given by (12)). Let us first present some old results.

Theorem 4 [18]-[20], [25]-[28], [41], [47], [54]-[56]. *Assume that the following conditions are fulfilled*:

1. *The function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the function $\psi_1(\tau)$ is two times continuously differentiable at the interval $[t,T]$.*

2. *$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or system of trigonometric functions in the space $L_2([t,T])$.*

Then, the iterated Stratonovich stochastic integral of multiplicity 2

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}\quad (i_1,i_2=1,\ldots,m)$$

is expanded into the converging in the mean-square sense multiple series

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where the meaning of notations introduced in the formulation of Theorem 1 *is remained.*

Proving the theorem 4 [18]-[20], [25]-[28], [41], [47], [54]-[56] we used Theorem 1 and double integration by parts. This procedure leads to the condition of double continuous differentiability of the function $\psi_1(\tau)$ at the interval $[t,T]$. The mentioned condition can be weakened [17], [35], [42], [49], [54]-[56] and Theorem 4 will be valid for continuously differentiable functions $\psi_l(\tau)$ $(l=1,2)$ at the interval $[t,T]$.

Theorem 5 [18]-[20], [25]-[28], [41], [47], [54]-[56]. *Assume, that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonomertic functions in the space $L_2([t,T])$. Furthermore, the function $\psi_2(\tau)$ is continuously differentiable at the interval $[t,T]$ and the functions $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable at the interval $[t,T]$. Then, for the iterated Stratonovich stochastic integral of multiplicity* 3

$$J^*[\psi^{(3)}]_{T,t}=\int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\quad (i_1,i_2,i_3=1,\ldots,m)$$

the following expansion

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \tag{35}$$

converging in the mean-square sense is valid, where

$$C_{j_3j_2j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3;$$

another notations are the same as in Theorems 1, 2.

Theorem 6 [18]–[20], [25]-[28], [41], [43], [54]-[56]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of multiplicity* 4

$$I_{T,t}^{*(i_1i_2i_3i_4)} = \int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)},$$

where $i_1, i_2, i_3, i_4 = 0, 1, \ldots, m$, *the following expansion*

$$I_{T,t}^{*(i_1i_2i_3i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

converging in the mean-square sense is valid, where

$$C_{j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\int\limits_t^{t_3} \phi_{j_2}(t_2)\int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

$\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$ *are independent standard Wiener processes and* $\mathbf{w}_\tau^{(0)} = \tau$; *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [54] (Sect. 2.10–2.16), [41] (Sect. 13–19), [43] (Sect. 5–11), [44] (Sect. 7–13), [71] (Sections 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 7 [41], [43], [44], [54], [71]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau), \psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T} \psi_3(t_3)\int\limits_t^{*t_3} \psi_2(t_2)\int\limits_t^{*t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \ldots, m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{36}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{37}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (36) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (37)*, constant* C *is independent of* p*,*

$$C_{j_3j_2j_1} = \int_t^T \psi_3(t_3)\phi_{j_3}(t_3)\int_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 8 [41], [43], [44], [54], [71]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau), \ldots, \psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = {\int_t^{*}}^{T} \psi_4(t_4){\int_t^{*}}^{t_4}\psi_3(t_3){\int_t^{*}}^{t_3}\psi_2(t_2){\int_t^{*}}^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{38}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{39}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{40}$$

are fulfilled, where $i_1, \ldots, i_4 = 0, 1, \ldots, m$ *in* (38), (39) *and* $i_1, \ldots, i_4 = 1, \ldots, m$ *in* (40)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$C_{j_4 j_3 j_2 j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 7*.*

Theorem 9 [41], [43], [44], [54], [71]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t} = \int\limits_t^{*T} \psi_5(t_5) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{41}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}, \tag{42}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{43}$$

are fulfilled, where $i_1, \ldots, i_5 = 0, 1, \ldots, m$ *in* (41), (42) *and* $i_1, \ldots, i_5 = 1, \ldots, m$ *in* (43)*, constant* C *is independent of* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5) \ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 \ldots dt_5;$$

another notations are the same as in Theorems 7, 8*.*

Theorem 10 [41], [43], [44], [54]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$I_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{44}$$

the following expansion

$$I_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 7–9*.*

On the base of Theorems 4–10 the folloing hypothesis was formulated in [18]-[20], [25]-[28], [44], [54]-[56].

Hypothesis 1 [18]-[20], [25]-[28], [44], [54]-[56]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Moreover, every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is an enough smooth nonrandom function on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(k)}]_{T,t}$ *defined by* (3) *the following expansion*

$$J^*[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{45}$$

converging in the mean-square sense is valid, where the notations are the same as in Theorems 1, 2*.*

Hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)},$$

where

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t}-J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\}=0.$$

Note that Hypothesis 1 is proved in [54] (Sect. 2.10) under the condition of convergence of trace series (also see [41], [43], [44]). In [44], [54]-[56] a more general hypothesis is formulated.

Applying Theorems 4–10, we obtain the following approximations of iterated Stratonovich stochastic integrals from (33)

$$I_{0_{\tau_{p+1},\tau_p}}^{*(i_1)}=\sqrt{\Delta}\zeta_0^{(i_1)},$$

$$I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}=\frac{\Delta}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right),$$

$$I_{1_{\tau_{p+1},\tau_p}}^{*(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{000_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{01_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} = -\frac{\Delta}{2}I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} - \frac{\Delta^2}{4}\Biggl[\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{q}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)}-(i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr],$$

$$I_{10_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} = -\frac{\Delta}{2}I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} - \frac{\Delta^2}{4}\Biggl[\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

$$+\sum_{i=0}^{q}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr],$$

$$I_{0000_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3i_4)q} = \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{2_{\tau_{p+1},\tau_p}}^{*(i_1)} = \frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

$$I_{100_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{010_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{001_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)},$$

$$I_{00000_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3i_4i_5)q} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)},$$

$$I_{02_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} = -\frac{\Delta^2}{4}I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} - \Delta I_{01_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q} + \frac{\Delta^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{q}\left(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\Bigg],$$

$$I_{20_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}=-\frac{\Delta^2}{4}I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}-\Delta I_{10_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}+\frac{\Delta^3}{8}\Bigg[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{q}\left(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\Bigg],$$

$$I_{11_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}=-\frac{\Delta^2}{4}I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}-\frac{\Delta}{2}\left(I_{10_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}+I_{01_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}\right)+\frac{\Delta^3}{8}\Bigg[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{q}\left(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+\right.$$

$$\left.+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\right)\Bigg],$$

$$I_{0001_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{0010_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{0100_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{1000_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3)q}=\sum_{j_1,j_2,j_3,j_4=0}^{q}C_{j_4j_3j_2j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)},$$

$$I_{000000_{\tau_{p+1},\tau_p}}^{*(i_1i_2i_3i_4i_5i_6)q}=\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_5}^{(i_5)},$$

where formulas for the Fourier–Legendre coefficients

$$C_{j_3j_2j_1},\ C_{j_4j_3j_2j_1},\ C_{j_3j_2j_1}^{001},\ C_{j_3j_2j_1}^{010},\ C_{j_3j_2j_1}^{100},\ C_{j_5j_4j_3j_2j_1}, C_{j_4j_3j_2j_1}^{0001},\ C_{j_4j_3j_2j_1}^{0010},$$

$$C_{j_4j_3j_2j_1}^{0100},\ C_{j_4j_3j_2j_1}^{1000},\ C_{j_6j_5j_4j_3j_2j_1}$$

can be found in Sect. 3.

On the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $\tau_{p+1}-\tau_p$ $(\tau_{p+1}-\tau_p \ll 1)$ in the mean-square sense for iterated Stratonovich stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals (see the numbers q in the approximations of iterated Stratonovich stochastic integrals from this section), which are required for achieving the acceptable accuracy of approximation.

From (29) $(i_1\ne i_2)$ we have

$$\mathsf{M}\left\{\left(I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)}-I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}\right)^2\right\}=\frac{\Delta^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1}\le$$

$$\le\frac{\Delta^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx=-\frac{\Delta^2}{8}\ln\left|1-\frac{2}{2q+1}\right|\le C_1\frac{\Delta^2}{q}, \tag{46}$$

where C_1 is a constant.

Since the value $\Delta=\tau_{p+1}-\tau_p$ plays the role of integration step in the numerical scheme (33), then this value is a sufficiently small.

Keeping in mind this circumstance, it is easy to notice that there exists such a constant C_2 that

$$\mathsf{M}\left\{\left(I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}-I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C_2\mathsf{M}\left\{\left(I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)}-I_{00_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}\right)^2\right\}, \tag{47}$$

where $I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}$ is an approximation of the iterated Stratonovich stochastic integral $I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}$.

From (46) and (47) we finally obtain

$$\mathsf{M}\left\{\left(I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)}-I_{l_1\ldots l_k\,\tau_{p+1},\tau_p}^{*(i_1\ldots i_k)q}\right)^2\right\}\le C\frac{\Delta^2}{q}, \tag{48}$$

where constant C does not depend on Δ.

The same idea can be found in [2] for the case of trigonometric functions. Note that, in contrast to the estimate (48), the constant C in Theorems 7–9 does not depend on q.

Since

$$J^*[\psi^{(k)}]_{T,t}=J[\psi^{(k)}]_{T,t}\quad \text{w. p. 1}$$

for pairwise different $i_1,\ldots,i_k=1,\ldots,m$, then we can write for pairwise different $i_1,\ldots,i_6=1,\ldots,m$ (see (24))

$$\mathsf{M}\left\{\left(I_{01_{\tau_{p+1},\tau_p}}^{*(i_1i_2)}-I_{01_{\tau_{p+1},\tau_p}}^{*(i_1i_2)q}\right)^2\right\}=\frac{\Delta^4}{4}-\sum_{j_1,j_2=0}^{q}\left(C_{j_2j_1}^{01}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{10_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)} - I_{10_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)q}\right)^2\right\} = \frac{\Delta^4}{12} - \sum_{j_1,j_2=0}^{q}\left(C_{j_2 j_1}^{10}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)} - I_{000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{\Delta^3}{6} - \sum_{j_3,j_2,j_1=0}^{q} C_{j_3 j_2 j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{0000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)} - I_{0000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{\Delta^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4 j_3 j_2 j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{100_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)} - I_{100_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{\Delta^5}{60} - \sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{010_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)} - I_{010_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{\Delta^5}{20} - \sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{001_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)} - I_{001_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q}\right)^2\right\} = \frac{\Delta^5}{10} - \sum_{j_1,j_2,j_3=0}^{q}\left(C_{j_3 j_2 j_1}^{001}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{00000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4 i_5)} - I_{00000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4 i_5)q}\right)^2\right\} = \frac{\Delta^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q} C_{j_5 i_4 i_3 i_2 j_1}^2,$$

$$\mathsf{M}\left\{\left(I_{20_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)} - I_{20_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)q}\right)^2\right\} = \frac{\Delta^6}{30} - \sum_{j_2,j_1=0}^{q}\left(C_{j_2 j_1}^{20}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{11_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)} - I_{11_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)q}\right)^2\right\} = \frac{\Delta^6}{18} - \sum_{j_2,j_1=0}^{q}\left(C_{j_2 j_1}^{11}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{02_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)} - I_{02_{\tau_{p+1},\tau_p}}^{*(i_1 i_2)q}\right)^2\right\} = \frac{\Delta^6}{6} - \sum_{j_2,j_1=0}^{q}\left(C_{j_2 j_1}^{02}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{1000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)} - I_{1000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{\Delta^6}{360} - \sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{1000}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{0100_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)} - I_{0100_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{\Delta^6}{120} - \sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0100}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{0010_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)} - I_{0010_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{\Delta^6}{60} - \sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0010}\right)^2,$$

$$\mathsf{M}\left\{\left(I_{0001_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)} - I_{0001_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3 i_4)q}\right)^2\right\} = \frac{\Delta^6}{36} - \sum_{j_1,j_2,j_3,j_4=0}^{q}\left(C_{j_4 j_3 j_2 j_1}^{0001}\right)^2,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4i_5i_6)}_{000000_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3i_4i_5i_6)q}_{000000_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^6}{720}-\sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q}C^2_{j_6j_5j_4j_3j_2j_1}.$$

For example [5] (also see [18]-[28], [54]-[56])

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{000_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3)6}_{000_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^3}{6}-\sum_{j_3,j_2,j_1=0}^{6}C^2_{j_3j_2j_1}\approx 0.01956000\Delta^3,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{100_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3)2}_{100_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^5}{60}-\sum_{j_1,j_2,j_3=0}^{2}\left(C^{100}_{j_3j_2j_1}\right)^2\approx 0.00815429\Delta^5,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{010_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3)2}_{010_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^5}{20}-\sum_{j_1,j_2,j_3=0}^{2}\left(C^{010}_{j_3j_2j_1}\right)^2\approx 0.01739030\Delta^5,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3)}_{001_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3)2}_{001_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^5}{10}-\sum_{j_1,j_2,j_3=0}^{2}\left(C^{001}_{j_3j_2j_1}\right)^2\approx 0.02528010\Delta^5,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4)}_{0000_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3i_4)2}_{0000_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^4}{24}-\sum_{j_1,j_2,j_3,j_4=0}^{2}C^2_{j_4j_3j_2j_1}\approx 0.02360840\Delta^4,$$

$$\mathsf{M}\left\{\left(I^{*(i_1i_2i_3i_4i_5)}_{00000_{\tau_{p+1},\tau_p}}-I^{*(i_1i_2i_3i_4i_5)1}_{00000_{\tau_{p+1},\tau_p}}\right)^2\right\}=\frac{\Delta^5}{120}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{1}C^2_{j_5i_4i_3i_2j_1}\approx 0.00759105\Delta^5.$$

The theory presented in this article was realized [57], [60] in the form of a software package in the Python programming language. The mentioned software package implements the strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with multidimensional non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions. At that for the numerical simulation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 we applied the formulas based on multiple Fourier–Legendre series [57], [60]. Moreover, we used [57], [60] the database with 270,000 exactly calculated Fourier–Legendre coefficients.

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.
[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.
[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.

[5] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[6] Platen E., Wagner W. On a Taylor formula for a class of Ito processes. Probab. Math. Statist. 3 (1982), 37-51.

[7] Kloeden P.E., Platen E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 90 pp. [In English].

[11] Kloeden, P.E., Platen, E., Schurz, H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[12] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl. 10, 4 (1992), 431-441.

[13] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Berlin, Heidelberg, Springer-Verlag, 2010. 868 pp.

[14] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[15] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77.
Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[16] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html
Hard Cover Edition: SPbGTU Publishing House, 1998, 204 pp. (ISBN 5-7422-0045-5)

[17] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 77 pp. [in English].

[18] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), 385 pp. (A.1-A.385).
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073.
Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228
Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[23] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[25] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257.

DOI: http://doi.org/10.18720/SPBPU/2/z17-7
Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[26] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[27] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233
Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[28] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[29] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[30] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888.
Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[31] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[32] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[33] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[34] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250.
DOI: http://doi.org/10.1134/S0965542519080116

[35] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[36] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[37] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62.
Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[38] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [in English].

[39] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 68 pp. [in English].

[40] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [in English].

[41] Kuznetsov D.F. Expansions of Iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 203 pp. [in English].

[42] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method arXiv:1801.01962 [math.PR]. 2018, 49 pp. [in English].

[43] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 126 pp. [in English].

[44] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [in English].

[45] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv: 1806.10705 [math.PR]. 2018, 28 pp. [In English].

[46] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv: 1802.04844 [math.PR]. 2018, 36 pp. [in English].

[47] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 65 pp. [in English].

[48] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combained approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 46 pp. [In English].

[49] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR]. 2018, 20 pp. [In English].

[50] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 56 pp. [in English].

[51] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 28 pp. [In English].

[52] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].

[53] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 40 pp. [In English].

[54] Kuznetsov D.F. Strong approximation of iterated Itô and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Itô SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 912 pp. [In English].

[55] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[56] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[57] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. arXiv:2009.14011 [math.PR], 2020, 342 pp. [In English].

[58] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[59] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].

[60] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[61] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[62] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[63] Kuznetsov D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[64] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[65] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:2006.16040 [math.PR], 2020, 33 pp. [In English].

[66] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[67] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 36, 5 (1965), 1560-1564.

[68] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[69] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[70] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[71] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Chapter 5.

Application to the High-Order Strong Numerical Methods for Non-Commutative Semilinear SPDEs

APPLICATION OF THE METHOD OF APPROXIMATION OF ITERATED ITÔ STOCHASTIC INTEGRALS BASED ON GENERALIZED MULTIPLE FOURIER SERIES TO THE HIGH-ORDER STRONG NUMERICAL METHODS FOR NON-COMMUTATIVE SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

DMITRIY F. KUZNETSOV

ABSTRACT. We consider a method for the approximation of iterated stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$) with respect to the infinite-dimensional Q-Wiener process using the mean-square approximation method of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes based on generalized multiple Fourier series. The case of multiple Fourier–Legendre series is considered in details. The results of the article can be applied to construction of high-order strong numerical methods (with respect to the temporal discretization) for the approximation of mild solution for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise.

1. INTRODUCTION

There exists a lot of publications on the subject of numerical integration of stochastic partial differential equations (SPDEs) (see, for example, [1]-[25]). One of the perspective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for SPDEs is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [12] (2015), [13] (2016). As shown in [12] (2015) and [17] (2007) the exponential Milstein type approximation method has the strong order of convergence $1.0 - \varepsilon$ (where ε is an arbitrary small posilive real number) [12] or 1.0 [17]. In [13] the exponential Wagner–Platen type numerical method for SPDEs with strong order $1.5 - \varepsilon$ (where ε is an arbitrary small posilive real number) has been considered. An important feature of these numerical methods is a presence in them of the so-called iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [19]. Approximation of these stochastic integrals is a complex problem. This problem can be significantly simplified if special commutativity conditions be fulfilled [12], [13]. In [25] (2019) two methods of the mean-square approximation of simplest iterated (double) stochastic integrals with respect to the infinite-dimensional Q-Wiener process are considered and theorems on the convergence of these methods are given (the basic idea about Karhunen–Loeve expansion of the Brownian bridge process was taken from monograph [26] (1988, In Russian)). It is important to note that the approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be reduced to the approximation of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes. In a lot of author's publications [27]-[65] the effective methods for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals with respect to the scalar standard Wiener processes were proposed and developed. One of these methods [30] (also see [31]-[65]) is based on generalized multiple Fourier series, in particular, on multiple Fourier–Legendre series. The purpose of this article is an adaptation of the method [30]-[65] for the mean-square approximation of iterated

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stochastic integrals of multiplicity k ($k \in \mathbb{N}$) with respect to the finite-dimensional approximation of the infinite-dimensional Q-Wiener process.

Let U, H be separable $\mathbb{R}$-Hilbert spaces and $L_{HS}(U,H)$ be a space of Hilbert–Schmidt operators mapping from U to H. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0,\bar{T}]\}$ [19], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0,\bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here $L(U)$ denotes all bounded linear operators mapping from U to U. Consider the semilinear parabolic SPDE

$$dX_t = (AX_t + F(X_t))\,dt + B(X_t)d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0,\bar{T}], \tag{1}$$

where nonlinear operators F, B ($F : H \to H$, $B : H \to L_{HS}(U_0, H)$), linear operator $A : D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the SPDE (1) mild solution [22] (see also [12], [13]). Here U_0 is an $\mathbb{R}$-Hilbert space defined by $U_0 = Q^{1/2}(U)$. The scalar product in U_0 is defined as follows $\langle u, w\rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w\right\rangle_U$ for all $u, w \in U_0$.

As it is known, strong numerical methods with high-orders of accuracy (with respect to the temporal discretization) for approximating the mild solution of the SPDE (1), which are based on the Taylor formula in Banach spaces and an exponential formula for the mild solution of SPDEs, contain iterated stochastic integrals with respect to the Q-Wiener process [8], [10]-[13], [17].

Note that the exponential Milstein type numerical scheme [12], [17], [24] and exponential Wagner–Platen type numerical scheme [13] contain, for example, the following iterated stochastic integrals

$$\int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \quad \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{2}$$

$$\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \quad \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \tag{3}$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} F(Z)dt_1\right)d\mathbf{W}_{t_2}, \quad \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{4}$$

where $0 \le t < T \le \bar{T}$, $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and F', B', B'' denote Frêchet derivatives. At that, the exponential Milstein type scheme [12] contains integrals (2) while the exponential Wagner–Platen type scheme [13] contains integrals (2)–(4). It is easy to notice that the numerical schemes for SPDEs with higher orders of convergence (with respect to the temporal discretization) in contrast with numerical schemes from [12], [13] will include iterated stochastic integrals (with respect to the Q-Wiener process) with multiplicities $k > 3$ [21] (2012). So, this work is partially devoted to the approximation of iterated stochastic integrals of the form

$$I[\Phi^{(k)}(Z)]_{T,t} = \int\limits_t^T \Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3} \Phi_2(Z)\left(\int\limits_t^{t_2} \Phi_1(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)\ldots\right)d\mathbf{W}_{t_k}, \tag{5}$$

where $Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v))\ldots\ ))$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0 \times \ldots \times U_0}_{k\ \text{times}}$ to H for all $v \in H$, and $0 \le t < T \le \bar{T}$.

In Sect. 5 we consider the approximation of more general iterated stochastic integrals than (5). In Sect. 6, 7 some other types of iterated stochastic integrals of multiplicities 2–4 with respect to the Q-Wiener process will be considered. In this paper, in all the integrals mentioned above, the infinite-dimensional Q-Wiener process will be replaced by its finite-dimensional approximation. In [59]-[61], (also see [44], Chapter 7) one can find a continuation of the studies begun in this work. In [44], [59]-[61] we consider the approximation of iterated stochastic integrals (2)–(4) with respect to the infinite-dimensional Q-Wiener process.

Note that the second stochastic integral in (4) is not a special case of the stochastic integral (5) for $k = 3$. Nevertheless, the expanded representation of the approximation of stochastic integral (4) has a close structure to (9) for $k = 3$ (see below). Moreover, the mentioned representation of stochastic integral (4) contains the same iterated Itô stochastic integrals of third multiplicity as in (9) for $k = 3$ (see Sect. 6). These conclusions mean that the main result of this article (Theorem 4, Sect. 5) for $k = 3$ can be reformulated naturally for the stochastic integral (4) (see Sect. 6).

It should be noted that by developing an approach from the work [13], which uses the Taylor formula in Banach spaces and a formula for the mild solution of the SPDE (1), we obviously obtain a number of other iterated stochastic integrals with respect to the Q-Wiener process. For example, the following stochastic integrals

$$\int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) d\mathbf{W}_{t_2},$$

$$\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) d\mathbf{W}_{t_2}\right) d\mathbf{W}_{t_3},$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_3} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) d\mathbf{W}_{t_2}\right) d\mathbf{W}_{t_3},$$

$$\int\limits_t^T F'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) d\mathbf{W}_{t_2}\right) dt_3,$$

$$\int\limits_t^T F''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) dt_2,$$

$$\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} F(Z)dt_1, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right) d\mathbf{W}_{t_2}$$

will be considered in Sect. 7. Here $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping and B', B'', B''', F', F'' are Frêchet derivatives.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i = (i_1,\ldots,i_d)$ $\in J$, $x = (x_1,\ldots,x_d)$, and $J = \{i:\ i\in\mathbb{N}^d,\ \text{and}\ \lambda_i > 0\}$.

The series representation of the Q-Wiener process has the following form [19]

$$\mathbf{W}(t,x) = \sum_{i\in J} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar{T}],$$

or in the shorter notations

$$\mathbf{W}_t=\sum_{i\in J} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar T],$$

where $\mathbf{w}_t^{(i)}$, $i\in J$ are independent standard Wiener processes. Note that eigenfunctions e_i, $i\in J$ form an orthonormal basis of U [19].

Consider the finite-dimensional approximation of $\mathbf{W}_t$ [19]

$$\mathbf{W}_t^M=\sum_{i\in J_M} e_i\sqrt{\lambda_i}\mathbf{w}_t^{(i)},\quad t\in[0,\bar T], \tag{6}$$

where $J_M=\{i:\ 1\le i_1,\ldots,i_d\le M,\ \text{and}\ \lambda_i>0\}$.

Using (6) and the relation [19]

$$\mathbf{w}_t^{(i)}=\frac{1}{\sqrt{\lambda_i}}\langle e_i,\mathbf{W}_t\rangle_U,\quad i\in J, \tag{7}$$

we obtain

$$\mathbf{W}_t^M=\sum_{i\in J_M} e_i\ \langle e_i,\mathbf{W}_t\rangle_U,\quad t\in[0,\bar T], \tag{8}$$

where $\langle\cdot,\cdot\rangle_U$ is a scalar product in U.

Taking into account (7), (8), we note that the approximation $I[\Phi^{(k)}(Z)]_{T,t}^M$ of iterated stochastic integral $I[\Phi^{(k)}(Z)]_{T,t}$ (see (5)) can be rewritten with probability 1 (further w. p. 1) in the following form

$$I[\Phi^{(k)}(Z)]_{T,t}^M=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\left(\int\limits_t^{t_2}\Phi_1(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)\ldots\right)d\mathbf{W}_{t_k}^M=$$

$$=\sum_{r_1,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\times$$

$$\times\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2}d\langle e_{r_1},\mathbf{W}_{t_1}\rangle_U\ d\langle e_{r_2},\mathbf{W}_{t_2}\rangle_U\ \ldots\ d\langle e_{r_k},\mathbf{W}_{t_k}\rangle_U=$$

$$=\sum_{r_1,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\times$$

$$\times\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}, \tag{9}$$

where $0\le t<T\le\bar T$.

Remark 1. *Obviously, without the loss of generality we can write* $J_M=\{1,2,\ldots,M\}$.

When special conditions of commutativity for SPDEs in the form (1) be fulfilled it is proposed to simulate numerically the stochastic integrals (2)–(4) using the simple formulas [12], [13]. In this case, the numerical simulation of mentioned stochastic integrals requires the use of increments of the Q-Wiener process only. However, if these commutativity conditions are not fulfilled (which often corresponds to SPDEs in numerous applications), the numerical simulation of stochastic integrals (2)–(4) becomes much more difficult. In [25] two methods for the mean-square approximation of simplest iterated (double) stochastic integrals with respect to the Q-Wiener process are proposed. In this article, we consider a substantially more general and effective method for the mean-square approximation of iterated stochastic integrals of multiplicity k ($k \in \mathbb{N}$) with respect to the Q-Wiener process. The convergence analysis in the transition from J_M to J, i.e. from $\mathbf{W}_t^M$ to $\mathbf{W}t$ is carried out in [44] (Sect.7.4.2), [45] (Sect.7.4.2), [46], [59], [60] for stochastic integrals of multiplicity k ($k = 1, 2, 3$) with respect to the Q-Wiener process (the cases $k = 1, 2$ is considered in Theorem 1 from [25]).

The monographs [43] (Chapters 5 and 6) and [44] or [45], [46] (Chapters 1, 2, and 5) (also see [30]-[42], [47]-[58]) are devoted to constructing of efficient methods of the mean-square approximation of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes. These results are adapted for iterated Stratonovich stochastic integrals [27]-[58]. Below (Sect. 2–4) we consider a very short review of results from monographs [43] (Chapters 5 and 6) and [44] or [45], [46] (Chapters 1, 2, and 5) and some new results (Sect. 5–7).

2. Method of Approximation of Iterated Itô Stochastic integrals Based on Generalized Multiple Fourier Series

Consider more general iterated Itô stochastic integrals than in (9)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{10}$$

where $0 \le t < T \le \bar{T}$ and every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a continuous non-random function on $[t, T]$; $\mathbf{w}_\tau^{(i)}$ ($i = 1, \ldots, m$) are independent standard Wiener processes (see Sect. 1) and $\mathbf{w}_\tau^{(0)} = \tau$; $i_1, \ldots, i_k = 0,\ 1, \ldots, m$. The case $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$ will be considered in Theorem 2 (see below).

Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in $L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k),\ t_1 < \ldots < t_k \\ \\ 0,\ \text{otherwise} \end{cases} = \prod_{l=1}^k \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}}, \tag{11}$$

where $t_1, \ldots, t_k \in [t, T]$ for $k \ge 2$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$. Here $\mathbf{1}_A$ is the indicator of the set A.

The function $K(t_1, \ldots, t_k)$ is piecewise continuous on the hypercube $[t, T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ converges to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0, \tag{12}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{13}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{14}$$

Theorem 1 [30] (2006) (also see [31]-[60]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\right.$$

$$\left.-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \text{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{15}$$

where

$$\text{G}_k=\text{H}_k\backslash\text{L}_k;\ \text{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\text{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{16}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (13), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is the discretization of* $[t,T]$, *which satisfies the condition* (14).

Note that in [30]-[57] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Another modifications and generalizations of Theorem 1 can be found in the monographs [44]-[46] (also see Theorem 2 below).

It is not difficult to see that for the case of pairwise different numbers $i_1,\ldots,i_k=1,\ldots,m$ from Theorem 1 we obtain

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}.$$

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [30]-[58] (the cases $k = 7$ and $k > 7$ can be found in [34], [39], [43]-[46])

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{17}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{18}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{19}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{20}$$

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{21}
$$

$$
\begin{aligned}
J[\psi^{(6)}]_{T,t}^{(i_1\ldots i_6)}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-&\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-&\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-&\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-&\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-&\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-&\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-&\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-&\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+&\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+&\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+&\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+&\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+&\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+&\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+&\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+&
\end{aligned}
$$

$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{22}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider the generalization of (17)–(22) for the case of an arbitrary k ($k\in\mathbb{N}$) as well as for the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{23}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (23) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{24}$$

Below there are several examples of sums in the form (24)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$

$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$

$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can formulate the following generalization of Theorem 1.

Theorem 2 [44] (Sect. 1.11), [47] (Sect. 15). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(25)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

In particular, from (25) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\sum_{j_1,\ldots,j_5=0}^{\infty}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}}\mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (21). Note that the correctness of formulas (17)–(22) can be verified by the fact that if $i_1 = \ldots = i_6 = i = 1, \ldots, m$ and $\psi_1(s), \ldots, \psi_6(s) \equiv \psi(s)$, then we can derive from (17)–(22) the well known equalities

$$J[\psi^{(1)}]_{T,t}^{(i)} = \frac{1}{1!}\delta_{T,t}^{(i)},$$

$$J[\psi^{(2)}]_{T,t}^{(ii)} = \frac{1}{2!}\left(\left(\delta_{T,t}^{(i)}\right)^2 - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}^{(iii)} = \frac{1}{3!}\left(\left(\delta_{T,t}^{(i)}\right)^3 - 3\delta_{T,t}^{(i)}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}^{(iiii)} = \frac{1}{4!}\left(\left(\delta_{T,t}^{(i)}\right)^4 - 6\left(\delta_{T,t}^{(i)}\right)^2\Delta_{T,t} + 3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}^{(iiiii)} = \frac{1}{5!}\left(\left(\delta_{T,t}^{(i)}\right)^5 - 10\left(\delta_{T,t}^{(i)}\right)^3\Delta_{T,t} + 15\delta_{T,t}^{(i)}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t}^{(iiiiii)} = \frac{1}{6!}\left(\left(\delta_{T,t}^{(i)}\right)^6 - 15\left(\delta_{T,t}^{(i)}\right)^4\Delta_{T,t} + 45\left(\delta_{T,t}^{(i)}\right)^2\Delta_{T,t}^2 - 15\Delta_{T,t}^3\right)$$

w. p. 1 [31]-[43], where

$$\delta_{T,t}^{(i)} = \int\limits_t^T \psi(s)d\mathbf{w}_s^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(s)ds.$$

The above equalities can be independently obtained using the Itô formula and Hermite polynomials [66].

3. Calculation of the Mean-Square Approximation Error of Iterated Itô Stochastic Integrals in Theorems 1, 2

Assume that $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1\ldots p_k}$ is an approximation of (10), which is the expression on the right-hand side of (25) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$. Let us denote

$$E^{(i_1\ldots i_k)p_1,\ldots,p_k} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}\right)^2\right\},$$

$$E^{(i_1\ldots i_k)p} = E_k^{(i_1\ldots i_k)p_1,\ldots,p_k}\Big|_{p_1=\ldots=p_k=p},$$

$$I_k = \|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{26}$$

In [39]-[46], [55]-[57] it was shown that

$$E_k^{(i_1\ldots i_k)p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right), \tag{27}$$

where $i_1,\ldots,i_k=1,\ldots,m$ for $0<T-t<\infty$ and $i_1,\ldots,i_k=0,1,\ldots,m$ for $0<T-t<1$. Note that the estimate (27) is valid under the conditions of Theorem 2.

The exact calcutation of $E^{(i_1\ldots i_k)p}$ is presented in the following theorem.

Theorem 3 [44] (Sect. 1.12). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. Then*

$$E^{(i_1\ldots i_k)p}=I_k-$$

$$-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\}, \tag{28}$$

where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p}$ is the expression on the right-hand side of (25) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for $p_1=\ldots=p_k=p$; the expression

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in Theorems 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}$$

for $i_1\ldots i_k=1,\ldots,m$.

Then from Theorem 3 for $i_1,\ldots,i_k=1,\ldots,m$ we obtain [40], [42]-[46]

$$E^{(i_1\ldots i_k)p}=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2\quad(\text{pairwise different } i_1,\ldots,i_k), \tag{29}$$

$$E^{(i_1i_2)p}=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}\quad(i_1=i_2),$$

$$E^{(i_1i_2i_3)p}=I_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}\quad(i_1=i_2\neq i_3),$$

$$E^{(i_1i_2i_3i_4)p}=I_4-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2)}C_{j_4j_3j_2j_1}\right)\right)\quad(i_1=i_2\neq i_3=i_4),$$

$$E^{(i_1 i_2 i_3 i_4 i_5)p} = I_5 - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{p} C_{j_5 j_4 j_3 j_2 j_1} \left(\sum_{(j_3,j_4)} \left(\sum_{(j_1,j_2,j_5)} C_{j_5 j_4 j_3 j_2 j_1} \right)\right)$$

$$(i_1 = i_2 = i_5 \neq i_3 = i_4).$$

4. Some Examples of the Mean-Square Approximations of Iterated Itô Stochastic Integrals Using Legendre Polynomials

Denote

$$I_{(1)T,t}^{(i_1)} = \int\limits_t^T d\mathbf{w}_{t_1}^{(i_1)},$$

$$I_{(10)T,t}^{(i_1 0)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2, \quad I_{(01)T,t}^{(0 i_2)} = \int\limits_t^T \int\limits_t^{t_2} dt_1 d\mathbf{w}_{t_2}^{(i_2)},$$

$$I_{(11)T,t}^{(i_1 i_2)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, \quad I_{(111)T,t}^{(i_1 i_2 i_3)} = \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)},$$

$$I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)} = \int\limits_t^T \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)},$$

$$I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)} = \int\limits_t^T \int\limits_t^{t_5} \int\limits_t^{t_4} \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)},$$

where $i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m$.

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2} \right) \frac{2}{T-t} \right); \ j = 0, 1, 2, \ldots, \tag{30}$$

where $P_j(x)$ is the Legendre polynomial.

Using the system of functions (30) and Theorems 1, 2 we obtain the following approximations of iterated Itô stochastic integrals [27]-[65]

$$I_{(1)T,t}^{(i_1)} = \sqrt{T-t}\,\zeta_0^{(i_1)},$$

$$I_{(01)T,t}^{(0 i_1)} = \frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right), \tag{31}$$

$$I_{(10)T,t}^{(i_1 0)} = \frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right), \tag{32}$$

$$I_{(11)T,t}^{(i_1 i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{33}$$

$$I_{(111)T,t}^{(i_1 i_1 i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right),$$

$$I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)q_2}=\sum_{j_1,j_2,j_3,j_4=0}^{q_2}C_{j_4 j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{34}$$

$$I_{(1111)T,t}^{(i_1 i_1 i_1 i_1)}=\frac{1}{24}(T-t)^2\left(\left(\zeta_0^{(i_1)}\right)^4-6\left(\zeta_0^{(i_1)}\right)^2+3\right),$$

$$I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_3}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_3}C_{j_5 j_4 j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$
$$-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$
$$+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$
$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$
$$+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{35}
$$

$$
I_{(11111)T,t}^{(i_1i_1i_1i_1i_1)}=\frac{1}{120}(T-t)^{5/2}\left(\left(\zeta_0^{(i_1)}\right)^5-10\left(\zeta_0^{(i_1)}\right)^3+15\zeta_0^{(i_1)}\right),
$$

$$
C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3j_2j_1},
$$

$$
C_{j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}(T-t)^{2}}{16}\bar{C}_{j_4j_3j_2j_1},
$$

$$
C_{j_5j_4j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}(T-t)^{5/2}}{32}\bar{C}_{j_5j_4j_3j_2j_1},
$$

$$
\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,
$$

$$
\bar{C}_{j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdu,
$$

$$
\bar{C}_{j_5j_4j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_5}(v)\int\limits_{-1}^{v}P_{j_4}(u)\int\limits_{-1}^{u}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydzdudv,
$$

random variables $\zeta_j^{(i)}$ are defined by (16), and

$$
I_{(11)T,t}^{(i_1i_2)}=\underset{q\to\infty}{\text{l.i.m.}}\ I_{(11)T,t}^{(i_1i_2)q},\quad I_{(111)T,t}^{(i_1i_2i_3)}=\underset{q_1\to\infty}{\text{l.i.m.}}\ I_{(111)T,t}^{(i_1i_2i_3)q_1},
$$

$$
I_{(1111)T,t}^{(i_1i_2i_3i_4)}=\underset{q_2\to\infty}{\text{l.i.m.}}\ I_{(1111)T,t}^{(i_1i_2i_3i_4)q_2},\quad I_{(11111)T,t}^{(i_1i_2i_3i_4i_5)}=\underset{q_3\to\infty}{\text{l.i.m.}}\ I_{(11111)T,t}^{(i_1i_2i_3i_4i_5)q_3}.
$$

Note that $T-t\ll 1$ ($T-t$ is an integration step with respect to the temporal variable). Thus $q_1\ll q$ (see Table 1 [30]-[39], [42]-[46]). Moreover, the values $\bar{C}_{j_3j_2j_1}$, $\bar{C}_{j_4j_3j_2j_1}$, $\bar{C}_{j_5j_4j_3j_2j_1}$ do not depend on $T-t$. This feature is important because we can use a variable integration step $T-t$. Coefficients $\bar{C}_{j_3j_2j_1}$, $\bar{C}_{j_4j_3j_2j_1}$, $\bar{C}_{j_5j_4j_3j_2j_1}$ are calculated once and before the start of the numerical scheme. Some examples of the exact calculation of coefficients $\bar{C}_{j_3j_2j_1}$, $\bar{C}_{j_4j_3j_2j_1}$, $\bar{C}_{j_5j_4j_3j_2j_1}$ via Python programming language can be found in Tables 2–4 (the database with 270,000 exactly calculated Fourier–Legendre coefficients was described in [62], [63]).

Denote

Table 1. Minimal numbers q, q_1 such that $E^{(i_1 i_2)q}$, $E^{(i_1 i_2 i_3)q_1} \le (T-t)^4$, $\quad q_1 \ll q$.

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 2. Coefficients $\bar{C}_{3jk}$.

$j \backslash k$	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

Table 3. Coefficients $\bar{C}_{21kl}$.

$k \backslash l$	0	1	2
0	$\frac{2}{21}$	$-\frac{2}{45}$	$\frac{2}{315}$
1	$\frac{2}{315}$	$\frac{2}{315}$	$-\frac{2}{225}$
2	$-\frac{2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 4. Coefficients $\bar{C}_{101lr}$.

$l \backslash r$	0	1
0	$\frac{4}{315}$	0
1	$\frac{4}{315}$	$-\frac{8}{945}$

$$E^{(i_1 i_2)q} = \mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)} - I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3)q_1} = \mathsf{M}\left\{\left(I_{(111)T,t}^{(i_1 i_2 i_3)} - I_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3 i_4)q_2} = \mathsf{M}\left\{\left(I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)} - I_{(1111)T,t}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3} = \mathsf{M}\left\{\left(I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)} - I_{(11111)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_3}\right)^2\right\}.$$

Then for pairwise different $i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m$ from Theorem 3 we obtain [27]-[65]

$$E^{(i_1 i_2)q} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2-1}\right), \tag{36}$$

$$E^{(i_1 i_2 i_3)q_1} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^2, \tag{37}$$

$$E^{(i_1 i_2 i_3 i_4)q_2} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C_{j_4 j_3 j_2 j_1}^2, \tag{38}$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3} = \frac{(T-t)^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_3} C_{j_5 j_4 j_3 j_2 j_1}^2. \tag{39}$$

On the basis of the presented approximations of iterated Itô stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ $(T-t \ll 1)$ in the mean-square sense for iterated Itô stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Itô stochastic integrals, which are required for achieving the acceptable accuracy of approximation $(q_1 \ll q)$.

From (37)–(39) we obtain [30]-[39], [42]-[46]

$$E^{(i_1 i_2 i_3)q_1}\Big|_{q_1=6} \approx 0.01956000(T-t)^3, \tag{40}$$

$$E^{(i_1 i_2 i_3 i_4)q_2}\Big|_{q_2=2} \approx 0.02360840(T-t)^4, \tag{41}$$

$$E^{(i_1 i_2 i_3 i_4 i_5)q_3}\Big|_{q_3=1} \approx 0.00759105(T-t)^5. \tag{42}$$

It is not difficult to see that the accuracy in (41) and (42) is significantly better than in (40) $(T-t \ll 1)$ even for $q_2 = 2$ and $q_3 = 1$. This means that in such situation in formulas (34), (35) the number of terms can be chosen significantly less than 3^4 $(q_2 = 2)$ and 2^5 $(q_3 = 1)$. So, in practice, we can leave only few terms in these formulas. For more details see [62]-[65].

5. Approximation of Iterated Stochastic Integrals of Multiplicity k with Respect to the Q-Wiener Process

Consider the iterated stochastic integral with respect to the Q-Wiener process in the form

$$I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} =$$

$$= \int_t^T \Phi_k(Z)\left(\ldots\left(\int_t^{t_3} \Phi_2(Z)\left(\int_t^{t_2} \Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\right)\psi_2(t_2)d\mathbf{W}_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}, \tag{43}$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v\in H$, and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}$ be an approximation of the stochastic integral (43)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}=$$

$$=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}^M\right)\psi_2(t_2)d\mathbf{W}_{t_2}^M\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}^M=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\ J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)},\tag{44}$$

where $0\le t<T\le\bar{T}$, and

$$J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated Itô stochastic integral (10), $r_1,r_2,\ldots,r_k\in J_M$.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}$ be an approximation of the stochastic integral (44)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\ J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k},\tag{45}$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as a prelimit expression on the right-hand side of (25)

$$J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^k\zeta_{j_l}^{(r_l)}+\sum_{m=1}^{[k/2]}(-1)^m\times\right.$$

$$\left.\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^m\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}\right).\tag{46}$$

Let U, H be separable $\mathbb{R}$-Hilbert spaces, $U_0 = Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let $L(U,H)_0 = \{T|_{U_0}:\ T\in L(U,H)\}$ (here $T|_{U_0}$ is the restriction of operator T to the space U_0). It is known [7] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 4 [44]-[46], [59], [60], [68]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Furthermore, let the following conditions be satisfied*:

1. *$Q\in L(U)$ is a nonnegative and symmetric trace class operator (λ_i and e_i ($i\in J$) are its eigenvalues and eigenfunctions (which form an orthonormal basis of U) correspondingly), and $\mathbf{W}_\tau$, $\tau\in[0,\bar{T}]$ is an U-valued Q-Wiener process.*
2. *$Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping.*
3. *$\Phi_1\in L(U,H)_0$, $\Phi_2\in L(H,L(U,H)_0)$, and $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ times}$ to H for all $v\in H$ such that*

$$
\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H^2\le L_k<\infty
$$

w. p. 1 for all $r_1,r_2,\ldots,r_k\in J_M$, $M\in\mathbb{N}$.

Then

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}-I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots p_k}\right\|_H^2\right\}\le
$$

$$
\le L_k(k!)^2\left(\mathrm{tr}\ Q\right)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right), \tag{47}
$$

where I_k is defined by (26)*,* $\mathrm{tr}\ Q=\sum\limits_{i\in J}\lambda_i$*, and*

$$
C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k,
$$

$$
K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ \\ 0,\ \text{otherwise}\end{cases}.
$$

Remark 2. *It should be noted that the right-hand side of the inequality* (47) *is independent of M and tends to zero if $p_1,\ldots,p_k\to\infty$ due to the Parseval equality.*

Proof. Using (27), we obtain

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}-I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots p_k}\right\|_H^2\right\}=
$$

$$= \mathsf{M}\left\{\left\|\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\times\right.\right.$$

$$\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\right\|_H^2\right\}= \tag{48}$$

$$=\left|\mathsf{M}\left\{\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}}\left\langle\ \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\ ,\right.\right.\right.$$

$$\left.\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{'}}\right)e_{r_2^{'}}\right)\ldots\right)e_{r_k^{'}}\ \right\rangle_H\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\sqrt{\lambda_{r_1^{'}}\lambda_{r_2^{'}}\ldots\lambda_{r_k^{'}}}\times$$

$$\times\ \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.$$

$$\left.\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})}-J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right\}\right|\le \tag{49}$$

$$\le\sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{'},r_2^{'},\ldots,r_k^{'}):\ \{r_1^{'},r_2^{'},\ldots,r_k^{'}\}=\{r_1,r_2,\ldots,r_k\}}\mathsf{M}\left\{\left\|\ \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H\times\right.$$

$$\times\left\|\ \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1^{'}}\right)e_{r_2^{'}}\right)\ldots\right)e_{r_k^{'}}\ \right\|_H\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\sqrt{\lambda_{r_1^{'}}\lambda_{r_2^{'}}\ldots\lambda_{r_k^{'}}}\times$$

$$\times\left|\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)\times\right.\right.$$

$$\left.\left.\left.\times\left(J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})}-J[\psi^{(k)}]_{T,t}^{(r_1^{'}r_2^{'}\ldots r_k^{'})p_1,\ldots,p_k}\right)\right|\mathbf{F}_t\right\}\right|\right\}\le$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\sqrt{\lambda_{r_1^{\prime}}\lambda_{r_2^{\prime}}\ldots\lambda_{r_k^{\prime}}}\times
$$

$$
\times\mathsf{M}\Biggl\{\Biggl|\Biggl(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\Biggr)\times
$$

$$
\times\Biggl(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\Biggr)\Biggr|\Biggr\}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\sqrt{\lambda_{r_1^{\prime}}\lambda_{r_2^{\prime}}\ldots\lambda_{r_k^{\prime}}}\times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})}-J[\psi^{(k)}]_{T,t}^{(r_1^{\prime}r_2^{\prime}\ldots r_k^{\prime})p_1,\ldots,p_k}\right)^2\right\}\right)^{1/2}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M}\ \sum_{(r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}):\ \{r_1^{\prime},r_2^{\prime},\ldots,r_k^{\prime}\}=\{r_1,r_2,\ldots,r_k\}} \sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\sqrt{\lambda_{r_1^{\prime}}\lambda_{r_2^{\prime}}\ldots\lambda_{r_k^{\prime}}}\times
$$

$$
\times\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)^{1/2}\le
$$

$$
\le L_k \sum_{r_1,r_2,\ldots,r_k\in J_M} k!\ \lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\right)=
$$

$$
=L_k\,(k!)^2\sum_{r_1,r_2,\ldots,r_k\in J_M}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\le
$$

$$\leq L_k\,(k!)^2\,(\text{tr}\ Q)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right),$$

where $\langle\cdot,\cdot\rangle_H$ is a scalar product in H, and

$$\sum_{(r_1^{\backprime},r_2^{\backprime},\ldots,r_k^{\backprime}):\ \{r_1^{\backprime},r_2^{\backprime},\ldots,r_k^{\backprime}\}=\{r_1,r_2,\ldots,r_k\}}$$

means the sum with respect to all possible permutations $(r_1^{\backprime},r_2^{\backprime},\ldots,r_k^{\backprime})$ such that

$$\{r_1^{\backprime},r_2^{\backprime},\ldots,r_k^{\backprime}\}=\{r_1,r_2,\ldots,r_k\}.$$

The transition from (48) to (49) is based on the following theorem.

Theorem 5 [44]-[46], [68]. *The following equality is true*

$$\mathsf{M}\Biggl\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\right)\times$$

$$\times\left(J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}-J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1\ldots p_k}\right)\Biggl|\mathbf{F}_t\Biggr\}=0 \tag{50}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ *such that* $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

Proof. Using the standard moment properties of the Itô stochastic integral, we obtain

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\Biggl|\mathbf{F}_t\right\}=0 \tag{51}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ such that $(r_1,\ldots,r_k)\neq(m_1,\ldots,m_k)$, $M\in\mathbb{N}$.

From the proof of Theorem 1.18 in [44] (Sect. 1.12) it follows that

$$\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}+\sum_{m=1}^{[k/2]}(-1)^m\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{m}\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\quad \text{w. p. 1,} \tag{52}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_l swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then r_l swapped with r_q in the permutation $(r_1,\ldots,r_k)$; another notations are the same as in Theorem 2.

Using (25) and (52), we get

$$J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}, \tag{53}$$

where notations are the same as in (52).

Then w. p. 1

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\Bigg|\mathbf{F}_t\right\}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\Bigg|\mathbf{F}_t\right\}.$$

From the standard moment properties of the Itô stochastic integral it follows that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\Bigg|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$, $M\in\mathbb{N}$.

Then

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)}J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\Bigg|\mathbf{F}_t\right\}=0 \tag{54}$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ such that $\{r_1,\ldots,r_k\}\ne\{m_1,\ldots,m_k\}$.

From (53) it follows that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1,\ldots,p_k}J[\psi^{(k)}]_{T,t}^{(m_1\ldots m_k)p_1,\ldots,p_k}\Big|\mathbf{F}_t\right\}=$$

$$=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\sum_{q_1=0}^{p_1}\ldots\sum_{q_k=0}^{p_k}C_{q_k\ldots q_1}\times$$

$$\times\mathsf{M}\left\{\left(\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(r_1)}\ldots d\mathbf{w}_{t_k}^{(r_k)}\right)\times\right.$$

$$\left.\times\left(\sum_{(q_1,\ldots,q_k)}\int\limits_t^T\phi_{q_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{q_1}(t_1)d\mathbf{w}_{t_1}^{(m_1)}\ldots d\mathbf{w}_{t_k}^{(m_k)}\right)\Big|\mathbf{F}_t\right\}=0 \qquad (55)$$

w. p. 1 for all $r_1,\ldots,r_k,m_1,\ldots,m_k\in J_M$ $(M\in\mathbb{N})$ such that $\{r_1,\ldots,r_k\}\neq\{m_1,\ldots,m_k\}$.

From (51), (54), and (55) we obtain (50). Theorem 5 is proved.

Corollary 1 [44]-[46], [68]. *The following equality is true*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)}-J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k}\right)\left(J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)}-J[\psi^{(l)}]_{T,t}^{(m_1\ldots m_l)q_1\ldots q_l}\right)\Big|\mathbf{F}_t\right\}=0$$

w. p. 1 for all $l=1,2,\ldots,k-1,$ *and* $r_1,\ldots,r_k,m_1,\ldots,m_l\in J_M,$ $p_1,\ldots,p_k,q_1,\ldots,q_l=0,1,2,\ldots$

6. Approximation of Some Iterated Stochastic Integrals of Second and Third Miltiplicity with Respect to the Q-Wiener Process

This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 1)

$$I_0[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2}F(Z)dt_1\right)d\mathbf{W}_{t_2}^M, \qquad (56)$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)dt_2, \qquad (57)$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M. \qquad (58)$$

Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v),B(v))$ be a 3-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0$ to H for all $v\in H$.

Then we have w. p. 1 (see (44))

$$I_0[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M}B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}I_{(01)T,t}^{(0r_1)},\tag{59}$$

$$I_1[B(Z),F(Z)]_{T,t}^M=\sum_{r_1\in J_M}F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}I_{(10)T,t}^{(r_10)},\tag{60}$$

$$I_2[B(Z)]_{T,t}^M=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$
$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}.\tag{61}$$

Using the Itô formula, we obtain

$$\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}=I_{(11)s,t}^{(r_1r_2)}+I_{(11)s,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}(s-t)\quad\text{w. p. 1.}\tag{62}$$

From (62) we have

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)}\int\limits_t^s d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}=I_{(111)T,t}^{(r_1r_2r_3)}+I_{(111)T,t}^{(r_2r_1r_3)}+\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(0r_3)}\quad\text{w. p. 1.}\tag{63}$$

Note that in (59), (60), (62), and (63) we use the notations from Sect. 4.

After substituting (63) into (61), we have

$$I_2[B(Z)]_{T,t}^M=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$
$$\times\left(I_{(111)T,t}^{(r_1r_2r_3)}+I_{(111)T,t}^{(r_2r_1r_3)}+\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(0r_3)}\right)\quad\text{w. p. 1.}\tag{64}$$

Taking into account (31) and (32), we put for $q=1$

$$I_{(01)T,t}^{(0r_3)q}=I_{(01)T,t}^{(0r_3)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(r_3)}\right)\quad(q=1)\quad\text{w. p. 1,}\tag{65}$$

$$I_{(10)T,t}^{(r_10)q}=I_{(10)T,t}^{(r_10)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(r_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right)\quad(q=1)\quad\text{w. p. 1,}\tag{66}$$

where $I_{(01)T,t}^{(0r_3)q}$, $I_{(10)T,t}^{(r_10)q}$ denote the approximations of corresponding iterated Itô stochastic integrals.

Denote by $I_0[B(Z),F(Z)]_{T,t}^{M,q}$, $I_1[B(Z),F(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$ the approximations of iterated stochastic integrals (59), (60), (64)

$$I_0[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1\in J_M}B'(Z)F(Z)e_{r_1}\sqrt{\lambda_{r_1}}I_{(01)T,t}^{(0r_1)q}, \tag{67}$$

$$I_1[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1\in J_M}F'(Z)(B(Z)e_{r_1})\sqrt{\lambda_{r_1}}I_{(10)T,t}^{(r_10)q}, \tag{68}$$

$$I_2[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(I_{(111)T,t}^{(r_1r_2r_3)q}+I_{(111)T,t}^{(r_2r_1r_3)q}+\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(0r_3)q}\right), \tag{69}$$

where $q\ge 1$, and the approximations $I_{(111)T,t}^{(r_1r_2r_3)q}$, $I_{(111)T,t}^{(r_2r_1r_3)q}$ are defined by (33).

From (59), (60), (64), (67)–(69) it follows that

$$I_0[B(Z),F(Z)]_{T,t}^{M}-I_0[B(Z),F(Z)]_{T,t}^{M,q}=0\quad \text{w. p. 1},$$

$$I_1[B(Z),F(Z)]_{T,t}^{M}-I_1[B(Z),F(Z)]_{T,t}^{M,q}=0\quad \text{w. p. 1},$$

$$I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(\left(I_{(111)T,t}^{(r_1r_2r_3)}-I_{(111)T,t}^{(r_1r_2r_3)q}\right)+\left(I_{(111)T,t}^{(r_2r_1r_3)}-I_{(111)T,t}^{(r_2r_1r_3)q}\right)\right)\quad \text{w. p. 1}.$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=3$, we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 4C(3!)^2\left(\text{tr }Q\right)^3\left(\frac{(T-t)^3}{6}-\sum_{j_1,j_2,j_3=0}^{q}C_{j_3j_2j_1}^2\right),$$

where here and further constant C has the same meaning as constant L_k in Theorem 4 (k is the multiplicity of the iterated stochastic integral), and

$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar C_{j_3j_2j_1},$$

$$\bar C_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_j(x)$ is the Legendre polynomial.

7. Approximation of Some Iterated Stochastic Integrals of Third and Fourth Miltiplicity with Respect to the Q-Wiener Process

In this section, we consider the approximation of iterated stochastic integrals of the following form (see Sect. 1)

$$I_3[B(Z)]_{T,t}^M=\int\limits_t^T B'''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M,$$

$$I_4[B(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3}B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M,$$

$$I_5[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_3}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_3}B'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M,$$

$$I_6[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_3}B'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)dt_3,$$

$$I_7[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T F''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)dt_2,$$

$$I_8[B(Z),F(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2}F(Z)dt_1,\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M.$$

Consider the stochastic integral $I_3[B(Z)]_{T,t}^M$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B'''(v)(B(v),B(v),B(v))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (44))

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right) e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times \int\limits_t^T \left(\int\limits_t^s d\mathbf{w}_\tau^{(r_1)} \int\limits_t^s d\mathbf{w}_\tau^{(r_2)} \int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\right) d\mathbf{w}_s^{(r_4)} \quad \text{w. p. 1.} \tag{70}$$

From [43] (pp. A.438–A.439) (also see [44]-[46]) or using the Itô formula we obtain

$$I_{(1)s,t}^{(r_1)} I_{(1)s,t}^{(r_2)} I_{(1)s,t}^{(r_3)} =$$

$$= I_{(111)s,t}^{(r_1 r_2 r_3)} + I_{(111)s,t}^{(r_1 r_3 r_2)} + I_{(111)s,t}^{(r_2 r_1 r_3)} + I_{(111)s,t}^{(r_2 r_3 r_1)} + I_{(111)s,t}^{(r_3 r_1 r_2)} + I_{(111)s,t}^{(r_3 r_2 r_1)} +$$

$$+ \mathbf{1}_{\{r_1=r_2\}}\left(I_{(10)s,t}^{(r_3 0)} + I_{(01)s,t}^{(0 r_3)}\right) + \mathbf{1}_{\{r_1=r_3\}}\left(I_{(10)s,t}^{(r_2 0)} + I_{(01)s,t}^{(0 r_2)}\right) +$$

$$+ \mathbf{1}_{\{r_2=r_3\}}\left(I_{(10)s,t}^{(r_1 0)} + I_{(01)s,t}^{(0 r_1)}\right) =$$

$$= \sum_{(r_1,r_2,r_3)} I_{(111)s,t}^{(r_1 r_2 r_3)} + (s-t)\left(\mathbf{1}_{\{r_2=r_3\}} I_{(1)s,t}^{(r_1)} + \mathbf{1}_{\{r_1=r_3\}} I_{(1)s,t}^{(r_2)} + \mathbf{1}_{\{r_1=r_2\}} I_{(1)s,t}^{(r_3)}\right) \quad \text{w. p. 1,} \tag{71}$$

where

$$\sum_{(r_1,r_2,r_3)}$$

means the sum with respect to all possible permutations (r_1, r_2, r_3) and we use the notations from Sect. 4.

After substituting (71) into (70), we obtain

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right) e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)} I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)} - \mathbf{1}_{\{r_1=r_2\}} J_{(01)T,t}^{(r_3 r_4)} - \mathbf{1}_{\{r_1=r_3\}} J_{(01)T,t}^{(r_2 r_4)} - \mathbf{1}_{\{r_2=r_3\}} J_{(01)T,t}^{(r_1 r_4)}\right) \quad \text{w. p. 1,} \tag{72}$$

where

$$J_{(01)T,t}^{(r_1 r_2)} = \int\limits_t^T (t-s) \int\limits_t^s d\mathbf{w}_\tau^{(r_1)} d\mathbf{w}_s^{(r_2)}. \tag{73}$$

Denote by $I_3[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (72), which has the following form

$$I_3[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M}B'''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2},B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(\sum_{(r_1,r_2,r_3)}I_{(1111)T,t}^{(r_1r_2r_3r_4)q}-\mathbf{1}_{\{r_1=r_2\}}J_{(01)T,t}^{(r_3r_4)q}-\mathbf{1}_{\{r_1=r_3\}}J_{(01)T,t}^{(r_2r_4)q}-\mathbf{1}_{\{r_2=r_3\}}J_{(01)T,t}^{(r_1r_4)q}\right), \tag{74}$$

where the approximations $I_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $J_{(01)T,t}^{(r_1r_2)q}$ are based on Theorems 1, 2 and Legendre polynomials.

The approximation $J_{(01)T,t}^{(r_1r_2)q}$ of the stochastic integral $J_{(01)T,t}^{(r_1r_2)}$ $(r_1,r_2=1,\ldots,M)$, which is based on Theorems 1, 2 and Legendre polynomials has the following form (see [43] (formula (6.91), p. A.544) or [39] (formula (5.7), p. A.249))

$$J_{(01)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}I_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_1)}\zeta_1^{(r_2)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+2)\zeta_i^{(r_1)}\zeta_{i+2}^{(r_2)}-(i+1)\zeta_{i+2}^{(r_1)}\zeta_i^{(r_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}-\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \tag{75}$$

$$I_{(11)T,t}^{(r_1r_2)q}=\frac{T-t}{2}\left(\zeta_0^{(r_1)}\zeta_0^{(r_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(r_1)}\zeta_i^{(r_2)}-\zeta_i^{(r_1)}\zeta_{i-1}^{(r_2)}\right)-\mathbf{1}_{\{r_1=r_2\}}\right), \tag{76}$$

where notations are the same as in Theorems 1, 2.

Moreover (see [43] (formula (6.106), p. A.551) or [39] (formula (5.19), p. A.252–A.253)),

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}=\frac{(T-t)^4}{16}\times$$

$$\times\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right)\quad(r_1\neq r_2). \tag{77}$$

From (27), (29) we obtain

$$\mathsf{M}\left\{\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)^2\right\}\le$$

$$\le\frac{(T-t)^4}{8}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right),$$

where $r_1,r_2=1,\ldots,M$.

From (72), (74) it follows that

$$I_3[B(Z)]_{T,t}^M - I_3[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B'''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\sum_{(r_1,r_2,r_3)}\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)} - I_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right) - \mathbf{1}_{\{r_1=r_2\}}\left(J_{(01)T,t}^{(r_3r_4)} - J_{(01)T,t}^{(r_3r_4)q}\right) -$$

$$-\mathbf{1}_{\{r_1=r_3\}}\left(J_{(01)T,t}^{(r_2r_4)} - J_{(01)T,t}^{(r_2r_4)q}\right) - \mathbf{1}_{\{r_2=r_3\}}\left(J_{(01)T,t}^{(r_1r_4)} - J_{(01)T,t}^{(r_1r_4)q}\right)\Biggr)\quad \text{w. p. 1.} \tag{78}$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k = 2, 4$, we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^M - I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\text{tr } Q)^4\left(6^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + 3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (77), and

$$C_{j_4j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}(T-t)^2}{16}\bar{C}_{j_4j_3j_2j_1}, \tag{79}$$

$$\bar{C}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu,$$

where $P_j(x)$ is the Legendre polynomial.

Consider the stochastic integral $I_4[B(Z)]_{T,t}^M$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B'(v)(B''(v)(B(v), B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0 \times U_0 \times U_0 \times U_0$ to H for all $v \in H$.

We have (see (44))

$$I_4[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\int\limits_t^T\int\limits_t^s\left(\int\limits_t^{\tau} d\mathbf{w}_u^{(r_1)}\int\limits_t^{\tau} d\mathbf{w}_u^{(r_2)}\right)d\mathbf{w}_{\tau}^{(r_3)}d\mathbf{w}_s^{(r_4)}\quad \text{w. p. 1.} \tag{80}$$

From (63) and (80) we obtain

$$I_4[B(Z)]_{T,t}^{M}=\sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)}+I_{(1111)T,t}^{(r_2r_1r_3r_4)}-\mathbf{1}_{\{r_1=r_2\}}J_{(10)T,t}^{(r_3r_4)}\right)\quad \text{w. p. 1,} \tag{81}$$

where

$$J_{(10)T,t}^{(r_3r_4)}=\int\limits_t^T\int\limits_t^s (t-\tau)d\mathbf{w}_\tau^{(r_3)}d\mathbf{w}_s^{(r_4)}. \tag{82}$$

Denote by $I_4[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (81), which has the following form

$$I_4[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)q}+I_{(1111)T,t}^{(r_2r_1r_3r_4)q}-\mathbf{1}_{\{r_1=r_2\}}J_{(10)T,t}^{(r_3r_4)q}\right)\quad \text{w. p. 1,} \tag{83}$$

where the approximations $I_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $J_{(10)T,t}^{(r_1r_2)q}$ are based on Theorems 1, 2 and Legendre polynomials.

The approximation $J_{(10)T,t}^{(r_1r_2)q}$ of the stochastic integral $J_{(10)T,t}^{(r_1r_2)}$ $(r_1,r_2=1,\ldots,M)$, which is based on Theorems 1, 2 and Legendre polynomials has the following form (see [43] (formula (6.92), p. A.544) or [39] (formula (5.8), p. A.249))

$$J_{(10)T,t}^{(r_1r_2)q}=-\frac{T-t}{2}I_{(11)T,t}^{(r_1r_2)q}-\frac{(T-t)^2}{4}\left(\frac{1}{\sqrt{3}}\zeta_0^{(r_2)}\zeta_1^{(r_1)}+\right.$$

$$\left.+\sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(r_2)}\zeta_i^{(r_1)}-(i+2)\zeta_i^{(r_2)}\zeta_{i+2}^{(r_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(r_1)}\zeta_i^{(r_2)}}{(2i-1)(2i+3)}\right)\right), \tag{84}$$

where the approximation $I_{(11)T,t}^{(r_1r_2)q}$ is defined by (76).

Moreover,

$$\mathsf{M}\left\{\left(J_{(10)T,t}^{(r_1r_2)}-J_{(10)T,t}^{(r_1r_2)q}\right)^2\right\}=E_q\quad (r_1\ne r_2), \tag{85}$$

where E_q is the right-hand side of (77) (see [43] (formula (6.106), p. A.551) or [39] (formula (5.19), p. A.252–A.253)).

From (81), (83) it follows that

$$I_4[B(Z)]_{T,t}^{M} - I_4[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4\in J_M} B'(Z)\left(B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\right)e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

$$\times\Biggl(\left(I_{(1111)T,t}^{(r_1r_2r_3r_4)} - I_{(1111)T,t}^{(r_1r_2r_3r_4)q}\right) + \left(I_{(1111)T,t}^{(r_2r_1r_3r_4)} - I_{(1111)T,t}^{(r_2r_1r_3r_4)q}\right) -$$

$$-\mathbf{1}_{\{r_1=r_2\}}\left(J_{(10)T,t}^{(r_3r_4)} - J_{(10)T,t}^{(r_3r_4)q}\right)\Biggr) \quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k = 2, 4$, we obtain

$$\mathsf{M}\left\{\left\|I_4[B(Z)]_{T,t}^{M} - I_4[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\mathrm{tr}\ Q)^4\left(2^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + (2!)^2E_q\right),$$

where E_q is the right-hand side of (77), and $C_{j_4j_3j_2j_1}$ is defined by (79).

Consider the stochastic integral $I_5[B(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v), B'(v)(B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0\times U_0\times U_0\times U_0$ to H for all $v\in H$.

We have (see (44))

$$I_5[B(Z)]_{T,t}^{M} = \sum_{r_1,r_2,r_3,r_4\in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}}\times$$

(86)
$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)} \quad \text{w. p. 1.}$$

Using the theorem on the integration order replacement in iterated Itô stochastic integrals (see [43] (p. A.150, p. A.163), [44]-[46], [67]) or the Itô formula, we obtain

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}_\tau^{(r_3)}\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_2)}d\mathbf{w}_\tau^{(r_1)}\right)d\mathbf{w}_s^{(r_4)} =$$

$$= I_{(1111)T,t}^{(r_2r_1r_3r_4)} + I_{(1111)T,t}^{(r_2r_3r_1r_4)} + I_{(1111)T,t}^{(r_3r_2r_1r_4)} +$$

$$+ \mathbf{1}_{\{r_1=r_3\}} \left(J_{(10)T,t}^{(r_2r_4)} - J_{(01)T,t}^{(r_2r_4)} \right) - \mathbf{1}_{\{r_2=r_3\}} J_{(10)T,t}^{(r_1r_4)} \quad \text{w. p. 1,} \tag{87}$$

where we use the notations from Sect. 4, and $J_{(01)T,t}^{(r_1r_2)}$, $J_{(10)T,t}^{(r_1r_2)}$ are defined by (73), (82).

After substituting (87) into (86), we obtain

$$I_5[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \times$$

$$\times \Biggl(I_{(1111)T,t}^{(r_2r_1r_3r_4)} + I_{(1111)T,t}^{(r_2r_3r_1r_4)} + I_{(1111)T,t}^{(r_3r_2r_1r_4)} +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(J_{(10)T,t}^{(r_2r_4)} - J_{(01)T,t}^{(r_2r_4)} \right) - \mathbf{1}_{\{r_2=r_3\}} J_{(10)T,t}^{(r_1r_4)} \Biggr) \quad \text{w. p. 1.} \tag{88}$$

Denote by $I_5[B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (88), which has the following form

$$I_5[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \times$$

$$\times \Biggl(I_{(1111)T,t}^{(r_2r_1r_3r_4)q} + I_{(1111)T,t}^{(r_2r_3r_1r_4)q} + I_{(1111)T,t}^{(r_3r_2r_1r_4)q} +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(J_{(10)T,t}^{(r_2r_4)q} - J_{(01)T,t}^{(r_2r_4)q} \right) - \mathbf{1}_{\{r_2=r_3\}} J_{(10)T,t}^{(r_1r_4)q} \Biggr) \quad \text{w. p. 1,} \tag{89}$$

where the approximations $I_{(1111)T,t}^{(r_1r_2r_3r_4)q}$, $J_{(01)T,t}^{(r_1r_2)q}$, and $J_{(10)T,t}^{(r_1r_2)q}$ are based on Theorems 1, 2 and Legendre polynomials.

From (88), (89) it follows that

$$I_5[B(Z)]_{T,t}^M - I_5[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1})e_{r_4} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}} \times$$

$$\times \Biggl(\left(I_{(1111)T,t}^{(r_2r_1r_3r_4)} - I_{(1111)T,t}^{(r_2r_1r_3r_4)q} \right) + \left(I_{(1111)T,t}^{(r_2r_3r_1r_4)} - I_{(1111)T,t}^{(r_2r_3r_1r_4)q} \right) + \left(I_{(1111)T,t}^{(r_3r_2r_1r_4)} - I_{(1111)T,t}^{(r_3r_2r_1r_4)q} \right) +$$

$$+\mathbf{1}_{\{r_1=r_3\}} \left(\left(J_{(10)T,t}^{(r_2r_4)} - J_{(10)T,t}^{(r_2r_4)q} \right) - \left(J_{(01)T,t}^{(r_2r_4)} - J_{(01)T,t}^{(r_2r_4)q} \right) \right) - \mathbf{1}_{\{r_2=r_3\}} \left(J_{(10)T,t}^{(r_1r_4)} - J_{(10)T,t}^{(r_1r_4)q} \right) \Biggr) \quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k = 2, 4$ and taking into account (85), we obtain

$$\mathsf{M}\left\{\left\|I_5[B(Z)]_{T,t}^{M} - I_5[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le C\,(\mathrm{tr}\ Q)^4\left(3^2(4!)^2\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_4j_3j_2j_1}^2\right) + 3^2(2!)^2E_q\right),$$

where E_q is the right-hand side of (77), and $C_{j_4j_3j_2j_1}$ is defined by (79).

Consider the stochastic integral $I_6[B(Z), F(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. We have (see (44))

$$I_6[B(Z), F(Z)]_{T,t}^{M} = \sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

(90) $$\times\int\limits_t^T\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_1)}d\mathbf{w}_\tau^{(r_2)}ds \quad \text{w. p. 1.}$$

Using the theorem on the integration order replacement in iterated Itô stochastic integrals (see [43] (p. A.150, p. A.163), [44]-[46], [67]) or the Itô formula, we obtain

(91) $$\int\limits_t^T\int\limits_t^s\int\limits_t^\tau d\mathbf{w}_u^{(r_1)}d\mathbf{w}_\tau^{(r_2)}ds = (T-t)I_{(11)T,t}^{(r_1r_2)} + J_{(01)T,t}^{(r_1r_2)} \quad \text{w. p. 1.}$$

After substituting (91) into (90) we have

$$I_6[B(Z), F(Z)]_{T,t}^{M} = \sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

(92) $$\times\left((T-t)I_{(11)T,t}^{(r_1r_2)} + J_{(01)T,t}^{(r_1r_2)}\right) \quad \text{w. p. 1.}$$

Denote by $I_6[B(Z), F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (92), which has the following form

$$I_6[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2\in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

(93) $$\times\left((T-t)I_{(11)T,t}^{(r_1r_2)q} + J_{(01)T,t}^{(r_1r_2)q}\right),$$

where the approximations $J^{(r_1r_2)q}_{(01)T,t}$, $I^{(r_1r_2)q}_{(11)T,t}$ are defined by (75), (76).

From (92), (93) it follows that

$$I_6[B(Z),F(Z)]^M_{T,t}-I_6[B(Z),F(Z)]^{M,q}_{T,t}=\sum_{r_1,r_2\in J_M}F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2})\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\Biggl((T-t)\left(I^{(r_1r_2)}_{(11)T,t}-I^{(r_1r_2)q}_{(11)T,t}\right)+\left(J^{(r_1r_2)}_{(01)T,t}-J^{(r_1r_2)q}_{(01)T,t}\right)\Biggr)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_6[B(Z),F(Z)]^M_{T,t}-I_6[B(Z),F(Z)]^{M,q}_{T,t}\right\|^2_H\right\}\le$$

$$\le 2C(2!)^2\left(\text{tr } Q\right)^2\Bigl((T-t)^2G_q+E_q\Bigr),$$

where G_q and E_q are the right-hand sides of (36) and (77) correspondingly.

Consider the stochastic integral $I_7[B(Z),F(Z)]^M_{T,t}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Then we have (see (44)) w. p. 1

$$I_7[B(Z),F(Z)]^M_{T,t}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}^{(r_1)}_\tau\int\limits_t^s d\mathbf{w}^{(r_2)}_\tau\right)ds. \tag{94}$$

Using the Itô formula, we obtain

$$\int\limits_t^s d\mathbf{w}^{(r_1)}_\tau\int\limits_t^s d\mathbf{w}^{(r_2)}_\tau=I^{(r_1r_2)}_{(11)s,t}+I^{(r_2r_1)}_{(11)s,t}+\mathbf{1}_{\{r_1=r_2\}}(s-t)\quad \text{w. p. 1,} \tag{95}$$

where we use the notations from Sect. 4.

From (95) and (91) we have

$$\int\limits_t^T\left(\int\limits_t^s d\mathbf{w}^{(r_1)}_\tau\int\limits_t^s d\mathbf{w}^{(r_2)}_\tau\right)ds=$$

$$=\int\limits_t^T I^{(r_1r_2)}_{(11)s,t}ds+\int\limits_t^T I^{(r_2r_1)}_{(11)s,t}ds+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=$$

$$
= (T-t)\left(I_{(11)T,t}^{(r_1r_2)}+I_{(11)T,t}^{(r_2r_1)}\right)+J_{(01)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=
$$

$$
= (T-t)\left(I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}-\mathbf{1}_{\{r_1=r_2\}}(T-t)\right)+J_{(01)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_2r_1)}+\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}=
$$

$$
= (T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}+J_{(01)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_2r_1)}-
$$

$$
-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\quad \text{w. p. 1.} \tag{96}
$$

After substituting (96) into (94) we obtain

$$
I_7[B(Z),F(Z)]_{T,t}^{M}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left((T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}+J_{(01)T,t}^{(r_1r_2)}+J_{(01)T,t}^{(r_2r_1)}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right)\quad \text{w. p. 1.} \tag{97}
$$

Denote by $I_7[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (97), which has the following form

$$
I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left((T-t)I_{(1)T,t}^{(r_1)}I_{(1)T,t}^{(r_2)}+J_{(01)T,t}^{(r_1r_2)q}+J_{(01)T,t}^{(r_2r_1)q}-\mathbf{1}_{\{r_1=r_2\}}\frac{(T-t)^2}{2}\right), \tag{98}
$$

where the approximation $J_{(01)T,t}^{(r_1r_2)q}$ is defined by (75).

From (97), (98) it follows that

$$
I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}F''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)\sqrt{\lambda_{r_1}\lambda_{r_2}}\times
$$

$$
\times\left(\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)+\left(J_{(01)T,t}^{(r_2r_1)}-J_{(01)T,t}^{(r_2r_1)q}\right)\right)\quad \text{w. p. 1.}
$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_7[B(Z),F(Z)]_{T,t}^{M}-I_7[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2^2 C(2!)^2\,(\mathrm{tr}\ Q)^2\,E_q,$$

where E_q is the right-hand side of (77).

Consider the stochastic integral $I_8[B(Z),F(Z)]_{T,t}^{M}$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Then we have (see (44)) w. p. 1

$$I_8[B(Z),F(Z)]_{T,t}^{M}=-\sum_{r_1,r_2\in J_M}B''(Z)\,(F(Z),B(Z)e_{r_1})\,e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(01)T,t}^{(r_1r_2)}. \tag{99}$$

Denote by $I_8[B(Z),F(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (99), which has the following form

$$I_8[B(Z),F(Z)]_{T,t}^{M,q}=-\sum_{r_1,r_2\in J_M}B''(Z)\,(F(Z),B(Z)e_{r_1})\,e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}J_{(01)T,t}^{(r_1r_2)q}, \tag{100}$$

where the approximation $J_{(01)T,t}^{(r_1r_2)q}$ is defined by (75).

From (99), (100) it follows that

$$I_8[B(Z),F(Z)]_{T,t}^{M}-I_8[B(Z),F(Z)]_{T,t}^{M,q}=$$

$$=-\sum_{r_1,r_2\in J_M}B''(Z)\,(F(Z),B(Z)e_{r_1})\,e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}\times$$

$$\times\left(J_{(01)T,t}^{(r_1r_2)}-J_{(01)T,t}^{(r_1r_2)q}\right)\quad \text{w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the case $k=2$, we obtain

$$\mathsf{M}\left\{\left\|I_8[B(Z),F(Z)]_{T,t}^{M}-I_8[B(Z),F(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le C(2!)^2\,(\mathrm{tr}\ Q)^2\,E_q,$$

where E_q is the right-hand side of (77).

Using computational experiments it was shown in [64], [65] (also see [44], Sect. 5.4) that we can neglect the multiplier factor $k!$ in the estimate (27). As a result, the computational costs for the approximation of iterated Itô stochastic integrals are significantly reduced. For the same reason, we can replace the multiplier factor $(k!)^2$ by $k!$ in the formula (47) in practical calculations.

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References

[1] Gyöngy I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. Potential Anal. 9, 1 (1998), 1-25.

[2] Gyöngy I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. Potential Anal. 11, 1 (1999), 1-37.

[3] Gyöngy I. and Krylov N. An accelerated splitting-up method for parabolic equations. SIAM J. Math. Anal. 37, 4 (2005), 1070-1097.

[4] Hausenblas E. Numerical analysis of semilinear stochastic evolution equations in Banach spaces. J. Comp. Appl. Math. 147, 2 (2002), 485-516.

[5] Hausenblas E. Approximation for semilinear stochastic evolution equations. Potential Anal. 18, 2 (2003), 141-186.

[6] Hutzenthaler M. and Jentzen A. Non-globally Lipschitz counterexamples for the stochastic Euler scheme. arXiv:0905.0273 [math.NA] (2009), 22 pp.

[7] Jentzen A. Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Anal. 31, 4 (2009), 375-404.

[8] Jentzen A. Taylor expansions of solutions of stochastic partial differential equations. arXiv:0904.2232 [math.NA] (2009), 32 pp.

[9] Jentzen A. and Kloeden P.E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. Proc. R. Soc. Lond. Ser. Math. Phys. Eng. Sci. 465, 2102 (2009), 649-667.

[10] Jentzen A. and Kloeden P.E. Taylor expansions of solutions of stochastic partial differential equations with additive noise. Ann. Prob. 38, 2 (2010), 532-569.

[11] Jentzen A. and Kloeden P.E. Taylor approximations for stochastic partial differential equations. SIAM, Philadelphia, 2011, 224 pp.

[12] Jentzen A. and Röckner M. A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.

[13] Becker S., Jentzen A. and Kloeden P.E. An exponential Wagner-Platen type scheme for SPDEs. SIAM J. Numer. Anal. 54, 4 (2016), 2389-2426.

[14] Zhang Z. and Karniadakis G. Numerical methods for stochastic partial differential equations with white noise. Springer, 2017, 398 pp.

[15] Jentzen A. and Röckner M. Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. J. Differ. Eq. 252, 1 (2012), 114-136.

[16] Lord G.J. and Tambue A. A modified semi-implict Euler-Maruyama scheme for finite element discretization of SPDEs. arXiv:1004.1998 [math.NA] (2010), 23 pp.

[17] Mishura Y.S. and Shevchenko G.M. Approximation schemes for stochastic differential equations in a Hilbert space. Theor. Prob. Appl. 51, 3 (2007), 442-458.

[18] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. In Monte Carlo and quasi-Monte Carlo methods 2006. Springer, Berlin, 2007, pp. 577-589.

[19] Prévôt C. and Röckner M. A concise course on stochastic partial differential equations, V. 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007, 148 pp.

[20] Shardlow T. Numerical methods for stochastic parabolic PDEs. Numer. Funct. Anal. Optim. 20, 1-2 (1999), 121-145.

[21] Da Prato G., Jentzen A. and Röckner M. A mild Itô formula for SPDEs. arXiv:1009.3526 [math.PR] (2012), 39 pp.

[22] Da Prato G. and Zabczyk J. Stochastic equations in infinite dimensions. 2nd Ed. Cambridge Univ. Press, Cambridge, 2014, 493 pp.

[23] Kruse R. Optimal Error Estimates of Galerkin Finite Element Methods for Stochastic Partial Differential Equations with Multiplicative Noise. IMA J. Numer. Anal. 34, 1 (2014), 217-251.

[24] Kruse R. Consistency and stability of a Milstein-Galerkin finite element scheme for semilinear SPDE. Stoch. PDE: Anal. Comp. 2, 4 (2014), 471-516.

[25] Leonhard C. and Rößler A. Iterated stochastic integrals in infinite dimensions: approximation and error estimates. Stoch. PDE: Anal. Comp. 7, 2 (2019), 209-239.

[26] Milstein G.N. Numerical integration of stochastic differential equations. Ural Univ. Press, Sverdlovsk, 1988, 225 pp.

[27] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[28] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publ., 204 pp. (ISBN 5-7422-0045-5)

[29] Kuznetsov D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[30] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[31] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[32] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[33] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[34] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: St.-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[35] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[36] Dmitriy F. Kuznetsov. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, St.-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[37] Dmitriy F. Kuznetsov. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, St.-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[38] Dmitriy F. Kuznetsov. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House: St.-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[39] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2017, no. 1, 385 pp. (A.1–A.385). DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[40] Kuznetsov D.F. Development and Application of the Fourier Method for the Numerical Solution of Ito Stochastic Differential Equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[41] Kuznetsov D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 1.5 and 2.0 Orders of Strong Convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[42] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[43] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[44] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2020, 859 pp. [In English].

[45] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[46] Kuznetsov, D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[47] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2017, 107 pp.

[48] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. [In English]. arXiv:1712.09516 [math.PR], 2022, 173 pp.

[49] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 100 pp.

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 74 pp.

[51] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 39 pp.

[52] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2018, 91 pp.

[53] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 57 pp.

[54] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604.

[55] Kuznetsov D.F. On Numerical Modeling of the Multidimentional Dynamic Systems Under Random Perturbations With the 2.5 Order of Strong Convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[56] Kuznetsov D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[57] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 34 pp.

[58] Kuznetsov, D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

[59] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes, 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[60] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.

[61] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[62] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre

series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[63] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 336 pp. [In English].

[64] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 59 pp. [In English].

[65] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[66] Kloeden P.E. and Platen E. Numerical solution of stochastic differential equations. Springer-Verlag, Berlin, 1992, 632 pp.

[67] Kuznetsov D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [In English]. arXiv:1801.04634 [math.PR], 2018, 27 pp.

[68] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes. 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Institute of Applied Mathematics and Mechanics,
Department of Mathematics,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

APPLICATION OF MULTIPLE FOURIER–LEGENDRE SERIES TO IMPLEMENTATION OF STRONG EXPONENTIAL MILSTEIN AND WAGNER–PLATEN METHODS FOR NON-COMMUTATIVE SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise. These methods have strong orders of convergence $1.0-\varepsilon$ and $1.5-\varepsilon$ correspondingly (here ε is an arbitrary small positive real number) with respect to the temporal discretization. The theorem on mean-square convergence of approximations of iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process is formulated and proved. The results of the article can be applied to implementation of exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise.

1. Introduction

It is well-known that one of the effective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for stochastic partial differential equations (SPDEs) is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [1]-[6]. A significant step in this direction was made in [2], [3], where the exponential Milstein and Wagner–Platen methods for semilinear SPDEs were constructed. Under the appropriate conditions [2], [3] these methods have strong orders of convergence $1.0-\varepsilon$ and $1.5-\varepsilon$ correspondingly (where ε is an arbitrary small posilive real number) with respect to the temporal discretization.

An important feature of the mentioned numerical methods is the presence in them of the so-called iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process [7]. The problem of numerical modeling of these stochastic integrals was solved in [2], [3] for the case when special commutativity conditions are fulfilled.

If the mentioned commutativity conditions are not fulfilled, which often corresponds to SPDEs in numerous applications, the numerical simulation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process becomes much more difficult. Note that the exponential Milstein scheme [2] contains the iterated stochastic integrals of multiplicities 1 and 2 with respect to the infinite-dimensional Q-Wiener process and the exponential Wagner–Platen scheme [3] contains the mentioned stochastic integrals of multiplicities 1 to 3. In [8] two methods of the mean-square approximation of iterated stochastic integrals from the Milstein scheme [2] have been considered. Note that the mean-square error of approximation of these stochastic integrals consists of two components [8].

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Keywords: Non-commutative semilinear Stochastic partial differential equation, Multiplicative trace class noise, Infinite-dimensional Q-Wiener process, Iterated Itô stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Exponential Milstein scheme, Exponential Wagner–Platen scheme, Legendre polynomials, Mean-square approximation, Expansion.

The first component is related with the finite-dimentional approximation of the infinite-dimentional Q-Wiener process while the second one is connected with the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions. In the author's publications [9], [10], [23], [24] the problem of the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process in the sense of second component of approximation error (see above) has been solved for an arbitrary multiplicity k ($k \in \mathbb{N}$) of stochastic integrals. More precisely, in [9], [10] (also see [23], [24]) the method of generalized multiple Fourier series [11]-[54] for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process (in the sense of second component of the approximation error).

In this article, we extend the method for estimating the first component of approximation error from [8] for iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. In addition, we combine the obtained results with the results from [9], [10]. Thus, the results of the article can be applied to the implementation of exponential Milstein and Wagner–Platen schemes for semilinear SPDEs with multiplicative trace class noise and without the conditions of commutativity for SPDEs.

2. Exponential Milstein and Wagner–Platen Numerical Schemes for Non-Commutative Semilinear SPDEs

Let U, H be separable $\mathbb{R}$-Hilbert spaces and $L_{HS}(U,H)$ be a space of Hilbert–Schmidt operators mapping from U to H. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$ [7], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here $L(U)$ denotes all bounded linear operators mapping from U to U. Consider an $\mathbb{R}$-Hilbert space $U_0 = Q^{1/2}(U)$ with a scalar product

$$\langle u, w \rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w \right\rangle_U$$

for all $u, w \in U_0$.

Consider the semilinear parabolic SPDE

$$dX_t = (AX_t + F(X_t))\,dt + B(X_t)d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}], \tag{1}$$

where nonlinear operators F, B ($F: H \to H$, $B: H \to L_{HS}(U_0, H)$), linear operator $A: D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the SPDE (1) mild solution (see [3], Assumptions A1–A4).

It is well-known [6] that Assumptions A1–A4 [3] guarantee the existence and uniqueness (up to modifications) of the mild solution $X_t : [0, \bar{T}] \times \Omega \to H$ of the SPDE (1)

$$X_t = \exp(At)\xi + \int\limits_0^t \exp(A(t-\tau))F(X_\tau)d\tau + \int\limits_0^t \exp(A(t-\tau))B(X_\tau)d\mathbf{W}_\tau \tag{2}$$

with probability 1 (further w. p. 1) for all $t \in [0, \bar{T}]$, where $\exp(At)$, $t \geq 0$ is the semigroup generated by the operator A.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i = (i_1, \ldots, i_d) \in J$, $x = (x_1, \ldots, x_d)$, and $J = \{i : \ i \in \mathbb{N}^d, \ \text{and} \ \lambda_i > 0\}$.

The series representation of the Q-Wiener process has the following form [7]

$$\mathbf{W}_t = \sum_{i\in J} e_i \sqrt{\lambda_i}\mathbf{w}_t^{(i)} \quad \text{or} \quad \mathbf{W}_t = \sum_{i\in J_M} e_i \ \langle e_i, \mathbf{W}_t\rangle_U,$$

where $t \in [0, \bar{T}]$, $\mathbf{w}_t^{(i)}$ $(i \in J)$ are independent standard Wiener processes and $\langle\cdot,\cdot\rangle_U$ is a scalar product in U.

Note that eigenfunctions e_i, $i \in J$ form an orthonormal basis of U [7]. Consider the finite-dimensional approximation of $\mathbf{W}_t$ [7]

$$\mathbf{W}_t^M = \sum_{i\in J_M} e_i \sqrt{\lambda_i}\mathbf{w}_t^{(i)}, \quad t \in [0, \bar{T}], \tag{3}$$

where

$$J_M = \{i: \ 1 \le i_1, \ldots, i_d \le M, \ \text{and} \ \lambda_i > 0\}. \tag{4}$$

Remark 1. *Obviously, without the loss of generality we can write* $J_M = \{1, 2, \ldots, M\}$.

Let $\Delta > 0$, $\tau_p = p\Delta$ $(p = 0, 1, \ldots, N)$, and $N\Delta = \bar{T}$. Consider the following exponential Milstein and Wagner–Platen numerical schemes for the SPDE (1) [2], [3]

$$Y_{p+1} = \exp\left(A\Delta\right)\left(Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$\left. + \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s\right), \tag{5}$$

(an exponential Milstein scheme)

$$Y_{p+1} = \exp\left(\frac{A\Delta}{2}\right)\times$$

$$\times\left(\exp\left(\frac{A\Delta}{2}\right)Y_p + \Delta F(Y_p) + \frac{\Delta^2}{2}F'(Y_p)\Bigl(AY_p + F(Y_p)\Bigr) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s + \right.$$

$$+\frac{\Delta^2}{4}\sum_{i\in J}\lambda_i F''(Y_p)\Bigl(B(Y_p)e_i, B(Y_p)e_i\Bigr) + \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s +$$

$$+A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau ds - \frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s\right) +$$

$$
+\Delta\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_s-\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_\tau ds+
$$

$$
+\frac{1}{2}\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)d\mathbf{W}_s+
$$

$$
+\int\limits_{\tau_p}^{\tau_{p+1}} F'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau\right)ds+
$$

$$
+\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B'(Y_p)\left(\int\limits_{\tau_p}^{\tau} B(Y_p)d\mathbf{W}_\theta\right)d\mathbf{W}_\tau\right)d\mathbf{W}_s\Biggr), \tag{6}
$$

(an exponential Wagner–Platen scheme)

where Y_p is an approximation of X_{τ_p} (mild solution (2) at the time moment τ_p), $p=0,1,\ldots,N$, and B', B'', F', F'' are Frêchet derivatives. In addition to the temporal discretization, the implementation of numerical schemes (5) and (6) also requires a finite-dimensional approximation of the spaces H, U. Further, we will consider this approximation only for the space U.

Let us consider the following iterated Itô stochastic integrals

$$
I_{(1)T,t}^{(r_1)}=\int\limits_t^T d\mathbf{w}_{t_1}^{(r_1)},
$$

$$
I_{(10)T,t}^{(r_10)}=\int\limits_t^T\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)}dt_2,\quad I_{(01)T,t}^{(0r_2)}=\int\limits_t^T\int\limits_t^{t_2} dt_1 d\mathbf{w}_{t_2}^{(r_2)},
$$

$$
I_{(11)T,t}^{(r_1r_2)}=\int\limits_t^T\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)},
$$

$$
I_{(111)T,t}^{(r_1r_2r_3)}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}d\mathbf{w}_{t_3}^{(r_3)},
$$

where $r_1,r_2,r_3\in J_M,\ \ 0\le t<T\le\bar{T}$, and J_M is defined by (4).

Let us replace the infinite-dimensional Q-Wiener process in the iterated stochastic integrals from (5), (6) by its finite-dimensional approximation (3). Then we have w. p. 1

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)d\mathbf{W}_s^M=\sum_{r_1\in J_M} B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}I_{(1)\tau_{p+1},\tau_p}^{(r_1)}, \tag{7}
$$

$$A\left(\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}^{M}ds-\frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}}B(Y_p)d\mathbf{W}_{s}^{M}\right)=A\int\limits_{\tau_p}^{\tau_{p+1}}B(Y_p)\left(\frac{\tau_{p+1}}{2}-s+\frac{\tau_p}{2}\right)d\mathbf{W}_{s}^{M}=$$

$$=\sum_{r_1\in J_M}AB(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\frac{\Delta}{2}I_{(1)\tau_{p+1},\tau_p}^{(r_1)}-I_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{8}$$

$$\Delta\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_{s}^{M}-\int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_{\tau}^{M}ds=$$

$$=\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\int\limits_{\tau_p}^{s}\Bigl(AY_p+F(Y_p)\Bigr)d\tau d\mathbf{W}_{s}^{M}=$$

$$=\sum_{r_1\in J_M}B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}I_{(01)\tau_{p+1},\tau_p}^{(0r_1)}, \tag{9}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}}F'(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}^{M}\right)ds=$$

$$=\sum_{r_1\in J_M}F'(Y_p)B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\Delta I_{(1)\tau_{p+1},\tau_p}^{(r_1)}-I_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \tag{10}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\left(\int\limits_{\tau_p}^{s}B(Y_p)d\mathbf{W}_{\tau}^{M}\right)d\mathbf{W}_{s}^{M}=$$

$$=\sum_{r_1,r_2\in J_M}B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(11)\tau_{p+1},\tau_p}^{(r_1r_2)}, \tag{11}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}}B'(Y_p)\left(\int\limits_{\tau_p}^{s}B'(Y_p)\left(\int\limits_{\tau_p}^{\tau}B(Y_p)d\mathbf{W}_{\theta}^{M}\right)d\mathbf{W}_{\tau}^{M}\right)d\mathbf{W}_{s}^{M}=$$

$$=\sum_{r_1,r_2,r_3\in J_M}B'(Y_p)\left(B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}I_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)}, \tag{12}$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1}, B(Y_p)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times \int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)}. \tag{13}$$

Note that in (8)–(10) we used the Itô formula. Moreover, using the Itô formula we obtain

$$\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)} = I_{(11)s,\tau_p}^{(r_1r_2)} + I_{(11)s,\tau_p}^{(r_2r_1)} + \mathbf{1}_{\{r_1=r_2\}}(s-\tau_p) \quad \text{w. p. 1.} \tag{14}$$

From (14) we have

$$\int\limits_{\tau_p}^{\tau_{p+1}}\left(\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_1)}\int\limits_{\tau_p}^{s} d\mathbf{w}_\tau^{(r_2)}\right)d\mathbf{w}_s^{(r_3)} = I_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + I_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}I_{(01)\tau_{p+1},\tau_p}^{(0r_3)} \quad \text{w. p. 1.} \tag{15}$$

After substituting (15) into (13), we obtain

$$\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(I_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)} + I_{(111)\tau_{p+1},\tau_p}^{(r_2r_1r_3)} + \mathbf{1}_{\{r_1=r_2\}}I_{(01)\tau_{p+1},\tau_p}^{(0r_3)}\right) \quad \text{w. p. 1.} \tag{16}$$

Thus, for the implementation of numerical schemes (5) and (6) we need to approximate the following collection of iterated Itô stochastic integrals

$$I_{(1)T,t}^{(r_1)}, \quad I_{(01)T,t}^{(0r_1)}, \quad I_{(11)T,t}^{(r_1r_2)}, \quad I_{(111)T,t}^{(r_1r_2r_3)},$$

where $r_1, r_2, r_3 \in J_M$, $0 \le t < T \le \bar{T}$.

The monographs [22] (Chapters 5, 6) and [23] or [24] (Chapters 1, 2, and 5) (also see [11]-[21], [26]-[54]) are devoted to the constructing of efficient methods (based on generalized multiple Fourier series) of the mean-square approximation of iterated Itô stochastic integrals with respect to components of the finite-dimensional Wiener process. These results are also adapted for iterated Stratonovich stochastic integrals [16]-[24], [28], [33], [35]-[42], [46], [49], [50], [52], [54]. In Sect. 3, we consider a very short review of the results from monographs [22] (Chapters 5, 6) and [23] or [24] (Chapters 1, 2, and 5).

3. Method of Approximation of Iterated Itô Stohastic integrals Based on Generalized Multiple Fourier Series. The Case of Multiple Fourier–Legendre Series

Consider the following iterated Itô stochastic integrals

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{17}$$

where $0\le t<T\le\bar{T}$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)}$ $(i=1,\ldots,m)$ are independent standard Wiener processes, $\mathbf{w}_\tau^{(0)}=\tau$, $i_1,\ldots,i_k=0,1,\ldots,m$. The case $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered in Theorem 2 (see below).

Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ \\ 0,\ \text{otherwise}\end{cases}=\prod_{l=1}^k\psi_l(t_l)\ \prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}, \tag{18}$$

where $t_1,\ldots,t_k\in[t,T]$ for $k\ge 2$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$. Here $\mathbf{1}_A$ is the indicator of the set A.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous on the hypercube $[t,T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ converges to $K(t_1,\ldots,t_k)$ on the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^k\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)}=0, \tag{19}$$

where

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{20}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)}=\left(\int\limits_{[t,T]^k}f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{21}$$

Theorem 1 [11] (2006) (also see [9], [10], [12]-[54]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\ \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{22}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\Bigr\},$$

$$\mathrm{L}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\Bigr\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{23}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (20), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is the discretization of* $[t,T]$, *which satisfies the condition* (21).

Note that in [11]-[24], [32] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Another version of Theorem 1 related to the application of complete orthonormal with weight $r(t_1)\ldots r(t_k)\ge 0$ systems of functions in $L_2([t,T]^k)$ has been considered in [22]-[24], [44]. A generalization of Theorem 1 to the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ as well as $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ will be considered below (see Theorem 2).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,5$ [11]-[54] (cases $k=6,\ 7$ and $k>7$ can be found in [12]-[54])

$$J[\psi^{(1)}]_{T,t}^{(i_1)}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{24}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{25}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{26}$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{27}$$

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{28}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider the generalization of (24)–(28) for the case of an arbitrary k ($k \in \mathbb{N}$) as well as for the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$.

In order to do this, let us consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\underbrace{\{q_1,\ldots,q_{k-2r}\}}_{\text{part 2}}), \tag{29}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set and parentheses mean an ordered set.

We will say that (29) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}. \tag{30}$$

Theorem 2 [23] (Sect. 1.11), [32] (Sect. 15). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{31}$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

In particular, from (31) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\,i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\,j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\,i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\,j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (28). Note that the correctness of formulas (24)–(28) can be verified by the fact that if $i_1 = \ldots = i_5 = i = 1, \ldots, m$ and $\psi_1(s), \ldots, \psi_6(s) \equiv \psi(s)$, then we can derive from (24)–(28) the well known equalities (the cases $k = 2, 3$ were discussed in details in [12]-[24], [32])

$$J[\psi^{(1)}]_{T,t}^{(i)} = \frac{1}{1!}\delta_{T,t}^{(i)},$$

$$J[\psi^{(2)}]_{T,t}^{(ii)} = \frac{1}{2!}\left(\left(\delta_{T,t}^{(i)}\right)^2 - \Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}^{(iii)} = \frac{1}{3!}\left(\left(\delta_{T,t}^{(i)}\right)^3 - 3\delta_{T,t}^{(i)}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}^{(iiii)} = \frac{1}{4!}\left(\left(\delta_{T,t}^{(i)}\right)^4 - 6\left(\delta_{T,t}^{(i)}\right)^2\Delta_{T,t} + 3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}^{(iiiii)} = \frac{1}{5!}\left(\left(\delta_{T,t}^{(i)}\right)^5 - 10\left(\delta_{T,t}^{(i)}\right)^3\Delta_{T,t} + 15\delta_{T,t}^{(i)}\Delta_{T,t}^2\right)$$

w. p. 1, where

$$\delta_{T,t}^{(i)} = \int\limits_t^T \psi(s)d\mathbf{w}_s^{(i)}, \quad \Delta_{T,t} = \int\limits_t^T \psi^2(s)ds.$$

The above equalities can be independently obtained using the Itô formula and Hermite polynomials [61].

Assume that $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1\ldots p_k}$ is an approximation of the stochastic integral (17), which is the expression on the right-hand side of (31) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$. Let us denote

$$E^{(i_1\ldots i_k)p_1,\ldots,p_k} = \mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} - J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}\right)^2\right\},$$

$$I_k = \|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k, \qquad E^{(i_1\ldots i_k)p_1,\ldots,p_k}\Big|_{p_1=\ldots=p_k=p} \stackrel{\mathsf{def}}{=} E^{(i_1\ldots i_k)p}. \tag{32}$$

In [21], [22]-[24], [32], [34] it was shown that

$$E_k^{(i_1\ldots i_k)p_1,\ldots,p_k} \le k!\left(I_k - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}^2\right), \tag{33}$$

where $i_1, \ldots, i_k = 1, \ldots, m$ for $0 < T - t < \infty$ and $i_1, \ldots, i_k = 0, 1, \ldots, m$ for $0 < T - t < 1$. Note that the estimate (33) is valid under the conditions of Theorem 2.

Let us consider some approximations of iterated Itô stochastic integrals using Theorems 1, 2 and multiple Fourier–Legendre series.

The complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{34}$$

where $P_j(x)$ is the Legendre polynomial.

Using the system of functions (34) and Theorems 1, 2 we obtain the following approximations of iterated Itô stochastic integrals [9]-[58] (also see early publications [62] (1997), [63] (1998), [64] (2000))

$$I_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{35}$$

$$I_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{36}$$

$$I_{(10)T,t}^{(i_10)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{37}$$

$$I_{(11)T,t}^{(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{38}$$

$$I_{(111)T,t}^{(i_1i_2i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{39}$$

$$I_{(111)T,t}^{(i_1i_1i_1)}=\frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right),$$

$$C_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

Table 1. Minimal numbers q, q_1 such that $E^{(i_1 i_2)q}$, $E^{(i_1 i_2 i_3)q_1} \le (T-t)^4$, $\quad q_1 \ll q$.

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

Table 2. Coefficients $\bar{C}_{3jk}$.

$j \backslash k$	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

standard Gaussian random variables $\zeta_j^{(i)}$ $(i \ne 0)$ are defined by (23), and

$$I_{(11)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}}\ I_{(11)T,t}^{(i_1 i_2)q},$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)} = \underset{q_1\to\infty}{\text{l.i.m.}}\ I_{(111)T,t}^{(i_1 i_2 i_3)q_1}.$$

Note that $T-t \ll 1$ ($T-t$ is an integration step with respect to the temporal variable). Thus $q_1 \ll q$ (see Table 1 [9]-[24], [35]). Moreover, the values $\bar{C}_{j_3 j_2 j_1}$ do not depend on $T-t$. This feature is important because we can use a variable integration step $T-t$. Coefficients $\bar{C}_{j_3 j_2 j_1}$ are calculated once and before the start of the numerical scheme. Some examples of exact calculation of coefficients $\bar{C}_{j_3 j_2 j_1}$ via Python programming language can be found in Table 2 (the database with 270,000 exactly calculated Fourier–Legendre coefficients was described in [55], [56]).

According to the notations introduced above, we have

$$E^{(i_1 i_2)q} = \mathsf{M}\left\{\left(I_{(11)T,t}^{(i_1 i_2)} - I_{(11)T,t}^{(i_1 i_2)q}\right)^2\right\},$$

$$E^{(i_1 i_2 i_3)q_1} = \mathsf{M}\left\{\left(I_{(111)T,t}^{(i_1 i_2 i_3)} - I_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2\right\}.$$

Then for pairwise different $i_1, i_2, i_3 = 1, \ldots, m$ we obtain [9]-[24], [34]

$$E^{(i_1 i_2)q} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2-1}\right), \tag{40}$$

$$E^{(i_1 i_2 i_3)q_1} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C^2_{j_3 j_2 j_1}. \qquad (41)$$

On the basis of the presented approximations of iterated Itô stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to $T-t$ $(T-t \ll 1)$ in the mean-square sense for iterated Itô stochastic integrals. This leads to sharp decrease of member quantities in the approximations of iterated Itô stochastic integrals, which are required for achieving the acceptable accuracy of approximation $(q_1 \ll q)$.

From (41) we obtain [9]-[24], [35], [54] (for more details see [55]-[58])

$$E^{(i_1 i_2 i_3)q_1}\Big|_{q_1=6} \approx 0.01956000(T-t)^3. \qquad (42)$$

4. Approximation of Iterated Stochastic Integrals of Multiplicity k with Respect to the Q-Wiener Process

Consider the iterated Itô stochastic integral with respect to the Q-Wiener process in the following form

$$I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} =$$

$$= \int_t^T \Phi_k(Z)\left(\ldots\left(\int_t^{t_3} \Phi_2(Z)\left(\int_t^{t_2} \Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\right)\psi_2(t_2)d\mathbf{W}_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}, \qquad (43)$$

where $Z: \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0 \times \ldots \times U_0}_{k \text{ times}}$ to H for all $v \in H$, and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$.

Let $I[\Phi^{(k)}(Z), \psi^{(k)}]^M_{T,t}$ be an approximation of the stochastic integral (43)

$$I[\Phi^{(k)}(Z), \psi^{(k)}]^M_{T,t} =$$

$$= \int_t^T \Phi_k(Z)\left(\ldots\left(\int_t^{t_3} \Phi_2(Z)\left(\int_t^{t_2} \Phi_1(Z)\psi_1(t_1)d\mathbf{W}^M_{t_1}\right)\psi_2(t_2)d\mathbf{W}^M_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}^M_{t_k} =$$

$$= \sum_{r_1,r_2,\ldots,r_k \in J_M} \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k} \times$$

$$\times \sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\ J[\psi^{(k)}]^{(r_1 r_2 \ldots r_k)}_{T,t}, \qquad (44)$$

where $0 \le t < T \le \bar{T}$ and

$$J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_3} \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated Itô stochastic integral (17), $r_1, r_2, \ldots, r_k \in J_M$.

Let $I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}$ be an approximation of the stochastic integral (44)

$$I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k} = \sum_{r_1,r_2,\ldots,r_k\in J_M} \Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_k}}\ J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}, \tag{45}$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as a prelimit expression on the right-hand side of (31)

$$J[\psi^{(k)}]_{T,t}^{(r_1\ldots r_k)p_1\ldots p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)} + \sum_{m=1}^{[k/2]}(-1)^m\times\right.$$

$$\left.\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{m}\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}\right). \tag{46}$$

Let U, H be separable $\mathbb{R}$-Hilbert spaces, $U_0 = Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let $L(U,H)_0 = \{T|_{U_0} :\ T \in L(U,H)\}$ (here $T|_{U_0}$ is the restriction of operator T to the space U_0). It is known [7] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 3 [9], [10], [23]-[25]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. Furthermore, let the following conditions be satisfied*:

1. *$Q \in L(U)$ is a nonnegative and symmetric trace class operator (λ_i and e_i ($i \in J$) are its eigenvalues and eigenfunctions (which form an orthonormal basis of U) correspondingly), and $\mathbf{W}_\tau$, $\tau \in [0,\bar{T}]$ is an U-valued Q-Wiener process.*
2. *$Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping.*
3. *$\Phi_1 \in L(U,H)_0$, $\Phi_2 \in L(H, L(U,H)_0)$, and $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v \in H$ such that*

$$\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H^2 \le L_k < \infty$$

w. p. 1 for all $r_1, r_2, \ldots, r_k \in J_M$, $M \in \mathbb{N}$. Then

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M}-I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots p_k}\right\|_H^2\right\}\le
$$

$$
\le L_k(k!)^2\left(\text{tr }Q\right)^k\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right), \tag{47}
$$

where I_k *is defined by* (32), $\text{tr }Q=\sum\limits_{i\in J}\lambda_i$, *and*

$$
C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k,
$$

$$
K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ \\ 0,\ \text{otherwise}\end{cases}.
$$

Remark 2. *It should be noted that the right-hand side of the inequality* (47) *is independent of* M *and tends to zero if* $p_1,\ldots,p_k\to\infty$ *due to the Parseval equality.*

5. Approximation of Iterated Stochastic Integrals From the Exponential Milstein and Wagner–Platen Schemes for SPDEs

This section is devoted to the approximation of iterated stochastic integrals from the Milstein scheme (5) and Wagner–Platen scheme (6) for SPDEs. These integrals have the following form

$$
J_1[B(Z)]_{T,t}=\int\limits_t^T B(Z)d\mathbf{W}_s, \tag{48}
$$

$$
J_2[B(Z)]_{T,t}=A\left(\int\limits_t^T\int\limits_t^s B(Z)d\mathbf{W}_\tau ds-\frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_s\right), \tag{49}
$$

$$
J_3[B(Z),F(Z)]_{T,t}=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_2}\Bigl(AZ+F(Z)\Bigr)dt_1\right)d\mathbf{W}_{t_2}, \tag{50}
$$

$$
J_4[B(Z),F(Z)]_{T,t}=\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right)dt_2, \tag{51}
$$

$$I_1[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^s B(Z)d\mathbf{W}_\tau\right)d\mathbf{W}_s, \tag{52}$$

$$I_2[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \tag{53}$$

$$I_3[B(Z)]_{T,t} = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{54}$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $0\le t<T\le\bar{T}$.

Note that according to (7)–(10), (35), and (36) we can write w. p. 1

$$J_1[B(Z)]_{T,t}^M = \int\limits_t^T B(Z)d\mathbf{W}_s^M = (T-t)^{1/2}\sum_{r_1\in J_M} B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_0^{(r_1)},$$

$$J_2[B(Z)]_{T,t}^M = A\left(\int\limits_t^T\int\limits_t^s B(Z)d\mathbf{W}_\tau^M ds - \frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_s^M\right) =$$

$$= -\frac{(T-t)^{3/2}}{2\sqrt{3}}\sum_{r_1\in J_M} AB(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_1^{(r_1)}, \tag{55}$$

$$J_3[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2}\Bigl(AZ+F(Z)\Bigr)dt_1\right)d\mathbf{W}_{t_2}^M =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} B'(Z)\Bigl(AZ+F(Z)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \tag{56}$$

$$J_4[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2 =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} F'(Z)B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \tag{57}$$

where $\zeta_0^{(r_1)}$, $\zeta_1^{(r_1)}$ $(r_1\in J_M)$ are independent standard Gaussian random variables.

Let $I_1[B(Z)]_{T,t}^M$, $I_2[B(Z)]_{T,t}^M$, $I_3[B(Z)]_{T,t}^M$ be approximations of stochastic integrals (52)–(54), which have the following form (see (11), (12), and (16))

$$I_1[B(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^s B(Z)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M=$$

$$=\sum_{r_1,r_2\in J_M}B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(11)T,t}^{(r_1r_2)}, \tag{58}$$

$$I_2[B(Z)]_{T,t}^M=\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\mathbf{W}_{t_3}^M=$$

$$=\sum_{r_1,r_2,r_3\in J_M}B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}I_{(111)T,t}^{(r_1r_2r_3)}, \tag{59}$$

$$I_3[B(Z)]_{T,t}^M=\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M=$$

$$=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(I_{(111)T,t}^{(r_1r_2r_3)}+I_{(111)T,t}^{(r_2r_1r_3)}+\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(0r_3)}\right). \tag{60}$$

Let $I_1[B(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$, $I_3[B(Z)]_{T,t}^{M,q}$ be approximations of stochastic integrals (58)–(60), which are represented as follows

$$I_1[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2\in J_M}B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(11)T,t}^{(r_1r_2)q},$$

$$I_2[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M}B'(Z)\left(B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\right)e_{r_3}\times$$

$$\times\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}I_{(111)T,t}^{(r_1r_2r_3)q}, \tag{61}$$

$$I_3[B(Z)]_{T,t}^{M,q}=\sum_{r_1,r_2,r_3\in J_M}B''(Z)\left(B(Z)e_{r_1},B(Z)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times$$

$$\times\left(I_{(111)T,t}^{(r_1r_2r_3)q}+I_{(111)T,t}^{(r_2r_1r_3)q}+\mathbf{1}_{\{r_1=r_2\}}I_{(01)T,t}^{(0r_3)q}\right), \tag{62}$$

where $q\ge 1$ and the approximations $I_{(11)T,t}^{(r_1r_2)q}$, $I_{(111)T,t}^{(r_1r_2r_3)q}$, $I_{(111)T,t}^{(r_2r_1r_3)q}$ are defined by (38), (39).

Recall that $L_{HS}(U_0,H)$ is a space of Hilbert–Schmidt operators mapping from U_0 to H. Let $L_{HS}^{(2)}(U_0,H)$ and $L_{HS}^{(3)}(U_0,H)$ be spaces of bilinear and 3-linear Hilbert–Schmidt operators mapping from $U_0\times U_0$ to H and from $U_0\times U_0\times U_0$ to H correspondingly. Furthermore, let $\|\cdot\|_{L_{HS}(U_0,H)}$, $\|\cdot\|_{L_{HS}^{(2)}(U_0,H)}$, and $\|\cdot\|_{L_{HS}^{(3)}(U_0,H)}$ be operator norms in these spaces.

Theorem 4 [59] (also see [23]-[25], [60]). *Let the conditions* 1, 2 *of Theorem* 3 *be fulfilled. Let* $B(v)$ *be a Hilbert–Schmidt operator mapping from* U_0 *to* H *for all* $v\in H$, $B'(v)(B(v))$ *be a bilinear Hilbert–Schmidt operator mapping from* $U_0\times U_0$ *to* H *for all* $v\in H$, *and* $B'(v)(B'(v)(B(v)))$, $B''(v)(B(v),B(v))$ *be* 3*-linear Hilbert–Schmidt operators mapping from* $U_0\times U_0\times U_0$ *to* H *for all* $v\in H$ (*we suppose that Frêchet derivatives* B', B'' *exist* (*see Sect.* 2)). *Moreover, let there exists a constant* C *such that w. p.* 1

$$\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}<C,\quad \left\|B'(Z)(B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)}<C$$

$$\left\|B'(Z)(B'(Z)(B(Z)))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C,$$

$$\left\|B''(Z)(B(Z),B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)}<C,$$

for some $\alpha>0$. *Then*

$$\mathsf{M}\left\{\left\|I_1[B(Z)]_{T,t}-I_1[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le (T-t)^2\left(C_0\,(\text{tr }Q)^2\left(\frac{1}{2}-\sum_{j=1}^{q}\frac{1}{4j^2-1}\right)+K_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \tag{63}$$

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le (T-t)^3\left(C_1\,(\text{tr }Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+L_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \tag{64}$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le (T-t)^3\left(C_2\,(\mathrm{tr}\ Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+M_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right), \tag{65}$$

where $q\in\mathbb{N},\ C_0,C_1,C_2,K_Q,L_Q,M_Q<\infty,$ *and*

$$\hat{C}_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_j(x)$ $(j=0,1,2,\ldots)$ *is the Legendre polynomial.*

Remark 3. *Note that the estimate like* (63) *has been derived in* [8] (*also see* [2]) *with the difference connected with the first term on the right-hand side of* (63). *In* [8] *the authors used the Karhunen–Loeve expansion of the Brownian bridge process for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes. In this article we apply Theorem* 1 *and the system of Legendre polynomials for obtainment the first term on the right-hand side of* (63).

Proof. The estimate (63) directly follows from Theorem 3 of this article (the first term on the right-hand side of (63)) and Theorem 1 from [8] (the second term on the right-hand side of (63)). Further C_3, C_4, ... denote various constants.

Let us prove the estimates (64), (65). Using the elementary inequality $(a+b)^2\le 2(a^2+b^2)$ and Theorem 3, we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2\left(\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\right)\le$$

$$\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+C_3(T-t)^3\,(\mathrm{tr}\ Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \tag{66}$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2\left(\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\right). \tag{67}$$

Repeating with an insignificant modification the proof of Theorem 3 for the case $k=3$ (see for delails [9] (pp. 39–44) or [10], [23]-[25]), we have

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}^{M}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 4C(3!)^2\left(\text{tr }Q\right)^3(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right), \tag{68}$$

where constant C has the same meaning as constant L_k in Theorem 3 (k is the multiplicity of the iterated stochastic integral).

Combining (67) and (68), we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+C_4(T-t)^3\left(\text{tr }Q\right)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right). \tag{69}$$

Let us evaluate the values

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\},\quad \mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}.$$

Using the elementary inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$, we have

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(E_{T,t}^{1,M}+E_{T,t}^{2,M}+E_{T,t}^{3,M}\right), \tag{70}$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M}\right\|_H^2\right\}\le 3\left(G_{T,t}^{1,M}+G_{T,t}^{2,M}+G_{T,t}^{3,M}\right), \tag{71}$$

where

$$E_{T,t}^{1,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{2,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{3,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\left(\mathbf{W}_{t_3}-\mathbf{W}_{t_3}^M\right)\right\|_H^2\right\},$$

$$G_{T,t}^{1,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1},\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{2,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right),\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{3,M}=\mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M,\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right\|_H^2\right\}.$$

We have

$$E_{T,t}^{1,M}=\int\limits_t^T\mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3\le$$

$$\le C_5\int\limits_t^T\mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\}dt_3=$$

$$=C_5\int\limits_t^T\int\limits_t^{t_3}\mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2dt_3\le$$

$$\leq C_6 \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\} dt_2 dt_3 \leq \tag{72}$$

$$\leq C_6 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_1 dt_2 dt_3 \leq \tag{73}$$

$$\leq C_7 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \tag{74}$$

Note that the transition from (72) to (73) was made by analogy with the proof of Theorem 1 in [8] (also see [2]). More precisely, taking into account the relation $Q^\alpha e_i = \lambda_i^\alpha e_i$, we have (see [8], Sect. 3.1)

$$\mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\} =$$

$$= \mathsf{M}\left\{\left\|\sum_{i\in J\backslash J_M} \sqrt{\lambda_i} \int\limits_t^{t_2} B(Z)e_i d\mathbf{w}_{t_1}^{(i)}\right\|_H^2\right\} =$$

$$= \sum_{i\in J\backslash J_M} \lambda_i \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}Q^\alpha e_i\right\|_H^2\right\} dt_1 =$$

$$= \sum_{i\in J\backslash J_M} \lambda_i^{1+2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\} dt_1 =$$

$$= \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{\sum_{i\in J\backslash J_M} \lambda_i \left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\} dt_1 \leq$$

$$\leq \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{\sum_{i\in J} \lambda_i \left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\} dt_1 =$$

$$= \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_1. \tag{75}$$

Further we also will use the estimate like (75). We have

$$E^{2,M}_{T,t} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}^M_{t_2}\right)\right)\right\|^2_{L_{HS}(U_0,H)}\right\}dt_3 \le$$

$$\le C_8\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}^M_{t_2}\right)\right\|^2_{H}\right\}dt_3 \le$$

$$\le C_8\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}dt_2dt_3 \le$$

$$\le C_8\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(B(Z)\right)Q^{-\alpha}\right\|^2_{L^{(2)}_{HS}(U_0,H)}\right\}(t_2-t)dt_2dt_3 \le$$

$$\le C_9\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3, \tag{76}$$

$$E^{3,M}_{T,t} \le \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\times$$

$$\times\int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}^M_{t_1}\right)d\mathbf{W}^M_{t_2}\right)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}dt_3 \le$$

$$\le C_{10}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(B'(Z)\left(B(Z)\right)\right)Q^{-\alpha}\right\|^2_{L^{(3)}_{HS}(U_0,H)}\right\}\frac{(t_3-t)^2}{2}dt_3 \le$$

$$\le C_{11}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \tag{77}$$

Combining (66), (70), (74)–(77), we obtain (64). We have

$$G_{T,t}^{1,M}=\int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1},\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3\le$$

$$\le C_{12}\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^2\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_3\le$$

$$\le C_{12}\int\limits_t^T\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^4\right\}\right)^{1/2}\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\le$$

$$\le C_{13}\int\limits_t^T\int\limits_t^{t_2}\left(\mathsf{M}\left\{\left\|B(Z)\right\|_{L_{HS}(U_0,H)}^4\right\}\right)^{1/2}dt_1\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3\le$$

$$\le C_{14}\int\limits_t^T(t_2-t)\left(\mathsf{M}\left\{\left\|\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}\right)^{1/2}dt_3. \tag{78}$$

Let us estimate the right-hand side of (78). Let $s>t$. For fixed $M\in\mathbb{N}$ and for some $N>M$ ($N\in\mathbb{N}$) we have

$$\mathsf{M}\left\{\left\|\int\limits_t^{s}B(Z)d\left(\mathbf{W}_{t_1}^N-\mathbf{W}_{t_1}^M\right)\right\|_H^4\right\}=$$

$$=\mathsf{M}\left\{\left\langle\sum_{j\in J_N\setminus J_M}\sqrt{\lambda_j}B(Z)e_j\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right),\sum_{j'\in J_N\setminus J_M}\sqrt{\lambda_{j'}}B(Z)e_{j'}\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\right\rangle_H^2\right\}=$$

$$=\sum_{j,j',l,l'\in J_N\setminus J_M}\sqrt{\lambda_j\lambda_{j'}\lambda_l\lambda_{l'}}\ \ \mathsf{M}\left\{\left\langle B(Z)e_j,B(Z)e_{j'}\right\rangle_H\left\langle B(Z)e_l,B(Z)e_{l'}\right\rangle_H\times\right.$$

$$\left.\times\mathsf{M}\left\{\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right)\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\left(\mathbf{w}_s^{(l)}-\mathbf{w}_t^{(l)}\right)\left(\mathbf{w}_s^{(l')}-\mathbf{w}_t^{(l')}\right)\Big|\mathbf{F}_t\right\}\right\}=$$

$$= 3(s-t)^2 \sum_{j\in J_N\backslash J_M} \lambda_j^2 \mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^4\right\} +$$

$$+(s-t)^2 \sum_{j,j'\in J_N\backslash J_M (j\ne j')} \lambda_j\lambda_{j'}\left(\mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^2\Bigl\|B(Z)e_{j'}\Bigr\|_H^2\right\}+2\Bigl\langle B(Z)e_j, B(Z)e_{j'}\Bigr\rangle_H^2\right) \le$$

$$\le 3(s-t)^2\left(\sum_{j\in J_N\backslash J_M} \lambda_j^2 \mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^4\right\} +\right.$$

$$\left.+\sum_{j,j'\in J_N\backslash J_M (j\ne j')} \lambda_j\lambda_{j'} \mathsf{M}\left\{\Bigl\|B(Z)e_j\Bigr\|_H^2\Bigl\|B(Z)e_{j'}\Bigr\|_H^2\right\}\right) =$$

$$= 3(s-t)^2 \mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M} \lambda_j \Bigl\|B(Z)e_j\Bigr\|_H^2\right)^2\right\} \le$$

$$\le 3(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha} \mathsf{M}\left\{\left(\sum_{j\in J_N\backslash J_M} \lambda_j \Bigl\|B(Z)Q^{-\alpha}e_j\Bigr\|_H^2\right)^2\right\} \le$$

$$\le C_{15}(s-t)^2\left(\sup_{i\in J_N\backslash J_M}\lambda_i\right)^{4\alpha} \mathsf{M}\left\{\Bigl\|B(Z)Q^{-\alpha}\Bigr\|_{L_{HS}(U_0,H)}^4\right\}. \tag{79}$$

Performing the passage to the limit $\lim\limits_{N\to\infty}$ in (79) and using (78), we have

$$G_{T,t}^{1,M} \le C_{16}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \tag{80}$$

Absolutely analogously we obtain

$$G_{T,t}^{2,M} \le C_{17}\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}(T-t)^3. \tag{81}$$

Let us estimate $G_{T,t}^{3,M}$. We have

$$G_{T,t}^{3,M} \le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times$$

$$\times \int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2 \le$$

$$\le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \sum_{i\in J}\sum_{j,l\in J_M} \lambda_i\lambda_j\lambda_l \int\limits_t^T (t_2-t)^2 \times$$

$$\times\Biggl(\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H^2\right\}+$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_j)Q^{-\alpha}e_i\right\|_H \times\right.$$

$$\left.\times\left\|B''(Z)(B(Z)e_l, B(Z)e_l)Q^{-\alpha}e_i\right\|_H\right\}+$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H \times\right.$$

$$\left.\times\left\|B''(Z)(B(Z)e_l, B(Z)e_j)Q^{-\alpha}e_i\right\|_H\right\}\Biggr)dt_2 \le$$

$$\le C_{18}\left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \tag{82}$$

Combaining (69), (71), and (80)–(82), we get (65). Theorem 4 is proved.

Let us consider the convergence analysis for the stochastic integrals (49)–(51) (convergence of the stochastic integral (48) follows from (75) (see Theorem 1 in [8] or [2])).

Using the Itô formula, we obtain w. p. 1 [3]

$$J_2[B(Z)]_{T,t} = \int\limits_t^T \left(\frac{T}{2} - s + \frac{t}{2}\right) AB(Z)d\mathbf{W}_s,$$

$$J_3[B(Z), F(Z)]_{T,t} = \int\limits_t^T (s-t)B'(Z)\Bigl(AZ + F(Z)\Bigr)d\mathbf{W}_s.$$

Suppose that

$$\mathsf{M}\left\{\left\|B'(Z)\Bigl(AZ+F(Z)\Bigr)Q^{-\alpha}\right\|^2_{L_{HS(U_0,H)}}\right\}<\infty,$$

$$\mathsf{M}\left\{\left\|AB(Z)Q^{-\alpha}\right\|^2_{L_{HS(U_0,H)}}\right\}<\infty$$

for some $\alpha>0$.

Then by analogy with (75) we get

$$\mathsf{M}\left\{\left\|J_2[B(Z)]_{T,t}-J_2[B(Z)]^M_{T,t}\right\|^2_H\right\}\le$$

$$\le C_{19}(T-t)^3\left(\sup_{i\in J\setminus J_M}\lambda_i\right)^{2\alpha},$$

$$\mathsf{M}\left\{\left\|J_3[B(Z),F(Z)]_{T,t}-J_3[B(Z),F(Z)]^M_{T,t}\right\|^2_H\right\}\le$$

$$\le C_{20}(T-t)^3\left(\sup_{i\in J\setminus J_M}\lambda_i\right)^{2\alpha},$$

where $J_2[B(Z)]^M_{T,t}$, $J_3[B(Z),F(Z)]^M_{T,t}$ are defined by (55), (56).

Moreover, in conditions of Theorem 4 we obtain

$$\mathsf{M}\left\{\left\|J_4[B(Z),F(Z)]_{T,t}-J_4[B(Z),F(Z)]^M_{T,t}\right\|^2_H\right\}=$$

$$=\mathsf{M}\left\{\left\|\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}^M_{t_1}\right)\right)dt_2\right\|^2_H\right\}\le$$

$$\le (T-t)\int\limits_t^T\mathsf{M}\left\{\left\|F'(Z)\left(\int\limits_t^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}^M_{t_1}\right)\right)\right\|^2_H\right\}dt_2\le$$

$$\leq C_{21}(T-t)\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2 \leq$$

$$\leq C_{21}(T-t)\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1 dt_2 \leq$$

$$\leq C_{22}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},$$

where $J_4[B(Z),F(Z)]_{T,t}^M$ is defined by (57).

References

[1] Jentzen A., Kloeden P.E. Taylor approximations for stochastic partial differential equations. SIAM, Philadelphia, 2011, 224 pp.

[2] Jentzen A., Röckner M. A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.

[3] Becker S., Jentzen A., Kloeden P.E. An exponential Wagner-Platen type scheme for SPDEs. SIAM J. Numer. Anal. 54, 4 (2016), 2389-2426.

[4] Jentzen A., Röckner M. Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. J. Differ. Eq. 252, 1 (2012), 114-136.

[5] Da Prato G., Jentzen A., Röckner M. A mild Itô formula for SPDEs. arXiv:1009.3526 [math.PR] (2012), 39 pp.

[6] Da Prato G., Zabczyk J. Stochastic equations in infinite dimensions. 2nd Ed. Cambridge Univ. Press, Cambridge, 2014, 493 pp.

[7] Prévôt C., Röckner M. A concise course on stochastic partial differential equations, V. 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007, 148 pp.

[8] Leonhard C., Rößler A. Iterated stochastic integrals in infinite dimensions: approximation and error estimates. Stoch. PDE: Anal. Comp. 7, 2 (2018), 209-239.

[9] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[10] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[11] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House: Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[16] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[18] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House: Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[20] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1-A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[23] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 859 pp. [In English].

[24] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[25] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[26] Kuznetsov D.F. Development and Application of the Fourier Method for the Numerical Solution of Ito Stochastic Differential Equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[27] Kuznetsov D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 1.5 and 2.0 Orders of Strong Convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[28] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals, based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 50-79. DOI: 10.13108/2019-11-4-49, Available at: http://matem.anrb.ru/en/article?art_id=604

[29] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[30] Kuznetsov D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[31] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

[32] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2017, 107 pp. [in English].

[33] Kuznetsov D.F. Developement and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 46 pp. [in English].

[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 57 pp. [in English].

[35] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 100 pp. [in English].

[36] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. arXiv:1712.09516 [math.PR], 2022, 173 pp. [in English].

[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 74 pp. [in English].

[38] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 59 pp. [in English].

[39] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combained approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 41 pp. [In English].

[40] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR]. 2018, 18 pp. [In English].

[41] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2018, 91 pp. [in English].

[42] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2018, 104 pp. [in English].

[43] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 34 pp. [In English].

[44] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 37 pp. [In English].

[45] Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 22 pp. [In English].

[46] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1806.10705 [math.PR]. 2018, 24 pp. [In English].

[47] Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1805.12527 [math.PR]. 2018, 23 pp. [In English].

[48] Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv: 1802.04844 [math.PR]. 2018, 31 pp. [in English].

[49] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR], 2018, 39 pp. [in English].

[50] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method arXiv:1801.01962 [math.PR]. 2018, 30 pp. [in English].

[51] Kuznetsov D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. arXiv:1801.04634 [math.PR], 2018, 27 pp. [in English].

[52] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor-Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. arXiv:1801.08862 [math.PR], 2018, 30 p. [in English].

[53] Kuznetsov D.F. New simple method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on expansion of the Brownian motion using Legendre polynomials and trigonometric functions. arXiv:1807.00409 [math.PR], 2019, 20 pp. [in English].

[54] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 80 pp. [in English].

[55] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[56] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 336 pp. [In English].

[57] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 59 pp. [In English].

[58] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[59] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[60] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[61] Kloeden P.E. and Platen E. Numerical solution of stochastic differential equations. Springer-Verlag, Berlin, 1992, 632 pp.

[62] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[63] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publ., 204 pp. (ISBN 5-7422-0045-5)

[64] Kuznetsov D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Institute of Applied Mathematics and Mechanics,
Department of Mathematics,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Chapter 6.

Integration Order Replacement Technique for Iterated Ito Stochastic Integrals

INTEGRATION ORDER REPLACEMENT TECHNIQUE FOR ITERATED ITO STOCHASTIC INTEGRALS AND ITERATED STOCHASTIC INTEGRALS WITH RESPECT TO MARTINGALES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. We consider the class of iterated Ito stochastic integrals, for which with probability 1 the formulas of integration order replacement corresponding to the rules of classical integral calculus are correct. The theorems on integration order replacement for the class of iterated Ito stochastic integrals are proved. Many examples of this theorems usage have been considered. These results are generalized for the class of iterated stochastic integrals with respect to martingales.

Contents

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Keywords: Iterated Ito stochastic integral, Iterated stochastic integral with respect to martingales, Integration order replacement technique, Ito formula.

1. Introduction

In this article, we performed rather laborious work connected with the theorems on integration order replacement for iterated Ito stochastic integrals. However, there may appear a question about a practical usefulness of this theory, since the significant part of its conclusions directly follows from the Ito formula [1].

It is not difficult to see that to obtain various relations for iterated Ito stochastic integrals (see, for example, Sect. 6) using the Ito formula, first of all these relations should be guessed. Then it is necessary to introduce corresponding Ito processes and afterwards to use the Ito formula. It is clear that this process requires intellectual expenses and it is not always trivial.

On the other hand, the technique on integration order replacement introduced in this article is formally comply with the similar technique for Riemann integrals, although it is related to Ito integrals, and it provides a possibility to perform transformations naturally (as with Riemann integrals) with iterated Ito stochastic integrals and to obtain various relations for them.

So, in order to implementation of transformations of the specific class of Ito processes, which is represented by iterated Ito stochastic integrals, it is more naturally and easier to use the theorems on integration order replacement, than the Ito formula.

Many examples of these theorems usage are presented in Sect. 6.

Note that in a lot of publications of the author [2]-[18] the integration order replacement technique for iterated Ito stochastic integrals has been successfully applied for the proof and development of the method of approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series as well as for the construction of the so-called unified Taylor–Ito and Taylor–Stratonovich expansions.

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a complete probability space and let $f(t,\omega) : [0,T]\times\Omega\to\mathbb{R}$ be the standard Wiener process defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$. Further, we will use the following notation: $f(t,\omega) \stackrel{\rm def}{=} f_t$.

Let us consider the family of σ-algebras $\{\mathrm{F}_t,\ t\in[0,T]\}$ defined on the probability space $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the Wiener process f_t in such a way that

1. $\mathrm{F}_s\subset \mathrm{F}_t\subset \mathrm{F}$ for $s<t$.
2. The Wiener process f_t is F_t-measurable for all $t\in[0,T]$.
3. The process $f_{t+\Delta}-f_t$ for all $t\ge 0,$ $\Delta>0$ is independent with the events of σ-algebra F_t.

Let us introduce the class $\mathrm{M}_2([0,T])$ of functions $\xi : [0,T]\times\Omega\to\mathbb{R}$, which satisfy the conditions:

1. The function $\xi(t,\omega)$ is measurable with respect to the pair of variables (t,ω).
2. The function $\xi(t,\omega)$ is F_t-measurable for all $t\in[0,T]$ and $\xi(\tau,\omega)$ is independent with increments $f_{t+\Delta}-f_t$ for $t\ge\tau,$ $\Delta>0$.
3. The following relation is fulfilled

$$\int\limits_0^T \mathsf{M}\left\{\left(\xi(t,\omega)\right)^2\right\}dt<\infty.$$

4. $\mathsf{M}\left\{\left(\xi(t,\omega)\right)^2\right\}<\infty$ for all $t\in[0,T]$.

For any partition $\tau_j^{(N)},$ $j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \ \text{if}\ \ N\to\infty \tag{1}$$

we will define the sequense of step functions

$$\xi^{(N)}(t,\omega)=\xi_j(\omega)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),$$

where $j=0,1,\ldots,N-1,\ \ N=1,2,\ldots$. Here and further, w. p. 1 means with probability 1.

Let us define the Ito stochastic integral for $\xi(t,\omega)\in \mathrm{M}_2([0,T])$ as the following mean-square limit [1]

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(f\left(\tau_{j+1}^{(N)},\omega\right)-f\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\xi_\tau df_\tau, \tag{2}$$

where $\xi^{(N)}(t,\omega)$ is any step function, which converges to the function $\xi(t,\omega)$ in the following sense

$$\lim_{N\to\infty}\int\limits_0^T\mathsf{M}\left\{\left|\xi^{(N)}(t,\omega)-\xi(t,\omega)\right|^2\right\}dt=0. \tag{3}$$

It is well known [1] that the Ito stochastic integral exists as the limit (2) and it does not depend on the selection of sequence $\xi^{(N)}(t,\omega)$. We suppose that standard properties of the Ito stochastic integral are well known to the reader (see, for example, [1]).

Let us define the stochastic integral for $\xi_\tau\in\mathrm{M}_2([0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(\tau_{j+1}^{(N)}-\tau_j^{(N)}\right)\stackrel{\text{def}}{=}\int\limits_0^T\xi_\tau d\tau,$$

where $\xi^{(N)}(t,\omega)$ is any step function from the class $\mathrm{M}_2([0,T])$, which converges in the sense (3) to the function $\xi(t,\omega)$.

We will introduce the class $\mathrm{S}_2([0,T])$ of functions $\xi:[0,T]\times\Omega\to\mathbb{R}$, which satisfy the conditions:

1. $\xi(\tau,\omega)\in\mathrm{M}_2([0,T])$.
2. $\xi(\tau,\omega)$ is the mean-square continuous random process at the interval $[0,T]$.

As we noted above, the Ito stochastic integral exists in the mean-square sense (see (2)), if the random process $\xi(\tau,\omega)\in\mathrm{M}_2([0,T])$, i.e., perhaps this process does not satisfy the property of the mean-square continuity on the interval $[0,T]$. In this article we will formulate and prove the theorems on integration order replacement for the special class of iterated Ito stochastic integrals. At the same time, the condition of the mean-square continuity of integrand in the innermost stochastic integral will be significant.

Let us introduce the following class of iterated stochastic integrals

$$J[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\phi_\tau dw_\tau^{(k+1)}dw_{t_k}^{(k)}\ldots dw_{t_1}^{(1)},$$

where $\phi(\tau,\omega)\stackrel{\text{def}}{=}\phi_\tau$, $\phi_\tau\in\mathrm{S}_2([t,T])$, every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continous nonrandom function at the interval $[t,T]$, here and further $w_\tau^{(l)}=f_\tau$ or $w_\tau^{(l)}=\tau$ for $\tau\in[t,T]$ $(l=1,\ldots,k+1)$, $(\psi_1,\ldots,\psi_k)\stackrel{\text{def}}{=}\psi^{(k)}$, $\psi^{(1)}\stackrel{\text{def}}{=}\psi_1$.

We will call the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$ as the iterated Ito stochastic integral.

It is well known that for the iterated Riemann integral in the case of specific conditions the formula on integration order replacement is correct. In particular, if the nonrandom functions $f(x)$ and $g(x)$ are continuous at the interval $[a,b]$, then

$$\int\limits_a^b f(x)\int\limits_a^x g(y)dydx=\int\limits_a^b g(y)\int\limits_y^b f(x)dxdy. \tag{4}$$

If we suppose that for the Ito stochastic integral

$$J[\phi,\psi_1]_{T,t}=\int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)}$$

the formula on integration order replacement, which is similar to (4), is valid, then we will have

$$\int\limits_t^T \psi_1(s)\int\limits_t^s \phi_\tau dw_\tau^{(2)}dw_s^{(1)}=\int\limits_t^T \phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}dw_\tau^{(2)}. \tag{5}$$

If, in addition $w_s^{(1)},\ w_s^{(2)}=f_s\ (s\in[t,T])$ in (5), then the stochastic process

$$\eta_\tau=\phi_\tau\int\limits_\tau^T \psi_1(s)dw_s^{(1)}$$

does not belong to the class $\mathrm{M}_2([t,T])$, and, consequently, for the Ito stochastic integral

$$\int\limits_t^T \eta_\tau dw_\tau^{(2)}$$

on the right-hand side of (5) the conditions of its existence are not fulfilled.

At the same time

$$\int\limits_t^T df_s\int\limits_t^T ds=\int\limits_t^T (s-t)df_s+\int\limits_t^T (f_s-f_t)ds\quad \text{w. p. 1}, \tag{6}$$

and we can obtain this equality, for example, using the Ito formula, but (6) can be considered as a result of integration order replacement (see below).

Actually, we can demonstrate that

$$\int\limits_t^T (f_s-f_t)ds=\int\limits_t^T\int\limits_t^s df_\tau ds=\int\limits_t^T\int\limits_\tau^T dsdf_\tau\quad \text{w. p. 1}.$$

Then

$$\int\limits_t^T (s-t)df_s+\int\limits_t^T (f_s-f_t)ds=\int\limits_t^T\int\limits_t^\tau dsdf_\tau+\int\limits_t^T\int\limits_\tau^T dsdf_\tau=\int\limits_t^T df_s\int\limits_t^T ds\quad \text{w. p. 1}.$$

The aim of this article is to establish the strict mathematical sense of the formula (5) for the case $w_s^{(1)},\ w_s^{(2)}=f_s\ (s\in[t,T])$ as well as its analogue corresponding to the iterated Ito stochastic integral $J[\phi,\psi^{(k)}]_{T,t},\ k\ge 2$. At that, we will use the definition of the Ito stochastic integral which is more general than (2).

Let us consider the partition $\tau_j^{(N)},\ j=0,1,\ldots,N$ of the interval $[t,T]$ such that

$$t=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \text{ if }\ N\to\infty. \tag{7}$$

In [19] Stratonovich R.L. introduced the definition of the so-called combined stochastic integral for the specific class of integrated processes. Taking this definition as a foundation, let us consider the following construction of stochastic integral

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\phi_{\tau_j}\left(f_{\tau_{j+1}}-f_{\tau_j}\right)\theta_{\tau_{j+1}}\stackrel{\text{def}}{=}\int\limits_t^T\phi_\tau df_\tau\theta_\tau, \tag{8}$$

where $\phi_\tau,\ \theta_\tau\in\mathrm{S}_2([t,T])$, $\{\tau_j\}_{j=0}^N$ is the partition of the interval $[t,T]$, which satisfies to the condition (1) (for simplicity we write here and sometimes further τ_j instead of $\tau_j^{(N)}$).

Further, we will prove existence of the integral (8) for $\phi_\tau\in\mathrm{S}_2([t,T])$ and θ_τ from a little bit narrower class of processes than $\mathrm{S}_2([t,T])$. In addition, the integral defined by (8) will be used for the formulation and proof of the theorem on integration order replacement for the iterated Ito stochastic integrals $J[\phi,\psi^{(k)}]_{T,t},\ k\ge 1$.

Note that under the appropriate conditions the following properties of stochastic integrals defined by the formula (8) can be proved

$$\int\limits_t^T\phi_\tau df_\tau g(\tau)=\int\limits_t^T\phi_\tau g(\tau)df_\tau\quad\text{w. p. 1},$$

where $g(\tau)$ is a continuous nonrandom function at the interval $[t,T]$,

$$\int\limits_t^T\left(\alpha\phi_\tau+\beta\psi_\tau\right)df_\tau\theta_\tau=\alpha\int\limits_t^T\phi_\tau df_\tau\theta_\tau+\beta\int\limits_t^T\psi_\tau df_\tau\theta_\tau\quad\text{w. p. 1},$$

$$\int\limits_t^T\phi_\tau df_\tau\left(\alpha\theta_\tau+\beta\psi_\tau\right)=\alpha\int\limits_t^T\phi_\tau df_\tau\theta_\tau+\beta\int\limits_t^T\phi_\tau df_\tau\psi_\tau\quad\text{w. p. 1},$$

where $\alpha,\ \beta\in\mathbb{R}$.

At that, we suppose that the stochastic processes ϕ_τ, θ_τ, and ψ_τ are such that the integrals included in the mentioned properties exist.

2. Formulation of the Theorem on Integration Order Replacement for Iterated Ito Stochastic Integrals of Arbitrary Multiplicity

Let us define the stochastic integrals $\hat{I}[\psi^{(k)}]_{T,s},\ k\ge 1$ of the form

$$\hat{I}[\psi^{(k)}]_{T,s}=\int\limits_s^T\psi_k(t_k)dw_{t_k}^{(k)}\int\limits_{t_k}^T\psi_{k-1}(t_{k-1})dw_{t_{k-1}}^{(k-1)}\ldots\int\limits_{t_2}^T\psi_1(t_1)dw_{t_1}^{(1)}$$

in accordance with the definition (8) by the following recurrence relation

$$\hat{I}[\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\psi_k(\tau_l)\Delta w_{\tau_l}^{(k)}\hat{I}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \tag{9}$$

where $k\ge 1$, $\hat{I}[\psi^{(0)}]_{T,s}\stackrel{\rm def}{=}1$, $[s,T]\subseteq[t,T]$, here and further $\Delta w_{\tau_l}^{(i)}=w_{\tau_{l+1}}^{(i)}-w_{\tau_l}^{(i)},\ \ i=1,\ldots,k+1$, $l=0,1,\ldots,N-1$.

Then, we will define the iterated stochastic integral $\hat{J}[\phi,\psi^{(k)}]_{T,t}$, $k\ge 1$

$$\hat{J}[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\phi_s dw_s^{(k+1)}\hat{I}[\psi^{(k)}]_{T,s}$$

similarly in accordance with the definition (8)

$$\hat{J}[\phi,\psi^{(k)}]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}.$$

Let us formulate the theorem on integration order replacement for iterated Ito stochastic integrals.

Theorem 1 [20], [21] (also see [2]-[7], [16]-[18]). *Suppose that $\phi_\tau\in{\rm S}_2([t,T])$ and every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$. Then, the stochastic integral $\hat{J}[\phi,\psi^{(k)}]_{T,t}$ $(k\ge 1)$ exists and*

$$J[\phi,\psi^{(k)}]_{T,t}=\hat{J}[\phi,\psi^{(k)}]_{T,t}\quad w.\ p.\ 1.$$

3. Proof of Theorem 1 for the Case of Iterated Ito Stochastic Integrals of Multiplicity 2

First, let us prove Theorem 1 for the case $k=1$. We have

$$J[\phi,\psi_1]_{T,t}\stackrel{\rm def}{=}\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\psi_1(\tau_l)\Delta w_{\tau_l}^{(1)}\int\limits_t^{\tau_l}\phi_\tau dw_\tau^{(2)}=$$

$$=\underset{N\to\infty}{\rm l.i.m.}\sum_{l=0}^{N-1}\psi_1(\tau_l)\Delta w_{\tau_l}^{(1)}\sum_{j=0}^{l-1}\int\limits_{\tau_j}^{\tau_{j+1}}\phi_\tau dw_\tau^{(2)}, \tag{10}$$

$$\hat{J}[\phi,\psi_1]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.} \sum_{j=0}^{N-1} \phi_{\tau_j} \Delta w_{\tau_j}^{(2)} \int\limits_{\tau_{j+1}}^{T} \psi_1(s) dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\rm l.i.m.} \sum_{j=0}^{N-1} \phi_{\tau_j} \Delta w_{\tau_j}^{(2)} \sum_{l=j+1}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s) dw_s^{(1)} =$$

$$= \underset{N\to\infty}{\rm l.i.m.} \sum_{l=0}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \psi_1(s) dw_s^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j} \Delta w_{\tau_j}^{(2)}. \qquad (11)$$

It is clear that if the difference ε_N of prelimit expressions on the right-hand sides of (10) and (11) tends to zero when $N \to \infty$ in the mean-square sense, then the stochastic integral $\hat{J}[\phi,\psi_1]_{T,t}$ exists and

$$J[\phi,\psi_1]_{T,t} = \hat{J}[\phi,\psi_1]_{T,t} \quad \text{w. p. 1.}$$

The difference ε_N can be presented in the form $\varepsilon_N = \tilde{\varepsilon}_N + \hat{\varepsilon}_N$, where

$$\tilde{\varepsilon}_N = \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w_{\tau_l}^{(1)} \sum_{j=0}^{l-1} \int\limits_{\tau_j}^{\tau_{j+1}} \left(\phi_\tau - \phi_{\tau_j}\right) dw_\tau^{(2)};$$

$$\hat{\varepsilon}_N = \sum_{l=0}^{N-1} \int\limits_{\tau_l}^{\tau_{l+1}} \left(\psi_1(\tau_l) - \psi_1(s)\right) dw_s^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j} \Delta w_{\tau_j}^{(2)}.$$

We will demonstrate that

$$\underset{N\to\infty}{\rm l.i.m.}\ \varepsilon_N = 0.$$

In order to do it we will analyze four cases:

1. $w_\tau^{(2)} = f_\tau,\ \Delta w_{\tau_l}^{(1)} = \Delta f_{\tau_l}$.
2. $w_\tau^{(2)} = \tau,\ \Delta w_{\tau_l}^{(1)} = \Delta f_{\tau_l}$.
3. $w_\tau^{(2)} = f_\tau,\ \Delta w_{\tau_l}^{(1)} = \Delta \tau_l$.
4. $w_\tau^{(2)} = \tau,\ \Delta w_{\tau_l}^{(1)} = \Delta \tau_l$.

Consider the well known standard moment properties of stochastic integrals [1]

$$\mathsf{M}\left\{\left|\int\limits_{t_0}^{t} \xi_\tau df_\tau\right|^2\right\} = \int\limits_{t_0}^{t} \mathsf{M}\left\{|\xi_\tau|^2\right\} d\tau,$$

$$\mathsf{M}\left\{\left|\int\limits_{t_0}^{t}\xi_\tau d\tau\right|^2\right\} \le (t-t_0)\int\limits_{t_0}^{t}\mathsf{M}\left\{|\xi_\tau|^2\right\}d\tau, \tag{12}$$

where $\xi_\tau \in \mathrm{M}_2([t_0, t])$.

For Case 1 using standard moment properties for the Ito stochastic integral as well as mean-square continuity (which means uniform mean-square continuity) of the process ϕ_τ on the interval $[t, T]$, we obtain

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} = \sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}\mathsf{M}\left\{\left|\phi_\tau - \phi_{\tau_j}\right|^2\right\}d\tau <$$

$$< C^2\varepsilon\sum_{k=0}^{N-1}\Delta\tau_k\sum_{j=0}^{k-1}\Delta\tau_j < C^2\varepsilon\frac{(T-t)^2}{2},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Let us consider Case 2. Using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (12) for the stochastic integral, we have

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} = \sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\mathsf{M}\left\{\left(\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau - \phi_{\tau_j})d\tau\right)^2\right\} \le$$

$$\le \sum_{k=0}^{N-1}\psi_1^2(\tau_k)\Delta\tau_k\left(\sum_{j=0}^{k-1}\left(\mathsf{M}\left\{\left(\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau - \phi_{\tau_j})d\tau\right)^2\right\}\right)^{1/2}\right)^2 <$$

$$< C^2\varepsilon\sum_{k=0}^{N-1}\Delta\tau_k\left(\sum_{j=0}^{k-1}\Delta\tau_j\right)^2 < C^2\varepsilon\frac{(T-t)^3}{3},$$

i.e. $\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

For Case 3 using the Minkowski inequality, standard moment properties for the Ito stochastic integral as well as uniform mean-square continuity of the process ϕ_τ, we find

$$\mathsf{M}\left\{|\tilde{\varepsilon}_N|^2\right\} \le \left(\sum_{k=0}^{N-1}|\psi_1(\tau_k)|\Delta\tau_k\left(\mathsf{M}\left\{\left(\sum_{j=0}^{k-1}\int\limits_{\tau_j}^{\tau_{j+1}}(\phi_\tau - \phi_{\tau_j})df_\tau\right)^2\right\}\right)^{1/2}\right)^2 =$$

$$= \left(\sum_{k=0}^{N-1} |\psi_1(\tau_k)| \Delta\tau_k \left(\sum_{j=0}^{k-1} \int\limits_{\tau_j}^{\tau_{j+1}} \mathsf{M}\left\{ |\phi_\tau - \phi_{\tau_j}|^2 \right\} d\tau \right)^{1/2} \right)^2 <$$

$$< C^2 \varepsilon \left(\sum_{k=0}^{N-1} \Delta\tau_k \left(\sum_{j=0}^{k-1} \Delta\tau_j \right)^{1/2} \right)^2 < C^2 \varepsilon \frac{4(T-t)^3}{9},$$

i.e. $\mathsf{M}\left\{ |\tilde{\varepsilon}_N|^2 \right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Finally, for Case 4 using the Minkowski inequality, uniform mean-square continuity of the process ϕ_τ as well as the estimate (12) for the stochastic integral, we obtain

$$\mathsf{M}\left\{ |\tilde{\varepsilon}_N|^2 \right\} \le \left(\sum_{k=0}^{N-1} \sum_{j=0}^{k-1} |\psi_1(\tau_k)| \Delta\tau_k \left(\mathsf{M}\left\{ \left(\int\limits_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_{\tau_j}) d\tau \right)^2 \right\} \right)^{1/2} \right)^2 <$$

$$< C^2 \varepsilon \left(\sum_{k=0}^{N-1} \Delta\tau_k \sum_{j=0}^{k-1} \Delta\tau_j \right)^2 < C^2 \varepsilon \frac{(T-t)^4}{4},$$

i.e. $\mathsf{M}\left\{ |\tilde{\varepsilon}_N|^2 \right\} \to 0$ when $N \to \infty$. Here $\Delta\tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N-1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on τ), $|\psi_1(\tau)| < C$.

Thus, we have proved that

$$\underset{N\to\infty}{\text{l.i.m.}}\ \tilde{\varepsilon}_N = 0.$$

Analogously, taking into account the uniform continuity of the function $\psi_1(\tau)$ on the interval $[t, T]$, we can demonstrate that

$$\underset{N\to\infty}{\text{l.i.m.}}\ \hat{\varepsilon}_N = 0.$$

Consequently,

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0.$$

Theorem 1 is proved for the case $k = 1$.

Remark 1. *Proving Theorem* 1, *we used the fact that if the stochastic process ϕ_t is mean-square continuous at the interval $[t, T]$, then it is uniformly mean-square continuous at this interval, i.e. $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ such that for all $t_1, t_2 \in [t, T]$ satisfying the condition $|t_1 - t_2| < \delta(\varepsilon)$ the inequality*

$$\mathsf{M}\left\{ |\phi_{t_1} - \phi_{t_2}|^2 \right\} < \varepsilon$$

is fulfilled (here $\delta(\varepsilon)$ does not depend on t_1 and t_2).

Proof. *Suppose that the stochastic process ϕ_t is mean-square continuous at the interval $[t, T]$, but not uniformly mean-square continuous at this interval. Then for some $\varepsilon > 0$ and $\forall\ \delta(\varepsilon) > 0\ \exists\ t_1, t_2 \in [t, T]$ such that $|t_1 - t_2| < \delta(\varepsilon)$, but*

$$\mathsf{M}\left\{|\phi_{t_1}-\phi_{t_2}|^2\right\}\ge\varepsilon.$$

Consequently, for $\delta=\delta_n=1/n$ $(n\in\mathbf{N})$ $\exists$ $t_1^{(n)}$, $t_2^{(n)}\in[t,T]$ *such that*

$$\left|t_1^{(n)}-t_2^{(n)}\right|<\frac{1}{n},$$

but

$$\mathsf{M}\left\{\left|\phi_{t_1^{(n)}}-\phi_{t_2^{(n)}}\right|^2\right\}\ge\varepsilon.$$

The sequence $t_1^{(n)}$ $(n\in\mathbf{N})$ *is bounded, consequently, according to the Bolzano–Weierstrass Theorem, we can choose from it the subsequence* $t_1^{(k_n)}$ $(n\in\mathbf{N})$ *that converges to a certain number* $\tilde t$ *(it is simple to demonstrate that* $\tilde t\in[t,T]$*). Similarly to it and in virtue of the inequality*

$$\left|t_1^{(n)}-t_2^{(n)}\right|<\frac{1}{n}$$

we have $t_2^{(k_n)}\to\tilde t$ *when* $n\to\infty$.

According to the mean-square continuity of the process ϕ_t *at the moment* $\tilde t$ *and the elementary inequality* $(a+b)^2\le 2(a^2+b^2)$, *we obtain*

$$0\le\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{t_2^{(k_n)}}\right|^2\right\}\le$$

$$\le 2\left(\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{\tilde t}\right|^2\right\}+\mathsf{M}\left\{\left|\phi_{t_2^{(k_n)}}-\phi_{\tilde t}\right|^2\right\}\right)\to 0$$

when $n\to\infty$. *Then*

$$\lim_{n\to\infty}\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{t_2^{(k_n)}}\right|^2\right\}=0.$$

It is impossible by virtue of the fact that

$$\mathsf{M}\left\{\left|\phi_{t_1^{(k_n)}}-\phi_{t_2^{(k_n)}}\right|^2\right\}\ge\varepsilon>0.$$

The obtained contradiction proves the required statement.

4. Proof of Theorem 1 for the Case of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity

Let us prove Theorem 1 for the case $k>1$. In order to do it we will introduce the following notations

$$I[\psi_q^{(r+1)}]_{\theta,s}\stackrel{\rm def}{=}\int\limits_s^{\theta}\psi_q(t_1)\ldots\int\limits_s^{t_r}\psi_{q+r}(t_{r+1})dw_{t_{r+1}}^{(q+r)}\ldots dw_{t_1}^{(q)},$$

$$J[\phi,\psi_q^{(r+1)}]_{\theta,s} \stackrel{\rm def}{=} \int\limits_s^{\theta} \psi_q(t_1)\ldots \int\limits_s^{t_r} \psi_{q+r}(t_{r+1}) \int\limits_s^{t_{r+1}} \phi_\tau dw_\tau^{(q+r+1)} dw_{t_{r+1}}^{(q+r)} \ldots dw_{t_1}^{(q)},$$

$$G[\psi_q^{(r+1)}]_{n,m} = \sum_{j_q=m}^{n-1} \sum_{j_{q+1}=m}^{j_q-1} \ldots \sum_{j_{q+r}=m}^{j_{q+r-1}-1} \prod_{l=q}^{r+q} I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$(\psi_q,\ldots,\psi_{q+r}) \stackrel{\rm def}{=} \psi_q^{(r+1)}, \quad \psi_q^{(1)} \stackrel{\rm def}{=} \psi_q,$$

$$(\psi_1,\ldots,\psi_{r+1}) \stackrel{\rm def}{=} \psi_1^{(r+1)}, \quad \psi_1^{(r+1)} \stackrel{\rm def}{=} \psi^{(r+1)}$$

Note that according to notations introduced above

$$I[\psi_l]_{s,\theta} = \int\limits_{\theta}^{s} \psi_l(\tau) dw_\tau^{(l)}.$$

To prove Theorem 1 for $k>1$ it is enough to show that

$$J[\phi,\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\rm l.i.m.}\ S[\phi,\psi^{(k)}]_N = \hat{J}[\phi,\psi^{(k)}]_{T,t} \quad \text{w. p. 1}, \tag{13}$$

where

$$S[\phi,\psi^{(k)}]_N = G[\psi^{(k)}]_{N,0} \sum_{l=0}^{j_k-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)},$$

where $\Delta w_{\tau_l}^{(k+1)} = w_{\tau_{l+1}}^{(k+1)} - w_{\tau_l}^{(k+1)}$.

First, let us prove the right equality in (13). We have

$$\hat{J}[\phi,\psi^{(k)}]_{T,t} \stackrel{\rm def}{=} \underset{N\to\infty}{\rm l.i.m.} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}. \tag{14}$$

On the basis of the inductive hypothesis we obtain that

$$I[\psi^{(k)}]_{T,\tau_{l+1}} = \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}} \quad \text{w. p. 1}, \tag{15}$$

where $\hat{I}[\psi^{(k)}]_{T,s}$ is defined in accordance with (9) and

$$I[\psi^{(k)}]_{T,s} = \int\limits_s^T \psi_1(t_1)\ldots \int\limits_s^{t_{k-2}} \psi_{k-1}(t_{k-1}) \int\limits_s^{t_{k-1}} \psi_k(t_k) dw_{t_k}^{(k)} dw_{t_{k-1}}^{(k-1)} \ldots dw_{t_1}^{(1)}.$$

Let us note that when $k \ge 4$ (for $k=2,\ 3$ the arguments are similar) due to additivity of the Ito stochastic integral the following equalities are correct

$$I[\psi^{(k)}]_{T,\tau_{l+1}}=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(t_1)\int\limits_{\tau_{l+1}}^{t_1}\psi_2(t_2)I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}}dw_{t_2}^{(2)}dw_{t_1}^{(1)}=$$

$$=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(t_1)\left(\sum_{j_2=l+1}^{j_1-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}+\int\limits_{\tau_{j_1}}^{t_1}\right)\psi_2(t_2)I[\psi_3^{(k-2)}]_{t_2,\tau_{l+1}}dw_{t_2}^{(2)}dw_{t_1}^{(1)}=$$

$$=\ldots=G[\psi^{(k)}]_{N,l+1}+H[\psi^{(k)}]_{N,l+1}\quad \text{w. p. 1,} \tag{16}$$

where

$$H[\psi^{(k)}]_{N,l+1}=\sum_{j_1=l+1}^{N-1}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}\psi_1(s)\int\limits_{\tau_{j_1}}^{s}\psi_2(\tau)I[\psi_3^{(k-2)}]_{\tau,\tau_{l+1}}dw_{\tau}^{(2)}dw_{s}^{(1)}+$$

$$+\sum_{r=2}^{k-2}G[\psi^{(r-1)}]_{N,l+1}\sum_{j_r=l+1}^{j_{r-1}-1}\int\limits_{\tau_{j_r}}^{\tau_{j_r+1}}\psi_r(s)\int\limits_{\tau_{j_r}}^{s}\psi_{r+1}(\tau)I[\psi_{r+2}^{(k-r-1)}]_{\tau,\tau_{l+1}}dw_{\tau}^{(r+1)}dw_{s}^{(r)}+$$

$$+G[\psi^{(k-2)}]_{N,l+1}\sum_{j_{k-1}=l+1}^{j_{k-2}-1}I[\psi_{k-1}^{(2)}]_{\tau_{j_{k-1}+1},\tau_{j_{k-1}}}. \tag{17}$$

Let us substitute (16) into (15) and (15) into (14). Then w. p. 1

$$\hat{J}[\phi,\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}\left(G[\psi^{(k)}]_{N,l+1}+H[\psi^{(k)}]_{N,l+1}\right). \tag{18}$$

Since

$$\sum_{j_1=0}^{N-1}\sum_{j_2=0}^{j_1-1}\ldots\sum_{j_k=0}^{j_{k-1}-1}a_{j_1\ldots j_k}=\sum_{j_k=0}^{N-1}\sum_{j_{k-1}=j_k+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1}a_{j_1\ldots j_k}, \tag{19}$$

where $a_{j_1\ldots j_k}$ are scalars, then

$$G[\psi^{(k)}]_{N,l+1}=\sum_{j_k=l+1}^{N-1}\ldots\sum_{j_1=j_2+1}^{N-1}\prod_{l=1}^{k}I[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}. \tag{20}$$

Let us substitute (20) into

$$\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)}G[\psi^{(k)}]_{N,l+1}$$

and again use the formula (19). Then

$$\sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} G[\psi^{(k)}]_{N,l+1} = S[\phi, \psi^{(k)}]_N. \tag{21}$$

Let us suppose that the limit

$$\underset{N\to\infty}{\text{l.i.m.}}\ S[\phi, \psi^{(k)}]_N \tag{22}$$

exists (its existence will be proved further).

Then from (21) and (18) it follows that for proof of the right equality in (13) we have to demonstrate that w. p. 1

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} H[\psi^{(k)}]_{N,l+1} = 0. \tag{23}$$

Analyzing the second moment of the prelimit expression on the left-hand side of (23) and taking into account (17), the independence of ϕ_{τ_l}, $\Delta w_{\tau_l}^{(k+1)}$, and $H[\psi^{(k)}]_{N,l+1}$ as well as the standard estimates for second moments of stochastic integrals and the Minkowski inequality, we find that (23) is correct. Thus, by the assumption of existence of the limit (22) we obtain that the right equality in (13) is fulfilled.

Let us demonstrate that the left equality in (13) is also fulfilled.

We have

$$J[\phi, \psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w_{\tau_l}^{(1)} J[\phi, \psi_2^{(k-1)}]_{\tau_l,t}. \tag{24}$$

Let us use for the integral $J[\phi, \psi_2^{(k-1)}]_{\tau_l,t}$ in (24) the same arguments, which resulted to the relation (16) for the integral $I[\psi^{(k)}]_{T,\tau_{l+1}}$. After that let us substitute the expression obtained for the integral $J[\phi, \psi_2^{(k-1)}]_{\tau_l,t}$ into (24).

Further, using the Minkowski inequality and standard estimates for second moments of stochastic integrals it is easy to obtain that

$$J[\phi, \psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}}\ R[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}, \tag{25}$$

where

$$R[\phi, \psi^{(k)}]_N = \sum_{j_1=0}^{N-1} \psi_1(\tau_{j_1}) \Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} \int\limits_{\tau_l}^{\tau_{l+1}} \phi_\tau dw_\tau^{(k+1)}.$$

We will demonstrate that

$$\underset{N\to\infty}{\text{l.i.m.}}\ R[\phi, \psi^{(k)}]_N = \underset{N\to\infty}{\text{l.i.m.}}\ S[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}. \tag{26}$$

It is easy to see that

$$R[\phi, \psi^{(k)}]_N = U[\phi, \psi^{(k)}]_N + V[\phi, \psi^{(k)}]_N + S[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}, \tag{27}$$

where

$$U[\phi,\psi^{(k)}]_N=\sum_{j_1=0}^{N-1}\psi_1(\tau_{j_1})\Delta w_{\tau_{j_1}}^{(1)}G[\psi_2^{(k-1)}]_{j_1,0}\sum_{l=0}^{j_k-1}I[\Delta\phi]_{\tau_{l+1},\tau_l},$$

$$V[\phi,\psi^{(k)}]_N=\sum_{j_1=0}^{N-1}I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}}G[\psi_2^{(k-1)}]_{j_1,0}\sum_{l=0}^{j_k-1}\phi_{\tau_l}\Delta w_{\tau_l}^{(k+1)},$$

$$I[\Delta\psi_1]_{\tau_{j_1+1},\tau_{j_1}}=\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}}(\psi_1(\tau_{j_1})-\psi_1(\tau))dw_{\tau}^{(1)},$$

$$I[\Delta\phi]_{\tau_{l+1},\tau_l}=\int\limits_{\tau_l}^{\tau_{l+1}}(\phi_\tau-\phi_{\tau_l})dw_{\tau}^{(k+1)}.$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals, the condition that the process ϕ_τ belongs to the class $\mathrm{S}_2([t,T])$ as well as continuity (which means uniform continuity) of the function $\psi_1(\tau)$, we obtain that

$$\underset{N\to\infty}{\text{l.i.m.}}\ V[\phi,\psi^{(k)}]_N=\underset{N\to\infty}{\text{l.i.m.}}\ U[\phi,\psi^{(k)}]_N=0\quad \text{w. p. 1}.$$

Then, considering (27), we obtain (26). From (26) and (25) it follows that the left equality in (13) is fulfilled.

Note that the limit (22) exists because it is equal to the stochastic integral $J[\phi,\psi^{(k)}]_{T,t}$, which exists under the conditions of Theorem 1. So, the chain of equalities (13) is proved. Theorem 1 is proved.

5. Corollaries and Generalizations of Theorem 1

Denote $D_k=\{(t_1,\ldots,t_k): t\le t_1<\ldots<t_k\le T\}$. We will use the same symbol D_k to denote the open and closed domains corresponding to the domain D_k. However, we always specify what domain we consider (open or closed).

Suppose that the following conditions are fulfilled:

AI. $\xi_\tau\in\mathrm{S}_2([t,T])$.

AII. $\Phi(t_1,\ldots,t_{k-1})$ is a continuous nonrandom function in the closed domain D_{k-1}.

Let us define the following stochastic integrals

$$\hat{J}[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\xi_{t_k}d\mathbf{w}_{t_k}^{(i_k)}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)}\stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta\mathbf{w}_{\tau_l}^{(i_k)}\int\limits_{\tau_{l+1}}^T d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\ldots\int\limits_{t_3}^T d\mathbf{w}_{t_2}^{(i_2)}\int\limits_{t_2}^T\Phi(t_1,t_2,\ldots,t_{k-1})d\mathbf{w}_{t_1}^{(i_1)}$$

for $k\ge 3$ and

$$\hat J[\xi,\Phi]_{T,t}^{(2)}=\int\limits_t^T \xi_{t_2}d{\bf w}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi(t_1)d{\bf w}_{t_1}^{(i_1)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr {\rm l.i.m.}\cr \hbox{$\scriptstyle N\to\infty$}\cr}}}\sum_{l=0}^{N-1}\xi_{\tau_l}\Delta{\bf w}_{\tau_l}^{(i_2)}\int\limits_{\tau_{l+1}}^T \Phi(t_1)d{\bf w}_{t_1}^{(i_1)}$$

for $k=2$. Here ${\bf w}_{\tau}^{(i)}={\bf f}_{\tau}^{(i)}$ if $i=1,\ldots,m$ and ${\bf w}_{\tau}^{(0)}=\tau,$ ${\bf f}_{\tau}^{(i)}$ $(i=1,\ldots,m)$ are ${\rm F}_{\tau}$-measurable for all $\tau\in[0,T]$ $(0\le t<T)$ independent standard Wiener processes, $i_1,\ldots,i_k=0,1,\ldots,m.$

Let us denote

$$J[\xi,\Phi]_{T,t}^{(k)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-1}}\Phi(t_1,\ldots,t_{k-1})\xi_{t_k}d{\bf w}_{t_k}^{(i_k)}\ldots d{\bf w}_{t_1}^{(i_1)},\quad k\ge 2, \tag{28}$$

where the right-hand side of (28) is the iterated Ito stochastic integral.

Let us introduce the following iterated stochastic integrals

$$\tilde J[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T d{\bf w}_{t_{k-1}}^{(i_{k-1})}\ldots\int\limits_{t_3}^T d{\bf w}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})d{\bf w}_{t_1}^{(i_1)}\stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=}\hbox{\vtop{\offinterlineskip\halign{\hfil#\hfil\cr {\rm l.i.m.}\cr \hbox{$\scriptstyle N\to\infty$}\cr}}}\sum_{l=0}^{N-1}\Delta{\bf w}_{\tau_l}^{(i_{k-1})}\int\limits_{\tau_{l+1}}^T d{\bf w}_{t_{k-2}}^{(i_{k-2})}\ldots\int\limits_{t_3}^T d{\bf w}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1})d{\bf w}_{t_1}^{(i_1)},$$

$$J'[\Phi]_{T,t}^{(k-1)}=\int\limits_t^T\ldots\int\limits_t^{t_{k-2}}\Phi(t_1,\ldots,t_{k-1})d{\bf w}_{t_{k-1}}^{(i_{k-1})}\ldots d{\bf w}_{t_1}^{(i_1)},\quad k\ge 2.$$

Similarly to the proof of Theorem 1 it is easy to demonstrate that under the condition AII the stochastic integral $\tilde J[\Phi]_{T,t}^{(k-1)}$ exists and

$$J'[\Phi]_{T,t}^{(k-1)}=\tilde J[\Phi]_{T,t}^{(k-1)}\quad \hbox{w. p. 1}. \tag{29}$$

Moreover, using (29) the following generalization of Theorem 1 can be proved similarly to the proof of Theorem 1.

Theorem 2 [20], [21] (also see [2]-[7], [16]-[18]). *Suppose that the conditions* AI, AII *of this section are fulfilled. Then, the stochastic integral* $\hat J[\xi,\Phi]_{T,t}^{(k)}$ *exists and for* $k\ge 2$

$$J[\xi,\Phi]_{T,t}^{(k)}=\hat J[\xi,\Phi]_{T,t}^{(k)}\quad \hbox{w. p. 1}.$$

Let us consider the following stochastic integrals

$$I=\int\limits_t^T d\mathbf{f}_{t_2}^{(i_2)}\int\limits_{t_2}^T \Phi_1(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)},\quad J=\int\limits_t^T\int\limits_t^{t_2}\Phi_2(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}.$$

If we consider

$$\int\limits_{t_2}^T \Phi_1(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of I and

$$\int\limits_{t}^{t_2} \Phi_2(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}$$

as the integrand of J, then, due to independence of these integrands we may mistakenly think that $\mathsf{M}\{IJ\}=0$.

But it is not the fact. Actually, using the integration order replacement technique in the stochastic integral I, we have w. p. 1

$$I=\int\limits_t^T\int\limits_t^{t_1}\Phi_1(t_1,t_2)d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_1}^{(i_1)}=\int\limits_t^T\int\limits_t^{t_2}\Phi_1(t_2,t_1)d\mathbf{f}_{t_1}^{(i_2)}d\mathbf{f}_{t_2}^{(i_1)}.$$

So, using the standard properties of the Ito stochastic integral [1], we get

$$\mathsf{M}\{IJ\}=\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{t_2}\Phi_1(t_2,t_1)\Phi_2(t_1,t_2)dt_1dt_2,$$

where $\mathbf{1}_{\{A\}}$ is the indicator of the set A.

Let us consider the following statement.

Theorem 3 [20], [21] (also see [2]-[7], [16]-[18]). *Let the conditions of Theorem* 1 *are fulfilled and $h(\tau)$ is a continuous nonrandom function at the interval $[t,T]$. Then*

$$\int\limits_t^T \phi_\tau dw_\tau^{(k+1)}h(\tau)\hat{I}[\psi^{(k)}]_{T,\tau}=\int\limits_t^T \phi_\tau h(\tau)dw_\tau^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau}\quad \text{w. p. 1}, \tag{30}$$

where stochastic integrals on the left-hand side of (30) *as well as on the right-hand side of* (30) *exist.*

Proof. According to Theorem 1, the iterated stochastic integral on the right-hand side of (30) exists. In addition

$$\int\limits_t^T \phi_\tau h(\tau)dw_\tau^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau}=\int\limits_t^T \phi_\tau dw_\tau^{(k+1)}h(\tau)\hat{I}[\psi^{(k)}]_{T,\tau}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\phi_{\tau_j}\Delta h(\tau_j)\Delta w_{\tau_j}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{j+1}}\quad \text{w. p. 1},$$

where $\Delta h(\tau_j)=h(\tau_{j+1})-h(\tau_j)$.

Using the arguments which resulted to the right equality in (13), we obtain

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{l=0}^{N-1}\phi_{\tau_l}\Delta h(\tau_l)\Delta w_{\tau_l}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\ G[\psi^{(k)}]_{N,0}\sum_{l=0}^{j_k-1}\phi_{\tau_l}\Delta h(\tau_l)\Delta w_{\tau_l}^{(k+1)}\quad \text{w. p. 1.} \tag{31}$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals as well as continuity of the function $h(\tau)$, we obtain that the second moment of the prelimit expression on the right-hand side of (31) tends to zero when $N\to\infty$. Theorem is proved.

Let us consider one corollary of Theorem 1.

Theorem 4 [20], [21] (also see [2]-[7], [16]-[18]). *Under the conditions of Theorem* 3 *the following equality is fulfilled*

$$\int\limits_t^T h(t_1)\int\limits_t^{t_1}\phi_\tau dw_\tau^{(k+2)}dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}=$$

$$=\int\limits_t^T\phi_\tau dw_\tau^{(k+2)}\int\limits_\tau^T h(t_1)dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}\quad \textit{w. p.}\ 1. \tag{32}$$

Moreover, the stochastic integrals in (32) *exist.*

Proof. Using Theorem 1 two times, we obtain

$$\int\limits_t^T\phi_\tau dw_\tau^{(k+2)}\int\limits_\tau^T h(t_1)dw_{t_1}^{(k+1)}\hat{I}[\psi^{(k)}]_{T,t_1}=$$

$$=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\rho_\tau dw_\tau^{(k+1)}dw_{t_k}^{(k)}\ldots dw_{t_1}^{(1)}=$$

$$=\int\limits_t^T\rho_\tau dw_\tau^{(k+1)}\int\limits_\tau^T\psi_k(t_k)dw_{t_k}^{(k)}\ldots\int\limits_{t_2}^T\psi_1(t_1)dw_{t_1}^{(1)}\quad \text{w. p. 1},$$

where

$$\rho_\tau\stackrel{\text{def}}{=}h(\tau)\int\limits_t^\tau\phi_s dw_s^{(k+2)}.$$

Theorem 4 is proved.

6. Examples of Integration Order Replacement Technique for the Concrete Iterated Ito Stochastic Integrals

As we mentioned above, the formulas from this section could be obtained using the Ito formula. However, the method based on Theorem 1 is more simple and familiar, since it deals with usual rules of the integration order replacement for Riemann integrals.

Using the integration order replacement technique for iterated Ito stochastic integrals (Theorem 1), we obtain the following equalities which are fulfilled w. p. 1

$$\int_t^T \int_t^{t_2} df_{t_1} dt_2 = \int_t^T (T-t_1) df_{t_1},$$

$$\int_t^T \cos(t_2-T) \int_t^{t_2} df_{t_1} dt_2 = \int_t^T \sin(T-t_1) df_{t_1},$$

$$\int_t^T \sin(t_2-T) \int_t^{t_2} df_{t_1} dt_2 = \int_t^T \left(\cos(T-t_1)-1\right) df_{t_1},$$

$$\int_t^T e^{\alpha(t_2-T)} \int_t^{t_2} df_{t_1} dt_2 = \frac{1}{\alpha}\int_t^T \left(1-e^{\alpha(t_1-T)}\right) df_{t_1}, \quad \alpha\neq 0,$$

$$\int_t^T (t_2-T)^{\alpha} \int_t^{t_2} df_{t_1} dt_2 = -\frac{1}{\alpha+1}\int_t^T (t_1-T)^{\alpha+1} df_{t_1}, \quad \alpha\neq -1,$$

$$J_{(100)T,t} = \frac{1}{2}\int_t^T (T-t_1)^2 df_{t_1},$$

$$J_{(010)T,t} = \int_t^T (t_1-t)(T-t_1) df_{t_1},$$

(33) $$J_{(110)T,t} = \int_t^T (T-t_2)\int_t^{t_2} df_{t_1} df_{t_2},$$

$$J_{(101)T,t} = \int_t^T \int_t^{t_2} (t_2-t_1) df_{t_1} df_{t_2},$$

$$J_{(1011)T,t} = \int_t^T \int_t^{t_3} \int_t^{t_2} (t_2-t_1) df_{t_1} df_{t_2} df_{t_3},$$

$$J_{(1101)T,t}=\int\limits_t^T\int\limits_t^{t_3}(t_3-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1110)T,t}=\int\limits_t^T(T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2}df_{t_1}df_{t_2}df_{t_3},$$

$$J_{(1100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_2)^2\int\limits_t^{t_2}df_{t_1}df_{t_2},$$

$$J_{(1001)T,t}=\frac{1}{2}\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)^2df_{t_1}df_{t_2},$$

(34)
$$J_{(1010)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2},$$

$$J_{(0110)T,t}=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0101)T,t}=\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)(t_1-t)df_{t_1}df_{t_2},$$

$$J_{(0010)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)(t_1-t)^2df_{t_1},$$

$$J_{(0100)T,t}=\frac{1}{2}\int\limits_t^T(T-t_1)^2(t_1-t)df_{t_1},$$

$$J_{(1000)T,t}=\frac{1}{3!}\int\limits_t^T(T-t_1)^3df_{t_1},$$

$$J_{(1\underbrace{0\ldots0}_{k-1})T,t}=\frac{1}{(k-1)!}\int\limits_t^T(T-t_1)^{k-1}df_{t_1},$$

$$J_{(11\underbrace{0\ldots0}_{k-2})T,t}=\frac{1}{(k-2)!}\int\limits_t^T(T-t_2)^{k-2}\int\limits_t^{t_2}df_{t_1}df_{t_2},$$

$$J_{(\underbrace{1\ldots 1}_{k-1}0)T,t}=\int\limits_t^T(T-t_1)J_{(\underbrace{1\ldots 1}_{k-2})t_1,t}df_{t_1},$$

$$J_{(1\underbrace{0\ldots 0}_{k-2}1)T,t}=\frac{1}{(k-2)!}\int\limits_t^T\int\limits_t^{t_2}(t_2-t_1)^{k-2}df_{t_1}df_{t_2},$$

$$J_{(10\underbrace{1\ldots 1}_{k-2})T,t}=\int\limits_t^T\ldots\int\limits_t^{t_3}\int\limits_t^{t_2}(t_2-t_1)df_{t_1}df_{t_2}\ldots df_{t_{k-1}},$$

$$J_{(\underbrace{1\ldots 1}_{k-2}01)T,t}=\int\limits_t^T\int\limits_t^{t_{k-1}}(t_{k-1}-t_{k-2})\int\limits_t^{t_{k-2}}\ldots\int\limits_t^{t_2}df_{t_1}\ldots df_{t_{k-3}}df_{t_{k-2}}df_{t_{k-1}},$$

$$J_{(10)T,t}+J_{(01)T,t}=(T-t)J_{(1)T,t},$$

$$J_{(110)T,t}+J_{(101)T,t}+J_{(011)T,t}=(T-t)J_{(11)T,t},$$

$$J_{(001)T,t}+J_{(010)T,t}+J_{(100)T,t}=\frac{(T-t)^2}{2}J_{(1)T,t},$$

$$J_{(1100)T,t}+J_{(1010)T,t}+J_{(1001)T,t}+J_{(0110)T,t}+$$

$$+J_{(0101)T,t}+J_{(0011)T,t}=\frac{(T-t)^2}{2}J_{(11)T,t},$$

$$J_{(1000)T,t}+J_{(0100)T,t}+J_{(0010)T,t}+J_{(0001)T,t}=\frac{(T-t)^3}{3!}J_{(1)T,t},$$

$$J_{(1110)T,t}+J_{(1101)T,t}+J_{(1011)T,t}+J_{(0111)T,t}=(T-t)J_{(111)T,t},$$

$$\sum_{l=1}^{k}J_{(\underbrace{0\ldots 0}_{l-1}1\underbrace{0\ldots 0}_{k-l})T,t}=\frac{1}{(k-1)!}(T-t)^{k-1}J_{(1)T,t},$$

$$\sum_{l=1}^{k}J_{(\underbrace{1\ldots 1}_{l-1}0\underbrace{1\ldots 1}_{k-l})T,t}=(T-t)J_{(\underbrace{1\ldots 1}_{k-1})T,t},$$

$$\sum_{\substack{l_1+\ldots+l_k=m\\ l_i\in\{0,\ 1\},\ i=1,\ldots,k}} J_{(l_1\ldots l_k)T,t}=\frac{(T-t)^{k-m}}{(k-m)!}J_{(\underbrace{1\ldots 1}_{m})T,t},$$

where

$$J_{(l_1\ldots l_k)T,t}=\int\limits_t^T\ldots\int\limits_t^{t_2}dw_{t_1}^{(1)}\ldots dw_{t_k}^{(k)},$$

$l_i=1$ when $w_{t_i}^{(i)}=f_{t_i}$ and $l_i=0$ when $w_{t_i}^{(i)}=t_i$ $(i=1,\ldots,k)$, f_τ is a standard Wiener process.

Let us consider two examples and show explicitly the technique on integration order replacement for iterated Ito stochastic integrals.

Example 1. *Let us prove the equality* (33). *Using Theorems* 1 *and* 3, *we obtain*

$$J_{(110)T,t}\stackrel{\rm def}{=}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}df_{t_1}df_{t_2}dt_3=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}\int\limits_{t_2}^T dt_3=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T df_{t_2}(T-t_2)=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T (T-t_2)df_{t_2}=$$

$$=\int\limits_t^T(T-t_2)\int\limits_t^{t_2}df_{t_1}df_{t_2}\quad \text{w. p. 1.}$$

Example 2. *Let us prove the equality* (34). *Using Theorems* 1 *and* 3, *we obtain*

$$J_{(1010)T,t}\stackrel{\rm def}{=}\int\limits_t^T\int\limits_t^{t_4}\int\limits_t^{t_3}\int\limits_t^{t_2}df_{t_1}dt_2df_{t_3}dt_4=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}\int\limits_{t_3}^T dt_4=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T df_{t_3}(T-t_3)=$$

$$=\int\limits_t^T df_{t_1}\int\limits_{t_1}^T dt_2\int\limits_{t_2}^T (T-t_3)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2 df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3}\int\limits_t^{t_2} df_{t_1}dt_2\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}\int\limits_{t_1}^{t_3} dt_2\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} df_{t_1}(t_3-t_1)\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_3)\left(\int\limits_t^{t_3} (t_3-t_1)df_{t_1}\right)df_{t_3}=$$

$$=\int\limits_t^T (T-t_2)\int\limits_t^{t_2} (t_2-t_1)df_{t_1}df_{t_2}\quad \text{w. p. 1.}$$

7. Integration Order Replacement Technique for Iterated Stochastic Integrals With Respect to Martingale

In this section, we will generalize the theorems on integration order replacement for iterated Ito stochastic integrals to the class of iterated stochastic integrals with respect to martingale.

Let $(\Omega, \text{F}, \mathsf{P})$ be a complete probability space and let $\{\text{F}_t, t\in[0,T]\}$ be a nondecreasing family of σ-algebras defined on the probability space $(\Omega, \text{F}, \mathsf{P})$. Suppose that $M_t,\ t\in[0,T]$ is an F_t-measurable martingale for all $t\in[0,T]$, which satisfies the condition $\mathsf{M}\left\{|M_t|\right\}<\infty.$ Moreover, for all $t\in[0,T]$ there exists an F_t- measurable and nonnegative w. p. 1 stochastic process $\rho_t,\ t\in[0,T]$ such that

$$\mathsf{M}\left\{\left(M_s-M_t\right)^2 \mid \text{F}_t\right\}=\mathsf{M}\left\{\int\limits_t^s \rho_\tau d\tau\bigg|\text{F}_t\right\}\quad \text{w. p. 1},$$

where $0\le t<s\le T.$

Let us consider the class $H_2(\rho,[0,T])$ of stochastic processes $\varphi_t,\ t\in[0,T]$, which are F_t-measurable for all $t\in[0,T]$ and satisfy the condition

$$\mathsf{M}\left\{\int\limits_0^T \varphi_t^2\rho_t dt\right\}<\infty.$$

For any partition $\tau_j^{(N)}$, $j=0,1,\ldots,N$ of the interval $[0,T]$ such that

$$0=\tau_0^{(N)}<\tau_1^{(N)}<\ldots<\tau_N^{(N)}=T,\qquad \max_{0\le j\le N-1}\left|\tau_{j+1}^{(N)}-\tau_j^{(N)}\right|\to 0\ \text{ if }\ N\to\infty \tag{35}$$

we will define the sequence of step functions

$$\varphi^{(N)}(t,\omega)=\varphi_j(\omega)\quad \text{w. p. 1}\quad \text{for}\quad t\in\left[\tau_j^{(N)},\tau_{j+1}^{(N)}\right),$$

where $j=0,1,\ldots,N-1$, $N=1,2,\ldots$

Let us define the stochastic integral with respect to martingale for $\varphi(t,\omega)\in H_2(\rho,[0,T])$ as the following mean-square limit [1]

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j=0}^{N-1}\varphi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(M\left(\tau_{j+1}^{(N)},\omega\right)-M\left(\tau_j^{(N)},\omega\right)\right)\stackrel{\text{def}}{=}\int\limits_0^T\varphi_\tau dM_\tau,$$

where $\varphi^{(N)}(t,\omega)$ is any step function from the class $H_2(\rho,[0,T])$, which converges to the function $\varphi(t,\omega)$ in the following sense

$$\lim_{N\to\infty}\int\limits_0^T\mathsf{M}\left\{\left|\varphi^{(N)}(t,\omega)-\varphi(t,\omega)\right|^2\right\}\rho_t dt=0.$$

It is well known [1] that the stochastic integral

$$\int\limits_0^T\varphi_\tau dM_\tau$$

exists and it does not depend on the selection of sequence $\varphi^{(N)}(t,\omega)$.

Let $\tilde{H}_2(\rho,[0,T])$ be the class of stochastic processes φ_τ, $\tau\in[0,T]$, which are mean-square continuous for all $\tau\in[0,T]$ and belong to the class $H_2(\rho,[0,T])$.

Let us consider the following iterated stochastic integrals

$$S[\phi,\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)\int\limits_t^{t_k}\phi_\tau dM_\tau^{(k+1)}dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}, \tag{36}$$

$$S[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_1(t_1)\ldots\int\limits_t^{t_{k-1}}\psi_k(t_k)dM_{t_k}^{(k)}\ldots dM_{t_1}^{(1)}. \tag{37}$$

Here $\phi_\tau\in\tilde{H}_2(\rho,[t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)$ are continuous nonrandom functions at the interval $[t,T]$, $M_\tau^{(l)}=M_\tau$ or $M_\tau^{(l)}=\tau$ if $\tau\in[t,T]$, $l=1,\ldots,k+1$, M_τ is the martingale defined above.

Let us define the iterated stochastic integral $\hat{S}[\psi^{(k)}]_{T,s}$, $0\le t\le s\le T$, $k\ge 1$ with respect to martingale

$$\hat{S}[\psi^{(k)}]_{T,s}=\int\limits_s^T\psi_k(t_k)dM_{t_k}^{(k)}\ldots\int\limits_{t_2}^T\psi_1(t_1)dM_{t_1}^{(1)}$$

by the following recurrence relation

$$\hat{S}[\psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \psi_k(\tau_l)\Delta M_{\tau_l}^{(k)} \hat{S}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \tag{38}$$

where $k \ge 1$, $\hat{S}[\psi^{(0)}]_{T,s} \stackrel{\text{def}}{=} 1$, $[s,T] \subseteq [t,T]$, here and further $\Delta M_{\tau_l}^{(i)} = M_{\tau_{l+1}}^{(i)} - M_{\tau_l}^{(i)}$, $i = 1,\ldots,k+1$, $l = 0,1,\ldots,N-1$, $\{\tau_l\}_{l=0}^{N}$ is the partition of the interval $[t,T]$, which satisfies the condition similar to (35), another notations are the same as in (36), (37).

Further, let us define the iterated stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$, $k \ge 1$ of the form

$$\hat{S}[\phi,\psi^{(k)}]_{T,t} = \int\limits_t^T \phi_s dM_s^{(k+1)} \hat{S}[\psi^{(k)}]_{T,s}$$

by the equality

$$\hat{S}[\phi,\psi^{(k)}]_{T,t} \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \sum_{l=0}^{N-1} \phi_{\tau_l}\Delta M_{\tau_l}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,\tau_{l+1}},$$

where the sense of notations included in (36)–(38) is saved.

Let us formulate the theorem on integration order replacement for the iterated stochastic integrals with respect to martingale, which is the generalization of Theorem 1.

Theorem 5 [20], [21] (also see [2]-[7], [16]-[18]). *Let $\phi_\tau \in \tilde{H}_2(\rho,[t,T])$, every $\psi_l(\tau)$ $(l = 1,\ldots,k)$ is a continuous nonrandom function at the interval $[t,T]$, and $|\rho_\tau| \le K < \infty$ w. p. 1 for all $\tau \in [t,T]$. Then, the stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ exists and*

$$S[\phi,\psi^{(k)}]_{T,t} = \hat{S}[\phi,\psi^{(k)}]_{T,t} \quad \text{w. p. 1}.$$

The proof of Theorem 5 is similar to the proof of Theorem 1.

Remark 2. *Let us note that we can propose another variant of the conditions in Theorem* 5. *For example, if we not require the boundedness of the process ρ_τ, then it is necessary to require the fulfillment of the following additional conditions*:

1. $\mathsf{M}\{|\rho_\tau|\} < \infty$ *for all* $\tau \in [t,T]$.
2. *The process ρ_τ is independent with the processes ϕ_τ and M_τ.*

Remark 3. *Note that it is well known the construction of stochastic integral with respect to the Wiener process with integrable process, which is not an F_τ-measurable stochastic process — the so-called Stratonovich stochastic integral* [19].

The stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ is also the stochastic integral with integrable process, which is not an F_τ-measurable stochastic process. However, under the conditions of Theorem 5

$$S[\phi,\psi^{(k)}]_{T,t} = \hat{S}[\phi,\psi^{(k)}]_{T,t} \quad \text{w. p. 1},$$

where $S[\phi,\psi^{(k)}]_{T,t}$ is a usual iterated stochastic integral with respect to martingale. If, for example, $M_\tau, \tau \in [t,T]$ is the Wiener process, then the question on connection between stochastic integral $\hat{S}[\phi,\psi^{(k)}]_{T,t}$ and Stratonovich stochastic integral is solving as a standard question on connection between Stratonovich and Ito stochastic integrals [19].

Let us consider several statements, which are the generalizations of theorems formulated in the previous sections.

Assume that $D_k = \{(t_1,\ldots,t_k):\ t \le t_1 < \ldots < t_k \le T\}$ and the following conditions are met:

BI. $\xi_\tau \in \tilde{H}_2(\rho,[t,T])$.

BII. $\Phi(t_1,\ldots,t_{k-1})$ is a continuous nonrandom function in the closed domain D_{k-1} (recall that we use the same symbol D_{k-1} to denote the open and closed domains corresponding to the domain D_{k-1}; however, we always specify what domain we consider (open or closed)).

Let us define the following stochastic integrals with respect to martingale

$$\hat{S}[\xi,\Phi]^{(k)}_{T,t} = \int\limits_t^T \xi_{t_k} dM^{(k)}_{t_k} \ldots \int\limits_{t_3}^T dM^{(2)}_{t_2} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) dM^{(1)}_{t_1} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \mathop{\hbox{\rm l.i.m.}}\limits_{N\to\infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta M^{(k)}_{\tau_l} \int\limits_{\tau_{l+1}}^T dM^{(k-1)}_{t_{k-1}} \ldots \int\limits_{t_3}^T dM^{(2)}_{t_2} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) dM^{(1)}_{t_1}$$

for $k \ge 3$ and

$$\hat{S}[\xi,\Phi]^{(2)}_{T,t} = \int\limits_t^T \xi_{t_2} dM^{(2)}_{t_2} \int\limits_{t_2}^T \Phi(t_1) dM^{(1)}_{t_1} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \mathop{\hbox{\rm l.i.m.}}\limits_{N\to\infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta M^{(2)}_{\tau_l} \int\limits_{\tau_{l+1}}^T \Phi(t_1) dM^{(1)}_{t_1}$$

for $k = 2$, where the sense of notations included in (36)–(38) is saved. Moreover, the stochastic process ξ_τ, $\tau \in [t,T]$ belongs to the class $\tilde{H}_2(\rho,[t,T])$.

In addition, let

$$S[\xi,\Phi]^{(k)}_{T,t} = \int\limits_t^T \ldots \int\limits_t^{t_{k-1}} \Phi(t_1,\ldots,t_{k-1}) \xi_{t_k} dM^{(k)}_{t_k} \ldots dM^{(1)}_{t_1}, \quad k \ge 2, \tag{39}$$

where the right-hand side of (39) is the iterated stochastic integral with respect to martingale.

Let us introduce the following iterated stochastic integrals with respect to martingale

$$\tilde{S}[\Phi]^{(k-1)}_{T,t} = \int\limits_t^T dM^{(k-1)}_{t_{k-1}} \ldots \int\limits_{t_3}^T dM^{(2)}_{t_2} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) dM^{(1)}_{t_1} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \mathop{\hbox{\rm l.i.m.}}\limits_{N\to\infty} \sum_{l=0}^{N-1} \Delta M^{(k-1)}_{\tau_l} \int\limits_{\tau_{l+1}}^T dM^{(k-2)}_{t_{k-2}} \ldots \int\limits_{t_3}^T dM^{(2)}_{t_2} \int\limits_{t_2}^T \Phi(t_1,t_2,\ldots,t_{k-1}) dM^{(1)}_{t_1},$$

$$S'[\Phi]^{(k-1)}_{T,t} = \int\limits_t^T \ldots \int\limits_t^{t_{k-2}} \Phi(t_1,\ldots,t_{k-1}) dM^{(k-1)}_{t_{k-1}} \ldots dM^{(1)}_{t_1}, \quad k \ge 2.$$

It is easy to demonstrate similarly to the proof of Theorem 5 that under the condition BII the stochastic integral $\tilde{S}[\Phi]_{T,t}^{(k-1)}$ exists and

$$S'[\Phi]_{T,t}^{(k-1)} = \tilde{S}[\Phi]_{T,t}^{(k-1)} \quad \text{w. p. } 1.$$

In its turn, using this fact we can prove the following theorem similarly to the proof of Theorem 5.

Theorem 6 [20], [21] (also see [2]-[7], [16]-[18]). *Let the conditions* BI, BII *of this section are fulfilled and* $|\rho_\tau| \le K < \infty$ *w. p.* 1 *for all* $\tau \in [t, T]$. *Then, the stochastic integral* $\hat{S}[\xi, \Phi]_{T,t}^{(k)}$ *exists and for* $k \ge 2$

$$S[\xi, \Phi]_{T,t}^{(k)} = \hat{S}[\xi, \Phi]_{T,t}^{(k)} \quad \textit{w. p. } 1.$$

Theorem 6 is the generalization of Theorem 2 for the case of iterated stochastic integrals with respect to martingale.

Let us consider two statements.

Theorem 7 [20], [21] (also see [2]-[7], [16]-[18]). *Let the conditions of Theorem* 5 *are fulfilled and* $h(\tau)$ *is a continuous nonrandom function at the interval* $[t, T]$. *Then*

$$\int\limits_t^T \phi_\tau dM_\tau^{(k+1)} h(\tau) \hat{S}[\psi^{(k)}]_{T,\tau} = \int\limits_t^T \phi_\tau h(\tau) dM_\tau^{(k+1)} \hat{S}[\psi^{(k)}]_{T,\tau} \quad \textit{w. p. } 1 \tag{40}$$

and stochastic integrals on the left-hand side of (40) *as well as on the right-hand side of* (40) *exist.*

Theorem 8 [20], [21] (also see [2]-[7], [16]-[18]). *Under the conditions of Theorem* 7

$$\int\limits_t^T h(t_1) \int\limits_t^{t_1} \phi_\tau dM_\tau^{(k+2)} dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} =$$

$$= \int\limits_t^T \phi_\tau dM_\tau^{(k+2)} \int\limits_\tau^T h(t_1) dM_{t_1}^{(k+1)} \hat{S}[\psi^{(k)}]_{T,t_1} \ \text{w. p. } 1. \tag{41}$$

Moreover, the stochastic integrals in (41) *exist.*

The proofs of Theorems 7 and 8 are similar to the proofs of Theorems 3 and 4 correspondingly.

Remark 4. *The integration order replacement technique for iterated Ito stochastic integrals* (*Theorems* 1–4) [2]-[15], [16]-[18] *has been successfully applied for construction of the so-called unified Taylor–Ito and Taylor–Stratonovich expansions* [16]-[18] (*see references therein*) *as well as for proof and development of the mean-square approximation method for iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series* [16]-[18] (*see references therein*).

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations. Kiev, Naukova Dumka Publ., 1968. 354 pp.

[2] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[3] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[4] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234
Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[5] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385.
DOI: http://doi.org/10.18720/SPBPU/2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[6] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[7] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Ito expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[10] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[11] Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. [In English]. arXiv:1712.08991 [math.PR]. 2022, 56 pp.

[12] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 67 pp.

[13] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[14] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[15] Kuznetsov D.F. Expansions of Iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 174 pp.

[16] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2022, 869 pp.

[17] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at:
http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[18] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[19] Stratonovich R.L. Conditional Markov's Processes and its Applications to the Theory of Optimal Control. Moscow, State University Publ., 1966. 320 pp.

[20] Kuznetsov D.F. Theorems about integration order replacement in iterated stochastic integrals. Dep. VINITI. 3607-V97, 1997, 31 pp.

[21] Kuznetsov D.F. Integration order replacement in iterated stochastic integrals with respect to martingale. Preprint. St.-Petersburg: SPbGTU Publ., 1999 , 11 pp. Available at: http://www.sde-kuznetsov.spb.ru/99c.pdf

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Chapter 7.

Implementation to the High-Order Strong Numerical Methods for Ito SDEs

OPTIMIZATION OF THE MEAN-SQUARE APPROXIMATION PROCEDURES FOR ITERATED ITO STOCHASTIC INTEGRALS OF MULTIPLICITIES 1 TO 5 FROM THE UNIFIED TAYLOR–ITO EXPANSION BASED ON MULTIPLE FOURIER–LEGENDRE SERIES

MIKHAIL D. KUZNETSOV AND DMITRIY F. KUZNETSOV

Abstract. The article is devoted to optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5. The mentioned stochastic integrals are part of strong numerical methods with convergence orders 1.0, 1.5, 2.0, and 2.5 for Ito stochastic differential equations with multidimensional non-commutative noise based on the unified Taylor–Ito expansion and multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L_2([t,T]^k)$ $(k = 1,\ldots,5)$. In this article we use multiple Fourier–Legendre series within the framework of the method of expansion and mean-square approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. We show that the lengths of sequences of independent standard Gaussian random variables required for the mean-square approximation of iterated Ito stochastic integrals of multiplicities 1 to 5 can be significantly reduced without the loss of the mean-square accuracy of approximation for these stochastic integrals.

Contents

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.

Keywords: Iterated Ito stochastic integral, Iterated Stratonovich stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Multiple trigonometric Fourier series, Ito stochastic differential equation, Taylor–Ito expansion, Numerical solution, Mean-square approximation, Convergence with propability 1, Expansion.

1. Explicit One-Step Strong Numerical Methods with Convergence Orders 1.0, 1.5, 2.0, and 2.5 for Ito SDEs Based on the Unified Taylor–Ito Expansion

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F, and let $\mathbf{f}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,T]$. We assume that the components $\mathbf{f}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^m\int\limits_0^t B_i(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)},\quad \mathbf{x}_0=\mathbf{x}(0,\omega). \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the Ito SDE (1). The nonrandom functions $\mathbf{a}(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $B(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the Ito SDE (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral and $B_i(\mathbf{x},t)$ is the ith colomn of the matrix function $B(\mathbf{x},t)$. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\bigl\{|\mathbf{x}_0|^2\bigr\}<\infty$ (M is an expectation operator). We assume that $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ are independent when $t>0$.

It is well known that one of the promising approaches to the numerical integration of Ito SDEs is an approach based on the stochastic Taylor expansions [2]-[10]. The essential feature of such stochastic expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$,

$$\int\ \text{and}\ \int\limits^{*}$$

denote Ito and Stratonovich stochastic integrals, respectively; $i_1,\ldots,i_k=0,1,\ldots,m$ (in this paper, we use the definition of the Stratonovich stochastic integral from [3]).

Note that $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ in the classical Taylor–Ito and Taylor–Stratonovich expansions [3], [4]. At the same time $\psi_l(\tau)\equiv(t-\tau)^{q_l}$ $(l=1,\ldots,k;\ q_1,\ldots,q_k=0,1,2,\ldots)$ and $i_1,\ldots,i_k=1,\ldots,m$ in the unified Taylor–Ito and Taylor–Stratonovich expansions [5]-[14].

Let $C^{2,1}(\mathbb{R}^n\times[0,T])$ be the space of functions $R(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^1$ with the following property: these functions are twice continuously differentiable in $\mathbf{x}$ and have one continuous derivative in t. Let us consider the following differential operators on the space $C^{2,1}(\mathbb{R}^n\times[0,T])$

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}a^{(i)}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}^{(i)}}+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2}{\partial\mathbf{x}^{(l)}\partial\mathbf{x}^{(i)}},\qquad(4)$$

$$G_0^{(i)}=\sum_{j=1}^{n}B^{(ji)}(\mathbf{x},t)\frac{\partial}{\partial\mathbf{x}^{(j)}},\quad i=1,\ldots,m,\qquad(5)$$

where $a^{(i)}(\mathbf{x},t)$ is the ith component of the vector function $a(\mathbf{x},t)$ and $B^{(ij)}(\mathbf{x},t)$ is the ijth element of the matrix function $B(\mathbf{x},t)$.

Consider the following sequence of differential operators

$$G_p^{(i)}=\frac{1}{p}\left(G_{p-1}^{(i)}L-LG_{p-1}^{(i)}\right),\quad p=1,2,\ldots,\quad i=1,\ldots,m,$$

where L and $G_0^{(i)}$, $i=1,\ldots,m$ are defined by the equalities (4), (5).

For the further consideration, we need to introduce the following set of iterated Ito stochastic integrals

$$I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)}=\int\limits_t^s(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},\qquad(6)$$

where $l_1,\ldots,l_k=0,1,\ldots$ and $i_1,\ldots,l_k=1,\ldots,m$.

Assume that $R(\mathbf{x},t)$, $\mathbf{a}(\mathbf{x},t)$, and $B_i(\mathbf{x},t)$, $i=1,\ldots,m$ are enough smooth functions with respect to the variables $\mathbf{x}$ and t. Then for all $s,t\in[0,T]$ such that $s>t$ we can write the following unified Taylor–Ito expansion [5]-[14]

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)}+$$

$$+\left(H_{r+1}\right)_{s,t}\quad\text{w. p. 1},\qquad(7)$$

where

$$L^jR(\mathbf{x},t)\stackrel{\sf def}{=}\begin{cases}\underbrace{L\ldots L}_{j}R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases},$$

(8) $$\mathrm{D}_q=\left\{(k,j,l_1,\ldots,l_k):\ k+2\left(j+\sum_{p=1}^{k}l_p\right)=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\},$$

and $(H_{r+1})_{s,t}$ is the remainder term in integral form [5]-[14].

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0,T]$ such that

(9) $$0=\tau_0<\tau_1<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}|\tau_{j+1}-\tau_j|.$$

Let $\mathbf{y}_{\tau_j}\stackrel{\mathsf{def}}{=}\mathbf{y}_j$, $j=0,1,\ldots,N$ be a time discrete approximation of the process $\mathbf{x}_t$, $t\in[0,T]$, which is a solution of the Ito SDE (1).

Definiton 1 [3]. *We will say that a time discrete approximation* $\mathbf{y}_j$, $j=0,1,\ldots,N$, *corresponding to the maximal step of discretization* Δ_N, *converges strongly with order* $\gamma>0$ *at time moment* T *to the process* $\mathbf{x}_t$, $t\in[0,T]$, *if there exists a constant* $C>0$, *which does not depend on* Δ_N, *and a* $\delta>0$ *such that*

$$\mathsf{M}\{|\mathbf{x}_T-\mathbf{y}_T|\}\le C(\Delta_N)^{\gamma}$$

for each $\Delta_N\in(0,\delta)$.

From (7) for $s=\tau_{p+1}$ and $t=\tau_p$ we obtain the following representation for family of explicit one-step strong numerical schemes for the Ito SDE (1)

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(\tau_{p+1}-\tau_p)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^j\,\mathbf{y}_p\ \hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}+$$

(10) $$+\mathbf{1}_{\{r=2d-1,d\in\mathbb{N}\}}\frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}\mathbf{y}_p,$$

where $\hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}$ is an approximation of the iterated Ito stochastic integral (6). The equality (10) should be understood componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Let for simplicity $\tau_p=p\Delta$, $\Delta=T/N$, $T=\tau_N$, $p=0,1,\ldots,N$.

It is known [3] that under the standard conditions the numerical scheme (10) has strong order of convergence $r/2$ $(r\in\mathbb{N})$.

Further, we consider particular cases of the numerical scheme (10) for $r=2,3,4,$ and 5, i.e. explicit one-step strong numerical schemes with convergence orders $1.0, 1.5, 2.0,$ and 2.5 for the Ito SDE (1). At that for simplicity we will write $\mathbf{a}$, $L\mathbf{a}$, B_i, $G_0^{(i)}B_j$ etc. instead of $\mathbf{a}(\mathbf{y}_p,\tau_p)$, $L\mathbf{a}(\mathbf{y}_p,\tau_p)$, $B_i(\mathbf{y}_p,\tau_p)$, $G_0^{(i)}B_j(\mathbf{y}_p,\tau_p)$ etc. correspondingly. Here L and $G_0^{(i)}$, $i=1,\ldots,m$ are defined by the equalities (4), (5). Thus, we obtain the following numerical schemes.

Milstein scheme (Scheme with strong order 1.0)

(11) $$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}.$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}+\frac{\Delta^2}{2}L\mathbf{a}. \tag{12}$$

Scheme with strong order 2.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat I_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}+\frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_1)}LB_{i_2}\left(\hat I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}-\hat I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}\right)-LG_0^{(i_1)}B_{i_2}\hat I_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}+\right.$$

$$\left.+G_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\hat I_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}+\Delta\hat I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat I_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}. \tag{13}$$

Scheme with strong order 2.5

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat I_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat I_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat I_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_1)}LB_{i_2}\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right) - LG_0^{(i_1)}B_{i_2}\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)}+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} + \frac{\Delta^2}{2}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}\right)+$$

$$+\frac{1}{2}LLB_{i_1}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} - LG_0^{(i_1)}\mathbf{a}\left(\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_1)}LG_0^{(i_2)}B_{i_3}\left(\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}LB_{i_3}\left(\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\mathbf{a}\left(\Delta\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)-$$

$$-LG_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}B_{i_5}\hat{I}_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \tag{14}$$

It is well known [3] that under the standard conditions the numerical schemes (12)–(14) have strong orders of convergence 1.0, 1.5, 2.0, and 2.5 correspondingly. Among these conditions we consider only the condition for approximations of iterated Ito stochastic integrals from the numerical schemes (12)–(14) [3] (also see [6])

$$\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)} - \hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\right)^2\right\} \le C\Delta^{r+1}, \tag{15}$$

where constant C is independent of Δ and $r/2$ is the strong order of convergence for the numerical schemes (12)–(14), i.e. $r/2 = 1.0, 1.5, 2.0,$ and 2.5.

Note that the numerical schemes (11)–(14) are unrealizable in practice without effective procedures for the numerical simulation of iterated Ito stochastic integrals from (10). That is why in the next section, we consider the effective method of the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$).

2. Method of Expansion and Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series

Let us consider the effective approach to expansion and mean-square approximation of iterated Ito stochastic integrals [6] (2006), [7]-[14], [17]-[47] (the so-called method of generalized multiple Fourier series).

The idea of this method is as follows: the iterated Ito stochastic integral (2) of the multiplicity k ($k \in \mathbb{N}$) is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of k variables defined on the hypercube $[t,T]^k$. Here $[t,T]$ is the interval of integration of the iterated Ito stochastic integral (2). Then, the mentioned nonrandom function of k variables is expanded in the hypercube $[t,T]^k$ into the generalized multiple Fourier series converging in the mean-square sense in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come to the mean-square converging expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of k variables, which can be calculated using the explicit formula regardless of the multiplicity k of the iterated Ito stochastic integral (2).

Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$. Define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k) & \text{for} \quad t_1 < \ldots < t_k \\ \\ 0 & \text{otherwise} \end{cases}, \quad t_1,\ldots,t_k \in [t,T], \quad k \geq 2, \tag{16}$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. The function $K(t_1,\ldots,t_k)$ belongs to the space $L_2([t,T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \left\| K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)} = 0,$$

where

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{17}$$

is the Fourier coefficient,

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{18}$$

Theorem 1 [6] (2006), [7]-[14], [17]-[47]. *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} \ = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} - \right.$$

$$\left. - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \text{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right), \tag{19}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\text{G}_k=\text{H}_k\backslash\text{L}_k,\quad \text{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\text{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \tag{20}$$

are independent standard Gaussian random variables for various i or j (if $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (17), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^N$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (18).

Note that the adaptation of Theorem 1 for complete orthonormal systems of Haar and Rademacher–Walsh functions in the space $L_2([t,T])$ can be found in [6]-[14], [17]-[21], [24]-[26].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$ [6]-[14], [17]-[47]

$$J[\psi^{(1)}]_{T,t} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{21}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{22}$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{23}$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{24}$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} -$$
$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} +$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} +$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} +$$

$$
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{25}
$$

$$
\begin{gathered}
J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-\\
-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\\
-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\\
-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\\
+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\\
+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+\\
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\\
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
\end{gathered}
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+
$$
$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$
$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$
$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \tag{26}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Thus, we obtain the following advantages and new possibilities of the method of generalized multiple Fourier series (Theorem 1) in comparison with the well known methods of approximation of iterated stochastic integrals [2]-[4], [48]-[58].

1. There is an explicit formula (see (17)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity k ($k \in \mathbb{N}$).

2. We have new possibilities for exact calculation and effective estimation of the mean-square approximation error of iterated Ito stochastic integral [12]-[14], [24], [25], [32] (see Sect. 3, 4).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space $L_2([t,T])$, then we have new possibilities for approximation — we can use not only trigonometric functions as in [2]-[4], [48], [54], [55], [57], [58], but Legendre polynomials.

4. As it turned out [12]-[14], [28] (also see [5]-[11], [15]-[25], [30], [34]-[36], [38]-[41], [45])-[47]) it is more convenient to work with the Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions (see [2]-[4], [48]-[58]). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [12]-[14] (Sect. 5.3), [35], [36].

5. The approach to expansion of iterated stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process [2]-[4], [48], [54], [55], [57], [58] leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1) starting from the second multiplicity (in the general case) and third multiplicity (for the case $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$; $i_1,i_2,i_3=0,1,\ldots,m$) of the iterated Ito stochastic integrals (2). The same problem (iterated application of the operation of limit transition) also appears in the method of expansion of iterated stochastic integrals based on the Wiener process series expansion using various complete orthonormal system of functions in the space $L_2([t,T])$ [49], [50]. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the

iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_1, \ldots, p_k$). For example, when $p_1 = \ldots = p_k = p \to \infty$. For iterated series, the condition $p_1 = \ldots = p_k = p \to \infty$ obviously does not guarantee the convergence of this series. However, in [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [48] (pp. 438-439), [55] (pp. 263-264) the authors use (without rigorous proof) the condition $p_1 = p_2 = p_3 = p \to \infty$ within the frames of the mentioned approach based on the Karhunen–Loeve expansion of the Brownian bridge process [2] together with the Wong–Zakai approximation [59]-[61] (see discussions in [12] (Sect. 2.18, 6.2), [13] (Sect. 2.6.2, 6.2), [14] (Sect. 2.6.2, 6.2), [26] (Sect. 11), [27] (Sect. 6), [28] (Sect. 8), [29] (Sect. 6) for detail).

6. Constructing the expansions of iterated Ito stochastic integrals from Theorem 1, we saved all information about these integrals. That is why it is natural to expect that the mentioned expansions will converge with probability 1 and in the mean of degree $2n$ ($n \in \mathbb{N}$). The convergence with probability 1 in Theorem 1 is proved [12]-[14], [26], [28], [32], [45] for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Furthermore, the convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) in Theorem 1 is proved in [8]-[14], [17]-[21], [24]-[26].

7. The versions of Theorem 1 for complete orthonormal with weight $r(t_1)\ldots r(t_k)$ systems of functions in the space $L_2([t,T]^k)$ ($k \in \mathbb{N}$) as well as for some other types of iterated stochastic integrals (iterated stochastic integrals with respect to martingale Poisson measures and iterated stochastic integrals with respect to martingales) were obtained in [12]-[14], [37] (also see [6]-[10], [17]-[21], [24], [25]).

8. The adaptation of Theorem 1 for iterated Stratonovich stochastic integrals of multiplicities 1 to 6 is realized in [12]-[14], [17]-[21], [24], [25], [27], [29], [31], [33] (see Theorems 3–7 below).

9. Application of Theorem 1 and Theorem 2 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process can be found in the monographs [12]-[14] (Chapter 7) and in [38]-[41].

For further consideration, let us consider the generalization of formulas (21)–(26) for the case of an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2) as well as for the case of an arbitrary complete orthonormal systems of functions in $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. In order to do this, let us introduce some notations. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$
(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{27}
$$

where

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},
$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (27) is a partition and consider the sum with respect to all possible partitions

$$
\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1 \ldots q_{k-2r}}. \tag{28}
$$

Below there are several examples of sums in the form (28)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2}=a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4}=a_{1234}+a_{1324}+a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2}=$$

$$=a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3}=$$

$$=a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+$$
$$+a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1}=$$

$$=a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+$$
$$+a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+$$
$$+a_{52,34,1}+a_{53,24,1}+a_{54,23,1}.$$

Now we can formulate the following generalization of Theorem 1.

Theorem 2 [12] (Sect. 1.11), [26] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \quad (29)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [62]. Note that we use another notations [12] (Sect. 1.11), [26] (Sect. 15) in comparison with [62]. Moreover, the proof of an analogue of Theorem 2 from [62] is somewhat different from the proof given in [12] (Sect. 1.11), [26] (Sect. 15).

As we mentioned above, in a number of works [12]-[14], [17]-[21], [24], [25], [27], [29], [31], [33] Theorem 1 is adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6 (the case of multiplicity 1 is given by (21)). Let as collect some old results in the following theorem.

Theorem 3 [12]-[14], [17]-[21], [24], [25], [27], [29], [31], [33]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *At the same time* $\psi_2(\tau)$ *is a continuously differentiable function on* $[t,T]$ *and* $\psi_1(\tau)$, $\psi_3(\tau)$ *are twice continuously differentiable functions on* $[t,T]$. *Then*

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} \quad (i_1,i_2=1,\ldots,m), \tag{30}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m), \tag{31}$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{32}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \quad (i_1,\ldots,i_4=0,1,\ldots,m), \tag{33}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (3), *and* $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ *in* (31), (33); *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [12] (Sect. 2.10–2.16), [27] (Sect. 13–19), [31] (Sect. 5–11), [44] (Sect. 7–13), [63] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

Theorem 4 [12], [27], [31], [44], [63]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \quad (i_1,i_2,i_3=0,1,\ldots,m)$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{34}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{35}$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (34) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (35), *constant* C *is independent of* p,

$$C_{j_3j_2j_1}=\int\limits_t^T\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$); another notations are the same as in Theorems 1, 2.

Theorem 5 [12], [27], [31], [44], [63]. *Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Furthermore, let $\psi_1(\tau),\ldots,\psi_4(\tau)$ be continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t}=\int\limits_t^{*T}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} \tag{36}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{37}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right)^2\right\}\le\frac{C}{p^{1-\varepsilon}} \tag{38}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ in (36), (37) *and $i_1,\ldots,i_4=1,\ldots,m$ in* (38), *constant C does not depend on p, ε is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t,T])$,*

$$C_{j_4j_3j_2j_1}=\int\limits_t^T\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4;$$

another notations are the same as in Theorem 4.

Theorem 6 [12], [27], [31], [44], [63]. *Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=\int\limits_t^{*T}\psi_5(t_5)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_5}^{(i_5)} \tag{39}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{40}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{41}$$

are fulfilled, where $i_1,\ldots,i_5 = 0,1,\ldots,m$ *in* (39), (40) *and* $i_1,\ldots,i_5 = 1,\ldots,m$ *in* (41), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5\ldots j_1} = \int\limits_t^T \psi_5(t_5)\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_5;$$

another notations are the same as in Theorems 4, 5.

Theorem 7 [12], [27], [31], [44]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m$,

$$C_{j_6\ldots j_1} = \int\limits_t^T \phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6;$$

another notations are the same as in Theorems 4–6.

3. Estimate for the Mean-Square Approximation Error in the Method of Approximation of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

Assume that $J[\psi^{(k)}]_{T,t}^p$ is the approximation of (2), which is the expression on the right-hand side of (29) before passing to the limit $\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}$ for the case $p_1 = \ldots = p_k = p$, i.e.

$$
J[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times
$$

$$
\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr).
$$

Let us denote

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^p\right)^2\right\} \stackrel{\rm def}{=} E_k^p,
$$

$$
\|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k \stackrel{\rm def}{=} I_k.
$$

When proving Theorems 1 and 2 [12]-[14] (also see [21]-[26]), we have proved the following estimate

$$
E_k^p \le k!\left(I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2\right), \tag{42}
$$

where $i_1,\ldots,i_k = 1,\ldots,m$ for $T-t\in(0,\infty)$ and $i_1,\ldots,i_k = 0,1,\ldots,m$ for $T-t\in(0,1)$; another notations are the same as in Theorems 1, 2.

Combining the estimates (15) and (42), we obtain

$$
k!\left(I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2\right) \le C(T-t)^{r+1}, \tag{43}
$$

where constant C is independent of $T-t$.

It is not difficult to see that the multiplier factor $k!$ on the left-hand side of (43) leads to a significant increase of the minimal natural number p satisfying the estimate (43). For example, for the numerical methods (11)–(14) we will have the following multiplier factors on the left-hand side of the inequality (43): $2!=2$, $3!=6$, $4!=24$, $5!=120$.

As we will see in the next section, the mentioned problem can be partially overcome if we calculate the mean-square approximation error E_k^p exactly.

4. Exact Formulas for the Mean-Square Approximation Error in the Method of Approximation of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

This section is devoted to exact expressions for the mean-square approximation error in Theorems 1, 2 for iterated Ito stochastic integrals of arbitrary multiplicity k ($k\in\mathbb{N}$).

As it turned out, the value E_k^p can be calculated exactly.

Theorem 8 [12] (Sect. 1.12), [32] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$E_k^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{44}$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\} = C_{j_k\ldots j_1}.$$

Then we can obtain the following particular cases of Theorem 8 for $k=1,\ldots,5$ and $i_1,\ldots,i_5=1,\ldots,m$ [12]-[14], [22], [25], [32].

The case $k=1$

$$E_1^p = I_1 - \sum_{j_1=0}^{p} C_{j_1}^2.$$

The case $k=2$

(I). $i_1\ne i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2. \tag{45}$$

(II). $i_1=i_2$:

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}C_{j_1j_2}. \tag{46}$$

The case $k=3$

(I). $i_1\ne i_2, i_1\ne i_3, i_2\ne i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2. \tag{47}$$

(II). $i_1 = i_2 = i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\Biggl(\sum_{(j_1,j_2,j_3)} C_{j_3j_2j_1}\Biggr). \tag{48}$$

(III).1. $i_1 = i_2 \neq i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}. \tag{49}$$

(III).2. $i_1 \neq i_2 = i_3$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_1}C_{j_3j_2j_1}. \tag{50}$$

(III).3. $i_1 = i_3 \neq i_2$:

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3}. \tag{51}$$

The case $k = 4$

(I). $i_1, \ldots, i_4$ are pairwise different:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}^2. \tag{52}$$

(II). $i_1 = i_2 = i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\Biggl(\sum_{(j_1,\ldots,j_4)} C_{j_4\ldots j_1}\Biggr). \tag{53}$$

(III).1. $i_1 = i_2 \neq i_3, i_4;\ i_3 \neq i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\Biggl(\sum_{(j_1,j_2)} C_{j_4\ldots j_1}\Biggr). \tag{54}$$

(III).2. $i_1 = i_3 \neq i_2, i_4;\ i_2 \neq i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\Biggl(\sum_{(j_1,j_3)} C_{j_4\ldots j_1}\Biggr). \tag{55}$$

(III).3. $i_1 = i_4 \neq i_2, i_3;\ i_2 \neq i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\Biggl(\sum_{(j_1,j_4)} C_{j_4\ldots j_1}\Biggr). \tag{56}$$

(III).4. $i_2 = i_3 \neq i_1, i_4;\ i_1 \neq i_4:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right). \tag{57}$$

(III).5. $i_2 = i_4 \neq i_1, i_3;\ i_1 \neq i_3:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1} \right). \tag{58}$$

(III).6. $i_3 = i_4 \neq i_1, i_2;\ i_1 \neq i_2:$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \right). \tag{59}$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1} \right). \tag{60}$$

(IV).2. $i_2 = i_3 = i_4 \neq i_1$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1} \right). \tag{61}$$

(IV).3. $i_1 = i_2 = i_4 \neq i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1} \right). \tag{62}$$

(IV).4. $i_1 = i_3 = i_4 \neq i_2$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1} \right). \tag{63}$$

(V).1. $i_1 = i_2 \neq i_3 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} C_{j_4\ldots j_1} \right) \right). \tag{64}$$

(V).2. $i_1 = i_3 \neq i_2 = i_4$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} C_{j_4\ldots j_1} \right) \right). \tag{65}$$

(V).3. $i_1 = i_4 \neq i_2 = i_3$:

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right) \right). \tag{66}$$

The case $k = 5$

(I). $i_1, \ldots, i_5$ are pairwise different:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}^2. \tag{67}$$

(II). $i_1 = i_2 = i_3 = i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,\ldots,j_5)} C_{j_5\ldots j_1} \right). \tag{68}$$

(III).1. $i_1 = i_2 \neq i_3, i_4, i_5$ (i_3, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} C_{j_5\ldots j_1} \right). \tag{69}$$

(III).2. $i_1 = i_3 \neq i_2, i_4, i_5$ (i_2, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1} \right). \tag{70}$$

(III).3. $i_1 = i_4 \neq i_2, i_3, i_5$ (i_2, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \right). \tag{71}$$

(III).4. $i_1 = i_5 \neq i_2, i_3, i_4$ (i_2, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right). \tag{72}$$

(III).5. $i_2 = i_3 \neq i_1, i_4, i_5$ (i_1, i_4, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right). \tag{73}$$

(III).6. $i_2 = i_4 \neq i_1, i_3, i_5$ (i_1, i_3, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \Biggr). \tag{74}$$

(III).7. $i_2 = i_5 \neq i_1, i_3, i_4$ (i_1, i_3, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{75}$$

(III).8. $i_3 = i_4 \neq i_1, i_2, i_5$ (i_1, i_2, i_5 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \Biggr). \tag{76}$$

(III).9. $i_3 = i_5 \neq i_1, i_2, i_4$ (i_1, i_2, i_4 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{77}$$

(III).10. $i_4 = i_5 \neq i_1, i_2, i_3$ (i_1, i_2, i_3 are pairwise different):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{78}$$

(IV).1. $i_1 = i_2 = i_3 \neq i_4, i_5$ ($i_4 \neq i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1} \Biggr). \tag{79}$$

(IV).2. $i_1 = i_2 = i_4 \neq i_3, i_5$ ($i_3 \neq i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1} \Biggr). \tag{80}$$

(IV).3. $i_1 = i_2 = i_5 \neq i_3, i_4$ ($i_3 \neq i_4$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{81}$$

(IV).4. $i_2 = i_3 = i_4 \neq i_1, i_5$ ($i_1 \neq i_5$):

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1} \Biggr). \tag{82}$$

(IV).5. $i_2 = i_3 = i_5 \neq i_1, i_4$ $(i_1 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\right). \tag{83}$$

(IV).6. $i_2 = i_4 = i_5 \neq i_1, i_3$ $(i_1 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\right). \tag{84}$$

(IV).7. $i_3 = i_4 = i_5 \neq i_1, i_2$ $(i_1 \neq i_2)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\right). \tag{85}$$

(IV).8. $i_1 = i_3 = i_5 \neq i_2, i_4$ $(i_2 \neq i_4)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\right). \tag{86}$$

(IV).9. $i_1 = i_3 = i_4 \neq i_2, i_5$ $(i_2 \neq i_5)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1}\right). \tag{87}$$

(IV).10. $i_1 = i_4 = i_5 \neq i_2, i_3$ $(i_2 \neq i_3)$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1}\right). \tag{88}$$

(V).1. $i_1 = i_2 = i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_3,j_4)} C_{j_5\ldots j_1}\right). \tag{89}$$

(V).2. $i_1 = i_2 = i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_3,j_5)} C_{j_5\ldots j_1}\right). \tag{90}$$

(V).3. $i_1 = i_2 = i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2,j_4,j_5)} C_{j_5\ldots j_1}\right). \tag{91}$$

(V).4. $i_1 = i_3 = i_4 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_3,j_4,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{92}$$

(V).5. $i_2 = i_3 = i_4 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_3,j_4,j_5)} C_{j_5\ldots j_1} \Biggr). \tag{93}$$

(VI).1. $i_5 \neq i_1 = i_2 \neq i_3 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2)} \Biggl(\sum_{(j_3,j_4)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{94}$$

(VI).2. $i_5 \neq i_1 = i_3 \neq i_2 = i_4 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_3)} \Biggl(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{95}$$

(VI).3. $i_5 \neq i_1 = i_4 \neq i_2 = i_3 \neq i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_4)} \Biggl(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{96}$$

(VI).4. $i_4 \neq i_1 = i_2 \neq i_3 = i_5 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_2)} \Biggl(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{97}$$

(VI).5. $i_4 \neq i_1 = i_5 \neq i_2 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_1,j_5)} \Biggl(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{98}$$

(VI).6. $i_4 \neq i_2 = i_5 \neq i_1 = i_3 \neq i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_5)} \Biggl(\sum_{(j_1,j_3)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{99}$$

(VI).7. $i_3 \neq i_2 = i_5 \neq i_1 = i_4 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \Biggl(\sum_{(j_2,j_5)} \Biggl(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \Biggr)\Biggr). \tag{100}$$

(VI).8. $i_3 \neq i_1 = i_2 \neq i_4 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{101}$$

(VI).9. $i_3 \neq i_2 = i_4 \neq i_1 = i_5 \neq i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{102}$$

(VI).10. $i_2 \neq i_1 = i_4 \neq i_3 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{103}$$

(VI).11. $i_2 \neq i_1 = i_3 \neq i_4 = i_5 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{104}$$

(VI).12. $i_2 \neq i_1 = i_5 \neq i_3 = i_4 \neq i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right). \tag{105}$$

(VI).13. $i_1 \neq i_2 = i_3 \neq i_4 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)}\left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{106}$$

(VI).14. $i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{107}$$

(VI).15. $i_1 \neq i_2 = i_5 \neq i_3 = i_4 \neq i_1$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_3,j_4)} C_{j_5\ldots j_1}\right)\right). \tag{108}$$

(VII).1. $i_1 = i_2 = i_3 \neq i_4 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_4,j_5)}\left(\sum_{(j_1,j_2,j_3)} C_{j_5\ldots j_1}\right)\right). \tag{109}$$

(VII).2. $i_1 = i_2 = i_4 \neq i_3 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_5)}\left(\sum_{(j_1,j_2,j_4)} C_{j_5\ldots j_1}\right)\right). \tag{110}$$

(VII).3. $i_1 = i_2 = i_5 \neq i_3 = i_4$:

$$E_p = I - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{111}$$

(VII).4. $i_2 = i_3 = i_4 \neq i_1 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_5)}\left(\sum_{(j_2,j_3,j_4)} C_{j_5\ldots j_1}\right)\right). \tag{112}$$

(VII).5. $i_2 = i_3 = i_5 \neq i_1 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_4)}\left(\sum_{(j_2,j_3,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{113}$$

(VII).6. $i_2 = i_4 = i_5 \neq i_1 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_3)}\left(\sum_{(j_2,j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{114}$$

(VII).7. $i_3 = i_4 = i_5 \neq i_1 = i_2$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_1,j_2)}\left(\sum_{(j_3,j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{115}$$

(VII).8. $i_1 = i_3 = i_5 \neq i_2 = i_4$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_4)}\left(\sum_{(j_1,j_3,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{116}$$

(VII).9. $i_1 = i_3 = i_4 \neq i_2 = i_5$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_5)}\left(\sum_{(j_1,j_3,j_4)} C_{j_5\ldots j_1}\right)\right). \tag{117}$$

(VII).10. $i_1 = i_4 = i_5 \neq i_2 = i_3$:

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\left(\sum_{(j_2,j_3)}\left(\sum_{(j_1,j_4,j_5)} C_{j_5\ldots j_1}\right)\right). \tag{118}$$

Obviously, the above formulas do not contain multiplier factors 2!, 3!, 4!, and 5! in contrast to the estimate (42). However, the number of the mentioned conditions is quite large, which is inconvenient for practical calculations.

In the papers [46] and [47], it was proposed the hypothesis that all the formulas (45)-(118) can be replaced by the following equalities

$$E_2^p = I_2 - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}^2,$$

$$E_3^p = I_3 - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1}^2,$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}^2,$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}^2,$$

where $i_1,\ldots,i_5 = 1,\ldots,m$.

At that, the specified replacement will not lead to a noticeable loss of the mean-square accuracy of approximation of iterated Ito stochastic integrals from the family (6).

This paper is devoted to the detailed confirmation of the hypothesis from [46], [47] for the case of multiple Fourier–Legendre series.

It should be noted that unlike the method based on Theorems 1 and 2, existing approaches to the mean-square approximation of iterated stochastic integrals (see, for example, [2]-[4], [48]-[58]) do not allow choosing different numbers p for approximations of different iterated stochastic integrals. Moreover, the noted approaches [2]-[4], [48]-[58] exclude the possibility for obtaining of approximate and exact expressions for the mean-square approximation error similar to the formulas (42), (44).

5. Approximations of Iterated Ito Stochastic Integrals from the Numerical Schemes (11)–(14) Using Legendre Polynomials

In this section, we consider the approximations of the iterated Ito stochastic integrals (6) of multiplicities 1 to 5 based on Theorems 1, 2 and multiple Fourier–Legendre series.

The numerical schemes (11)–(14) contain the following set (see (6)) of iterated Ito stochastic integrals

$$I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(2)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1 i_2)},\quad I_{(10)T,t}^{(i_1 i_2)},\quad I_{(01)T,t}^{(i_1 i_2)},\quad I_{(000)T,t}^{(i_1 i_2 i_3)},\quad I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}, \tag{119}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)},\quad I_{(100)T,t}^{(i_1 i_2 i_3)},\quad I_{(010)T,t}^{(i_1 i_2 i_3)},\quad I_{(001)T,t}^{(i_1 i_2 i_3)}. \tag{120}$$

Let us consider the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$

$$\phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right),\quad j=0,1,2,\ldots, \tag{121}$$

where $P_j(x)$ is the Legendre polynomial

$$P_j(x)=\frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

Using Theorems 1, 2 and the system of functions (121), we obtain the following formulas for numerical modeling of the stochastic integrals (119), (120) [5]-[47]

$$I_{(0)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{(1)T,t}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{(2)T,t}^{(i_1)}=\frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)}+\frac{\sqrt{3}}{2}\zeta_1^{(i_1)}+\frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

$$I_{(00)T,t}^{(i_1i_2)q}=\frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{122}$$

$$I_{(000)T,t}^{(i_1i_2i_3)q_1}=\sum_{j_1,j_2,j_3=0}^{q_1}C_{j_3j_2j_1}^{000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$I_{(10)T,t}^{(i_1i_2)q_2}=\sum_{j_1,j_2=0}^{q_2}C_{j_2j_1}^{10}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{123}$$

$$I_{(01)T,t}^{(i_1i_2)q_2}=\sum_{j_1,j_2=0}^{q_2}C_{j_2j_1}^{01}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{124}$$

$$I_{(0000)T,t}^{(i_1i_2i_3i_4)q_3}=\sum_{j_1,j_2,j_3,j_4=0}^{q_3}C_{j_4j_3j_2j_1}^{0000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
\left.+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\right),
$$

$$
I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)q_4}=\sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}C_{j_5j_4j_3j_2j_1}^{00000}\left(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-\right.
$$

$$
-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right),
$$

$$
I_{(001)T,t}^{(i_1i_2i_3)q_5}=\sum_{j_1,j_2,j_3=0}^{q_5}C_{j_3j_2j_1}^{001}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\right.
$$

$$
\left.-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right),
$$

$$
I_{(010)T,t}^{(i_1i_2i_3)q_6}=\sum_{j_1,j_2,j_3=0}^{q_6}C_{j_3j_2j_1}^{010}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\right.
$$

$$
\left.-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right),
$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)q_7} = \sum_{j_1,j_2,j_3=0}^{q_7} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr),$$

where $\mathbf{1}_A$ is the indicator of the set A,

(125) $$C_{j_3 j_2 j_1}^{000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8} (T-t)^{3/2} \bar{C}_{j_3 j_2 j_1}^{000},$$

(126) $$C_{j_2 j_1}^{01} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^{2} \bar{C}_{j_2 j_1}^{01},$$

(127) $$C_{j_2 j_1}^{10} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^{2} \bar{C}_{j_2 j_1}^{10},$$

(128) $$C_{j_4 j_3 j_2 j_1}^{0000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16} (T-t)^{2} \bar{C}_{j_4 j_3 j_2 j_1}^{0000},$$

(129) $$C_{j_3 j_2 j_1}^{001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{001},$$

(130) $$C_{j_3 j_2 j_1}^{010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{010},$$

(131) $$C_{j_3 j_2 j_1}^{100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16} (T-t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{100},$$

(132) $$C_{j_5 j_4 j_3 j_2 j_1}^{00000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32} (T-t)^{5/2} \bar{C}_{j_5 j_4 j_3 j_2 j_1}^{00000},$$

where

(133) $$\bar{C}_{j_3 j_2 j_1}^{000} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz,$$

(134) $$\bar{C}_{j_2 j_1}^{01} = - \int\limits_{-1}^{1} (1+y) P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy,$$

$$\bar{C}_{j_2j_1}^{10} = -\int\limits_{-1}^{1} P_{j_2}(y)\int\limits_{-1}^{y}(1+x)P_{j_1}(x)dxdy, \tag{135}$$

$$\bar{C}_{j_4j_3j_2j_1}^{0000} = \int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu, \tag{136}$$

$$\bar{C}_{j_3j_2j_1}^{001} = -\int\limits_{-1}^{1} P_{j_3}(z)(z+1)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz, \tag{137}$$

$$\bar{C}_{j_3j_2j_1}^{010} = -\int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)(y+1)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz, \tag{138}$$

$$\bar{C}_{j_3j_2j_1}^{100} = -\int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)(x+1)dxdydz, \tag{139}$$

$$\bar{C}_{j_5j_4j_3j_2j_1}^{00000} = \int\limits_{-1}^{1} P_{j_5}(v)\int\limits_{-1}^{v} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdudv; \tag{140}$$

another notations are the same as in Theorems 1, 2.

6. Optimization of Approximations of Iterated Ito Stochastic Integrals from the Numerical Schemes (11)–(14)

This section is devoted to the optimization of approximations of iterated Ito stochastic integrals from the numerical schemes (11)–(14). More precisely, we discuss how minimize the numbers $q, q_1, q_2, \ldots, q_7$ from Sect. 5.

Let us combine the relations (122), (125)-(140) with (45)-(118). Thus, we have the following formulas.

The case $k=2$ for the integral $I_{(00)T,t}^{(i_1i_2)}$

2.1.a. $i_1 \neq i_2$:

$$E_2^p = (T-t)^2\left(\frac{1}{2}-\frac{1}{4}-\frac{1}{2}\sum_{i=1}^{p}\frac{1}{4i^2-1}\right) = \frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{p}\frac{1}{4i^2-1}\right). \tag{141}$$

2.2.a. $i_1 = i_2$:

$$E_2^p = (T-t)^2 \left(\frac{1}{2} - \frac{1}{16} \sum_{j_1,j_2=0}^{p} (2j_1+1)(2j_2+1) \bar{C}_{j_2j_1}^{00} \left(\sum_{(j_1,j_2)} \bar{C}_{j_2j_1}^{00} \right) \right) \equiv 0.$$

The case $k = 2$ for the integral $I_{(01)T,t}^{(i_1i_2)}$

2.1.b. $i_1 \neq i_2$:

$$E_2^p = (T-t)^4 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1,j_2=0}^{p} (2j_1+1)(2j_2+1) \left(\bar{C}_{j_2j_1}^{01} \right)^2 \right). \tag{142}$$

2.2.b. $i_1 = i_2$:

$$E_2^p = (T-t)^4 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1,j_2=0}^{p} (2j_1+1)(2j_2+1) \bar{C}_{j_2j_1}^{01} \left(\sum_{(j_1,j_2)} \bar{C}_{j_2j_1}^{01} \right) \right). \tag{143}$$

The case $k = 2$ for the integral $I_{(10)T,t}^{(i_1i_2)}$

2.1.c. $i_1 \neq i_2$:

$$E_2^p = (T-t)^4 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1,j_2=0}^{p} (2j_1+1)(2j_2+1) \left(\bar{C}_{j_2j_1}^{10} \right)^2 \right). \tag{144}$$

2.2.c. $i_1 = i_2$:

$$E_2^p = (T-t)^4 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1,j_2=0}^{p} (2j_1+1)(2j_2+1) \bar{C}_{j_2j_1}^{10} \left(\sum_{(j_1,j_2)} \bar{C}_{j_2j_1}^{10} \right) \right). \tag{145}$$

The case $k = 3$ for the integral $I_{(000)T,t}^{(i_1i_2i_3)}$

3.1.a. $i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3$:

$$E_3^p = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\bar{C}_{j_3j_2j_1}^{000} \right)^2 \right). \tag{146}$$

3.2.a. $i_1 = i_2 = i_3$:

$$E_3^p = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \bar{C}_{j_3j_2j_1}^{000} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_3j_2j_1}^{000} \right) \right) \equiv 0.$$

3.3.1.a. $i_1 = i_2 \ne i_3$:

$$E_3^p = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{000} \right)^2 + \bar{C}_{j_3j_1j_2}^{000} \bar{C}_{j_3j_2j_1}^{000} \right) \right). \tag{147}$$

3.3.2.a. $i_1 \ne i_2 = i_3$:

$$E_3^p = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{000} \right)^2 + \bar{C}_{j_2j_3j_1}^{000} \bar{C}_{j_3j_2j_1}^{000} \right) \right). \tag{148}$$

3.3.3.a. $i_1 = i_3 \ne i_2$:

$$E_3^p = (T-t)^3 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{000} \right)^2 + \bar{C}_{j_3j_2j_1}^{000} \bar{C}_{j_1j_2j_3}^{000} \right) \right). \tag{149}$$

The case $k = 3$ for the integral $I_{(001)T,t}^{(i_1i_2i_3)}$

3.1.b. $i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3$:

$$E_3^p = (T-t)^5 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\bar{C}_{j_3j_2j_1}^{001} \right)^2 \right). \tag{150}$$

3.2.b. $i_1 = i_2 = i_3$:

$$E_3^p = (T-t)^5 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \bar{C}_{j_3j_2j_1}^{001} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_3j_2j_1}^{001} \right) \right). \tag{151}$$

3.3.1.b. $i_1 = i_2 \ne i_3$:

$$E_3^p = (T-t)^5 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{001} \right)^2 + \bar{C}_{j_3j_1j_2}^{001} \bar{C}_{j_3j_2j_1}^{001} \right) \right). \tag{152}$$

3.3.2.b. $i_1 \ne i_2 = i_3$:

$$E_3^p = (T-t)^5 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{001} \right)^2 + \bar{C}_{j_2j_3j_1}^{001} \bar{C}_{j_3j_2j_1}^{001} \right) \right). \tag{153}$$

3.3.3.b. $i_1 = i_3 \ne i_2$:

$$E_3^p = (T-t)^5 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{001} \right)^2 + \bar{C}_{j_3j_2j_1}^{001} \bar{C}_{j_1j_2j_3}^{001} \right) \right). \tag{154}$$

The case $k=3$ for the integral $I_{(010)T,t}^{(i_1i_2i_3)}$

3.1.c. $i_1\ne i_2, i_1\ne i_3, i_2\ne i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2\right). \tag{155}$$

3.2.c. $i_1=i_2=i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\bar{C}_{j_3j_2j_1}^{010}\left(\sum_{(j_1,j_2,j_3)}\bar{C}_{j_3j_2j_1}^{010}\right)\right). \tag{156}$$

3.3.1.c. $i_1=i_2\ne i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2+\bar{C}_{j_3j_1j_2}^{010}\bar{C}_{j_3j_2j_1}^{010}\right)\right). \tag{157}$$

3.3.2.c. $i_1\ne i_2=i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2+\bar{C}_{j_2j_3j_1}^{010}\bar{C}_{j_3j_2j_1}^{010}\right)\right). \tag{158}$$

3.3.3.c. $i_1=i_3\ne i_2$:

$$E_3^p=(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2+\bar{C}_{j_3j_2j_1}^{010}\bar{C}_{j_1j_2j_3}^{010}\right)\right). \tag{159}$$

The case $k=3$ for the integral $I_{(100)T,t}^{(i_1i_2i_3)}$

3.1.d. $i_1\ne i_2, i_1\ne i_3, i_2\ne i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2\right). \tag{160}$$

3.2.d. $i_1=i_2=i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\bar{C}_{j_3j_2j_1}^{100}\left(\sum_{(j_1,j_2,j_3)}\bar{C}_{j_3j_2j_1}^{100}\right)\right). \tag{161}$$

3.3.1.d. $i_1=i_2\ne i_3$:

$$E_3^p=(T-t)^5\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2+\bar{C}_{j_3j_1j_2}^{100}\bar{C}_{j_3j_2j_1}^{100}\right)\right). \tag{162}$$

3.3.2.d. $i_1 \ne i_2 = i_3$:

$$E_3^p = (T-t)^5 \left(\frac{1}{60} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{100} \right)^2 + \bar{C}_{j_2j_3j_1}^{100} \bar{C}_{j_3j_2j_1}^{100} \right) \right). \quad (163)$$

3.3.3.d. $i_1 = i_3 \ne i_2$:

$$E_3^p = (T-t)^5 \left(\frac{1}{60} - \frac{1}{256} \sum_{j_1,j_2,j_3=0}^{p} (2j_1+1)(2j_2+1)(2j_3+1) \left(\left(\bar{C}_{j_3j_2j_1}^{100} \right)^2 + \bar{C}_{j_3j_2j_1}^{100} \bar{C}_{j_1j_2j_3}^{100} \right) \right). \quad (164)$$

The case $k = 4$ for the integral $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}$

4.1. $i_1, \ldots, i_4$ are pairwise different:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \left(\bar{C}_{j_4\ldots j_1}^{0000} \right)^2 \right). \quad (165)$$

4.2. $i_1 = i_2 = i_3 = i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,\ldots,j_4)} \bar{C}_{j_4\ldots j_1}^{0000} \right) \right) \equiv 0.$$

4.3.1. $i_1 = i_2 \ne i_3, i_4;\ i_3 \ne i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_2)} \bar{C}_{j_4\ldots j_1}^{0000} \right) \right). \quad (166)$$

4.3.2. $i_1 = i_3 \ne i_2, i_4;\ i_2 \ne i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_3)} \bar{C}_{j_4\ldots j_1}^{0000} \right) \right). \quad (167)$$

4.3.3. $i_1 = i_4 \ne i_2, i_3;\ i_2 \ne i_3$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_4)} \bar{C}_{j_4\ldots j_1}^{0000} \right) \right). \quad (168)$$

4.3.4. $i_2 = i_3 \ne i_1, i_4;\ i_1 \ne i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4} (2j_l+1) \right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_2,j_3)} \bar{C}_{j_4\ldots j_1}^{0000} \right) \right). \quad (169)$$

4.3.5. $i_2 = i_4 \neq i_1, i_3;\ i_1 \neq i_3:$

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_2,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{170}$$

4.3.6. $i_3 = i_4 \neq i_1, i_2;\ i_1 \neq i_2:$

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_3,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{171}$$

4.4.1. $i_1 = i_2 = i_3 \neq i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{172}$$

4.4.2. $i_2 = i_3 = i_4 \neq i_1$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_2,j_3,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{173}$$

4.4.3. $i_1 = i_2 = i_4 \neq i_3$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_2,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{174}$$

4.4.4. $i_1 = i_3 = i_4 \neq i_2$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_3,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right). \tag{175}$$

4.5.1. $i_1 = i_2 \neq i_3 = i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right)\right). \tag{176}$$

4.5.2. $i_1 = i_3 \neq i_2 = i_4$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right)\right). \tag{177}$$

4.5.3. $i_1 = i_4 \neq i_2 = i_3$:

$$E_4^p = (T-t)^4 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1,\ldots,j_4=0}^{p} \left(\prod_{l=1}^{4}(2j_l+1)\right) \cdot \bar{C}_{j_4\ldots j_1}^{0000} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} \bar{C}_{j_4\ldots j_1}^{0000}\right)\right)\right). \tag{178}$$

The case $k=5$ for the integral $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$

5.1. $i_1,\ldots,i_5$ are pairwise different:

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\left(\bar C_{j_5\ldots j_1}^{00000}\right)^2\right). \tag{179}$$

5.2. $i_1=i_2=i_3=i_4=i_5$:

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_1,\ldots,j_5)}\bar C_{j_5\ldots j_1}^{00000}\right)\right)\equiv 0.$$

5.3.1. $i_1=i_2\ne i_3,i_4,i_5$ (i_3,i_4,i_5 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_1,j_2)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{180}$$

5.3.2. $i_1=i_3\ne i_2,i_4,i_5$ (i_2,i_4,i_5 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_1,j_3)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{181}$$

5.3.3. $i_1=i_4\ne i_2,i_3,i_5$ (i_2,i_3,i_5 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_1,j_4)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{182}$$

5.3.4. $i_1=i_5\ne i_2,i_3,i_4$ (i_2,i_3,i_4 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_1,j_5)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{183}$$

5.3.5. $i_2=i_3\ne i_1,i_4,i_5$ (i_1,i_4,i_5 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_2,j_3)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{184}$$

5.3.6. $i_2=i_4\ne i_1,i_3,i_5$ (i_1,i_3,i_5 are pairwise different):

$$E_5^p=(T-t)^5\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{p}\left(\prod_{l=1}^{5}(2j_l+1)\right)\cdot\bar C_{j_5\ldots j_1}^{00000}\left(\sum_{(j_2,j_4)}\bar C_{j_5\ldots j_1}^{00000}\right)\right). \tag{185}$$

5.3.7. $i_2 = i_5 \neq i_1, i_3, i_4$ (i_1, i_3, i_4 are pairwise different):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{186}$$

5.3.8. $i_3 = i_4 \neq i_1, i_2, i_5$ (i_1, i_2, i_5 are pairwise different):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{187}$$

5.3.9. $i_3 = i_5 \neq i_1, i_2, i_4$ (i_1, i_2, i_4 are pairwise different):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{188}$$

5.3.10. $i_4 = i_5 \neq i_1, i_2, i_3$ (i_1, i_2, i_3 are pairwise different):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{189}$$

5.4.1. $i_1 = i_2 = i_3 \neq i_4, i_5$ ($i_4 \neq i_5$):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{190}$$

5.4.2. $i_1 = i_2 = i_4 \neq i_3, i_5$ ($i_3 \neq i_5$):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{191}$$

5.4.3. $i_1 = i_2 = i_5 \neq i_3, i_4$ ($i_3 \neq i_4$):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{192}$$

5.4.4. $i_2 = i_3 = i_4 \neq i_1, i_5$ ($i_1 \neq i_5$):

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{193}$$

5.4.5. $i_2 = i_3 = i_5 \neq i_1, i_4$ $(i_1 \neq i_4)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{194}$$

5.4.6. $i_2 = i_4 = i_5 \neq i_1, i_3$ $(i_1 \neq i_3)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{195}$$

5.4.7. $i_3 = i_4 = i_5 \neq i_1, i_2$ $(i_1 \neq i_2)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_3,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{196}$$

5.4.8. $i_1 = i_3 = i_5 \neq i_2, i_4$ $(i_2 \neq i_4)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{197}$$

5.4.9. $i_1 = i_3 = i_4 \neq i_2, i_5$ $(i_2 \neq i_5)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{198}$$

5.4.10. $i_1 = i_4 = i_5 \neq i_2, i_3$ $(i_2 \neq i_3)$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{199}$$

5.5.1. $i_1 = i_2 = i_3 = i_4 \neq i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{200}$$

5.5.2. $i_1 = i_2 = i_3 = i_5 \neq i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{201}$$

5.5.3. $i_1 = i_2 = i_4 = i_5 \neq i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{202}$$

5.5.4. $i_1 = i_3 = i_4 = i_5 \neq i_2$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{203}$$

5.5.5. $i_2 = i_3 = i_4 = i_5 \neq i_1$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_3,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right). \tag{204}$$

5.6.1. $i_5 \neq i_1 = i_2 \neq i_3 = i_4 \neq i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{205}$$

5.6.2. $i_5 \neq i_1 = i_3 \neq i_2 = i_4 \neq i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{206}$$

5.6.3. $i_5 \neq i_1 = i_4 \neq i_2 = i_3 \neq i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{207}$$

5.6.4. $i_4 \neq i_1 = i_2 \neq i_3 = i_5 \neq i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{208}$$

5.6.5. $i_4 \neq i_1 = i_5 \neq i_2 = i_3 \neq i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_2,j_3)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{209}$$

5.6.6. $i_4 \ne i_2 = i_5 \ne i_1 = i_3 \ne i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_1,j_3)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{210}$$

5.6.7. $i_3 \ne i_2 = i_5 \ne i_1 = i_4 \ne i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_1,j_4)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{211}$$

5.6.8. $i_3 \ne i_1 = i_2 \ne i_4 = i_5 \ne i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{212}$$

5.6.9. $i_3 \ne i_2 = i_4 \ne i_1 = i_5 \ne i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_1,j_5)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{213}$$

5.6.10. $i_2 \ne i_1 = i_4 \ne i_3 = i_5 \ne i_2$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{214}$$

5.6.11. $i_2 \ne i_1 = i_3 \ne i_4 = i_5 \ne i_2$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{215}$$

5.6.12. $i_2 \ne i_1 = i_5 \ne i_3 = i_4 \ne i_2$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{216}$$

5.6.13. $i_1 \ne i_2 = i_3 \ne i_4 = i_5 \ne i_1$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5}(2j_l+1)\right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_3)} \left(\sum_{(j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000}\right)\right)\right). \tag{217}$$

5.6.14. $i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{218}$$

5.6.15. $i_1 \neq i_2 = i_5 \neq i_3 = i_4 \neq i_1$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{219}$$

5.7.1. $i_1 = i_2 = i_3 \neq i_4 = i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_4,j_5)} \left(\sum_{(j_1,j_2,j_3)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{220}$$

5.7.2. $i_1 = i_2 = i_4 \neq i_3 = i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_3,j_5)} \left(\sum_{(j_1,j_2,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{221}$$

5.7.3. $i_1 = i_2 = i_5 \neq i_3 = i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_3,j_4)} \left(\sum_{(j_1,j_2,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{222}$$

5.7.4. $i_2 = i_3 = i_4 \neq i_1 = i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_5)} \left(\sum_{(j_2,j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{223}$$

5.7.5. $i_2 = i_3 = i_5 \neq i_1 = i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{224}$$

5.7.6. $i_2 = i_4 = i_5 \neq i_1 = i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_3)} \left(\sum_{(j_2,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{225}$$

5.7.7. $i_3 = i_4 = i_5 \neq i_1 = i_2$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_1,j_2)} \left(\sum_{(j_3,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{226}$$

5.7.8. $i_1 = i_3 = i_5 \neq i_2 = i_4$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_1,j_3,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{227}$$

5.7.9. $i_1 = i_3 = i_4 \neq i_2 = i_5$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_5)} \left(\sum_{(j_1,j_3,j_4)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{228}$$

5.7.10. $i_1 = i_4 = i_5 \neq i_2 = i_3$:

$$E_5^p = (T-t)^5 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1,\ldots,j_5=0}^{p} \left(\prod_{l=1}^{5} (2j_l+1) \right) \cdot \bar{C}_{j_5\ldots j_1}^{00000} \left(\sum_{(j_2,j_3)} \left(\sum_{(j_1,j_4,j_5)} \bar{C}_{j_5\ldots j_1}^{00000} \right) \right) \right). \tag{229}$$

Denote $q(\alpha)$ the numbers p from the formulas (141)-(229), where α are the numbers of the cases corresponding to the formulas (141)-(229). For example, $q(\mathbf{2}.2.b)$ is the number p from the formula (143), $q(\mathbf{5}.7.10)$ is the number p from the formula (229), etc.

Let

$$E_2^p \le (T-t)^4, \quad E_3^p \le (T-t)^4 \tag{230}$$

where E_2^p is defined by (141) and E_3^p is defined by (146)-(149).

Let

$$E_2^p \le (T-t)^5, \quad E_3^p \le (T-t)^5, \quad E_4^p \le (T-t)^5 \tag{231}$$

where E_2^p is defined by (141)-(145), E_3^p is defined by (146)-(149), and E_4^p is defined by (165)-(178).

Let

$$E_2^p \le (T-t)^6, \quad E_3^p \le (T-t)^6, \quad E_4^p \le (T-t)^6, \quad E_5^p \le (T-t)^6, \tag{232}$$

where E_2^p is defined by (141)-(145), E_3^p is defined by (146)-(164), E_4^p is defined by (165)-(178), and E_5^p is defined by (179)-(229).

Note that the conditions (230)-(232) are particular cases of (15) for $r = 3, 4$, and 5.

Let us show by numerical experiments (see Tables 1–13) that in most situations the following inequalities are fullfilled (under conditions (230)-(232))

TABLE 1. Stochastic integral $I_{(000)T,t}^{(i_1 i_2 i_3)}$. The condition (230).

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q(\mathbf{3}.1.a)$	$\boxed{12}$	$\boxed{16}$	$\boxed{28}$	$\boxed{36}$	$\boxed{47}$	$\boxed{50}$
$q(\mathbf{3}.3.1.a)$	6	8	14	18	23	25
$q(\mathbf{3}.3.2.a)$	6	8	14	18	23	25
$q(\mathbf{3}.3.3.a)$	12	16	28	36	47	$\boxed{51}$

TABLE 2. Stochastic integrals $I_{(01)T,t}^{(i_1 i_2)}$, $I_{(10)T,t}^{(i_1 i_2)}$. The condition (231).

$T-t$	0.010	0.005	0.0025
$q(\mathbf{2}.1.b)$	$\boxed{4}$	$\boxed{8}$	$\boxed{16}$
$q(\mathbf{2}.2.b)$	1	1	1
$q(\mathbf{2}.1.c)$	$\boxed{4}$	$\boxed{8}$	$\boxed{16}$
$q(\mathbf{2}.2.c)$	1	1	1

$$q(\mathbf{2}.1.b) \geq q(\mathbf{2}.2.b), \quad q(\mathbf{2}.1.c) \geq q(\mathbf{2}.2.c), \tag{233}$$

$$q(\mathbf{3}.1.a) \geq q(\mathbf{3}.3.1.a),\ q(\mathbf{3}.3.2.a),\ q(\mathbf{3}.3.3.a), \tag{234}$$

$$q(\mathbf{3}.1.b) \geq q(\mathbf{3}.2.b),\ q(\mathbf{3}.3.1.b),\ q(\mathbf{3}.3.2.b),\ q(\mathbf{3}.3.3.b), \tag{235}$$

$$q(\mathbf{3}.1.c) \geq q(\mathbf{3}.2.c),\ q(\mathbf{3}.3.1.c),\ q(\mathbf{3}.3.2.c),\ q(\mathbf{3}.3.3.c), \tag{236}$$

$$q(\mathbf{3}.1.d) \geq q(\mathbf{3}.2.d),\ q(\mathbf{3}.3.1.d),\ q(\mathbf{3}.3.2.d),\ q(\mathbf{3}.3.3.d), \tag{237}$$

$$q(\mathbf{4}.1) \geq q(\mathbf{4}.3.1),\ \ldots,\ q(\mathbf{4}.3.6),\ q(\mathbf{4}.4.1),\ \ldots,\ q(\mathbf{4}.4.4),\ q(\mathbf{4}.5.1),\ \ldots,\ q(\mathbf{4}.5.3), \tag{238}$$

$$q(\mathbf{5}.1) \geq q(\mathbf{5}.3.1),\ \ldots,\ q(\mathbf{5}.3.10),\ q(\mathbf{5}.4.1),\ \ldots,\ q(\mathbf{5}.4.10),\ q(\mathbf{5}.5.1),\ \ldots,\ q(\mathbf{5}.5.5), \tag{239}$$

$$q(\mathbf{5}.1) \geq q(\mathbf{5}.6.1),\ \ldots,\ q(\mathbf{5}.6.15),\ q(\mathbf{5}.7.1),\ \ldots,\ q(\mathbf{5}.7.10), \tag{240}$$

where all numbers in the inequalities (233)-(240) are minimal natural numbers satisfying the conditions (230)-(232).

In Tables 1–13, we can see the results of numerical experiments. These results confirm (in most situations) the inequalities (233)-(240)).

Let us show by numerical experiments that we can choose the minimal natural numbers p satisfying the inequalities (230)-(232) only for the values E_2^p, E_3^p, E_4^p, E_5^p defined by the relations (141), (142),

TABLE 3. $T - t = 0.01$. The condition (232).

$I^{(i_1 i_2 i_3)}_{(001)T,t}$	$I^{(i_1 i_2 i_3)}_{(010)T,t}$	$I^{(i_1 i_2 i_3)}_{(100)T,t}$
$q(\mathbf{3}.1.b) = \boxed{6}$	$q(\mathbf{3}.1.c) = \boxed{4}$	$q(\mathbf{3}.1.d) = \boxed{2}$
$q(\mathbf{3}.2.b) = 0$	$q(\mathbf{3}.2.c) = 0$	$q(\mathbf{3}.2.d) = 0$
$q(\mathbf{3}.3.1.b) = 3$	$q(\mathbf{3}.3.1.c) = 3$	$q(\mathbf{3}.3.1.d) = 1$
$q(\mathbf{3}.3.2.b) = 3$	$q(\mathbf{3}.3.2.c) = 1$	$q(\mathbf{3}.3.2.d) = 1$
$q(\mathbf{3}.3.3.b) = 6$	$q(\mathbf{3}.3.3.c) = 4$	$q(\mathbf{3}.3.3.d) = 2$

TABLE 4. Stochastic integral $I^{(i_1 i_2 i_3 i_4)}_{(0000)T,t}$. The condition (231).

$T - t$	0.011	0.008	0.0045	0.0042	0.0040
$q(\mathbf{4}.1)$	$\boxed{6}$	$\boxed{8}$	$\boxed{14}$	$\boxed{15}$	$\boxed{16}$
$q(\mathbf{4}.3.1)$	4	5	10	11	11
$q(\mathbf{4}.3.2)$	6	8	14	15	16
$q(\mathbf{4}.3.3)$	6	8	14	15	16
$q(\mathbf{4}.3.4)$	3	5	9	9	10
$q(\mathbf{4}.3.5)$	6	8	14	15	16
$q(\mathbf{4}.3.6)$	4	5	10	11	11
$q(\mathbf{4}.4.1)$	2	3	4	5	5
$q(\mathbf{4}.4.2)$	2	3	4	5	5
$q(\mathbf{4}.4.3)$	4	6	10	11	11
$q(\mathbf{4}.4.4)$	4	6	10	11	11
$q(\mathbf{4}.5.1)$	2	3	5	6	6
$q(\mathbf{4}.5.2)$	6	8	14	15	16
$q(\mathbf{4}.5.3)$	3	5	9	9	10

(144), (146), (150), (155), (160), (165), (179). At that, we can suppose $i_1, \ldots, i_5 = 1, \ldots, m$ in these relations and use the above numbers p for all remaining cases. This means that we can ignore all the formulas (143), (145), (147)-(149), (151)-(154), (156)-(159), (161)-(164), (166)-(178), (180)-(229). As a result, we will not get a noticeable loss of the mean-square approximation accuracy for iterated Ito stochastic integrals. The detailed numerical confirmation of the above hypothesis can be found in Tables 14–22.

Taking into account the results of this article, we can recommend the following conditions for choosing the minimal natural numbers $q, q_1, q_2, \ldots, q_7$ (see Sect. 5) for the numerical schemes (11)–(14) (constant C (see below) has the same meaning as in the condition (15)).

Milstein scheme (11)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1}\right) \le C(T-t)^3.$$

TABLE 5. $T-t=0.011$. Stochastic integral $I^{(i_1 i_2 i_3 i_4 i_5)}_{(00000)T,t}$. The condition (232).

$q(\mathbf{5}.1)=\boxed{0}$	$q(\mathbf{5}.3.1)=0$	$q(\mathbf{5}.4.1)=0$	$q(\mathbf{5}.5.1)=0$	$q(\mathbf{5}.6.1)=0$	$q(\mathbf{5}.7.1)=0$
	$q(\mathbf{5}.3.2)=0$	$q(\mathbf{5}.4.2)=0$	$q(\mathbf{5}.5.2)=0$	$q(\mathbf{5}.6.2)=0$	$q(\mathbf{5}.7.2)=0$
	$q(\mathbf{5}.3.3)=0$	$q(\mathbf{5}.4.3)=0$	$q(\mathbf{5}.5.3)=0$	$q(\mathbf{5}.6.3)=0$	$q(\mathbf{5}.7.3)=0$
	$q(\mathbf{5}.3.4)=0$	$q(\mathbf{5}.4.4)=0$	$q(\mathbf{5}.5.4)=0$	$q(\mathbf{5}.6.4)=0$	$q(\mathbf{5}.7.4)=0$
	$q(\mathbf{5}.3.5)=0$	$q(\mathbf{5}.4.5)=0$	$q(\mathbf{5}.5.5)=0$	$q(\mathbf{5}.6.5)=0$	$q(\mathbf{5}.7.5)=0$
	$q(\mathbf{5}.3.6)=0$	$q(\mathbf{5}.4.6)=0$		$q(\mathbf{5}.6.6)=0$	$q(\mathbf{5}.7.6)=0$
	$q(\mathbf{5}.3.7)=0$	$q(\mathbf{5}.4.7)=0$		$q(\mathbf{5}.6.7)=0$	$q(\mathbf{5}.7.7)=0$
	$q(\mathbf{5}.3.8)=0$	$q(\mathbf{5}.4.8)=0$		$q(\mathbf{5}.6.8)=0$	$q(\mathbf{5}.7.8)=0$
	$q(\mathbf{5}.3.9)=0$	$q(\mathbf{5}.4.9)=0$		$q(\mathbf{5}.6.9)=0$	$q(\mathbf{5}.7.9)=0$
	$q(\mathbf{5}.3.10)=0$	$q(\mathbf{5}.4.10)=0$		$q(\mathbf{5}.6.10)=0$	$q(\mathbf{5}.7.10)=0$
				$q(\mathbf{5}.6.11)=0$	
				$q(\mathbf{5}.6.12)=0$	
				$q(\mathbf{5}.6.13)=0$	
				$q(\mathbf{5}.6.14)=0$	
				$q(\mathbf{5}.6.15)=0$	

TABLE 6. $T-t=0.008$. Stochastic integral $I^{(i_1 i_2 i_3 i_4 i_5)}_{(00000)T,t}$. The condition (232).

$q(\mathbf{5}.1)=\boxed{1}$	$q(\mathbf{5}.3.1)=1$	$q(\mathbf{5}.4.1)=0$	$q(\mathbf{5}.5.1)=0$	$q(\mathbf{5}.6.1)=1$	$q(\mathbf{5}.7.1)=0$
	$q(\mathbf{5}.3.2)=1$	$q(\mathbf{5}.4.2)=0$	$q(\mathbf{5}.5.2)=0$	$q(\mathbf{5}.6.2)=1$	$q(\mathbf{5}.7.2)=0$
	$q(\mathbf{5}.3.3)=1$	$q(\mathbf{5}.4.3)=0$	$q(\mathbf{5}.5.3)=0$	$q(\mathbf{5}.6.3)=1$	$q(\mathbf{5}.7.3)=0$
	$q(\mathbf{5}.3.4)=1$	$q(\mathbf{5}.4.4)=0$	$q(\mathbf{5}.5.4)=0$	$q(\mathbf{5}.6.4)=1$	$q(\mathbf{5}.7.4)=0$
	$q(\mathbf{5}.3.5)=1$	$q(\mathbf{5}.4.5)=0$	$q(\mathbf{5}.5.5)=0$	$q(\mathbf{5}.6.5)=1$	$q(\mathbf{5}.7.5)=0$
	$q(\mathbf{5}.3.6)=1$	$q(\mathbf{5}.4.6)=0$		$q(\mathbf{5}.6.6)=1$	$q(\mathbf{5}.7.6)=0$
	$q(\mathbf{5}.3.7)=1$	$q(\mathbf{5}.4.7)=0$		$q(\mathbf{5}.6.7)=1$	$q(\mathbf{5}.7.7)=0$
	$q(\mathbf{5}.3.8)=1$	$q(\mathbf{5}.4.8)=0$		$q(\mathbf{5}.6.8)=1$	$q(\mathbf{5}.7.8)=0$
	$q(\mathbf{5}.3.9)=1$	$q(\mathbf{5}.4.9)=0$		$q(\mathbf{5}.6.9)=1$	$q(\mathbf{5}.7.9)=0$
	$q(\mathbf{5}.3.10)=1$	$q(\mathbf{5}.4.10)=0$		$q(\mathbf{5}.6.10)=1$	$q(\mathbf{5}.7.10)=0$
				$q(\mathbf{5}.6.11)=1$	
				$q(\mathbf{5}.6.12)=1$	
				$q(\mathbf{5}.6.13)=1$	
				$q(\mathbf{5}.6.14)=1$	
				$q(\mathbf{5}.6.15)=1$	

Strong Taylor–Ito scheme with convergence order 1.5 (12)

$$
\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^4,
$$

TABLE 7. $T-t=0.0045$. Stochastic integral $I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}$. The condition (232).

$q(\mathbf{5}.1)=\boxed{4}$	$q(\mathbf{5}.3.1)=4$	$q(\mathbf{5}.4.1)=2$	$q(\mathbf{5}.5.1)=1$	$q(\mathbf{5}.6.1)=2$	$q(\mathbf{5}.7.1)=1$
	$q(\mathbf{5}.3.2)=4$	$q(\mathbf{5}.4.2)=3$	$q(\mathbf{5}.5.2)=2$	$q(\mathbf{5}.6.2)=4$	$q(\mathbf{5}.7.2)=3$
	$q(\mathbf{5}.3.3)=4$	$q(\mathbf{5}.4.3)=3$	$q(\mathbf{5}.5.3)=2$	$q(\mathbf{5}.6.3)=3$	$q(\mathbf{5}.7.3)=2$
	$q(\mathbf{5}.3.4)=4$	$q(\mathbf{5}.4.4)=2$	$q(\mathbf{5}.5.4)=2$	$q(\mathbf{5}.6.4)=3$	$q(\mathbf{5}.7.4)=2$
	$q(\mathbf{5}.3.5)=3$	$q(\mathbf{5}.4.5)=3$	$q(\mathbf{5}.5.5)=1$	$q(\mathbf{5}.6.5)=3$	$q(\mathbf{5}.7.5)=3$
	$q(\mathbf{5}.3.6)=4$	$q(\mathbf{5}.4.6)=3$		$q(\mathbf{5}.6.6)=4$	$q(\mathbf{5}.7.6)=3$
	$q(\mathbf{5}.3.7)=4$	$q(\mathbf{5}.4.7)=2$		$q(\mathbf{5}.6.7)=4$	$q(\mathbf{5}.7.7)=1$
	$q(\mathbf{5}.3.8)=3$	$q(\mathbf{5}.4.8)=4$		$q(\mathbf{5}.6.8)=2$	$q(\mathbf{5}.7.8)=4$
	$q(\mathbf{5}.3.9)=4$	$q(\mathbf{5}.4.9)=3$		$q(\mathbf{5}.6.9)=4$	$q(\mathbf{5}.7.9)=3$
	$q(\mathbf{5}.3.10)=3$	$q(\mathbf{5}.4.10)=3$		$q(\mathbf{5}.6.10)=4$	$q(\mathbf{5}.7.10)=2$
				$q(\mathbf{5}.6.11)=3$	
				$q(\mathbf{5}.6.12)=3$	
				$q(\mathbf{5}.6.13)=2$	
				$q(\mathbf{5}.6.14)=4$	
				$q(\mathbf{5}.6.15)=3$	

TABLE 8. $T-t=0.0042$. Stochastic integral $I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}$. The condition (232).

$q(\mathbf{5}.1)=\boxed{5}$	$q(\mathbf{5}.3.1)=5$	$q(\mathbf{5}.4.1)=2$	$q(\mathbf{5}.5.1)=1$	$q(\mathbf{5}.6.1)=2$	$q(\mathbf{5}.7.1)=1$
	$q(\mathbf{5}.3.2)=5$	$q(\mathbf{5}.4.2)=4$	$q(\mathbf{5}.5.2)=2$	$q(\mathbf{5}.6.2)=4$	$q(\mathbf{5}.7.2)=3$
	$q(\mathbf{5}.3.3)=5$	$q(\mathbf{5}.4.3)=4$	$q(\mathbf{5}.5.3)=2$	$q(\mathbf{5}.6.3)=3$	$q(\mathbf{5}.7.3)=2$
	$q(\mathbf{5}.3.4)=5$	$q(\mathbf{5}.4.4)=2$	$q(\mathbf{5}.5.4)=2$	$q(\mathbf{5}.6.4)=4$	$q(\mathbf{5}.7.4)=2$
	$q(\mathbf{5}.3.5)=3$	$q(\mathbf{5}.4.5)=3$	$q(\mathbf{5}.5.5)=1$	$q(\mathbf{5}.6.5)=3$	$q(\mathbf{5}.7.5)=3$
	$q(\mathbf{5}.3.6)=4$	$q(\mathbf{5}.4.6)=4$		$q(\mathbf{5}.6.6)=5$	$q(\mathbf{5}.7.6)=3$
	$q(\mathbf{5}.3.7)=5$	$q(\mathbf{5}.4.7)=2$		$q(\mathbf{5}.6.7)=5$	$q(\mathbf{5}.7.7)=1$
	$q(\mathbf{5}.3.8)=3$	$q(\mathbf{5}.4.8)=5$		$q(\mathbf{5}.6.8)=2$	$q(\mathbf{5}.7.8)=4$
	$q(\mathbf{5}.3.9)=5$	$q(\mathbf{5}.4.9)=3$		$q(\mathbf{5}.6.9)=4$	$q(\mathbf{5}.7.9)=3$
	$q(\mathbf{5}.3.10)=4$	$q(\mathbf{5}.4.10)=4$		$q(\mathbf{5}.6.10)=5$	$q(\mathbf{5}.7.10)=2$
				$q(\mathbf{5}.6.11)=4$	
				$q(\mathbf{5}.6.12)=3$	
				$q(\mathbf{5}.6.13)=2$	
				$q(\mathbf{5}.6.14)=4$	
				$q(\mathbf{5}.6.15)=3$	

$$(T-t)^3\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le$$

$$\le C(T-t)^4.$$

TABLE 9. $T-t=0.0035$. Stochastic integral $I_{(00000)T,t}^{(i_1i_2i_3i_4i_5)}$. The condition (232).

$q(\mathbf{5}.1)=\boxed{6}$	$q(\mathbf{5}.3.1)=6$	$q(\mathbf{5}.4.1)=3$	$q(\mathbf{5}.5.1)=1$	$q(\mathbf{5}.6.1)=3$	$q(\mathbf{5}.7.1)=3$
	$q(\mathbf{5}.3.2)=6$	$q(\mathbf{5}.4.2)=4$	$q(\mathbf{5}.5.2)=3$	$q(\mathbf{5}.6.2)=6$	$q(\mathbf{5}.7.2)=4$
	$q(\mathbf{5}.3.3)=6$	$q(\mathbf{5}.4.3)=4$	$q(\mathbf{5}.5.3)=3$	$q(\mathbf{5}.6.3)=4$	$q(\mathbf{5}.7.3)=3$
	$q(\mathbf{5}.3.4)=6$	$q(\mathbf{5}.4.4)=2$	$q(\mathbf{5}.5.4)=3$	$q(\mathbf{5}.6.4)=4$	$q(\mathbf{5}.7.4)=2$
	$q(\mathbf{5}.3.5)=4$	$q(\mathbf{5}.4.5)=4$	$q(\mathbf{5}.5.5)=1$	$q(\mathbf{5}.6.5)=4$	$q(\mathbf{5}.7.5)=4$
	$q(\mathbf{5}.3.6)=6$	$q(\mathbf{5}.4.6)=4$		$q(\mathbf{5}.6.6)=6$	$q(\mathbf{5}.7.6)=4$
	$q(\mathbf{5}.3.7)=6$	$q(\mathbf{5}.4.7)=3$		$q(\mathbf{5}.6.7)=6$	$q(\mathbf{5}.7.7)=1$
	$q(\mathbf{5}.3.8)=4$	$q(\mathbf{5}.4.8)=6$		$q(\mathbf{5}.6.8)=3$	$q(\mathbf{5}.7.8)=6$
	$q(\mathbf{5}.3.9)=6$	$q(\mathbf{5}.4.9)=4$		$q(\mathbf{5}.6.9)=6$	$q(\mathbf{5}.7.9)=4$
	$q(\mathbf{5}.3.10)=4$	$q(\mathbf{5}.4.10)=4$		$q(\mathbf{5}.6.10)=6$	$q(\mathbf{5}.7.10)=3$
				$q(\mathbf{5}.6.11)=4$	
				$q(\mathbf{5}.6.12)=4$	
				$q(\mathbf{5}.6.13)=3$	
				$q(\mathbf{5}.6.14)=6$	
				$q(\mathbf{5}.6.15)=4$	

TABLE 10. Milstein scheme. Stochastic integral $I_{(00)T,t}^{(i_1i_2)}$. The condition (230).

$T-t$	2^{-1}	2^{-4}	2^{-8}	2^{-12}
$q(\mathbf{2}.1.a)$	1	2	32	512

TABLE 11. Scheme with strong order 1.5. Stochastic integrals $I_{(00)T,t}^{(i_1i_2)}$, $I_{(000)T,t}^{(i_1i_2i_3)}$. The condition (230).

$T-t$	2^{-1}	2^{-3}	2^{-5}	2^{-8}
$q(\mathbf{2}.1.a)$	1	8	128	8192
$q(\mathbf{3}.1.a)$	$\boxed{0}$	$\boxed{1}$	$\boxed{4}$	$\boxed{32}$
$q(\mathbf{3}.3.1.a)$	0	0	2	16
$q(\mathbf{3}.3.2.a)$	0	0	2	16
$q(\mathbf{3}.3.3.a)$	0	0	4	$\boxed{33}$

Strong Taylor–Ito scheme with convergence order 2.0 (13)

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^5,$$

TABLE 12. Scheme with strong order 2.0. Stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$, $I_{(01)T,t}^{(i_1 i_2)}$, $I_{(10)T,t}^{(i_1 i_2)}$, $I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}$. The condition (231).

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}
$q(\mathbf{2}.1.a)$	1	8	64	512
$q(\mathbf{3}.1.a)$	$\boxed{0}$	$\boxed{2}$	$\boxed{8}$	$\boxed{32}$
$q(\mathbf{3}.3.1.a)$	0	1	4	16
$q(\mathbf{3}.3.2.a)$	0	1	4	16
$q(\mathbf{3}.3.3.a)$	0	2	8	$\boxed{33}$
$q(\mathbf{2}.1.b)$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$
$q(\mathbf{2}.2.b)$	0	0	0	0
$q(\mathbf{2}.1.c)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
$q(\mathbf{2}.2.c)$	0	0	0	0
$q(\mathbf{4}.1)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
$q(\mathbf{4}.3.1)$	0	0	0	0
...	...	...	...	...
$q(\mathbf{4}.3.6)$	0	0	0	0
$q(\mathbf{4}.4.1)$	0	0	0	0
...	...	...	...	...
$q(\mathbf{4}.4.4)$	0	0	0	0
$q(\mathbf{4}.5.1)$	0	0	0	0
$q(\mathbf{4}.5.2)$	0	0	0	0
$q(\mathbf{4}.5.3)$	0	0	0	0

$$(T-t)^3\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le C(T-t)^5,$$

$$(T-t)^4\left(\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2\right)\le C(T-t)^5,$$

$$(T-t)^4\left(\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2\right)\le C(T-t)^5,$$

$$(T-t)^4\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le$$

$$\le C(T-t)^5.$$

TABLE 13. Scheme with strong order 2.5. Stochastic integrals $I_{(00)T,t}^{(i_1 i_2)}$, $I_{(000)T,t}^{(i_1 i_2 i_3)}$, $I_{(01)T,t}^{(i_1 i_2)}$, $I_{(10)T,t}^{(i_1 i_2)}$, $I_{(0000)T,t}^{(i_1 i_2 i_4)}$, $I_{(001)T,t}^{(i_1 i_2 i_3)}$, $I_{(010)T,t}^{(i_1 i_2 i_3)}$, $I_{(100)T,t}^{(i_1 i_2 i_3)}$, $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$. The condition (232).

$T-t$	2^{-1}	$2^{-3/2}$	2^{-2}	$2^{-5/2}$
$q(\mathbf{2}.1.a)$	2	8	32	128
$q(\mathbf{3}.1.a)$	$\boxed{1}$	$\boxed{3}$	$\boxed{8}$	$\boxed{23}$
$q(\mathbf{3}.3.1.a)$	0	1	4	11
$q(\mathbf{3}.3.2.a)$	0	1	4	11
$q(\mathbf{3}.3.3.a)$	0	3	8	23
$q(\mathbf{2}.1.b)$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$
$q(\mathbf{2}.2.b)$	0	0	0	0
$q(\mathbf{2}.1.c)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$
$q(\mathbf{2}.2.c)$	0	0	0	0
$q(\mathbf{4}.1)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$
$q(\mathbf{4}.3.1)$	0	0	0	1
$q(\mathbf{4}.3.2)$	0	0	0	1
$q(\mathbf{4}.3.3)$	0	0	0	2
$q(\mathbf{4}.3.4)$	0	0	0	1
$q(\mathbf{4}.3.5)$	0	0	0	1
$q(\mathbf{4}.3.6)$	0	0	0	1
$q(\mathbf{4}.4.1)$	0	0	0	0
$q(\mathbf{4}.4.2)$	0	0	0	0
$q(\mathbf{4}.4.3)$	0	0	0	0
$q(\mathbf{4}.4.4)$	0	0	0	0
$q(\mathbf{4}.5.1)$	0	0	0	1
$q(\mathbf{4}.5.2)$	0	0	0	1
$q(\mathbf{4}.5.3)$	0	0	0	1
$q(\mathbf{3}.1.b)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
$q(\mathbf{3}.2.b)$	0	0	0	0
$q(\mathbf{3}.3.1.b)$	0	0	0	0
$q(\mathbf{3}.3.2.b)$	0	0	0	0
$q(\mathbf{3}.3.3.b)$	0	0	0	0
$q(\mathbf{3}.1.c)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
$q(\mathbf{3}.2.c)$	0	0	0	0
$q(\mathbf{3}.3.1.c)$	0	0	0	0
$q(\mathbf{3}.3.2.c)$	0	0	0	0
$q(\mathbf{3}.3.3.c)$	0	0	0	0
$q(\mathbf{3}.1.d)$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
$q(\mathbf{3}.2.d)$	0	0	0	0
$q(\mathbf{3}.3.1.d)$	0	0	0	0
$q(\mathbf{3}.3.2.d)$	0	0	0	0
$q(\mathbf{3}.3.3.d)$	0	0	0	0

All numbers $q(\alpha)$ for $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$ are equal to zero

TABLE 14. Stochastic integral $I_{(000)T,t}^{(i_1 i_2 i_3)}$. The values $E_3^p/(T-t)^3 \stackrel{\sf def}{=} E$.

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q(\mathbf{3}.1.a)$	12	16	28	36	47	50
E	0.010154	0.007681	0.004433	0.003456	0.002652	0.002494
$q(\mathbf{3}.3.1.a)$	12	16	28	36	47	50
E	0.005077	0.003841	0.002216	0.001728	0.001326	0.001247
$q(\mathbf{3}.3.2.a)$	12	16	28	36	47	50
E	0.005077	0.003841	0.002216	0.001728	0.001326	0.001247
$q(\mathbf{3}.3.3.a)$	12	16	28	36	47	50
E	0.010308	0.007787	0.004480	0.003488	0.002673	0.002513

Strong Taylor–Ito scheme with convergence order 2.5 ([14])

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^6,$$

$$(T-t)^3\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^4\left(\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^4\left(\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^4\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^4\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{q_4}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^5\left(\frac{1}{10}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_5}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{001}\right)^2\right)\le C(T-t)^6,$$

TABLE 15. Stochastic integral $I^{(i_1 i_2 i_3 i_4)}_{(0000)T,t}$. The values $E^p_4/(T-t)^4 \stackrel{\sf def}{=} E$.

$T-t$	0.011	0.008	0.0045	0.0042
$q(\mathbf{4}.1)$	6	8	14	15
E	0.009636	0.007425	0.004378	0.004096
$q(\mathbf{4}.3.1)$	6	8	14	15
E	0.006771	0.005191	0.003041	0.002843
$q(\mathbf{4}.3.2)$	6	8	14	15
E	0.009722	0.007502	0.004424	0.004139
$q(\mathbf{4}.3.3)$	6	8	14	15
E	0.009641	0.007427	0.004379	0.004097
$q(\mathbf{4}.3.4)$	6	8	14	15
E	0.005997	0.004614	0.002720	0.002545
$q(\mathbf{4}.3.5)$	6	8	14	15
E	0.009722	0.007502	0.004424	0.004139
$q(\mathbf{4}.3.6)$	6	8	14	15
E	0.006771	0.005191	0.003041	0.002843
$q(\mathbf{4}.4.1)$	6	8	14	15
E	0.003095	0.002364	0.001379	0.001290
$q(\mathbf{4}.4.2)$	6	8	14	15
E	0.003095	0.002364	0.001379	0.001290
$q(\mathbf{4}.4.3)$	6	8	14	15
E	0.006885	0.005282	0.003090	0.002889
$q(\mathbf{4}.4.4)$	6	8	14	15
E	0.006885	0.005282	0.003090	0.002889
$q(\mathbf{4}.5.1)$	6	8	14	15
E	0.003690	0.002834	0.001663	0.001555
$q(\mathbf{4}.5.2)$	6	8	14	15
E	0.009756	0.007545	0.004457	0.004170
$q(\mathbf{4}.5.3)$	6	8	14	15
E	0.006010	0.004621	0.002722	0.002547

TABLE 16. Stochastic integrals $I^{(i_1 i_2)}_{(01)T,t}$, $I^{(i_1 i_2)}_{(10)T,t}$. The values $E^p_2/(T-t)^4 \stackrel{\sf def}{=} E$.

$T-t$	0.010	0.005	0.0025
$q(\mathbf{2}.1.b)$	4	8	16
E	0.008950	0.004660	0.002383
$q(\mathbf{2}.2.b)$	4	8	16
E	0.000042	0.000006	0.000001
$q(\mathbf{2}.1.c)$	4	8	16
E	0.008950	0.004660	0.002383
$q(\mathbf{2}.2.c)$	4	8	16
E	0.000042	0.000006	0.000001

TABLE 17. $T-t=0.011$. Stochastic integral $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$. The values $E_5^p/(T-t)^5 \stackrel{\mathsf{def}}{=} E$.

$q(\mathbf{5}.1)=0$ E=0.008264	$q(\mathbf{5}.3.1)=0$ E=0.008195	$q(\mathbf{5}.4.1)=0$ E=0.007917	$q(\mathbf{5}.5.1)=0$ E=0.006667	$q(\mathbf{5}.6.1)=0$ E=0.008056	$q(\mathbf{5}.7.1)=0$ E=0.007500
	$q(\mathbf{5}.3.2)=0$ E=0.008195	$q(\mathbf{5}.4.2)=0$ E=0.007917	$q(\mathbf{5}.5.2)=0$ E=0.006667	$q(\mathbf{5}.6.2)=0$ E=0.008056	$q(\mathbf{5}.7.2)=0$ E=0.007500
	$q(\mathbf{5}.3.3)=0$ E=0.008195	$q(\mathbf{5}.4.3)=0$ E=0.007917	$q(\mathbf{5}.5.3)=0$ E=0.006667	$q(\mathbf{5}.6.3)=0$ E=0.008056	$q(\mathbf{5}.7.3)=0$ E=0.007500
	$q(\mathbf{5}.3.4)=0$ E=0.008195	$q(\mathbf{5}.4.4)=0$ E=0.007917	$q(\mathbf{5}.5.4)=0$ E=0.006667	$q(\mathbf{5}.6.4)=0$ E=0.008056	$q(\mathbf{5}.7.4)=0$ E=0.007500
	$q(\mathbf{5}.3.5)=0$ E=0.008195	$q(\mathbf{5}.4.5)=0$ E=0.007917	$q(\mathbf{5}.5.5)=0$ E=0.006667	$q(\mathbf{5}.6.5)=0$ E=0.008056	$q(\mathbf{5}.7.5)=0$ E=0.007500
	$q(\mathbf{5}.3.6)=0$ E=0.008195	$q(\mathbf{5}.4.6)=0$ E=0.007917		$q(\mathbf{5}.6.6)=0$ E=0.008056	$q(\mathbf{5}.7.6)=0$ E=0.007500
	$q(\mathbf{5}.3.7)=0$ E=0.008195	$q(\mathbf{5}.4.7)=0$ E=0.007917		$q(\mathbf{5}.6.7)=0$ E=0.008056	$q(\mathbf{5}.7.7)=0$ E=0.007500
	$q(\mathbf{5}.3.8)=0$ E=0.008195	$q(\mathbf{5}.4.8)=0$ E=0.007917		$q(\mathbf{5}.6.8)=0$ E=0.008056	$q(\mathbf{5}.7.8)=0$ E=0.007500
	$q(\mathbf{5}.3.9)=0$ E=0.008195	$q(\mathbf{5}.4.9)=0$ E=0.007917		$q(\mathbf{5}.6.9)=0$ E=0.008056	$q(\mathbf{5}.7.9)=0$ E=0.007500
	$q(\mathbf{5}.3.10)=0$ E=0.008195	$q(\mathbf{5}.4.10)=0$ E=0.007917		$q(\mathbf{5}.6.10)=0$ E=0.008056	$q(\mathbf{5}.7.10)=0$ E=0.007500
				$q(\mathbf{5}.6.11)=0$ E=0.008056	
				$q(\mathbf{5}.6.12)=0$ E=0.008056	
				$q(\mathbf{5}.6.13)=0$ E=0.008056	
				$q(\mathbf{5}.6.14)=0$ E=0.008056	
				$q(\mathbf{5}.6.15)=0$ E=0.008056	

TABLE 18. $T-t=0.008$. Stochastic integral $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$. The values $E_5^p/(T-t)^5 \stackrel{\sf def}{=} E$.

$q(\mathbf{5}.1)=1$ E=0.007590	$q(\mathbf{5}.3.1)=1$ E=0.007570	$q(\mathbf{5}.4.1)=1$ E=0.005488	$q(\mathbf{5}.5.1)=1$ E=0.003272	$q(\mathbf{5}.6.1)=1$ E=0.006052	$q(\mathbf{5}.7.1)=1$ E=0.004175
	$q(\mathbf{5}.3.2)=1$ E=0.007300	$q(\mathbf{5}.4.2)=1$ E=0.006701	$q(\mathbf{5}.5.2)=1$ E=0.005292	$q(\mathbf{5}.6.2)=1$ E=0.007058	$q(\mathbf{5}.7.2)=1$ E=0.006105
	$q(\mathbf{5}.3.3)=1$ E=0.007558	$q(\mathbf{5}.4.3)=1$ E=0.006976	$q(\mathbf{5}.5.3)=1$ E=0.005774	$q(\mathbf{5}.6.3)=1$ E=0.007014	$q(\mathbf{5}.7.3)=1$ E=0.006072
	$q(\mathbf{5}.3.4)=1$ E=0.007570	$q(\mathbf{5}.4.4)=1$ E=0.005995	$q(\mathbf{5}.5.4)=1$ E=0.005292	$q(\mathbf{5}.6.4)=1$ E=0.006467	$q(\mathbf{5}.7.4)=1$ E=0.005955
	$q(\mathbf{5}.3.5)=1$ E=0.007084	$q(\mathbf{5}.4.5)=1$ E=0.006679	$q(\mathbf{5}.5.5)=1$ E=0.003272	$q(\mathbf{5}.6.5)=1$ E=0.007054	$q(\mathbf{5}.7.5)=1$ E=0.006576
	$q(\mathbf{5}.3.6)=1$ E=0.007432	$q(\mathbf{5}.4.6)=1$ E=0.006701		$q(\mathbf{5}.6.6)=1$ E=0.007260	$q(\mathbf{5}.7.6)=1$ E=0.006105
	$q(\mathbf{5}.3.7)=1$ E=0.007558	$q(\mathbf{5}.4.7)=1$ E=0.005488		$q(\mathbf{5}.6.7)=1$ E=0.007521	$q(\mathbf{5}.7.7)=1$ E=0.003236
	$q(\mathbf{5}.3.8)=1$ E=0.007084	$q(\mathbf{5}.4.8)=1$ E=0.007134		$q(\mathbf{5}.6.8)=1$ E=0.005819	$q(\mathbf{5}.7.8)=1$ E=0.006797
	$q(\mathbf{5}.3.9)=1$ E=0.007300	$q(\mathbf{5}.4.9)=1$ E=0.006679		$q(\mathbf{5}.6.9)=1$ E=0.007412	$q(\mathbf{5}.7.9)=1$ E=0.006576
	$q(\mathbf{5}.3.10)=1$ E=0.006962	$q(\mathbf{5}.4.10)=1$ E=0.006976		$q(\mathbf{5}.6.10)=1$ E=0.007260	$q(\mathbf{5}.7.10)=1$ E=0.006072
				$q(\mathbf{5}.6.11)=1$ E=0.006467	
				$q(\mathbf{5}.6.12)=1$ E=0.007054	
				$q(\mathbf{5}.6.13)=1$ E=0.006052	
				$q(\mathbf{5}.6.14)=1$ E=0.007058	
				$q(\mathbf{5}.6.15)=1$ E=0.007014	

TABLE 19. $T-t=0.0045$. Stochastic integral $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$. The values $E_5^p/(T-t)^5 \stackrel{\text{def}}{=} E$.

$q(\mathbf{5}.1)=4$ E=0.004209	$q(\mathbf{5}.3.1)=4$ E=0.004208	$q(\mathbf{5}.4.1)=4$ E=0.002351	$q(\mathbf{5}.5.1)=4$ E=0.001055	$q(\mathbf{5}.6.1)=4$ E=0.002247	$q(\mathbf{5}.7.1)=4$ E=0.002065
	$q(\mathbf{5}.3.2)=4$ E=0.004204	$q(\mathbf{5}.4.2)=4$ E=0.003461	$q(\mathbf{5}.5.2)=4$ E=0.002379	$q(\mathbf{5}.6.2)=4$ E=0.004149	$q(\mathbf{5}.7.2)=4$ E=0.003428
	$q(\mathbf{5}.3.3)=4$ E=0.004212	$q(\mathbf{5}.4.3)=4$ E=0.003460	$q(\mathbf{5}.5.3)=4$ E=0.002624	$q(\mathbf{5}.6.3)=4$ E=0.003168	$q(\mathbf{5}.7.3)=4$ E=0.002256
	$q(\mathbf{5}.3.4)=4$ E=0.004208	$q(\mathbf{5}.4.4)=4$ E=0.001982	$q(\mathbf{5}.5.4)=4$ E=0.002379	$q(\mathbf{5}.6.4)=4$ E=0.003451	$q(\mathbf{5}.7.4)=4$ E=0.001982
	$q(\mathbf{5}.3.5)=4$ E=0.003161	$q(\mathbf{5}.4.5)=4$ E=0.003189	$q(\mathbf{5}.5.5)=4$ E=0.001055	$q(\mathbf{5}.6.5)=4$ E=0.003160	$q(\mathbf{5}.7.5)=4$ E=0.003191
	$q(\mathbf{5}.3.6)=4$ E=0.004180	$q(\mathbf{5}.4.6)=4$ E=0.003461		$q(\mathbf{5}.6.6)=4$ E=0.004206	$q(\mathbf{5}.7.6)=4$ E=0.003428
	$q(\mathbf{5}.3.7)=4$ E=0.004212	$q(\mathbf{5}.4.7)=4$ E=0.002351		$q(\mathbf{5}.6.7)=4$ E=0.004214	$q(\mathbf{5}.7.7)=4$ E=0.001318
	$q(\mathbf{5}.3.8)=4$ E=0.003161	$q(\mathbf{5}.4.8)=4$ E=0.004201		$q(\mathbf{5}.6.8)=4$ E=0.002590	$q(\mathbf{5}.7.8)=4$ E=0.004124
	$q(\mathbf{5}.3.9)=4$ E=0.004204	$q(\mathbf{5}.4.9)=4$ E=0.003189		$q(\mathbf{5}.6.9)=4$ E=0.004180	$q(\mathbf{5}.7.9)=4$ E=0.003191
	$q(\mathbf{5}.3.10)=4$ E=0.003456	$q(\mathbf{5}.4.10)=4$ E=0.003460		$q(\mathbf{5}.6.10)=4$ E=0.004206	$q(\mathbf{5}.7.10)=4$ E=0.002256
				$q(\mathbf{5}.6.11)=4$ E=0.003451	
				$q(\mathbf{5}.6.12)=4$ E=0.003160	
				$q(\mathbf{5}.6.13)=4$ E=0.002247	
				$q(\mathbf{5}.6.14)=4$ E=0.00414	
				$q(\mathbf{5}.6.15)=4$ E=0.003168	

TABLE 20. $T-t=0.0042$. Stochastic integral $I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}$. The values $E_5^p/(T-t)^5 \stackrel{\sf def}{=} E$.

$q(\mathbf{5}.1)=5$ E=0.003557	$q(\mathbf{5}.3.1)=5$ E=0.003556	$q(\mathbf{5}.4.1)=5$ E=0.001940	$q(\mathbf{5}.5.1)=5$ E=0.000863	$q(\mathbf{5}.6.1)=5$ E=0.001863	$q(\mathbf{5}.7.1)=5$ E=0.001728
	$q(\mathbf{5}.3.2)=5$ E=0.003564	$q(\mathbf{5}.4.2)=5$ E=0.002910	$q(\mathbf{5}.5.2)=5$ E=0.001969	$q(\mathbf{5}.6.2)=5$ E=0.003539	$q(\mathbf{5}.7.2)=5$ E=0.002897
	$q(\mathbf{5}.3.3)=5$ E=0.003559	$q(\mathbf{5}.4.3)=5$ E=0.002897	$q(\mathbf{5}.5.3)=5$ E=0.002188	$q(\mathbf{5}.6.3)=5$ E=0.002639	$q(\mathbf{5}.7.3)=5$ E=0.001869
	$q(\mathbf{5}.3.4)=5$ E=0.003556	$q(\mathbf{5}.4.4)=5$ E=0.001642	$q(\mathbf{5}.5.4)=5$ E=0.001969	$q(\mathbf{5}.6.4)=5$ E=0.002903	$q(\mathbf{5}.7.4)=5$ E=0.001641
	$q(\mathbf{5}.3.5)=5$ E=0.002634	$q(\mathbf{5}.4.5)=5$ E=0.002661	$q(\mathbf{5}.5.5)=5$ E=0.000863	$q(\mathbf{5}.6.5)=5$ E=0.002634	$q(\mathbf{5}.7.5)=5$ E=0.002664
	$q(\mathbf{5}.3.6)=5$ E=0.003552	$q(\mathbf{5}.4.6)=5$ E=0.002910		$q(\mathbf{5}.6.6)=5$ E=0.003566	$q(\mathbf{5}.7.6)=5$ E=0.002897
	$q(\mathbf{5}.3.7)=5$ E=0.003559	$q(\mathbf{5}.4.7)=5$ E=0.001940		$q(\mathbf{5}.6.7)=5$ E=0.003561	$q(\mathbf{5}.7.7)=5$ E=0.001090
	$q(\mathbf{5}.3.8)=5$ E=0.002634	$q(\mathbf{5}.4.8)=5$ E=0.003572		$q(\mathbf{5}.6.8)=5$ E=0.002155	$q(\mathbf{5}.7.8)=5$ E=0.003531
	$q(\mathbf{5}.3.9)=5$ E=0.003564	$q(\mathbf{5}.4.9)=5$ E=0.002661		$q(\mathbf{5}.6.9)=5$ E=0.003552	$q(\mathbf{5}.7.9)=5$ E=0.002664
	$q(\mathbf{5}.3.10)=5$ E=0.002894	$q(\mathbf{5}.4.10)=5$ E=0.002897		$q(\mathbf{5}.6.10)=5$ E=0.003566	$q(\mathbf{5}.7.10)=5$ E=0.001869
				$q(\mathbf{5}.6.11)=5$ E=0.002903	
				$q(\mathbf{5}.6.12)=5$ E=0.002634	
				$q(\mathbf{5}.6.13)=5$ E=0.001863	
				$q(\mathbf{5}.6.14)=5$ E=0.003539	
				$q(\mathbf{5}.6.15)=5$ E=0.002639	

TABLE 21. $T-t=0.0035$. Stochastic integral $I^{(i_1 i_2 i_3 i_4 i_5)}_{(00000)T,t}$. The values $E^p_5/(T-t)^5 \stackrel{\sf def}{=} E$.

$q(\mathbf{5}.1)=6$ E=0.003071	$q(\mathbf{5}.3.1)=6$ E=0.003071	$q(\mathbf{5}.4.1)=6$ E=0.001650	$q(\mathbf{5}.5.1)=6$ E=0.000729	$q(\mathbf{5}.6.1)=6$ E=0.001591	$q(\mathbf{5}.7.1)=6$ E=0.001591
	$q(\mathbf{5}.3.2)=6$ E=0.003083	$q(\mathbf{5}.4.2)=6$ E=0.002503	$q(\mathbf{5}.5.2)=6$ E=0.001676	$q(\mathbf{5}.6.2)=6$ E=0.003074	$q(\mathbf{5}.7.2)=6$ E=0.002500
	$q(\mathbf{5}.3.3)=6$ E=0.003073	$q(\mathbf{5}.4.3)=6$ E=0.002486	$q(\mathbf{5}.5.3)=6$ E=0.001872	$q(\mathbf{5}.6.3)=6$ E=0.002260	$q(\mathbf{5}.7.3)=6$ E=0.001596
	$q(\mathbf{5}.3.4)=6$ E=0.003071	$q(\mathbf{5}.4.4)=6$ E=0.001399	$q(\mathbf{5}.5.4)=6$ E=0.001676	$q(\mathbf{5}.6.4)=6$ E=0.002497	$q(\mathbf{5}.7.4)=6$ E=0.001399
	$q(\mathbf{5}.3.5)=6$ E=0.002256	$q(\mathbf{5}.4.5)=6$ E=0.002281	$q(\mathbf{5}.5.5)=6$ E=0.000729	$q(\mathbf{5}.6.5)=6$ E=0.002256	$q(\mathbf{5}.7.5)=6$ E=0.002284
	$q(\mathbf{5}.3.6)=6$ E=0.003077	$q(\mathbf{5}.4.6)=6$ E=0.002503		$q(\mathbf{5}.6.6)=6$ E=0.003085	$q(\mathbf{5}.7.6)=6$ E=0.002500
	$q(\mathbf{5}.3.7)=6$ E=0.003073	$q(\mathbf{5}.4.7)=6$ E=0.001650		$q(\mathbf{5}.6.7)=6$ E=0.003074	$q(\mathbf{5}.7.7)=6$ E=0.000928
	$q(\mathbf{5}.3.8)=6$ E=0.002256	$q(\mathbf{5}.4.8)=6$ E=0.003096		$q(\mathbf{5}.6.8)=6$ E=0.001841	$q(\mathbf{5}.7.8)=6$ E=0.003074
	$q(\mathbf{5}.3.9)=6$ E=0.003083	$q(\mathbf{5}.4.9)=6$ E=0.002281		$q(\mathbf{5}.6.9)=6$ E=0.003077	$q(\mathbf{5}.7.9)=6$ E=0.002284
	$q(\mathbf{5}.3.10)=6$ E=0.002484	$q(\mathbf{5}.4.10)=6$ E=0.002486		$q(\mathbf{5}.6.10)=6$ E=0.003085	$q(\mathbf{5}.7.10)=6$ E=0.001596
				$q(\mathbf{5}.6.11)=6$ E=0.002497	
				$q(\mathbf{5}.6.12)=6$ E=0.002256	
				$q(\mathbf{5}.6.13)=6$ E=0.001591	
				$q(\mathbf{5}.6.14)=6$ E=0.003074	
				$q(\mathbf{5}.6.15)=6$ E=0.002260	

Table 22. $T-t=0.01$. The values $E_3^p/(T-t)^5 \stackrel{\sf def}{=} E$.

$I_{(001)T,t}^{(i_1i_2i_3)}$	$I_{(010)T,t}^{(i_1i_2i_3)}$	$I_{(100)T,t}^{(i_1i_2i_3)}$
$q(\mathbf{3}.1.b)=6$ E=0.009425	$q(\mathbf{3}.1.c)=4$ E=0.009051	$q(\mathbf{3}.1.d)=2$ E=0.008154
$q(\mathbf{3}.2.b)=0$ E=0.000007	$q(\mathbf{3}.2.c)=4$ E=0.000049	$q(\mathbf{3}.2.d)=2$ E=0.000147
$q(\mathbf{3}.3.1.b)=6$ E=0.004361	$q(\mathbf{3}.3.1.c)=4$ E=0.006366	$q(\mathbf{3}.3.1.d)=2$ E=0.004142
$q(\mathbf{3}.3.2.b)=6$ E=0.005044	$q(\mathbf{3}.3.2.c)=4$ E=0.002731	$q(\mathbf{3}.3.2.d)=2$ E=0.004778
$q(\mathbf{3}.3.3.b)=6$ E=0.009557	$q(\mathbf{3}.3.3.c)=4$ E=0.009152	$q(\mathbf{3}.3.3.d)=2$ E=0.007963

Table 23. Comparison of the conditions (241), (242).

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
p	0	0	1	2	4	8
$(p+1)^3$	1	1	8	27	125	729
p'	1	3	6	12	24	48
$(p'+1)^3$	8	64	343	2197	15625	117649

$$(T-t)^5\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_6}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2\right)\le C(T-t)^6,$$

$$(T-t)^5\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_7}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2\right)\le C(T-t)^6.$$

Note that in this paper we use the database with 270,000 exactly calculated Fourier–Legendre coefficients (see [46], [47]) for detail).

Let us consider the minimal natural numbers p,p',q,q',r,r' satisfying the conditions

$$(T-t)^3\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le (T-t)^4, \tag{241}$$

TABLE 24. Comparison of the conditions (243), (244).

$T-t$	2^{-1}	$2^{-3/2}$	2^{-2}	$2^{-5/2}$	2^{-3}	$2^{-7/2}$
q	0	0	0	0	0	0
$(q+1)^4$	1	1	1	1	1	1
q'	3	4	6	9	12	17
$(q'+1)^4$	256	625	2401	10000	28561	104976

TABLE 25. Comparison of the conditions (245), (246).

$T-t$	$2^{-1/8}$	$2^{-1/4}$	$2^{-1/2}$	$2^{-3/4}$	2^{-1}
r	0	0	0	0	0
$(r+1)^5$	1	1	1	1	1
r'	1	2	3	4	5
$(r'+1)^5$	32	243	1024	3125	7776

$$6(T-t)^3\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p'}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le (T-t)^4, \tag{242}$$

$$(T-t)^4\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le (T-t)^5, \tag{243}$$

$$24(T-t)^4\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q'}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le (T-t)^5, \tag{244}$$

$$(T-t)^4\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{r}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2\right)\le$$

$$\le (T-t)^6, \tag{245}$$

$$120(T-t)^4\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{r'}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2\right)\le$$

$$\le (T-t)^6, \tag{246}$$

where the inequalities (242), (244), (246) are particular cases of the formula (43) for $r = 3, 4$, and 5.

In Tables 23–25, we can see the numerical comparison of the conditions (241), (243), (245) with the conditions (242), (244), (246), respectively. Obviously, the conditions (241), (243), (245) (i.e. conditions without the multiplier factors 3!, 4!, and 5!) essentially reduce the calculation costs for the mean-square approximations of iterated Ito stochastic integrals

$$I_{(000)T,t}^{(i_1 i_2 i_3)}, \quad I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}, \quad I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)} \quad (i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m).$$

References

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982. 612 pp.

[2] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[5] Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html Hard Cover Edition: 1998, SPbGTU Publ., 204 pp. (ISBN 5-7422-0045-5)

[6] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[7] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[8] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[9] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, 768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, 786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[11] Kuznetsov D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. [In English]. arXiv:2001.10192 [math.PR], 2020, 90 pp.

[12] Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [In English]. arXiv:2003.14184 [math.PR], 2022, 912 pp.

[13] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[14] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[15] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations

and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[16] Kuznetsov D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 32, Issue 12, (2000), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[17] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[18] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[19] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[20] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[21] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[22] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[23] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/z17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[25] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[26] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[27] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 203 pp.

[28] Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR]. 2017, 106 pp.

[29] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR]. 2018, 77 pp.

[30] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[31] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR]. 2022, 126 pp.

[32] Kuznetsov D.F. Exact calculation of mean-square error in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 68 pp.

[33] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604

[34] Kuznetsov D.F. On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[35] Kuznetsov D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[36] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02345 [math.GM], 2019, 40 pp.

[37] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[38] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[39] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[40] Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02612 [math.PR], 2019, 32 pp.

[41] Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[42] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev A.N., Samouylov K.E., Kozyrev D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[43] Kuznetsov D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[44] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [in English]. arXiv:1801.03195 [math.PR]. 2022, 138 pp.

[45] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117.
Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[46] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[47] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[48] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications 10, 4 (1992), 431-441.

[49] Averina T.A., Prigarin S.M. Calculation of stochastic integrals of Wiener processes. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[50] Prigarin S.M., Belov S.M. One application of series expansions of Wiener process. [In Russian]. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[51] Wiktorsson M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487.

[52] Ryden T., Wiktorsson M. On the simulation of iterated Ito integrals. Stochastic Processes and their Applications, 91, 1 (2001), 151-168.
[53] Gaines J. G., Lyons, T. J. Random generation of stochastic area integrals. SIAM J. Appl. Math. 54 (1994), 1132-1146.
[54] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.
[55] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010, 868 pp.
[56] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.
[57] Tang X., Xiao A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics. 45 (2019), 813-846.
[58] Zahri M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science. 8, 2 (2014), 186-198.
[59] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.
[60] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.
[61] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[62] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html
[63] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

Mikhail Dmitrievich Kuznetsov
Saint-Petersburg Electrotechnical University,
ul. Professora Popova, 5,
197376, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru

Implementation of Strong Numerical Methods of Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs with Multidimensional Non-Commutative Noise Based on the Unified Taylor–Itô and Taylor–Stratonovich Expansions and Multiple Fourier–Legendre Series

Mikhail D. Kuznetsov[1] and Dmitriy F. Kuznetsov[2]

[1]Faculty of Computer Technologies and Informatics, St. Petersburg Electrotechnical University, Saint-Petersburg, Russia
[2]Institute of Applied Mathematics and Mechanics, Peter the Great St. Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to the implementation of strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô stochastic differential equations with multidimensional non-commutative noise based on multiple Fourier–Legendre series and unified Taylor–Itô and Taylor–Stratonovich expansions. Algorithms for the implementation of these methods

are constructed and a package of programs in the Python programming language is presented. An important part of this software package concerning the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 with respect to components of the multidimensional Wiener process is based on the method of generalized multiple Fourier series. More precisely, we used multiple Fourier–Legendre series converging in the sense of norm in Hilbert space for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals.
Key words: Software package, Python programming language, numerical method, strong convergence, Itô stochastic differential equation, multidimensional Wiener process, non-commutative noise, unified Taylor–Itô expansion, unified Taylor–Stratonovich expansion, Milstein scheme, high-order strong numerical scheme, iterated Itô stochastic integral, iterated Stratonovich stochastic integral, mean-square approximation, generalized multiple Fourier series, multiple Fourier–Legendre series, Legendre polynomial.

Contents

1 Introduction

As known, Itô stochastic differential equations (SDEs) have appeared in the theory of random processes relatively recently [1] (1951). Nevertheless, to date, a large number of mathematical models for dynamical systems of different physical nature under the influence of random perturbations have been built on the basis of such equations [2]-[16]. Among them we note mathematical models in stochastic financial mathematics [5]-[7], [10]-[12], geophysics [2], [4], genetics [13], hydrology [2], epidemiology [9], chemical kinetics [2], [9], biology [8], [15], seismology [2], electrodynamics [16] and many other fields [2], [9], [14]. In addition, Itô SDEs arise when solving a number of mathematical problems, such as filtration [2], [3], [17]-[21], stochastic control [2], [17], stochastic stability [2], parameter estimation of stochastic systems [2], [3], [22].

Exact solutions of Itô SDEs are known in rare cases. For this reason, it becomes necessary to construct numerical methods for Itô SDEs. Moreover, the problem of numerical solution of Itô SDEs often occurs even in cases when the exact solution of Itô SDE is known. This means that in some cases, knowing the exact solution of the Itô SDE does not allow us to simulate it numerically in a simple way.

This article is devoted to the implementation of high-order strong numerical methods for systems of Itô SDEs with multidimensional non-commutative noise. More precisely, we consider strong numerical methods with convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. The article also considers the Euler method, which under suitable conditions [2] has the order 0.5 of strong convergence. To construct the mentioned numerical methods in this article, we use the so-called unified Taylor–Itô and Taylor–Stratonovich expansions [24]-[27], [68]. The important components of these expansions are the iterated Itô and Stratonovich stochastic integrals, which are functionals of a complex structure with respect to the components of a multidimensional Wiener process.

It should be noted that it is impossible to construct a numerical method for Itô SDE in a general case (multidimensional non-commutative noise) that includes only increments of the multidimensional Wiener processes, but has a higher order of convergence (in the mean-square sense) than the Euler method (simplest numerical method for Itô SDEs). This result is known as the "Clark–Cameron paradox" [23] (1980) and well explains the need to use high-order numerical methods for Itô SDEs, since the accuracy of the Euler method is insufficient for solving a number of practical problems related to Itô SDEs [2].

According to the "Clark–Cameron paradox" [23], avoidance of the problem of mean-square approximation of the mentioned iterated stochastic integrals is impossible in the general case when constructing high-order strong numerical methods for Itô SDEs.

The problem of mean-square approximation of iterated Itô and Stratonovich stochastic integrals in the context of the numerical integration of Itô SDEs was considered in a number of works [2], [3], [7], [8], [28]-[40].

It should be explained why the results of these works are insufficient for constructing effective procedures for the implementation of strong numerical methods of order 1.5 and higher for Itô SDEs.

There exists an approach to the mean-square approximation of iterated stochastic integrals based on integral sums [28], [35], [36]. Note that one of the variants of this method is based on reducing the problem of mean-square approximation of iterated stochastic integrals to the numerical integration of systems of linear Itô SDEs by the Euler method [40]. However, this approach [28], [35], [36], [40] implies the partitioning of the interval of integration for iterated stochastic integrals. It should be noted that the length of this interval is an integration step for numerical methods for Itô SDEs, which is already a fairly small value even without additional partitioning. Computational experiments show that the numerical modeling of iterated stochastic integrals by the method of integral sums [28], [35], [36], [40] leads to unacceptably high computational cost and accumulation of computation errors [43].

More efficient approach of the mean-square approximation of iterated stochastic integrals is based on the expansion of the so-called Brownian bridge process into the trigonometric Fourier series with random terms (version of the so-called Karhunen–Loève expansion) [2], [3], [7], [28], [29], [34], [35], [38], [39]. However, in [28], [34], [35], [39], this approach was used to approximate only iterated stochastic integrals of multiplicities 1 and 2, which makes it possible to implement numerical method with order 1.0 of strong convergence for Itô SDEs (Milstein method [28]). In papers [2], [3], [7], [29], the approximation of iterated stochastic integrals of multiplicities 1 to 3 was considered by the above approach, which makes it possible to implement numerical methods with orders 1.0 and 1.5 of strong convergence for Itô SDEs. However, formulas concerning integrals of multiplicity 3 turned out to be too complicated and did not find wide application in practice. Moreover, these formulas (for iterated stochastic integrals of multiplicity 3) were obtained without strict theoretical justification and exclude the possibility of effective estimation of the mean-square error of approximation (see discussion in [26] (Sect. 2.6.2, 6.2) for details).

It should be noted that in papers [30], [31], a similar approach was used to approximate iterated stochastic integrals of multiplicities 1 to 3 based on the series expansion of the Wiener process using trigonometric functions and Haar functions. In [41] orthonormal expansions of functions in terms of Walsh series were used to represent the iterated stochastic integrals.

Note that the iterated stochastic integrals under consideration are the random variables with unknown density functions. The only exception is the iterated Itô stochastic integral with multiplicity 2 [32]. However, the knowledge of density function of the mentioned stochastic integral gives no simple way of its approximation [32].

In this work, we use a more efficient method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals than the methods considered above. This method (the so-called method of generalized multiple Fourier series) is based on the theory constructed in Chapters 1, 2, and 5 of monographs [26], [27], [68] (also see bibliography therein). The method of generalized multiple Fourier series made it possible in this work to successfully implement the procedures for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6. In this case, we use multiple Fourier–Legendre series, that is, we have chosen Legendre polynomials as a basis system of functions for approximating iterated stochastic integrals. It is important to note that the Legendre polynomials were first applied in the context of this problem in [44] (1997), while in the works of other authors Legendre polynomials were not considered as a system of basis functions for approximating iterated stochastic integrals (an exception is work [37]). As shown in [45], the Legendre polynomials are an optimal system of basis functions for approximating iterated Itô and Stratonovich stochastic integrals.

In this article, to build the SDE-MATH software package in the Python programming language, we use a database with 270,000 exactly calculated Fourier–Legendre coefficients to approximate iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6. It should be noted that the procedures for the mean-square approximation of iterated stochastic integrals of multiplicities 4, 5, and 6 constructed in this work have no analogues in the literature. At the same time, we propose a much more convenient procedure for the mean-square approximation of iterated stochastic integrals of multiplicity 3 than in works [2], [3], [7], [29]. This procedure provides an accurate calculation of the mean-square error of approximation of the mentioned stochastic integrals.

Another important feature of the presented software package is the use of unified Taylor–Itô and Taylor–Stratonovich expansions [24]-[27], [68] for constructing strong numerical methods with convergence orders 1.5, 2.0, 2.5, and 3.0 for Itô SDEs. Unified Taylor–Itô and Taylor—Stratonovich expansions make it possible (in contrast with its classical analogues [2]) to use the minimal sets of iterated Itô and Stratonovich stochastic integrals. This property well explains the motive for using the mentioned unified expansions.

The results of this work on the approximation of iterated stochastic integrals can be used to numerically solve various types of SDEs. For example, for semilinear SPDEs with multiplicative trace class noise [26], [27], [68] (Chapter 7), [46], [47]. This is due to the fact that iterated stochastic integrals are a universal tool for constructing high-order strong numerical methods for various types of SDEs. In recent years, the mentioned numerical methods have been constructed for SDEs with jumps [7], SPDEs with multiplicative trace class noise [48]-[50], McKean SDEs [51], SDEs with switchings [52], mean-field SDEs [53], Itô–Volterra stochastic integral equations [50], etc.

There are many publications in which codes of programs in various programming languages are given for the numerical solution of SDEs [3], [9], [14], [54]-[62]. Among them, we note the software described in [3], [55], [57], [61]. Some of the mentioned works [3], [55], [57], [58], [61] are based on the results of monograph [2] on the approximation of iterated stochastic integrals (see above discusson on the disadvantages of approach [2]). Other publications [9], [14], [54], [56] do not use the modeling of iterated stochastic integrals for the case of multidimensional non-commutative noise at all.

In this article, we develop software for the numerical integration of Itô SDEs based on theoretical results and MATLAB codes from monographs [59], [62] for modeling iterated stochastic integrals of multiplicities 1 to 6 (the case of multidimensional non-commutative noise). In addition, we provide software (as a part of the SDE-MATH software package) for the numerical integration of linear stationary systems of Itô SDEs based on the results of article [63] and MATLAB codes from monographs [59], [62].

In Sect. 7 we discuss possible directions for the development of the SDE-MATH software package. In particular, the parallelization of computations, the implementation of methods of the Runge–Kutta type [2], [7], [43], [62] and multistep numerical methods for Itô SDEs [2], [7], [43], [62], the development of a part of the software package for solving filtering problem and stochastic optimal control problem [2], as well as improvement of the graphical user interface.

2 Theoretical Results Underlying the SDE-MATH Software Package

2.1 Strong Numerical Methods with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Itô Expansion

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space and let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F. Let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process with independent components $\mathbf{w}_t^{(i)}$ $(i=1,\ldots,m)$, which is F_t-measurable for any $t\in[0,T]$. Consider an Itô SDE in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\sum_{i=1}^m\int\limits_0^t B_i(\mathbf{x}_\tau,\tau)d\mathbf{w}_\tau^{(i)},\quad \mathbf{x}_0=\mathbf{x}(0,\omega), \tag{1}$$

where $\mathbf{x}_t\in\mathbb{R}^n$ is a solution of the Itô SDE (1), the nonrandom functions $\mathbf{a}(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $B(\mathbf{x},t):\mathbb{R}^n\times[0,T]\to\mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [64], the second integral on the right-hand side of (1) is interpreted as an Itô stochastic integral, $B_i(\mathbf{x},t)$ is the ith colomn of the matrix function $B(\mathbf{x},t)$, $\mathbf{x}_0$ is an n-dimensional and F_0-measurable random variable, $\mathsf{M}\bigl\{|\mathbf{x}_0|^2\bigr\}<\infty$ (M is an expectation operator). We assume that $\mathbf{x}_0$ and $\mathbf{w}_t-\mathbf{w}_0$ are independent when $t>0$.

It is well known that one of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions [2], [7], [43]. The essential feature of such expansions is the so-called iterated Itô and Stratonovich stochastic integrals, which have the form

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a nonrandom function on $[t,T]$, $\mathbf{w}_\tau^{(i)}$ $(i=$

$1,\ldots,m$) are independent standard Wiener processes and $\mathbf{w}_{\tau}^{(0)} \stackrel{\sf def}{=} \tau$,

$$\int \ \text{and} \ \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively; $i_1,\ldots,i_k = 0,1,\ldots,m$ (see definitions in [2]).

Note that $\psi_l(\tau) \equiv 1$ $(l=1,\ldots,k)$ and $i_1,\ldots,i_k = 0,1,\ldots,m$ in the classical Taylor–Itô and Taylor–Stratonovich expansions [2]. At the same time $\psi_l(\tau) \equiv (t-\tau)^{q_l}$ $(l=1,\ldots,k;\ q_1,\ldots,q_k = 0,1,2,\ldots)$ and $i_1,\ldots,i_k = 1,\ldots,m$ in the unified Taylor–Itô and Taylor–Stratonovich expansions [24], [25] (also see [26], [27], [68], Chapter 4).

Let $C^{2,1}(\mathbb{R}^n \times [0,T])$ be the space of functions $R(\mathbf{x},t): \mathbb{R}^n \times [0,T] \rightarrow \mathbb{R}^1$ with the following property: these functions are twice continuously differentiable in $\mathbf{x}$ and have one continuous derivative in t. Let us consider the following differential operators on the space $C^{2,1}(\mathbb{R}^n \times [0,T])$

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathbf{a}^{(i)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(i)}} + \frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n} B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}}, \qquad (4)$$

$$G_0^{(i)} = \sum_{j=1}^{n} B^{(ji)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(j)}}, \quad i=1,\ldots,m, \qquad (5)$$

where $\mathbf{a}^{(i)}(\mathbf{x},t)$ is the ith component of the vector function $\mathbf{a}(\mathbf{x},t)$ and $B^{(ij)}(\mathbf{x},t)$ is the ijth component of the matrix function $B(\mathbf{x},t)$.

Consider the following sequence of differential operators

$$G_p^{(i)} = \frac{1}{p}\left(G_{p-1}^{(i)}L - LG_{p-1}^{(i)}\right), \quad p=1,2,\ldots, \quad i=1,\ldots,m,$$

where L and $G_0^{(i)}$, $i=1,\ldots,m$ are defined by (4), (5).

For the further consideration, we need to introduce the following set of iterated Itô stochastic integrals

$$I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)} = \int\limits_t^s (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (6)$$

where $l_1,\ldots,l_k = 0,1,\ldots$ and $i_1,\ldots,i_k = 1,\ldots,m$.

Assume that $R(\mathbf{x},t)$, $\mathbf{a}(\mathbf{x},t)$, and $B_i(\mathbf{x},t)$, $i=1,\ldots,m$ are enough smooth functions with respect to the variables $\mathbf{x}$ and t. Then for all $s,t\in[0,T]$ such that $s>t$ we can write the following unified Taylor–Itô expansion [24] (also see [26], [27], [68], Chapter 4)

$$R(\mathbf{x}_s,s)=$$

$$=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(s-t)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)\ I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)}+$$

$$+\left(H_{r+1}\right)_{s,t}\quad \text{w. p. 1}, \tag{7}$$

where

$$L^jR(\mathbf{x},t)\stackrel{\mathsf{def}}{=}\begin{cases}\underbrace{L\ldots L}_{j}R(\mathbf{x},t) & \text{for } j\ge 1\\ \\ R(\mathbf{x},t) & \text{for } j=0\end{cases},$$

$$\mathrm{D}_q=\left\{(k,j,l_1,\ldots,l_k):\ k+2\left(j+\sum_{p=1}^{k}l_p\right)=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\right\}, \tag{8}$$

and $\left(H_{r+1}\right)_{s,t}$ is the remainder term in integral form [26], [27], [68].

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of the interval $[0,T]$ such that

$$0=\tau_0<\tau_1<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\left|\tau_{j+1}-\tau_j\right|. \tag{9}$$

Let $\mathbf{y}_{\tau_j}\stackrel{\mathsf{def}}{=}\mathbf{y}_j$, $j=0,1,\ldots,N$ be a time discrete approximation of the process $\mathbf{x}_t$, $t\in[0,T]$, which is a solution of the Itô SDE (1).

Definiton 1 [2]. *We will say that a time discrete approximation* $\mathbf{y}_j$, $j=0,1,\ldots,N$, *corresponding to the maximal step of discretization* Δ_N, *converges strongly with order* $\gamma>0$ *at time moment* T *to the process* $\mathbf{x}_t$, $t\in[0,T]$, *if there exists a constant* $C>0$, *which does not depend on* Δ_N, *and a* $\delta>0$ *such that* $\mathsf{M}\{|\mathbf{x}_T-\mathbf{y}_T|\}\le C(\Delta_N)^{\gamma}$ *for each* $\Delta_N\in(0,\delta)$.

From (7) for $s=\tau_{p+1}$ and $t=\tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (1)

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{D}_q}\frac{(\tau_{p+1}-\tau_p)^j}{j!}\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^j\mathbf{y}_p\ \hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}+$$

$$+\mathbf{1}_{\{r=2d-1,d\in\mathbb{N}\}}\frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!}L^{(r+1)/2}\mathbf{y}_p, \qquad (10)$$

where $\hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}$ is an approximation of the iterated Itô stochastic integral (6) and $\mathbf{1}_A$ is the indicator of the set A. Note that we understand the equality (10) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $p = 0, 1, \ldots, N$.

Under the appropriate conditions [2] the numerical scheme (10) has strong order $r/2$ ($r \in \mathbb{N}$) of convergence.

Below we consider particular cases of the numerical scheme (10) for $r = 1, 2, 3, 4, 5,$ and 6, i.e. explicit one-step strong numerical schemes with convergence orders $0.5, 1.0, 1.5, 2.0, 2.5,$ and 3.0 for the Itô SDE (1) [26], [27], [65], [66]. At that for simplicity we will write $\mathbf{a}$, $L\mathbf{a}$, B_i, $G_0^{(i)}B_j$ etc. instead of $\mathbf{a}(\mathbf{y}_p,\tau_p)$, $L\mathbf{a}(\mathbf{y}_p,\tau_p)$, $B_i(\mathbf{y}_p,\tau_p)$, $G_0^{(i)}B_j(\mathbf{y}_p,\tau_p)$ etc. correspondingly. Moreover, the operators L and $G_0^{(i)}$, $i = 1,\ldots,m$ are defined by (4), (5).

Scheme with strong order 0.5 (Euler scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}^{(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\mathbf{a}. \qquad (11)$$

Scheme with strong order 1.0 (Milstein scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}^{(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)}B_{i_2}\hat{I}^{(i_1i_2)}_{(00)\tau_{p+1},\tau_p}. \qquad (12)$$

Scheme with strong order 1.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}^{(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)}B_{i_2}\hat{I}^{(i_1i_2)}_{(00)\tau_{p+1},\tau_p} +$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}^{(i_1)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{(i_1)}_{(1)\tau_{p+1},\tau_p}\right) - LB_{i_1}\hat{I}^{(i_1)}_{(1)\tau_{p+1},\tau_p}\right] +$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}^{(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p} + \frac{\Delta^2}{2}L\mathbf{a}. \qquad (13)$$

Scheme with strong order 2.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} +$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right) - LB_{i_1}\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right] +$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2}L\mathbf{a} +$$

$$+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_1)}LB_{i_2}\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right) - LG_0^{(i_1)}B_{i_2}\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} +\right.$$

$$\left.+G_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)\right] +$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}. \tag{14}$$

Scheme with strong order 2.5

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\mathbf{a} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} +$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right) - LB_{i_1}\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right] +$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2}L\mathbf{a} +$$

$$+\sum_{i_1,i_2=1}^{m}\left[G_0^{(i_1)}LB_{i_2}\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right) - LG_0^{(i_1)}B_{i_2}\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} +\right.$$

$$\left.+G_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)\right] +$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)} +$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}+\frac{\Delta^2}{2}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}\right)+$$

$$+\frac{1}{2}LLB_{i_1}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}-LG_0^{(i_1)}\mathbf{a}\Bigl(\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\Bigr)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Bigl[G_0^{(i_1)}LG_0^{(i_2)}B_{i_3}\left(\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}LB_{i_3}\left(\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}-\hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\mathbf{a}\left(\Delta\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}+\hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}\right)-$$

$$-LG_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}\Bigr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}B_{i_5}\hat{I}_{(00000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \qquad (15)$$

Scheme with strong order 3.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}+$$

$$+\sum_{i_1=1}^{m}\Bigl[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}+\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\right)-LB_{i_1}\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\Bigr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}+\frac{\Delta^2}{2}L\mathbf{a}+$$

$$+\sum_{i_1,i_2=1}^{m}\Bigl[G_0^{(i_1)}LB_{i_2}\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}-\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}\right)-LG_0^{(i_1)}B_{i_2}\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1i_2)}+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1i_2)}+\Delta\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1i_2)}\right)\Bigr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \qquad (16)$$

where

$$\mathbf{q}_{p+1,p} = \sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} + \frac{\Delta^2}{2}\hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)}\right)+$$

$$+\frac{1}{2}LLB_{i_1}\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} - LG_0^{(i_1)}\mathbf{a}\Bigl(\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta\hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)}\Bigr)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Bigl[G_0^{(i_1)}LG_0^{(i_2)}B_{i_3}\left(\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}LB_{i_3}\left(\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\mathbf{a}\left(\Delta\hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\right)-$$

$$-LG_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}\Bigr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}B_{i_5}\hat{I}_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a},$$

and

$$\mathbf{r}_{p+1,p} = \sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_1)}G_0^{(i_2)}L\mathbf{a}\left(\frac{1}{2}\hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \frac{\Delta^2}{2}\hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)+$$

$$+\frac{1}{2}LLG_0^{(i_1)}B_{i_2}\hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)}+$$

$$+G_0^{(i_1)}LG_0^{(i_2)}\mathbf{a}\left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta\left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)\right)+$$

$$+LG_0^{(i_1)}LB_{i_2}\left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)+$$

$$+G_0^{(i_1)}LLB_{i_2}\left(\frac{1}{2}\hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \frac{1}{2}\hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)-$$

$$-LG_0^{(i_1)}G_0^{(i_2)}\mathbf{a}\left(\Delta\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)}\right)\Biggr]+$$

$$
+\sum_{i_1,i_2,i_3,i_4=1}^{m}\Biggl[G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}\mathbf{a}\left(\Delta\hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}+\hat{I}_{(0001)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}\right)+
$$

$$
+G_0^{(i_1)}G_0^{(i_2)}LG_0^{(i_3)}B_{i_4}\left(\hat{I}_{(0100)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-\hat{I}_{(0010)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}\right)-
$$

$$
-LG_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}_{(1000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}+
$$

$$
+G_0^{(i_1)}LG_0^{(i_2)}G_0^{(i_3)}B_{i_4}\left(\hat{I}_{(1000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-\hat{I}_{(0100)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}\right)+
$$

$$
+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}LB_{i_4}\left(\hat{I}_{(0010)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}-\hat{I}_{(0001)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4)}\right)\Biggr]+
$$

$$
+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}G_0^{(i_5)}B_{i_6}\hat{I}_{(000000)\tau_{p+1},\tau_p}^{(i_1i_2i_3i_4i_5i_6)}.
$$

Under the suitable conditions [2] the numerical schemes (12)–(16) have strong orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence correspondingly. Among these conditions we consider only the condition for approximations of iterated Itô stochastic integrals from (12)–(16) [2] (also see [43])

$$
\mathsf{M}\left\{\left(I_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}-\hat{I}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}^{(i_1\ldots i_k)}\right)^2\right\}\le C\Delta^{r+1}, \tag{17}
$$

where constant C is independent of Δ and $r/2$ are the strong convergence orders for the numerical schemes (12)–(16), i.e. $r/2=1.0, 1.5,\ 2.0, 2.5,$ and 3.0.

Note that the numerical schemes (12)–(16) are unrealizable in practice without procedures for the numerical simulation of iterated Itô stochastic integrals from (10). In Sect. 2.3 we give a brief overview of the effective method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k ($k\in\mathbb{N}$).

2.2 Strong Numerical Methods with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Stratonovich Expansion

Let us consider the following differential operator on the space $C^{2,1}(\mathbb{R}^n\times[0,T])$

$$\bar{L} = L - \frac{1}{2}\sum_{i=1}^{m} G_0^{(i)} G_0^{(i)}, \tag{18}$$

where operators L and $G_0^{(i)}$, $i = 1, \ldots, m$ are defined by (4), (5).

Define the following sequence of differential operators

$$\bar{G}_p^{(i)} \stackrel{\textsf{def}}{=} \frac{1}{p}\left(\bar{G}_{p-1}^{(i)}\bar{L} - \bar{L}\bar{G}_{p-1}^{(i)}\right), \quad p = 1, 2, \ldots, \quad i = 1, \ldots, m, \tag{19}$$

where $\bar{G}_0^{(i)} \stackrel{\textsf{def}}{=} G_0^{(i)}$, $i = 1, \ldots, m$. The operators $\bar{L}$ and $G_0^{(i)}$, $i = 1, \ldots, m$ are defined by (18) and (5) correspondingly.

For the further consideration, we need to introduce the following set of iterated Stratonovich stochastic integrals

$$I_{(l_1\ldots l_k)s,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*s} (t - t_k)^{l_k} \ldots \int\limits_t^{*t_2} (t - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{20}$$

where $l_1, \ldots, l_k = 0, 1, \ldots$ and $i_1, \ldots, i_k = 1, \ldots, m$.

Assume that $R(\mathbf{x}, t)$, $\mathbf{a}(\mathbf{x}, t)$, and $B_i(\mathbf{x}, t)$, $i = 1, \ldots, m$ are enough smooth functions with respect to the variables $\mathbf{x}$ and t. Then for all $s, t \in [0, T]$ such that $s > t$ we can write the following unified Taylor–Stratonovich expansion [25] (also see [26], [27], [68], Chapter 4)

$$R(\mathbf{x}_s, s) =$$

$$= R(\mathbf{x}_t, t) + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(s-t)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t)\ I_{(l_1\ldots l_k)s,t}^{*(i_1\ldots i_k)} +$$

$$+ \left(\bar{H}_{r+1}\right)_{s,t} \quad \text{w. p. 1}, \tag{21}$$

where

$$\bar{L}^j R(\mathbf{x}, t) \stackrel{\textsf{def}}{=} \begin{cases} \underbrace{\bar{L} \ldots \bar{L}}_{j} R(\mathbf{x}, t) & \text{for } j \geq 1 \\ \\ R(\mathbf{x}, t) & \text{for } j = 0 \end{cases},$$

the set D_q is defined by the equality (8) and $\left(\bar{H}_{r+1}\right)_{s,t}$ is the remainder term in integral form [25] (also see [26], [27], [68], Chapter 4).

Consider the partition (9) of the interval $[0, T]$. From (21) for $s = \tau_{p+1}$ and $t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (1)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k)\in \mathrm{D}_q} \frac{(\tau_{p+1}-\tau_p)^j}{j!} \sum_{i_1,\ldots,i_k=1}^{m} \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} \bar{L}^j \mathbf{y}_p \ \hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p} +$$

$$+ \mathbf{1}_{\{r=2d-1, d\in N\}} \frac{(\tau_{p+1}-\tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \qquad (22)$$

where $\hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}$ is an approximation of the iterated Stratonovich stochastic integral (20) and $\mathbf{1}_A$ is the indicator of the set A. Note that we understand the equality (22) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column $\mathbf{y}_p$. Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $p = 0, 1, \ldots, N$.

Under the appropriate conditions [2] the numerical scheme (22) has strong order $r/2$ ($r \in \mathbb{N}$) of convergence.

Denote

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t) - \frac{1}{2} \sum_{j=1}^{m} G_0^{(j)} B_j(\mathbf{x}, t),$$

where $B_j(\mathbf{x}, t)$ is the jth column of the matrix function $B(\mathbf{x}, t)$.

It is not difficult to show that (see (18))

$$\bar{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \bar{\mathbf{a}}^{(i)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(i)}}, \qquad (23)$$

where $\bar{\mathbf{a}}^{(i)}(\mathbf{x}, t)$ is the ith component of the vector function $\bar{\mathbf{a}}(\mathbf{x}, t)$.

Below we consider particular cases of the numerical scheme (22) for $r = 2, 3, 4, 5$, and 6, i.e. explicit one-step strong numerical schemes with convergence orders $1.0, 1.5, 2.0, 2.5$, and 3.0 for the Itô SDE (1) [26], [27], [67], [68]. At that for simplicity we will write $\bar{\mathbf{a}}$, $\bar{L}\bar{\mathbf{a}}$, $L\mathbf{a}$, B_i, $G_0^{(i)} B_j$ etc. instead of $\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $\bar{L}\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p)$, $L\mathbf{a}(\mathbf{y}_p, \tau_p)$, $B_i(\mathbf{y}_p, \tau_p)$, $G_0^{(i)} B_j(\mathbf{y}_p, \tau_p)$ etc. correspondingly.

Scheme with strong order 1.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta \bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2} \hat{I}^{*(i_1 i_2)}_{(00)\tau_{p+1},\tau_p}. \qquad (24)$$

Scheme with strong order 1.5

$$
\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2} \hat{I}^{*(i_1 i_2)}_{(00)\tau_{p+1},\tau_p} +
$$

$$
+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right) - \bar{L} B_{i_1} \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right] +
$$

$$
+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}^{*(i_1 i_2 i_3)}_{(000)\tau_{p+1},\tau_p} + \frac{\Delta^2}{2} L\mathbf{a}. \qquad (25)
$$

Scheme with strong order 2.0

$$
\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2} \hat{I}^{*(i_1 i_2)}_{(00)\tau_{p+1},\tau_p} +
$$

$$
+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right) - \bar{L} B_{i_1} \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right] +
$$

$$
+ \sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}^{*(i_1 i_2 i_3)}_{(000)\tau_{p+1},\tau_p} + \frac{\Delta^2}{2} \bar{L}\bar{\mathbf{a}} +
$$

$$
+ \sum_{i_1,i_2=1}^{m} \left[G_0^{(i_1)} \bar{L} B_{i_2} \left(\hat{I}^{*(i_1 i_2)}_{(10)\tau_{p+1},\tau_p} - \hat{I}^{*(i_1 i_2)}_{(01)\tau_{p+1},\tau_p} \right) - \bar{L} G_0^{(i_1)} B_{i_2} \hat{I}^{*(i_1 i_2)}_{(10)\tau_{p+1},\tau_p} + \right.
$$

$$
\left. + G_0^{(i_1)} G_0^{(i_2)} \bar{\mathbf{a}} \left(\hat{I}^{*(i_1 i_2)}_{(01)\tau_{p+1},\tau_p} + \Delta \hat{I}^{*(i_1 i_2)}_{(00)\tau_{p+1},\tau_p} \right) \right] +
$$

$$
+ \sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}^{*(i_1 i_2 i_3 i_4)}_{(0000)\tau_{p+1},\tau_p}. \qquad (26)
$$

Scheme with strong order 2.5

$$
\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^{m} B_{i_1} \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \Delta\bar{\mathbf{a}} + \sum_{i_1,i_2=1}^{m} G_0^{(i_1)} B_{i_2} \hat{I}^{*(i_1 i_2)}_{(00)\tau_{p+1},\tau_p} +
$$

$$
+ \sum_{i_1=1}^{m} \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right) - \bar{L} B_{i_1} \hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p} \right] +
$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}B_{i_2}\left(\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}\right)-\bar{L}G_0^{(i_1)}B_{i_2}\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1i_2)}_{(00)\tau_{p+1},\tau_p}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}^{*(i_1i_2i_3i_4)}_{(0000)\tau_{p+1},\tau_p}+$$

$$+\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}G_0^{(i_2)}B_{i_3}\left(\hat{I}^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\bar{L}B_{i_3}\left(\hat{I}^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$-\bar{L}G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}B_{i_5}\hat{I}^{*(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}+$$

$$+\frac{\Delta^3}{6}LL\mathbf{a}. \qquad (27)$$

Scheme with strong order 3.0

$$\mathbf{y}_{p+1}=\mathbf{y}_p+\sum_{i_1=1}^{m}B_{i_1}\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^{m}G_0^{(i_1)}B_{i_2}\hat{I}^{*(i_1i_2)}_{(00)\tau_{p+1},\tau_p}+$$

$$+\sum_{i_1=1}^{m}\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)-\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m} G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\bar{L}\bar{\mathbf{a}}+$$

$$+\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}B_{i_2}\left(\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}\right)-\bar{L}G_0^{(i_1)}B_{i_2}\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1i_2)}_{(00)\tau_{p+1},\tau_p}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}^{*(i_1i_2i_3i_4)}_{(0000)\tau_{p+1},\tau_p}+\mathbf{q}_{p+1,p}+\mathbf{r}_{p+1,p}, \qquad (28)$$

where

$$\mathbf{q}_{p+1,p}=\sum_{i_1=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{*(i_1)}_{(0)\tau_{p+1},\tau_p}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}B_{i_1}\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}-\bar{L}G_0^{(i_1)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1)}_{(2)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1)}_{(1)\tau_{p+1},\tau_p}\right)\Biggr]+$$

$$+\sum_{i_1,i_2,i_3=1}^{m}\Biggl[G_0^{(i_1)}\bar{L}G_0^{(i_2)}B_{i_3}\left(\hat{I}^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}\bar{L}B_{i_3}\left(\hat{I}^{*(i_1i_2i_3)}_{(010)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}\right)+$$

$$+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1i_2i_3)}_{(000)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1i_2i_3)}_{(001)\tau_{p+1},\tau_p}\right)-$$

$$-\bar{L}G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}^{*(i_1i_2i_3)}_{(100)\tau_{p+1},\tau_p}\Biggr]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}B_{i_5}\hat{I}^{*(i_1i_2i_3i_4i_5)}_{(00000)\tau_{p+1},\tau_p}+$$

$$+\frac{\Delta^3}{6}\bar{L}\bar{L}\bar{\mathbf{a}},$$

and

$$\mathbf{r}_{p+1,p}=\sum_{i_1,i_2=1}^{m}\Biggl[G_0^{(i_1)}G_0^{(i_2)}\bar{L}\bar{\mathbf{a}}\left(\frac{1}{2}\hat{I}^{*(i_1i_2)}_{(02)\tau_{p+1},\tau_p}+\Delta\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}+\frac{\Delta^2}{2}\hat{I}^{*(i_1i_2)}_{(00)\tau_{p+1},\tau_p}\right)+$$

$$+\frac{1}{2}\bar{L}\bar{L}G_0^{(i_1)}B_{i_2}\hat{I}^{*(i_1i_2)}_{(20)\tau_{p+1},\tau_p}+$$

$$+G_0^{(i_1)}\bar{L}G_0^{(i_2)}\bar{\mathbf{a}}\left(\hat{I}^{*(i_1i_2)}_{(11)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(02)\tau_{p+1},\tau_p}+\Delta\left(\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(01)\tau_{p+1},\tau_p}\right)\right)+$$

$$+\bar{L}G_0^{(i_1)}\bar{L}B_{i_2}\left(\hat{I}^{*(i_1i_2)}_{(11)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(20)\tau_{p+1},\tau_p}\right)+$$

$$+G_0^{(i_1)}\bar{L}\bar{L}B_{i_2}\left(\frac{1}{2}\hat{I}^{*(i_1i_2)}_{(02)\tau_{p+1},\tau_p}+\frac{1}{2}\hat{I}^{*(i_1i_2)}_{(20)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2)}_{(11)\tau_{p+1},\tau_p}\right)-$$

$$\left.-\bar{L}G_0^{(i_1)}G_0^{(i_2)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1i_2)}_{(10)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1i_2)}_{(11)\tau_{p+1},\tau_p}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4=1}^{m}\left[G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}\bar{\mathbf{a}}\left(\Delta\hat{I}^{*(i_1i_2i_3i_4)}_{(0000)\tau_{p+1},\tau_p}+\hat{I}^{*(i_1i_2i_3i_4)}_{(0001)\tau_{p+1},\tau_p}\right)+\right.$$

$$+G_0^{(i_1)}G_0^{(i_2)}\bar{L}G_0^{(i_3)}B_{i_4}\left(\hat{I}^{*(i_1i_2i_3i_4)}_{(0100)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3i_4)}_{(0010)\tau_{p+1},\tau_p}\right)-$$

$$-\bar{L}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\hat{I}^{*(i_1i_2i_3i_4)}_{(1000)\tau_{p+1},\tau_p}+$$

$$+G_0^{(i_1)}\bar{L}G_0^{(i_2)}G_0^{(i_3)}B_{i_4}\left(\hat{I}^{*(i_1i_2i_3i_4)}_{(1000)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3i_4)}_{(0100)\tau_{p+1},\tau_p}\right)+$$

$$\left.+G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}\bar{L}B_{i_4}\left(\hat{I}^{*(i_1i_2i_3i_4)}_{(0010)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1i_2i_3i_4)}_{(0001)\tau_{p+1},\tau_p}\right)\right]+$$

$$+\sum_{i_1,i_2,i_3,i_4,i_5,i_6=1}^{m}G_0^{(i_1)}G_0^{(i_2)}G_0^{(i_3)}G_0^{(i_4)}G_0^{(i_5)}B_{i_6}\hat{I}^{*(i_1i_2i_3i_4i_5i_6)}_{(000000)\tau_{p+1},\tau_p}.$$

Under the suitable conditions [2] the numerical schemes (24)–(28) have strong orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence correspondingly. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from (24)–(28) [2] (also see [43])

$$\mathsf{M}\left\{\left(I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}-\hat{I}^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p}\right)^2\right\}\le C\Delta^{r+1}, \tag{29}$$

where constant C is independent of Δ and $r/2$ are the strong convergence orders for the numerical schemes (24)–(28), i.e. $r/2 = 1.0, 1.5,\ 2.0, 2.5,$ and 3.0.

Note that the numerical schemes (24)–(28) are unrealizable in practice without procedures for the numerical simulation of iterated Stratonovich stochastic integrals from (22). The next section is devoted to the effective method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$).

2.3 Method of Expansion and Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series

Let us consider the effective approach to expansion of iterated Itô stochastic integrals [43] (2006) (also see [26], [27], [45]-[47], [59], [62], [68], [69]). This method is reffered to as the method of generalized multiple Fourier series.

Suppose that every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a nonrandom function from the space $L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k) & \text{for} \ \ t_1 < \ldots < t_k \\ \\ 0 & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t, T], \tag{30}$$

where $k \ge 2$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$.

The function $K(t_1, \ldots, t_k)$ belongs to the space $L_2([t, T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0,$$

where

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \tag{31}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\quad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \text{ if }\ N\to\infty,\quad \Delta\tau_j=\tau_{j+1}-\tau_j. \tag{32}$$

Theorem 1 [43] (2006) (also see [26], [27], [45]-[47], [59], [62], [68], [69]). *Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous nonrandom function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} - $$
$$- \underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{33}$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{34}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\ne 0$), $C_{j_k\ldots j_1}$ is the Fourier coefficient (31), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ is a partition of the interval $[t,T]$, which satisfies the condition (32).

Note that a number of modifications and generalizations of Theorem 1 can be found in [26], [27], [68].

Consider transformed particular cases of (33) for $k=1,\ldots,6$ [26], [27], [59], [62], [68], [69]

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{35}$$

$$J[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \qquad (36)$$

$$J[\psi^{(3)}]_{T,t} = \underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \qquad (37)$$

$$J[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (38)$$

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$\left.+\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{39}$$

$$J[\psi^{(6)}]_{T,t}=\underset{p_1,\ldots,p_6\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_6=0}^{p_6}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_6\neq 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\neq 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_6\neq 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\neq 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_5=i_6\neq 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_1=i_5\neq 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}-
$$

$$
-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}-
$$

$$
\left.-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\right), \tag{40}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

A detailed discussion of advantages of the method based on Theorem 1 over the approximation methods from works [2], [3], [7], [8], [28]-[36], [38]-[40] can be found in [26] (Sect. 1.1.10) or in [27], [68].

For further consideration, let us consider the generalization of formulas (35)–(40) for the case of an arbitrary multiplicity k ($k \in \mathbb{N}$) of the iterated Ito stochastic integral $J[\psi^{(k)}]_{T,t}$ defined by (2). In order to do this, let us introduce some notations. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{41}$$

where

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},$$

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (41) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}}. \tag{42}$$

Below there are several examples of sums in the form (42)

$$\sum_{\substack{(\{g_1,g_2\}) \\ \{g_1,g_2\}=\{1,2\}}} a_{g_1 g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\}) \\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\}) \\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1 g_2,q_1 q_2} =$$

$$= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$

$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +$$

$$+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +$$

$$+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can write (33) as

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \quad (43)$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

In particular, from (43) for $k=5$ we obtain

$$J[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$- \sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}} \prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})} +$$

$$+ \sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\Biggr).$$

The last equality obviously agrees with (39).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$.

Theorem 2 [68] (Sect. 1.11), [89] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)\qquad(44)$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number x; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [123]. Note that we use another notations [68] (Sect. 1.11), [89] (Sect. 15) in comparison with [123]. Moreover, the proof of an analogue of Theorem 2 from [123] is somewhat different from the proof given in [68] (Sect. 1.11), [89] (Sect. 15).

As it turned out, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 [26], [27], [68]-[70] (also see bibliography therein). Let as collect some old results in the following theorem.

Theorem 3 [26], [27], [68]-[70]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. At the same time $\psi_2(\tau)$ is a continuously differentiable function on $[t,T]$ and $\psi_1(\tau)$, $\psi_3(\tau)$ are twice continuously differentiable functions on $[t,T]$. Then*

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\quad(i_1,i_2=1,\ldots,m),\qquad(45)$$

$$J^*[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\quad(i_1,i_2,i_3=0,1,\ldots,m),\qquad(46)$$

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} \quad (i_1,i_2,i_3=1,\ldots,m), \tag{47}$$

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} \quad (i_1,\ldots,i_4=0,1,\ldots,m), \tag{48}$$

where $J^*[\psi^{(k)}]_{T,t}$ *is defined by* (3), *and* $\psi_l(\tau)\equiv 1$ $(l=1,\ldots,4)$ *in* (46), (48); *another notations are the same as in Theorems* 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [68] (Sect. 2.10–2.16), [93] (Sect. 13–19), [98] (Sect. 5–11), [99] (Sect. 7–13), [100] (Sect. 4–9).

Let us formulate four theorems that were obtained using this approach.

Theorem 4 [68], [93], [98], [99], [100]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Furthermore, let* $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of third multiplicity*

$$J^*[\psi^{(3)}]_{T,t} = \int\limits_t^{*T}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)} \tag{49}$$

the following relations

$$J^*[\psi^{(3)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{50}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{51}$$

are fulfilled, where $i_1,i_2,i_3=0,1,\ldots,m$ *in* (49), (50) *and* $i_1,i_2,i_3=1,\ldots,m$ *in* (51), *constant* C *is independent of* p,

$$C_{j_3j_2j_1} = \int\limits_t^{T}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

and

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{f}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*); another notations are the same as in Theorems* 1, 2.

Theorem 5 [68], [93], [98], [99], [100]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity*

$$J^*[\psi^{(4)}]_{T,t} = \int\limits_t^{*T} \psi_4(t_4) \int\limits_t^{*t_4} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} \tag{52}$$

the following relations

$$J^*[\psi^{(4)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \tag{53}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{54}$$

are fulfilled, where $i_1,\ldots,i_4 = 0,1,\ldots,m$ *in* (52), (53) *and* $i_1,\ldots,i_4 = 1,\ldots,m$ *in* (54)*, constant* C *does not depend on* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon = 0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$*,*

$$C_{j_4 j_3 j_2 j_1} =$$

$$= \int\limits_t^T \psi_4(t_4)\phi_{j_4}(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;$$

another notations are the same as in Theorem 4.

Theorem 6 [68], [93], [98], [99], [100]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$J^*[\psi^{(5)}]_{T,t}=$$

$$=\int\limits_t^{*T}\psi_5(t_5)\int\limits_t^{*t_5}\psi_4(t_4)\int\limits_t^{*t_4}\psi_3(t_3)\int\limits_t^{*t_3}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)}d\mathbf{w}_{t_5}^{(i_5)} \tag{55}$$

the following relations

$$J^*[\psi^{(5)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}, \tag{56}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}-\sum_{j_1,j_2,j_3,j_4,j_5=0}^{p}C_{j_5j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\right)^2\right\}\le$$

$$\le\frac{C}{p^{1-\varepsilon}} \tag{57}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (55), (56) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (57), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$,

$$C_{j_5j_4j_3j_2j_1}=\int\limits_t^T\psi_5(t_5)\phi_{j_5}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5;$$

another notations are the same as in Theorems 4, 5.

Theorem 7 [68], [93], [98], [99]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$I_{T,t}^{*(i_1 i_2 i_3 i_4 i_5 i_6)} =$$

$$= \int\limits_t^{*T} \int\limits_t^{*t_6} \int\limits_t^{*t_5} \int\limits_t^{*t_4} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} d\mathbf{w}_{t_5}^{(i_5)} d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$I_{T,t}^{*(i_1 i_2 i_3 i_4 i_5 i_6)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{p} C_{j_6 j_5 j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6 = 0,1,\ldots,m,$

$$C_{j_6 j_5 j_4 j_3 j_2 j_1} =$$

$$= \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 dt_5 dt_6;$$

another notations are the same as in Theorems 4–6.

Consider the following Hypothesis on expansion of the iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbb{N}$).

Hypothesis 1 [26], [27], [68], [69]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Moreover, every* $\psi_l(\tau)$ ($l = 1,\ldots,k$) *is an enough smooth nonrandom function on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* (3) *of multiplicity* k *the following expansion*

$$J^*[\psi^{(k)}]_{T,t} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \tag{58}$$

that converges in the mean-square sense is valid, where notations are the same as in Theorems 1, 2.

Hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{59}$$

where

$$\lim_{p\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - J^*[\psi^{(k)}]_{T,t}^p\right)^2\right\} = 0.$$

Note that Hypothesis 1 is proved in [68] (Sect. 2.10) under the condition of convergence of trace series (also see [93], [98], [99]).

Assume that $J[\psi^{(k)}]_{T,t}^p$ is the approximation of (2), which is the expression on the right-hand side of (44) before passing to the limit for the case $p_1 = \ldots = p_k = p$, i.e.

$$J[\psi^{(k)}]_{T,t}^p = \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})}\Biggr), \tag{60}$$

where $[x]$ is an integer part of a real number x; another notations are the same as in Theorems 1, 2.

Let us denote

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^p\right)^2\right\} \stackrel{\text{def}}{=} E_k^p,$$

$$\|K\|_{L_2([t,T]^k)}^2 = \int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k \stackrel{\text{def}}{=} I_k.$$

For the futher consideration, we need the following useful estimate [26], [27], [68]

$$E_k^p \le k!\left(I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}^2\right), \tag{61}$$

where $i_1,\ldots,i_k = 1,\ldots,m$ for $T-t \in (0,\infty)$ and $i_1,\ldots,i_k = 0,1,\ldots,m$ for $T-t\in(0,1)$; another notations are the same as in Theorems 1, 2.

The value E_k^p can be calculated exactly.

Theorem 8 [68] (Sect. 1.12), [91] (Sect. 6). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$, $i_1,\ldots,i_k=1,\ldots,m$. *Then*

$$E_k^p=I_k-$$

$$=\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}, \tag{62}$$

where $i_1,\ldots,i_k=1,\ldots,m$; *the expression*

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$; *another notations are the same as in Theorems* 1, 2.

Note that

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}\right\}=C_{j_k\ldots j_1}.$$

Then from Theorem 8 we obtain

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2\quad (i_1,\ldots,i_k \text{ are pairwise different}), \tag{63}$$

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\left(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\right)\quad (i_1=\ldots=i_k). \tag{64}$$

Consider some examples of the application of Theorem 8 ($i_1,\ldots,i_5=1,\ldots,m$):

$$E_2^p=I_2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2-\sum_{j_1,j_2=0}^{p}C_{j_2j_1}C_{j_1j_2}\quad (i_1=i_2), \tag{65}$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1} \quad (i_1 = i_2 \neq i_3), \tag{66}$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_2j_3j_1}C_{j_3j_2j_1} \quad (i_1 \neq i_2 = i_3), \tag{67}$$

$$E_3^p = I_3 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}^2 - \sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3} \quad (i_1 = i_3 \neq i_2), \tag{68}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_2 \neq i_3, i_4;\ i_3 \neq i_4), \tag{69}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_3 \neq i_2, i_4;\ i_2 \neq i_4), \tag{70}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right) \quad (i_2 = i_3 \neq i_1, i_4;\ i_1 \neq i_4), \tag{71}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_4 \neq i_2, i_3;\ i_2 \neq i_3), \tag{72}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_4)} \left(\sum_{(j_2,j_3)} C_{j_4\ldots j_1} \right) \right) \quad (i_1 = i_4 \neq i_2 = i_3), \tag{73}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_3)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_2 = i_3 \neq i_4), \tag{74}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_2,j_3,j_4)} C_{j_4\ldots j_1} \right) \quad (i_2 = i_3 = i_4 \neq i_1), \tag{75}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_2,j_4)} C_{j_4\ldots j_1} \right), \quad (i_1 = i_2 = i_4 \ne i_3), \tag{76}$$

$$E_4^p = I_4 - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1} \left(\sum_{(j_1,j_3,j_4)} C_{j_4\ldots j_1} \right) \quad (i_1 = i_3 = i_4 \ne i_2), \tag{77}$$

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_2)} C_{j_5\ldots j_1} \right), \tag{78}$$

where $i_1 = i_2 \ne i_3, i_4, i_5$ and i_3, i_4, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_3)} C_{j_5\ldots j_1} \right), \tag{79}$$

where $i_2 = i_3 \ne i_1, i_4, i_5$ and i_1, i_4, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_3)} C_{j_5\ldots j_1} \right), \tag{80}$$

where $i_1 = i_3 \ne i_2, i_4, i_5$ and i_2, i_4, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_4)} C_{j_5\ldots j_1} \right), \tag{81}$$

where $i_1 = i_4 \ne i_2, i_3, i_5$ and i_2, i_3, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_1,j_5)} C_{j_5\ldots j_1} \right), \tag{82}$$

where $i_1 = i_5 \ne i_2, i_3, i_4$ and i_2, i_3, i_4 are pairwise different.

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} C_{j_5\ldots j_1} \right), \tag{83}$$

where $i_2 = i_4 \ne i_1, i_3, i_5$ and i_1, i_3, i_5 are pairwise different.

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_4,j_5)} C_{j_5\ldots j_1} \right), \tag{84}$$

where $i_4 = i_5 \neq i_1, i_2, i_3$ and i_1, i_2, i_3 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \left(\sum_{(j_2,j_4)} \left(\sum_{(j_3,j_5)} C_{j_5\ldots j_1} \right)\right) \quad (i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1). \tag{85}$$

2.4 Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (11)–(16) Using Legendre Polynomials

This section is devoted to approximation of the iterated Itô stochastic integrals (6) of multiplicities 1 to 6 based on Theorems 1, 2. At that we will use multiple Fourier–Legendre series for approximation of the mentioned stochastic integrals.

The numerical schemes (11)–(16) contain the following set (see (6)) of iterated Itô stochastic integrals

$$I_{(0)T,t}^{(i_1)},\quad I_{(1)T,t}^{(i_1)},\quad I_{(2)T,t}^{(i_1)},\quad I_{(00)T,t}^{(i_1 i_2)},\quad I_{(10)T,t}^{(i_1 i_2)},\quad I_{(01)T,t}^{(i_1 i_2)},\quad I_{(000)T,t}^{(i_1 i_2 i_3)},\quad I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)}, \tag{86}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)},\quad I_{(02)T,t}^{(i_1 i_2)},\quad I_{(20)T,t}^{(i_1 i_2)},\quad I_{(11)T,t}^{(i_1 i_2)},\quad I_{(100)T,t}^{(i_1 i_2 i_3)},\quad I_{(010)T,t}^{(i_1 i_2 i_3)},\quad I_{(001)T,t}^{(i_1 i_2 i_3)}, \tag{87}$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)},\quad I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)},\quad I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)},\quad I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)},\quad I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)}. \tag{88}$$

Let us consider the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j \left(\left(x - \frac{T+t}{2} \right) \frac{2}{T-t} \right), \quad j = 0, 1, 2, \ldots, \tag{89}$$

where $P_j(x)$ is the Legendre polynomial

$$P_j(x) = \frac{1}{2^j j!} \frac{d^j}{dx^j} \left(x^2 - 1 \right)^j.$$

Using Theorems 1, 2 and well known properties of the Legendre polynomials, we obtain the following formulas for numerical modeling of the stochastic integrals (86)–(88) [26], [27], [43]-[47], [68], [69], [71]-[73]

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t}\,\zeta_0^{(i_1)}, \tag{90}$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{91}$$

$$I_{(2)T,t}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{92}$$

$$I_{(00)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{93}$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^{000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{94}$$

$$I_{(10)T,t}^{(i_1 i_2)q_2} = \sum_{j_1,j_2=0}^{q_2} C_{j_2 j_1}^{10}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{95}$$

$$I_{(01)T,t}^{(i_1 i_2)q_2} = \sum_{j_1,j_2=0}^{q_2} C_{j_2 j_1}^{01}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{96}$$

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_3} = \sum_{j_1,j_2,j_3,j_4=0}^{q_3} C_{j_4 j_3 j_2 j_1}^{0000}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{97}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_4} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4} C_{j_5 j_4 j_3 j_2 j_1}^{00000}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} -$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$
\left.+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \tag{98}
$$

$$
I_{(20)T,t}^{(i_1i_2)q_5}=\sum_{j_1,j_2=0}^{q_5}C_{j_2j_1}^{20}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{99}
$$

$$
I_{(11)T,t}^{(i_1i_2)q_6}=\sum_{j_1,j_2=0}^{q_6}C_{j_2j_1}^{11}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{100}
$$

$$
I_{(02)T,t}^{(i_1i_2)q_7}=\sum_{j_1,j_2=0}^{q_7}C_{j_2j_1}^{02}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\right), \tag{101}
$$

$$
I_{(001)T,t}^{(i_1i_2i_3)q_8}=\sum_{j_1,j_2,j_3=0}^{q_8}C_{j_3j_2j_1}^{001}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\right.
$$

$$
\left.-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \tag{102}
$$

$$
I_{(010)T,t}^{(i_1 i_2 i_3)q_9} = \sum_{j_1,j_2,j_3=0}^{q_9} C_{j_3 j_2 j_1}^{010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -
$$

$$
- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{103}
$$

$$
I_{(100)T,t}^{(i_1 i_2 i_3)q_{10}} = \sum_{j_1,j_2,j_3=0}^{q_{10}} C_{j_3 j_2 j_1}^{100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -
$$

$$
- \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr), \tag{104}
$$

$$
I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)q_{11}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{11}} C_{j_4 j_3 j_2 j_1}^{0001} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$

$$
+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \tag{105}
$$

$$
I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)q_{12}} \sum_{j_1,j_2,j_3,j_4=0}^{q_{12}} C_{j_4 j_3 j_2 j_1}^{0010} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
- \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$

$$
+ \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \tag{106}
$$

$$
I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)q_{13}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{13}} C_{j_4 j_3 j_2 j_1}^{0100} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \tag{107}
$$

$$
I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)q_{14}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{14}} C_{j_4 j_3 j_2 j_1}^{1000} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} -
$$

$$
-\mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
$$

$$
+\mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} +
$$

$$
+ \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \Biggr), \tag{108}
$$

$$
I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)q_{15}} = \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q_{15}} C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000} \Biggl(\prod_{l=1}^{6} \zeta_{j_l}^{(i_l)} -
$$

$$
-\mathbf{1}_{\{j_1=j_6\}} \mathbf{1}_{\{i_1=i_6\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_2=j_6\}} \mathbf{1}_{\{i_2=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_3=j_6\}} \mathbf{1}_{\{i_3=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_4=j_6\}} \mathbf{1}_{\{i_4=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} -
$$

$$
-\mathbf{1}_{\{j_5=j_6\}} \mathbf{1}_{\{i_5=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_1=i_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_1=i_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_1=i_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_1=i_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_2=i_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} -
$$

$$
-\mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{i_2=i_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_5\}} \mathbf{1}_{\{i_2=i_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} -
$$

$$-\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-$$

$$-\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+$$

$$+\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$
$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$
$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}-$$
$$-\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$
$$-\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$
$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}}-$$
$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}-$$
$$-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}}-$$
$$\left.-\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\right), \qquad (109)$$

where $\mathbf{1}_A$ is the indicator of the set A and

$$C_{j_3j_2j_1}^{000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}(T-t)^{3/2}\bar{C}_{j_3j_2j_1}^{000}, \qquad (110)$$

$$C_{j_2j_1}^{01}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^{2}\bar{C}_{j_2j_1}^{01}, \qquad (111)$$

$$C_{j_2j_1}^{10}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{8}(T-t)^{2}\bar{C}_{j_2j_1}^{10}, \qquad (112)$$

$$C_{j_4j_3j_2j_1}^{0000}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16}(T-t)^{2}\bar{C}_{j_4j_3j_2j_1}^{0000}, \qquad (113)$$

$$C_{j_2j_1}^{02}=\frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^{3}\bar{C}_{j_2j_1}^{02}, \qquad (114)$$

$$C_{j_2j_1}^{20} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{20}, \tag{115}$$

$$C_{j_2j_1}^{11} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16}(T-t)^3\bar{C}_{j_2j_1}^{11}, \tag{116}$$

$$C_{j_3j_2j_1}^{001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{001}, \tag{117}$$

$$C_{j_3j_2j_1}^{010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{010}, \tag{118}$$

$$C_{j_3j_2j_1}^{100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{16}(T-t)^{5/2}\bar{C}_{j_3j_2j_1}^{100}, \tag{119}$$

$$C_{j_5j_4j_3j_2j_1}^{00000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}}{32}(T-t)^{5/2}\bar{C}_{j_5j_4j_3j_2j_1}^{00000}, \tag{120}$$

$$C_{j_4j_3j_2j_1}^{0001} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0001}, \tag{121}$$

$$C_{j_3j_2j_1}^{0010} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{0010}, \tag{122}$$

$$C_{j_4j_3j_2j_1}^{0100} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_3j_2j_1}^{0100}, \tag{123}$$

$$C_{j_4j_3j_2j_1}^{1000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{32}(T-t)^3\bar{C}_{j_4j_3j_2j_1}^{1000}, \tag{124}$$

$$C_{j_6j_5j_4j_3j_2j_1}^{000000} =$$

$$= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}}{64}(T-t)^3\bar{C}_{j_6j_5j_4j_3j_2j_1}^{000000}, \tag{125}$$

where

$$\bar{C}^{000}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz, \tag{126}$$

$$\bar{C}^{01}_{j_2j_1} = -\int\limits_{-1}^{1} (1+y) P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy, \tag{127}$$

$$\bar{C}^{10}_{j_2j_1} = -\int\limits_{-1}^{1} P_{j_2}(y) \int\limits_{-1}^{y} (1+x) P_{j_1}(x) dx dy, \tag{128}$$

$$\bar{C}^{0000}_{j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du, \tag{129}$$

$$\bar{C}^{02}_{j_2j_1} = \int\limits_{-1}^{1} P_{j_2}(y) (y+1)^2 \int\limits_{-1}^{y} P_{j_1}(x) dx dy, \tag{130}$$

$$\bar{C}^{20}_{j_2j_1} = \int\limits_{-1}^{1} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) (x+1)^2 dx dy, \tag{131}$$

$$\bar{C}^{11}_{j_2j_1} = \int\limits_{-1}^{1} P_{j_2}(y) (y+1) \int\limits_{-1}^{y} P_{j_1}(x) (x+1) dx dy, \tag{132}$$

$$\bar{C}^{001}_{j_3j_2j_1} = -\int\limits_{-1}^{1} P_{j_3}(z) (z+1) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz, \tag{133}$$

$$\bar{C}^{010}_{j_3j_2j_1} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) (y+1) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz, \tag{134}$$

$$\bar{C}^{100}_{j_3j_2j_1} = -\int\limits_{-1}^{1} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) (x+1) dx dy dz, \tag{135}$$

$$\bar{C}^{00000}_{j_5j_4j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_5}(v) \int\limits_{-1}^{v} P_{j_4}(u) \int\limits_{-1}^{u} P_{j_3}(z) \int\limits_{-1}^{z} P_{j_2}(y) \int\limits_{-1}^{y} P_{j_1}(x) dx dy dz du dv, \tag{136}$$

$$\bar{C}_{j_4j_3j_2j_1}^{1000} = -\int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)(x+1)dxdydzdu, \qquad (137)$$

$$\bar{C}_{j_4j_3j_2j_1}^{0100} = -\int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)(y+1)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu, \qquad (138)$$

$$\bar{C}_{j_4j_3j_2j_1}^{0010} = -\int\limits_{-1}^{1} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)(z+1)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu, \qquad (139)$$

$$\bar{C}_{j_4j_3j_2j_1}^{0001} = -\int\limits_{-1}^{1} P_{j_4}(u)(u+1)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdu, \qquad (140)$$

$$\bar{C}_{j_6j_5j_4j_3j_2j_1}^{000000} =$$

$$= \int\limits_{-1}^{1} P_{j_6}(w)\int\limits_{-1}^{w} P_{j_5}(v)\int\limits_{-1}^{v} P_{j_4}(u)\int\limits_{-1}^{u} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydzdudvdw; \qquad (141)$$

another notations are the same as in Theorems 1, 2.

2.5 Optimization of Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (12)–(16)

This section is devoted to the optimization of approximations of iterated Itô stochastic integrals from the numerical schemes (12)–(16). More precisely, we discuss how to minimize the numbers $q, q_1, q_2, \ldots, q_{15}$ from Sect. 2.4.

Suppose that $\varepsilon > 0$ is the mean-square accuracy of approximation of the iterated Itô stochastic integrals (6), i.e.

$$E_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p} \stackrel{\mathsf{def}}{=} \mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)} - I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}\right)^2\right\} \le \varepsilon,$$

where $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)p}$, $p \in \mathbb{N}$ is the approximation of the iterated Itô stochastic integral $I_{(l_1\ldots l_k)T,t}^{(i_1\ldots i_k)}$. Then from (93) and (61) we obtain the following conditions for choosing the numbers $q, q_1, q_2, \ldots, q_{15}$ for approximations of the iterated Itô stochastic integrals (86)–(88) [26], [27], [68]

$$E_{(00)T,t}^{(i_1 i_2)q} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le \varepsilon \quad (i_1 \ne i_2), \tag{142}$$

$$E_{(000)T,t}^{(i_1 i_2 i_3)q_1} \le 6\left(\frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1}\left(C_{j_3 j_2 j_1}^{000}\right)^2\right) \le \varepsilon, \tag{143}$$

$$E_{(01)T,t}^{(i_1 i_2)q_2} \le 2\left(\frac{(T-t)^4}{4} - \sum_{j_1,j_2=0}^{q_2}\left(C_{j_2 j_1}^{01}\right)^2\right) \le \varepsilon, \tag{144}$$

$$E_{(10)T,t}^{(i_1 i_2)q_2} \le 2\left(\frac{(T-t)^4}{12} - \sum_{j_1,j_2=0}^{q_2}\left(C_{j_2 j_1}^{10}\right)^2\right) \le \varepsilon, \tag{145}$$

$$E_{(0000)T,t}^{(i_1 \ldots i_4)q_3} \le 24\left(\frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_3}\left(C_{j_4 j_3 j_2 j_1}^{0000}\right)^2\right) \le \varepsilon, \tag{146}$$

$$E_{(00000)T,t}^{(i_1 \ldots i_5)q_4} \le 120\left(\frac{(T-t)^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4}\left(C_{j_5 j_4 j_3 j_2 j_1}^{00000}\right)^2\right) \le \varepsilon, \tag{147}$$

$$E_{(20)T,t}^{(i_1 i_2)q_5} \le 2\left(\frac{(T-t)^6}{30} - \sum_{j_2,j_1=0}^{q_5}\left(C_{j_2 j_1}^{20}\right)^2\right) \le \varepsilon, \tag{148}$$

$$E_{(11)T,t}^{(i_1 i_2)q_6} \le 2\left(\frac{(T-t)^6}{18} - \sum_{j_2,j_1=0}^{q_6}\left(C_{j_2 j_1}^{11}\right)^2\right) \le \varepsilon, \tag{149}$$

$$E_{(02)T,t}^{(i_1 i_2)q_7} \le 2\left(\frac{(T-t)^6}{6} - \sum_{j_2,j_1=0}^{q_7}\left(C_{j_2 j_1}^{02}\right)^2\right) \le \varepsilon, \tag{150}$$

$$E_{(001)T,t}^{(i_1 i_2 i_3)q_8} \le 6\left(\frac{(T-t)^5}{10} - \sum_{j_1,j_2,j_3=0}^{q_8}\left(C_{j_3 j_2 j_1}^{001}\right)^2\right) \le \varepsilon, \tag{151}$$

$$E_{(010)T,t}^{(i_1 i_2 i_3)q_9} \le 6\left(\frac{(T-t)^5}{20} - \sum_{j_1,j_2,j_3=0}^{q_9}\left(C_{j_3 j_2 j_1}^{010}\right)^2\right) \le \varepsilon, \tag{152}$$

$$E_{(100)T,t}^{(i_1 i_2 i_3)q_{10}} \le 6\left(\frac{(T-t)^5}{60} - \sum_{j_1,j_2,j_3=0}^{q_{10}}\left(C_{j_3 j_2 j_1}^{100}\right)^2\right) \le \varepsilon, \tag{153}$$

$$E_{(0001)T,t}^{(i_1\ldots i_4)q_{11}} \le 24\left(\frac{(T-t)^6}{36} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{11}} \left(C_{j_4j_3j_2j_1}^{0001}\right)^2\right) \le \varepsilon, \tag{154}$$

$$E_{(0010)T,t}^{(i_1\ldots i_4)q_{12}} \le 24\left(\frac{(T-t)^6}{60} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{12}} \left(C_{j_4j_3j_2j_1}^{0010}\right)^2\right) \le \varepsilon, \tag{155}$$

$$E_{(0100)T,t}^{(i_1\ldots i_4)q_{13}} \le 24\left(\frac{(T-t)^6}{120} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{13}} \left(C_{j_4j_3j_2j_1}^{0100}\right)^2\right) \le \varepsilon, \tag{156}$$

$$E_{(1000)T,t}^{(i_1\ldots i_4)q_{14}} \le 24\left(\frac{(T-t)^6}{360} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{14}} \left(C_{j_4j_3j_2j_1}^{1000}\right)^2\right) \le \varepsilon, \tag{157}$$

$$E_{(000000)T,t}^{(i_1\ldots i_6)q_{15}} \le 720\left(\frac{(T-t)^6}{720} - \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q_{15}} \left(C_{j_6j_5j_4j_3j_2j_1}\right)^2\right) \le \varepsilon. \tag{158}$$

Taking into account (17) and (110)–(141), (142)–(158), we obtain the following conditions for choosing the numbers $q, q_1, q_2, \ldots, q_{15}$ for the numerical schemes (12)–(16) (constant C is independent of $T-t$ (see below)).

Milstein scheme (12)

$$\frac{1}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le C(T-t).$$

Strong Taylor–Itô scheme (13) with convergence order 1.5

$$\frac{1}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le C(T-t)^2,$$

$$6\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right) \le C(T-t). \tag{159}$$

Strong Taylor–Itô scheme (14) with convergence order 2.0

$$\frac{1}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le C(T-t)^3,$$

$$6\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le C(T-t)^2, \quad (160)$$

$$2\left(\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2\right)\le C(T-t), \quad (161)$$

$$2\left(\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2\right)\le C(T-t), \quad (162)$$

$$24\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le$$
$$\le C(T-t). \quad (163)$$

Strong Taylor–Itô scheme (15) with convergence order 2.5

$$\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^4,$$

$$6\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le C(T-t)^3, \quad (164)$$

$$2\left(\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2\right)\le C(T-t)^2, \quad (165)$$

$$2\left(\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2\right)\le C(T-t)^2, \quad (166)$$

$$24\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le$$
$$\le C(T-t)^2, \quad (167)$$

$$6\left(\frac{1}{10}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_8}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{001}\right)^2\right)\le$$
$$\le C(T-t), \quad (168)$$

$$6\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_9}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2\right)\le$$

$$\le C(T-t), \tag{169}$$

$$6\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_{10}}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2\right)\le$$

$$\le C(T-t), \tag{170}$$

$$120\Biggl(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{q_4}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\times$$

$$\times\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2\Biggr)\le C(T-t). \tag{171}$$

Strong Taylor–Itô scheme (16) with convergence order 3.0

$$\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^5,$$

$$6\left(\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2\right)\le C(T-t)^4, \tag{172}$$

$$2\left(\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2\right)\le C(T-t)^3, \tag{173}$$

$$2\left(\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2\right)\le C(T-t)^3, \tag{174}$$

$$24\left(\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2\right)\le$$

$$\le C(T-t)^3, \tag{175}$$

$$6\left(\frac{1}{10}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_8}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{001}\right)^2\right)\le$$

$$\le C(T-t)^2, \tag{176}$$

$$6\left(\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_9}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2\right)\le$$

$$\le C(T-t)^2, \qquad (177)$$

$$6\left(\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_{10}}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2\right)\le$$

$$\le C(T-t)^2, \qquad (178)$$

$$120\Biggl(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{q_4}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\times$$

$$\times\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2\Biggr)\le C(T-t)^2, \qquad (179)$$

$$2\left(\frac{1}{30}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_5}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{20}\right)^2\right)\le C(T-t), \qquad (180)$$

$$2\left(\frac{1}{18}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_6}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{11}\right)^2\right)\le C(T-t), \qquad (181)$$

$$2\left(\frac{1}{6}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_7}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{02}\right)^2\right)\le C(T-t), \qquad (182)$$

$$24\left(\frac{1}{36}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{11}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0001}\right)^2\right)\le$$

$$\le C(T-t), \qquad (183)$$

$$24\left(\frac{1}{60}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{12}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0010}\right)^2\right)\le$$

$$\le C(T-t), \qquad (184)$$

$$24\left(\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{13}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0100}\right)^2\right)\le$$

$$\le C(T-t), \qquad (185)$$

$$
24\left(\frac{1}{360}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{14}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{1000}\right)^2\right)\le
$$
$$
\le C(T-t), \tag{186}
$$

$$
720\Biggl(\frac{1}{720}-\frac{1}{64^2}\sum_{j_1,\ldots,j_6=0}^{q_{15}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)\times
$$
$$
\times\left(\bar C_{j_6\ldots j_1}^{000000}\right)^2\Biggr)\le C(T-t). \tag{187}
$$

Taking into account Theorem 8 and the results of Listings 5 and 6 (see Sect. 5) we decided to exclude the multiplier factors $k!$ from the left-hand sides of (159), (160)–(163), (164)–(171), (172)–(187). The detailed numerical confirmation of the mentioned possibility can be found in [74]. This means that we will use the following conditions for choosing the numbers $q, q_1, q_2, \ldots, q_{15}$ for the numerical schemes (12)–(16) (constant C is independent of $T-t$ (see below)).

Milstein scheme (12)

$$
\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t). \tag{188}
$$

Strong Taylor–Itô scheme (13) with convergence order 1.5

$$
\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^2, \tag{189}
$$

$$
\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{000}\right)^2\le C(T-t). \tag{190}
$$

Strong Taylor–Itô scheme (14) with convergence order 2.0

$$
\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^3, \tag{191}
$$

$$
\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{000}\right)^2\le C(T-t)^2, \tag{192}
$$

$$\frac{1}{4} - \frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2 \le C(T-t), \tag{193}$$

$$\frac{1}{12} - \frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2 \le C(T-t), \tag{194}$$

$$\frac{1}{24} - \frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2 \le C(T-t). \tag{195}$$

Strong Taylor–Itô scheme (15) with convergence order 2.5

$$\frac{1}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right) \le C(T-t)^4, \tag{196}$$

$$\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2 \le C(T-t)^3, \tag{197}$$

$$\frac{1}{4} - \frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{01}\right)^2 \le C(T-t)^2, \tag{198}$$

$$\frac{1}{12} - \frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar{C}_{j_2j_1}^{10}\right)^2 \le C(T-t)^2, \tag{199}$$

$$\frac{1}{24} - \frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar{C}_{j_4\ldots j_1}^{0000}\right)^2 \le C(T-t)^2, \tag{200}$$

$$\frac{1}{10} - \frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_8}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{001}\right)^2 \le C(T-t), \tag{201}$$

$$\frac{1}{20} - \frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_9}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{010}\right)^2 \le C(T-t), \tag{202}$$

$$\frac{1}{60} - \frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_{10}}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar{C}_{j_3j_2j_1}^{100}\right)^2 \le C(T-t), \tag{203}$$

$$\frac{1}{120} - \frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{q_4}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\left(\bar{C}_{j_5\ldots j_1}^{00000}\right)^2 \le$$

$$\le C(T-t). \tag{204}$$

Strong Taylor–Itô scheme (16) with convergence order 3.0

$$\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le C(T-t)^5, \tag{205}$$

$$\frac{1}{6}-\frac{1}{64}\sum_{j_1,j_2,j_3=0}^{q_1}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{000}\right)^2\le C(T-t)^4, \tag{206}$$

$$\frac{1}{4}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar C_{j_2j_1}^{01}\right)^2\le C(T-t)^3, \tag{207}$$

$$\frac{1}{12}-\frac{1}{64}\sum_{j_1,j_2=0}^{q_2}(2j_1+1)(2j_2+1)\left(\bar C_{j_2j_1}^{10}\right)^2\le C(T-t)^3, \tag{208}$$

$$\frac{1}{24}-\frac{1}{256}\sum_{j_1,\ldots,j_4=0}^{q_3}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{0000}\right)^2\le C(T-t)^3, \tag{209}$$

$$\frac{1}{10}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_8}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{001}\right)^2\le C(T-t)^2, \tag{210}$$

$$\frac{1}{20}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_9}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{010}\right)^2\le C(T-t)^2, \tag{211}$$

$$\frac{1}{60}-\frac{1}{256}\sum_{j_1,j_2,j_3=0}^{q_{10}}(2j_1+1)(2j_2+1)(2j_3+1)\left(\bar C_{j_3j_2j_1}^{100}\right)^2\le C(T-t)^2, \tag{212}$$

$$\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_5=0}^{q_4}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)\left(\bar C_{j_5\ldots j_1}^{00000}\right)^2\le$$

$$\le C(T-t)^2, \tag{213}$$

$$\frac{1}{30}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_5}(2j_1+1)(2j_2+1)\left(\bar C_{j_2j_1}^{20}\right)^2\le C(T-t), \tag{214}$$

$$\frac{1}{18}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_6}(2j_1+1)(2j_2+1)\left(\bar C_{j_2j_1}^{11}\right)^2\le C(T-t), \tag{215}$$

$$\frac{1}{6}-\frac{1}{256}\sum_{j_1,j_2=0}^{q_7}(2j_1+1)(2j_2+1)\left(\bar C_{j_2j_1}^{02}\right)^2\le C(T-t), \tag{216}$$

$$\frac{1}{36}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{11}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{0001}\right)^2\le C(T-t), \quad (217)$$

$$\frac{1}{60}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{12}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{0010}\right)^2\le C(T-t), \quad (218)$$

$$\frac{1}{120}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{13}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{0100}\right)^2\le C(T-t), \quad (219)$$

$$\frac{1}{360}-\frac{1}{32^2}\sum_{j_1,\ldots,j_4=0}^{q_{14}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\left(\bar C_{j_4\ldots j_1}^{1000}\right)^2\le C(T-t), \quad (220)$$

$$\frac{1}{720}-\frac{1}{64^2}\sum_{j_1,\ldots,j_6=0}^{q_{15}}(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)\times$$
$$\times\left(\bar C_{j_6\ldots j_1}^{000000}\right)^2\le C(T-t). \quad (221)$$

2.6 Approximations of Iterated Stratonovich Stochastic Integrals from the Numerical Schemes (24)–(28) Using Legendre Polynomials

This section is devoted to approximation of the Stratonovich stochastic integrals (20) of multiplicities 1 to 6 based on Theorems 3–7. At that we will use multiple Fourier–Legendre series for approximation of the mentioned stochastic integrals.

The numerical schemes (24)–(28) contain the following set (see (20)) of iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)},\quad I_{(1)T,t}^{*(i_1)},\quad I_{(2)T,t}^{*(i_1)},\quad I_{(00)T,t}^{*(i_1i_2)},\quad I_{(10)T,t}^{*(i_1i_2)},\quad I_{(01)T,t}^{*(i_1i_2)},\quad I_{(000)T,t}^{*(i_1i_2i_3)},\quad I_{(0000)T,t}^{*(i_1i_2i_3i_4)}, \quad (222)$$

$$I_{(00000)T,t}^{*(i_1i_2i_3i_4i_5)},\quad I_{(02)T,t}^{*(i_1i_2)},\quad I_{(20)T,t}^{*(i_1i_2)},\quad I_{(11)T,t}^{*(i_1i_2)},\quad I_{(100)T,t}^{*(i_1i_2i_3)},\quad I_{(010)T,t}^{*(i_1i_2i_3)},\quad I_{(001)T,t}^{*(i_1i_2i_3)}, \quad (223)$$

$$I_{(0001)T,t}^{*(i_1i_2i_3i_4)},\quad I_{(0010)T,t}^{*(i_1i_2i_3i_4)},\quad I_{(0100)T,t}^{*(i_1i_2i_3i_4)},\quad I_{(1000)T,t}^{*(i_1i_2i_3i_4)},\quad I_{(000000)T,t}^{*(i_1i_2i_3i_4i_5i_6)}. \quad (224)$$

Using Theorems 3–7 and well known properties of the Legendre polynomials, we obtain the following formulas for numerical modeling of the stochastic integrals (222)–(224) [26], [27], [43]-[47], [68], [69], [71]-[73]

$$I_{(0)T,t}^{*(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \quad (225)$$

$$I_{(1)T,t}^{*(i_1)} = -\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{226}$$

$$I_{(2)T,t}^{*(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{227}$$

$$I_{(00)T,t}^{*(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right)\right), \tag{228}$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}^{000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{229}$$

$$I_{(10)T,t}^{*(i_1 i_2)q_2} = \sum_{j_1,j_2=0}^{q_2} C_{j_2 j_1}^{10}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{230}$$

$$I_{(01)T,t}^{*(i_1 i_2)q_2} = \sum_{j_1,j_2=0}^{q_2} C_{j_2 j_1}^{01}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{231}$$

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_3} = \sum_{j_1,j_2,j_3,j_4=0}^{q_3} C_{j_4 j_3 j_2 j_1}^{0000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \tag{232}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_4} = \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4} C_{j_5 j_4 j_3 j_2 j_1}^{00000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}, \tag{233}$$

$$I_{(20)T,t}^{*(i_1 i_2)q_5} = \sum_{j_1,j_2=0}^{q_5} C_{j_2 j_1}^{20}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{234}$$

$$I_{(11)T,t}^{(i_1 i_2)q_6} = \sum_{j_1,j_2=0}^{q_6} C_{j_2 j_1}^{11}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{235}$$

$$I_{(02)T,t}^{(i_1 i_2)q_7} = \sum_{j_1,j_2=0}^{q_7} C_{j_2 j_1}^{02}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{236}$$

$$I_{(001)T,t}^{*(i_1 i_2 i_3)q_8} = \sum_{j_1,j_2,j_3=0}^{q_8} C_{j_3 j_2 j_1}^{001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{237}$$

$$I_{(010)T,t}^{(i_1 i_2 i_3)q_9} = \sum_{j_1,j_2,j_3=0}^{q_9} C_{j_3 j_2 j_1}^{010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{238}$$

$$I_{(100)T,t}^{(i_1i_2i_3)q_{10}} = \sum_{j_1,j_2,j_3=0}^{q_{10}} C_{j_3j_2j_1}^{100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \qquad (239)$$

$$I_{(0001)T,t}^{(i_1i_2i_3i_4)q_{11}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{11}} C_{j_4j_3j_2j_1}^{0001}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \qquad (240)$$

$$I_{(0010)T,t}^{(i_1i_2i_3i_4)q_{12}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{12}} C_{j_4j_3j_2j_1}^{0010}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \qquad (241)$$

$$I_{(0100)T,t}^{(i_1i_2i_3i_4)q_{13}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{13}} C_{j_4j_3j_2j_1}^{0100}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \qquad (242)$$

$$I_{(1000)T,t}^{(i_1i_2i_3i_4)q_{14}} = \sum_{j_1,j_2,j_3,j_4=0}^{q_{14}} C_{j_4j_3j_2j_1}^{1000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}, \qquad (243)$$

$$I_{(000000)T,t}^{(i_1i_2i_3i_4i_5i_6)q_{15}} = \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q_{15}} C_{j_6j_5j_4j_3j_2j_1}^{000000}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}, \qquad (244)$$

where $\mathbf{1}_A$ is the indicator of the set A; another notations are the same as in Sect. 2.4.

The question on choosing the numbers $q_1, q_2, \ldots, q_{15}$ in (229)–(244) turned out to be nontrivial [26], [27], [68] (Chapter 5). The expansions (229)–(244) for iterated Stratonovich stochastic integrals are simpler than their analogues (94)–(109) for iterated Itô stochastic integrals. However, the calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Itô stochastic integrals [26], [27], [68] (Chapter 5). Below we give some reasoning regarding this problem.

Denote

$$E_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p} \stackrel{\text{def}}{=} \mathsf{M}\left\{\left(I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)} - I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}\right)^2\right\},$$

where $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)p}$, $p \in \mathbb{N}$ is the approximation of the iterated Stratonovich stochastic integral $I_{(l_1\ldots l_k)T,t}^{*(i_1\ldots i_k)}$.

From (228) for $i_1 \neq i_2$ we obtain [26], [27], [68]

$$E_{(00)T,t}^{*(i_1i_2)q} = \frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1} \leq \frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx =$$

$$= -\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right| \le C_1\frac{(T-t)^2}{q}, \tag{245}$$

where constant C_1 is independent of q.

It is easy to notice that for a sufficiently small $T-t$ (recall that $T-t \ll 1$ since it is a step of integration for numerical schemes for Itô SDEs) there exists a constant C_2 such that

$$E^{*(i_1\ldots i_k)q}_{(l_1\ldots l_k)T,t} \le C_2 E^{*(i_1 i_2)q}_{(00)T,t}. \tag{246}$$

From (245) and (246) we finally obtain

$$E^{*(i_1\ldots i_k)q}_{(l_1\ldots l_k)T,t} \le C\frac{(T-t)^2}{q}, \tag{247}$$

where constant C does not depend on $T-t$. The same idea can be found in [2] in the framework of the method based on the trigonometric expansion of the Brownian bridge process. Note that, in contrast to (247), the constant C in Theorems 4–6 does not depend on q.

Obviously, we can get more information about the numbers $q_1, q_2, \ldots, q_{15}$ (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} \quad \text{w. p. 1}$$

for pairwise different $i_1, \ldots, i_k = 1, \ldots, m$, where $J[\psi^{(k)}]_{T,t}$, $J^*[\psi^{(k)}]_{T,t}$ are defined by (2) and (3) correspondingly, then for pairwise different $i_1, \ldots, i_6 = 1, \ldots, m$ from (63) we obtain [26], [27], [68]

$$E^{*(i_1 i_2)q}_{(00)T,t} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{248}$$

$$E^{*(i_1 i_2 i_3)q_1}_{(000)T,t} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1}\left(C^{000}_{j_3 j_2 j_1}\right)^2, \tag{249}$$

$$E^{*(i_1 i_2)q_2}_{(01)T,t} = \frac{(T-t)^4}{4} - \sum_{j_1,j_2=0}^{q_2}\left(C^{01}_{j_2 j_1}\right)^2, \tag{250}$$

$$E^{*(i_1 i_2)q_2}_{(10)T,t} = \frac{(T-t)^4}{12} - \sum_{j_1,j_2=0}^{q_2}\left(C^{10}_{j_2 j_1}\right)^2, \tag{251}$$

$$E^{*(i_1\ldots i_4)q_3}_{(0000)T,t} = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_3} \left(C^{0000}_{j_4j_3j_2j_1}\right)^2, \tag{252}$$

$$E^{*(i_1\ldots i_5)q_4}_{(00000)T,t} = \frac{(T-t)^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_4} \left(C^{00000}_{j_5i_4i_3i_2j_1}\right)^2, \tag{253}$$

$$E^{*(i_1i_2)q_5}_{(20)T,t} = \frac{(T-t)^6}{30} - \sum_{j_2,j_1=0}^{q_5} \left(C^{20}_{j_2j_1}\right)^2, \tag{254}$$

$$E^{*(i_1i_2)q_6}_{(11)T,t} = \frac{(T-t)^6}{18} - \sum_{j_2,j_1=0}^{q_6} \left(C^{11}_{j_2j_1}\right)^2, \tag{255}$$

$$E^{*(i_1i_2)q_7}_{(02)T,t} = \frac{(T-t)^6}{6} - \sum_{j_2,j_1=0}^{q_7} \left(C^{02}_{j_2j_1}\right)^2, \tag{256}$$

$$E^{*(i_1i_2i_3)q_8}_{(001)T,t} = \frac{(T-t)^5}{10} - \sum_{j_1,j_2,j_3=0}^{q_8} \left(C^{001}_{j_3j_2j_1}\right)^2, \tag{257}$$

$$E^{*(i_1i_2i_3)q_9}_{(010)T,t} = \frac{(T-t)^5}{20} - \sum_{j_1,j_2,j_3=0}^{q_9} \left(C^{010}_{j_3j_2j_1}\right)^2, \tag{258}$$

$$E^{*(i_1i_2i_3)q_{10}}_{(100)T,t} = \frac{(T-t)^5}{60} - \sum_{j_1,j_2,j_3=0}^{q_{10}} \left(C^{100}_{j_3j_2j_1}\right)^2, \tag{259}$$

$$E^{*(i_1\ldots i_4)q_{11}}_{(0001)T,t} = \frac{(T-t)^6}{36} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{11}} \left(C^{0001}_{j_4j_3j_2j_1}\right)^2, \tag{260}$$

$$E^{*(i_1\ldots i_4)q_{12}}_{(0010)T,t} = \frac{(T-t)^6}{60} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{12}} \left(C^{0010}_{j_4j_3j_2j_1}\right)^2, \tag{261}$$

$$E^{*(i_1\ldots i_4)q_{13}}_{(0100)T,t} = \frac{(T-t)^6}{120} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{13}} \left(C^{0100}_{j_4j_3j_2j_1}\right)^2, \tag{262}$$

$$E^{*(i_1\ldots i_4)q_{14}}_{(1000)T,t} = \frac{(T-t)^6}{360} - \sum_{j_1,j_2,j_3,j_4=0}^{q_{14}} \left(C^{1000}_{j_4j_3j_2j_1}\right)^2, \tag{263}$$

$$E^{*(i_1\ldots i_6)q_{15}}_{(000000)T,t} = \frac{(T-t)^6}{720} - \sum_{j_1,j_2,j_3,j_4,j_5,j_6=0}^{q_{15}} \left(C^{000000}_{j_6j_5j_4j_3j_2j_1}\right)^2. \tag{264}$$

Table 1. High-order strong Taylor–Stratonovich schemes.

Order of convergence	Scheme	Conditions for choosing the numbers $q, q_1, \ldots, q_{15}$
1.0	(24)	(188)
1.5	(25)	(189), (190)
2.0	(26)	(191)–(195)
2.5	(27)	(196)–(204)
3.0	(28)	(205)–(221)

Taking into account (248)–(264) and the results of paper [74], we use in the SDE-MATH software package the conditions from Table 1 for choosing the numbers $q, q_1, q_2, \ldots, q_{15}$ for the numerical schemes (24)–(28).

Note that in the SDE-MATH software package, which is presented in the following sections, we use the following upper bounds b on the numbers $q_1, \ldots, q_{15}$

$$b = 56 \quad \text{for} \quad q_1, \qquad b = 15 \quad \text{for} \quad q_2,\ q_3, \qquad b = 6 \quad \text{for} \quad q_4,\ q_8,\ q_9,\ q_{10},$$

$$b = 2 \quad \text{for} \quad q_5,\ q_6,\ q_7,\ q_{11},\ q_{12},\ q_{13},\ q_{14},\ q_{15}.$$

This means that for the implementing of the numerical methods (13)–(16) and (25)–(28) we use in the SDE-MATH software package the following quantities of the exactly calculated Fourier–Legendre coefficients

$$57^3 = 185{,}193 \quad \text{for} \quad C_{j_3j_2j_1}^{000},$$

$$16^3 = 4{,}096 \quad \text{for each of} \quad C_{j_2j_1}^{10},\ C_{j_2j_1}^{01},$$

$$16^4 = 65{,}536 \quad \text{for} \quad C_{j_4j_3j_2j_1}^{0000},$$

$$7^3 = 343 \quad \text{for each of} \quad C_{j_3j_2j_1}^{100},\ C_{j_3j_2j_1}^{010},\ C_{j_3j_2j_1}^{001},$$

$$7^5 = 16{,}807 \quad \text{for} \quad C_{j_5j_4j_3j_2j_1}^{00000},$$

$$3^2 = 9 \quad \text{for each of} \quad C_{j_2j_1}^{20},\ C_{j_2j_1}^{02},\ C_{j_2j_1}^{11},$$

$$3^4 = 81 \quad \text{for each of} \quad C_{j_4j_3j_2j_1}^{1000},\ C_{j_4j_3j_2j_1}^{0100},\ C_{j_4j_3j_2j_1}^{0010},\ C_{j_4j_3j_2j_1}^{0001},$$

$$3^6 = 729 \quad \text{for} \quad C_{j_6j_5j_4j_3j_2j_1}^{000000}.$$

It should be noted that unlike the method based on Theorems 1–8, existing and well-known approaches to the mean-square approximation of iterated stochastic integrals based on the trigonometric basis functions [2], [3], [7], [28],

[29], [35], [38] do not allow choosing theoretically different numbers q for approximations of different iterated stochastic integrals (starting form the multiplicity 2 of stochastic integrals). Moreover, the noted approaches [2], [3], [7], [28], [29], [35], [38] exclude the possibility for obtaining of approximate and exact expressions for the mean-square approximation error similar to the formulas (61), (62).

2.7 Numerical Algorithm for Linear Stationary Systems of Itô SDEs Based on Spectral Decomposition

Consider the following linear stationary system of Itô SDEs

$$d\mathbf{x}_t = (A\mathbf{x}_t + B\mathbf{u}(t))\,dt + F\,d\mathbf{w}_t, \quad \mathbf{x}_0 = \mathbf{x}(0), \quad t \in [0, T], \tag{265}$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is a solution of the system (265), $\mathbf{u}(t) : [0, T] \to \mathbb{R}^k$ is a nonrandom function, $A \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times k}$, and $\mathbf{w}_t$ is a standard m-dimensional Wiener process with independent components $\mathbf{w}_t^{(i)}$, $i = 1, \ldots, m$. Also we suppose that n, m, $k \ge 1$. The process $\mathbf{y}_t = H\mathbf{x}_t \in \mathbb{R}^1$ is interpreted as an output process of the system (265), where $H \in \mathbb{R}^{1\times n}$.

It is well-known that the solution of (265) has the form [4]

$$\mathbf{x}_t = e^{A(t-t_0)}\mathbf{x}_{t_0} + \int\limits_{t_0}^{t} e^{A(t-s)}B\mathbf{u}(s)ds + \int\limits_{t_0}^{t} e^{A(t-s)}F\,d\mathbf{w}_s, \quad 0 \le t_0 \le t \le T, \tag{266}$$

where e^C is a matrix exponent

$$\sum_{j=0}^{\infty} \frac{C^j}{j!} \stackrel{\mathsf{def}}{=} e^C,$$

C is a square matrix, and $C^0 \stackrel{\mathsf{def}}{=} I$ is a unity matrix.

Consider the partition $\{\tau_p\}_{p=0}^{N}$ of $[0, T]$ such that $\tau_p = p\Delta$, $\Delta > 0$. For simplicity, we will suppose that $\mathbf{u}(s)$, $s \in [0, T]$ can be approximated by the step function, i.e. $\mathbf{u}(s) \approx \hat{\mathbf{u}}(s)$, $s \in [0, T]$, where $\hat{\mathbf{u}}(s) = \mathbf{u}(\tau_p)$ for $s \in [\tau_p, \tau_{p+1})$, $p = 0, 1, \ldots, N-1$ (more accurate approximations of $\mathbf{u}(s)$ are discussed in [63] (also see [59], [62])). Substituting $t = \tau_{p+1}$, $t_0 = \tau_p$, and $\hat{\mathbf{u}}(s)$ instead of $\mathbf{u}(s)$ into (266), we obtain

$$\hat{\mathbf{x}}_{p+1} = e^{A\Delta}\hat{\mathbf{x}}_p + A^{-1}\bigl(e^{A\Delta} - I\bigr)B\mathbf{u}(p\Delta) + \tilde{\mathbf{w}}_{p+1}(\Delta), \quad \mathbf{x}_0 = \mathbf{x}(0), \tag{267}$$

where $\hat{\mathbf{x}}_p$ is the approximation of $\mathbf{x}_{\tau_p}$ and

$$\int_0^{\Delta} e^{A(\Delta-s)} F d\mathbf{w}_{s+p\Delta} \stackrel{\mathsf{def}}{=} \tilde{\mathbf{w}}_{p+1}(\Delta).$$

Also we assume that $\hat{\mathbf{y}}_p = H\hat{\mathbf{x}}_p$, where $\hat{\mathbf{y}}_p$ is the approximation of $\mathbf{y}_{\tau_p}$. The random column $\tilde{\mathbf{w}}_{p+1}(\Delta)$ admits the following representation [4]

$$\tilde{\mathbf{w}}_{p+1}(\Delta) = S_D(\Delta)\Lambda_D(\Delta)\bar{\mathbf{w}}_{p+1}, \tag{268}$$

where $\bar{\mathbf{w}}_p \in \mathbb{R}^n$ is a column of independent standard Gaussian random variables such that $\mathsf{M}\left\{\bar{\mathbf{w}}_p\bar{\mathbf{w}}_q^{\mathsf{T}}\right\} = \mathcal{O}$ for $p \ne q$, $\mathcal{O}$ is a zero matrix of size $n \times n$, $S_D(\Delta)$ is a matrix of orthonormal eigenvectors of the matrix $D_f(\Delta)$ and $\Lambda_D^2(\Delta)$ is a diagonal matrix on the main diagonal of which are the eigenvalues of the matrix $D_f(\Delta)$, the matrix $D_f(\Delta)$ is defined by

$$D_f(\Delta) = \mathsf{M}\left\{\tilde{\mathbf{w}}_{p+1}(\Delta)\tilde{\mathbf{w}}_{p+1}^{\mathsf{T}}(\Delta)\right\} = \int_0^{\Delta} \exp(A(\Delta-s))FF^{\mathsf{T}}\exp(A^{\mathsf{T}}(\Delta-s))ds,$$

where C^{T} is a transposed matrix C. Moreover, $D_f(\Delta) = D_f(t)\big|_{t=\Delta}$, where $D_f(t)$ is a solution of the following Cauchy problem [4]

$$\frac{dD_f}{dt}(t) = AD_f(t) + D_f(t)A^{\mathsf{T}} + FF^{\mathsf{T}}, \quad D_f(0) = \mathcal{O}.$$

In the SDE-MATH software package, we implement the numerical modeling of the system (265) by the formulas (267), (268). At that we use Algorithms 2.3–2.6 from [63] (also see [62], Chapter 11) for the implemetation of (268).

3 The Structure of the SDE-MATH Software Package

3.1 Development Tools

The software package was implemented with Python programming language. The main reason to use it is a huge community and significant amount of helpful libraries for calculations and mathematics. The development was performed in free to use Atom text editor[1].

[1]All programs in Python programming language from this paper were written by the first author

3.2 Dependency Libraries

In the development of the SDE-MATH software package such libraries as SymPy, NumPy, PyQt5, and Matplotlib were involved. All these libraries and tools are free and open source.

- SymPy is a Python library able to perform symbolic algebra calculations.
- NumPy is a library which specialization is efficient mathematical calculations. Most part of this library is written in C programming language that guarantees high calculation performance.
- The database is SQLite3. This is a tiny database for a local usage on one machine.
- Matplotlib library is a piece of software used to present obtained results in a best way.
- PyQt5 is a library used to build graphical user interface for the SDE-MATH software package.

3.3 Architecture

Taking into account, that the SDE-MATH software package is oriented on a numerical modeling its architecture is clear. There are two main statements. The first is that mathematical formulas are strongly integrated with SymPy library. By that we mean that they completely rely on SymPy. And the second is usage of database to make some calculations able for caching. The architecture itself is provided on Figure 1. Here all parts of the software package can be seen.

The main package is responsible for startup, so it decides which part of the software package must be started. The software package has several modes of operation. The objectives now are

- Run program to calculate and store the Fourier–Legendre coefficients in few text files with further loading in database.
- Run program with graphical user interface. This is the main program entry for the SDE-MATH software package.

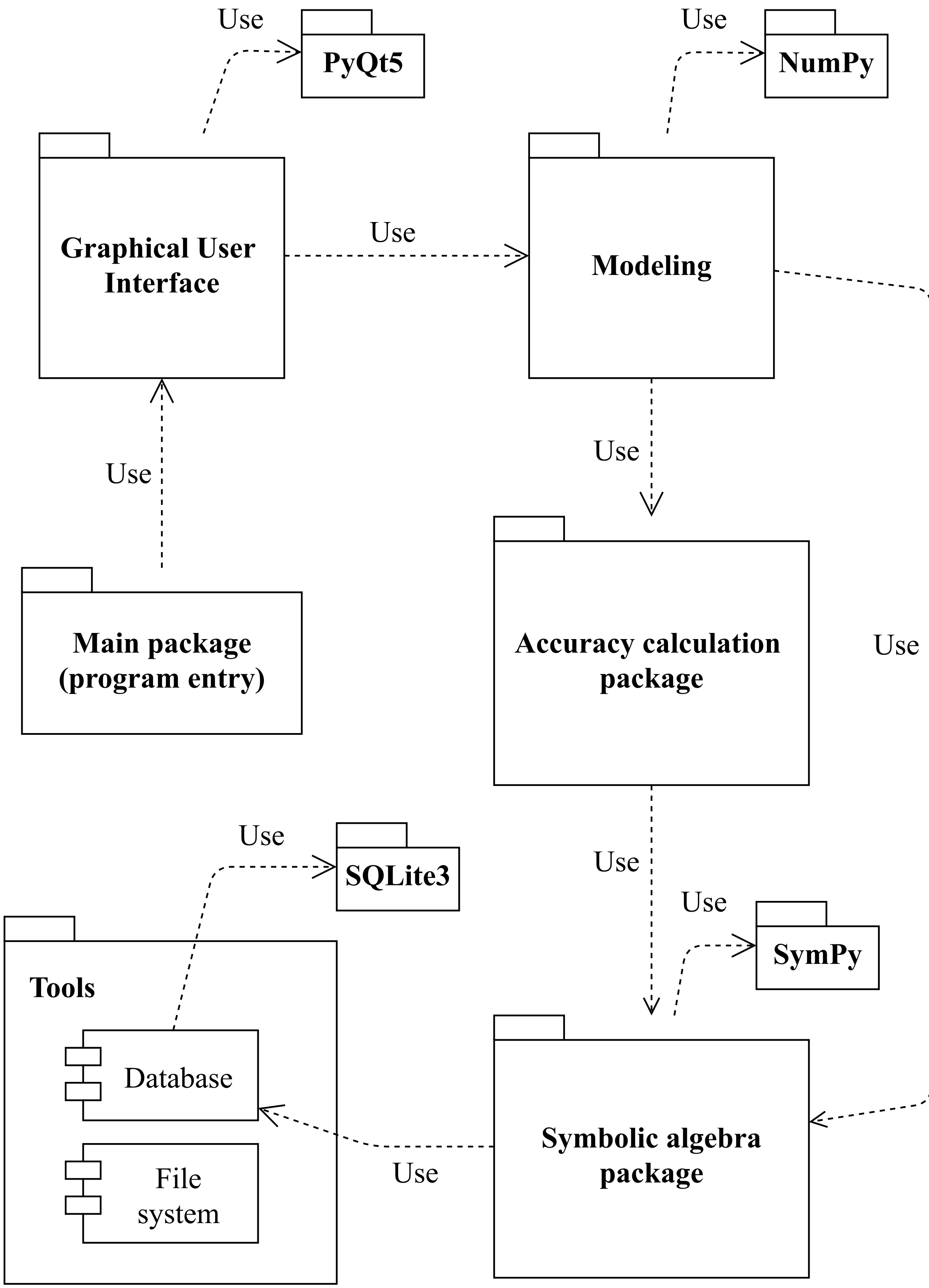


Figure 1: The SDE-MATH software package architecture

On the current state of development the main entry package booting up PyQt5 library with all necessary widgets. More detailed description of this process will be provided later.

Moving further, the modeling package comes up. This package responsible for all work referenced to modeling including initialization of modeling environment, calculations loops and more. Also, it depends on accuracy calculation module deciding which amount of members in each approximation of iterated stochastic integral should be used in modeling of the Itô SDE (1) solution.

Accuracy calculation module accepts the order of strong numerical scheme for the Itô SDE (1) and its integration step and then calculates necessary amount of members in approximations of iterated Itô and Stratonovich stochastic integrals.

Symbolic algebra module is the construction part which combines many supplementary differential operators with strong numerical schemes for the Itô SDE (1). Having these components combined this module performs simplification of resulting formula so the modeling package can do its modeling work.

Tools module provides some functionality related to bootstrap of runtime environment and external instruments such as database and file system.

3.3.1 Integration with SymPy

Class inheritance tree was extended to implement strong numerical schemes for Itô SDEs. While numerical schemes for Itô SDEs were being implemented it was also necessary to implement supplementary subprograms. SymPy is a Python library able to perform symbolic algebra calculations. This is a core part of the project since it is differentiates input functions, builds and simplifies strong numerical schemes for Itô SDEs to model the Itô SDE (1) solution. Without this part the program package cannot be able to provide such flexible input of data.

3.3.2 Purpose of NumPy

NumPy is a library that helps with calculation optimizations in this project. The library specialization is efficient mathematical calculations. The main usage case is to calculate compiled symbolic formulas with it. It has integration with SymPy to replace symbolic functions with high performance numerical functions.

3.3.3 Purpose of SQLite Database

The database was used to store the precalculated Fourier–Legendre coefficients, so getting them from there made numerical modeling much faster, because calculation process for these Fourier–Legendre coefficients involve high-cost symbolic operations. The database contains only one table, and might be thought redundant, but modeling needs hundreds (or even thousands) of precalculated coefficients. Obviously, calculation of them at runtime is terribly inefficient, but text files also not the best choice. Text files provide a sequential access memory and combining different accuracy values $q_1, \ldots, q_{15}$ it causes sequential search which extends time to give the result. That is where database comes up. The random access allows to get any Fourier–Legendre coefficient or any quantity of them which makes solution as flexible as it possible.

The download of precalculated Fourier–Legendre coefficients is built in supplemental subprograms to provide fluent calculation pipeline. Having the precalculated Fourier–Legendre coefficient not found, subprogram initiates calculation for it with following store in the database.

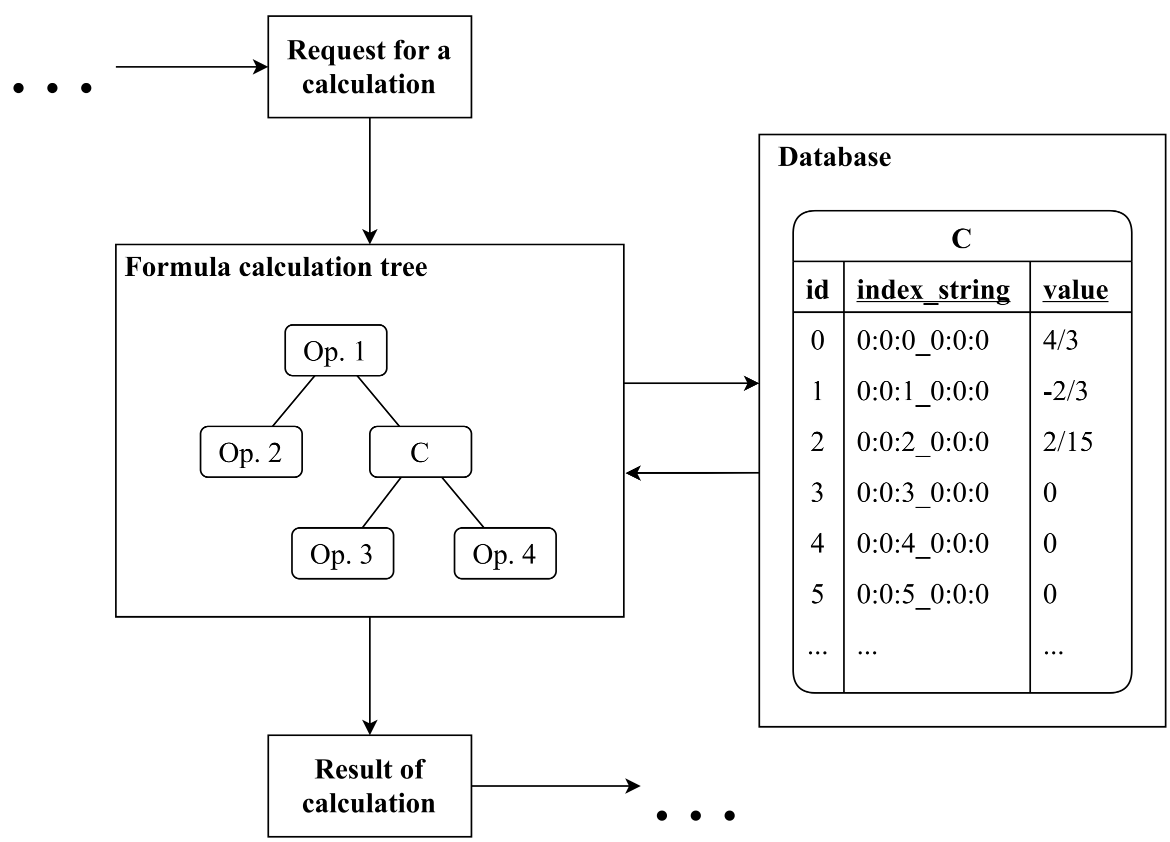


Figure 2: Fourier–Legendre coefficients calculations explanation

It is an interesting note that having mentioned earlier optimization done, the calculations performance were increased in several times. Now the most heavy operation is symbolic simplifications before modeling. The actual modeling takes seconds, for thousands of iterations on m components of stochastic process, so it is not such important how long modeling period of time, as the accuracy that needs to be accomplished.

The scheme of calculation process is presented on Figure 2.

3.3.4 Purpose of Matplotlib

Matplotlib library is a piece of software used to present obtained results in a best way. This library has many features, but feature that needed in this project is to print charts with modeling results in a PyQt5 widget. Thus the data visualization is integrated in graphical user interface.

3.4 Implementation Plan

The implementation of SDE-MATH software package was performed sequentially. The components of SDE-MATH software package were implemented in order of their necessity for calculation pipeline completion.

3.4.1 Calculation of the Fourier–Legendre Coefficients

The Fourier–Legendre coefficients for the approximations of iterated Itô and Stratonovich stochastic integrals were implemented and placed in Listings 43–61. This was the first step since the Fourier–Legendre coefficients involved almost in every strong numerical scheme for the Itô SDE (1).

Also it is important to note that the SDE-MATH software package contains a Python script intended for generating of Fourier–Legendre coefficients using multiprocessing. This script placed in Listing 62 and already contains tasks that were performed to generate about 300,000 Fourier–Legendre coefficients. Similarly, user can run and calculate additional Fourier–Legendre coefficients if they are needed. To determine which Fourier–Legendre coefficients will be calculated user must specify pairs of starting and ending values of components in lower multi-index and specify upper multi-index of the Fourier–Legendre coefficient. For example $(((0,15),(0,15),(0,15)),(0,1,0))$. This means that program calculates the Fourier–Legendre coefficients $C^{010}_{j_3 j_2 j_1}$, where $j_1, j_2, j_3 \in \{0, 1, \ldots, 14\}$.

3.4.2 Differential Operators L, $\bar{L}$, $G_0^{(i)}$, $i = 1, \ldots, m$

Moving further, strong numerical schemes for Itô SDEs rely on the differential operators (4), (5), and (23). They were implemented and placed in Listings 64–67.

3.4.3 Approximations of Iterated Stochastic Integrals

The next step is implementation of approximations of iterated Itô and Stratonovich stochastic integrals for the numerical schemes (12)–(16), (24)–(28). They are implemented and definition of their classes are placed in Listings 69–108.

3.4.4 Strong Numerical Schemes for Itô SDEs

The strong numerical schemes (12)–(16), (24)–(28) for Itô SDEs were implemented. They are placed in Listings 110–131.

3.4.5 Graphical User Interface

Finally, the graphical user interface was implemented. The source codes referenced to graphical user interface are placed in Listings 7–42.

4 Software Package Graphical User Interface

For the SDE-MATH software package mentioned above the graphical user interface was developed. The graphical user interface is important and massive part of SDE-MATH software package because it allows user to perform modeling experiments without programming skills and understanding of program package architecture and principles of work.

4.1 Information Model of The Graphical User Interface

The development of graphical user interface was started from consideration of

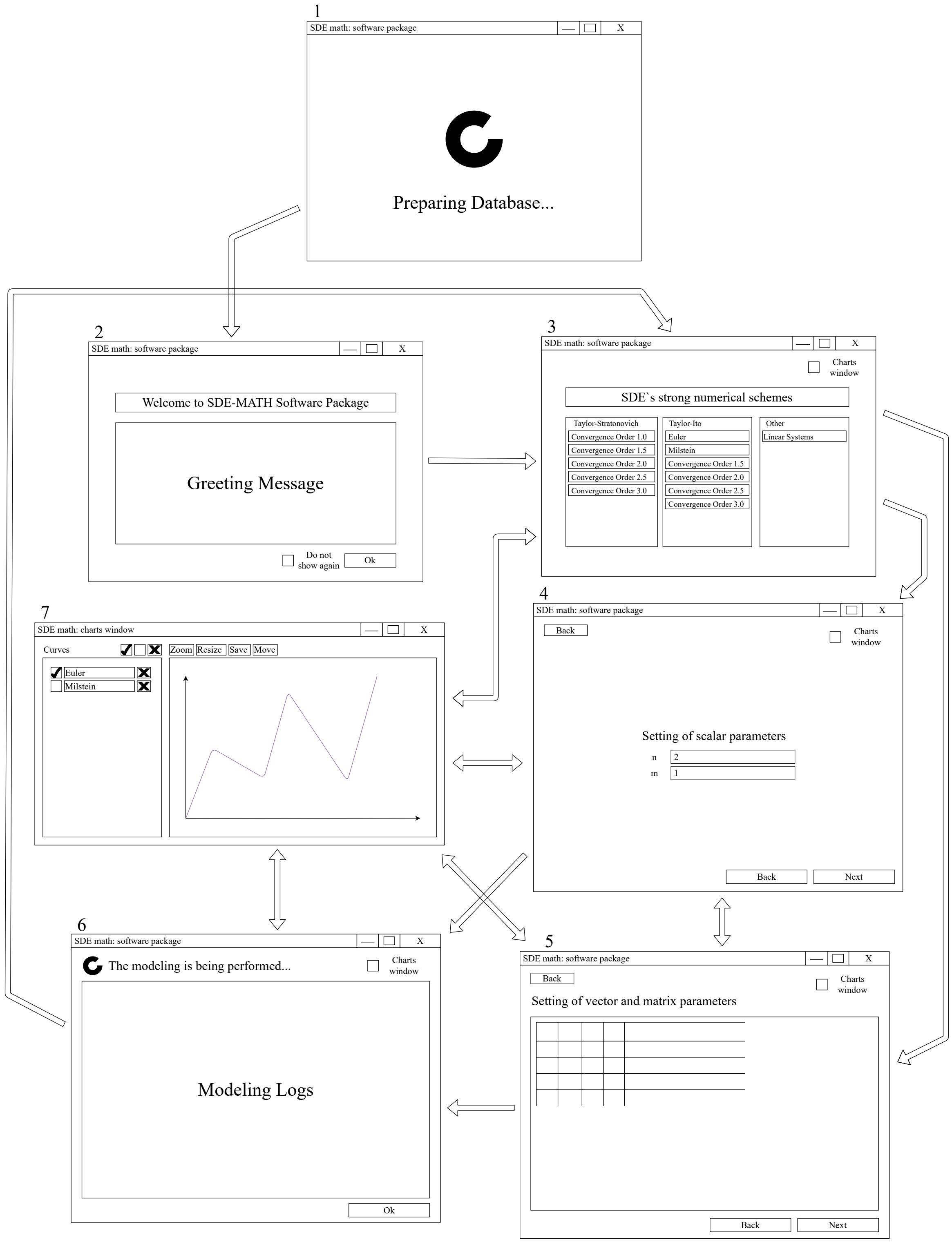


Figure 3: Information model of graphical user interface

experiments and routines which can be performed with the SDE-MATH software package. The graphical user interface is aimed on provision of user capabilities to perform nonlinear and linear systems of Itô SDEs modeling experiments. The information model which schematically describes the graphical user interface structure is presented on Figure 3.

4.1.1 Processing Screens

To represent long duration processes the graphical user interface has two dialogs which can be seen on Figure 3, in Windows 1 and 6. The first one represents database preparing process on application very first run. During this process the Fourier–Legendre coefficients are being loaded into the SQLite database. This screen appears also when user calculates new Fourier–Legendre coefficients. The other screen shows logs during modeling experiment.

4.1.2 Greetings Dialog

After the SDE-MATH software package has completed the database preparation, it shows greeting dialog which represents short information about its purposes. The greeting dialog can be seen on Figure 3 in Window 2.

4.1.3 Main Menu Dialog

In the main menu of the SDE-MATH software package user can choose one of strong numerical schemes for Itô SDEs to perform modeling experiments. The main menu dialog can be seen on Figure 3 in Window 3.

4.1.4 Visualization Tool

It is important to note that the main SDE-MATH software package window has a checkbox in right upper corner which do switching on and off charts window. In any time user can call this window or hide it if it is not needed. The charts window is universal utility for modeling experiments results visualization. This window has few instruments on it. The left side bar contains all curves labels, and control elements for hiding, showing, and deleting curves. On the right side of the window there are plot which draws the curves. The charts window can be seen on Figure 3, it is Window 3.

4.1.5 Data Input Dialogs

Since the software package has options to perform linear Itô SDEs modeling experiments it is necessary to provide user with input fields for numerical data both scalar and matrix. On the other side, for nonlinear Itô SDEs it is necessary to provide symbolic input. The choice of control elements is conditioned by the above obstacles. On Figure 3, and especially in Windows 4 and 5, these input controls can be seen. There are "LineEditWidget" and "TableWidget" which are sufficient to provide input abilities. The topic of input data validation is also important but to be more accurate referenced to user experience rather than to information model, so it will be described further.

4.2 The User Experience and Implementation Results

The above part represents the structure of software package but not the dynamics and user experience of it. Let us discuss the SDE-MATH software package user experience on few examples provided further on Figures 4–36. This examples represent two scenarios of the SDE-MATH software package use.

The database preparation screen is presented on Figure 4. During the database preparation this screen displays informational message and spinning visualizer of process continuation.

The screen that presented on Figure 5 appears every time when software package runs unless user presses "Ok" button with marked checkbox. In such case this message screen will not be shown again.

On Figure 6 the main menu dialog is presented. In this dialog user can choose any strong numerical scheme for Itô SDEs to perform modeling experiment.

The tooltip example can be seen on Figure 7. Such tooltips displayed with characteristic icon are placed all over software package interface to help user with explanations.

As noted earlier, the dedicated charts window is universal tool for visualization. The specific examples of such visualization are presented on Figure 8,19, 34–36.

The initial state of input dialogs for nonlinear and linear Itô SDEs are displayed on Figures 9 and 20. At that moment user can start to input the data.

The example of wrong scalar data input is presented on Figures 10, 15, 21, and 31. When user input wrong data the error message appears and "Next" or "Perform modeling" button is blocked. The input field is being checked all the user data input process, and as soon as wrong character is entered notification pops up.

If scalar data is correct the "Next" button is automatically unblocked. On Figures 11, 16, 22, and 32 the examples of scenario are displayed.

On Figures 12, 13, and 28 the example of correct matrix data input is presented. In this particular case the input is symbolic. Symbolic algebra input errors are much harder to determine so this is done on further stages, in modeling runtime.

In the other case when matrix input data are numerical, the validation is performed right after user has finished input. The examples of incorrect matrix numerical input can be found on Figure 24.

When user finishes input with a success the "Next" or "Perform modeling" button is automatically unblocked. On Figures 23, 25–27, 29, and 30 that can be clearly seen.

The Figures 17, 18, and 33 displays sequence of log messages emerged during the modeling process.

After modeling has been done the focus moves to the charts window where obtained modeling results can be seen. The results of modeling is displayed on Figures 19, 34–36. On Figures 35 and 36 the expectations and variances of obtained components of solution are displayed.

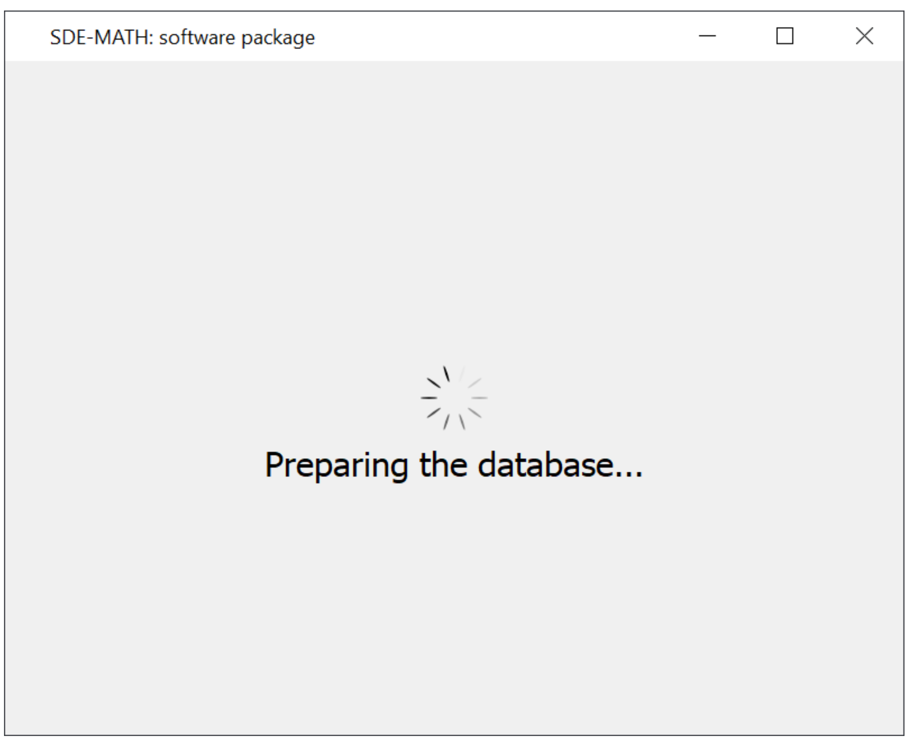


Figure 4: Fourier–Legendre coefficients database preparation screen

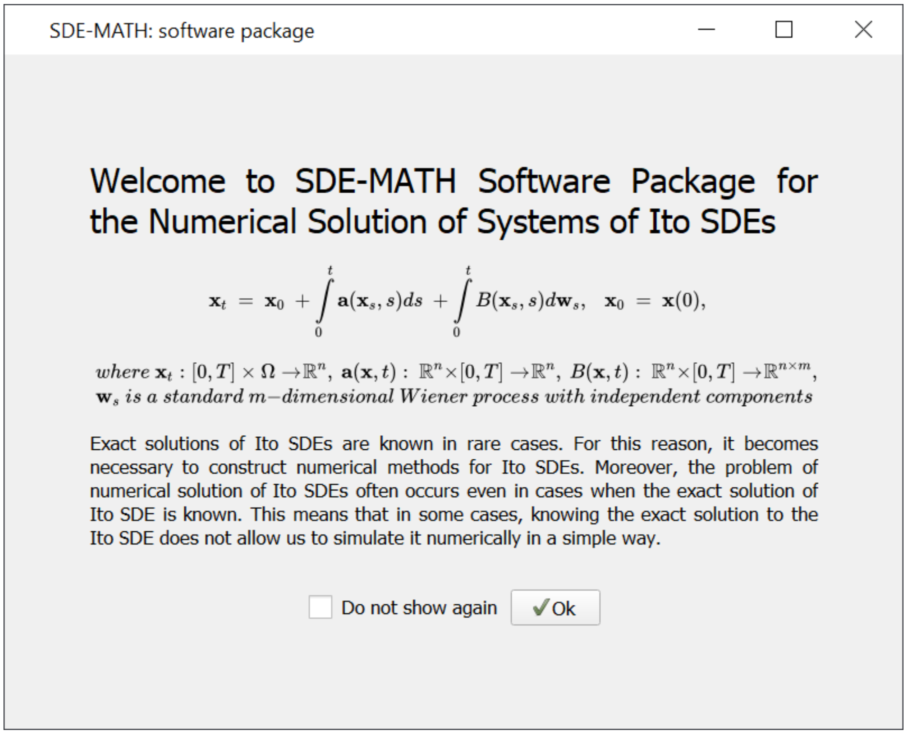


Figure 5: Greetings screen

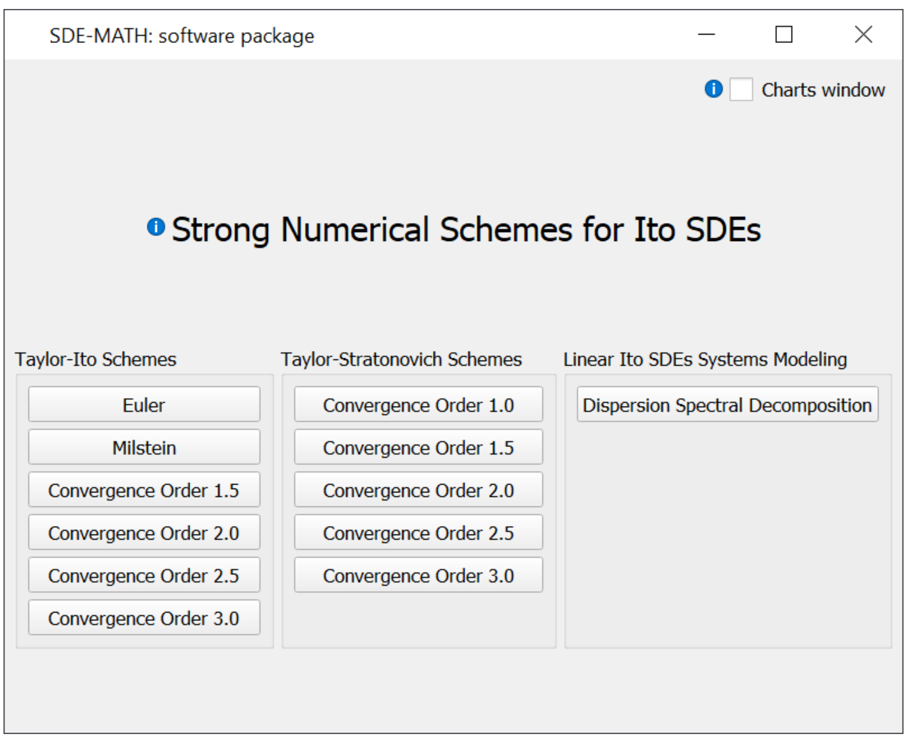


Figure 6: Main menu dialog

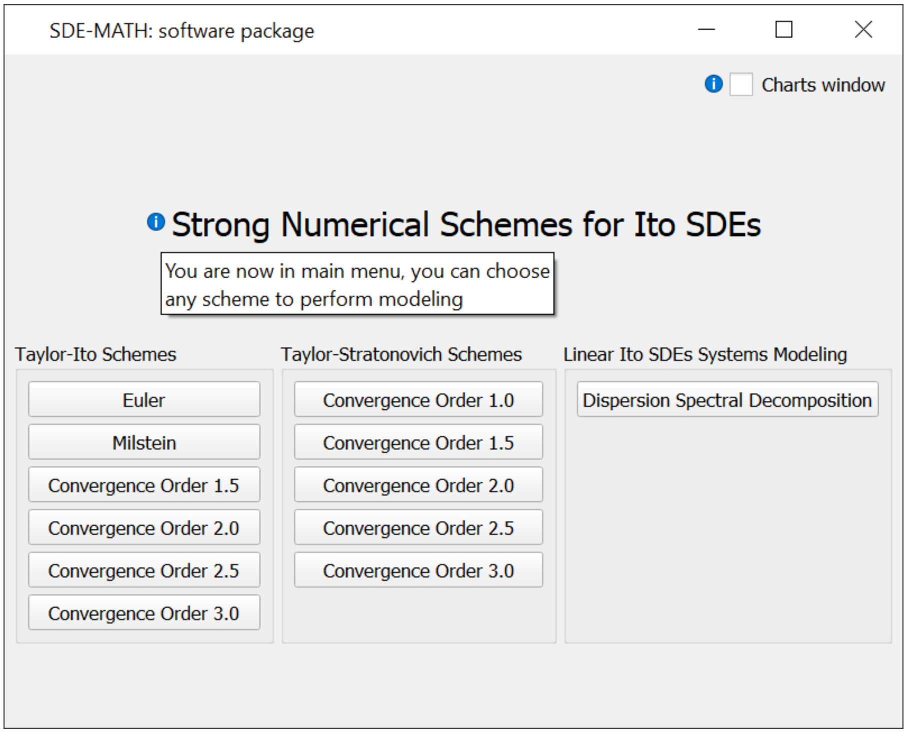


Figure 7: Tooltip

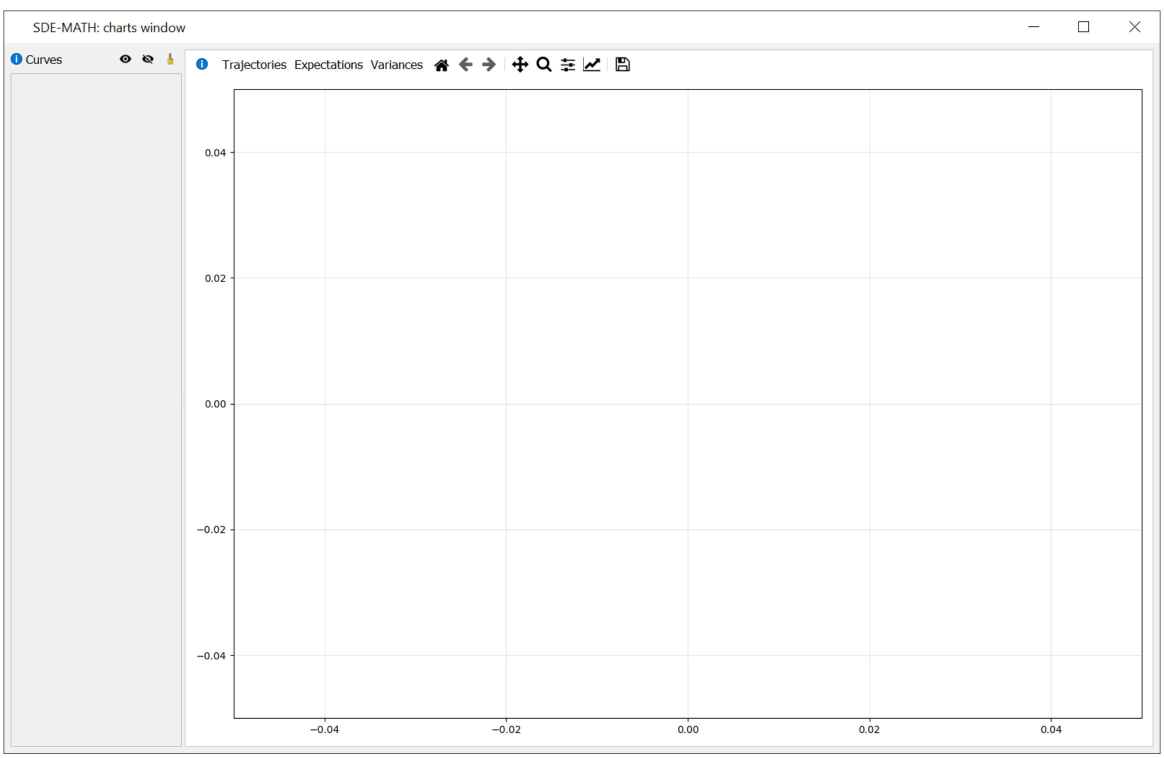


Figure 8: Charts window

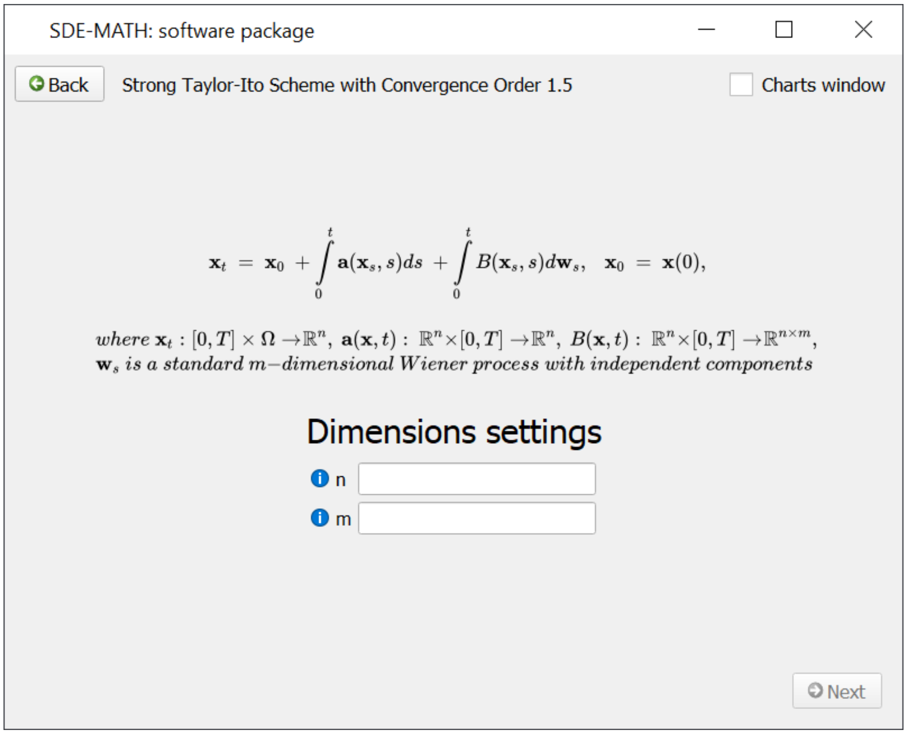


Figure 9: Nonlinear system of Itô SDEs data input

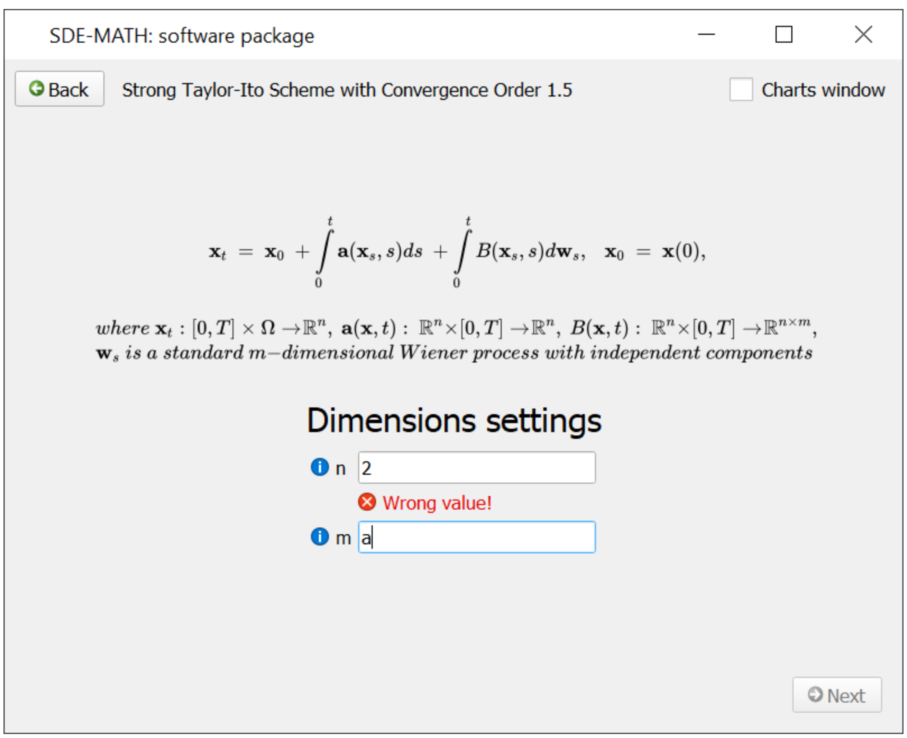


Figure 10: Wrong data input

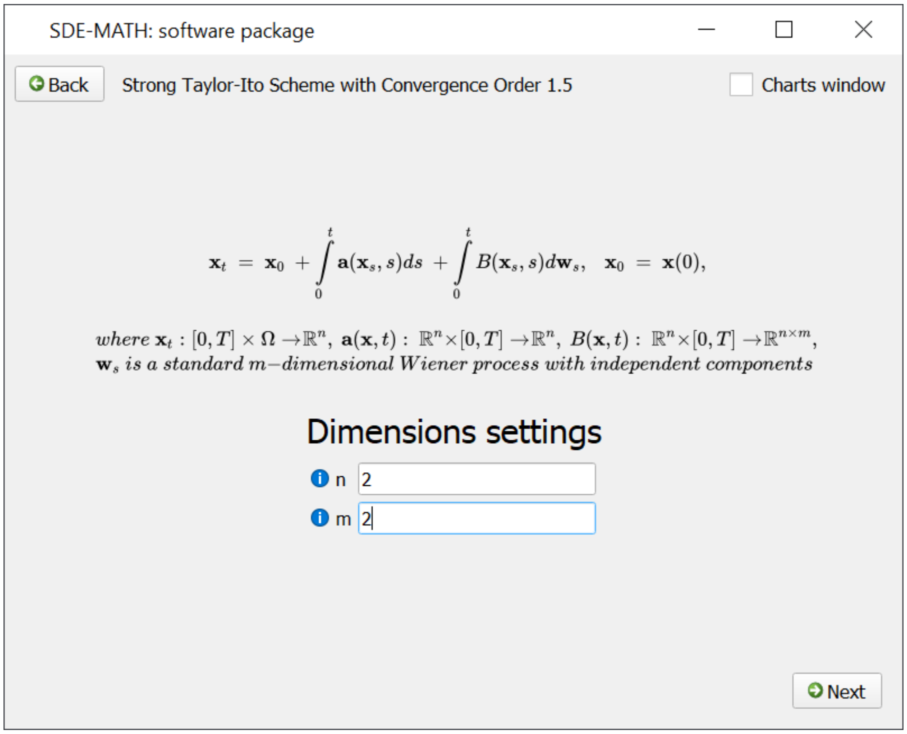


Figure 11: Correct data input

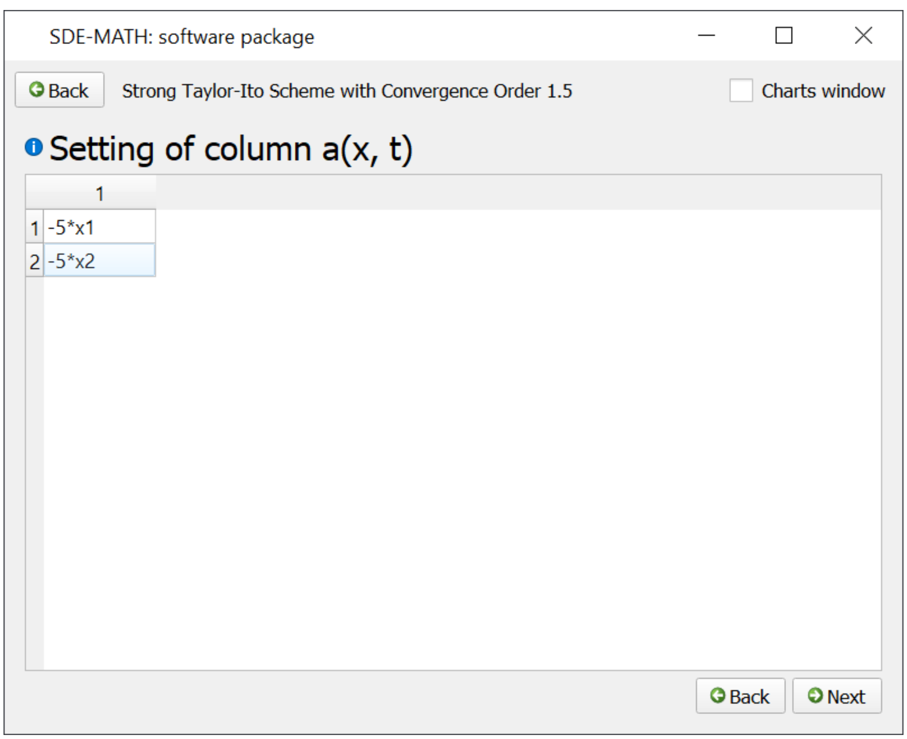


Figure 12: Vector function $\mathbf{a}(\mathbf{x}, t)$ input

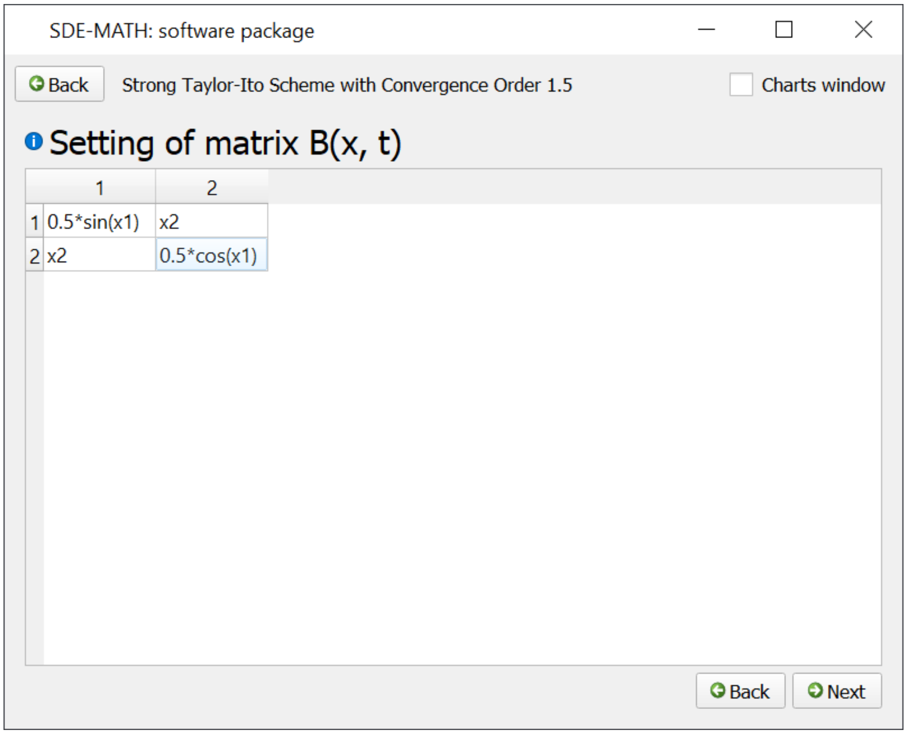


Figure 13: Matrix function $B(\mathbf{x}, t)$ input

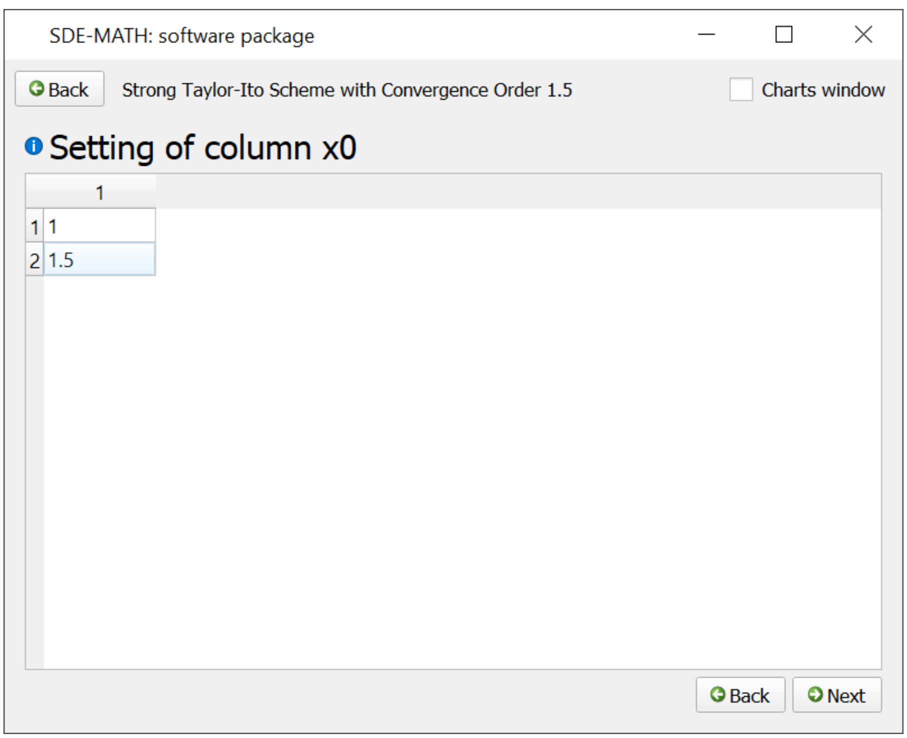


Figure 14: Initial data input

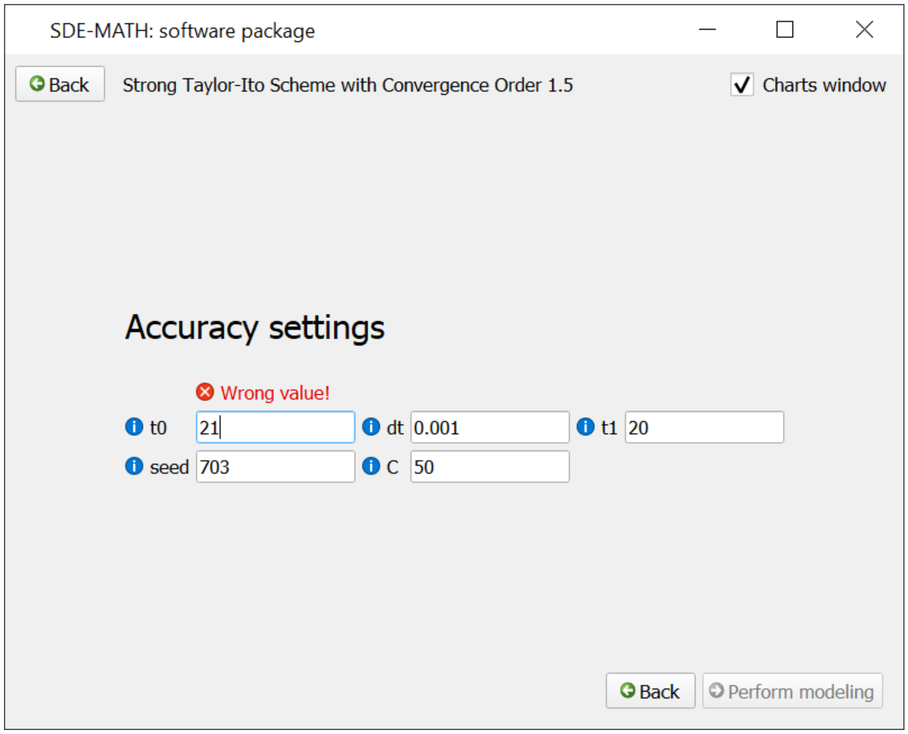


Figure 15: Wrong data input

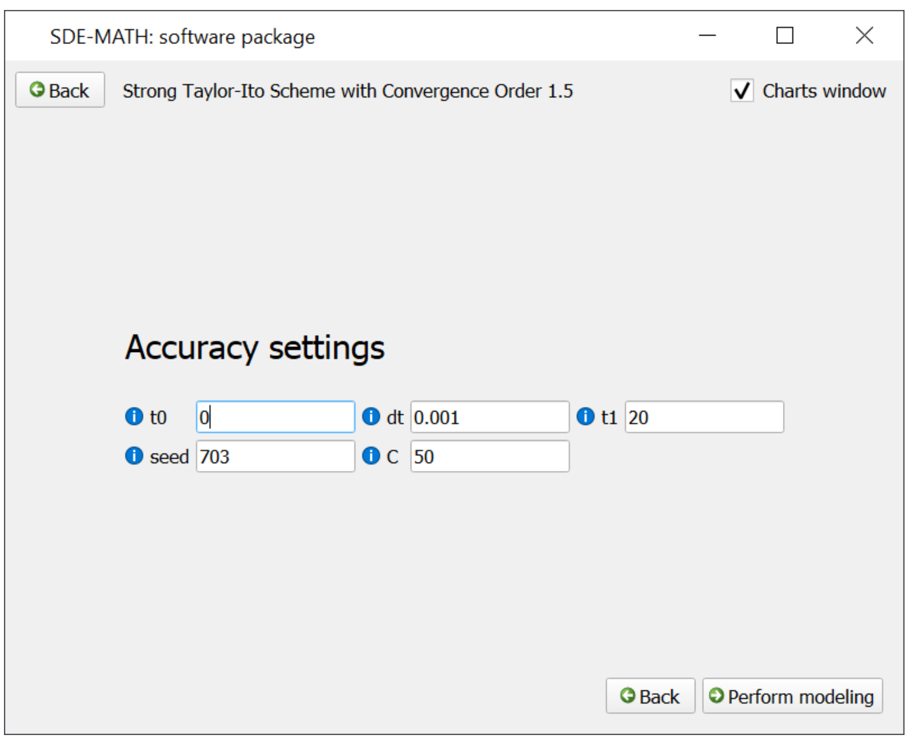


Figure 16: Correct data input

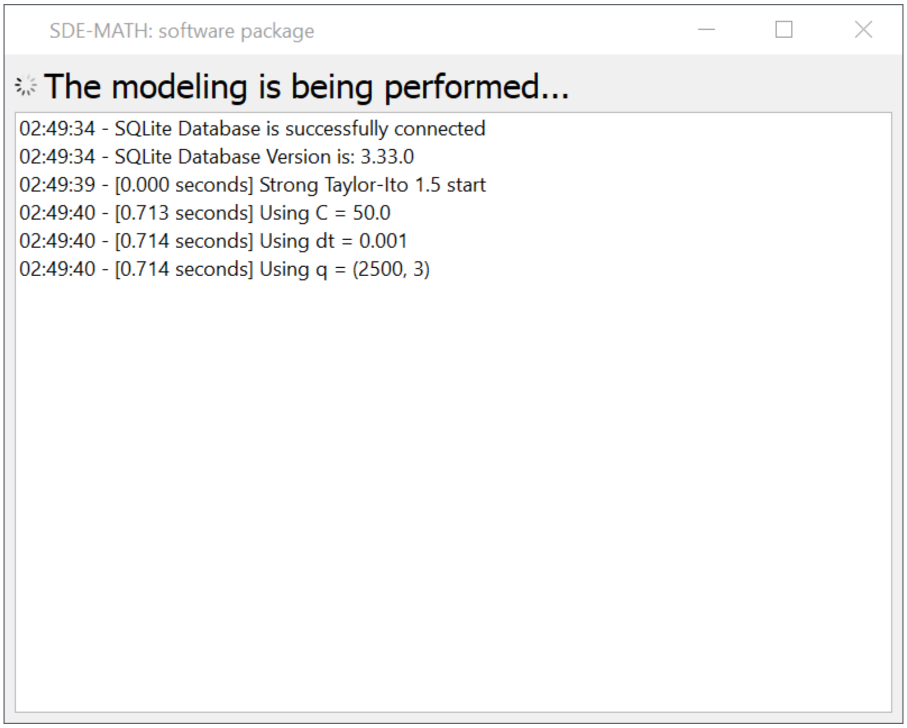


Figure 17: Modeling logs

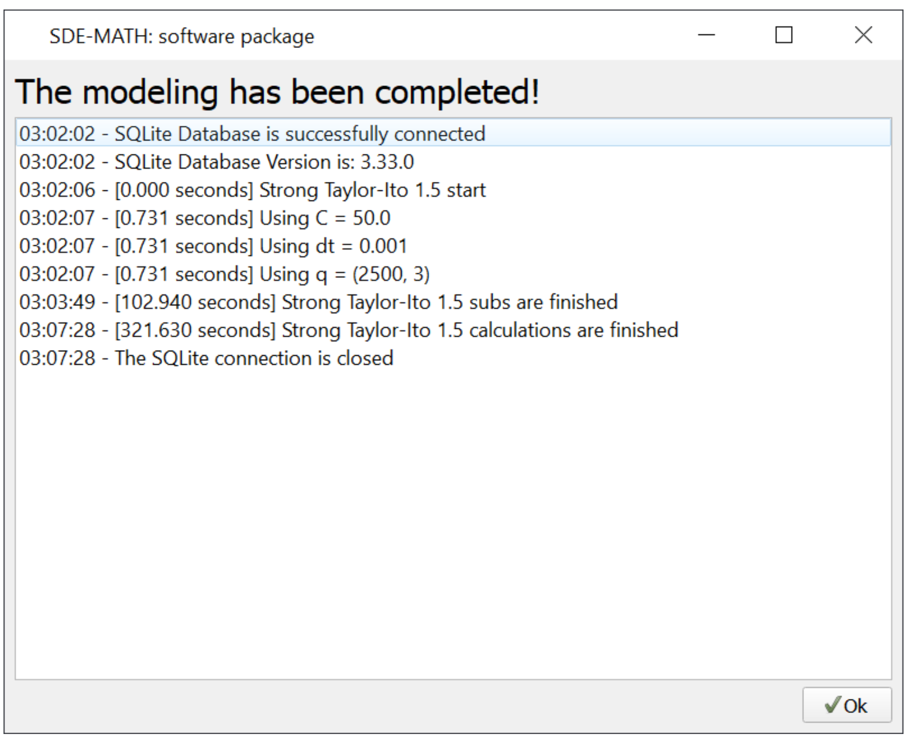


Figure 18: Modeling logs

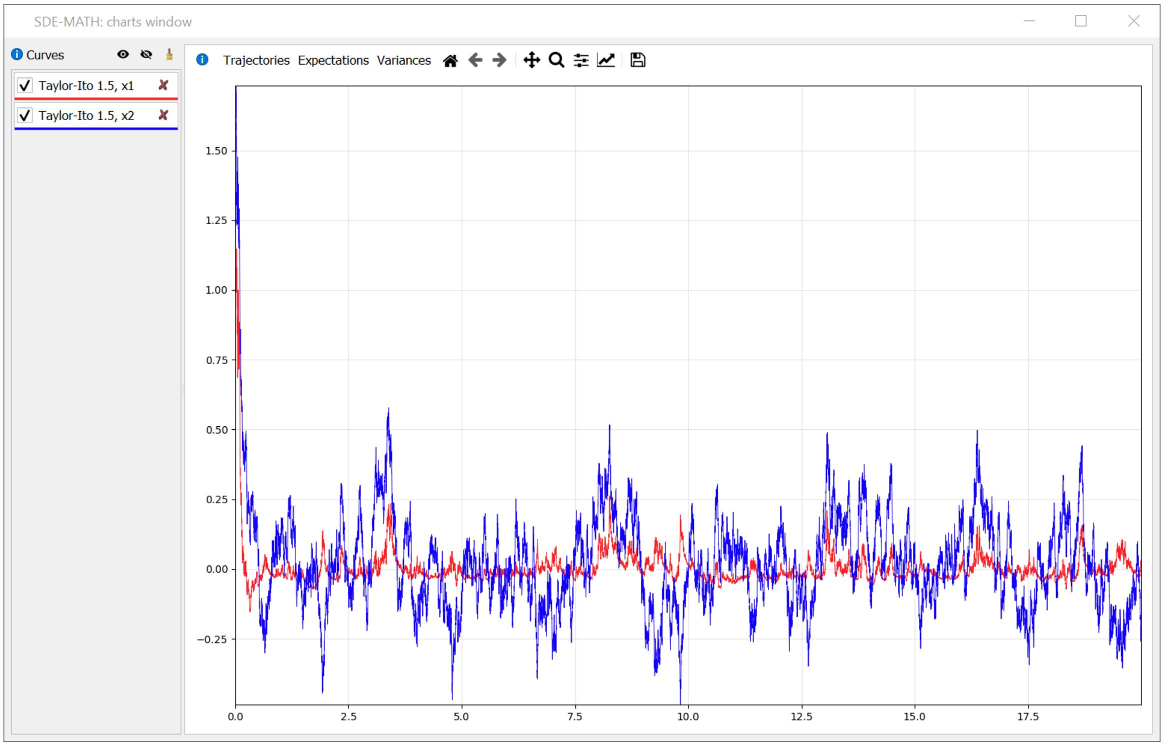


Figure 19: Modeling results

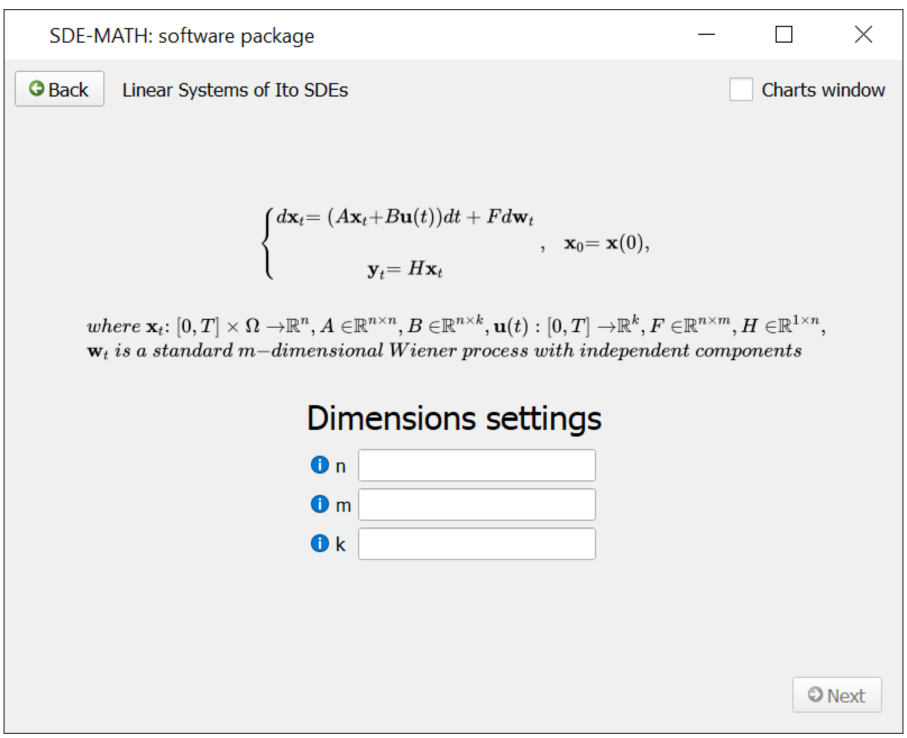


Figure 20: Linear system of Itô SDEs data input

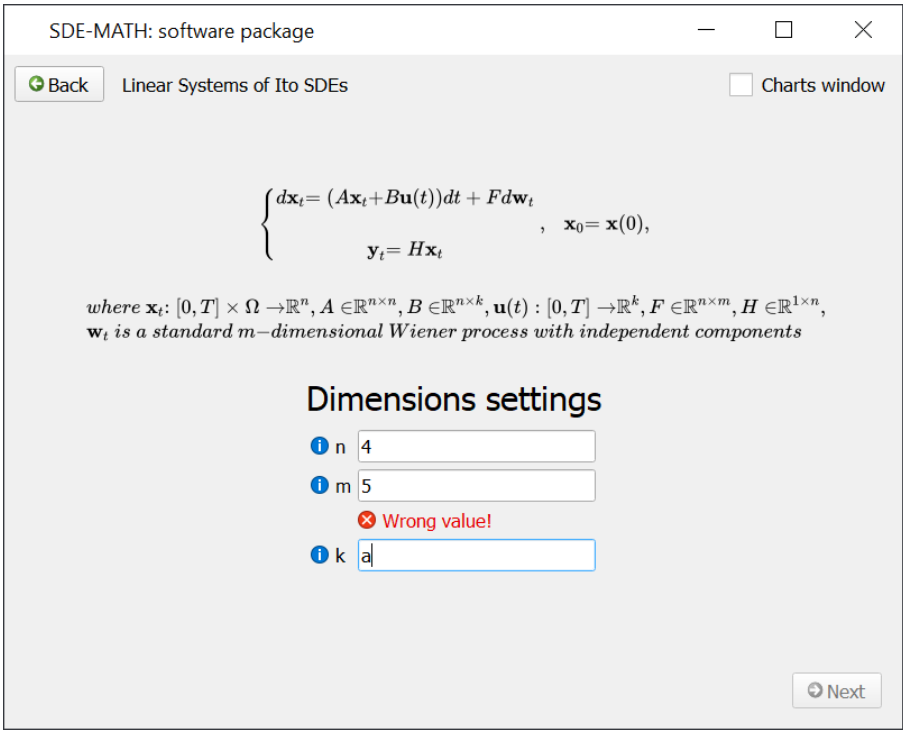


Figure 21: Wrong data input

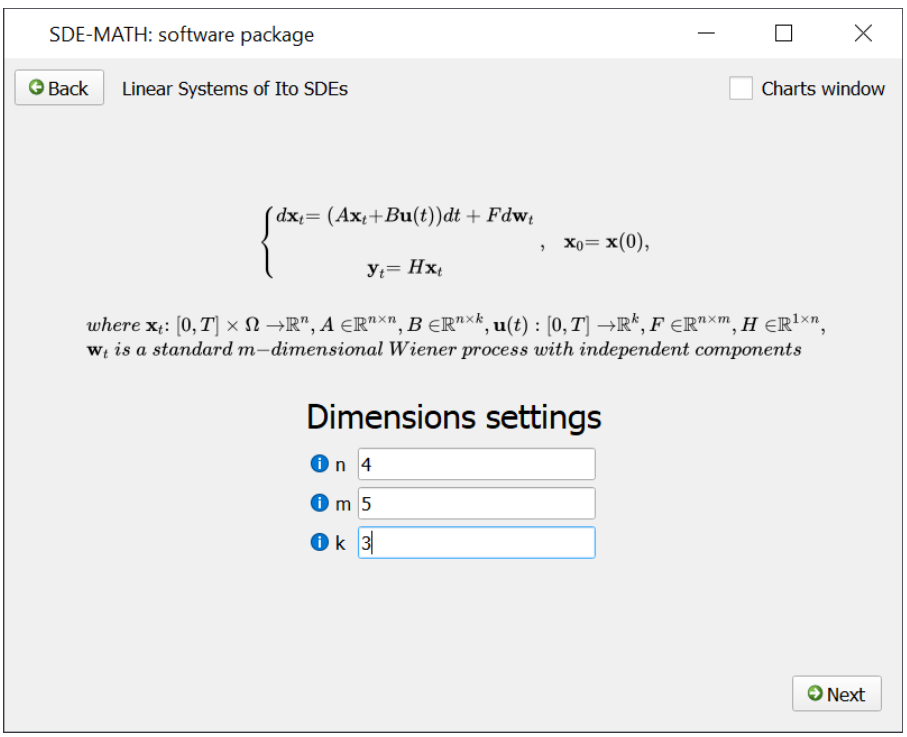


Figure 22: Correct data input

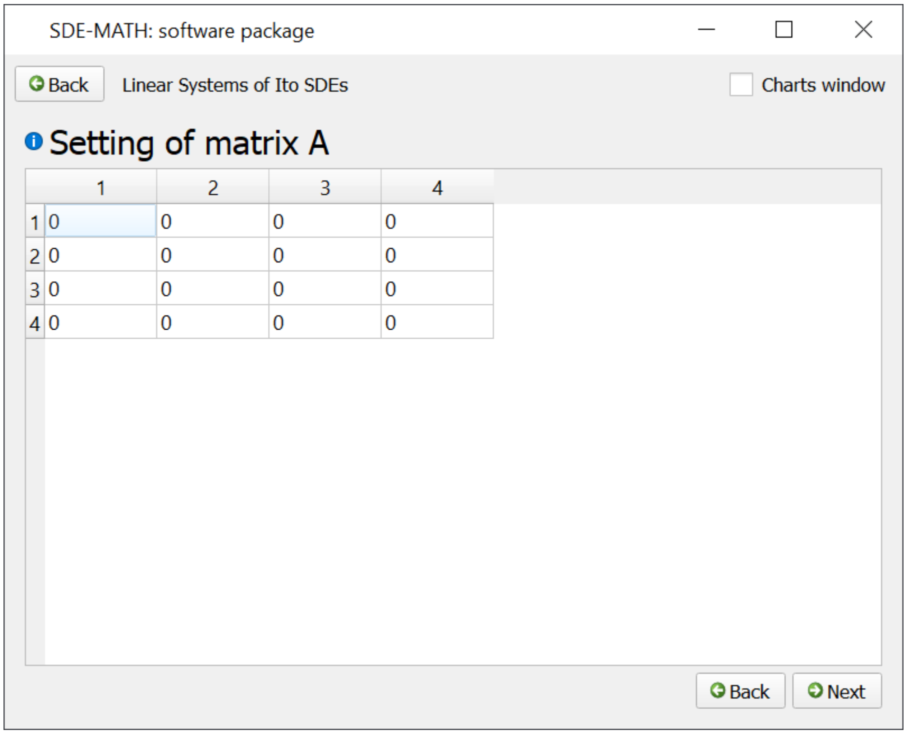


Figure 23: Matrix A input

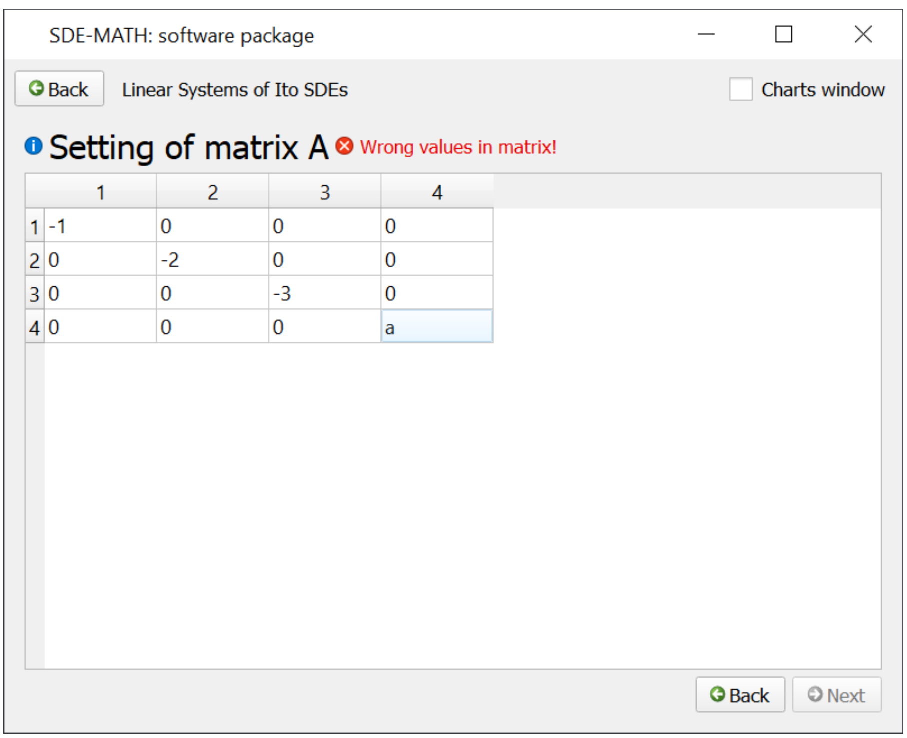


Figure 24: Wrong matrix A input

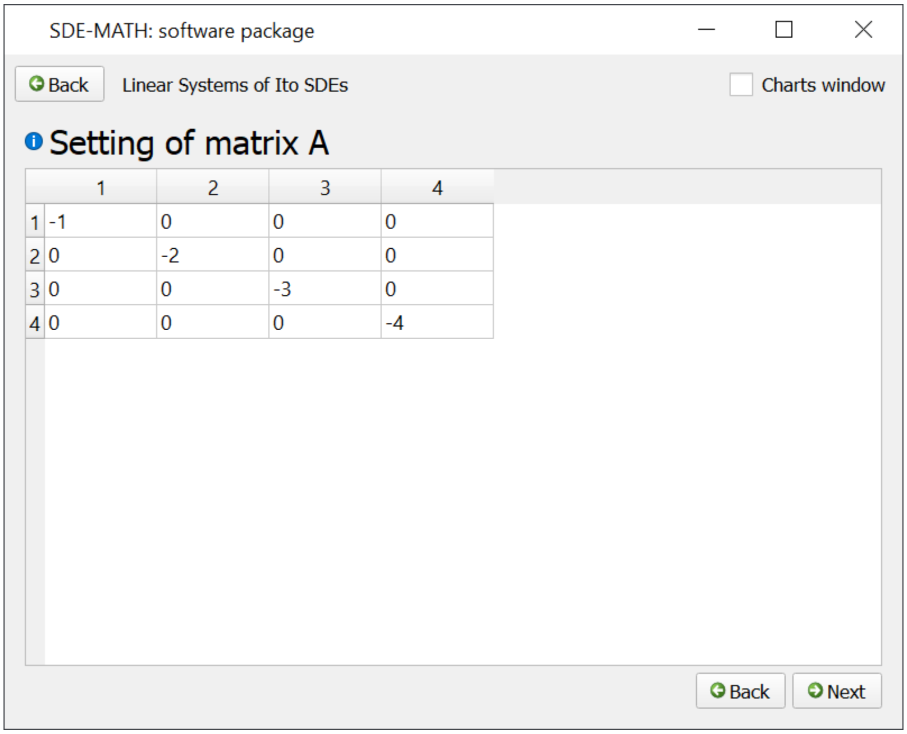


Figure 25: Correct matrix A input

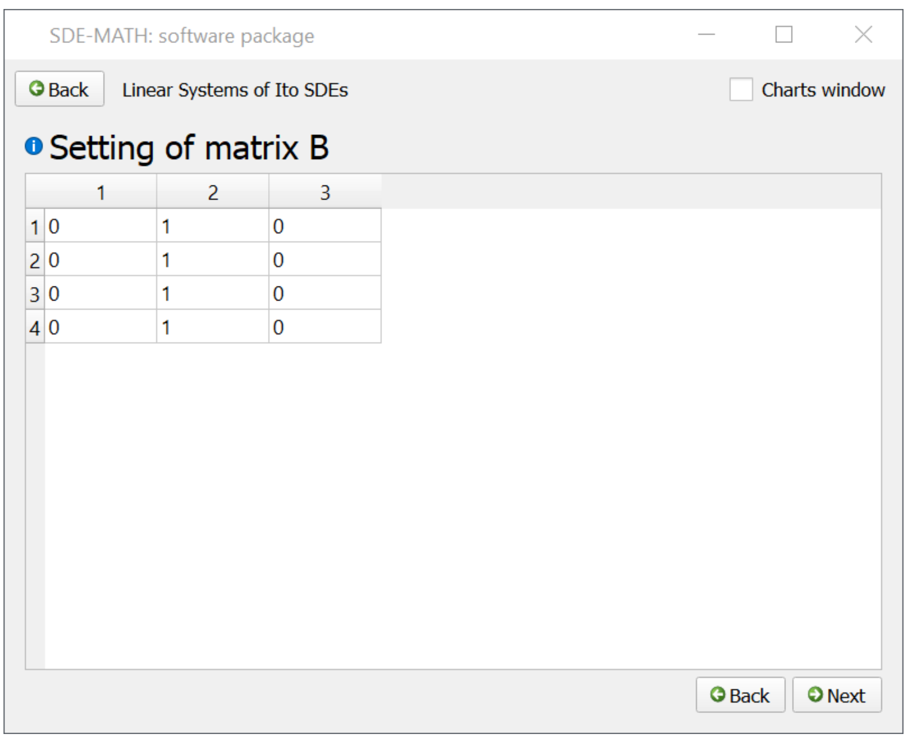


Figure 26: Matrix B input

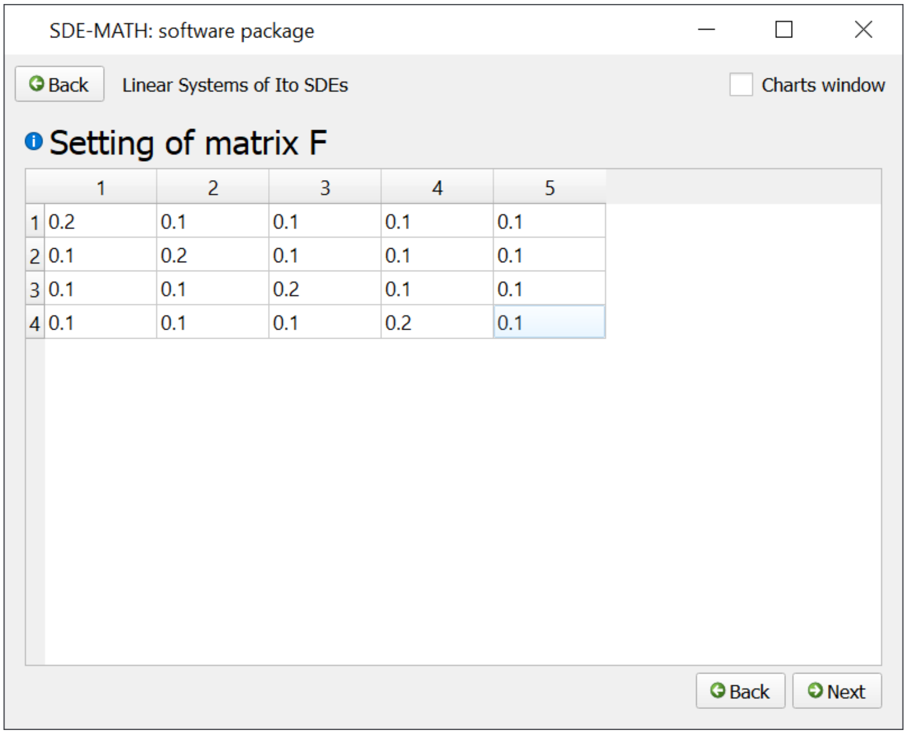


Figure 27: Matrix F input

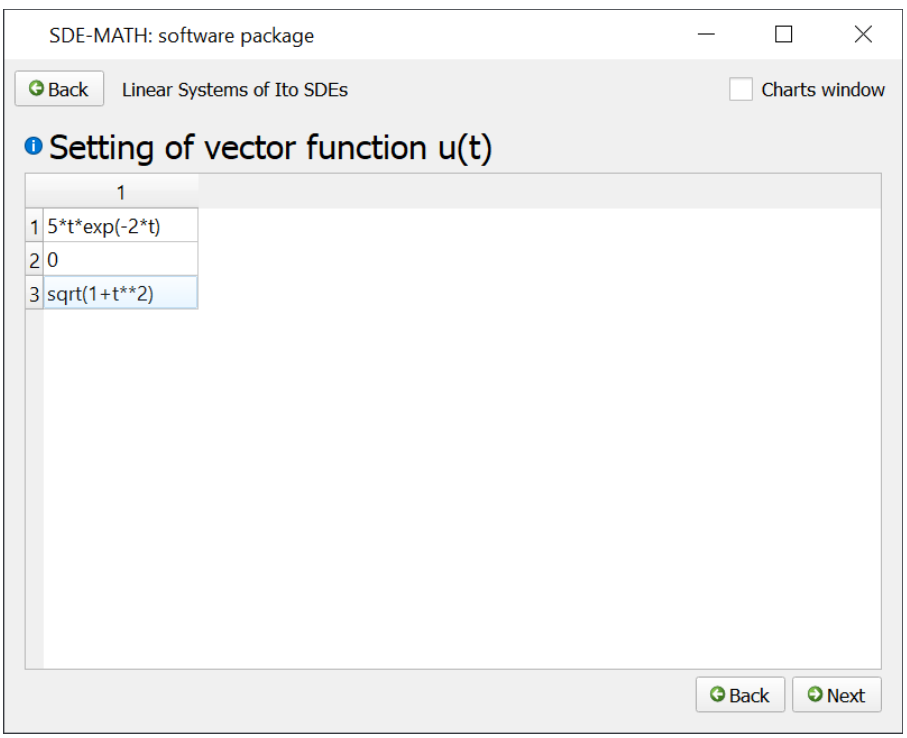


Figure 28: Vector function $\mathbf{u}(t)$ input

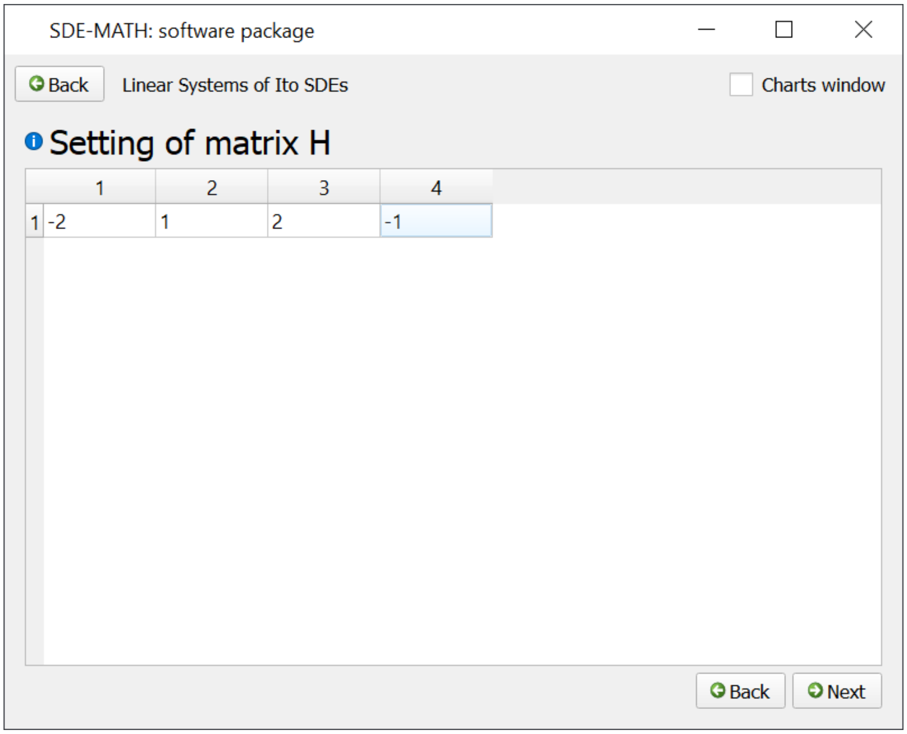


Figure 29: Matrix H input

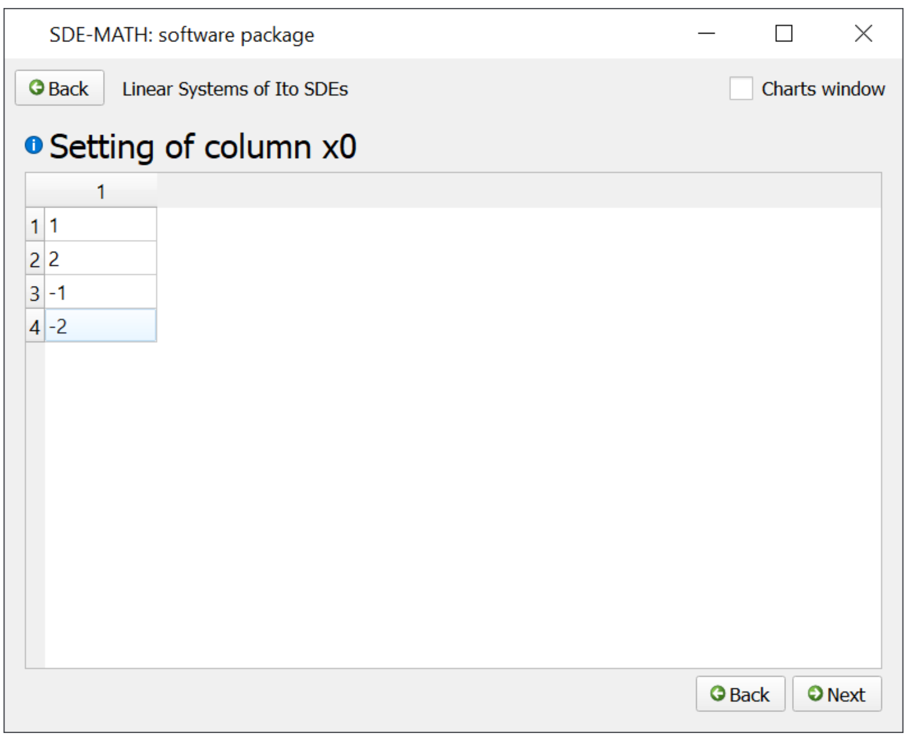


Figure 30: Initial data input

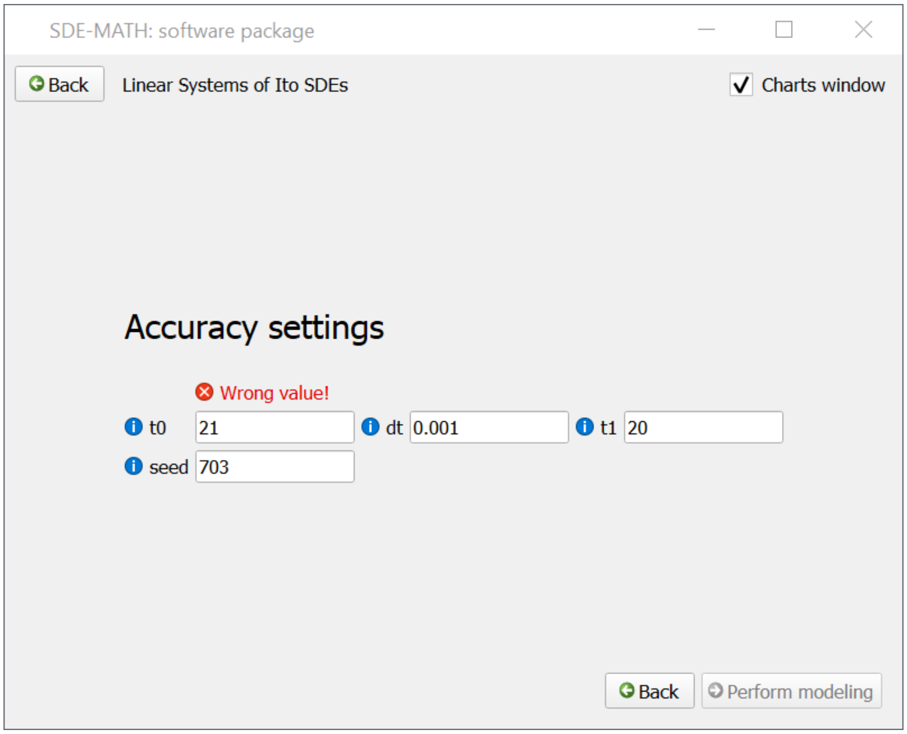


Figure 31: Wrong data input

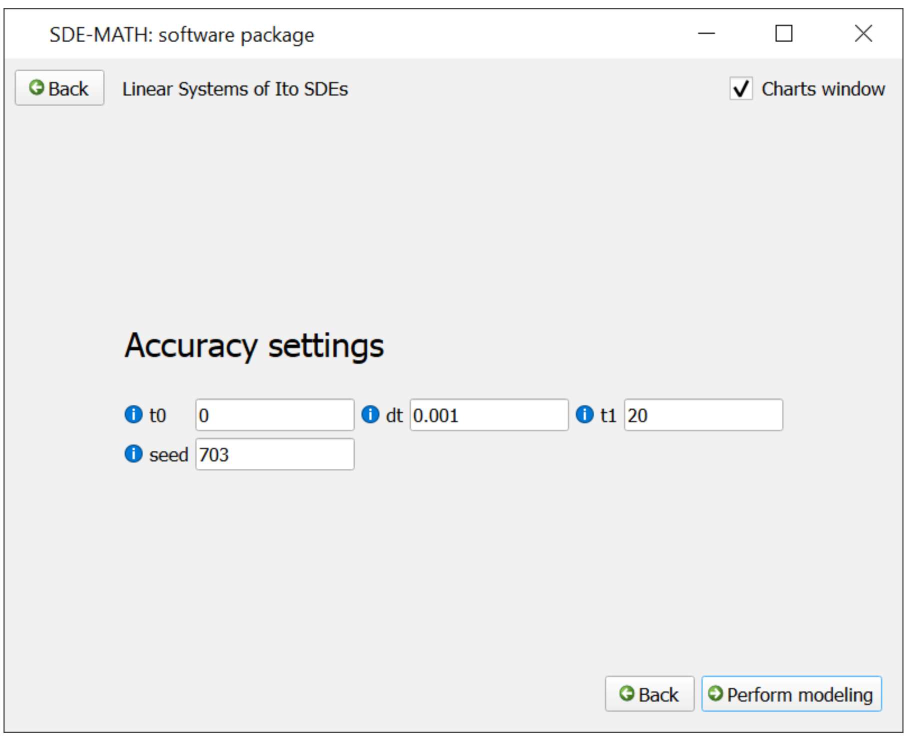


Figure 32: Correct data input

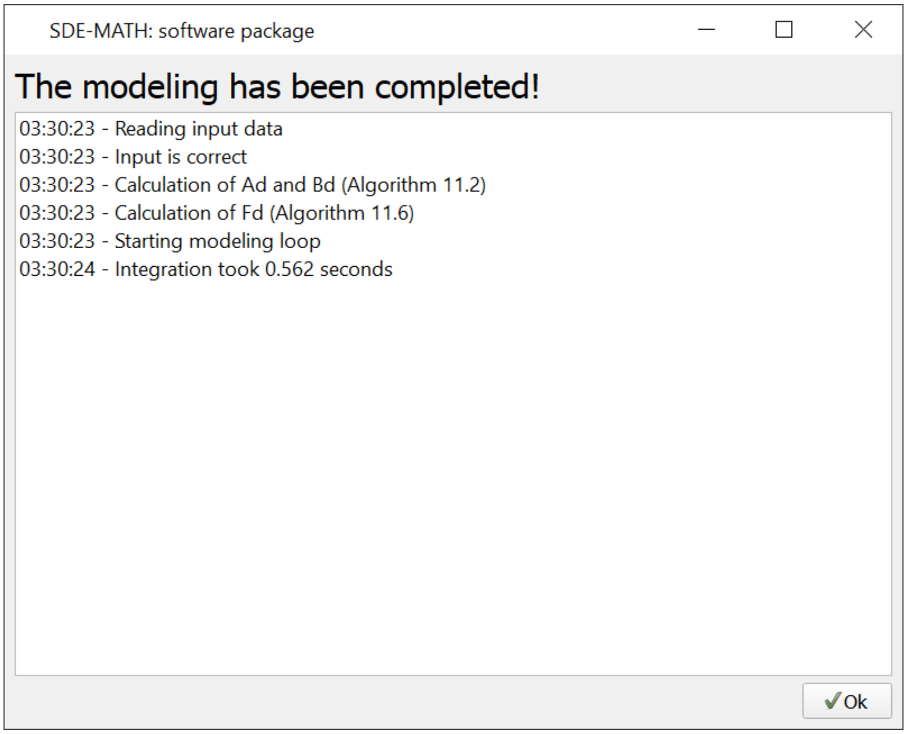


Figure 33: Modeling logs

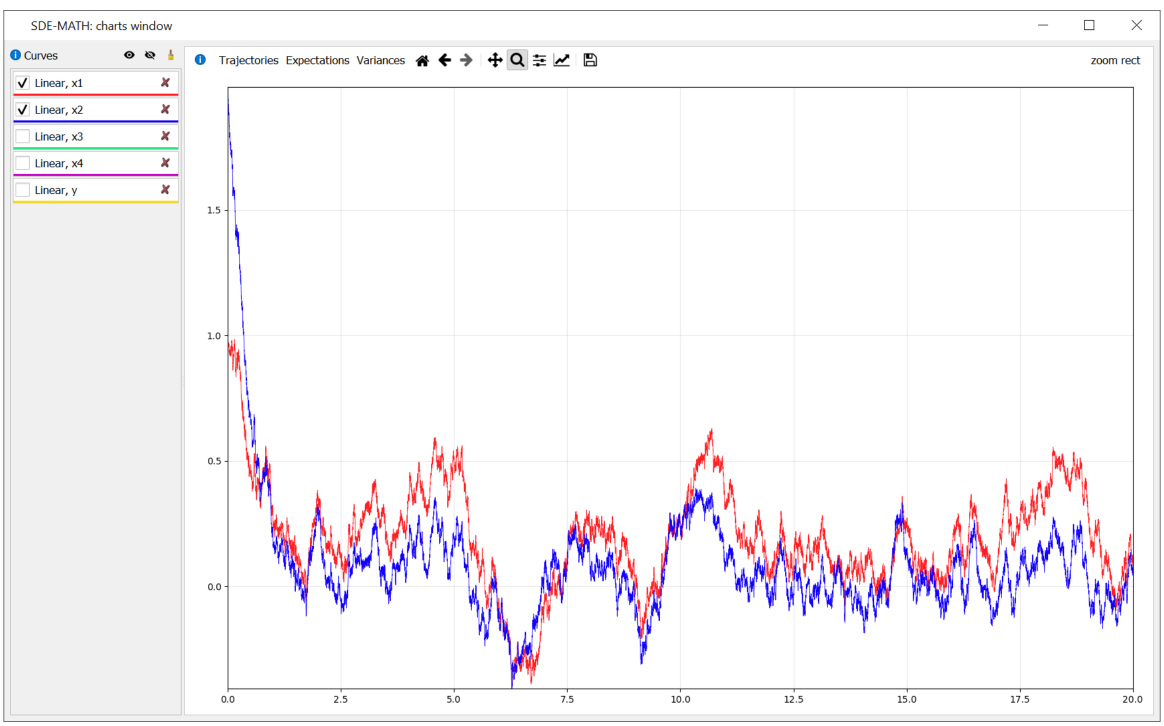


Figure 34: Modeling results (components of solution)

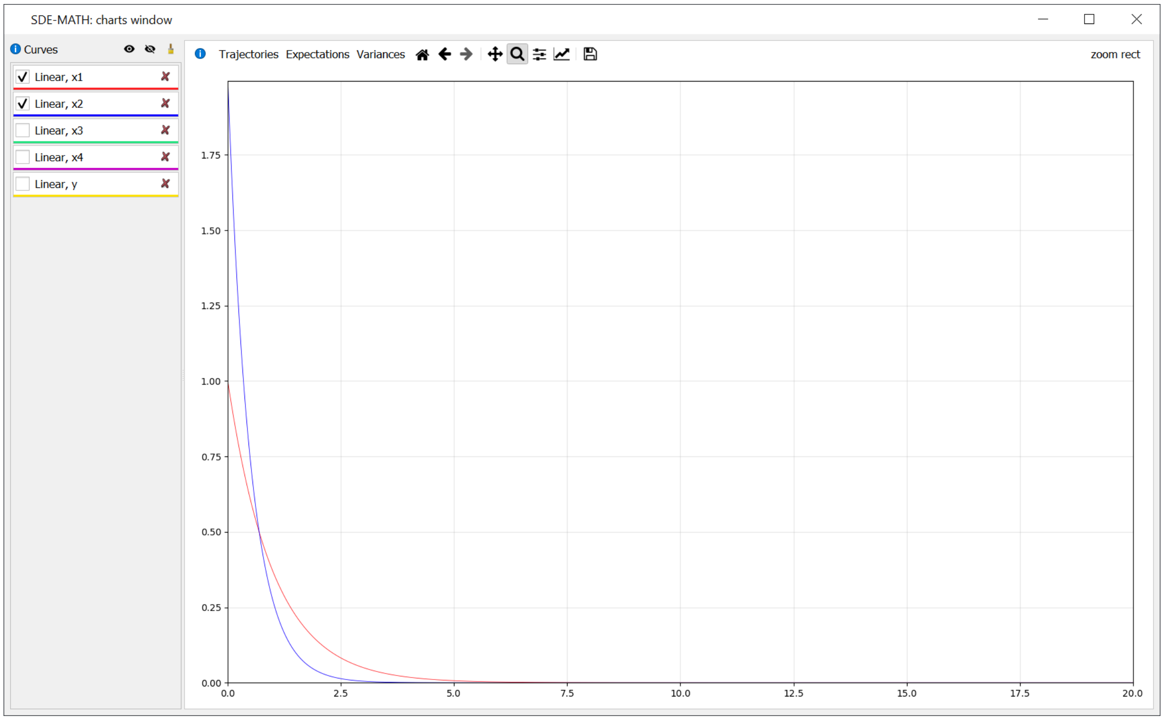


Figure 35: Modeling results (expectations)

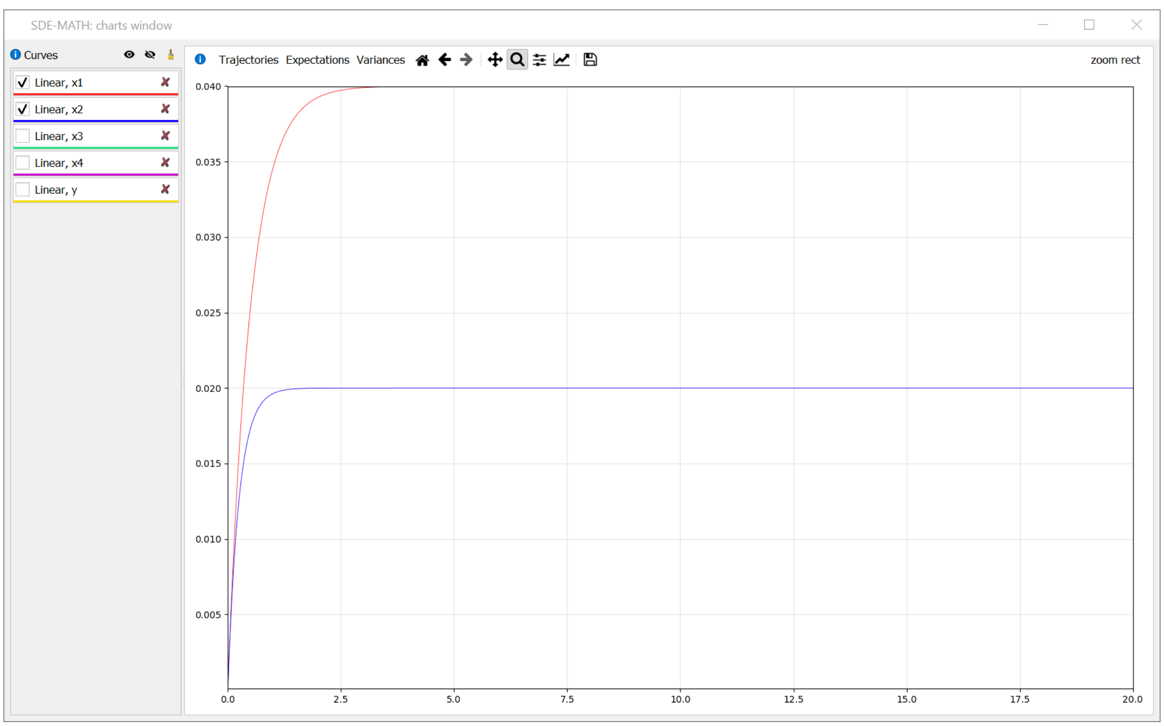


Figure 36: Modeling results (variances)

5 The Results Obtained Using the SDE-MATH Software Package

This section represents the results that were obtained with the SDE-MATH software package at the current stage of the development.

5.1 The Calculated Fourier–Legendre Coefficients

When application runs first time it performs loading of Fourier–Legendre coefficients basic pack in the database from the files. Further, in Listings 1–4 few examples of them can be seen.

Listing 1: **The Fourier–Legendre coefficients $C^{000}_{j_3j_2j_1}$ examples**

```
C_0:0:0 = 4/3
C_0:0:1 = -2/3
C_0:0:2 = 2/15
C_0:0:3 = 0
...
C_0:6:4 = -4/429
C_0:6:5 = 2/143
C_0:6:6 = 2/2145
C_1:0:0 = 2/3
...
C_47:33:44 = 3874457388633368/31334948307735906710660485
C_47:33:45 = 0
C_47:33:46 = 52892292737827468/2224781329849249376456894435
C_47:34:0 = 0
C_47:34:1 = 0
```

Listing 2: **The Fourier–Legendre coefficients $C^{0000}_{j_4j_3j_2j_1}$ examples**

```
C_0:0:0:0 = 2/3
C_0:0:0:1 = -2/5
C_0:0:0:2 = 2/15
C_0:0:1:0 = -2/15
C_0:0:1:1 = 2/15
...
C_1:1:0:1 = -2/35
C_1:1:0:2 = 0
C_1:1:1:0 = 2/105
C_1:1:1:1 = 0
...
C_20:20:20:1 = -2401828/165607444685315115
C_20:20:20:2 = 0
C_20:20:20:3 = -1241929832/77669891557412788935
C_20:20:20:4 = 0
```

Listing 3: **The Fourier–Legendre coefficients $C_{j_5 j_4 j_3 j_2 j_1}^{00000}$ examples**

```
C_0:0:0:0:0 = 4/15
C_0:0:0:0:1 = −8/45
C_0:0:0:1:0 = −4/45
C_0:0:0:1:1 = 8/105
C_0:0:1:0:0 = 0
C_0:0:1:0:1 = 4/315
...
C_1:1:0:1:0 = −4/315
C_1:1:0:1:1 = 4/315
C_1:1:1:0:0 = 2/105
C_1:1:1:0:1 = −8/945
C_1:1:1:1:0 = 2/945
C_1:1:1:1:1 = 0
...
C_20:20:20:20:1 = 0
C_20:20:20:20:2 = −249877207023610010/10969028984480026752856704371
C_20:20:20:20:3 = 0
C_20:20:20:20:4 = −307937246954575016/571102494076952592887484313076115
```

Listing 4: **The Fourier–Legendre coefficients $C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000}$ examples**

```
C_0:0:0:0:0:0 = 4/45
C_0:0:0:0:0:1 = −4/63
C_0:0:0:0:0:2 = 2/63
C_0:0:0:0:1:0 = −4/105
...
C_2:1:0:1:0:2 = −2/1575
C_2:1:0:1:1:0 = 38/22275
C_2:1:0:1:1:1 = −2/1575
C_2:1:0:1:1:2 = 68/81081
...
C_15:15:15:15:15:15 = 0
C_15:15:15:15:15:16 = −798538765964/243076352242280511713913783475
C_15:15:15:15:15:17 = 0
C_15:15:15:15:15:18 = −59075427603328/17302616709609603697454044769175
```

5.2 Accuracy Settings

From Theorem 7 (see formulas (63)–(85)) it follows that the number p in the formula (60) should be chosen individually for various combinations of indices $i_1, \ldots, i_k \in \{1, \ldots, m\}$. As follows from Listing 5 (see below) and the results of work [74], these numbers p in the overwhelming majority of cases do not exceed the number p from the formula (63). Moreover, all the mentioned numbers p are many times less than the number p selected using the formula (61) (due to the presence of the multiplier factor $k!$ on the left-hand side of (61)).

In this work, we have replaced the mentioned numbers p for all possible combinations of indices $i_1, \ldots, i_k \in \{1, \ldots, m\}$ with the number p according to the formula (63). This is possible due to the results of Listing 6. This listing shows that the above replacement does not lead to noticeable accuracy loss of the mean square approximation of iterated Itô stochastic integrals (for more details see [74]).

Thus, in this paper we decided to exclude the multiplier factor $k!$ in the conditions for choosing the numbers $q_1, \ldots, q_{15}$ (see (188)–(221)). Recall that these numbers are used to construct the approximations of iterated Itô and Stratonovich stochastic integrals from the numerical schemes (13)–(16), (25)–(28). The test script was written. The results of its work are presented in Listings 5 and 6, where

1. dt is the integration step;
2. q1(1,2) means p from (66), q1(2,3) means p from (67), q1(1,3) means p from (68), q1 means p from (63) for $k = 3$;
3. $C = 1$ (see (17) and (29));
4. error 1 means the left-hand side of (190);
5. error 2 means the left-hand side of (66) divided by $(T - t)^3$;
6. error 3 means the left-hand side of (68) divided by $(T - t)^3$;
7. error 4 means the left-hand side of (67) divided by $(T - t)^3$.

The above idea of calculation of the numbers $q_1, \ldots, q_{15}$ is described in Listing 109.

Listing 5: **Accuracy calculation module**

```

dt = 0.011
   q1      = 12
   q1 (1, 2) = 6
   q1 (1, 3) = 12
   q1 (2, 3) = 6

dt = 0.008
   q1      = 16
   q1 (1, 2) = 8
   q1 (1, 3) = 16
   q1 (2, 3) = 8
```

```

dt = 0.0045
  q1      = 28
  q1 (1, 2) = 14
  q1 (1, 3) = 28
  q1 (2, 3) = 14

dt = 0.0035
  q1      = 36
  q1 (1, 2) = 18
  q1 (1, 3) = 36
  q1 (2, 3) = 18

dt = 0.0027
  q1      = 47
  q1 (1, 2) = 23
  q1 (1, 3) = 47
  q1 (2, 3) = 23

dt = 0.0025
  q1      = 50
  q1 (1, 2) = 25
  q1 (1, 3) = 51
  q1 (2, 3) = 25


Process finished with exit code 0
```

Listing 6: **Accuracy calculation module**

```

dt = 0.011
  error 1 = 0.010153888451696458
  q1      = 12
  error 2 = 0.005076944225848201
  q1 (1, 2) = 12
  error 3 = 0.010307776903394072
  q1 (1, 3) = 12
  error 4 = 0.005076944225848284
  q1 (2, 3) = 12

dt = 0.008
  error 1 = 0.007681193827577537
  q1      = 16
  error 2 = 0.003840596913789046
  q1 (1, 2) = 16
  error 3 = 0.0077866300793989485
  q1 (1, 3) = 16
  error 4 = 0.003840596913789157
  q1 (2, 3) = 16

dt = 0.0045
```

```
  error 1 = 0.004432832059862973
  q1     = 28
  error 2 = 0.0022164160299319446
  q1 (1, 2) = 28
  error 3 = 0.004479699207443705
  q1 (1, 3) = 28
  error 4 = 0.002216416029932139
  q1 (2, 3) = 28

dt = 0.0035
  error 1 = 0.0034564405520411956
  q1     = 36
  error 2 = 0.0017282202760207088
  q1 (1, 2) = 36
  error 3 = 0.003488223569838411
  q1 (1, 3) = 36
  error 4 = 0.0017282202760210141
  q1 (2, 3) = 36

dt = 0.0027
  error 1 = 0.0026523659377455377
  q1     = 47
  error 2 = 0.00132618296887127
  q1 (1, 2) = 47
  error 3 = 0.0026731529281250332
  q1 (1, 3) = 47
  error 4 = 0.0013261829688714366
  q1 (2, 3) = 47

dt = 0.0025
  error 1 = 0.002494053620431952
  q1     = 50
  error 2 = 0.0012470268102122428
  q1 (1, 2) = 50
  error 3 = 0.0025128597161119537
  q1 (1, 3) = 50
  error 4 = 0.0012470268102123538
  q1 (2, 3) = 50


Process finished with exit code 0
```

5.3 Testing Example (Nonlinear System of Itô SDEs)

The input data for testing of the SDE-MATH software package correspond to the autonomous variant of nonlinear system of Itô SDE (1) with multidimensional non-commutative noise. More precisely, we choose $n = 2$, $m = 2$, $\mathbf{x}_0^{(1)} = 1$, $\mathbf{x}_0^{(2)} = 1.5$,

$$\mathbf{a}\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right)=\begin{pmatrix}-5\mathbf{x}^{(1)}\\ \\ -5\mathbf{x}^{(2)}\end{pmatrix},$$

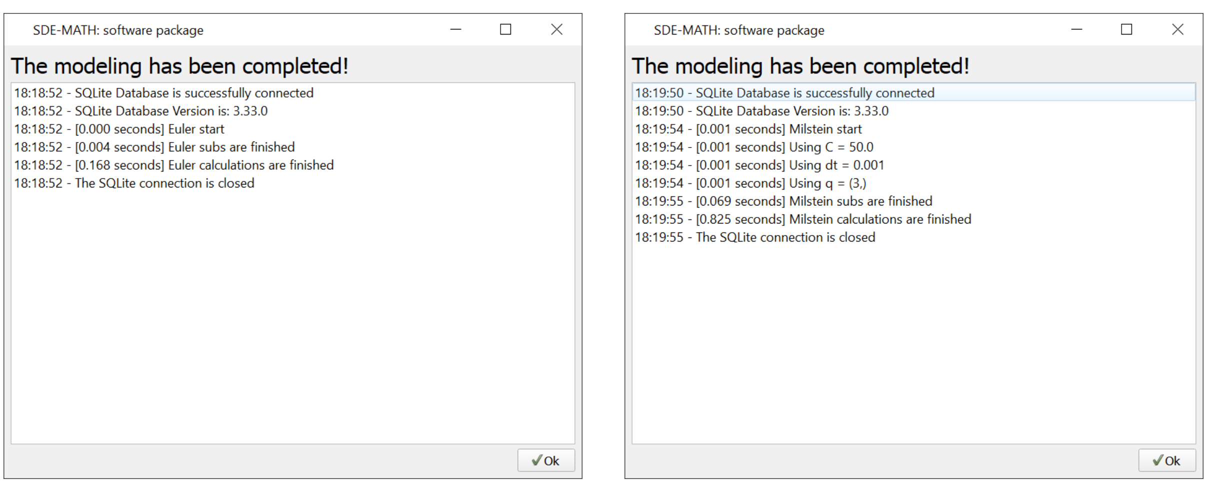
$$B\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right)=\begin{pmatrix}0.5\cdot\sin\left(\mathbf{x}^{(1)}\right) & \mathbf{x}^{(2)}\\ \\ \mathbf{x}^{(2)} & 0.5\cdot\cos\left(\mathbf{x}^{(1)}\right)\end{pmatrix}.$$

Figures 37–92 related to the strong high-order Taylor–Itô and Taylor–Stratonovich schemes (12)–(16), (24)–(28) for the Itô SDE (1) represent modeling results.

Test machine specifications are CPU with maximum core frequency 4.2 GHz and 16GB of RAM.

5.4 Visualization and Numerical Results for Nonlinear System of Itô SDEs Obtained via the SDE-MATH Software Package

This subsection is fully devoted to modeling logs and results visualization. They are presented on Figures 37–91.



Euler scheme ($dt = 0.001$) Milstein scheme ($C = 50$, $dt = 0.001$)

Figure 37: Modeling logs

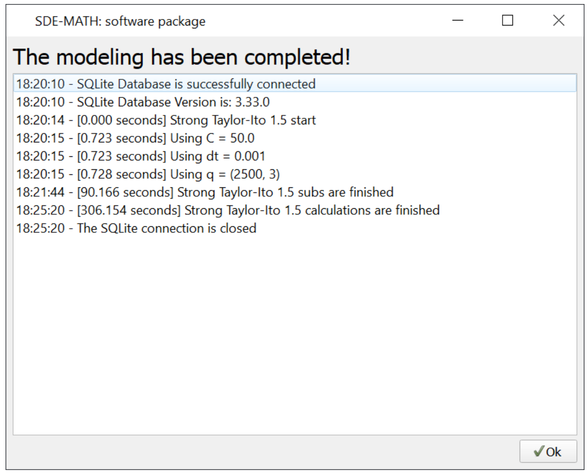


Figure 38: Strong Taylor–Itô scheme of order 1.5 ($C = 50$, $dt = 0.001$)

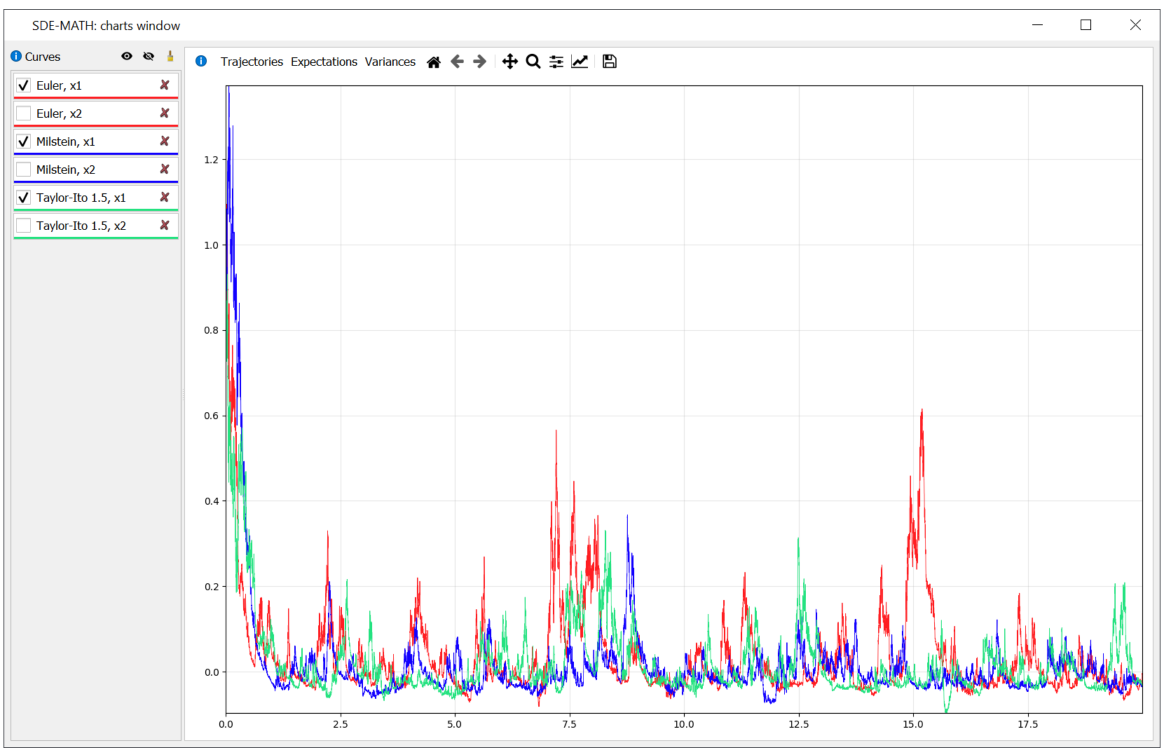


Figure 39: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 50$, $dt = 0.001$)

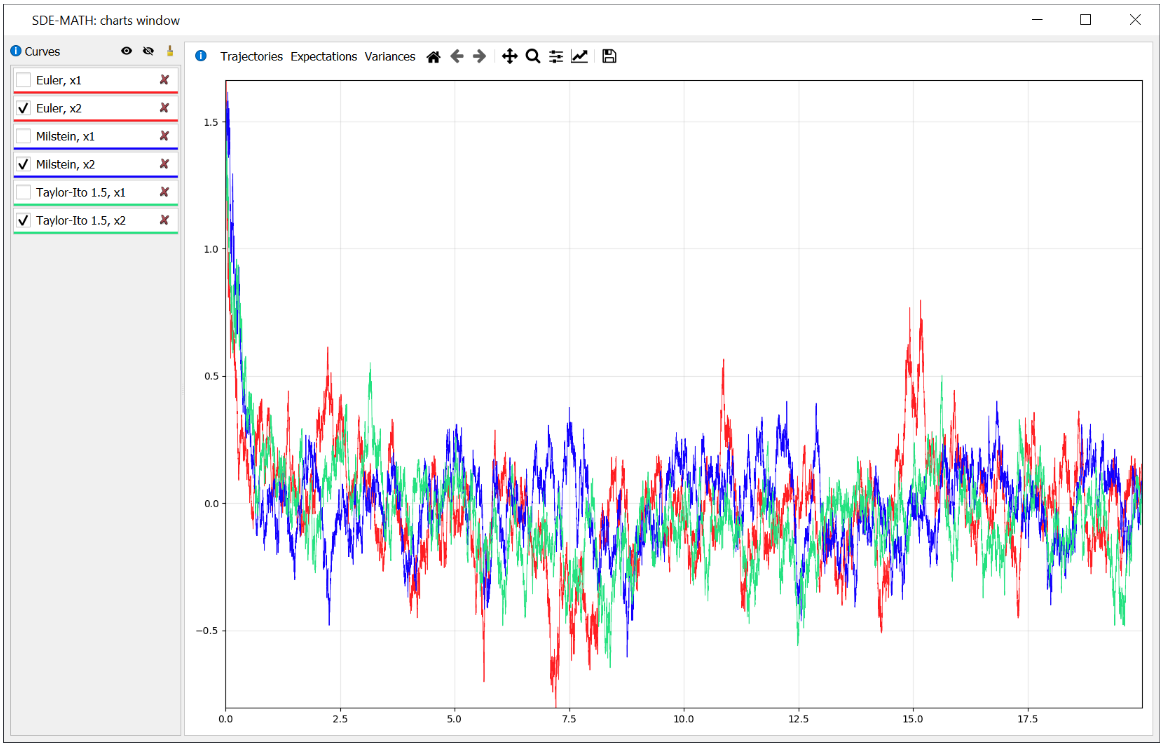
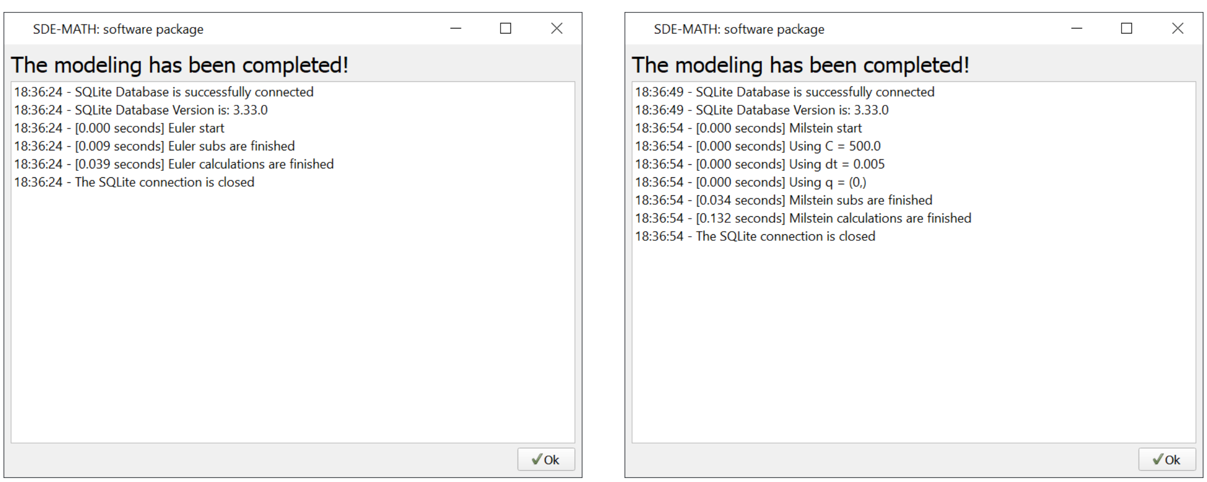
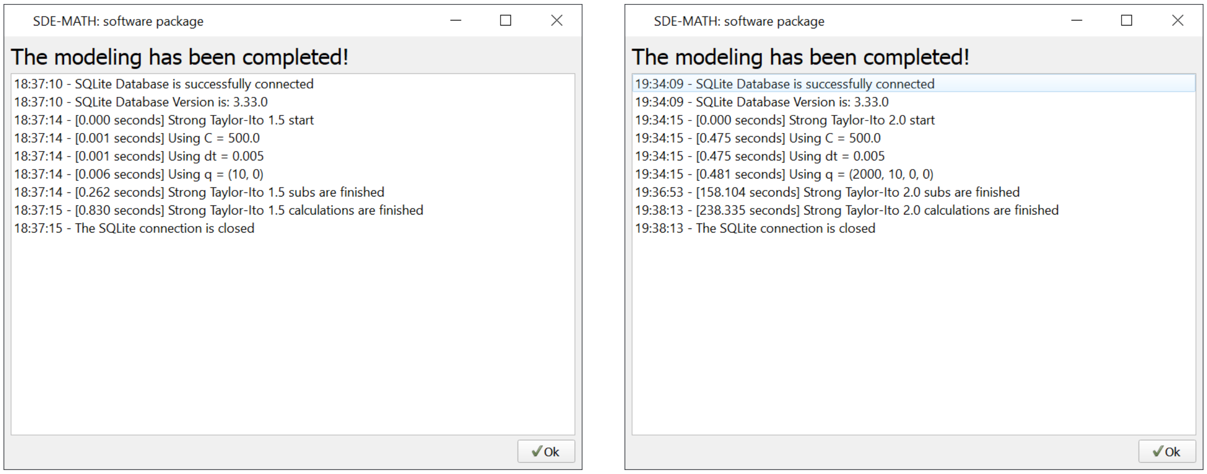


Figure 40: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 50$, $dt = 0.001$)



Euler scheme ($dt = 0.005$) Milstein scheme ($C = 500$, $dt = 0.005$)

Figure 41: Modeling logs



Strong Taylor–Itô scheme of order 1.5 ($C = 500$, $dt = 0.005$)

Strong Taylor–Itô scheme of order 2.0 ($C = 500$, $dt = 0.005$)

Figure 42: Modeling logs

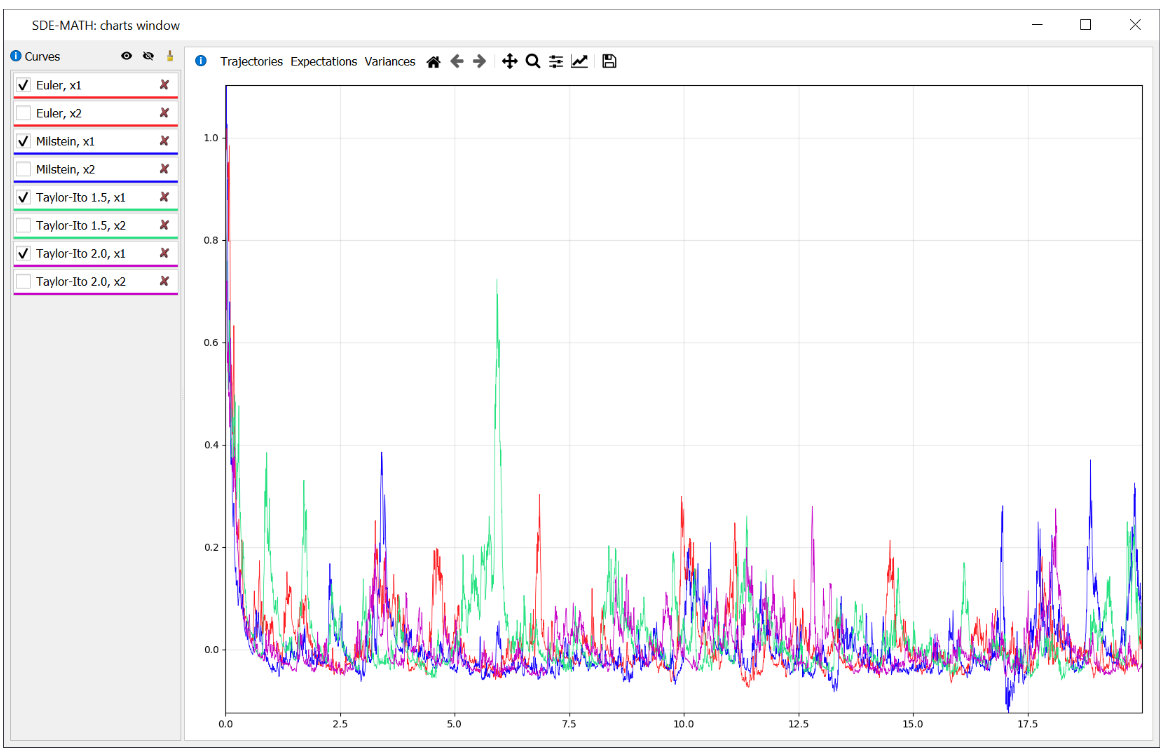


Figure 43: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 500$, $dt = 0.005$)

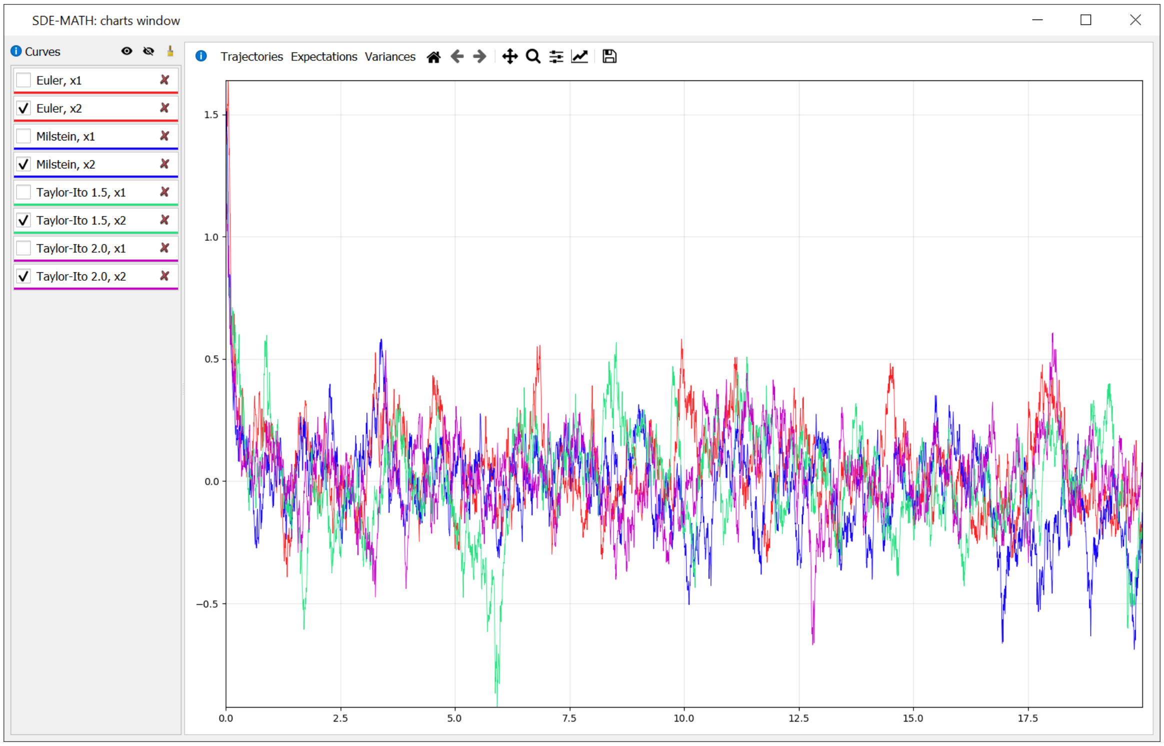


Figure 44: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 500$, $dt = 0.005$)

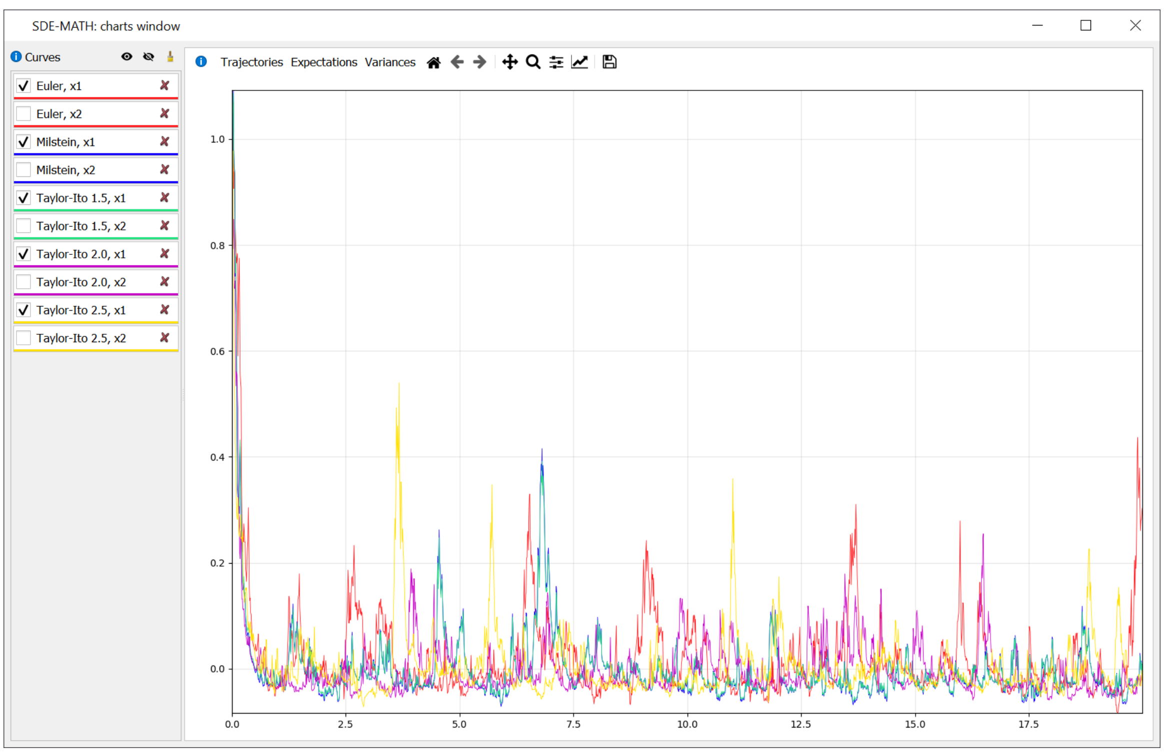
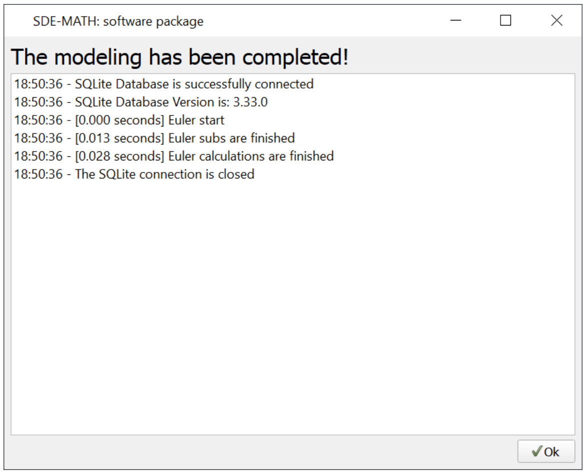
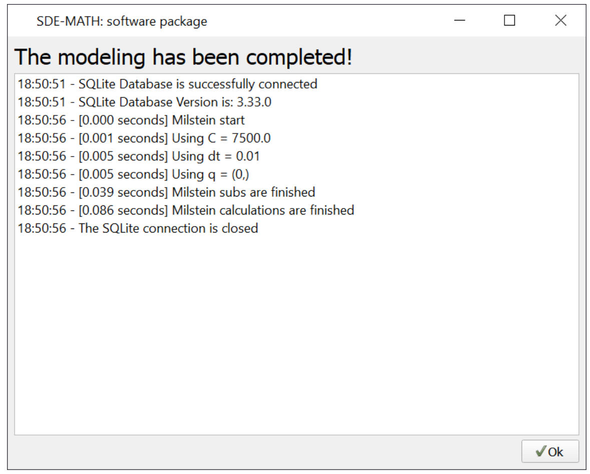


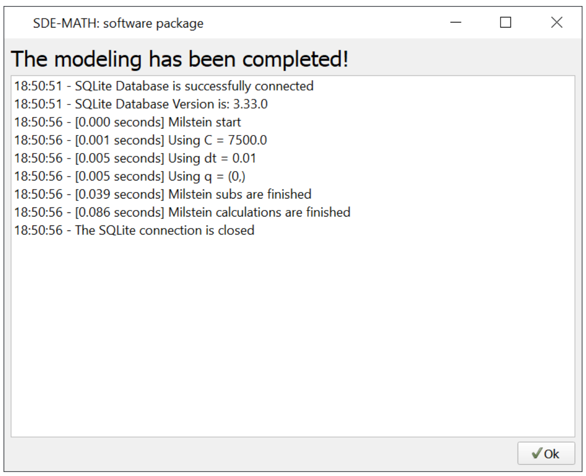
Figure 45: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 7500$, $dt = 0.01$)



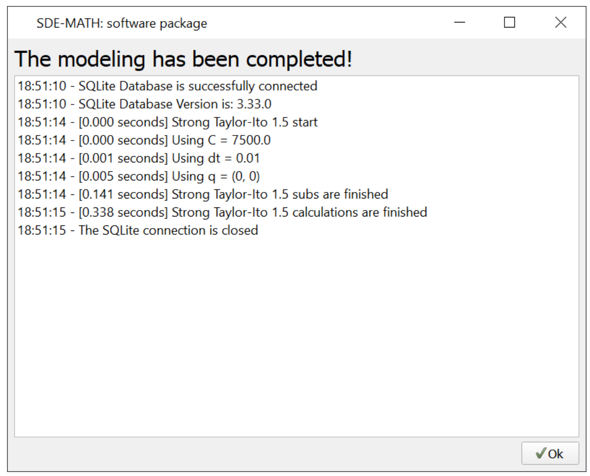
Euler scheme ($dt = 0.01$)



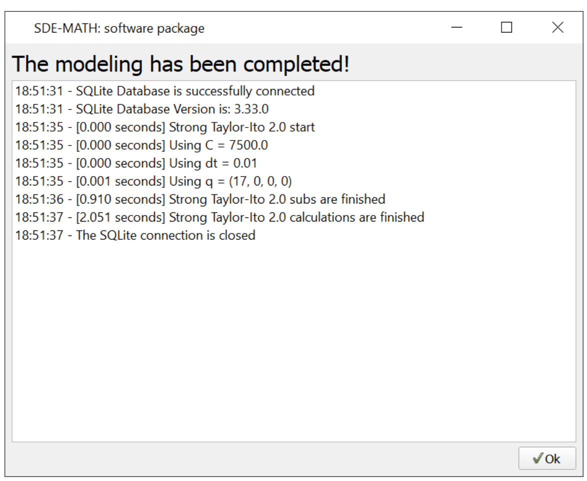
Milstein scheme ($C = 7500$, $dt = 0.01$)



Strong Taylor–Itô scheme of order 1.5 ($C = 7500$, $dt = 0.01$)



Strong Taylor–Itô scheme of order 2.0 ($C = 7500$, $dt = 0.01$)



Strong Taylor–Itô scheme of order 2.5 ($C = 7500$, $dt = 0.01$)

Figure 46: Modeling logs

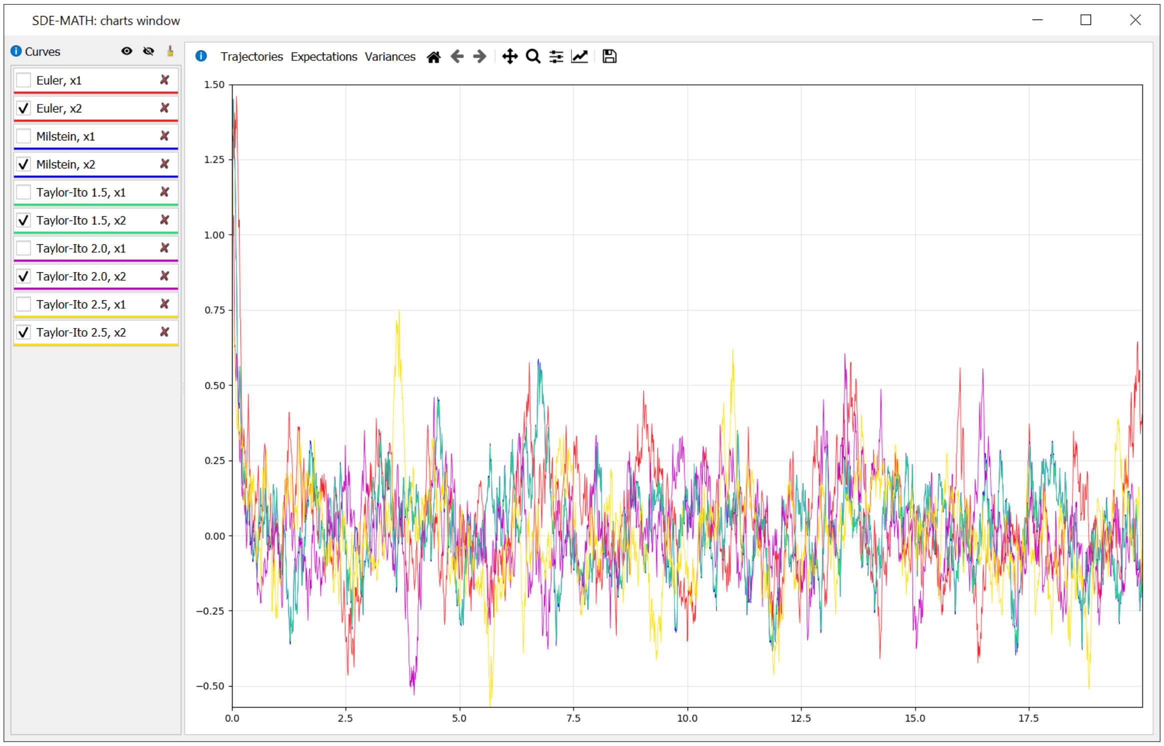


Figure 47: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 7500$, $dt = 0.01$)

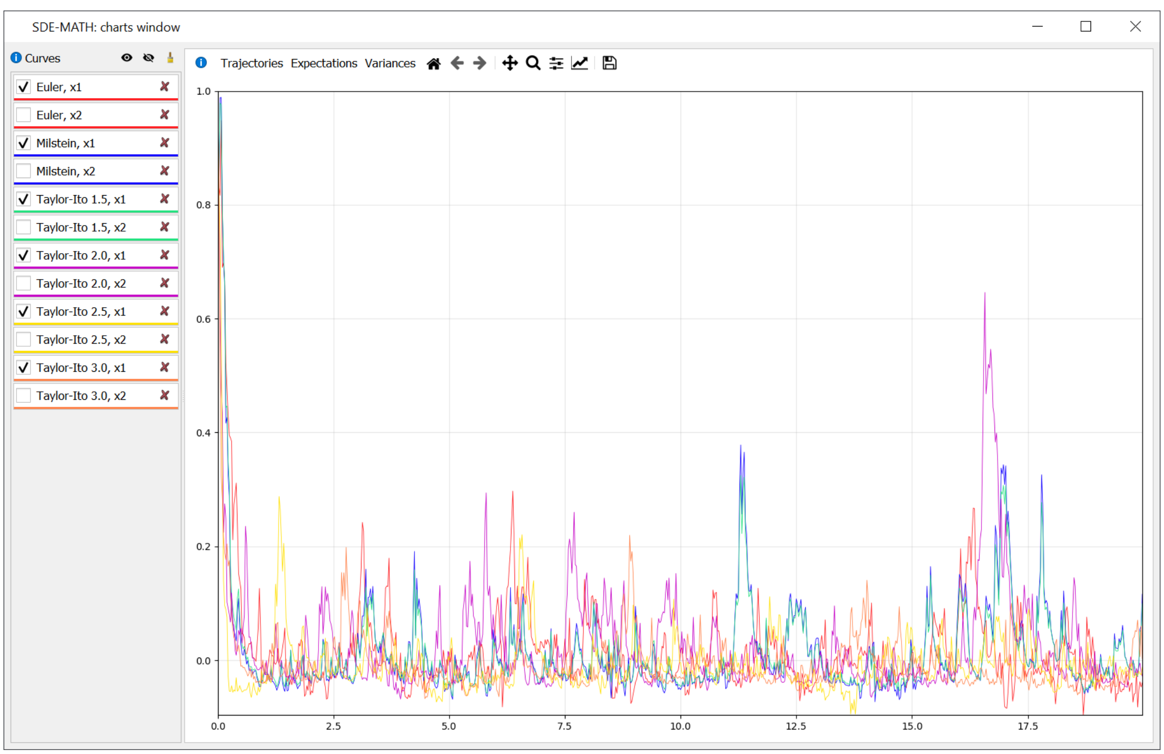
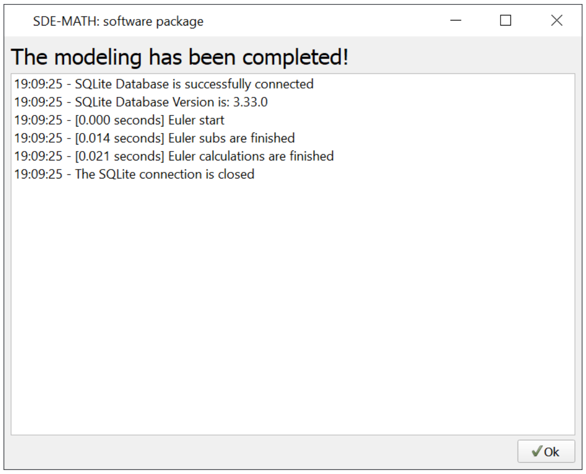
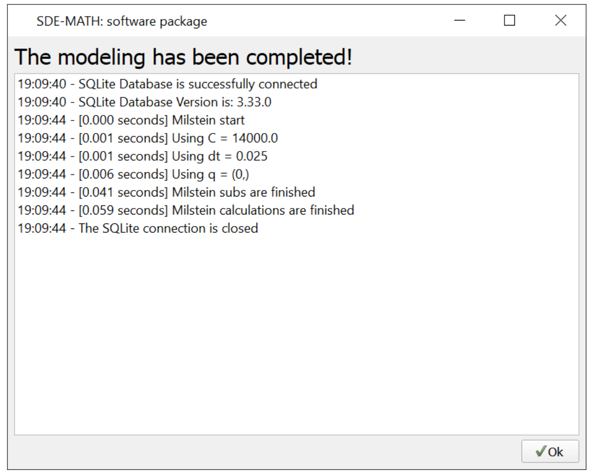


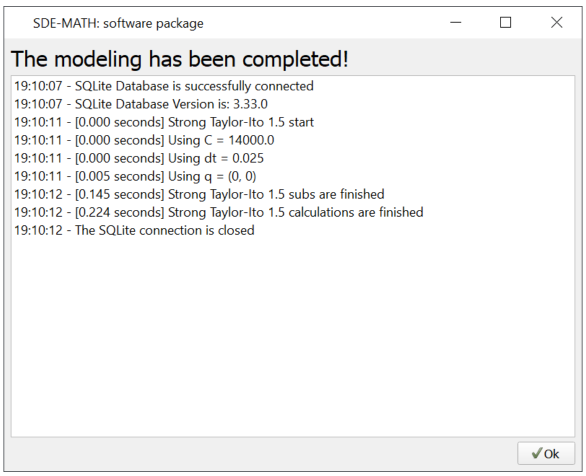
Figure 48: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 14000$, $dt = 0.025$)



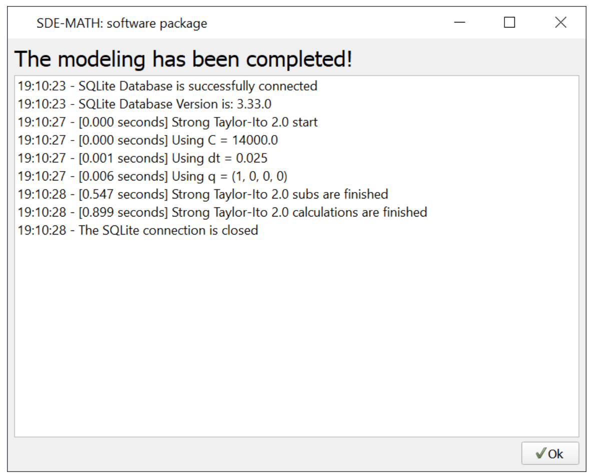
Euler scheme ($dt = 0.025$)



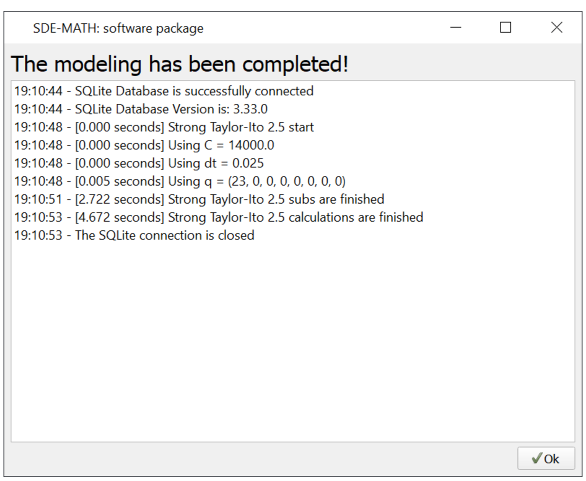
Milstein scheme ($C = 14000$, $dt = 0.025$)



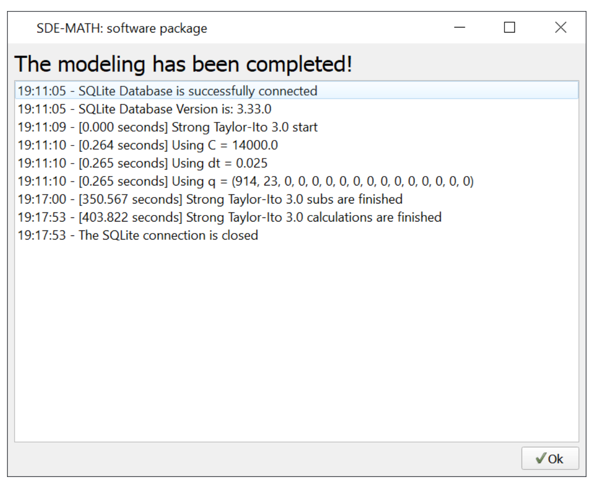
Strong Taylor–Itô scheme of order 1.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Itô scheme of order 2.0 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Itô scheme of order 2.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Itô scheme of order 3.0 ($C = 14000$, $dt = 0.025$)

Figure 49: Modeling logs

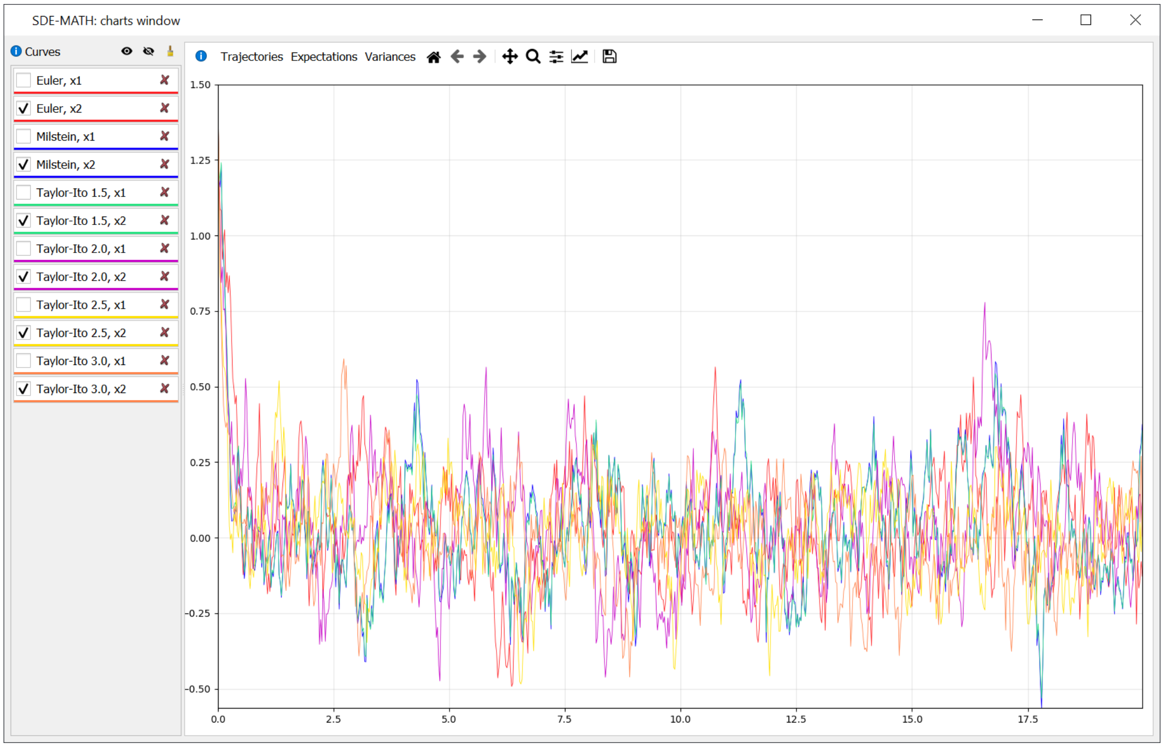
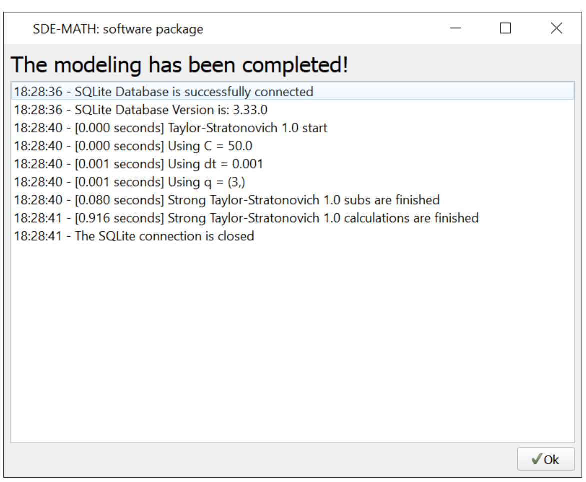
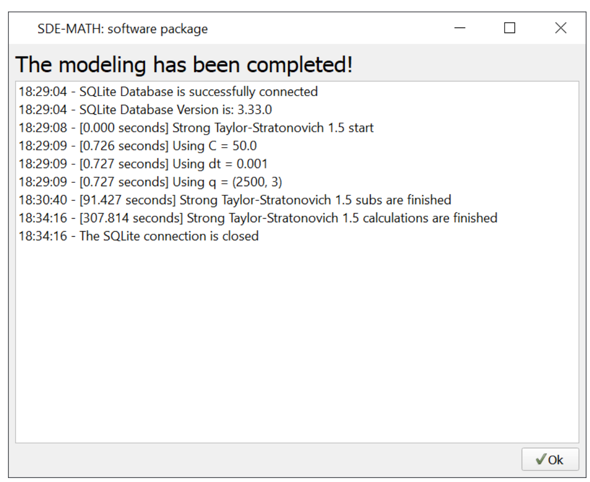


Figure 50: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 14000$, $dt = 0.025$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 50$, $dt = 0.001$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 50$, $dt = 0.001$)

Figure 51: Modeling logs

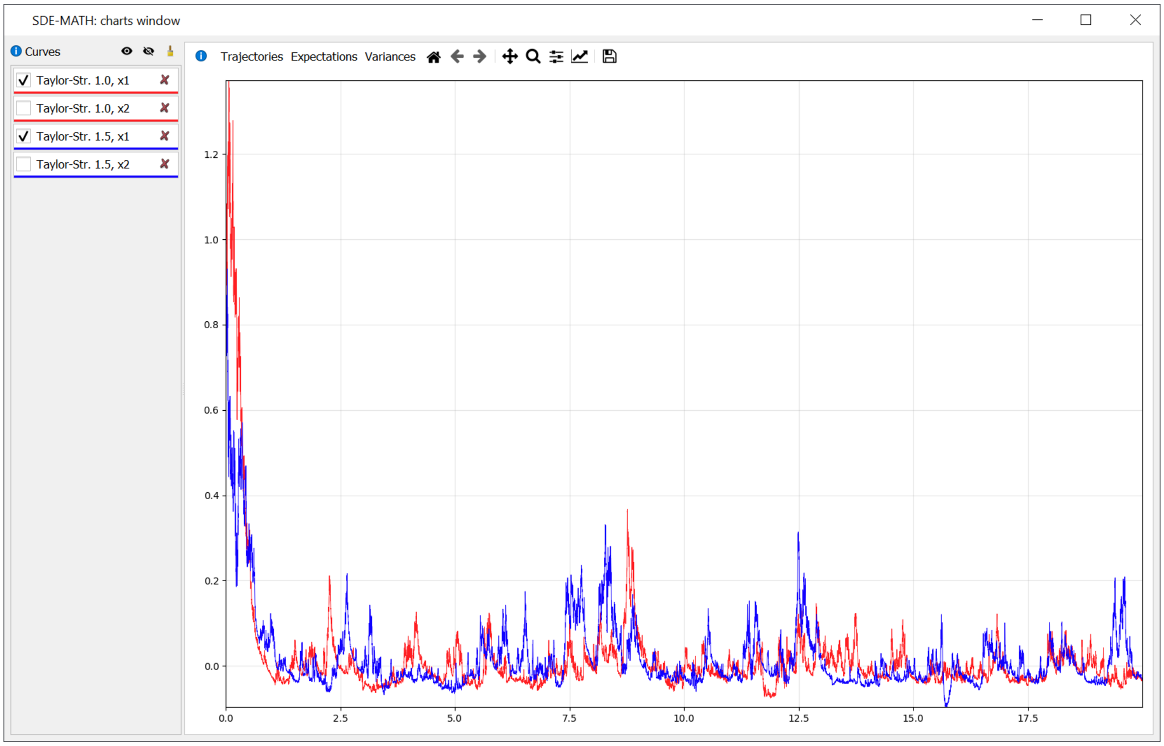


Figure 52: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 50$, $dt = 0.001$)

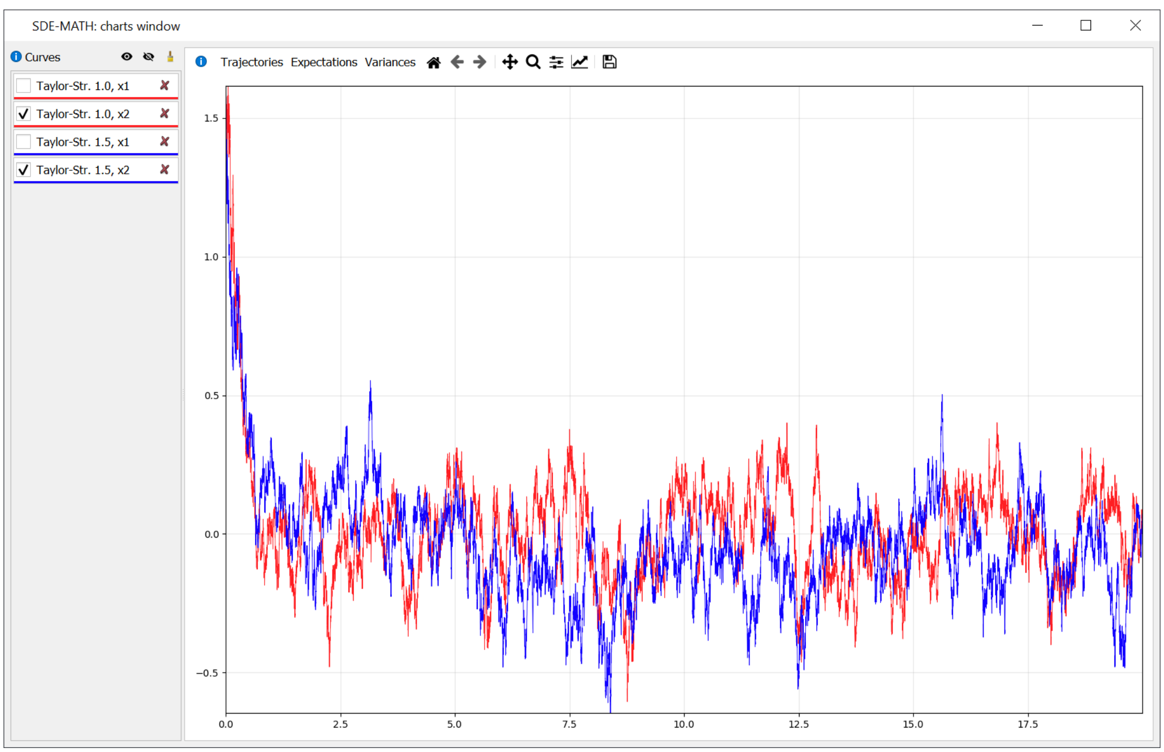


Figure 53: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 50$, $dt = 0.001$)

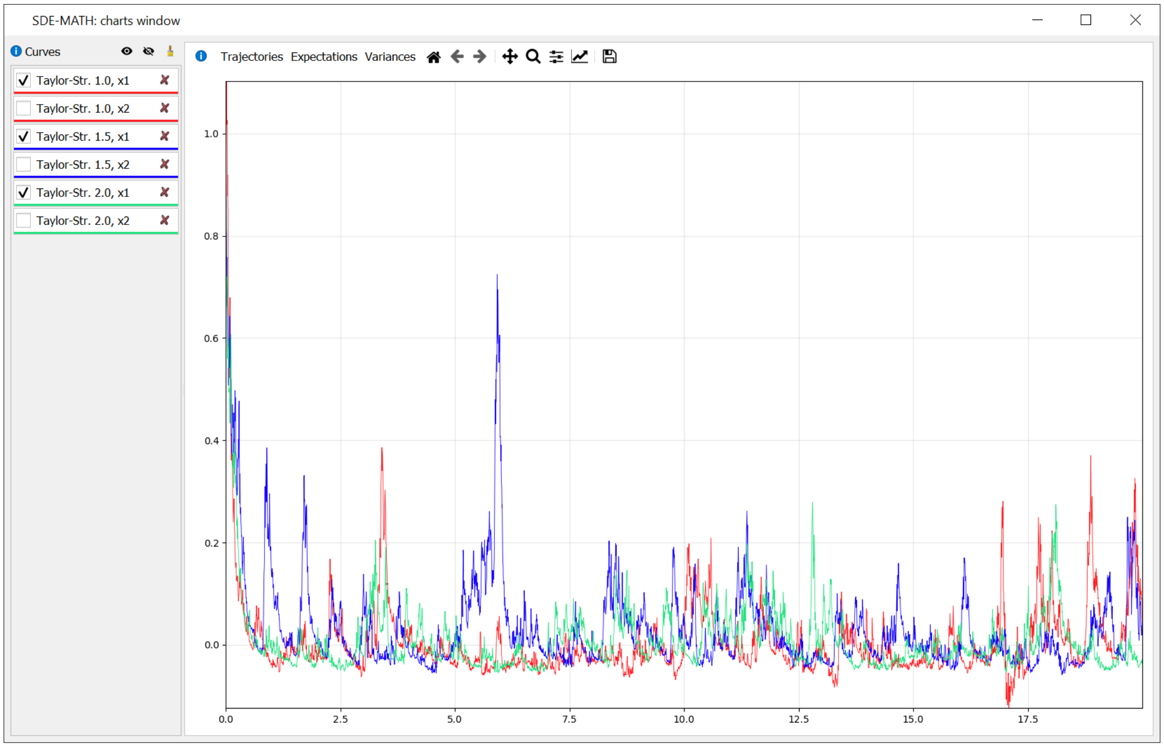
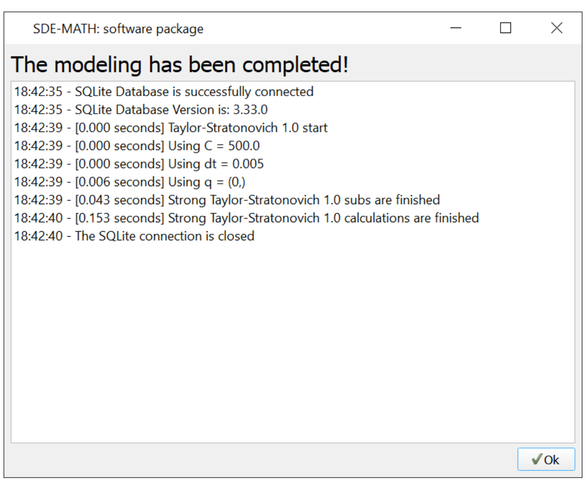
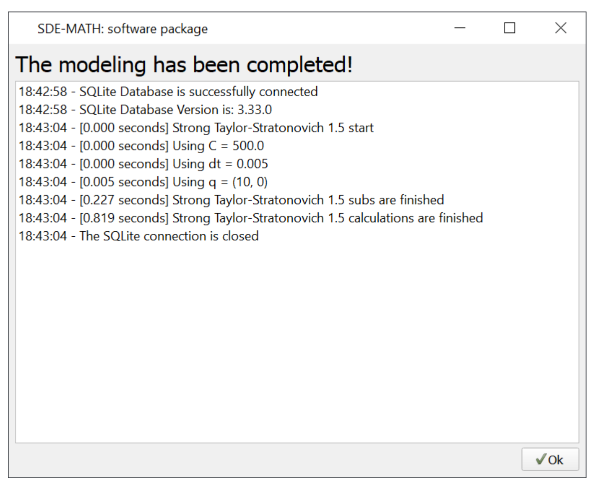


Figure 54: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 500$, $dt = 0.005$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 500$, $dt = 0.005$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 500$, $dt = 0.005$)

Figure 55: Modeling logs

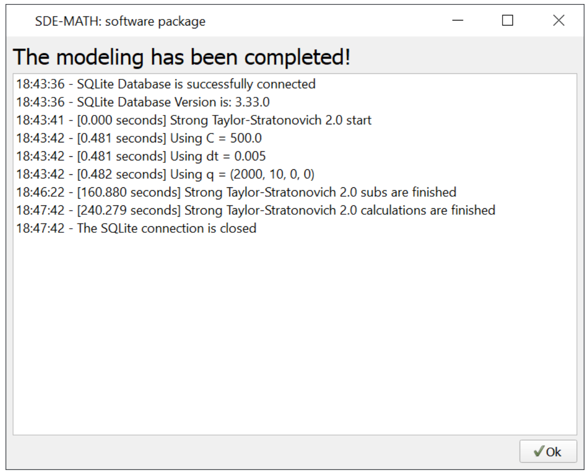


Figure 56: Strong Taylor–Stratonovich scheme of order 2.0 ($C = 500$, $dt = 0.005$)

Figure 57: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 500$, $dt = 0.005$)

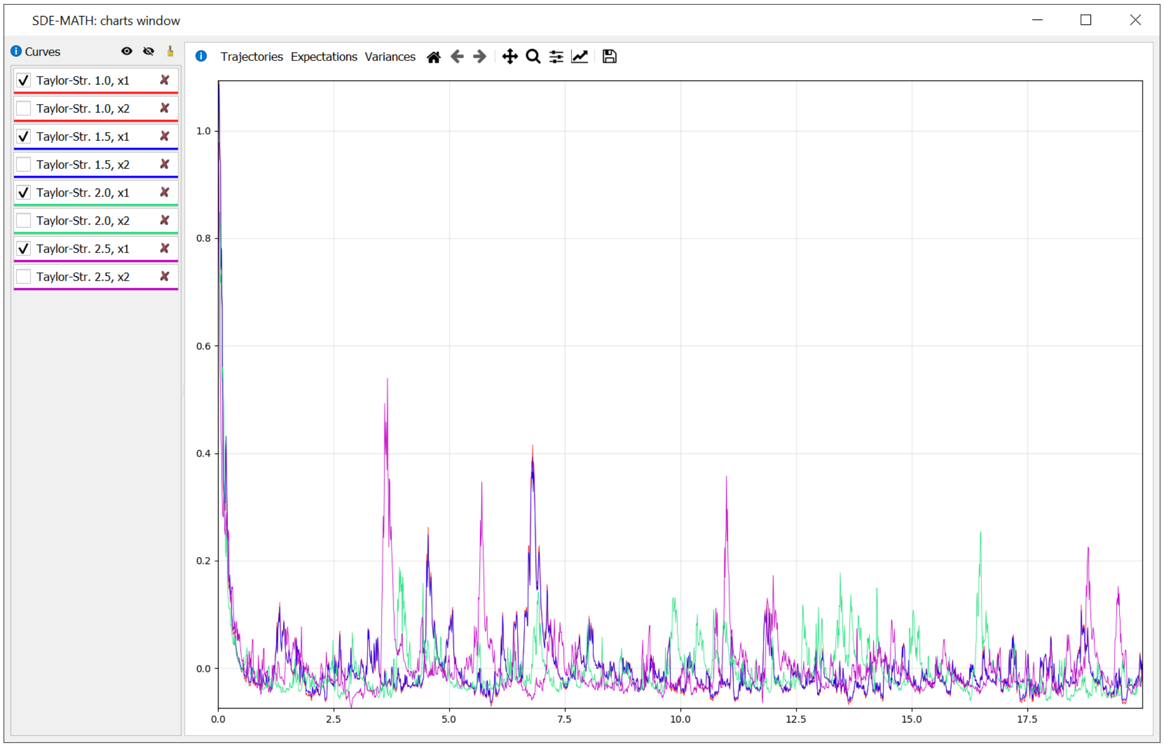
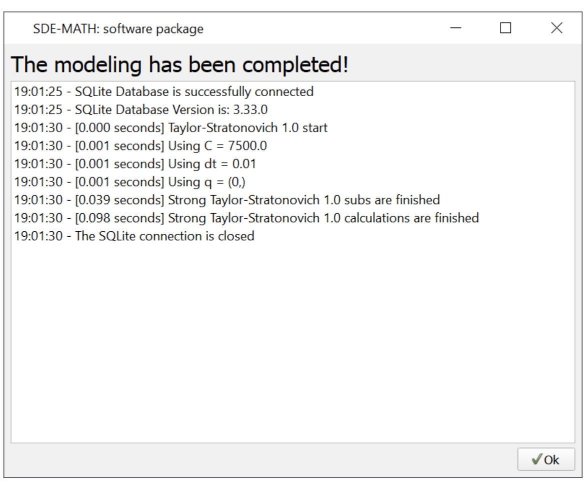
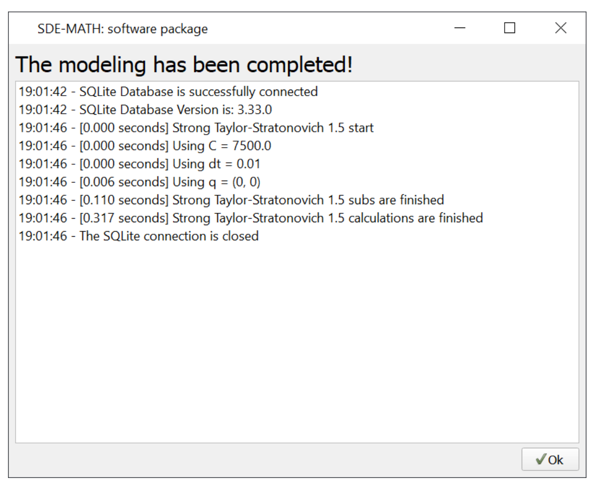


Figure 58: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 7500$, $dt = 0.01$)

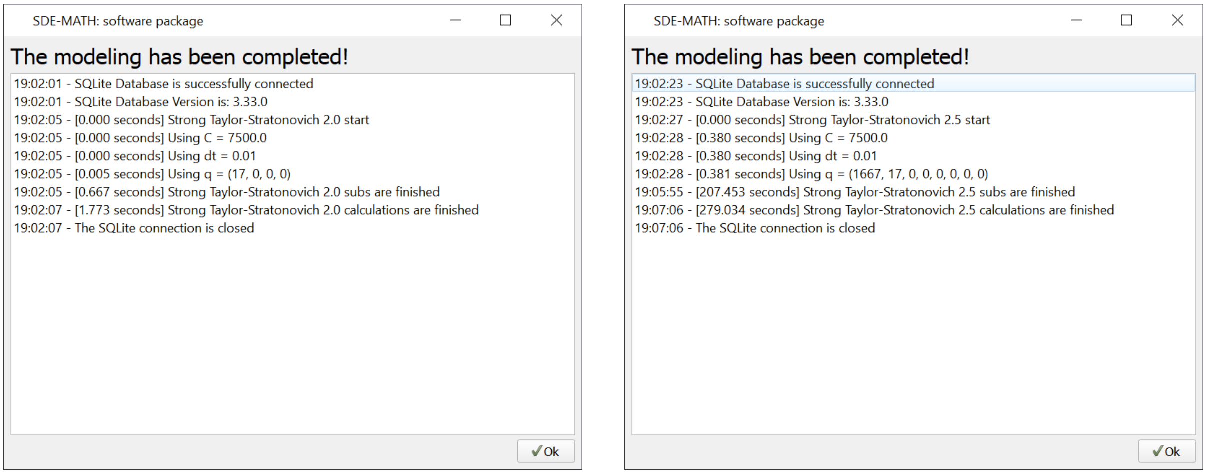


Strong Taylor–Stratonovich scheme of order 1.0 ($C = 7500$, $dt = 0.01$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 7500$, $dt = 0.01$)

Figure 59: Modeling logs

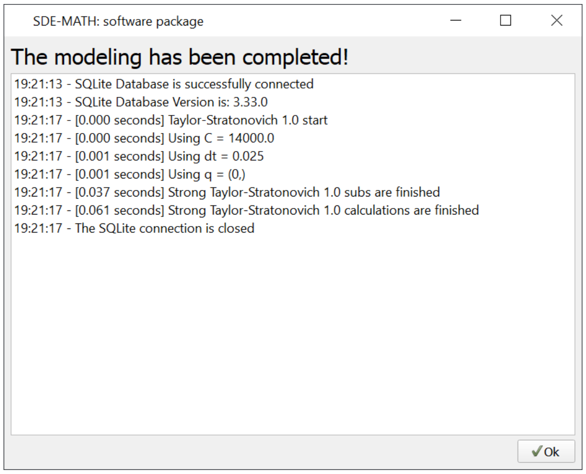


Strong Taylor–Stratonovich scheme of order 2.0 ($C = 7500$, $dt = 0.01$)

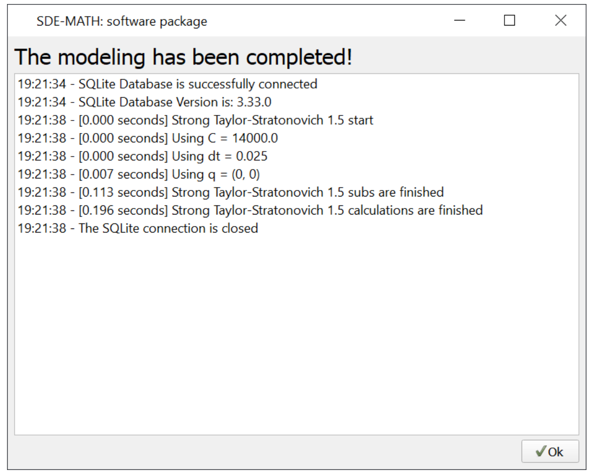
Strong Taylor–Stratonovich scheme of order 2.5 ($C = 7500$, $dt = 0.01$)

Figure 60: Modeling logs

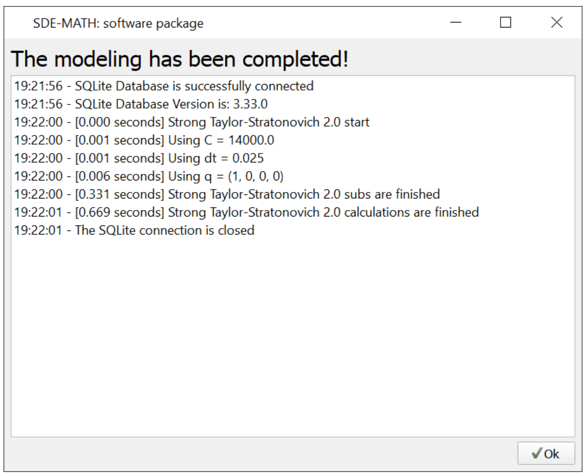
Figure 61: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 7500$, $dt = 0.01$)



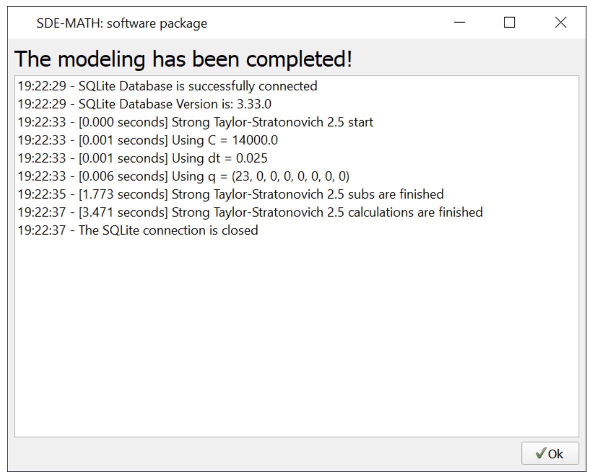
Strong Taylor–Stratonovich scheme of order 1.0 ($C = 14000$, $dt = 0.025$)



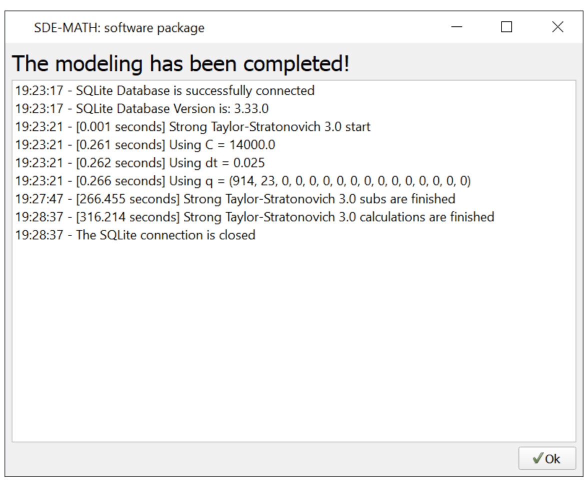
Strong Taylor–Stratonovich scheme of order 1.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Stratonovich scheme of order 2.0 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Stratonovich scheme of order 2.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor–Stratonovich scheme of order 3.0 ($C = 14000$, $dt = 0.025$)

Figure 62: Modeling logs

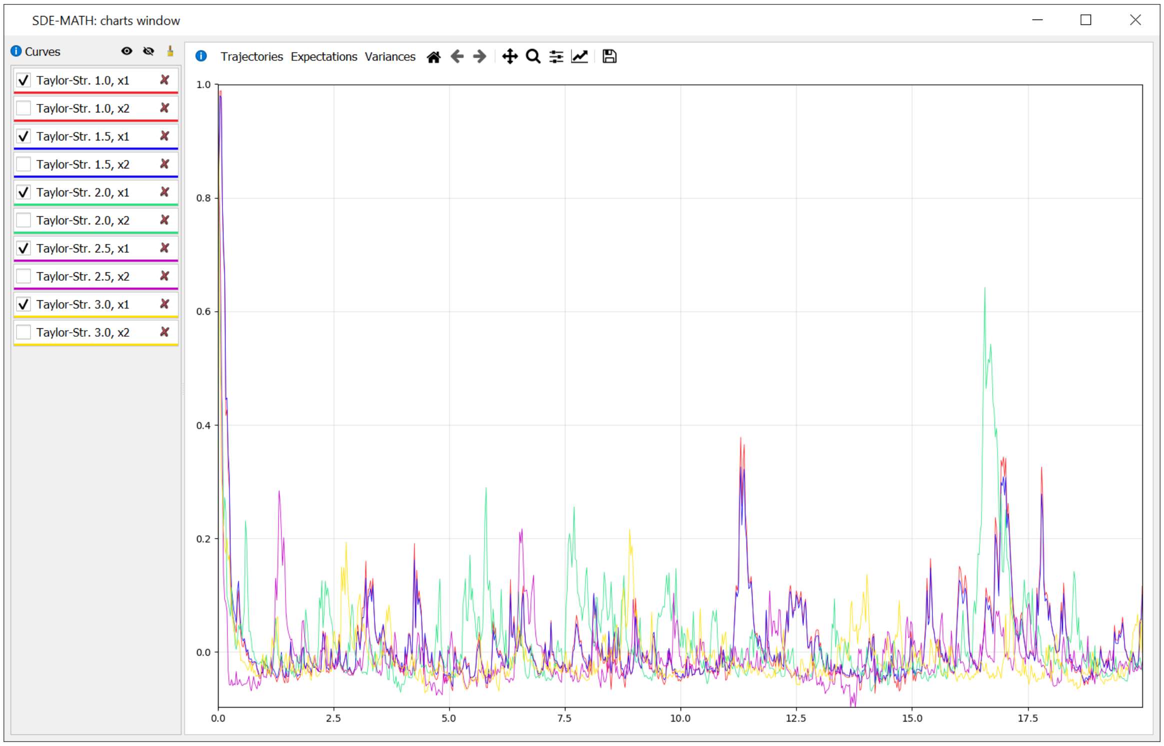


Figure 63: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 14000$, $dt = 0.025$)

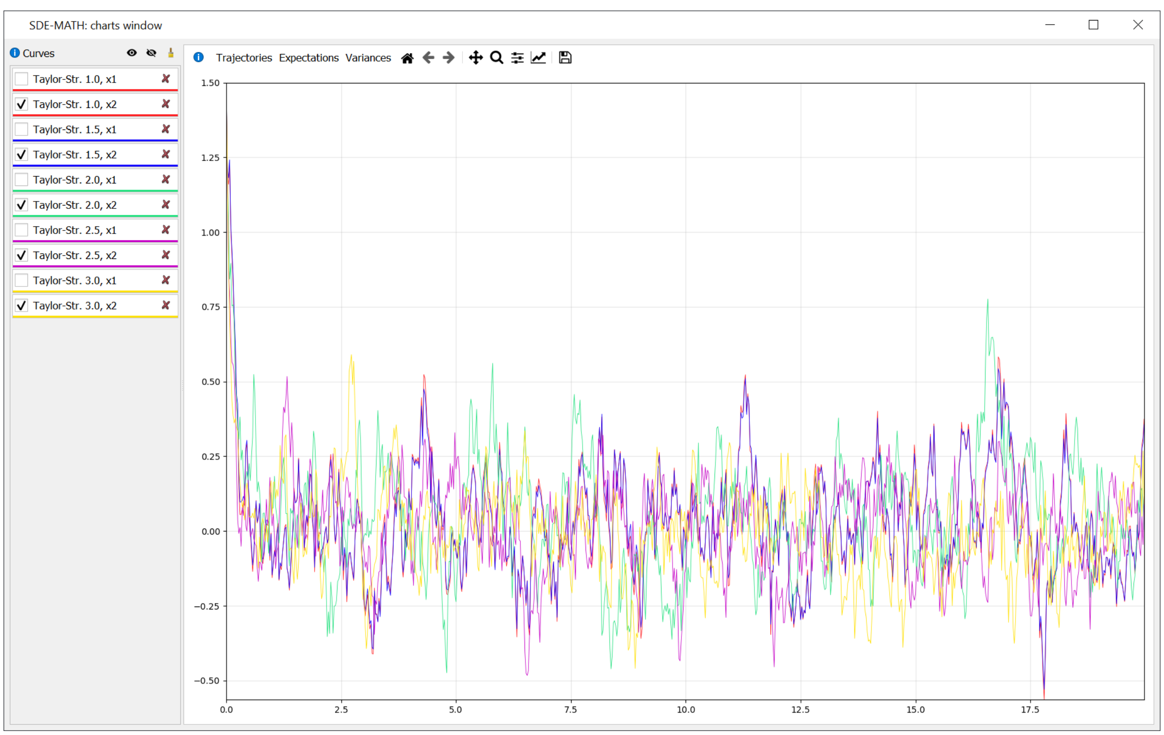


Figure 64: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 14000$, $dt = 0.025$)

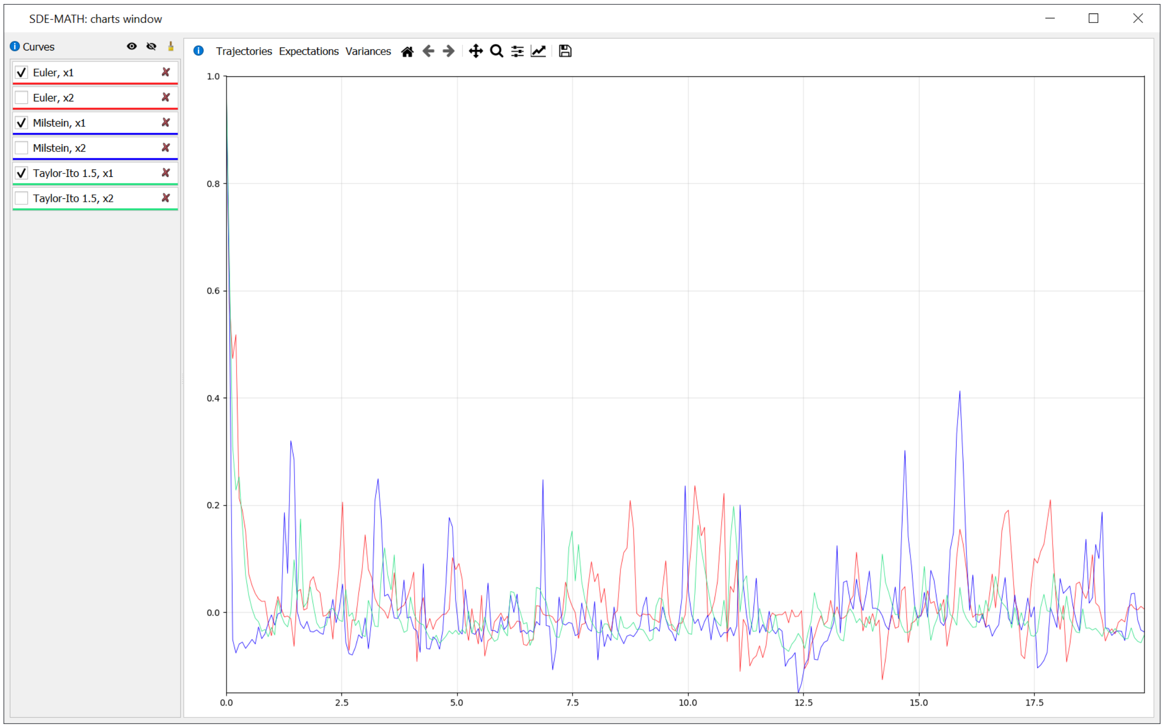
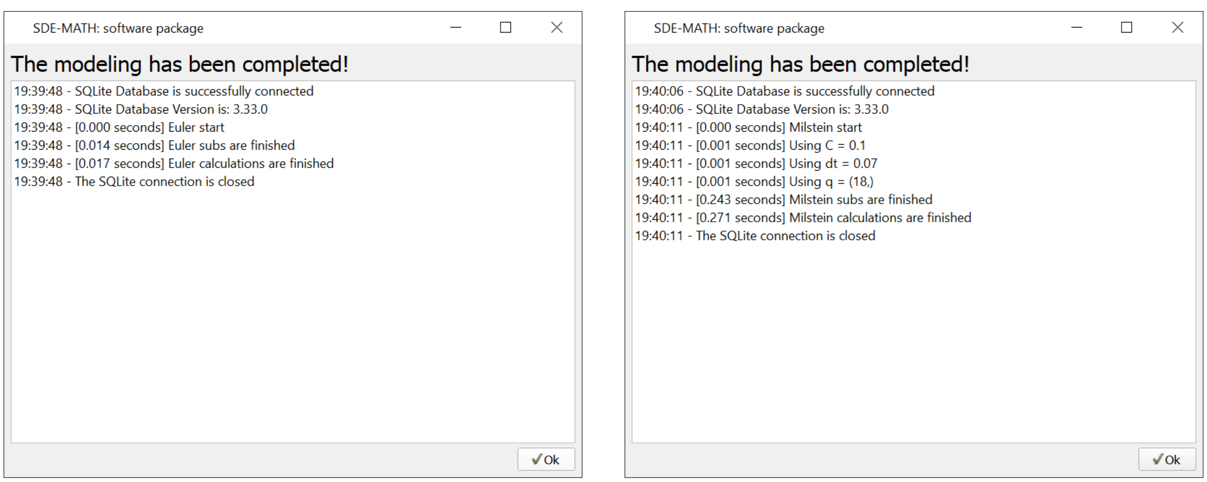


Figure 65: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.1$, $dt = 0.07$)



Euler scheme ($dt = 0.07$) Milstein scheme ($C = 0.1$, $dt = 0.07$)

Figure 66: Modeling logs

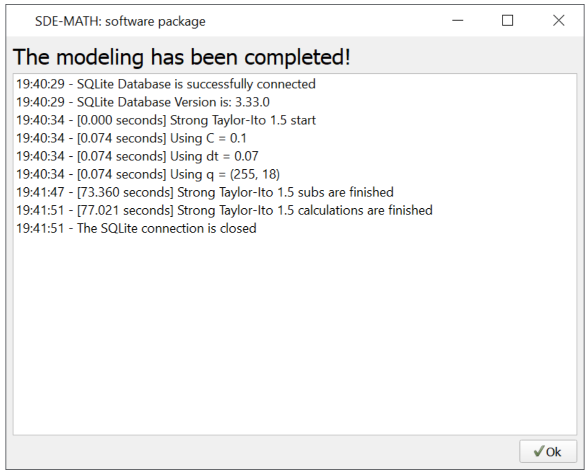


Figure 67: Strong Taylor–Itô scheme of order 1.5 ($C = 0.1$, $dt = 0.07$)

Figure 68: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.1$, $dt = 0.07$)

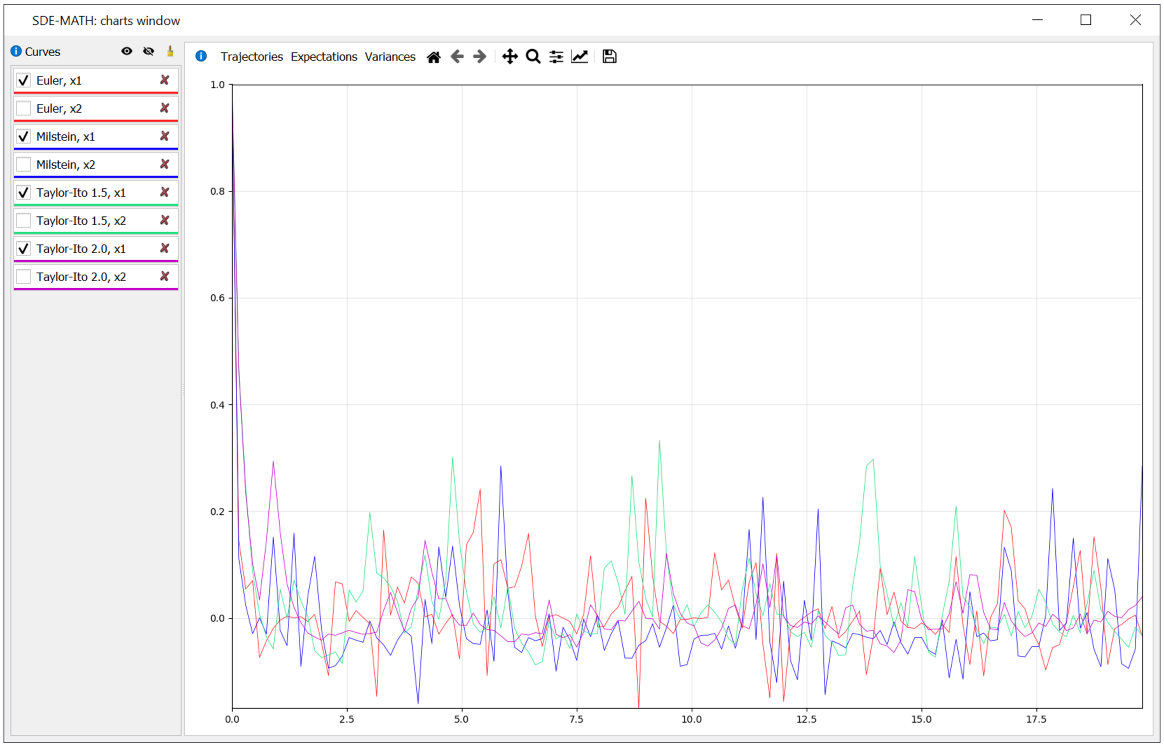
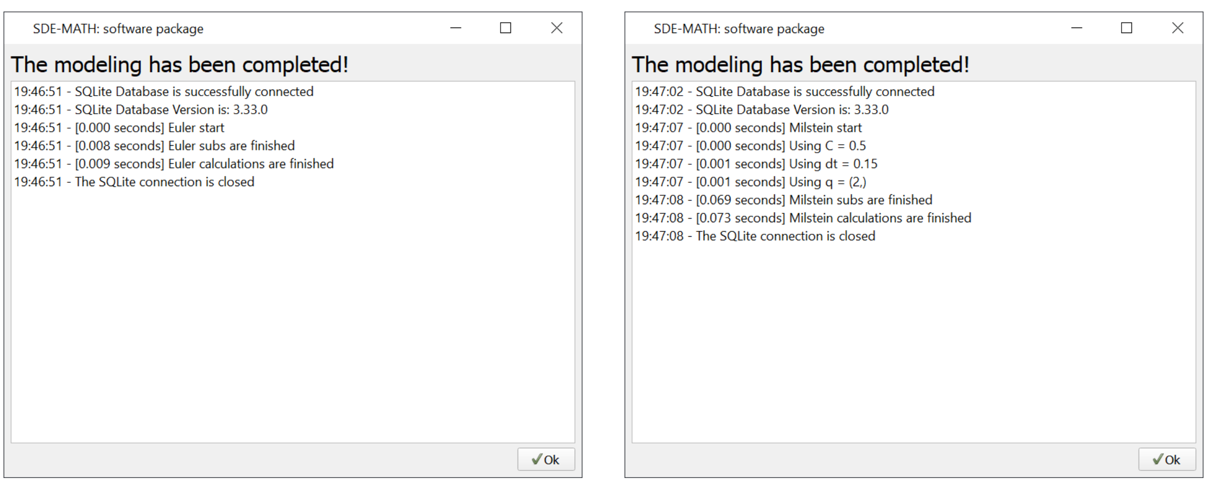
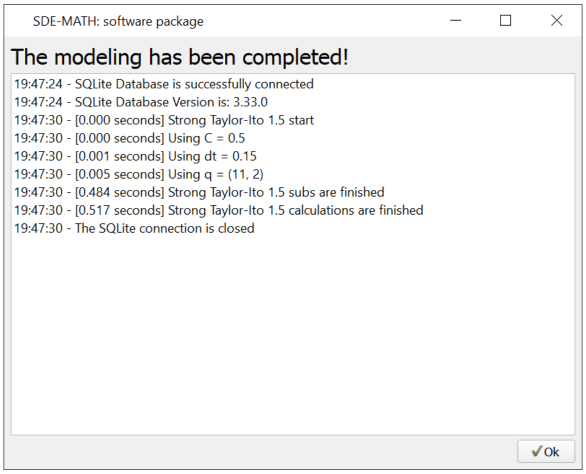


Figure 69: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 0.5$, $dt = 0.15$)

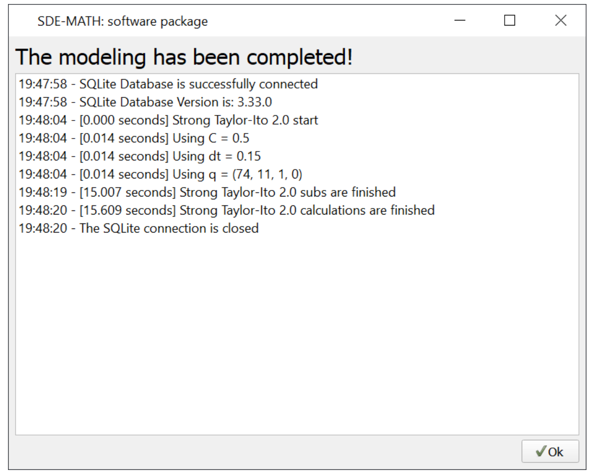


Euler scheme ($dt = 0.15$) Milstein scheme ($C = 0.5$, $dt = 0.15$)

Figure 70: Modeling logs



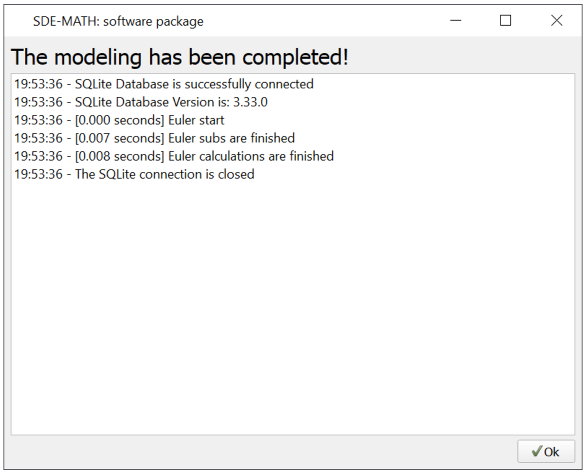
Strong Taylor–Itô scheme of order 1.5 ($C = 0.5$, $dt = 0.15$)



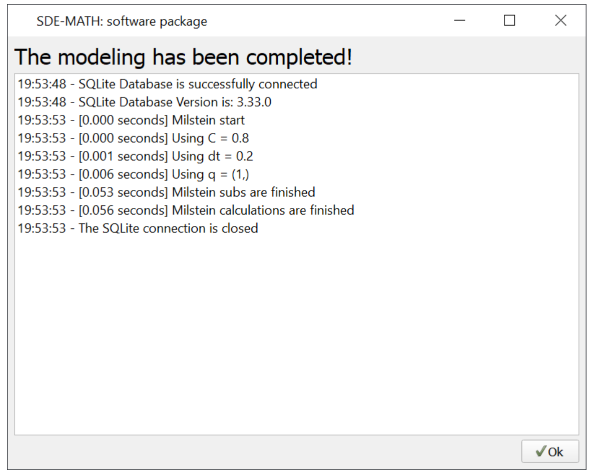
Strong Taylor–Itô scheme of order 2.0 ($C = 0.5$, $dt = 0.15$)

Figure 71: Modeling logs

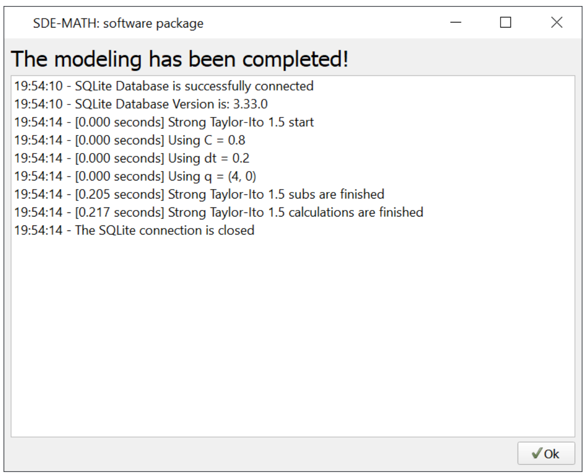
Figure 72: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 0.5$, $dt = 0.15$)



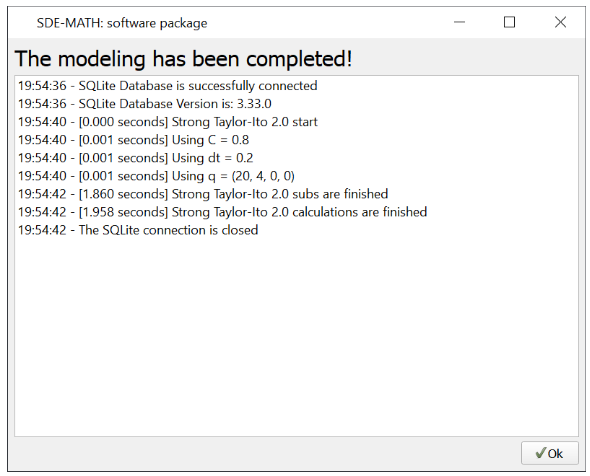
Euler scheme ($dt = 0.2$)



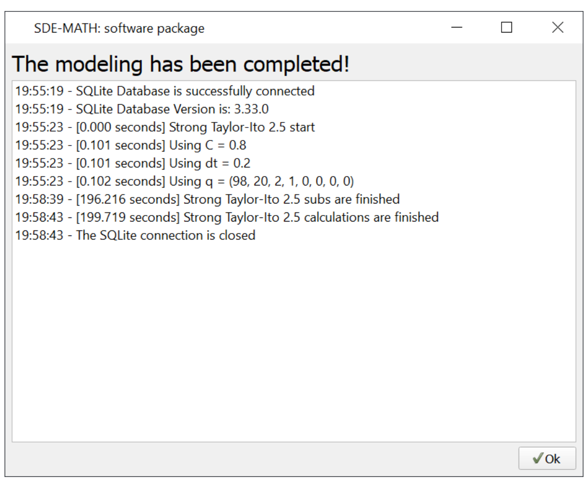
Milstein scheme ($C = 0.8$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 1.5 ($C = 0.8$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 2.0 ($C = 0.8$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 2.5 ($C = 0.8$, $dt = 0.2$)

Figure 73: Modeling logs

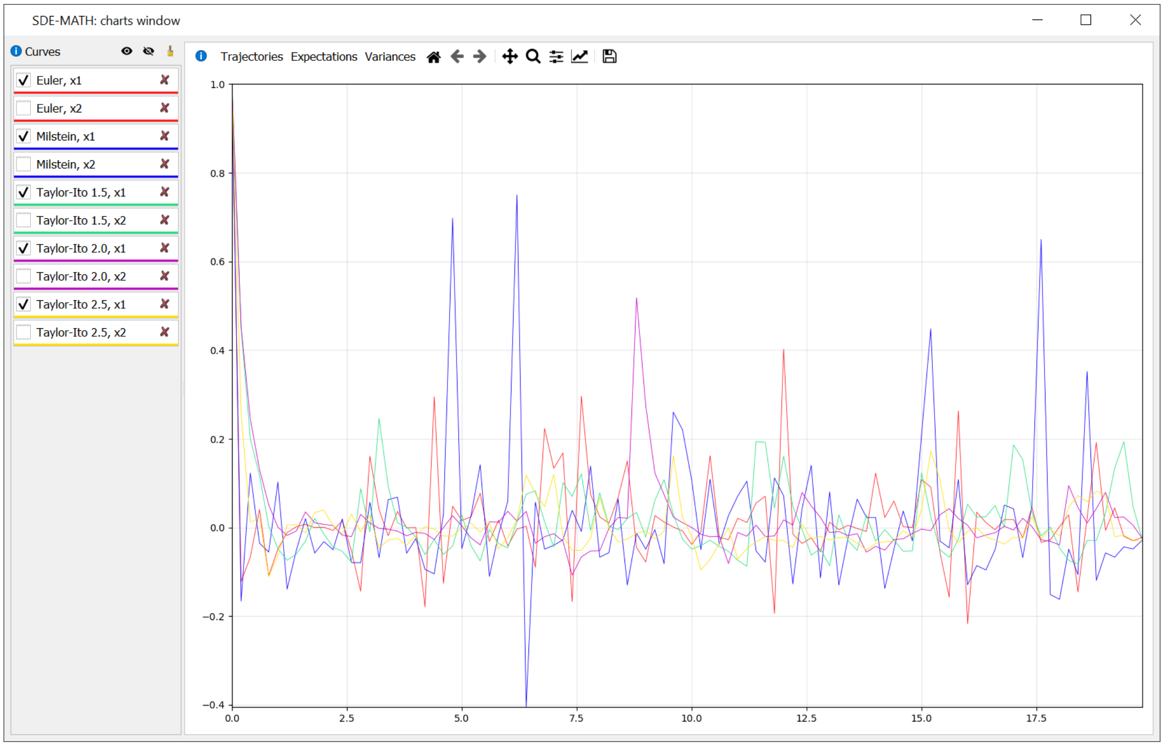


Figure 74: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.8$, $dt = 0.2$)

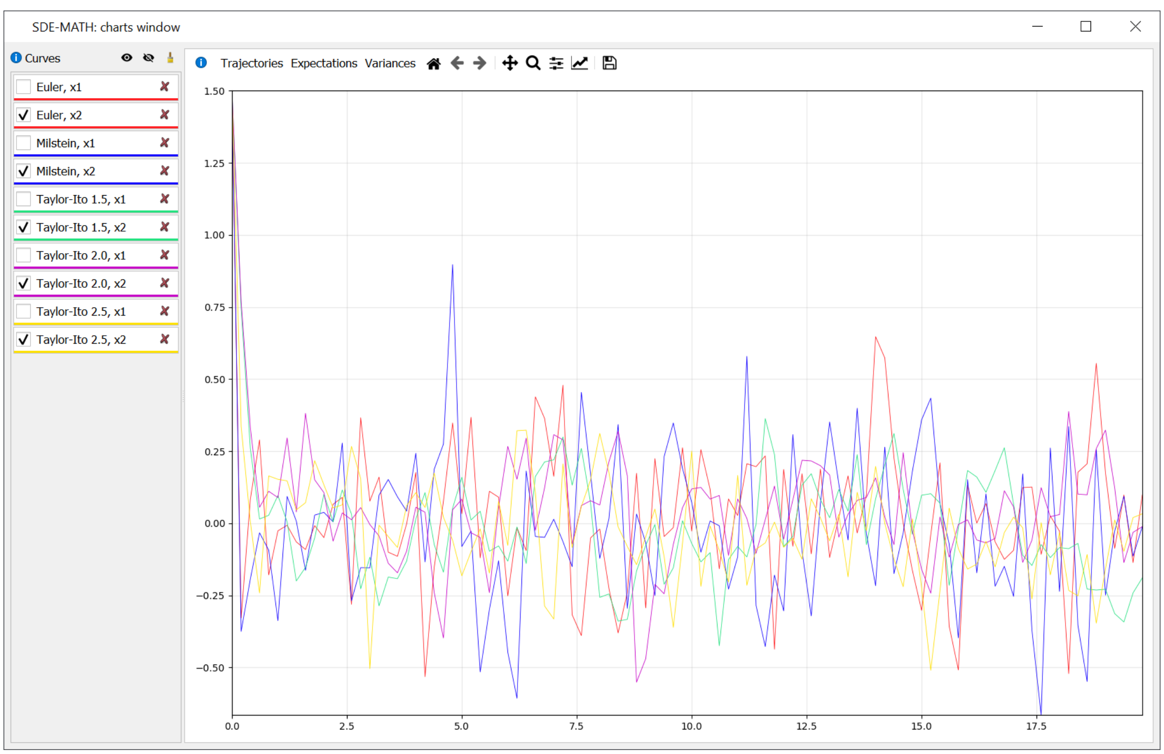
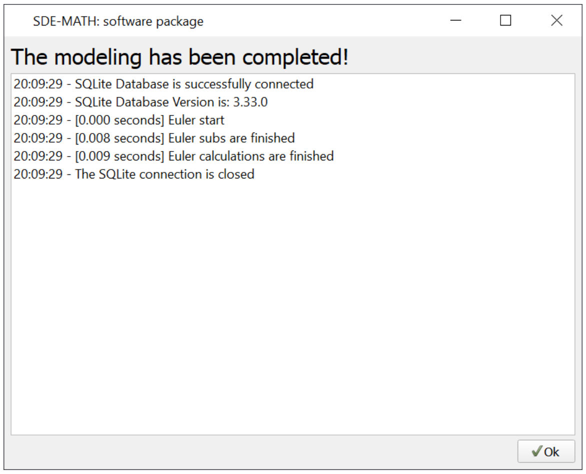
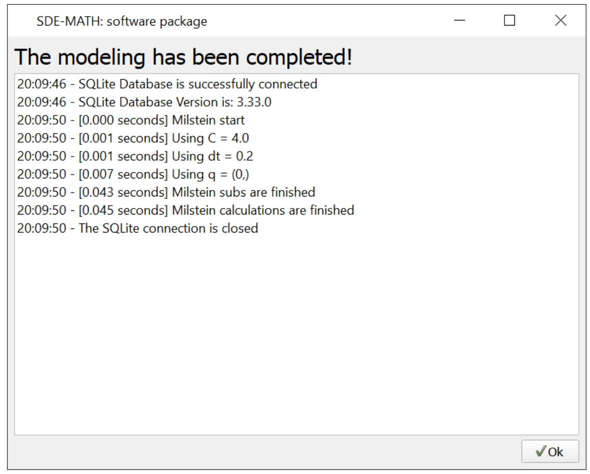


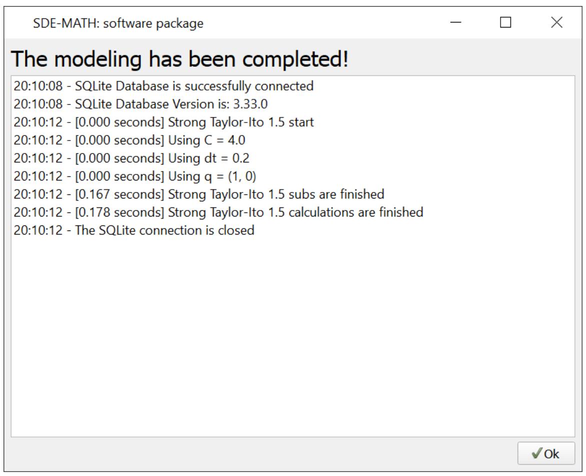
Figure 75: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.8$, $dt = 0.2$)



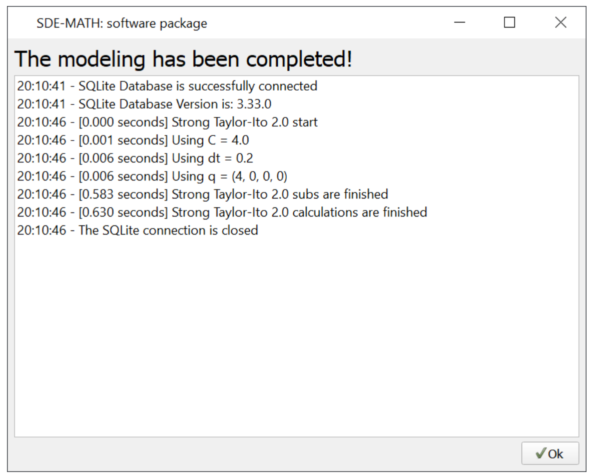
Euler scheme ($dt = 0.2$)



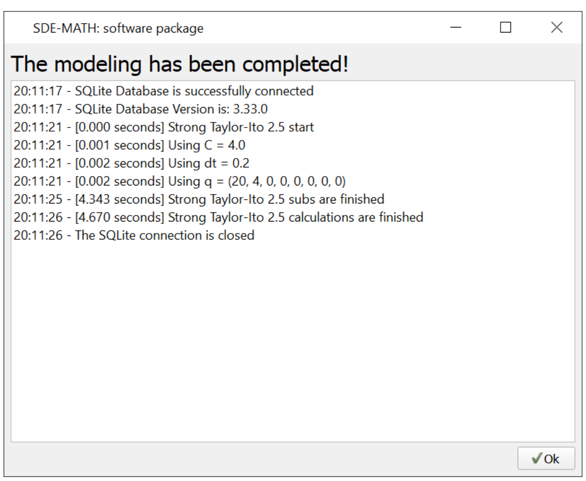
Milstein scheme ($C = 4$, $dt = 0.2$)



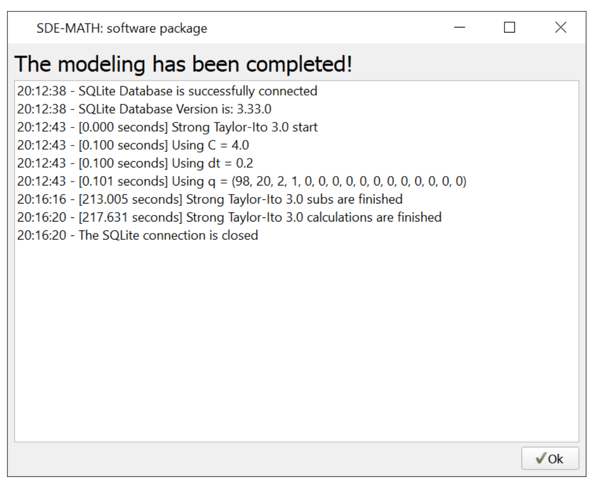
Strong Taylor–Itô scheme of order 1.5 ($C = 4$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 2.0 ($C = 4$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 2.5 ($C = 4$, $dt = 0.2$)



Strong Taylor–Itô scheme of order 3.0 ($C = 4$, $dt = 0.2$)

Figure 76: Modeling logs

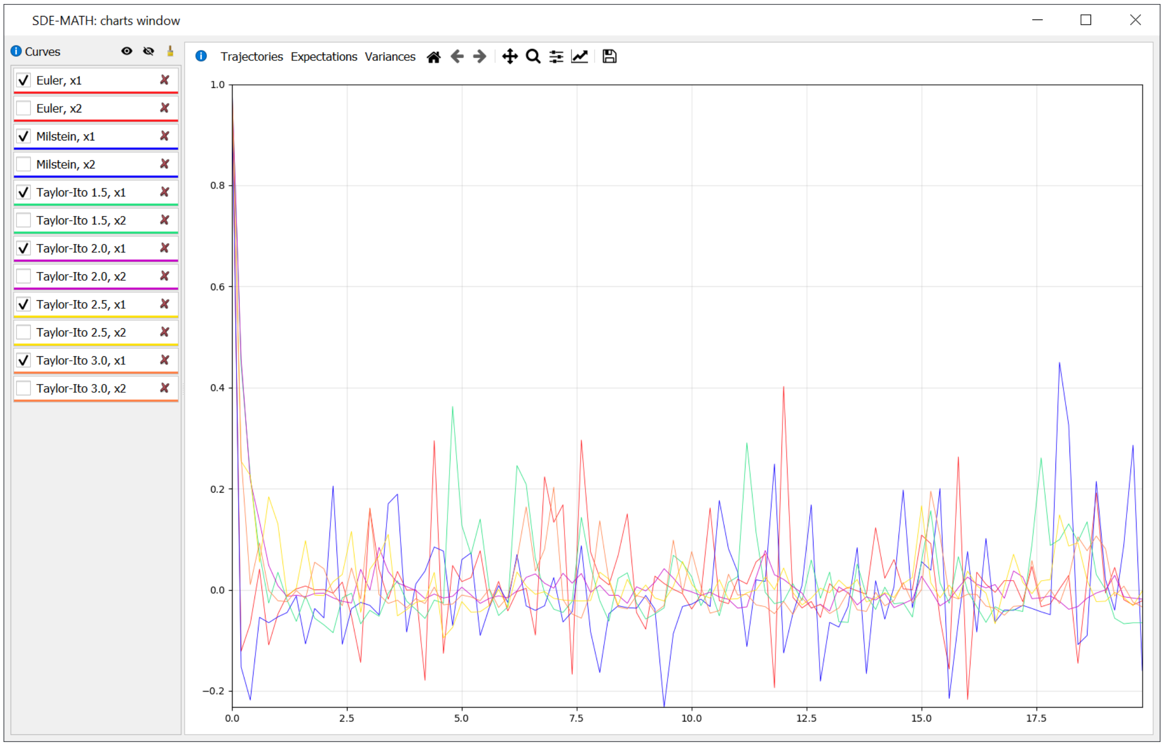


Figure 77: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 4$, $dt = 0.2$)

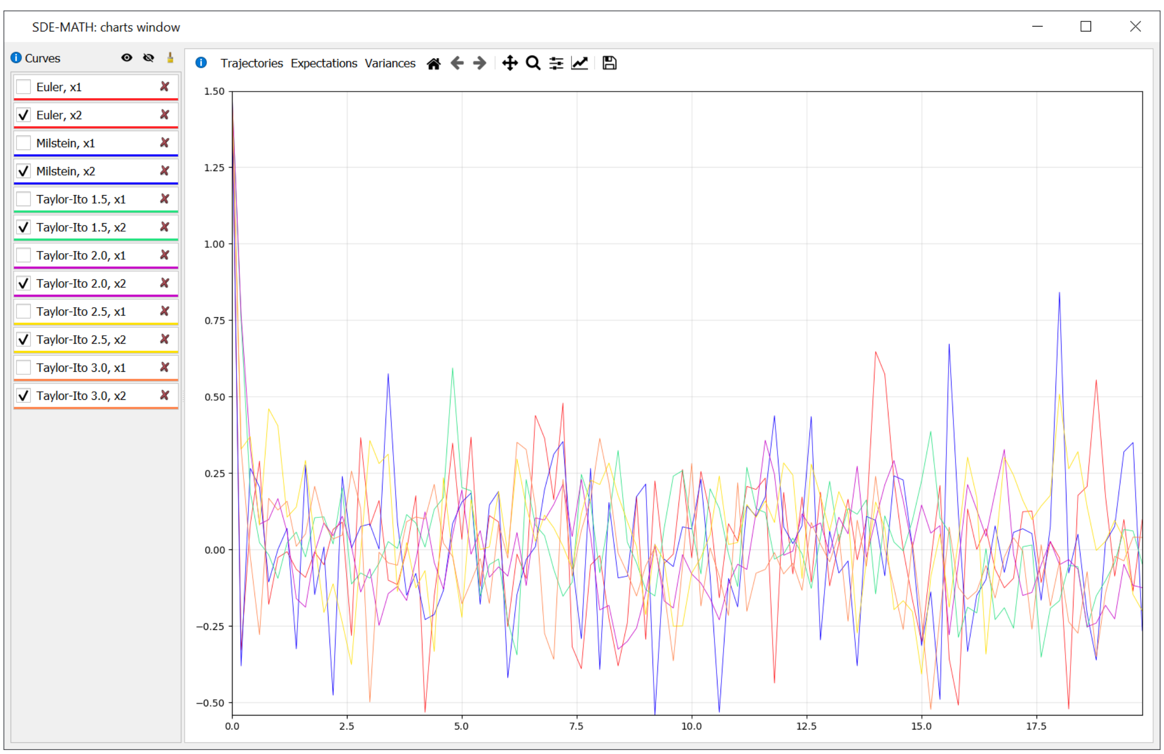
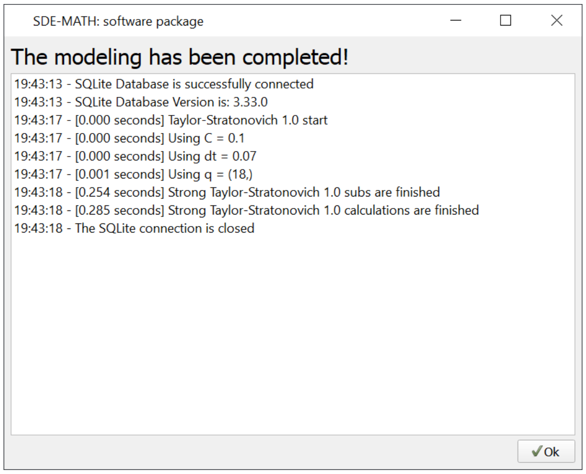
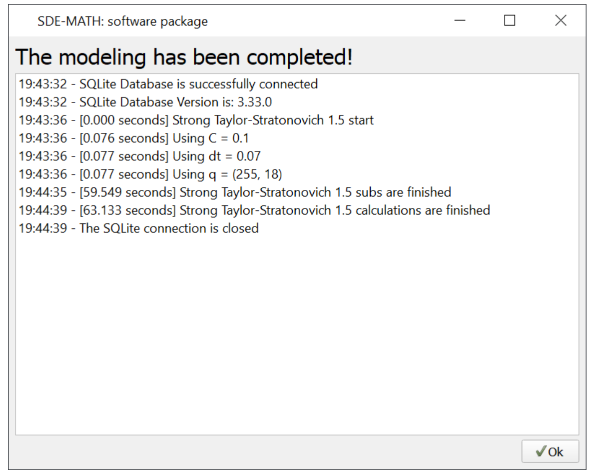


Figure 78: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 4$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.1$, $dt = 0.07$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.1$, $dt = 0.07$)

Figure 79: Modeling logs

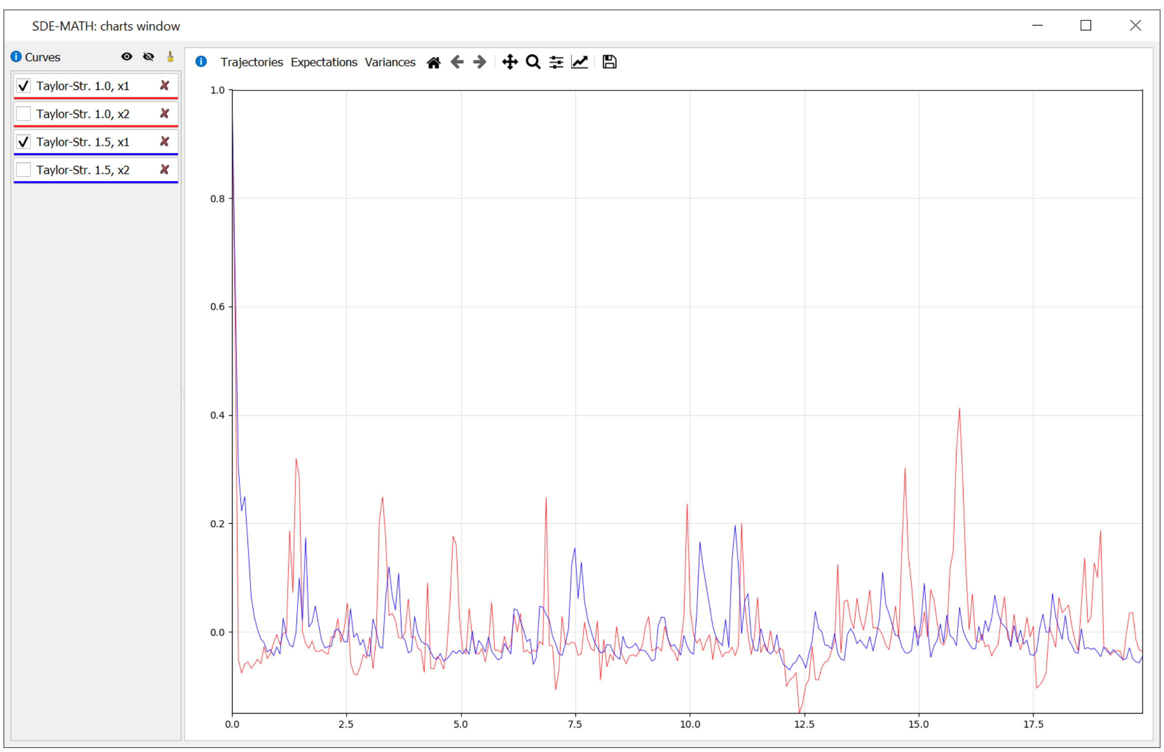


Figure 80: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.1$, $dt = 0.07$)

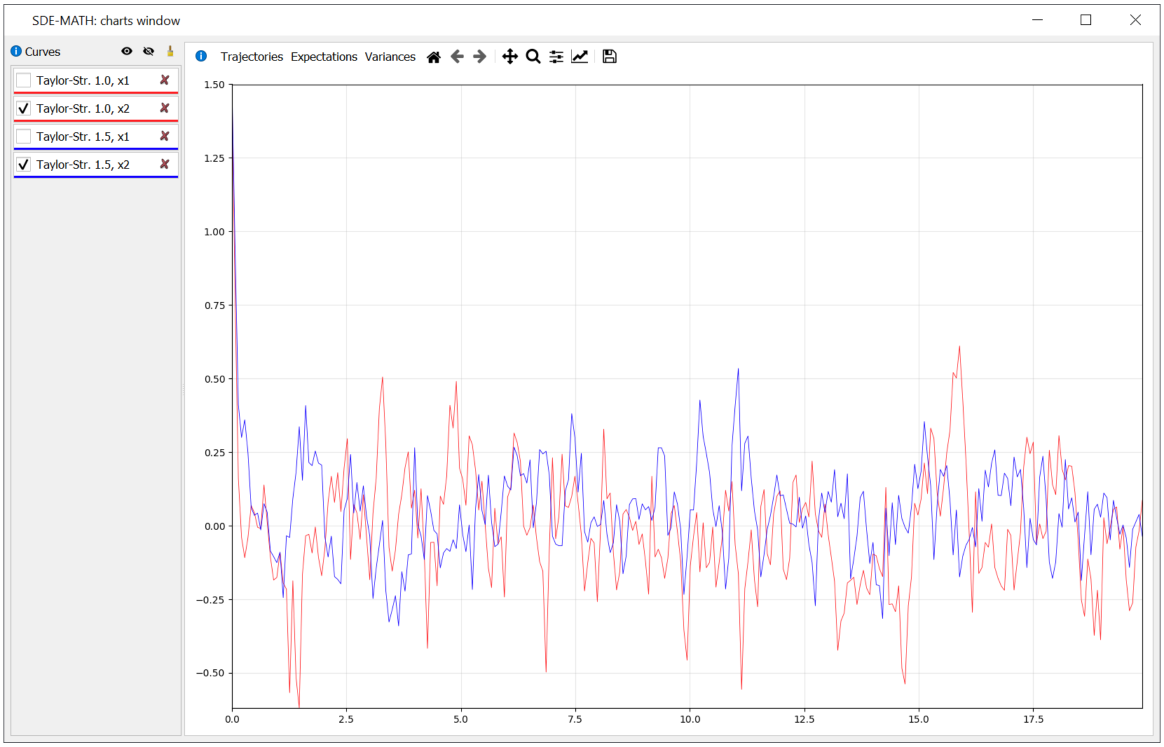
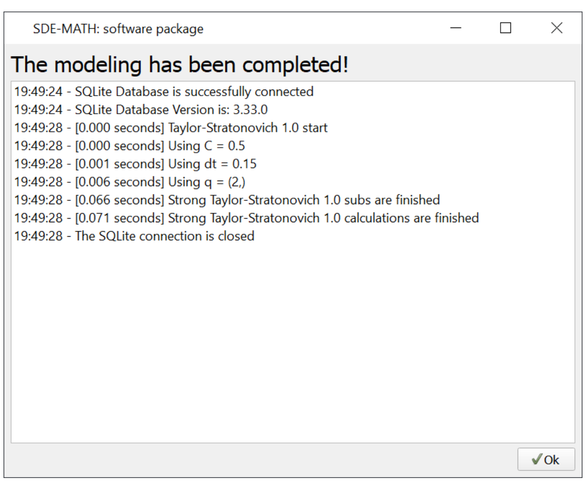
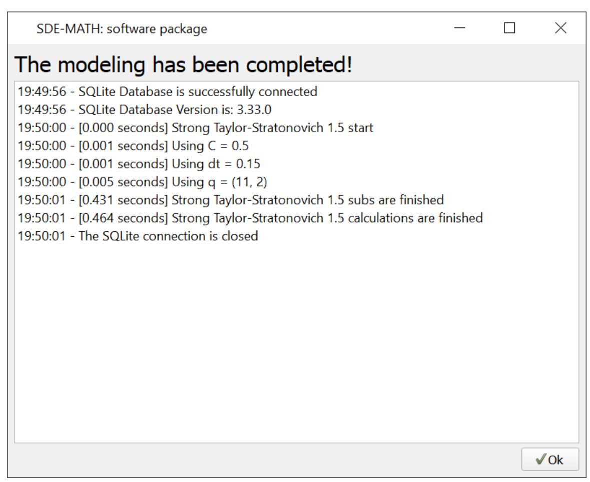


Figure 81: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.1$, $dt = 0.07$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.5$, $dt = 0.15$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.5$, $dt = 0.15$)

Figure 82: Modeling logs

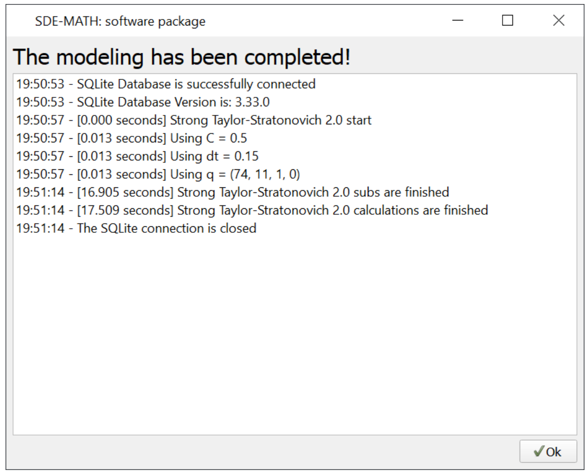


Figure 83: Strong Taylor–Stratonovich scheme of order 2.0 ($C = 0.5$, $dt = 0.15$)

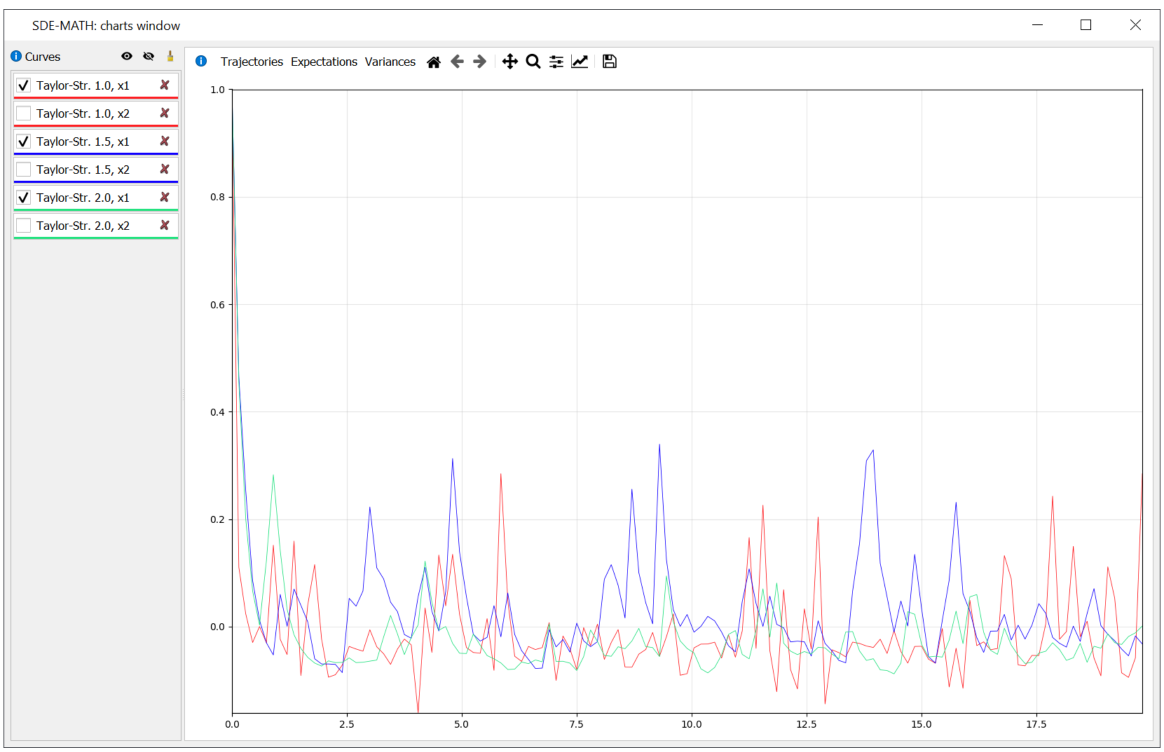
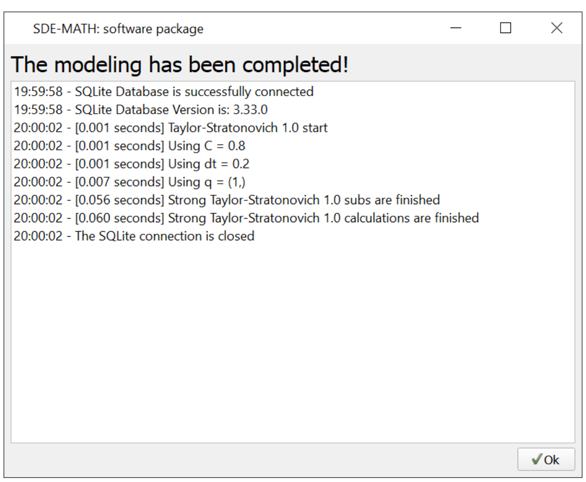
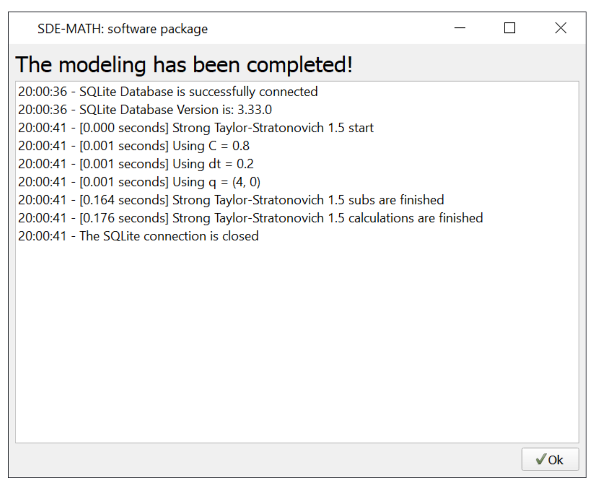


Figure 84: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 0.5$, $dt = 0.15$)

Figure 85: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 0.5$, $dt = 0.15$)

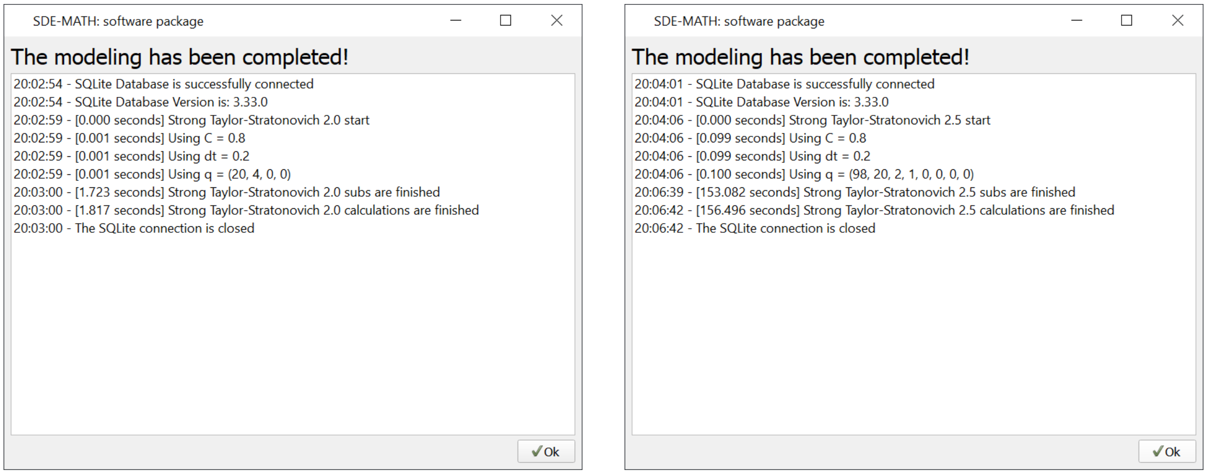


Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.8$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.8$, $dt = 0.2$)

Figure 86: Modeling logs



Strong Taylor–Stratonovich scheme of order 2.0 ($C = 0.8$, $dt = 0.2$)

Strong Taylor–Stratonovich scheme of order 2.5 ($C = 0.8$, $dt = 0.2$)

Figure 87: Modeling logs

Figure 88: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.8$, $dt = 0.2$)

Figure 89: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.8$, $dt = 0.2$)

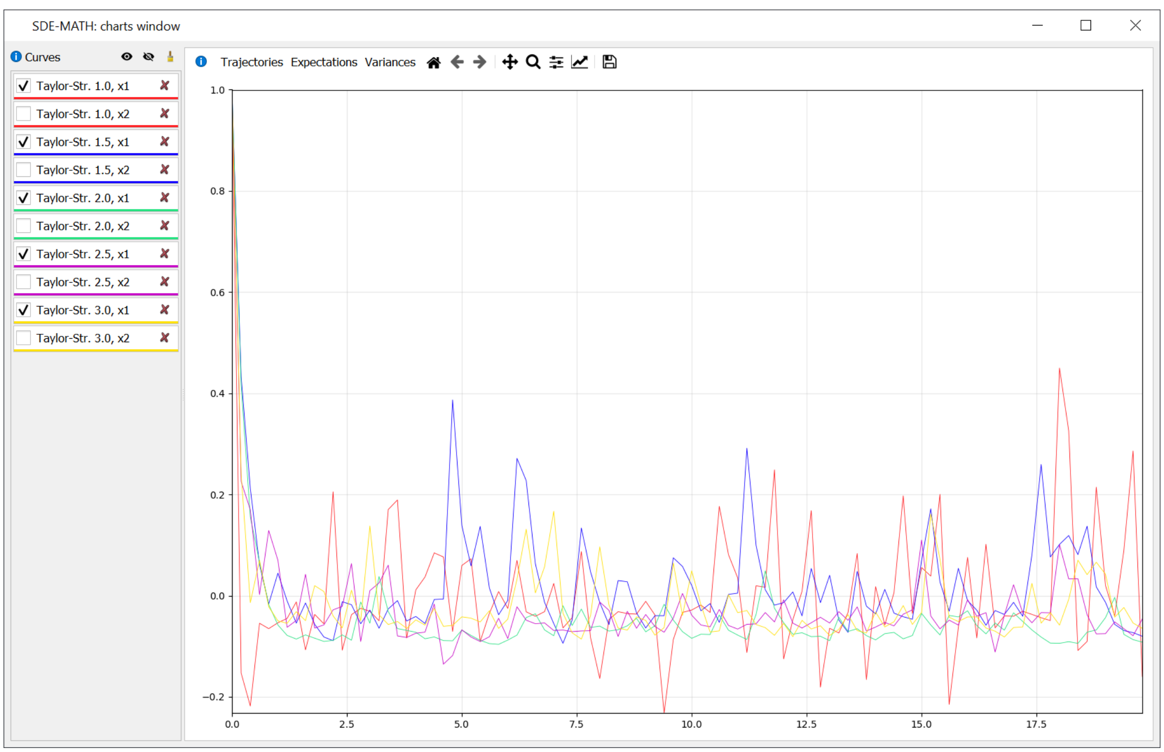
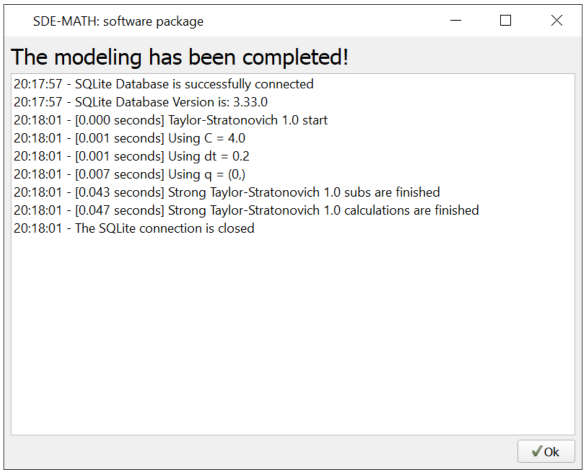
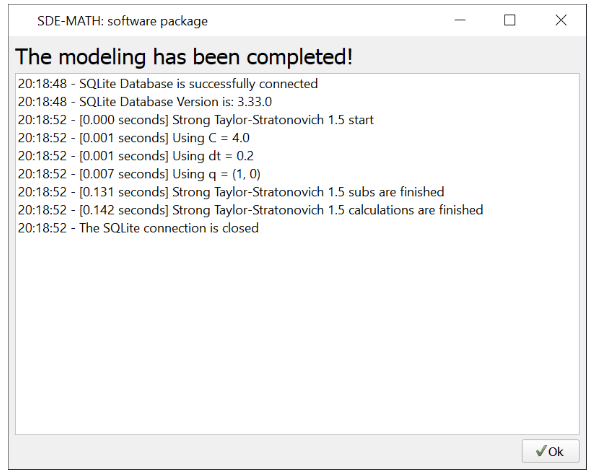


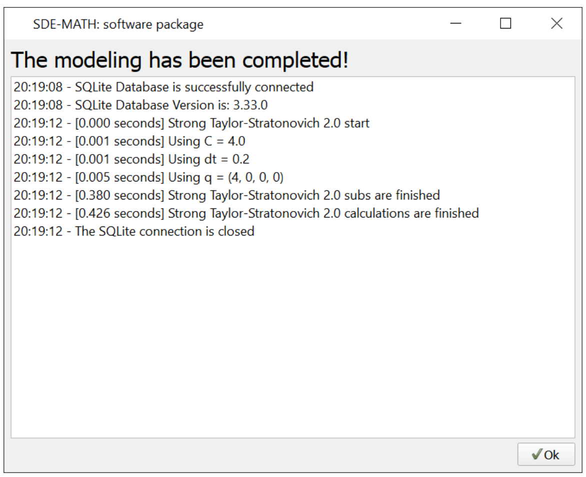
Figure 90: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 4$, $dt = 0.2$)



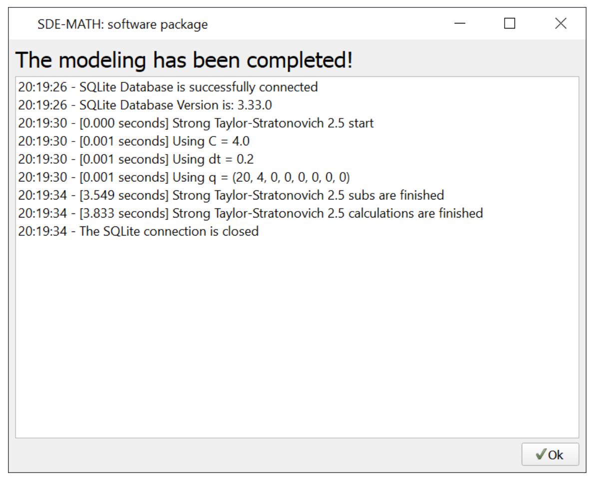
Strong Taylor–Stratonovich scheme of order 1.0 ($C = 4$, $dt = 0.2$)



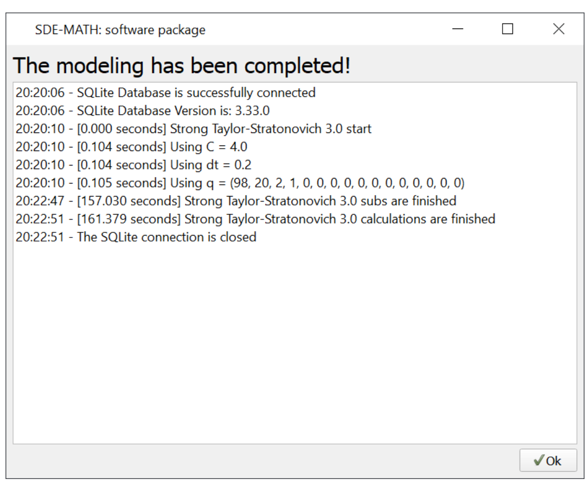
Strong Taylor–Stratonovich scheme of order 1.5 ($C = 4$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 2.0 ($C = 4$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 2.5 ($C = 4$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 3.0 ($C = 4$, $dt = 0.2$)

Figure 91: Modeling logs

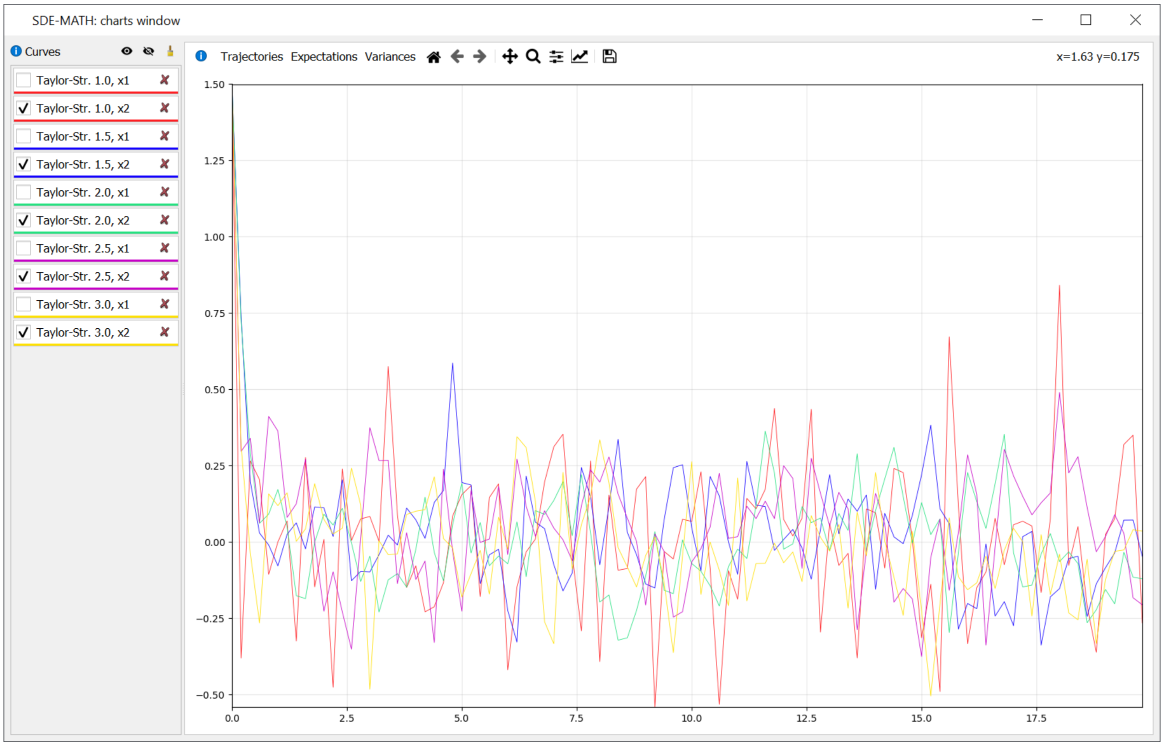


Figure 92: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C=4$, $dt=0.2$)

5.5 Example of Linear System of Itô SDEs (Solar Activity)

Consider a mathematical model of the solar activity without its average value in a form of the system of linear Itô SDEs (265) [4]. In (265) we choose [4] $n=2$, $m=1$, $k=2$, $\mathbf{x}_0^{(1)}=7$, $\mathbf{x}_0^{(2)}=-0.25$,

$$A=\begin{pmatrix}0 & 1\\ \\ -0.3205 & -0.14\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ \\ 0 & 0\end{pmatrix},\tag{269}$$

$$\mathbf{u}(t)\equiv\begin{pmatrix}0\\ \\ 0\end{pmatrix},\quad F=\begin{pmatrix}0\\ \\ 5.08\end{pmatrix}.\tag{270}$$

5.6 Visualization and Numerical Results for Solar Activity Model

This subsection is devoted to the visualization and numerical results for the model of solar activity (265), (269), (270).

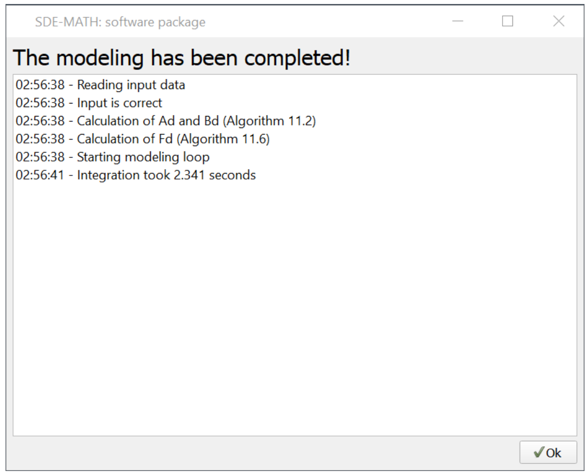


Figure 93: Modeling logs for solar activity model

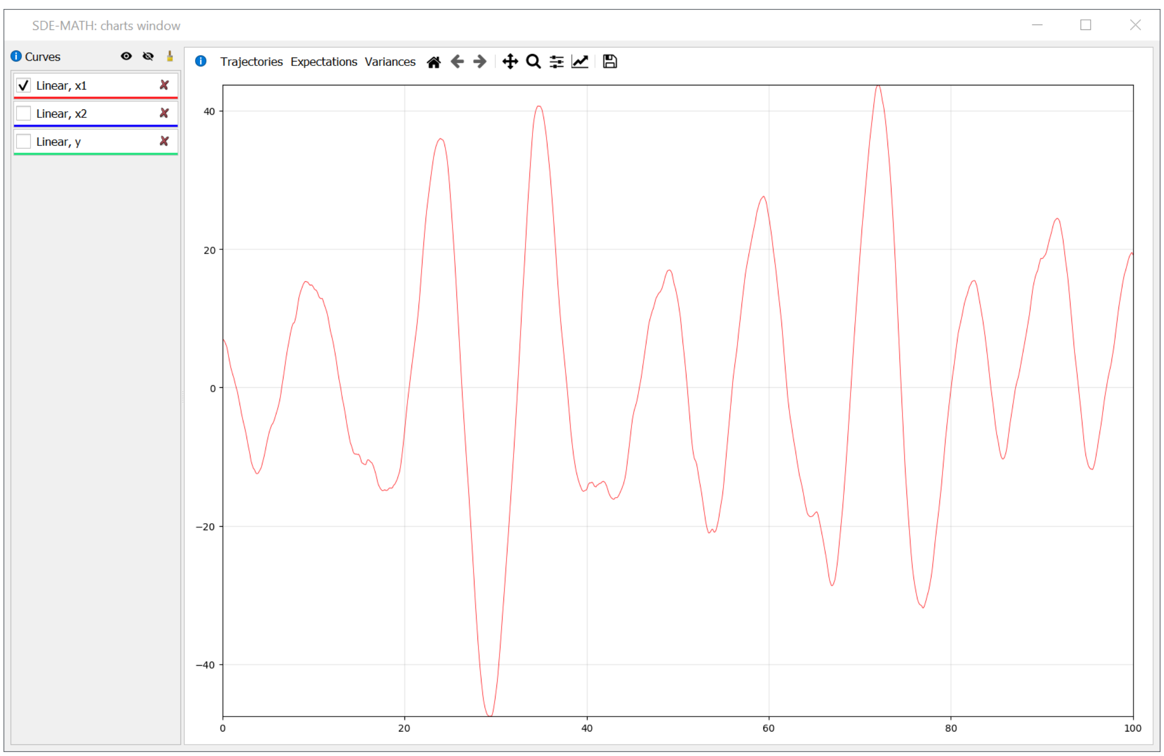


Figure 94: Solar activity model ($x_t^{(1)}$ component)

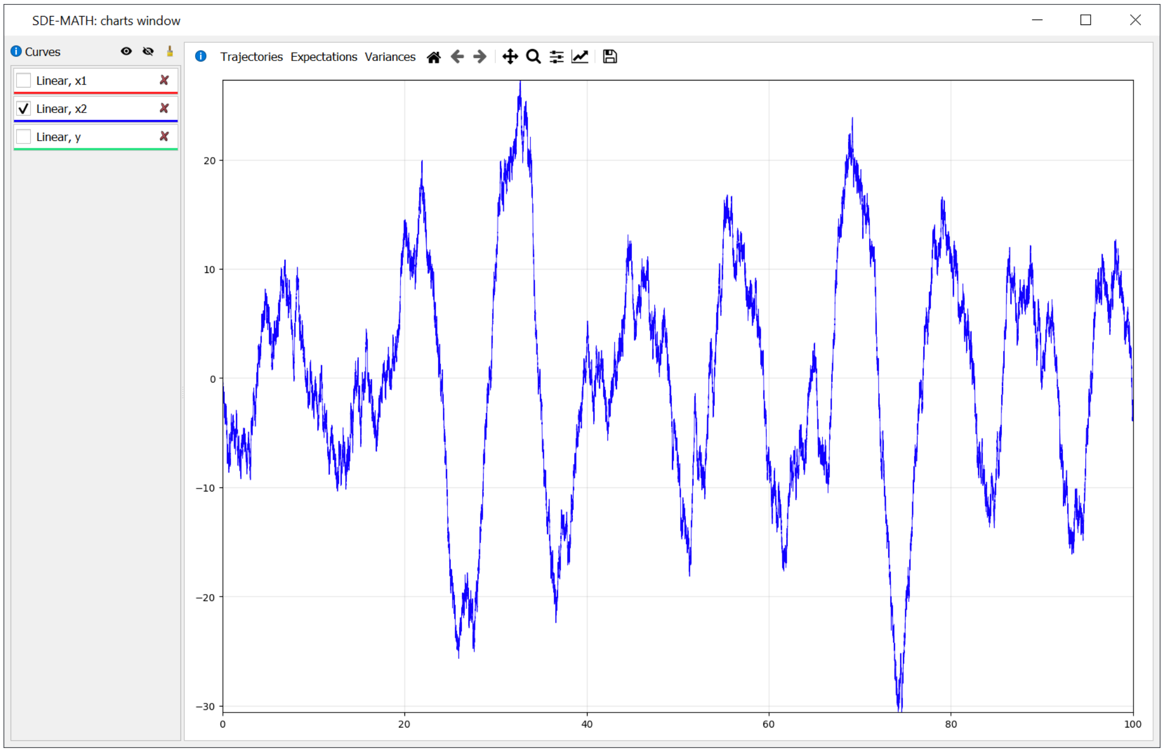


Figure 95: Solar activity model ($x_t^{(2)}$ component)

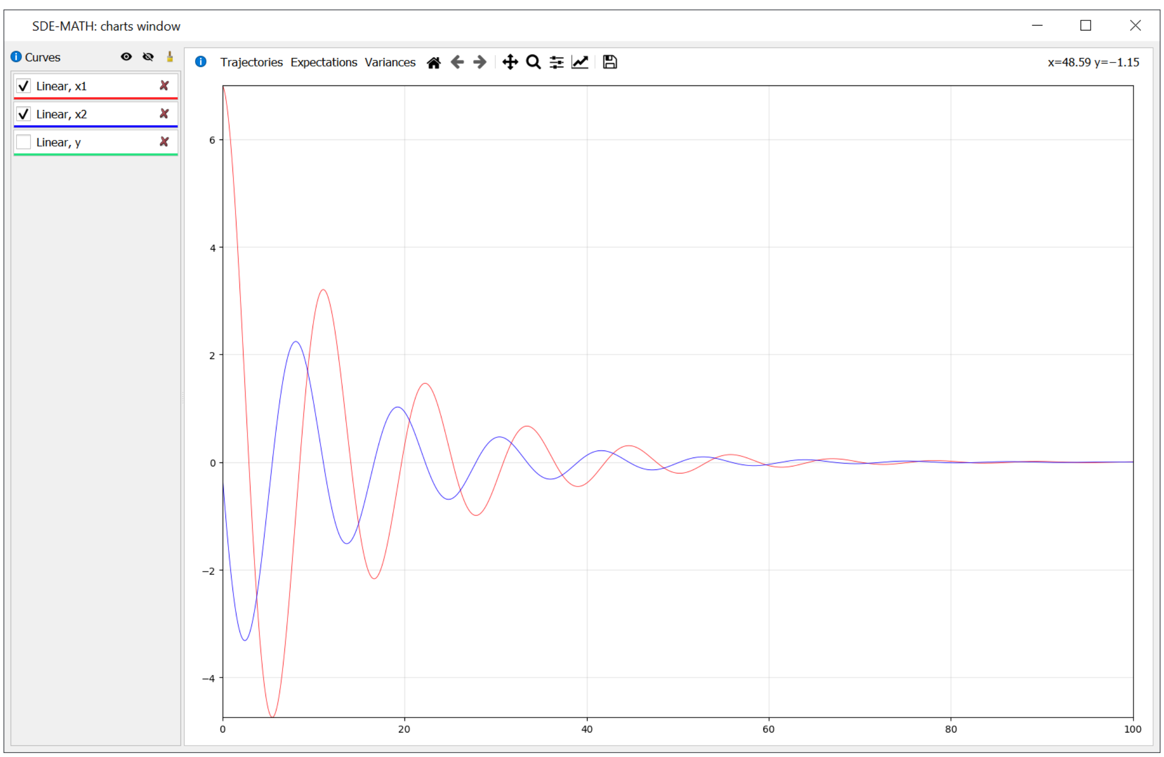


Figure 96: Solar activity model (expectations)

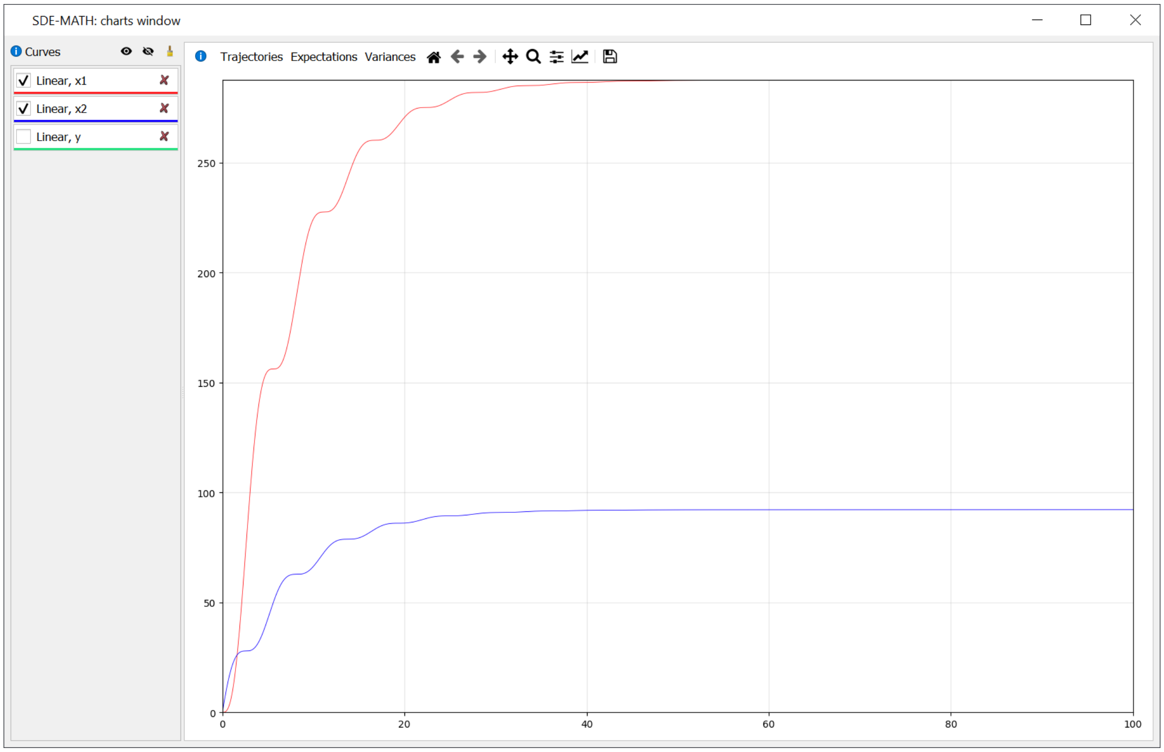


Figure 97: Solar activity model (variances)

5.7 Example of Abstract Linear System of Itô SDEs

Now consider the system of linear Itô SDEs (265) with the following data

$$n = 4, \quad m = 5, \quad k = 3, \quad \mathbf{x}_0^{(1)} = 1, \quad \mathbf{x}_0^{(2)} = 2, \quad \mathbf{x}_0^{(3)} = -1, \quad \mathbf{x}_0^{(4)} = -2, \tag{271}$$

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.2 & 0.1 \end{pmatrix}, \tag{272}$$

$$\mathbf{u}(t) \equiv \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^{\mathsf{T}}, \quad H = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}. \tag{273}$$

5.8 Visualization and Numerical Results for Abstract Linear System of Itô SDEs Obtained via the SDE-MATH Software Package

This subsection is devoted to the visualization and numerical results for the model (265), (271)–(273).

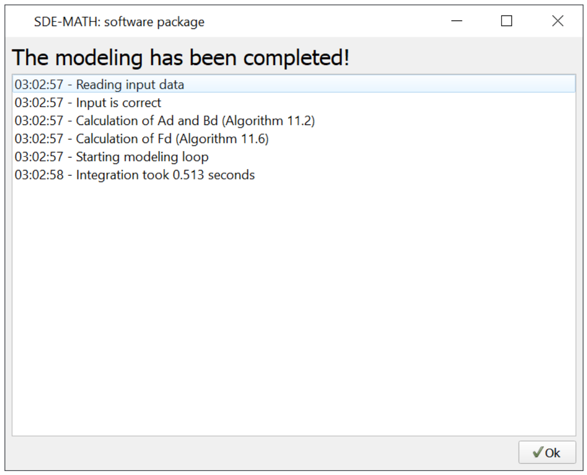


Figure 98: Modeling logs (linear system of Itô SDEs (265), (271)–(273))

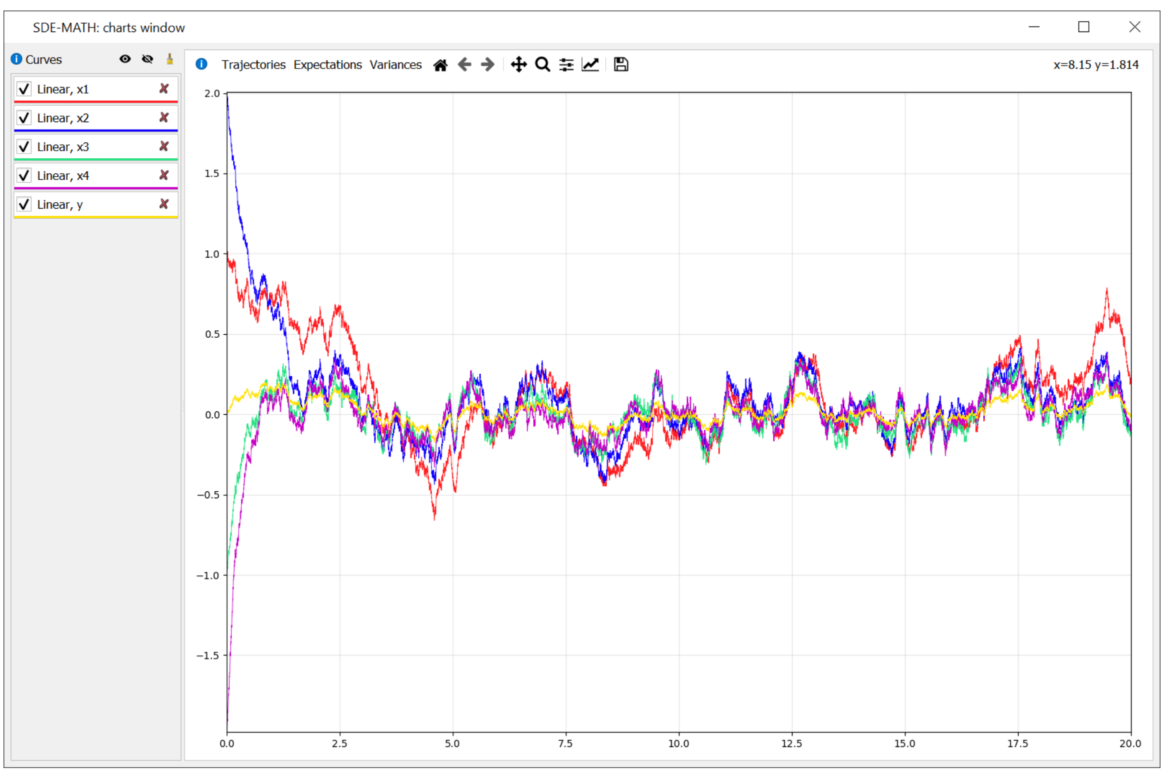


Figure 99: Linear system of Itô SDEs (265), (271)–(273) (components of solution)

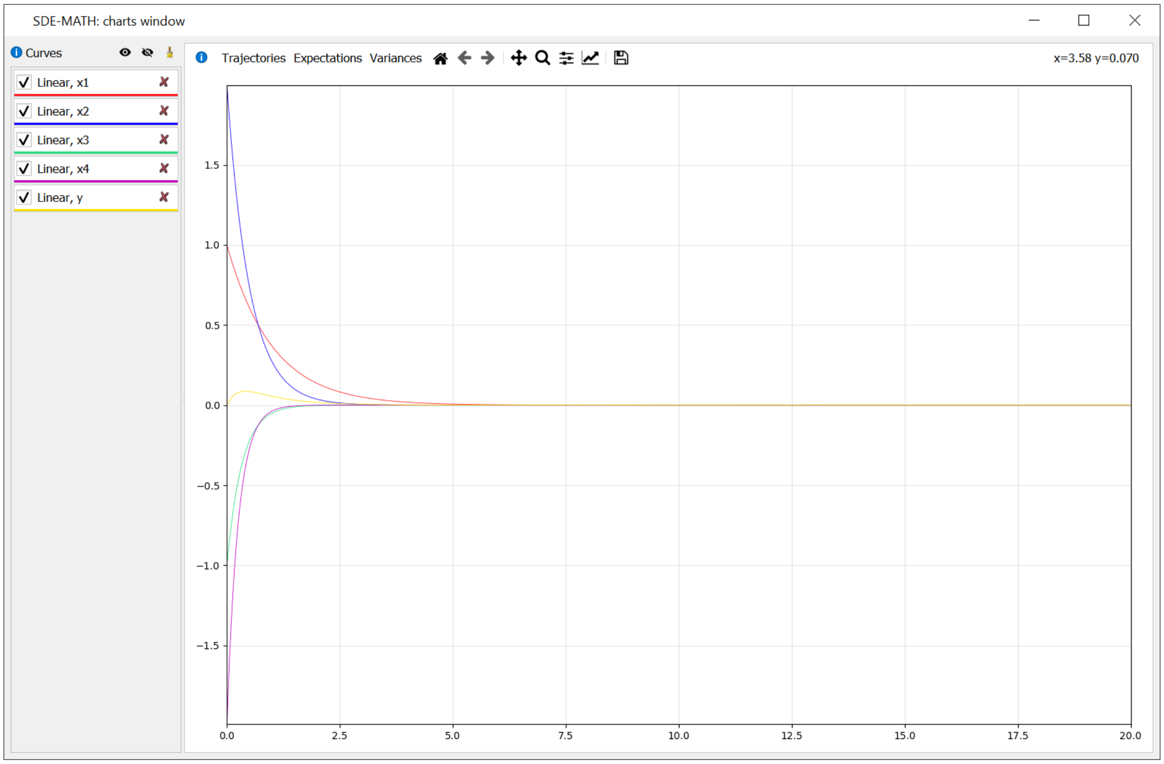


Figure 100: Linear system of Itô SDEs (265), (271)–(273) (expectations)

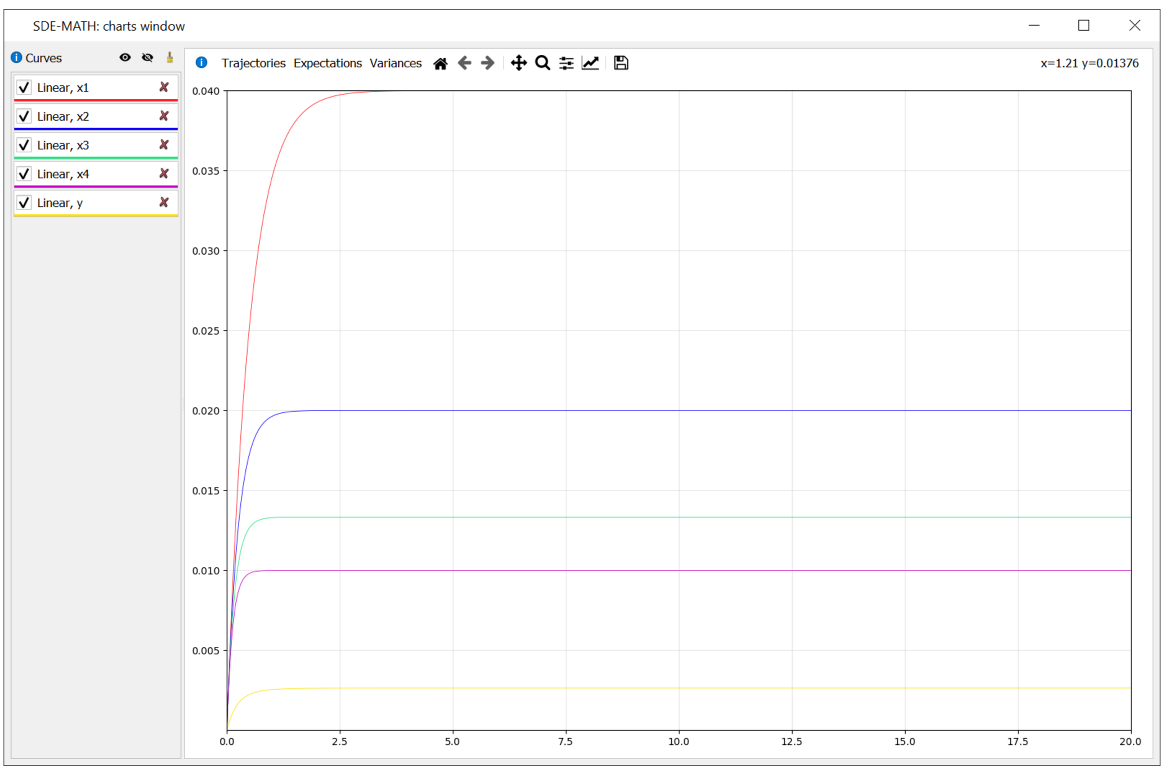


Figure 101: Linear system of Itô SDEs (265), (271)–(273) (variances)

6 Source Codes of the SDE-MATH Software Package in the Python Programming Language

6.1 Source Codes of Graphical User Interface

6.1.1 Source Codes of Main Menu

Listing 7: **Configuration file example**

```
"""
Configuration file example, change this paths to yours
"""

# Paths to resources
resources = "./resources/"

# Path to database
database = "./resources/database.db"

# Size of read buffer for the Fourier-Legendre coefficients
read_buffer_size = 8192

# Recursion limit for the Fourier-Legendre calculations
recursion_limit = 10 ** 8
```

Listing 8: **Program entry**

```
#!/usr/bin/env python
import logging
import os
import sys

from PyQt5 import QtGui, QtWidgets
from PyQt5.QtWidgets import QApplication
from PyQt5.QtWinExtras import QWinTaskbarButton

from config import database, images
from mathematics.sde.nonlinear.symbolic.coefficients.c import C
from tools.database import connect, disconnect
from ui.main.main_window import MainWindow


def main():

    logging.basicConfig(
        level=logging.INFO,
        format="%(asctime)s - %(levelname)s - %(message)s",
        datefmt="%H:%M:%S"
```

```
    )

    app = QApplication(sys.argv)
    app.setWindowIcon(QtGui.QIcon(os.path.join(images, "function.png")))
    app.setStyle(QtWidgets.QStyleFactory.create('Fusion'))

    main_window = MainWindow()

    main_window.taskbar_button = QWinTaskbarButton()
    main_window.taskbar_button.setOverlayIcon(QtGui.QIcon("resources/function.svg"))

    exit(app.exec())


if __name__ == "__main__":
    main()
```

Listing 9: **Main window**

```
from PyQt5.QtCore import QThreadPool, pyqtSignal
from PyQt5.QtWidgets import QStackedWidget, QMainWindow
from sympy.physics.mechanics.tests.test_system import lam

from init.initialization import initialization
from tools.fsys import is_locked
from ui.async_calls.worker import Worker
from ui.charts.charts_window import PlotWindow
from ui.main.greetings import GreetingsWidget
from ui.main.menu.base import MainMenuWidget
from ui.main.modeling.linear.base import LinearModelingWidget
from ui.main.modeling.nonliear.base import NonlinearModelingWidget
from ui.main.progress.complex_progress import ComplexProgressWidget
from ui.main.progress.simple_progress import SimpleProgressWidget


class MainWindow(QMainWindow):

    main_window_close = pyqtSignal()
    start_simple_progress = pyqtSignal(str)
    stop_simple_progress = pyqtSignal(str)

    def __init__(self):
        super(QMainWindow, self).__init__()

        self.plot_window = PlotWindow()

        self.stack_widget = QStackedWidget(self)

        self.main_menu = MainMenuWidget(self.stack_widget)
        self.complex_progress = ComplexProgressWidget(self.stack_widget)
        self.simple_progress = SimpleProgressWidget(self.stack_widget)
        self.greetings = GreetingsWidget(self.stack_widget)
        self.linear_modeling = LinearModelingWidget(self.stack_widget)
```

```
        self.nonlinear_modeling = NonlinearModelingWidget(self.stack_widget)

        self.stack_widget.addWidget(self.main_menu)
        self.stack_widget.addWidget(self.nonlinear_modeling)
        self.stack_widget.addWidget(self.linear_modeling)
        self.stack_widget.addWidget(self.greetings)
        self.stack_widget.addWidget(self.simple_progress)
        self.stack_widget.addWidget(self.complex_progress)

        self.setCentralWidget(self.stack_widget)

        self.exec_init()

        self.setWindowTitle("SDE-MATH: software package")
        self.setMinimumSize(640, 480)
        self.resize(800, 600)
        self.show()

        self.main_menu.group1.show_nonlinear_dialog.connect(self.show_nonlinear)
        self.main_menu.group2.show_nonlinear_dialog.connect(self.show_nonlinear)
        self.main_menu.group3.show_linear_dialog.connect(
          lambda: self.stack_widget.setCurrentWidget(self.linear_modeling))

        self.nonlinear_modeling.show_main_menu.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))
        self.nonlinear_modeling.start_progress.connect(
          lambda: self.stack_widget.setCurrentWidget(self.complex_progress))

        self.linear_modeling.show_main_menu.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))
        self.linear_modeling.start_progress.connect(
          lambda: self.stack_widget.setCurrentWidget(self.complex_progress))

        self.greetings.show_main_menu.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))

        self.greetings.show_main_menu.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))

        self.greetings.show_main_menu.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))

        self.complex_progress.back_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.main_menu))

        # plot events

        self.main_window_close.connect(self.plot_window.close)

        self.main_menu.charts_check.clicked.connect(self.plot_window.setVisible)
        self.plot_window.charts_show.connect(
          lambda: self.main_menu.charts_check.setChecked(True))
        self.plot_window.charts_hide.connect(
          lambda: self.main_menu.charts_check.setChecked(False))

```

```
        self.nonlinear_modeling.draw_chart.connect(self.plot_window.charts_list.new_items)
        self.nonlinear_modeling.draw_chart.connect(self.plot_window.plot_widget.new_items)
        self.nonlinear_modeling.draw_chart.connect(self.plot_window.show)

        self.nonlinear_modeling.charts_check.stateChanged.connect(self.plot_window.setVisible
        )
        self.plot_window.charts_show.connect(
            lambda: self.nonlinear_modeling.charts_check.setChecked(True))
        self.plot_window.charts_hide.connect(
            lambda: self.nonlinear_modeling.charts_check.setChecked(False))

        self.linear_modeling.draw_chart.connect(self.plot_window.charts_list.new_items)
        self.linear_modeling.draw_chart.connect(self.plot_window.plot_widget.new_items)
        self.linear_modeling.draw_chart.connect(self.plot_window.show)

        self.linear_modeling.charts_check.clicked.connect(self.plot_window.setVisible)
        self.plot_window.charts_show.connect(
            lambda: self.linear_modeling.charts_check.setChecked(True))
        self.plot_window.charts_hide.connect(
            lambda: self.linear_modeling.charts_check.setChecked(False))

        self.linear_modeling.start_progress.connect(self.complex_progress.spin)
        self.nonlinear_modeling.start_progress.connect(self.complex_progress.spin)
        self.linear_modeling.stop_progress.connect(self.complex_progress.stop)
        self.nonlinear_modeling.stop_progress.connect(self.complex_progress.stop)

    def closeEvent(self, event):
        self.main_window_close.emit()

    def exec_init(self):
        self.simple_progress.spin("Preparing the database...")
        self.stack_widget.setCurrentWidget(self.simple_progress)

        worker = Worker(initialization)
        worker.signals.finished.connect(self.init_done)

        QThreadPool.globalInstance().start(worker)

    def init_done(self):
        if not is_locked(".welcome.lock"):
            self.stack_widget.setCurrentWidget(self.greetings)
        else:
            self.stack_widget.setCurrentWidget(self.main_menu)
        self.simple_progress.stop()

    def show_nonlinear(self, scheme_id):
        self.nonlinear_modeling.set_scheme(scheme_id)
        self.stack_widget.setCurrentWidget(self.nonlinear_modeling)
```

Listing 10: **Greetings window**

```
from PyQt5.QtCore import pyqtSignal, Qt
from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QWidget, QSizePolicy, \
```

```
    QSpacerItem, QHBoxLayout, QLabel, QCheckBox, QApplication, QStyle

from tools.fsys import lock, unlock
from ui.main.svg import SVG


class GreetingsWidget(QWidget):

    show_main_menu = pyqtSignal()

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        header = QLabel("Welcome to SDE-MATH Software Package for "
                        "the Numerical Solution of Systems of Ito SDEs")
        font = header.font()
        font.setPointSize(15)
        header.setAlignment(Qt.AlignJustify)
        header.setWordWrap(True)
        header.setFont(font)

        welcome = QLabel(
            "Exact solutions of Ito SDEs are known in rare cases. For this "
            "reason, it becomes necessary to construct numerical methods for "
            "Ito SDEs. Moreover, the problem of numerical solution of Ito SDEs "
            "often occurs even in cases when the exact solution of Ito SDE is known. "
            "This means that in some cases, knowing the exact solution to the Ito "
            "SDE does not allow us to simulate it numerically in a simple way.", self
        )

        font = welcome.font()
        welcome.setFont(font)

        welcome.setAlignment(Qt.AlignJustify)
        welcome.setWordWrap(True)
        welcome.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Minimum))

        check_again = QCheckBox("Do not show again", self)
        next_btn = QPushButton("Ok", self)
        next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogApplyButton))

        check_again.clicked.connect(self.check_lock)
        next_btn.clicked.connect(lambda: self.show_main_menu.emit())

        controls = QHBoxLayout()
        controls.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        controls.addWidget(check_again)
        controls.addWidget(next_btn)
        controls.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        eq1 = QHBoxLayout()
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        eq1.addWidget(SVG("equation1.svg", scale_factor=1.))
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

```

```
    eq2 = QHBoxLayout()
    eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    eq2.addWidget(SVG("equation2.svg", scale_factor=1.))
    eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    column = QVBoxLayout()
    column.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    column.addWidget(header)
    column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
    column.addLayout(eq1)
    column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
    column.addLayout(eq2)
    column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
    column.addWidget(welcome)
    column.addItem(QSpacerItem(0, 30, QSizePolicy.Expanding, QSizePolicy.Minimum))
    column.addLayout(controls)
    column.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    layout = QHBoxLayout()
    layout.addItem(QSpacerItem(50, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    layout.addLayout(column)
    layout.addItem(QSpacerItem(50, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    self.setLayout(layout)

  def check_lock(self):
    if self.sender().isChecked():
      lock(".welcome.lock")
    else:
      unlock(".welcome.lock")
```

Listing 11: **Info icon**

```
from PyQt5.QtCore import QSize
from PyQt5.QtWidgets import QWidget, QApplication, QStyle, QLabel


class InfoIcon(QLabel):

  def __init__(self, text: str, parent=None):
    super(QWidget, self).__init__(parent)

    self.setToolTip(text)
    self.setStyleSheet("QToolTip {background: white;}")
    self.setPixmap(QApplication.style().standardIcon(
      QStyle.SP_MessageBoxInformation).pixmap(QSize(16, 16)))
```

Listing 12: **Error widget**

```
from PyQt5.QtCore import QSize
from PyQt5.QtWidgets import QWidget, QApplication, QStyle, QLabel, QHBoxLayout,
    QSpacerItem, QSizePolicy
```

```python


class ErrorWidget(QWidget):

  def __init__(self, text: str, parent=None):
    super(QWidget, self).__init__(parent)

    msg_m = QLabel(text)
    msg_m.setStyleSheet("QLabel { color: rgb(230, 0, 0); }")

    msg_i = QLabel()
    msg_i.setStyleSheet("QToolTip { background: white; }")
    msg_i.setPixmap(QApplication.style().standardIcon(
      QStyle.SP_MessageBoxCritical).pixmap(QSize(16, 16)))

    layout = QHBoxLayout()
    layout.setContentsMargins(0, 0, 0, 0)
    layout.addWidget(msg_i)
    layout.addWidget(msg_m)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    self.setLayout(layout)
```

Listing 13: **Svg picture**

```python
import os

from PyQt5.QtCore import QSize
from PyQt5.QtSvg import QSvgWidget
from PyQt5.QtWidgets import QSizePolicy

from config import images


class SVG(QSvgWidget):

  def __init__(self, name: str, scale_factor=1.):
    super(QSvgWidget, self).__init__()

    self.load(os.path.join(images, name))

    self.scale_factor = scale_factor
    self.setSizePolicy(QSizePolicy.Fixed, QSizePolicy.Fixed)

  def sizeHint(self):
    size = self.renderer().defaultSize()
    return QSize(size.width() * self.scale_factor,
           size.height() * self.scale_factor)
```

Listing 14: **Main menu (base part)**

```python
from PyQt5.QtWidgets import QWidget, QSizePolicy, QSpacerItem, QHBoxLayout, QVBoxLayout,
```

```
    QCheckBox, QLabel

from ui.main.info import InfoIcon
from ui.main.menu.linear import LinearGroupWidget
from ui.main.menu.taylor_ito import ItoGroupWidget
from ui.main.menu.taylor_stratonovich import StratonovichGroupWidget


class MainMenuWidget(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.charts_check = QCheckBox("Charts window", self)

    icon = InfoIcon("This is charts window checkbox, it will\n"
            "follow you on every application dialog, so\n"
            "you can easily open or close window with available charts")

    bar_layout = QHBoxLayout()
    bar_layout.addItem(QSpacerItem(0, 35, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bar_layout.addWidget(icon)
    bar_layout.addWidget(self.charts_check)

    header = QLabel("Strong Numerical Schemes for Ito SDEs", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    icon = InfoIcon("You are now in main menu, you can choose\n"
            "any scheme to perform modeling")

    header_layout = QHBoxLayout()
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
    )
    header_layout.addWidget(icon)
    header_layout.addWidget(header)
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
    )

    self.group3 = LinearGroupWidget(self)
    self.group1 = ItoGroupWidget(self)
    self.group2 = StratonovichGroupWidget(self)

    menu_layout = QHBoxLayout()
    menu_layout.addWidget(self.group1)
    menu_layout.addWidget(self.group2)
    menu_layout.addWidget(self.group3)

    layout = QVBoxLayout()
    layout.addLayout(bar_layout)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    layout.addLayout(header_layout)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    layout.addLayout(menu_layout)
```

```
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.setLayout(layout)
```

Listing 15: **Main menu (linear part)**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox


class LinearGroupWidget(QGroupBox):

  show_linear_dialog = pyqtSignal()

  def __init__(self, parent=None):
    super(QGroupBox, self).__init__(parent)

    linear_btn = QPushButton("Dispersion Spectral Decomposition")

    layout = QVBoxLayout()
    layout.addWidget(linear_btn)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.setLayout(layout)

    self.setTitle("Linear Ito SDEs Systems Modeling")
    self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))

    linear_btn.clicked.connect(lambda: self.show_linear_dialog.emit())
```

Listing 16: **Main menu (Taylor-Itô part)**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox


class ItoGroupWidget(QGroupBox):

  show_nonlinear_dialog = pyqtSignal(int)

  def __init__(self, parent=None):
    super(QGroupBox, self).__init__(parent)

    btn1 = QPushButton("Euler")
    btn2 = QPushButton("Milstein")
    btn3 = QPushButton("Convergence Order 1.5")
    btn4 = QPushButton("Convergence Order 2.0")
    btn5 = QPushButton("Convergence Order 2.5")
    btn6 = QPushButton("Convergence Order 3.0")

    layout = QVBoxLayout()
    layout.addWidget(btn1)
```

```
    layout.addWidget(btn2)
    layout.addWidget(btn3)
    layout.addWidget(btn4)
    layout.addWidget(btn5)
    layout.addWidget(btn6)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.setLayout(layout)

    self.setTitle("Taylor-Ito Schemes")
    self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))

    btn1.clicked.connect(lambda: self.show_nonlinear_dialog.emit(0))
    btn2.clicked.connect(lambda: self.show_nonlinear_dialog.emit(1))
    btn3.clicked.connect(lambda: self.show_nonlinear_dialog.emit(2))
    btn4.clicked.connect(lambda: self.show_nonlinear_dialog.emit(3))
    btn5.clicked.connect(lambda: self.show_nonlinear_dialog.emit(4))
    btn6.clicked.connect(lambda: self.show_nonlinear_dialog.emit(5))
```

Listing 17: **Main menu (Taylor–Stratonovich part)**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox


class StratonovichGroupWidget(QGroupBox):

  show_nonlinear_dialog = pyqtSignal(int)

  def __init__(self, parent=None):
    super(QGroupBox, self).__init__(parent)

    btn1 = QPushButton("Convergence Order 1.0")
    btn2 = QPushButton("Convergence Order 1.5")
    btn3 = QPushButton("Convergence Order 2.0")
    btn4 = QPushButton("Convergence Order 2.5")
    btn5 = QPushButton("Convergence Order 3.0")

    layout = QVBoxLayout()
    layout.addWidget(btn1)
    layout.addWidget(btn2)
    layout.addWidget(btn3)
    layout.addWidget(btn4)
    layout.addWidget(btn5)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.setLayout(layout)

    self.setTitle("Taylor-Stratonovich Schemes")
    self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))

    btn1.clicked.connect(lambda: self.show_nonlinear_dialog.emit(6))
    btn2.clicked.connect(lambda: self.show_nonlinear_dialog.emit(7))
```

```
    btn3.clicked.connect(lambda: self.show_nonlinear_dialog.emit(8))
    btn4.clicked.connect(lambda: self.show_nonlinear_dialog.emit(9))
    btn5.clicked.connect(lambda: self.show_nonlinear_dialog.emit(10))
```

Listing 18: **Complex progress view**

```
import logging

from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QSpacerItem, QSizePolicy,
    \
  QListWidget, QLabel, QPushButton, QApplication, QStyle
from pyqtspinner.spinner import WaitingSpinner

from ui.main.progress.log_handler import LogHandler


class ComplexProgressWidget(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.spinner = WaitingSpinner(self,
                   radius=5.0,
                   lines=10,
                   line_length=5.0,
                   centerOnParent=False)
    self.list_widget = QListWidget(self)
    self.handler = LogHandler(self.handle_message)

    self.label = QLabel(self)
    font = self.label.font()
    font.setPointSize(15)
    self.label.setFont(font)

    self.back_btn = QPushButton("Ok")
    self.back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogApplyButton))
    self.back_btn.hide()

    spinner_layout = QHBoxLayout()
    spinner_layout.addWidget(self.spinner)
    spinner_layout.addWidget(self.label)
    spinner_layout.addSpacerItem(QSpacerItem(0, 0, QSizePolicy.Expanding,
                          QSizePolicy.Minimum))

    bottom_bar = QHBoxLayout()
    bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bottom_bar.addWidget(self.back_btn)

    layout = QVBoxLayout()
    layout.addLayout(spinner_layout)
    layout.addWidget(self.list_widget)
    layout.addLayout(bottom_bar)

```

```
    self.setLayout(layout)

  def handle_message(self, text):
    self.list_widget.addItem(text)
    self.list_widget.scrollToBottom()

  def spin(self, text):
    self.list_widget.clear()
    logging.getLogger().addHandler(self.handler)
    self.back_btn.hide()
    self.spinner.start()
    self.label.setText(text)

  def stop(self, text):
    self.back_btn.show()
    self.list_widget.scrollToBottom()
    self.label.setText(text)
    self.spinner.stop()
    logging.getLogger().removeHandler(self.handler)
```

Listing 19: **Simple progress view**

```
from PyQt5.QtCore import Qt
from PyQt5.QtWidgets import QVBoxLayout, QWidget, QLabel, QSpacerItem, QSizePolicy
from pyqtspinner.spinner import WaitingSpinner


class SimpleProgressWidget(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.spinner = WaitingSpinner(self, radius=15.0, lines=10, line_length=15.0)

    self.label = QLabel()
    self.label.setAlignment(Qt.AlignCenter)
    font = self.label.font()
    font.setPointSize(15)
    self.label.setFont(font)

    layout = QVBoxLayout()
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    layout.addWidget(self.spinner)
    layout.addItem(QSpacerItem(0, 50, QSizePolicy.Minimum, QSizePolicy.Minimum))
    layout.addWidget(self.label)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.setLayout(layout)

  def spin(self, text):
    self.spinner.start()
    self.label.setText(text)

```

```
  def stop(self):
    self.spinner.stop()
```

Listing 20: **Log handler for application**

```
import logging


class LogHandler(logging.Handler):

  def __init__(self, callback):
    super().__init__()
    self.callback = callback
    self.setFormatter(logging.Formatter("%(asctime)s - %(message)s", "%H:%M:%S"))

  def handle(self, record):
    self.callback(self.format(record))
```

Listing 21: **Matrix widget**

```
from PyQt5.QtWidgets import QTableWidgetItem, QTableWidget, QSizePolicy


class MatrixWidget(QTableWidget):

  def __init__(self, parent=None):
    super(QTableWidget, self).__init__(parent)

    self.itemChanged.connect(self.item_changed)
    self.m = [["0"]]

    self.setRowCount(1)
    self.setColumnCount(1)
    self.setItem(0, 0, CustomItem("0"))

    self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))

  def resize_w(self, w: int):

    self.blockSignals(True)

    old_w = self.columnCount()
    self.setColumnCount(w)
    h = self.rowCount()

    self.m = [[self.m[i][j] if i < h and j < old_w else "0"
           for j in range(w)]
          for i in range(h)]

    for i in range(h):
      for j in range(w):
        item = self.item(i, j)
```

```
        if item is not None:
          item.setText(self.m[i][j])
        else:
          self.setItem(i, j, CustomItem(self.m[i][j]))

    self.blockSignals(False)

  def resize_h(self, h: int):

    self.blockSignals(True)

    old_h = self.rowCount()
    w = self.columnCount()
    self.setRowCount(h)

    self.m = [[self.m[i][j] if i < old_h and j < w else "0"
          for j in range(w)]
         for i in range(h)]

    for i in range(h):
      for j in range(w):
        item = self.item(i, j)
        if item is not None:
          item.setText(self.m[i][j])
        else:
          self.setItem(i, j, CustomItem(self.m[i][j]))

    self.blockSignals(False)

  def item_changed(self, item):
    self.m[item.row()][item.column()] = item.text()


class CustomItem(QTableWidgetItem):

  def __init__(self, value: str):
    super(QTableWidgetItem, self).__init__(value)

    self.valid = True
```

6.1.2 Source Codes of Charts Window

Listing 22: **Charts window**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QMainWindow, QSplitter

from ui.charts.side.available_charts_widget import AvailableChartsWidget
from ui.charts.visuals.charts_widget import ChartsWidget


```

```
class PlotWindow(QMainWindow):

  charts_show = pyqtSignal()
  charts_hide = pyqtSignal()

  def __init__(self):
    super(QMainWindow, self).__init__()

    self.plot_widget = ChartsWidget(self)
    self.charts_list = AvailableChartsWidget(self)

    splitter = QSplitter()
    splitter.addWidget(self.charts_list)
    splitter.addWidget(self.plot_widget)
    splitter.setSizes([splitter.width() / 0.85,
                       splitter.width() / 0.15])

    layout = QHBoxLayout(self)
    layout.addWidget(splitter)

    central_widget = QWidget()
    central_widget.setLayout(layout)
    self.setCentralWidget(central_widget)

    self.setWindowTitle("SDE-MATH: charts window")
    self.resize(1200, 800)

    self.charts_list.on_show_all.connect(self.plot_widget.show_all)
    self.charts_list.on_hide_all.connect(self.plot_widget.hide_all)
    self.charts_list.on_remove_all.connect(self.plot_widget.delete_all)

  def showEvent(self, event):
    self.charts_show.emit()

  def closeEvent(self, event):
    self.charts_hide.emit()
```

Listing 23: **Curves list**

```
import os

from PyQt5 import QtGui
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QVBoxLayout, QWidget, QSizePolicy, QSpacerItem, \
    QScrollArea, QLabel, QHBoxLayout, QPushButton, QApplication, QStyle

from config import images
from ui.charts.side.item_widget import ItemWidget
from ui.main.info import InfoIcon


class AvailableChartsWidget(QWidget):

```

```
  on_hide_all = pyqtSignal()
  on_show_all = pyqtSignal()
  on_remove_all = pyqtSignal()

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.items = dict()

    self.spacer = QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding)
    self.plot_widget = self.parent().plot_widget

    remove_all = QPushButton()
    remove_all.setFlat(True)
    remove_all.setIcon(
      QApplication.style().standardIcon(QStyle.SP_DialogResetButton))

    hide_all = QPushButton()
    hide_all.setFlat(True)
    hide_all.setIcon(QtGui.QIcon(os.path.join(images, "crossed.png")))

    show_all = QPushButton()
    show_all.setFlat(True)
    show_all.setIcon(QtGui.QIcon(os.path.join(images, "eye.png")))

    header_layout = QHBoxLayout()
    header_layout.addWidget(
      InfoIcon("Here You will see all modeling series\n"
           "You can hide them or delete, if you need to"))
    header_layout.addItem(
      QSpacerItem(5, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
    header_layout.addWidget(QLabel("Curves"))
    header_layout.addItem(
      QSpacerItem(5, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
    header_layout.setContentsMargins(0, 0, 0, 0)
    header_layout.setSpacing(0)
    header_layout.addItem(
      QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    header_layout.addWidget(show_all)
    header_layout.addWidget(hide_all)
    header_layout.addWidget(remove_all)

    self.layout = QVBoxLayout()
    self.layout.setContentsMargins(3, 3, 3, 3)
    self.layout.setSpacing(2)

    scroll_widget = QWidget(self)
    scroll_widget.setLayout(self.layout)

    scroll_area = QScrollArea(self)
    scroll_area.setWidgetResizable(True)
    scroll_area.setWidget(scroll_widget)

    self.layout.addItem(self.spacer)

```

```
        layout = QVBoxLayout()
        layout.setContentsMargins(0, 0, 0, 0)
        layout.addLayout(header_layout)
        layout.addWidget(scroll_area)

        self.setLayout(layout)

        self.setSizePolicy(
            QSizePolicy(QSizePolicy.MinimumExpanding,
                        QSizePolicy.MinimumExpanding))

        show_all.clicked.connect(self.show_all)
        hide_all.clicked.connect(self.hide_all)
        remove_all.clicked.connect(self.delete_all)

    def new_items(self, lines: list):
        self.layout.removeItem(self.spacer)

        for i in range(len(lines)):
            item_widget = ItemWidget(lines[i].name, lines[i].color, parent=self)
            item_widget.uid = lines[i].uid
            item_widget.on_show.connect(self.plot_widget.show_item)
            item_widget.on_hide.connect(self.plot_widget.hide_item)
            item_widget.on_delete.connect(self.plot_widget.delete_item)
            item_widget.on_delete.connect(self.delete_item)
            self.items[lines[i].uid] = item_widget

            self.plot_widget.hide_label.connect(lambda uid: self.items[uid].hide())
            self.plot_widget.show_label.connect(lambda uid: self.items[uid].show())

            self.layout.addWidget(item_widget)

        self.layout.addItem(self.spacer)

    def delete_item(self):
        s = self.sender()
        s.setParent(None)
        self.items.pop(s.uid)
        self.layout.removeWidget(s)

    def delete_all(self):
        for item in self.items.values():
            item.setParent(None)
            self.layout.removeWidget(item)
        self.items.clear()
        self.on_remove_all.emit()

    def hide_all(self):
        for item in self.items.values():
            item.checkbox.blockSignals(True)
            item.checkbox.setChecked(False)
            item.checkbox.blockSignals(False)
        self.on_hide_all.emit()

    def show_all(self):
```

```
    for item in self.items.values():
      item.checkbox.blockSignals(True)
      item.checkbox.setChecked(True)
      item.checkbox.blockSignals(False)
    self.on_show_all.emit()
```

Listing 24: **Curves list item**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QPushButton, QSizePolicy, QHBoxLayout, QStyle, \
  QApplication, QCheckBox, QVBoxLayout, QLabel, QSpacerItem, QFrame


class ItemWidget(QFrame):

  on_delete = pyqtSignal(object)
  on_hide = pyqtSignal(int)
  on_show = pyqtSignal(int)

  def __init__(self, name, color, parent=None):
    super(QFrame, self).__init__(parent)

    self.uid = 0

    self.setFrameShape(QFrame.StyledPanel)
    self.setStyleSheet("QFrame { background: white; }")

    self.checkbox = QCheckBox(name)
    self.checkbox.setChecked(True)

    btn = QPushButton()
    btn.setFlat(True)
    btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogCancelButton))

    underline = QLabel()
    underline.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Maximum))
    underline.setMaximumHeight(3)
    underline.setStyleSheet(f"QLabel {{ background: {color}; }}")

    layout = QHBoxLayout()
    layout.setContentsMargins(3, 3, 3, 3)
    layout.setSpacing(0)
    layout.addWidget(self.checkbox)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    layout.addWidget(btn)

    layout_underlined = QVBoxLayout()
    layout_underlined.setContentsMargins(0, 0, 0, 0)
    layout_underlined.setSpacing(0)
    layout_underlined.addLayout(layout)
    layout_underlined.addWidget(underline)

    self.setLayout(layout_underlined)
```

```

      self.checkbox.stateChanged.connect(self.checkbox_changed)
      btn.clicked.connect(lambda: self.on_delete.emit(self.uid))

  def checkbox_changed(self, v):
    if v > 0:
      self.on_show.emit(self.uid)
    else:
      self.on_hide.emit(self.uid)
```

Listing 25: **Charts area**

```
import matplotlib.pyplot as plt
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QSizePolicy, QFrame, QHBoxLayout, QPushButton, \
  QSpacerItem
from PyQt5.QtWidgets import QVBoxLayout
from matplotlib.backends.backend_qt5agg import FigureCanvasQTAgg as FigureCanvas

from ui.charts.visuals.color import Color
from ui.charts.visuals.toolbar import ToolBar
from ui.main.info import InfoIcon


class ChartsWidget(QFrame):

  hide_label = pyqtSignal(int)
  show_label = pyqtSignal(int)

  def __init__(self, parent=None):
    super(QFrame, self).__init__(parent)

    self.plots = dict()
    self.mode = 0

    self.setFrameStyle(QFrame.StyledPanel)
    self.setStyleSheet("QFrame { background: white; }")

    self.figure = plt.figure()
    self.canvas = FigureCanvas(self.figure)
    self.canvas.mpl_connect('resize_event', self.on_resize)
    self.toolbar = ToolBar(self.canvas, self)

    self.ax = self.figure.add_subplot(111)
    self.ax.margins(0)
    self.ax.grid(axis='both', alpha=.3)
    self.ax.relim(visible_only=True)
    self.ax.autoscale()

    self.figure.tight_layout()
    self.canvas.draw()

    self.btn_to_fn = QPushButton("Trajectories")
```

```
    self.btn_to_fn.setFlat(True)
    self.btn_to_mx = QPushButton("Expectations")
    self.btn_to_mx.setFlat(True)
    self.btn_to_dx = QPushButton("Variances")
    self.btn_to_dx.setFlat(True)

    toolbar_layout = QHBoxLayout()
    toolbar_layout.setContentsMargins(0, 0, 0, 0)
    toolbar_layout.setSpacing(0)
    toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
    toolbar_layout.addWidget(InfoIcon("Click this buttons to switch plot modes\n"
                    "between trajectories, expectations and variances"))
    toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
    toolbar_layout.addWidget(self.btn_to_fn)
    toolbar_layout.addWidget(self.btn_to_mx)
    toolbar_layout.addWidget(self.btn_to_dx)
    toolbar_layout.addWidget(self.toolbar)
    toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))

    layout = QVBoxLayout()
    layout.setContentsMargins(0, 0, 0, 0)
    layout.setSpacing(0)
    layout.addLayout(toolbar_layout)
    layout.addWidget(self.canvas)

    self.setLayout(layout)

    self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))

    self.btn_to_fn.pressed.connect(self.fn_mode)
    self.btn_to_mx.pressed.connect(self.mx_mode)
    self.btn_to_dx.pressed.connect(self.dx_mode)

  def rescale(self):
    self.ax.relim(visible_only=True)
    self.ax.autoscale()
    self.canvas.draw()

  def clear(self):
    for f in self.plots.values():
      self.hide_label.emit(f.uid)
      if f.line_fn is not None:
        f.line_fn.set_visible(False)
      if f.line_mx is not None:
        f.line_mx.set_visible(False)
      if f.line_dx is not None:
        f.line_dx.set_visible(False)

  def fn_mode(self):

    self.mode = 0

    self.clear()

    for f in self.plots.values():
```

```
            if f.line_fn is not None:
                self.show_label.emit(f.uid)
                if f.visible:
                    f.line_fn.set_visible(True)

        self.rescale()

    def mx_mode(self):

        self.mode = 1

        self.clear()

        for f in self.plots.values():
            if f.line_mx is not None:
                self.show_label.emit(f.uid)
                if f.visible:
                    f.line_mx.set_visible(True)

        self.rescale()

    def dx_mode(self):

        self.mode = 2

        self.clear()

        for f in self.plots.values():
            if f.line_dx is not None:
                self.show_label.emit(f.uid)
                if f.visible:
                    f.line_dx.set_visible(True)

        self.rescale()

    def new_items(self, lines: list):

        for line in lines:
            self.plots[line.uid] = line

        for f in self.plots.values():
            if f.line_fn is None and f.fn is not None:
                f.line_fn = self.ax.plot(f.t, f.fn, linewidth=1, color=f.color)[0]
            if f.line_mx is None and f.mx is not None:
                f.line_mx = self.ax.plot(f.t, f.mx, linewidth=1, color=f.color)[0]
            if f.line_dx is None and f.dx is not None:
                f.line_dx = self.ax.plot(f.t, f.dx, linewidth=1, color=f.color)[0]

        if self.mode == 0:
            self.fn_mode()

        if self.mode == 1:
            self.mx_mode()

        if self.mode == 2:
```

```
      self.dx_mode()

  def delete_item(self, uid: int):
    item = self.plots.pop(uid)
    Color.free(item.color)
    if item.line_fn is not None:
      item.line_fn.remove()
    if item.line_mx is not None:
      item.line_mx.remove()
    if item.line_dx is not None:
      item.line_dx.remove()

    self.rescale()

  def hide_item(self, uid: int):
    item = self.plots[uid]
    if item.line_fn is not None and self.mode == 0:
      item.line_fn.set_visible(False)
    if item.line_mx is not None and self.mode == 1:
      item.line_mx.set_visible(False)
    if item.line_dx is not None and self.mode == 2:
      item.line_dx.set_visible(False)
    item.visible = False

    self.rescale()

  def show_item(self, uid: int):
    item = self.plots[uid]
    if item.line_fn is not None and self.mode == 0:
      item.line_fn.set_visible(True)
    if item.line_mx is not None and self.mode == 1:
      item.line_mx.set_visible(True)
    if item.line_dx is not None and self.mode == 2:
      item.line_dx.set_visible(True)
    item.visible = True

    self.rescale()

  def show_all(self):
    for item in self.plots.values():
      if item.line_fn is not None and self.mode == 0:
        item.line_fn.set_visible(True)
      if item.line_mx is not None and self.mode == 1:
        item.line_mx.set_visible(True)
      if item.line_dx is not None and self.mode == 2:
        item.line_dx.set_visible(True)
      item.visible = True

    self.rescale()

  def hide_all(self):
    for item in self.plots.values():
      if item.line_fn is not None and self.mode == 0:
        item.line_fn.set_visible(False)
      if item.line_mx is not None and self.mode == 1:
```

```
        item.line_mx.set_visible(False)
      if item.line_dx is not None and self.mode == 2:
        item.line_dx.set_visible(False)
      item.visible = False

    self.rescale()

  def delete_all(self):
    for item in reversed(self.plots.values()):
      Color.free(item.color)
      if item.line_fn is not None:
        item.line_fn.remove()
      if item.line_mx is not None:
        item.line_mx.remove()
      if item.line_dx is not None:
        item.line_dx.remove()

    self.plots.clear()

    self.rescale()

  def on_resize(self, event):
    self.figure.tight_layout()
    self.canvas.draw()
```

Listing 26: **Curves color**

```
from random import choice


class Color:

  reserved_colors = [
    "#ff834a",
    "#ffe100",
    "#c700c7",
    "#24e280",
    "#1100ff",
    "#ff1e22",
  ]
  available_colors = [
    "#ff834a",
    "#ffe100",
    "#c700c7",
    "#24e280",
    "#1100ff",
    "#ff1e22",
  ]

  def __new__(cls, *args, **kwargs):

    try:
      return cls.available_colors.pop()
```

```

    except IndexError:
      return f"#{''.join([choice('0123456789ABCDEF') for j in range(6)])}"

  @classmethod
  def free(cls, code: str):

    if code in cls.reserved_colors:
      cls.available_colors.append(code)
```

Listing 27: **Curve**

```
from ui.charts.visuals.color import Color


class Line:
  count = 0

  def __init__(self, name, t, fn, mx=None, dx=None):
    self.name = name
    self.t = t
    self.fn = fn
    self.mx = mx
    self.dx = dx
    self.line_fn = None
    self.line_mx = None
    self.line_dx = None
    self.visible = True
    self.color = Color()

    self.uid = Line.count
    Line.count += 1
```

6.1.3 Source Codes of Input for Nonlinear Systems of Itô SDEs

Listing 28: **Base part of data input for nonlinear systems**

```
import logging

import numpy as np
from PyQt5.QtCore import QThreadPool, pyqtSignal
from PyQt5.QtWidgets import QCheckBox, QPushButton, QStyle, QApplication,\
  QSizePolicy, QHBoxLayout, QSpacerItem, QVBoxLayout, QStackedWidget, \
  QWidget, QLabel
from sympy import Matrix

import config
from mathematics.sde.nonlinear.drivers.euler import euler
from mathematics.sde.nonlinear.drivers.milstein import milstein
from mathematics.sde.nonlinear.drivers.strong_taylor_ito_1p5 import strong_taylor_ito_1p5
```

```
from mathematics.sde.nonlinear.drivers.strong_taylor_ito_2p0 import strong_taylor_ito_2p0
from mathematics.sde.nonlinear.drivers.strong_taylor_ito_2p5 import strong_taylor_ito_2p5
from mathematics.sde.nonlinear.drivers.strong_taylor_ito_3p0 import strong_taylor_ito_3p0
from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_1p0 import strong_taylor_stratonovich_1p0
from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_1p5 import strong_taylor_stratonovich_1p5
from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_2p0 import strong_taylor_stratonovich_2p0
from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_2p5 import strong_taylor_stratonovich_2p5
from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_3p0 import strong_taylor_stratonovich_3p0
from mathematics.sde.nonlinear.symbolic.coefficients.c import C
from tools import database
from ui.async_calls.worker import Worker
from ui.charts.visuals.line import Line
from ui.main.modeling.nonliear.step1 import Step1
from ui.main.modeling.nonliear.step2 import Step2
from ui.main.modeling.nonliear.step3 import Step3
from ui.main.modeling.nonliear.step4 import Step4
from ui.main.modeling.nonliear.step5 import Step5


class NonlinearModelingWidget(QWidget):

  show_main_menu = pyqtSignal()
  start_progress = pyqtSignal(str)
  stop_progress = pyqtSignal(str)
  draw_chart = pyqtSignal(list)

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.logger = logging.getLogger(__name__)
    self.scheme_id = 0
    self.schemes = [
      (euler, "Euler", "Euler Scheme"),
      (milstein, "Milstein", "Milstein Scheme"),
      (strong_taylor_ito_1p5, "Taylor-Ito 1.5",
       "Strong Taylor-Ito Scheme with Convergence Order 1.5"),
      (strong_taylor_ito_2p0, "Taylor-Ito 2.0",
       "Strong Taylor-Ito Scheme with Convergence Order 2.0"),
      (strong_taylor_ito_2p5, "Taylor-Ito 2.5",
       "Strong Taylor-Ito Scheme with Convergence Order 2.5"),
      (strong_taylor_ito_3p0, "Taylor-Ito 3.0",
       "Strong Taylor-Ito Scheme with Convergence Order 3.0"),
      (strong_taylor_stratonovich_1p0, "Taylor-Str. 1.0",
       "Strong Taylor-Stratonovich Scheme with Convergence Order 1.0"),
      (strong_taylor_stratonovich_1p5, "Taylor-Str. 1.5",
       "Strong Taylor-Stratonovich Scheme with Convergence Order 1.5"),
      (strong_taylor_stratonovich_2p0, "Taylor-Str. 2.0",
       "Strong Taylor-Stratonovich Scheme with Convergence Order 2.0"),
      (strong_taylor_stratonovich_2p5, "Taylor-Str. 2.5",
       "Strong Taylor-Stratonovich Scheme with Convergence Order 2.5"),
```

```
          (strong_taylor_stratonovich_3p0, "Taylor-Str. 3.0",
           "Strong Taylor-Stratonovich Scheme with Convergence Order 3.0"),
        ]

        self.stack_widget = QStackedWidget(self)

        self.step1 = Step1()
        self.step2 = Step2()
        self.step3 = Step3()
        self.step4 = Step4()
        self.step5 = Step5()

        back_btn = QPushButton("Back", self)
        back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        self.charts_check = QCheckBox("Charts window", self)

        self.scheme_name = QLabel()
        self.scheme_name.setSizePolicy(QSizePolicy.Expanding, QSizePolicy.Minimum)

        bar_layout = QHBoxLayout()
        bar_layout.addWidget(back_btn)
        bar_layout.addItem(QSpacerItem(10, 35, QSizePolicy.Minimum, QSizePolicy.Minimum))
        bar_layout.addWidget(self.scheme_name)
        bar_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bar_layout.addWidget(self.charts_check)

        self.stack_widget.addWidget(self.step1)
        self.stack_widget.addWidget(self.step2)
        self.stack_widget.addWidget(self.step3)
        self.stack_widget.addWidget(self.step4)
        self.stack_widget.addWidget(self.step5)

        layout = QVBoxLayout()
        layout.addLayout(bar_layout)
        layout.addWidget(self.stack_widget)

        self.setLayout(layout)

        back_btn.clicked.connect(self.show_main_menu.emit)
        back_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step1))

        self.step1.next_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step2))
        self.step2.prev_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step1))
        self.step2.next_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step3))
        self.step3.prev_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step2))
        self.step3.next_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step4))
        self.step4.prev_btn.clicked.connect(
          lambda: self.stack_widget.setCurrentWidget(self.step3))
```

```python
    self.step4.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step5))
    self.step5.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step4))

    self.step5.run_btn.clicked.connect(
      lambda: self.run_modeling())
    self.step5.run_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step1))

    self.step1.n_valid.connect(self.step2.matrix.resize_h)
    self.step1.n_valid.connect(self.step3.matrix.resize_h)
    self.step1.n_valid.connect(self.step4.matrix.resize_h)

    self.step1.m_valid.connect(self.step3.matrix.resize_w)

  def set_scheme(self, scheme_id):
    self.scheme_id = scheme_id
    self.scheme_name.setText(self.schemes[scheme_id][2])
    if scheme_id == 0:
      self.step5.count_c(False)
    else:
      self.step5.count_c(True)

  def run_modeling(self):
    self.start_progress.emit("The modeling is being performed...")

    worker = Worker(self.routine)
    worker.signals.result.connect(self.on_modeling_finish)
    worker.signals.error.connect(self.on_modeling_corrupted)
    QThreadPool.globalInstance().start(worker)

  def routine(self):

    scheme = self.schemes[self.scheme_id]

    a = Matrix(self.step2.matrix.m)
    b = Matrix(self.step3.matrix.m)

    x0 = np.ndarray(shape=(self.step4.matrix.rowCount(),
               self.step4.matrix.columnCount()), dtype=float)
    for i in range(self.step4.matrix.rowCount()):
      for j in range(self.step4.matrix.columnCount()):
        x0[i][j] = float(self.step4.matrix.m[i][j])

    if self.step5.s != 0:
      np.random.seed(self.step5.s)

    database.connect(config.database)

    if self.scheme_id == 0:
      result = scheme[0](
        x0, a, b,
        (self.step5.t0,
         self.step5.dt,
```

```
            self.step5.t1)
        )
    else:
        C.preload(56, 56, 56, 56, 56)
        result = scheme[0](
            x0, a, b, self.step5.c,
            (self.step5.t0,
             self.step5.dt,
             self.step5.t1)
        )

    database.disconnect()

    lines = [Line(f"{scheme[1]}, x{i + 1}",
                  np.array(result[1]).astype(float),
                  np.array(result[0][i, :]).astype(float))
             for i in range(len(result[0]))]

    return lines

  def on_modeling_finish(self, result):
    self.stop_progress.emit("The modeling has been completed!")
    self.draw_chart.emit(result)

  def on_modeling_corrupted(self, result):

    self.logger.error(result[0])
    self.logger.error(result[1])
    self.logger.error(result[2])

    self.stop_progress.emit("The modeling failed!")
```

Listing 29: **Step 1 of data input for nonlinear systems**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QGridLayout, QLineEdit, QLabel, \
  QVBoxLayout, QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.svg import SVG


class Step1(QWidget):

  n_valid = pyqtSignal(int)
  m_valid = pyqtSignal(int)

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.n_is_valid = False
    self.m_is_valid = False
```

```
        
        self.input_stack = self.parent()

        info_n = InfoIcon("Dimension of linear system of Ito SDEs")
        info_m = InfoIcon("Dimension of vector Wiener process")

        label_n = QLabel("n")
        label_m = QLabel("m")

        self.lineedit_n = QLineEdit()
        self.lineedit_m = QLineEdit()

        self.msg_n = ErrorWidget("Wrong value!")
        self.msg_n.hide()

        self.msg_m = ErrorWidget("Wrong value!")
        self.msg_m.hide()

        grid_layout = QGridLayout()
        grid_layout.addWidget(self.msg_n, 0, 2)
        grid_layout.addWidget(self.msg_m, 2, 2)
        grid_layout.addWidget(info_n, 1, 0)
        grid_layout.addWidget(info_m, 3, 0)
        grid_layout.addWidget(label_n, 1, 1)
        grid_layout.addWidget(label_m, 3, 1)
        grid_layout.addWidget(self.lineedit_n, 1, 2)
        grid_layout.addWidget(self.lineedit_m, 3, 2)

        header = QLabel("Dimensions settings", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
        self.next_btn.setEnabled(False)

        header_layout = QHBoxLayout()
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        header_layout.addWidget(header)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.next_btn)

        eq1 = QHBoxLayout()
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        eq1.addWidget(SVG("equation1.svg", scale_factor=1.))
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        eq2 = QHBoxLayout()
        eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        eq2.addWidget(SVG("equation2.svg", scale_factor=1.))
        eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
```

```
    equalities_layout = QVBoxLayout()
    equalities_layout.addLayout(eq1)
    equalities_layout.addItem(QSpacerItem(0, 20, QSizePolicy.Minimum, QSizePolicy.Minimum
    ))
    equalities_layout.addLayout(eq2)

    equalities_wrap = QHBoxLayout()
    equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum)
    )
    equalities_wrap.addLayout(equalities_layout)
    equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum)
    )

    grid_wrap = QHBoxLayout()
    grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    grid_wrap.addLayout(grid_layout)
    grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    layout = QVBoxLayout()
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
    layout.addLayout(equalities_wrap)
    layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
    layout.addLayout(header_layout)
    layout.addItem(QSpacerItem(0, 5, QSizePolicy.Minimum, QSizePolicy.Minimum))
    layout.addLayout(grid_wrap)
    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
    layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
    layout.addLayout(bottom_bar)

    self.setLayout(layout)

    self.lineedit_n.textChanged.connect(self.validate_n)
    self.lineedit_m.textChanged.connect(self.validate_m)

  def validate_form(self):
    if self.n_is_valid and self.m_is_valid:
      self.next_btn.setEnabled(True)
    else:
      self.next_btn.setEnabled(False)

  def validate_n(self, value):
    try:
      typed_value = int(value)
      if typed_value <= 0:
        raise ValueError()

      self.n_is_valid = True
      self.n_valid.emit(typed_value)
      self.msg_n.hide()

    except ValueError:
      self.n_is_valid = False
      self.msg_n.show()

```

```
    finally:
      self.validate_form()

  def validate_m(self, value):
    try:
      typed_value = int(value)
      if typed_value <= 0:
        raise ValueError()

      self.m_is_valid = True
      self.m_valid.emit(typed_value)
      self.msg_m.hide()

    except ValueError:
      self.m_is_valid = False
      self.msg_m.show()

    finally:
      self.validate_form()
```

Listing 30: **Step 2 of data input for nonlinear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
  QVBoxLayout, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step2(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.errors = 0

    header = QLabel("Setting of column a(x, t)", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    info = InfoIcon("Elements of column a(x, t) are expected to be functions\n"
              "Size: n x 1\n"
              "Functions must be set in python and SymPy notation")

    self.msg = ErrorWidget("Wrong values in matrix!")
    self.msg.hide()

    self.matrix = MatrixWidget(self)

    self.next_btn = QPushButton("Next", self)
    self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
```

```
        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)
```

Listing 31: **Step 3 of data input for nonlinear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
    QVBoxLayout, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step3(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.errors = 0

        header = QLabel("Setting of matrix B(x, t)", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        info = InfoIcon("Elements of matrix B(x, t) are expected to be functions\n"
                        "Size: n x m\n"
                        "Functions must be set in python and SymPy notation")

        self.msg = ErrorWidget("Wrong values in matrix!")
        self.msg.hide()

        self.matrix = MatrixWidget(self)

```

```
    self.next_btn = QPushButton("Next", self)
    self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

    self.prev_btn = QPushButton("Back", self)
    self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

    header_layout = QHBoxLayout()
    header_layout.addWidget(info)
    header_layout.addWidget(header)
    header_layout.addWidget(self.msg)
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    bottom_bar = QHBoxLayout()
    bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bottom_bar.addWidget(self.prev_btn)
    bottom_bar.addWidget(self.next_btn)

    layout = QVBoxLayout()
    layout.addLayout(header_layout)
    layout.addWidget(self.matrix)
    layout.addLayout(bottom_bar)

    self.setLayout(layout)
```

Listing 32: **Step 4 of data input for nonlinear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, \
  QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step4(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.errors = 0

    header = QLabel("Setting of column x0", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    info = InfoIcon("Elements of column x0 are expected to be functions\n"
            "Size: n x 1\n"
            "Functions must be set in python and SymPy notation")

    self.msg = ErrorWidget("Wrong values in matrix!")
    self.msg.hide()

```

```
        self.matrix = MatrixWidget(self)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)
```

Listing 33: **Step 5 of data input for nonlinear systems**

```
import sys

from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QLineEdit, \
    QGridLayout, QVBoxLayout, QSizePolicy, QSpacerItem, QPushButton, \
    QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon


class Step5(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.t0 = sys.float_info.min
        self.dt = 0
        self.t1 = sys.float_info.max
        self.s = 0
        self.c = 0

        self.t0_is_valid = False
        self.dt_is_valid = False
        self.t1_is_valid = False
        self.s_is_valid = True
```

```
        self.c_is_valid = False

        self.input_stack = self.parent()

        info_t0 = InfoIcon("Start point of integration interval\n"
                           "Must be in [0, t1) range")
        info_dt = InfoIcon("Integration step\n"
                           "Must be set in (0, 1) interval")
        info_t1 = InfoIcon("Final point of integration interval\n"
                           "Must be more then t0")
        info_s = InfoIcon("This is random generator seed\n"
                          "If You do not want to use specific\n"
                          "seed just leave this field empty")
        self.info_c = InfoIcon("The constant which defines approximation accuracy")

        label_t0 = QLabel("t0")
        label_dt = QLabel("dt")
        label_t1 = QLabel("t1")
        label_s = QLabel("seed")
        self.label_c = QLabel("C")

        self.lineedit_t0 = QLineEdit()
        self.lineedit_dt = QLineEdit()
        self.lineedit_t1 = QLineEdit()
        self.lineedit_s = QLineEdit()
        self.lineedit_c = QLineEdit()

        self.msg_t0 = ErrorWidget("Wrong value!")
        self.msg_t0.hide()

        self.msg_dt = ErrorWidget("Wrong value!")
        self.msg_dt.hide()

        self.msg_t1 = ErrorWidget("Wrong value!")
        self.msg_t1.hide()

        self.msg_s = ErrorWidget("Wrong value!")
        self.msg_s.hide()

        self.msg_c = ErrorWidget("Wrong value!")
        self.msg_c.hide()

        header = QLabel("Accuracy settings", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        self.run_btn = QPushButton("Perform modeling", self)
        self.run_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
        self.run_btn.setEnabled(False)

        header_layout = QHBoxLayout()
```

```
        header_layout.addWidget(header)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.run_btn)

        grid_layout = QGridLayout()

        grid_layout.addWidget(self.msg_t0, 0, 2)

        grid_layout.addWidget(info_t0, 1, 0)
        grid_layout.addWidget(label_t0, 1, 1)
        grid_layout.addWidget(self.lineedit_t0, 1, 2)

        grid_layout.addWidget(self.msg_dt, 0, 5)

        grid_layout.addWidget(info_dt, 1, 3)
        grid_layout.addWidget(label_dt, 1, 4)
        grid_layout.addWidget(self.lineedit_dt, 1, 5)

        grid_layout.addWidget(self.msg_t1, 0, 8)

        grid_layout.addWidget(info_t1, 1, 6)
        grid_layout.addWidget(label_t1, 1, 7)
        grid_layout.addWidget(self.lineedit_t1, 1, 8)

        grid_layout.addWidget(self.msg_s, 2, 2)

        grid_layout.addWidget(info_s, 3, 0)
        grid_layout.addWidget(label_s, 3, 1)
        grid_layout.addWidget(self.lineedit_s, 3, 2)

        grid_layout.addWidget(self.msg_c, 2, 5)

        grid_layout.addWidget(self.info_c, 3, 3)
        grid_layout.addWidget(self.label_c, 3, 4)
        grid_layout.addWidget(self.lineedit_c, 3, 5)

        column_layout = QVBoxLayout()
        column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
        column_layout.addLayout(header_layout)
        column_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
        column_layout.addLayout(grid_layout)
        column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

        control_layout = QHBoxLayout()
        control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
        control_layout.addLayout(column_layout)
        control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
```

```
        layout = QVBoxLayout()
        layout.addLayout(control_layout)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.lineedit_t0.textChanged.connect(self.validate_t0)
        self.lineedit_t0.textChanged.connect(self.validate_t1)
        self.lineedit_t0.textChanged.connect(self.validate_dt)
        self.lineedit_dt.textChanged.connect(self.validate_dt)
        self.lineedit_t1.textChanged.connect(self.validate_t0)
        self.lineedit_t1.textChanged.connect(self.validate_t1)
        self.lineedit_t1.textChanged.connect(self.validate_dt)
        self.lineedit_s.textChanged.connect(self.validate_s)
        self.lineedit_c.textChanged.connect(self.validate_c)

    def count_c(self, flag: bool):
        if flag:
            self.info_c.show()
            self.label_c.show()
            self.lineedit_c.show()
            self.validate_c()
        else:
            self.info_c.hide()
            self.label_c.hide()
            self.lineedit_c.hide()
            self.c_is_valid = True

        self.msg_c.hide()

    def validate_form(self):
        if self.t0_is_valid \
                and self.dt_is_valid \
                and self.t1_is_valid \
                and self.s_is_valid \
                and self.c_is_valid:
            self.run_btn.setEnabled(True)
        else:
            self.run_btn.setEnabled(False)

    def validate_t0(self):
        try:
            typed_value = float(self.lineedit_t0.text())
            if typed_value >= self.t1:
                raise ValueError()

            self.t0_is_valid = True
            self.t0 = typed_value
            self.msg_t0.hide()

        except ValueError:
            self.t0_is_valid = False
            self.msg_t0.show()

```

```
  def validate_dt(self):
    try:
      typed_value = float(self.lineedit_dt.text())
      if typed_value <= 0 \
          or (self.t1 - self.t0) / typed_value < 1 \
          or typed_value >= 1:
        raise ValueError()

      self.dt_is_valid = True
      self.dt = typed_value
      self.msg_dt.hide()
      self.validate_form()

    except ValueError:
      self.dt_is_valid = False
      self.msg_dt.show()

  def validate_t1(self):
    try:
      typed_value = float(self.lineedit_t1.text())
      if typed_value <= self.t0:
        raise ValueError()

      self.t1_is_valid = True
      self.t1 = typed_value
      self.msg_t1.hide()

    except ValueError:
      self.t1_is_valid = False
      self.msg_t1.show()

  def validate_s(self):
    try:
      if self.lineedit_s.text() == "":
        self.s_is_valid = True
        self.s = 0
      else:
        typed_value = int(self.lineedit_s.text())
        if typed_value <= 0:
          raise ValueError()
        self.s = typed_value

      self.s_is_valid = True
      self.msg_s.hide()

    except ValueError:
      self.s_is_valid = False
      self.msg_s.show()

    finally:
      self.validate_form()

  def validate_c(self):
    try:
      typed_value = float(self.lineedit_c.text())
```

```
      if typed_value <= 0:
        raise ValueError()

      self.c_is_valid = True
      self.c = typed_value
      self.msg_c.hide()

    except ValueError:
      self.c_is_valid = False
      self.msg_c.show()

    finally:
      self.validate_form()
```

6.1.4 Source Codes of Input for Linear Systems of Itô SDEs

Listing 34: **Base part of data input for linear systems**

```
import logging
from time import time

import numpy as np
from PyQt5.QtCore import QThreadPool, pyqtSignal
from PyQt5.QtWidgets import QCheckBox, QPushButton, QStyle, QApplication, \
  QSizePolicy, QHBoxLayout, QSpacerItem, QVBoxLayout, QStackedWidget, \
  QWidget, QLabel

from mathematics.sde.linear.dindet import dindet
from mathematics.sde.linear.distortions import Symbolic, ComplexDistortion
from mathematics.sde.linear.integration import Integral
from mathematics.sde.linear.stoch import stoch
from ui.async_calls.worker import Worker
from ui.charts.visuals.line import Line
from ui.main.modeling.linear.step1 import Step1
from ui.main.modeling.linear.step2 import Step2
from ui.main.modeling.linear.step3 import Step3
from ui.main.modeling.linear.step4 import Step4
from ui.main.modeling.linear.step5 import Step5
from ui.main.modeling.linear.step6 import Step6
from ui.main.modeling.linear.step7 import Step7
from ui.main.modeling.linear.step8 import Step8


class LinearModelingWidget(QWidget):

  show_main_menu = pyqtSignal()
  start_progress = pyqtSignal(str)
  stop_progress = pyqtSignal(str)
  draw_chart = pyqtSignal(list)

  def __init__(self, parent=None):
```

```
    super(QWidget, self).__init__(parent)

    self.logger = logging.getLogger(__name__)

    self.stack_widget = QStackedWidget(self)

    self.step1 = Step1()
    self.step2 = Step2()
    self.step3 = Step3()
    self.step4 = Step4()
    self.step5 = Step5()
    self.step6 = Step6()
    self.step7 = Step7()
    self.step8 = Step8()

    back_btn = QPushButton("Back", self)
    back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

    self.charts_check = QCheckBox("Charts window", self)

    self.scheme_name = QLabel("Linear Systems of Ito SDEs")

    bar_layout = QHBoxLayout()
    bar_layout.addWidget(back_btn)
    bar_layout.addItem(QSpacerItem(10, 35, QSizePolicy.Minimum, QSizePolicy.Minimum))
    bar_layout.addWidget(self.scheme_name)
    bar_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bar_layout.addWidget(self.charts_check)

    self.stack_widget.addWidget(self.step1)
    self.stack_widget.addWidget(self.step2)
    self.stack_widget.addWidget(self.step3)
    self.stack_widget.addWidget(self.step4)
    self.stack_widget.addWidget(self.step5)
    self.stack_widget.addWidget(self.step6)
    self.stack_widget.addWidget(self.step7)
    self.stack_widget.addWidget(self.step8)

    layout = QVBoxLayout()
    layout.addLayout(bar_layout)
    layout.addWidget(self.stack_widget)

    self.setLayout(layout)

    back_btn.clicked.connect(self.show_main_menu.emit)
    back_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step1))

    self.step1.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step2))
    self.step2.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step1))
    self.step2.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step3))
    self.step3.prev_btn.clicked.connect(
```

```
      lambda: self.stack_widget.setCurrentWidget(self.step2))
    self.step3.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step4))
    self.step4.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step3))
    self.step4.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step5))
    self.step5.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step4))
    self.step5.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step6))
    self.step6.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step5))
    self.step6.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step7))
    self.step7.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step6))
    self.step7.next_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step8))
    self.step8.prev_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step7))

    self.step8.run_btn.clicked.connect(
      lambda: self.run_modeling())
    self.step8.run_btn.clicked.connect(
      lambda: self.stack_widget.setCurrentWidget(self.step1))

    self.step1.n_valid.connect(self.step2.matrix.resize_h)
    self.step1.n_valid.connect(self.step2.matrix.resize_w)

    self.step1.n_valid.connect(self.step3.matrix.resize_h)
    self.step1.k_valid.connect(self.step3.matrix.resize_w)

    self.step1.n_valid.connect(self.step4.matrix.resize_h)
    self.step1.m_valid.connect(self.step4.matrix.resize_w)

    self.step1.k_valid.connect(self.step5.matrix.resize_h)

    self.step1.n_valid.connect(self.step6.matrix.resize_w)

    self.step1.n_valid.connect(self.step7.matrix.resize_h)

  def run_modeling(self):
    self.start_progress.emit("The modeling is being performed...")

    worker = Worker(self.routine)
    worker.signals.result.connect(self.on_modeling_finish)
    worker.signals.error.connect(self.on_modeling_corrupted)
    QThreadPool.globalInstance().start(worker)

  def routine(self):
    n = int(self.step1.lineedit_n.text())
    m = int(self.step1.lineedit_m.text())
    k = int(self.step1.lineedit_k.text())
    t0 = float(self.step8.lineedit_t0.text())
```

```
    dt = float(self.step8.lineedit_dt.text())
    t1 = float(self.step8.lineedit_t1.text())

    self.logger.info("Reading input data")

    integral = Integral(n)

    integral.k, integral.m, integral.dt, integral.t0, integral.tk = \
      k, m, dt, t0, t1

    integral.m_a = np.array([[float(self.step2.matrix.m[i][j])
                   for j in range(self.step2.matrix.columnCount())]
                   for i in range(self.step2.matrix.rowCount())])

    integral.mat_b = np.array([[float(self.step3.matrix.m[i][j])
                    for j in range(self.step3.matrix.columnCount())]
                    for i in range(self.step3.matrix.rowCount())])

    integral.mat_f = np.array([[float(self.step4.matrix.m[i][j])
                    for j in range(self.step4.matrix.columnCount())]
                    for i in range(self.step4.matrix.rowCount())])

    integral.m_h = np.array([[float(self.step6.matrix.m[i][j])
                   for j in range(self.step6.matrix.columnCount())]
                   for i in range(self.step6.matrix.rowCount())])

    integral.m_x0 = np.array([[float(self.step7.matrix.m[i][j])
                    for j in range(self.step7.matrix.columnCount())]
                   for i in range(self.step7.matrix.rowCount())])

    integral.m_mx0 = np.array([[float(self.step7.matrix.m[i][j])
                    for j in range(self.step7.matrix.columnCount())]
                    for i in range(self.step7.matrix.rowCount())])

    integral.m_dx0 = np.zeros((integral.n, integral.n))

    self.logger.info("Input is correct")
    self.logger.info("Calculation of Ad and Bd (Algorithm 11.2)")

    integral.m_ad, integral.m_bd = dindet(
      integral.n, integral.k, integral.m_a, integral.mat_b, integral.dt)

    self.logger.info("Calculation of Fd (Algorithm 11.6)")

    integral.m_fd = stoch(integral.n, integral.m_a, integral.mat_f, integral.dt)

    mat_u = np.array([[object]] * self.step5.matrix.rowCount())
    for i in range(self.step5.matrix.rowCount()):
      mat_u[i][0] = Symbolic(self.step5.matrix.m[i][0])

    integral.distortion = ComplexDistortion(self.step5.matrix.rowCount(), mat_u)

    self.logger.info("Starting modeling loop")

    start_time = time()
```

```
    integral.integrate()

    self.logger.info(f"Integration took {(time() - start_time):.3f} seconds")

    lines = [Line(f"Linear, x{i + 1}",
                  np.array(integral.v_t).astype(float),
                  np.array(integral.m_xt[i, :]).astype(float),
                  mx=np.array(integral.m_mx[i, :]).astype(float),
                  dx=np.array(integral.m_dx[i, :]).astype(float))
              for i in range(integral.m_xt.shape[0])]

    name = f"Linear"

    lines.append(Line(f"Linear, y",
                      np.array(integral.v_t).astype(float),
                      np.array(integral.v_yt).astype(float),
                      mx=np.array(integral.v_my).astype(float),
                      dx=np.array(integral.v_dy).astype(float)))

    return lines

  def on_modeling_finish(self, result):
    self.stop_progress.emit("The modeling has been completed!")
    self.draw_chart.emit(result)

  def on_modeling_corrupted(self, result):
    self.logger.error(result[0])
    self.logger.error(result[1])
    self.logger.error(result[2])

    self.stop_progress.emit("The modeling failed!")
```

Listing 35: **Step 1 of data input for linear systems**

```
from PyQt5.QtCore import pyqtSignal
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QGridLayout, QLineEdit, QLabel, \
  QVBoxLayout, QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.svg import SVG


class Step1(QWidget):

  n_valid = pyqtSignal(int)
  m_valid = pyqtSignal(int)
  k_valid = pyqtSignal(int)

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.n_is_valid = False
```

```
        self.m_is_valid = False
        self.k_is_valid = False

        self.input_stack = self.parent()

        info_n = InfoIcon("Dimension of linear system of Ito SDEs")
        info_m = InfoIcon("Dimension of vector Wiener process")
        info_k = InfoIcon("Dimension of vector function u(t)")

        label_n = QLabel("n")
        label_m = QLabel("m")
        label_k = QLabel("k")

        self.lineedit_n = QLineEdit()
        self.lineedit_m = QLineEdit()
        self.lineedit_k = QLineEdit()

        self.msg_n = ErrorWidget("Wrong value!")
        self.msg_n.hide()

        self.msg_m = ErrorWidget("Wrong value!")
        self.msg_m.hide()

        self.msg_k = ErrorWidget("Wrong value!")
        self.msg_k.hide()

        grid_layout = QGridLayout()
        grid_layout.addWidget(self.msg_n, 0, 2)
        grid_layout.addWidget(self.msg_m, 2, 2)
        grid_layout.addWidget(self.msg_k, 4, 2)
        grid_layout.addWidget(info_n, 1, 0)
        grid_layout.addWidget(info_m, 3, 0)
        grid_layout.addWidget(info_k, 5, 0)
        grid_layout.addWidget(label_n, 1, 1)
        grid_layout.addWidget(label_m, 3, 1)
        grid_layout.addWidget(label_k, 5, 1)
        grid_layout.addWidget(self.lineedit_n, 1, 2)
        grid_layout.addWidget(self.lineedit_m, 3, 2)
        grid_layout.addWidget(self.lineedit_k, 5, 2)

        header = QLabel("Dimensions settings", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
        self.next_btn.setEnabled(False)

        header_layout = QHBoxLayout()
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        header_layout.addWidget(header)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
```

```
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.next_btn)

        eq1 = QHBoxLayout()
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        eq1.addWidget(SVG("equation3.svg", scale_factor=1.))
        eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        eq2 = QHBoxLayout()
        eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        eq2.addWidget(SVG("equation4.svg", scale_factor=1.))
        eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        equalities_layout = QVBoxLayout()
        equalities_layout.addLayout(eq1)
        equalities_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
        equalities_layout.addLayout(eq2)

        equalities_wrap = QHBoxLayout()
        equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        equalities_wrap.addLayout(equalities_layout)
        equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        grid_wrap = QHBoxLayout()
        grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        grid_wrap.addLayout(grid_layout)
        grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        layout = QVBoxLayout()
        layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
        layout.addLayout(equalities_wrap)
        layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
        layout.addLayout(header_layout)
        layout.addItem(QSpacerItem(0, 5, QSizePolicy.Minimum, QSizePolicy.Minimum))
        layout.addLayout(grid_wrap)
        layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
        layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.lineedit_n.textChanged.connect(self.validate_n)
        self.lineedit_m.textChanged.connect(self.validate_m)
        self.lineedit_k.textChanged.connect(self.validate_k)

    def validate_form(self):
        if self.n_is_valid and self.m_is_valid and self.k_is_valid:
            self.next_btn.setEnabled(True)
        else:
            self.next_btn.setEnabled(False)

    def validate_n(self, value):
```

```
    try:
      typed_value = int(value)
      if typed_value <= 0:
        raise ValueError()

      self.n_is_valid = True
      self.n_valid.emit(typed_value)
      self.msg_n.hide()

    except ValueError:
      self.n_is_valid = False
      self.msg_n.show()

    finally:
      self.validate_form()

  def validate_m(self, value):
    try:
      typed_value = int(value)
      if typed_value <= 0:
        raise ValueError()

      self.m_is_valid = True
      self.m_valid.emit(typed_value)
      self.msg_m.hide()

    except ValueError:
      self.m_is_valid = False
      self.msg_m.show()

    finally:
      self.validate_form()

  def validate_k(self, value):
    try:
      typed_value = int(value)
      if typed_value <= 0:
        raise ValueError()

      self.k_is_valid = True
      self.k_valid.emit(typed_value)
      self.msg_k.hide()

    except ValueError:
      self.k_is_valid = False
      self.msg_k.show()

    finally:
      self.validate_form()
```

Listing 36: **Step 2 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
```

```
    QVBoxLayout, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step2(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.errors = 0

        header = QLabel("Setting of matrix A", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        info = InfoIcon("Elements of matrix A are\n"
                        "expected to be real values\n"
                        "Size: n x n")

        self.msg = ErrorWidget("Wrong values in matrix!")
        self.msg.hide()

        self.matrix = MatrixWidget(self)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.matrix.itemChanged.connect(self.validate_item)

    def validate_item(self, item):
```

```

    value = item.text()
    try:
      float(value)
      if not item.valid:
        item.valid = True
        self.errors -= 1

    except ValueError:
      if item.valid:
        item.valid = False
        self.errors += 1

    finally:
      self.validate_form()

  def validate_form(self):

    if self.errors == 0:
      self.msg.hide()
      self.next_btn.setEnabled(True)
    else:
      self.msg.show()
      self.next_btn.setEnabled(False)
```

Listing 37: **Step 3 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
  QVBoxLayout, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step3(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.errors = 0

    header = QLabel("Setting of matrix B", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    info = InfoIcon("Elements of matrix B are\n"
            "expected to be real values\n"
            "Size: n x k")

    self.msg = ErrorWidget("Wrong values in matrix!")
    self.msg.hide()
```

```python

        self.matrix = MatrixWidget(self)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.matrix.itemChanged.connect(self.validate_item)

    def validate_item(self, item):

        value = item.text()
        try:
            float(value)
            if not item.valid:
                item.valid = True
                self.errors -= 1

        except ValueError:
            if item.valid:
                item.valid = False
                self.errors += 1

        finally:
            self.validate_form()

    def validate_form(self):

        if self.errors == 0:
            self.msg.hide()
            self.next_btn.setEnabled(True)
        else:
            self.msg.show()
            self.next_btn.setEnabled(False)
```

Listing 38: **Step 4 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
  QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step4(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.errors = 0

    header = QLabel("Setting of matrix F", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    info = InfoIcon("Elements of matrix F are\n"
            "expected to be real values\n"
            "Size: n x m")

    self.msg = ErrorWidget("Wrong values in matrix!")
    self.msg.hide()

    self.matrix = MatrixWidget(self)

    self.next_btn = QPushButton("Next", self)
    self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

    self.prev_btn = QPushButton("Back", self)
    self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

    header_layout = QHBoxLayout()
    header_layout.addWidget(info)
    header_layout.addWidget(header)
    header_layout.addWidget(self.msg)
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    bottom_bar = QHBoxLayout()
    bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bottom_bar.addWidget(self.prev_btn)
    bottom_bar.addWidget(self.next_btn)

    layout = QVBoxLayout()
    layout.addLayout(header_layout)
    layout.addWidget(self.matrix)
    layout.addLayout(bottom_bar)

    self.setLayout(layout)

```

```
        self.matrix.itemChanged.connect(self.validate_item)

    def validate_item(self, item):

        value = item.text()
        try:
            float(value)
            if not item.valid:
                item.valid = True
                self.errors -= 1

        except ValueError:
            if item.valid:
                item.valid = False
                self.errors += 1

        finally:
            self.validate_form()

    def validate_form(self):

        if self.errors == 0:
            self.msg.hide()
            self.next_btn.setEnabled(True)
        else:
            self.msg.show()
            self.next_btn.setEnabled(False)
```

Listing 39: **Step 5 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
    QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step5(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.errors = 0

        header = QLabel("Setting of vector function u(t)", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        info = InfoIcon("Elements of vector u(t) are expected to be functions\n"
                        "Size: k x 1\n"
                        "Functions must be set in python and SymPy notation")
```

```

    self.msg = ErrorWidget("Wrong values in matrix!")
    self.msg.hide()

    self.matrix = MatrixWidget(self)

    self.next_btn = QPushButton("Next", self)
    self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

    self.prev_btn = QPushButton("Back", self)
    self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

    header_layout = QHBoxLayout()
    header_layout.addWidget(info)
    header_layout.addWidget(header)
    header_layout.addWidget(self.msg)
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    bottom_bar = QHBoxLayout()
    bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bottom_bar.addWidget(self.prev_btn)
    bottom_bar.addWidget(self.next_btn)

    layout = QVBoxLayout()
    layout.addLayout(header_layout)
    layout.addWidget(self.matrix)
    layout.addLayout(bottom_bar)

    self.setLayout(layout)
```

Listing 40: **Step 6 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
  QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step6(QWidget):

  def __init__(self, parent=None):
    super(QWidget, self).__init__(parent)

    self.errors = 0

    header = QLabel("Setting of matrix H", parent=self)
    font = header.font()
    font.setPointSize(15)
    header.setFont(font)

    info = InfoIcon("Elements of matrix H are\n"
```

```
                "expected to be int values\n"
                "Size: 1 x n")

        self.msg = ErrorWidget("Wrong values in matrix!")
        self.msg.hide()

        self.matrix = MatrixWidget(self)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.matrix.itemChanged.connect(self.validate_item)

    def validate_item(self, item):

        value = item.text()
        try:
            float(value)
            if not item.valid:
                item.valid = True
                self.errors -= 1

        except ValueError:
            if item.valid:
                item.valid = False
                self.errors += 1

        finally:
            self.validate_form()

    def validate_form(self):

        if self.errors == 0:
            self.msg.hide()
```

```
            self.next_btn.setEnabled(True)
        else:
            self.msg.show()
            self.next_btn.setEnabled(False)
```

Listing 41: **Step 7 of data input for linear systems**

```
from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
    QSizePolicy, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon
from ui.main.modeling.matrix_widget import MatrixWidget


class Step7(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.errors = 0

        header = QLabel("Setting of column x0", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        info = InfoIcon("Elements of column x0 are\n"
                        "expected to be real values\n"
                        "Size: n x 1")

        self.msg = ErrorWidget("Wrong values in matrix!")
        self.msg.hide()

        self.matrix = MatrixWidget(self)

        self.next_btn = QPushButton("Next", self)
        self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        header_layout = QHBoxLayout()
        header_layout.addWidget(info)
        header_layout.addWidget(header)
        header_layout.addWidget(self.msg)
        header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

        bottom_bar = QHBoxLayout()
        bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
        bottom_bar.addWidget(self.prev_btn)
        bottom_bar.addWidget(self.next_btn)

```

```
        layout = QVBoxLayout()
        layout.addLayout(header_layout)
        layout.addWidget(self.matrix)
        layout.addLayout(bottom_bar)

        self.setLayout(layout)

        self.matrix.itemChanged.connect(self.validate_item)

    def validate_item(self, item):

        value = item.text()
        try:
            float(value)
            if not item.valid:
                item.valid = True
                self.errors -= 1

        except ValueError:
            if item.valid:
                item.valid = False
                self.errors += 1

        finally:
            self.validate_form()

    def validate_form(self):

        if self.errors == 0:
            self.msg.hide()
            self.next_btn.setEnabled(True)
        else:
            self.msg.show()
            self.next_btn.setEnabled(False)
```

Listing 42: **Step 8 of data input for linear systems**

```
import sys

from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QLineEdit, QGridLayout, \
    QVBoxLayout, QSizePolicy, QSpacerItem, QPushButton, QApplication, QStyle

from ui.main.error import ErrorWidget
from ui.main.info import InfoIcon


class Step8(QWidget):

    def __init__(self, parent=None):
        super(QWidget, self).__init__(parent)

        self.t0 = sys.float_info.min
        self.dt = 0
```

```
        self.t1 = sys.float_info.max
        self.s = 0

        self.t0_is_valid = False
        self.dt_is_valid = False
        self.t1_is_valid = False
        self.s_is_valid = True

        self.input_stack = self.parent()

        info_t0 = InfoIcon("Start point of integration interval\n"
                           "Must be in [0, t1) range")
        info_dt = InfoIcon("Integration step\n"
                           "Must be set in (0, 1) interval")
        info_t1 = InfoIcon("Final point of integration interval\n"
                           "Must be more then t0")
        info_s = InfoIcon("This is random generator seed\n"
                          "If You do not want to use specific\n"
                          "seed just leave this field empty")

        label_t0 = QLabel("t0")
        label_dt = QLabel("dt")
        label_t1 = QLabel("t1")
        label_s = QLabel("seed")

        self.lineedit_t0 = QLineEdit()
        self.lineedit_dt = QLineEdit()
        self.lineedit_t1 = QLineEdit()
        self.lineedit_s = QLineEdit()

        self.msg_t0 = ErrorWidget("Wrong value!")
        self.msg_t0.hide()

        self.msg_dt = ErrorWidget("Wrong value!")
        self.msg_dt.hide()

        self.msg_t1 = ErrorWidget("Wrong value!")
        self.msg_t1.hide()

        self.msg_s = ErrorWidget("Wrong value!")
        self.msg_s.hide()

        header = QLabel("Accuracy settings", parent=self)
        font = header.font()
        font.setPointSize(15)
        header.setFont(font)

        self.prev_btn = QPushButton("Back", self)
        self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))

        self.run_btn = QPushButton("Perform modeling", self)
        self.run_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
        self.run_btn.setEnabled(False)

        header_layout = QHBoxLayout()
```

```
    header_layout.addWidget(header)
    header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

    bottom_bar = QHBoxLayout()
    bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
    bottom_bar.addWidget(self.prev_btn)
    bottom_bar.addWidget(self.run_btn)

    grid_layout = QGridLayout()

    grid_layout.addWidget(self.msg_t0, 0, 2)

    grid_layout.addWidget(info_t0, 1, 0)
    grid_layout.addWidget(label_t0, 1, 1)
    grid_layout.addWidget(self.lineedit_t0, 1, 2)

    grid_layout.addWidget(self.msg_dt, 0, 5)

    grid_layout.addWidget(info_dt, 1, 3)
    grid_layout.addWidget(label_dt, 1, 4)
    grid_layout.addWidget(self.lineedit_dt, 1, 5)

    grid_layout.addWidget(self.msg_t1, 0, 8)

    grid_layout.addWidget(info_t1, 1, 6)
    grid_layout.addWidget(label_t1, 1, 7)
    grid_layout.addWidget(self.lineedit_t1, 1, 8)

    grid_layout.addWidget(self.msg_s, 2, 2)

    grid_layout.addWidget(info_s, 3, 0)
    grid_layout.addWidget(label_s, 3, 1)
    grid_layout.addWidget(self.lineedit_s, 3, 2)

    column_layout = QVBoxLayout()
    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    column_layout.addLayout(header_layout)
    column_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
    column_layout.addLayout(grid_layout)
    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    control_layout = QHBoxLayout()
    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
    control_layout.addLayout(column_layout)
    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))

    layout = QVBoxLayout()
    layout.addLayout(control_layout)
    layout.addLayout(bottom_bar)

    self.setLayout(layout)
```

```

        self.lineedit_t0.textChanged.connect(self.validate_t0)
        self.lineedit_t0.textChanged.connect(self.validate_t1)
        self.lineedit_t0.textChanged.connect(self.validate_dt)
        self.lineedit_dt.textChanged.connect(self.validate_dt)
        self.lineedit_t1.textChanged.connect(self.validate_t0)
        self.lineedit_t1.textChanged.connect(self.validate_t1)
        self.lineedit_t1.textChanged.connect(self.validate_dt)
        self.lineedit_s.textChanged.connect(self.validate_s)

    def count_c(self, flag: bool):
        if flag:
            self.info_c.show()
            self.label_c.show()
        else:
            self.info_c.hide()
            self.label_c.hide()

        self.msg_c.hide()

    def validate_form(self):
        if self.t0_is_valid \
              and self.dt_is_valid \
              and self.t1_is_valid \
              and self.s_is_valid:
            self.run_btn.setEnabled(True)
        else:
            self.run_btn.setEnabled(False)

    def validate_t0(self):
        try:
            typed_value = float(self.lineedit_t0.text())
            if typed_value >= self.t1:
                raise ValueError()

            self.t0_is_valid = True
            self.t0 = typed_value
            self.msg_t0.hide()

        except ValueError:
            self.t0_is_valid = False
            self.msg_t0.show()

    def validate_dt(self):
        try:
            typed_value = float(self.lineedit_dt.text())
            if typed_value <= 0 \
                  or (self.t1 - self.t0) / typed_value < 1 \
                  or typed_value >= 1:
                raise ValueError()

            self.dt_is_valid = True
            self.dt = typed_value
            self.msg_dt.hide()
            self.validate_form()
```

```

        except ValueError:
            self.dt_is_valid = False
            self.msg_dt.show()

    def validate_t1(self):
        try:
            typed_value = float(self.lineedit_t1.text())
            if typed_value <= self.t0:
                raise ValueError()

            self.t1_is_valid = True
            self.t1 = typed_value
            self.msg_t1.hide()

        except ValueError:
            self.t1_is_valid = False
            self.msg_t1.show()

    def validate_s(self):
        try:
            if self.lineedit_s.text() == "":
                self.s_is_valid = True
                self.s = 0
            else:
                typed_value = int(self.lineedit_s.text())
                if typed_value <= 0:
                    raise ValueError()
                self.s = typed_value

            self.s_is_valid = True
            self.msg_s.hide()

        except ValueError:
            self.s_is_valid = False
            self.msg_s.show()

        finally:
            self.validate_form()
```

6.2 Source Codes for Nonlinear Systems of Itô SDEs

6.2.1 Source Codes for Calculation of the Fourier–Legendre Coefficients

Listing 43: **Symbolic function of the Legendre polinomial**

```
from sympy import Rational, factorial, diff

```

```python

def polynomial(n: int):
    """
    Returns the Legendre polynomial in symbolic format
    Parameters
    ==========
    n : int
        degree of the Legendre polynomial
    Returns
    =======
    sympy.Expr
    """
    from sympy.abc import x
    return Rational(1, 2) ** n / factorial(n) * diff((x ** 2 - 1) ** n, x, n)
```

Listing 44: **Symbolic function of the Fourier–Legendre coefficient calculation**

```python
from sympy import S, integrate

from mathematics.sde.nonlinear.legendre_polynomial import polynomial


def get_c(indices: tuple, weights: tuple):
    """
    Calculates the Fourier-Legendre coefficient depending on indices and weights
    Parameters
    ==========
    indices : tuple
        indices of the Fourier-Legendre coefficient
    weights : tuple
        weights of the Fourier-Legendre coefficient
    Returns
    =======
    sympy.Rational
    """
    from sympy.abc import x, y
    # multiplicity of iterated integral which is the Fourier-Legendre coefficient
    n = len(indices)
    w = list(reversed(weights))
    c = S.One

    for i in reversed(range(1, n)):
        c = integrate(polynomial(indices[i]) * (x + 1) ** w[i] * c, (x, -1, y)).subs(y, x)
    c = integrate(polynomial(indices[0]) * (x + 1) ** w[0] * c, (x, -1, 1))

    if sum(w) % 2 == 0:
        return c
    else:
        return -c
```

Listing 45: **Symbolic function of the Fourier–Legendre coefficient**

```python
import logging
```

```

from sympy import sympify, Function

import tools.database as db
from mathematics.sde.nonlinear.c import get_c


class C(Function):
    """
    Gives the Fourier-Legendre coefficient with requested indices and weights
    """
    _preloaded = dict()

    def __new__(cls, indices: tuple, weights: tuple, to_float=True, **kwargs):
        """
        Creates the Fourier-Legendre coefficient object with needed indices and weights
        Parameters
        ==========
        indices: tuple
            requested indices
        weights: tuple
            requested weights
        Returns
        =======
        symbolic.Rational or C
            calculated value or symbolic expression
        """
        if not len(indices) == len(weights):
            return super(C, cls).__new__(cls, indices, weights, **kwargs)

        index = f"{':'.join([str(i) for i in indices])}_{':'.join([str(i) for i in weights])}"
        try:
            return cls._value(index, to_float)

        except KeyError:
            respond = cls._download_one(index)
            if len(respond) != 0:
                cls._preloaded[respond[0][0]] = respond[0][1]
                return cls._value(index, to_float)
            else:
                new_c = cls._calculate(index, indices, weights)
                cls._upload_one(new_c)
                cls._preloaded[new_c[0]] = new_c[1]
                return cls._value(index, to_float)

    @classmethod
    def _calculate(cls, index, indices, weights):
        new_c = get_c(indices, weights)
        return index, (new_c, sympify(new_c).evalf())

    @classmethod
    def _upload_one(cls, c):
        logging.info(f"C: ADDING NEW C_{c[0]} = {c[1][0]}")
        db.execute(f"INSERT INTO `C` (`index`, `value`, `value_f`) VALUES ('{c[0]}', '{c
```

```
        [1][0]}', {c[1][1]})")

    @classmethod
    def _unpack(cls, rows):
        return [(rows[i][0], (rows[i][1], rows[i][2])) for i in range(len(rows))]

    @classmethod
    def _value(cls, index, to_float):
        c = cls._preloaded[index]
        if to_float:
            return c[1]
        else:
            return sympify(c[0])

    @classmethod
    def _download_one(cls, index):
        logging.info(f"C: MISSING PRELOADED VERSION OF C_{index}")
        respond = db.execute(
            f"SELECT `index`, `value`, `value_f` FROM `C`"
            f"WHERE REGEXP(`index`, '^{index}$')"
        )
        return cls._unpack(respond)

    @classmethod
    def preload(cls, *args):
        """
        Updates dictionary of the preloaded Fourier-Legendre coefficients
        Note: weights are not accepted, such coefficients are loaded
        with all available weights
        Parameters
        ==========
        args
            Indices for the Fourier-Legendre coefficients
            to download them from database
        """
        logging.info(f"C: PRELOADING COEFFICIENTS {args}")

        query = []
        for q in range(len(args)):
            numbers = [int(char) for char in str(args[q] + 1)]
            pattern = []

            for i in range(1, len(numbers)):
                pattern.append("[0-9]" * i)

            for i in range(len(numbers)):
                p = []
                for j in range(len(numbers)):
                    if j < i:
                        p.append(str(numbers[j]))
                    elif i == j:
                        p.append(f"[0-{numbers[j] - 1}]")
                    elif j > i:
                        p.append("[0-9]")
                pattern.append("".join(p))
```

```

            regex = f"^{':'.join(['|'.join(pattern) for _ in range(q + 2)])}_.*$"
            query.append(
                f"SELECT `index`, `value`, `value_f` FROM `C`"
                f"WHERE REGEXP(`index`, '{regex}')"
            )

        result = db.execute("\nUNION\n".join(query))
        cls._preloaded.update(cls._unpack(result))

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        C
        """
        return C(*self.args, **hints)
```

Listing 46: **Calculation of the Fourier–Legendre coefficients** $C_{j_3 j_2 j_1}^{000}$

```
from math import sqrt

from sympy import sympify, Number, Function

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C000(Function):
    """
    Gives the Fourier-Legendre coefficient with requested indices and weights
    """
    nargs = 4

    def __new__(cls, *args, **kwargs):
        """
        Creates the Fourier-Legendre coefficient object with needed
        indices and weights and calculates it
        Parameters
        ==========
        indices: tuple
            requested indices
        weights: tuple
            requested weights
        Returns
        =======
        symbolic.Rational or C000
            calculated value or symbolic expression
        """
        j3, j2, j1, dt = sympify(args)

        if not (isinstance(j1, Number) and
                isinstance(j2, Number) and
```

```
        isinstance(j3, Number) and
        isinstance(dt, Number)):
      return super(C000, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1)) * \
        dt ** 1.5 * \
        C((j3, j2, j1), (0, 0, 0)) / 8

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C000
    """
    return C000(*self.args, **hints)
```

Listing 47: **Calculation of the Fourier–Legendre coefficients** $C_{j_2j_1}^{10}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C10(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C10
      calculated value or symbolic expression
    """
    j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
```

```
        isinstance(j2, Number) and
        isinstance(dt, Number)):
      return super(C10, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1)) * \
        dt ** 2 * \
        C((j2, j1), (1, 0)) / 8

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C10
    """
    return C10(*self.args, **hints)
```

Listing 48: **Calculation of the Fourier–Legendre coefficients** $C_{j_2j_1}^{01}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C01(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C01
      calculated value or symbolic expression
    """
    j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
```

```
        isinstance(dt, Number)):
      return super(C01, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1)) * \
        dt ** 2 * \
        C((j2, j1), (0, 1)) / 8

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C01
    """
    return C01(*self.args, **hints)
```

Listing 49: **Calculation of the Fourier–Legendre coefficients** $C_{j_4 j_3 j_2 j_1}^{0000}$

```
from math import sqrt

from sympy import sympify, Number, Function

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C0000(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C0000
      calculated value or symbolic expression
    """
    j4, j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
```

```
            isinstance(j4, Number) and
            isinstance(dt, Number)):
        return super(C0000, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1) *
      (j4 * 2 + 1)) * \
        dt ** 2 * \
        C((j4, j3, j2, j1), (0, 0, 0, 0)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C0000
    """
    return C0000(*self.args, **hints)
```

Listing 50: **Calculation of the Fourier–Legendre coefficients** $C_{j_3j_2j_1}^{100}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C100(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 4

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C100
      calculated value or symbolic expression
    """
    j3, j2, j1, dt = sympify(args)

```

```
    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(dt, Number)):
      return super(C100, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1)) * \
        dt ** 2.5 * \
        C((j3, j2, j1), (1, 0, 0)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C100
    """
    return C100(*self.args, **hints)
```

Listing 51: **Calculation of the Fourier–Legendre coefficients** $C_{j_3 j_2 j_1}^{010}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C010(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 4

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C010
      calculated value or symbolic expression
    """
    j3, j2, j1, dt = sympify(args)
```

```

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(dt, Number)):
      return super(C010, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1)) * \
        dt ** 2.5 * \
        C((j3, j2, j1), (0, 1, 0)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C010
    """
    return C010(*self.args, **hints)
```

Listing 52: **Calculation of the Fourier–Legendre coefficients** $C_{j_3j_2j_1}^{001}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C001(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 4

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C001
      calculated value or symbolic expression
    """
```

```
    j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(dt, Number)):
      return super(C001, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1)) * \
        dt ** 2.5 * \
        C((j3, j2, j1), (0, 0, 1)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C001
    """
    return C001(*self.args, **hints)
```

Listing 53: **Calculation of the Fourier–Legendre coefficients** $C_{j_5j_4j_3j_2j_1}^{00000}$

```
from math import sqrt

from sympy import sympify, Number, Function

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C00000(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C00000
      calculated value or symbolic expression
```

```
    """
    j5, j4, j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(j4, Number) and
        isinstance(j5, Number) and
        isinstance(dt, Number)):
      return super(C00000, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1) *
      (j4 * 2 + 1) *
      (j5 * 2 + 1)) * \
        dt ** 2.5 * \
        C((j5, j4, j3, j2, j1), (0, 0, 0, 0, 0)) / 32

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C00000
    """
    return C00000(*self.args, **hints)
```

Listing 54: **Calculation of the Fourier–Legendre coefficients** $C^{20}_{j_2 j_1}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C20(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
```

```
      requested weights
    Returns
    =======
    symbolic.Rational or C20
      calculated value or symbolic expression
    """
    j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(dt, Number)):
      return super(C20, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1)) * \
        dt ** 3 * \
        C((j2, j1), (2, 0)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C20
    """
    return C20(*self.args, **hints)
```

Listing 55: **Calculation of the Fourier–Legendre coefficients** $C_{j_2 j_1}^{11}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C11(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
```

```
        Returns
        =======
        symbolic.Rational or C11
            calculated value or symbolic expression
        """
        j2, j1, dt = sympify(args)

        if not (isinstance(j1, Number) and
                isinstance(j2, Number) and
                isinstance(dt, Number)):
            return super(C11, cls).__new__(cls, *args, **kwargs)

        return sqrt(
            (j1 * 2 + 1) *
            (j2 * 2 + 1)) * \
            dt ** 3 * \
            C((j2, j1), (1, 1)) / 16

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        C11
        """
        return C11(*self.args, **hints)
```

Listing 56: **Calculation of the Fourier–Legendre coefficients** $C_{j_2j_1}^{02}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C02(Function):
    """
    Gives the Fourier-Legendre coefficient with requested indices and weights
    """
    nargs = 3

    def __new__(cls, *args, **kwargs):
        """
        Creates the Fourier-Legendre coefficient object with needed
        indices and weights and calculates it
        Parameters
        ==========
        indices: tuple
            requested indices
        weights: tuple
            requested weights
        Returns
```

```
    =======
    symbolic.Rational or C02
      calculated value or symbolic expression
    """
    j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(dt, Number)):
      return super(C02, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1)) * \
        dt ** 3 * \
        C((j2, j1), (0, 2)) / 16

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C02
    """
    return C02(*self.args, **hints)
```

Listing 57: **Calculation of the Fourier–Legendre coefficients** $C_{j_4 j_3 j_2 j_1}^{1000}$

```
from math import sqrt

from sympy import sympify, Number, Function

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C1000(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
```

```
    symbolic.Rational or C1000
      calculated value or symbolic expression
    """
    j4, j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(j4, Number) and
        isinstance(dt, Number)):
      return super(C1000, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1) *
      (j4 * 2 + 1)) * \
        dt ** 3 * \
        C((j4, j3, j2, j1), (1, 0, 0, 0)) / 32

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C1000
    """
    return C1000(*self.args, **hints)
```

Listing 58: **Calculation of the Fourier–Legendre coefficients** $C_{j_4 j_3 j_2 j_1}^{0100}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C0100(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
```

```
        requested weights
      Returns
      =======
      symbolic.Rational or C0100
        calculated value or symbolic expression
      """
      j4, j3, j2, j1, dt = sympify(args)

      if not (isinstance(j1, Number) and
          isinstance(j2, Number) and
          isinstance(j3, Number) and
          isinstance(j4, Number) and
          isinstance(dt, Number)):
        return super(C0100, cls).__new__(cls, *args, **kwargs)

      return sqrt(
        (j1 * 2 + 1) *
        (j2 * 2 + 1) *
        (j3 * 2 + 1) *
        (j4 * 2 + 1)) * \
          dt ** 3 * \
          C((j4, j3, j2, j1), (0, 1, 0, 0)) / 32

    def doit(self, **hints):
      """
      Tries to expand or calculate function
      Returns
      =======
      C0100
      """
      return C0100(*self.args, **hints)
```

Listing 59: **Calculation of the Fourier–Legendre coefficients** $C_{j_4 j_3 j_2 j_1}^{0010}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C0010(Function):
    """
    Gives the Fourier-Legendre coefficient with requested indices and weights
    """
    nargs = 5

    def __new__(cls, *args, **kwargs):
      """
      Creates the Fourier-Legendre coefficient object with needed
      indices and weights and calculates it
      Parameters
      ==========
```

```
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C0010
      calculated value or symbolic expression
    """
    j4, j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(j4, Number) and
        isinstance(dt, Number)):
      return super(C0010, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1) *
      (j4 * 2 + 1)) * \
        dt ** 3 * \
        C((j4, j3, j2, j1), (0, 0, 1, 0)) / 32

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C0010
    """
    return C0010(*self.args, **hints)
```

Listing 60: **Calculation of the Fourier–Legendre coefficients** $C_{j_4 j_3 j_2 j_1}^{0001}$

```
from math import sqrt

from sympy import sympify, Function, Number

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C0001(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates the Fourier-Legendre coefficient object with needed
```

```
    indices and weights and calculates it
    Parameters
    ==========
    indices: tuple
      requested indices
    weights: tuple
      requested weights
    Returns
    =======
    symbolic.Rational or C0001
      calculated value or symbolic expression
    """
    j4, j3, j2, j1, dt = sympify(args)

    if not (isinstance(j1, Number) and
        isinstance(j2, Number) and
        isinstance(j3, Number) and
        isinstance(j4, Number) and
        isinstance(dt, Number)):
      return super(C0001, cls).__new__(cls, *args, **kwargs)

    return sqrt(
      (j1 * 2 + 1) *
      (j2 * 2 + 1) *
      (j3 * 2 + 1) *
      (j4 * 2 + 1)) * \
        dt ** 3 * \
        C((j4, j3, j2, j1), (0, 0, 0, 1)) / 32

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    C0001
    """
    return C0001(*self.args, **hints)
```

Listing 61: **Calculation of the Fourier–Legendre coefficients** $C_{j_6j_5j_4j_3j_2j_1}^{000000}$

```
from math import sqrt

from sympy import sympify, Number, Function

from mathematics.sde.nonlinear.symbolic.coefficients.c import C


class C000000(Function):
  """
  Gives the Fourier-Legendre coefficient with requested indices and weights
  """
  nargs = 7

```

```
    def __new__(cls, *args, **kwargs):
        """
        Creates the Fourier-Legendre coefficient object with needed
        indices and weights and calculates it
        Parameters
        ==========
        indices: tuple
            requested indices
        weights: tuple
            requested weights
        Returns
        =======
        symbolic.Rational or C000000
            calculated value or symbolic expression
        """
        j6, j5, j4, j3, j2, j1, dt = sympify(args)

        if not (isinstance(j1, Number) and
                isinstance(j2, Number) and
                isinstance(j3, Number) and
                isinstance(j4, Number) and
                isinstance(j5, Number) and
                isinstance(dt, Number)):
            return super(C000000, cls).__new__(cls, *args, **kwargs)

        return sqrt(
            (j1 * 2 + 1) *
            (j2 * 2 + 1) *
            (j3 * 2 + 1) *
            (j4 * 2 + 1) *
            (j5 * 2 + 1) *
            (j6 * 2 + 1)) * \
              dt ** 3 * \
              C((j6, j5, j4, j3, j2, j1), (0, 0, 0, 0, 0, 0)) / 64

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        C000000
        """
        return C000000(*self.args, **hints)
```

Listing 62: **Program entry for generation of new Fourier–Legendre coefficients**

```
#!/usr/bin/env python
import logging
import os
from datetime import datetime
from multiprocessing import cpu_count, Pool
from pprint import pprint

```

```python
from config import csv, new_c_portion_size
from mathematics.sde.nonlinear.new_c import thread_c, split_task


def main():
    logging.basicConfig(
        level=logging.INFO,
        format="%(asctime)s - %(levelname)s - %(message)s",
        datefmt="%H:%M:%S"
    )
    logger = logging.getLogger(__name__)

    filename = os.path.join(csv, f"c_{datetime.now().strftime('%d-%m-%Y_%H-%M-%S')}.csv")
    logging.info(f"Writing to file {filename}")

    tasks = [
        (((57, 58), (57, 58), (57, 58)), (0, 0, 0)),
        # (((0, 56), (0, 56), (0, 56)), (0, 0, 0)),
        #
        # (((0, 15), (0, 15)), (0, 1)),
        # (((0, 15), (0, 15)), (1, 0)),
        # (((0, 15), (0, 15), (0, 15)), (0, 0, 0, 0)),
        #
        # (((0, 6), (0, 6), (0, 6)), (0, 0, 1)),
        # (((0, 6), (0, 6), (0, 6)), (0, 1, 0)),
        # (((0, 6), (0, 6), (0, 6)), (1, 0, 0)),
        # (((0, 6), (0, 6), (0, 6), (0, 6), (0, 6)), (0, 0, 0, 0, 0)),
        #
        # (((0, 2), (0, 2)), (0, 2)),
        # (((0, 2), (0, 2)), (1, 1)),
        # (((0, 2), (0, 2)), (2, 0)),
        # (((0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 1)),
        # (((0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 1, 0)),
        # (((0, 2), (0, 2), (0, 2), (0, 2)), (0, 1, 0, 0)),
        # (((0, 2), (0, 2), (0, 2), (0, 2)), (1, 0, 0, 0)),
        # (((0, 2), (0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 0, 0)),
        # (((0, 2), (0, 2), (0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 0, 0, 0)),
    ]

    with Pool(cpu_count()) as p:

        for t in tasks:

            logger.info(f"Running task {t}")

            for chunk in chunks(split_task(t), new_c_portion_size):
                c = p.map(thread_c, chunk)
                c.append("")

                with open(filename, "a") as f:
                    f.write("\n".join(c))
                    f.close()

                logger.info(f"The portion of C has been written {chunk[0]}-{chunk[-1]}")

```

```
    logger.info("Generation has been done")


def chunks(lst, n):
    for i in range(0, len(lst), n):
        yield lst[i:i + n]


if __name__ == "__main__":
    main()
```

Listing 63: **Module for the Fourier–Legendre coefficients generation**

```
from mathematics.sde.nonlinear.c import get_c


def split_task(tasks):
    c = len(tasks[1])

    if c == 2:
        return split_2(tasks)

    if c == 3:
        return split_3(tasks)

    if c == 4:
        return split_4(tasks)

    if c == 5:
        return split_5(tasks)

    if c == 6:
        return split_6(tasks)


def split_2(ranges):
    return [((i, j), ranges[1])
            for j in range(ranges[0][1][1])
            for i in range(ranges[0][0][1])
            if i >= ranges[0][1][0]
            or j >= ranges[0][0][1]]


def split_3(ranges):
    return [((i, j, k), ranges[1])
            for k in range(ranges[0][2][1])
            for j in range(ranges[0][1][1])
            for i in range(ranges[0][0][1])
            if k >= ranges[0][2][0]
            or j >= ranges[0][1][0]
            or i >= ranges[0][0][0]]


```

```
def split_4(ranges):
  return [((i, j, k, l), ranges[1])
      for j in range(*ranges[0][3])
      for i in range(*ranges[0][2])
      for k in range(*ranges[0][1])
      for l in range(*ranges[0][0])
      if l >= ranges[0][3][0]
      or k >= ranges[0][2][0]
      or j >= ranges[0][1][0]
      or i >= ranges[0][0][0]]


def split_5(ranges):
  return [((i, j, k, l, m), ranges[1])
      for j in range(*ranges[0][4])
      for i in range(*ranges[0][3])
      for k in range(*ranges[0][2])
      for l in range(*ranges[0][1])
      for m in range(*ranges[0][0])
      if m >= ranges[0][4][0]
      if l >= ranges[0][3][0]
      or k >= ranges[0][2][0]
      or j >= ranges[0][1][0]
      or i >= ranges[0][0][0]]


def split_6(ranges):
  return [((i, j, k, l, m, n), ranges[1])
      for j in range(*ranges[0][5])
      for i in range(*ranges[0][4])
      for k in range(*ranges[0][3])
      for l in range(*ranges[0][2])
      for m in range(*ranges[0][1])
      for n in range(*ranges[0][0])
      if n >= ranges[0][5][0]
      if m >= ranges[0][4][0]
      if l >= ranges[0][3][0]
      or k >= ranges[0][2][0]
      or j >= ranges[0][1][0]
      or i >= ranges[0][0][0]]


def thread_c(ranges):
  c = len(ranges[1])

  if c == 2:
    return gen_2(*ranges[0], ranges[1])

  if c == 3:
    return gen_3(*ranges[0], ranges[1])

  if c == 4:
    return gen_4(*ranges[0], ranges[1])

  if c == 5:
```

```
    return gen_5(*ranges[0], ranges[1])

  if c == 6:
    return gen_6(*ranges[0], ranges[1])


def gen_2(i, j, w):
  return f"\"{i}:{j}_{w[0]}:{w[1]}\";\"{get_c((i, j), w)}\""


def gen_3(i, j, k, w):
  return f"\"{i}:{j}:{k}_{w[0]}:{w[1]}:{w[2]}\";\"{get_c((i, j, k), w)}\""


def gen_4(i, j, k, l, w):
  return f"\"{i}:{j}:{k}:{l}_{w[0]}:{w[1]}:{w[2]}:{w[3]}\";\"{get_c((i, j, k, l), w)}\""


def gen_5(i, j, k, l, m, w):
  return f"\"{i}:{j}:{k}:{l}:{m}_{w[0]}:{w[1]}:{w[2]}:{w[3]}:{w[4]}\";\"{get_c((i, j, k, l, m), w)}\""


def gen_6(i, j, k, l, m, n, w):
  return f"\"{i}:{j}:{k}:{l}:{m}_{w[0]}:{w[1]}:{w[2]}:{w[3]}:{w[4]}:{w[5]}\";\"{get_c((i, j, k, l, m, n), w)}\""
```

6.2.2 Source Codes for Supplementary Differential Operators and Functions

Listing 64: **Implementation of the differential operator L**

```
from sympy import Add, sympify, Number, diff

from mathematics.sde.nonlinear.symbolic.operator import Operator


class L(Operator):
  nargs = 4

  def __new__(cls, *args, **kwargs):
    """
    Creates new L object with given args
    Parameters
    ==========
    args
      bunch of necessary arguments
    Returns
    =======
    sympy.Expr
```

```
      formula to simplify and substitute
    """
    a, b, f, dxs = sympify(args)

    if not ((isinstance(f, Number) or f.has(*dxs)) and
        not f.has(Operator) and a.is_Matrix):
      return super(L, cls).__new__(cls, *args, **kwargs)

    n = b.shape[0]
    m = b.shape[1]
    from sympy.abc import t

    return Add(
      diff(f, t),
      *[a[i, 0] * diff(f, dxs[i]) for i in range(n)],
      *[0.5 * b[i, j] * b[k, j] * diff(f, dxs[i], dxs[k])
       for j in range(m)
       for i in range(n)
       for k in range(n)]
    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    L
    """
    return L(*self.args, **hints)
```

Listing 65: **Implementation of the differential operator** $G_0^{(i)}$

```
from sympy import Add, sympify, Number, diff

from mathematics.sde.nonlinear.symbolic.operator import Operator


class G(Operator):
  nargs = 3

  _dict = dict()

  def __new__(cls, *args, **kwargs):
    """
    Creates new G object with given args
    Parameters
    ==========
    args
      bunch of necessary arguments
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
```

```
    """
    c, f, dxs = sympify(args)

    if not ((isinstance(f, Number) or f.has(*dxs)) and
        not f.has(Operator)):
      return super(G, cls).__new__(cls, *args, **kwargs)

    return Add(*[c[i, 0] * diff(f, dxs[i])
          for i in range(len(dxs))])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    G
    """
    return G(*self.args, **hints)
```

Listing 66: **Implementation of the function $\bar{\mathbf{a}}(\mathbf{x}, t)$**

```
from sympy import sympify, Matrix, Add

from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.operator import Operator


class Aj(Operator):
  nargs = 4

  def __new__(cls, *args, **kwargs):
    """
    Creates new Aj object with given args
    Parameters
    ==========
    args
      bunch of necessary arguments
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, a, b, dxs = sympify(args)
    n = b.shape[0]
    m = b.shape[1]

    return Matrix([a[i, 0] -
            (Add(*[0.5 * G(b[:, j], b[i, j], dxs)
               for j in range(m)]))
            for i in range(n)])
```

Listing 67: **Implementation of the differential operator $\bar{L}$**

```
from sympy import sympify, Number, diff, Add
```

```

from mathematics.sde.nonlinear.symbolic.operator import Operator


class Lj(Operator):
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates new Lj object with given args
    Parameters
    ==========
    args
      bunch of necessary arguments
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    a, f, dxs = sympify(args)

    if not (isinstance(f, Number) or (f.has(*dxs)) and
        not f.has(Operator)):
      return super(Lj, cls).__new__(cls, *args, **kwargs)

    n = a.shape[0]
    from sympy.abc import t
    return Add(diff(f, t), *[a[i, 0] * diff(f, dxs[i])
                 for i in range(n)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    Lj
    """
    return Lj(*self.args, **hints)
```

Listing 68: **Implementation of the indicator function $\mathbf{1}_{\{i_1=i_2\}}$**

```
from sympy import sympify, Number, Function


class Ind(Function):
  """
  Indicator function
  """
  nargs = 2

  def __new__(cls, *args, **kwargs):
    """
    Creates new Ind object with given args
```

```
    Parameters
    ==========
    args
      bunch of necessary arguments
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2 = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(Ind, cls).__new__(cls, *args, **kwargs)

    if i1 == i2:
      return 1
    else:
      return 0

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    Ind
    """
    return Ind(*self.args, **hints)
```

6.2.3 Source Codes for Iterated Itô Stochastic Integrals Approximations Subprograms

Listing 69: **Approximation of Itô stochastic integral** $I_{(0)\tau_{p+1},\tau_p}^{(i_1)}$

```
from math import sqrt

from sympy import Function, sympify, Number


class I0(Function):
  """
  Ito stochastic integral
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates new I0 object with given args
    Parameters
    ==========
```

```
    i1 : int
      integral index
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, dt, ksi = sympify(args)

    if not isinstance(i1, Number):
      return super(I0, cls).__new__(cls, *args, **kwargs)

    return ksi[0, i1] * sqrt(dt)

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I0
    """
    return I0(*self.args, **hints)
```

Listing 70: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2)}_{(00)\tau_{p+1},\tau_p}$

```
from math import sqrt

from sympy import sympify, Number, Function, Add

from mathematics.sde.nonlinear.symbolic.ind import Ind


class I00(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new I00 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
```

```
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(q, Number)):
      return super(I00, cls).__new__(cls, *args, **kwargs)

    return \
      (ksi[0, i1] * ksi[0, i2] +
       Add(*[
         (ksi[j1 - 1, i1] * ksi[j1, i2] -
         ksi[j1, i1] * ksi[j1 - 1, i2]) /
         sqrt(j1 ** 2 * 4 - 1)
         for j1 in range(1, q + 1)]) -
       Ind(i1, i2)) * dt / 2

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I00
    """
    return I00(*self.args, **hints)
```

Listing 71: **Approximation of Itô stochastic integral** $I_{(1)\tau_{p+1},\tau_p}^{(i_1)}$

```
from math import sqrt

from sympy import Function, sympify, Number


class I1(Function):
  """
  Ito stochastic integral
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates new I1 object with given args
    Parameters
    ==========
```

```
    i1 : int
      integral index
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, dt, ksi = sympify(args)

    if not isinstance(i1, Number):
      return super(I1, cls).__new__(cls, *args, **kwargs)

    return -(ksi[0, i1] + ksi[1, i1] / sqrt(3)) * dt ** 1.5 / 2

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I1
    """
    return I1(*self.args, **hints)
```

Listing 72: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3)}_{(000)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c000 import C000
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I000(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new I000 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    q : int
```

```
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(q, Number)):
      return super(I000, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C000(j3, j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
       Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
       Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
       Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I000
    """
    return I000(*self.args, **hints)
```

Listing 73: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2)}_{(10)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c10 import C10
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I10(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
```

```
    """
    Creates new I10 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(q, Number)):
      return super(I10, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C10(j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] -
       Ind(i1, i2) * Ind(j1, j2))
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I10
    """
    return I10(*self.args, **hints)
```

Listing 74: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2)}_{(01)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c01 import C01
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I01(Function):
  """
```

```
    Iterated Ito stochastic integral
    """
    nargs = 5

    def __new__(cls, *args, **kwargs):
      """
      Creates new I01 object with given args
      Parameters
      ==========
      i1 : int
        integral index
      i2 : int
        integral index
      q : int
        amount of terms in approximation of
        iterated Ito stochastic integral
      dt : float
        delta time
      ksi : numpy.ndarray
        matrix of Gaussian random variables
      Returns
      =======
      sympy.Expr
        formula to simplify and substitute
      """
      i1, i2, q, dt, ksi = sympify(args)

      if not (isinstance(i1, Number) and
          isinstance(i2, Number) and
          isinstance(q, Number)):
        return super(I01, cls).__new__(cls, *args, **kwargs)

      return Add(*[
        C01(j2, j1, dt) *
        (ksi[j1, i1] * ksi[j2, i2] -
         Ind(i1, i2) * Ind(j1, j2))
        for j2 in range(q + 1)
        for j1 in range(q + 1)])

    def doit(self, **hints):
      """
      Tries to expand or calculate function
      Returns
      =======
      I01
      """
      return I01(*self.args, **hints)
```

Listing 75: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4)}_{(0000)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0000 import C0000
```

```python
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I0000(Function):
    """
    Iterated Ito stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
        """
        Creates new I0000 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(q, Number)):
            return super(I0000, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C0000(j4, j3, j2, j1, dt) *
            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
             Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
             Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
             Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
             Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
             Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
             Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
             Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
             Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
             Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
```

```
        for j4 in range(q + 1)
        for j3 in range(q + 1)
        for j2 in range(q + 1)
        for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I0000
        """
        return I0000(*self.args, **hints)
```

Listing 76: **Approximation of Itô stochastic integral** $I^{(i_1)}_{(2)\tau_{p+1},\tau_p}$

```
from math import sqrt

from sympy import sympify, Number, Function


class I2(Function):
    """
    Ito stochastic integral
    """
    nargs = 3

    def __new__(cls, *args, **kwargs):
        """
        Creates new I2 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, dt, ksi = sympify(args)

        if not isinstance(i1, Number):
            return super(I2, cls).__new__(cls, *args, **kwargs)

        return (ksi[0, i1] + ksi[1, i1] * sqrt(3) / 2 +
                ksi[2, i1] / sqrt(5) / 2) * dt ** 2.5 / 3

    def doit(self, **hints):
        """
```

```
    Tries to expand or calculate function
    Returns
    =======
    I2
    """
    return I2(*self.args, **hints)
```

Listing 77: **Approximation of iterated Itô stochastic integral** $I_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c100 import C100
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I100(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new I100 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(q, Number)):
      return super(I100, cls).__new__(cls, *args, **kwargs)

    return Add(*[
```

```
      C100(j3, j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
       Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
       Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
       Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I100
    """
    return I100(*self.args, **hints)
```

Listing 78: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3)}_{(010)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c010 import C010
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I010(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new I010 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
```

```
      formula to simplify and substitute
    """
    i1, i2, i3, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(q, Number)):
      return super(I010, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C010(j3, j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
       Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
       Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
       Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I010
    """
    return I010(*self.args, **hints)
```

Listing 79: **Approximation of iterated Itô stochastic integral** $I_{(001)\tau_{p+1},\tau_p}^{(i_1i_2i_3)}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c001 import C001
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I001(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new I001 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
```

```
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(q, Number)):
            return super(I001, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C001(j3, j2, j1, dt) *
            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
             Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
             Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
             Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I001
        """
        return I001(*self.args, **hints)
```

Listing 80: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4 i_5)}_{(00000)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c00000 import C00000
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I00000(Function):
    """
    Iterated Ito stochastic integral
    """
    nargs = 8
```

```
  def __new__(cls, *args, **kwargs):
    """
    Creates new I00000 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    i4 : int
      integral index
    i5 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, i4, i5, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(i5, Number) and
        isinstance(q, Number) and
        isinstance(dt, Number)):
      return super(I00000, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C00000(j5, j4, j3, j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
       Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
       Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] * ksi[j5, i5] -
       Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] * ksi[j5, i5] -
       Ind(i1, i5) * Ind(j1, j5) * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
       Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] * ksi[j5, i5] -
       Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] * ksi[j5, i5] -
       Ind(i2, i5) * Ind(j2, j5) * ksi[j1, i1] * ksi[j3, i3] * ksi[j4, i4] -
       Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] * ksi[j5, i5] -
       Ind(i3, i5) * Ind(j3, j5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j4, i4] -
       Ind(i4, i5) * Ind(j4, j5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] +
       Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) * ksi[j5, i5] +
       Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i5) * Ind(j3, j5) * ksi[j4, i4] +
       Ind(i1, i2) * Ind(j1, j2) * Ind(i4, i5) * Ind(j4, j5) * ksi[j3, i3] +
```

```
        Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) * ksi[j5, i5] +
        Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i5) * Ind(j2, j5) * ksi[j4, i4] +
        Ind(i1, i3) * Ind(j1, j3) * Ind(i4, i5) * Ind(j4, j5) * ksi[j2, i2] +
        Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3) * ksi[j5, i5] +
        Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i5) * Ind(j2, j5) * ksi[j3, i3] +
        Ind(i1, i4) * Ind(j1, j4) * Ind(i3, i5) * Ind(j3, j5) * ksi[j2, i2] +
        Ind(i1, i5) * Ind(j1, j5) * Ind(i2, i3) * Ind(j2, j3) * ksi[j4, i4] +
        Ind(i1, i5) * Ind(j1, j5) * Ind(i2, i4) * Ind(j2, j4) * ksi[j3, i3] +
        Ind(i1, i5) * Ind(j1, j5) * Ind(i3, i4) * Ind(j3, j4) * ksi[j2, i2] +
        Ind(i2, i3) * Ind(j2, j3) * Ind(i4, i5) * Ind(j4, j5) * ksi[j1, i1] +
        Ind(i2, i4) * Ind(j2, j4) * Ind(i3, i5) * Ind(j3, j5) * ksi[j1, i1] +
        Ind(i2, i5) * Ind(j2, j5) * Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1])
      for j5 in range(q + 1)
      for j4 in range(q + 1)
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I00000
    """
    return I00000(*self.args, **hints)
```

Listing 81: **Approximation of iterated Itô stochastic integral** $I_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c20 import C20
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I20(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new I20 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
```

```
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(I20, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C20(j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I20
    """
    return I20(*self.args, **hints)
```

Listing 82: **Approximation of iterated Itô stochastic integral** $I_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c02 import C02
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I02(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new I02 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
```

```
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(I02, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C02(j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I02
    """
    return I02(*self.args, **hints)
```

Listing 83: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2)}_{(11)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c11 import C11
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I11(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new I11 object with given args
    Parameters
    ==========
    i1 : int
```

```
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Ito stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(I11, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C11(j2, j1, dt) *
      (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    I11
    """
    return I11(*self.args, **hints)
```

Listing 84: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4)}_{(1000)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c1000 import C1000
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I1000(Function):
  """
  Iterated Ito stochastic integral
  """
  nargs = 7

  def __new__(cls, *args, **kwargs):
    """
    Creates new I1000 object with given args
```

```
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(q, Number)):
            return super(I1000, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C1000(j4, j3, j2, j1, dt) *
            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
             Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
             Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
             Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
             Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
             Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
             Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
             Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
             Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
             Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
            for j4 in range(q + 1)
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I1000
        """
```

```
    return I1000(*self.args, **hints)
```

Listing 85: **Approximation of iterated Itô stochastic integral** $I_{(0100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0100 import C0100
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I0100(Function):
    """
    Iterated Ito stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
        """
        Creates new I0100 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(q, Number)):
            return super(I0100, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C0100(j4, j3, j2, j1, dt) *
            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
```

```
                Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
                Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
                Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
                Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
                Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
                Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
                Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
                Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
                Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
              for j4 in range(q + 1)
              for j3 in range(q + 1)
              for j2 in range(q + 1)
              for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I0100
        """
        return I0100(*self.args, **hints)
```

Listing 86: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4)}_{(0010)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0010 import C0010
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I0010(Function):
    """
    Iterated Ito stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
        """
        Creates new I0010 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i3 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
```

```
      dt : float
        delta time
      ksi : numpy.ndarray
        matrix of Gaussian random variables
      Returns
      =======
      sympy.Expr
        formula to simplify and substitute
      """
      i1, i2, i3, i4, q, dt, ksi = sympify(args)

      if not (isinstance(i1, Number) and
          isinstance(i2, Number) and
          isinstance(i3, Number) and
          isinstance(i4, Number) and
          isinstance(q, Number)):
        return super(I0010, cls).__new__(cls, *args, **kwargs)

      return Add(*[
        C0010(j4, j3, j2, j1, dt) *
        (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
         Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
         Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
         Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
         Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
         Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
         Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
         Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
         Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
         Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
        for j4 in range(q + 1)
        for j3 in range(q + 1)
        for j2 in range(q + 1)
        for j1 in range(q + 1)])

    def doit(self, **hints):
      """
      Tries to expand or calculate function
      Returns
      =======
      I0010
      """
      return I0010(*self.args, **hints)
```

Listing 87: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4)}_{(0001)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0001 import C0001
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I0001(Function):
```

```
    """
    Iterated Ito stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
      """
      Creates new I0001 object with given args
      Parameters
      ==========
      i1 : int
        integral index
      i2 : int
        integral index
      i3 : int
        integral index
      i4 : int
        integral index
      q : int
        amount of terms in approximation of
        iterated Ito stochastic integral
      dt : float
        delta time
      ksi : numpy.ndarray
        matrix of Gaussian random variables
      Returns
      =======
      sympy.Expr
        formula to simplify and substitute
      """
      i1, i2, i3, i4, q, dt, ksi = sympify(args)

      if not (isinstance(i1, Number) and
          isinstance(i2, Number) and
          isinstance(i3, Number) and
          isinstance(i4, Number) and
          isinstance(q, Number)):
        return super(I0001, cls).__new__(cls, *args, **kwargs)

      return Add(*[
        C0001(j4, j3, j2, j1, dt) *
        (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
         Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
         Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
         Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
         Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
         Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
         Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
         Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
         Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
         Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
        for j4 in range(q + 1)
        for j3 in range(q + 1)
        for j2 in range(q + 1)
        for j1 in range(q + 1)])
```

```python

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I0001
        """
        return I0001(*self.args, **hints)
```

Listing 88: **Approximation of iterated Itô stochastic integral** $I^{(i_1 i_2 i_3 i_4 i_5 i_6)}_{(000000)\tau_{p+1},\tau_p}$

```python
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c000000 import C000000
from mathematics.sde.nonlinear.symbolic.ind import Ind


class I000000(Function):
    """
    Iterated Ito stochastic integral
    """
    nargs = 9

    def __new__(cls, *args, **kwargs):
        """
        Creates new I000000 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        i5 : int
            integral index
        i6 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Ito stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
```

```
    i1, i2, i3, i4, i5, i6, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(i5, Number) and
        isinstance(i6, Number) and
        isinstance(q, Number) and
        isinstance(dt, Number)):
      return super(I000000, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C000000(j6, j5, j4, j3, j2, j1, dt) *
      (ksi[j1,i1] * ksi[j2,i2] * ksi[j3,i3] *
       ksi[j4,i4] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j1,j6) * Ind(i1,i6) * ksi[j2,i2] * ksi[j3,i3] * ksi[j4,i4] * ksi[j5,i5] -
       Ind(j2,j6) * Ind(i2,i6) * ksi[j1,i1] * ksi[j3,i3] * ksi[j4,i4] * ksi[j5,i5] -
       Ind(j3,j6) * Ind(i3,i6) * ksi[j1,i1] * ksi[j2,i2] * ksi[j4,i4] * ksi[j5,i5] -
       Ind(j4,j6) * Ind(i4,i6) * ksi[j1,i1] * ksi[j2,i2] * ksi[j3,i3] * ksi[j5,i5] -
       Ind(j5,j6) * Ind(i5,i6) * ksi[j1,i1] * ksi[j2,i2] * ksi[j3,i3] * ksi[j4,i4] -
       Ind(j1,j2) * Ind(i1,i2) * ksi[j3,i3] * ksi[j4,i4] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j1,j3) * Ind(i1,i3) * ksi[j2,i2] * ksi[j4,i4] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j1,j4) * Ind(i1,i4) * ksi[j2,i2] * ksi[j3,i3] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j1,j5) * Ind(i1,i5) * ksi[j2,i2] * ksi[j4,i4] * ksi[j3,i3] * ksi[j6,i6] -
       Ind(j2,j3) * Ind(i2,i3) * ksi[j1,i1] * ksi[j4,i4] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j2,j4) * Ind(i2,i4) * ksi[j1,i1] * ksi[j3,i3] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j2,j5) * Ind(i2,i5) * ksi[j1,i1] * ksi[j3,i3] * ksi[j4,i4] * ksi[j6,i6] -
       Ind(j3,j4) * Ind(i3,i4) * ksi[j1,i1] * ksi[j2,i2] * ksi[j5,i5] * ksi[j6,i6] -
       Ind(j3,j5) * Ind(i3,i5) * ksi[j1,i1] * ksi[j2,i2] * ksi[j4,i4] * ksi[j6,i6] -
       Ind(j4,j5) * Ind(i4,i5) * ksi[j1,i1] * ksi[j2,i2] * ksi[j3,i3] * ksi[j6,i6] +
       Ind(j1,j2) * Ind(i1,i2) * Ind(j3,j4) * Ind(i3,i4) * ksi[j5,i5] * ksi[j6,i6] +
       Ind(j1,j2) * Ind(i1,i2) * Ind(j3,j5) * Ind(i3,i5) * ksi[j4,i4] * ksi[j6,i6] +
       Ind(j1,j2) * Ind(i1,i2) * Ind(j4,j5) * Ind(i4,i5) * ksi[j3,i3] * ksi[j6,i6] +
       Ind(j1,j3) * Ind(i1,i3) * Ind(j2,j4) * Ind(i2,i4) * ksi[j5,i5] * ksi[j6,i6] +
       Ind(j1,j3) * Ind(i1,i3) * Ind(j2,j5) * Ind(i2,i5) * ksi[j4,i4] * ksi[j6,i6] +
       Ind(j1,j3) * Ind(i1,i3) * Ind(j4,j5) * Ind(i4,i5) * ksi[j2,i2] * ksi[j6,i6] +
       Ind(j1,j4) * Ind(i1,i4) * Ind(j2,j3) * Ind(i2,i3) * ksi[j5,i5] * ksi[j6,i6] +
       Ind(j1,j4) * Ind(i1,i4) * Ind(j2,j5) * Ind(i2,i5) * ksi[j3,i3] * ksi[j6,i6] +
       Ind(j1,j4) * Ind(i1,i4) * Ind(j3,j5) * Ind(i3,i5) * ksi[j2,i2] * ksi[j6,i6] +
       Ind(j1,j5) * Ind(i1,i5) * Ind(j2,j3) * Ind(i2,i3) * ksi[j4,i4] * ksi[j6,i6] +
       Ind(j1,j5) * Ind(i1,i5) * Ind(j2,j4) * Ind(i2,i4) * ksi[j3,i3] * ksi[j6,i6] +
       Ind(j1,j5) * Ind(i1,i5) * Ind(j3,j4) * Ind(i3,i4) * ksi[j2,i2] * ksi[j6,i6] +
       Ind(j2,j3) * Ind(i2,i3) * Ind(j4,j5) * Ind(i4,i5) * ksi[j1,i1] * ksi[j6,i6] +
       Ind(j2,j4) * Ind(i2,i4) * Ind(j3,j5) * Ind(i3,i5) * ksi[j1,i1] * ksi[j6,i6] +
       Ind(j2,j5) * Ind(i2,i5) * Ind(j3,j4) * Ind(i3,i4) * ksi[j1,i1] * ksi[j6,i6] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j3,j4) * Ind(i3,i4) * ksi[j2,i2] * ksi[j5,i5] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j3,j5) * Ind(i3,i5) * ksi[j2,i2] * ksi[j4,i4] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j5) * Ind(i2,i5) * ksi[j3,i3] * ksi[j4,i4] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j4) * Ind(i2,i4) * ksi[j3,i3] * ksi[j5,i5] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j4,j5) * Ind(i4,i5) * ksi[j2,i2] * ksi[j3,i3] +
       Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j3) * Ind(i2,i3) * ksi[j4,i4] * ksi[j5,i5] +
       Ind(j6,j2) * Ind(i6,i2) * Ind(j3,j5) * Ind(i3,i5) * ksi[j1,i1] * ksi[j4,i4] +
       Ind(j6,j2) * Ind(i6,i2) * Ind(j4,j5) * Ind(i4,i5) * ksi[j1,i1] * ksi[j3,i3] +
       Ind(j6,j2) * Ind(i6,i2) * Ind(j3,j4) * Ind(i3,i4) * ksi[j1,i1] * ksi[j5,i5] +
```

```
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j5) * Ind(i1,i5) * ksi[j3,i3] * ksi[j4,i4] +
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j4) * Ind(i1,i4) * ksi[j3,i3] * ksi[j5,i5] +
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j3) * Ind(i1,i3) * ksi[j4,i4] * ksi[j5,i5] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j2,j5) * Ind(i2,i5) * ksi[j1,i1] * ksi[j4,i4] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j4,j5) * Ind(i4,i5) * ksi[j1,i1] * ksi[j2,i2] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j2,j4) * Ind(i2,i4) * ksi[j1,i1] * ksi[j5,i5] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j1,j5) * Ind(i1,i5) * ksi[j2,i2] * ksi[j4,i4] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j1,j4) * Ind(i1,i4) * ksi[j2,i2] * ksi[j5,i5] +
                Ind(j6,j3) * Ind(i6,i3) * Ind(j1,j2) * Ind(i1,i2) * ksi[j4,i4] * ksi[j5,i5] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j3,j5) * Ind(i3,i5) * ksi[j1,i1] * ksi[j2,i2] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j2,j5) * Ind(i2,i5) * ksi[j1,i1] * ksi[j3,i3] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j2,j3) * Ind(i2,i3) * ksi[j1,i1] * ksi[j5,i5] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j5) * Ind(i1,i5) * ksi[j2,i2] * ksi[j3,i3] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j3) * Ind(i1,i3) * ksi[j2,i2] * ksi[j5,i5] +
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j2) * Ind(i1,i2) * ksi[j3,i3] * ksi[j5,i5] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j3,j4) * Ind(i3,i4) * ksi[j1,i1] * ksi[j2,i2] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j2,j4) * Ind(i2,i4) * ksi[j1,i1] * ksi[j3,i3] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j2,j3) * Ind(i2,i3) * ksi[j1,i4] * ksi[j1,i4] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j4) * Ind(i1,i4) * ksi[j2,i2] * ksi[j3,i3] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j3) * Ind(i1,i3) * ksi[j2,i2] * ksi[j4,i4] +
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j2) * Ind(i1,i2) * ksi[j3,i3] * ksi[j4,i4] -
                Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j5) * Ind(i2,i5) * Ind(j3,j4) * Ind(i3,i4) -
                Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j4) * Ind(i2,i4) * Ind(j3,j5) * Ind(i3,i5) -
                Ind(j6,j1) * Ind(i6,i1) * Ind(j2,j3) * Ind(i2,i3) * Ind(j4,j5) * Ind(i4,i5) -
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j5) * Ind(i1,i5) * Ind(j3,j4) * Ind(i3,i4) -
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j4) * Ind(i1,i4) * Ind(j3,j5) * Ind(i3,i5) -
                Ind(j6,j2) * Ind(i6,i2) * Ind(j1,j3) * Ind(i1,i3) * Ind(j4,j5) * Ind(i4,i5) -
                Ind(j6,j3) * Ind(i6,i3) * Ind(j1,j5) * Ind(i1,i5) * Ind(j2,j4) * Ind(i2,i4) -
                Ind(j6,j3) * Ind(i6,i3) * Ind(j1,j4) * Ind(i1,i4) * Ind(j2,j5) * Ind(i2,i5) -
                Ind(j3,j6) * Ind(i3,i6) * Ind(j1,j2) * Ind(i1,i2) * Ind(j4,j5) * Ind(i4,i5) -
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j5) * Ind(i1,i5) * Ind(j2,j3) * Ind(i2,i3) -
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j3) * Ind(i1,i3) * Ind(j2,j5) * Ind(i2,i5) -
                Ind(j6,j4) * Ind(i6,i4) * Ind(j1,j2) * Ind(i1,i2) * Ind(j3,j5) * Ind(i3,i5) -
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j4) * Ind(i1,i4) * Ind(j2,j3) * Ind(i2,i3) -
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j2) * Ind(i1,i2) * Ind(j3,j4) * Ind(i3,i4) -
                Ind(j6,j5) * Ind(i6,i5) * Ind(j1,j3) * Ind(i1,i3) * Ind(j2,j4) * Ind(i2,i4))
            for j6 in range(q + 1)
            for j5 in range(q + 1)
            for j4 in range(q + 1)
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        I000000
        """
        return I000000(*self.args, **hints)
```

6.2.4 Source Codes for Iterated Stratonovich Stochastic Integrals Approximations Subprograms

Listing 89: **Approximation of Stratonovich stochastic integral** $I_{(0)\tau_{p+1},\tau_p}^{*(i_1)}$

```
from math import sqrt

from sympy import Function, sympify, Number


class J0(Function):
  """
  Stratonovich stochastic integral
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates new J0 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, dt, ksi = sympify(args)

    if not isinstance(i1, Number):
      return super(J0, cls).__new__(cls, *args, **kwargs)

    return ksi[0, i1] * sqrt(dt)

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J0
    """
    return J0(*self.args, **hints)
```

Listing 90: **Approximation of iterated Stratonovich stochastic integral** $I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)}$

```
from math import sqrt
```

```python

from sympy import sympify, Number, Function, Add


class J00(Function):
    """
    Iterated Stratonovich stochastic integral
    """
    nargs = 5

    def __new__(cls, *args, **kwargs):
        """
        Creates new J00 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(q, Number)):
            return super(J00, cls).__new__(cls, *args, **kwargs)

        return \
            (ksi[0, i1] * ksi[0, i2] +
             Add(*[
                 (ksi[j1 - 1, i1] * ksi[j1, i2] -
                 ksi[j1, i1] * ksi[j1 - 1, i2]) /
                 sqrt(j1 ** 2 * 4 - 1)
                 for j1 in range(1, q + 1)])) * dt / 2

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J00
        """
        return J00(*self.args, **hints)
```

Listing 91: **Approximation of Stratonovich stochastic integral** $I_{(1)T,t}^{*(i_1)}$

```
from math import sqrt

from sympy import Function, sympify, Number


class J1(Function):
    """
    Stratonovich stochastic integral
    """
    nargs = 3

    def __new__(cls, *args, **kwargs):
        """
        Creates new J1 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, dt, ksi = sympify(args)

        if not isinstance(i1, Number):
            return super(J1, cls).__new__(cls, *args, **kwargs)

        return -(ksi[0, i1] + ksi[1, i1] / sqrt(3)) * dt ** 1.5 / 2

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J1
        """
        return J1(*self.args, **hints)
```

Listing 92: **Approximation of iterated Stratonovich stochastic integral** $I_{(000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c000 import C000


class J000(Function):
    """
```

```
    Iterated Stratonovich stochastic integral
    """
    nargs = 6

    def __new__(cls, *args, **kwargs):
        """
        Creates new J000 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(q, Number)):
            return super(J000, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C000(j3, j2, j1, dt) *
            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J000
        """
        return J000(*self.args, **hints)
```

Listing 93: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2)}_{(10)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add
```

```python

from mathematics.sde.nonlinear.symbolic.coefficients.c10 import C10


class J10(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new J10 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(q, Number)):
      return super(J10, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C10(j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2]
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J10
    """
    return J10(*self.args, **hints)
```

Listing 94: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2)}_{(01)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c01 import C01


class J01(Function):
    """
    Iterated Stratonovich stochastic integral
    """

    nargs = 5

    def __new__(cls, *args, **kwargs):
        """
        Creates new J01 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(q, Number)):
            return super(J01, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C01(j2, j1, dt) *
            ksi[j1, i1] * ksi[j2, i2]
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J01
        """
```

```
    return J01(*self.args, **hints)
```

Listing 95: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3 i_4)}_{(0000)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0000 import C0000


class J0000(Function):
    """
    Iterated Stratonovich stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
        """
        Creates new J0000 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(q, Number)):
            return super(J0000, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C0000(j4, j3, j2, j1, dt) *
            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
            for j4 in range(q + 1)
```

```
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J0000
    """
    return J0000(*self.args, **hints)
```

Listing 96: **Approximation of Stratonovich stochastic integral** $I^{*(i_1)}_{(2)\tau_{p+1},\tau_p}$

```
from math import sqrt

from sympy import sympify, Number, Function


class J2(Function):
  """
  Stratonovich stochastic integral
  """
  nargs = 3

  def __new__(cls, *args, **kwargs):
    """
    Creates new J2 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, dt, ksi = sympify(args)

    if not isinstance(i1, Number):
      return super(J2, cls).__new__(cls, *args, **kwargs)

    return (ksi[0, i1] + ksi[1, i1] * sqrt(3) / 2 +
        ksi[2, i1] / sqrt(5) / 2) * dt ** 2.5 / 3

  def doit(self, **hints):
    """
    Tries to expand or calculate function
```

```
    Returns
    =======
    J2
    """
    return J2(*self.args, **hints)
```

Listing 97: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3)}_{(100)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c100 import C100


class J100(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new J100 object with given args

    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(q, Number)):
      return super(J100, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C100(j3, j2, j1, dt) *
```

```
            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J100
        """
        return J100(*self.args, **hints)
```

Listing 98: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3)}_{(010)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c010 import C010


class J010(Function):
    """
    Iterated Stratonovich stochastic integral
    """
    nargs = 6

    def __new__(cls, *args, **kwargs):
        """
        Creates new J010 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
```

```
          isinstance(i2, Number) and
          isinstance(i3, Number) and
          isinstance(q, Number)):
        return super(J010, cls).__new__(cls, *args, **kwargs)

    return Add(*[
        C010(j3, j2, j1, dt) *
        ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
        for j3 in range(q + 1)
        for j2 in range(q + 1)
        for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J010
    """
    return J010(*self.args, **hints)
```

Listing 99: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3)}_{(001)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c001 import C001


class J001(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 6

  def __new__(cls, *args, **kwargs):
    """
    Creates new J001 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
```

```
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(q, Number)):
      return super(J001, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C001(j3, j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J001
    """
    return J001(*self.args, **hints)
```

Listing 100: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3 i_4 i_5)}_{(00000)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c00000 import C00000


class J00000(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new J00000 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    i4 : int
```

```
            integral index
        i5 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, i5, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(i5, Number) and
                isinstance(q, Number) and
                isinstance(dt, Number)):
            return super(J00000, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C00000(j5, j4, j3, j2, j1, dt) *
            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5]
            for j5 in range(q + 1)
            for j4 in range(q + 1)
            for j3 in range(q + 1)
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J00000
        """
        return J00000(*self.args, **hints)
```

Listing 101: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2)}_{(20)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c20 import C20


class J20(Function):
    """
```

```python
    Iterated Stratonovich stochastic integral
    """
    nargs = 5

    def __new__(cls, *args, **kwargs):
        """
        Creates new J20 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number)):
            return super(J20, cls).__new__(cls, *args, **kwargs)

        return Add(*[
            C20(j2, j1, dt) *
            ksi[j1, i1] * ksi[j2, i2]
            for j2 in range(q + 1)
            for j1 in range(q + 1)])

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        J20
        """
        return J20(*self.args, **hints)
```

Listing 102: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2)}_{(02)\tau_{p+1},\tau_p}$

```python
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c02 import C02


```

```
class J02(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new J02 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(J02, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C02(j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2]
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J02
    """
    return J02(*self.args, **hints)
```

Listing 103: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2)}_{(11)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c11 import C11
```

```python


class J11(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 5

  def __new__(cls, *args, **kwargs):
    """
    Creates new J11 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number)):
      return super(J11, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C11(j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2]
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J11
    """
    return J11(*self.args, **hints)
```

Listing 104: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1i_2i_3i_4)}_{(1000)\tau_{p+1},\tau_p}$

```python
from sympy import Function, Number, sympify, Add
```

```

from mathematics.sde.nonlinear.symbolic.coefficients.c1000 import C1000


class J1000(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 7

  def __new__(cls, *args, **kwargs):
    """
    Creates new J1000 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    i4 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, i4, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(q, Number)):
      return super(J1000, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C1000(j4, j3, j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
      for j4 in range(q + 1)
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
```

```
        Returns
        =======
        J1000
        """
        return J1000(*self.args, **hints)
```

Listing 105: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3 i_4)}_{(0100)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0100 import C0100


class J0100(Function):
    """
    Iterated Stratonovich stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
        """
        Creates new J0100 object with given args
        Parameters
        ==========
        i1 : int
            integral index
        i2 : int
            integral index
        i3 : int
            integral index
        i4 : int
            integral index
        q : int
            amount of terms in approximation of
            iterated Stratonovich stochastic integral
        dt : float
            delta time
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i1, i2, i3, i4, q, dt, ksi = sympify(args)

        if not (isinstance(i1, Number) and
                isinstance(i2, Number) and
                isinstance(i3, Number) and
                isinstance(i4, Number) and
                isinstance(q, Number)):
            return super(J0100, cls).__new__(cls, *args, **kwargs)

```

```
      return Add(*[
        C0100(j4, j3, j2, j1, dt) *
        ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
        for j4 in range(q + 1)
        for j3 in range(q + 1)
        for j2 in range(q + 1)
        for j1 in range(q + 1)])

    def doit(self, **hints):
      """
      Tries to expand or calculate function
      Returns
      =======
      J0100
      """
      return J0100(*self.args, **hints)
```

Listing 106: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3 i_4)}_{(0010)\tau_{p+1},\tau_p}$

```
from sympy import Function, sympify, Number, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0010 import C0010


class J0010(Function):
    """
    Iterated Stratonovich stochastic integral
    """
    nargs = 7

    def __new__(cls, *args, **kwargs):
      """
      Creates new J0010 object with given args
      Parameters
      ==========
      i1 : int
        integral index
      i2 : int
        integral index
      i3 : int
        integral index
      i3 : int
        integral index
      q : int
        amount of terms in approximation of
        iterated Stratonovich stochastic integral
      dt : float
        delta time
      ksi : numpy.ndarray
        matrix of Gaussian random variables
      Returns
      =======
      sympy.Expr
```

```
      formula to simplify and substitute
    """
    i1, i2, i3, i4, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(q, Number)):
      return super(J0010, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C0010(j4, j3, j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
      for j4 in range(q + 1)
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J0010
    """
    return J0010(*self.args, **hints)
```

Listing 107: **Approximation of iterated Stratonovich stochastic integral** $I^{*(i_1 i_2 i_3 i_4)}_{(0001)\tau_{p+1},\tau_p}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c0001 import C0001


class J0001(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 7

  def __new__(cls, *args, **kwargs):
    """
    Creates new J0001 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    i4 : int
```

```
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, i4, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(q, Number)):
      return super(J0001, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C0001(j4, j3, j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
      for j4 in range(q + 1)
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    J0001
    """
    return J0001(*self.args, **hints)
```

Listing 108: **Approximation of iterated Stratonovich stochastic integral** $I_{(000000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4 i_5 i_6)}$

```
from sympy import Function, Number, sympify, Add

from mathematics.sde.nonlinear.symbolic.coefficients.c000000 import C000000


class J000000(Function):
  """
  Iterated Stratonovich stochastic integral
  """
  nargs = 9

  def __new__(cls, *args, **kwargs):
```

```
    """
    Creates new J000000 object with given args
    Parameters
    ==========
    i1 : int
      integral index
    i2 : int
      integral index
    i3 : int
      integral index
    i4 : int
      integral index
    i5 : int
      integral index
    i6 : int
      integral index
    q : int
      amount of terms in approximation of
      iterated Stratonovich stochastic integral
    dt : float
      delta time
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i1, i2, i3, i4, i5, i6, q, dt, ksi = sympify(args)

    if not (isinstance(i1, Number) and
        isinstance(i2, Number) and
        isinstance(i3, Number) and
        isinstance(i4, Number) and
        isinstance(i5, Number) and
        isinstance(i6, Number) and
        isinstance(q, Number) and
        isinstance(dt, Number)):
      return super(J000000, cls).__new__(cls, *args, **kwargs)

    return Add(*[
      C000000(j6, j5, j4, j3, j2, j1, dt) *
      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] *
      ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6]
      for j6 in range(q + 1)
      for j5 in range(q + 1)
      for j4 in range(q + 1)
      for j3 in range(q + 1)
      for j2 in range(q + 1)
      for j1 in range(q + 1)])

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
```

```
        =======
        J000000
        """
        return J000000(*self.args, **hints)
```

6.2.5 Source Codes for Calculation of the Numbers q, q_1,..., q_{15}

Listing 109: **Calculation of the numbers** q, q_1,..., q_{15}

```
from mathematics.sde.nonlinear.symbolic.coefficients.c import C


def solve_q(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 4 - 1 / 2 * sum([
        1 / (4 * j ** 2 - 1)
        for j in range(1, i + 1)
    ])


def solve_q1(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 6 - 1 / 64 * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        C((j1, j2, j3), (0, 0, 0)) ** 2
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
    ])
```

```


def solve_q2(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
    """
    return 1 / 4 - 1 / 64 * sum([
      (2 * j1 + 1) *
      (2 * j2 + 1) *
      (C((j1, j2), (0, 1)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
    ])


def solve_q2_optional(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
    """
    return 1 / 12 - 1 / 64 * sum([
      (2 * j1 + 1) *
      (2 * j2 + 1) *
      (C((j1, j2), (1, 0)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
    ])


def solve_q3(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
```

```
    """
    return 1 / 24 - 1 / 256 * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (C((j1, j2, j3, j4), (0, 0, 0, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
    ])


def solve_q4(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 120 - 1 / (32 ** 2) * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (2 * j5 + 1) *
        (C((j1, j2, j3, j4, j5), (0, 0, 0, 0, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
        for j5 in range(i + 1)
    ])


def solve_q5(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 60 - 1 / 256 * sum([
        (2 * j1 + 1) *
```

```
      (2 * j2 + 1) *
      (C((j1, j2), (2, 0)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
    ])


def solve_q6(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
    """
    return 1 / 18 - 1 / 256 * sum([
      (2 * j1 + 1) *
      (2 * j2 + 1) *
      (C((j1, j2), (1, 1)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
    ])


def solve_q7(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
    """
    return 1 / 6 - 1 / 256 * sum([
      (2 * j1 + 1) *
      (2 * j2 + 1) *
      (C((j1, j2), (0, 2)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
    ])


def solve_q8(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
```

```
    amount of members
  Returns
  =======
  values : float
    value for iteration of loop that calculates amount of q
  """
  return 1 / 10 - 1 / 256 * sum([
    (2 * j1 + 1) *
    (2 * j2 + 1) *
    (2 * j3 + 1) *
    (C((j1, j2, j3), (0, 0, 1)) ** 2)
    for j1 in range(i + 1)
    for j2 in range(i + 1)
    for j3 in range(i + 1)
  ])


def solve_q9(i):
  """
  Calculates value for iteration of loop
  Parameters
  ==========
  i : int
    amount of members
  Returns
  =======
  values : float
    value for iteration of loop that calculates amount of q
  """
  return 1 / 20 - 1 / 256 * sum([
    (2 * j1 + 1) *
    (2 * j2 + 1) *
    (2 * j3 + 1) *
    (C((j1, j2, j3), (0, 1, 0)) ** 2)
    for j1 in range(i + 1)
    for j2 in range(i + 1)
    for j3 in range(i + 1)
  ])


def solve_q10(i):
  """
  Calculates value for iteration of loop
  Parameters
  ==========
  i : int
    amount of members
  Returns
  =======
  values : float
    value for iteration of loop that calculates amount of q
  """
  return 1 / 60 - 1 / 256 * sum([
    (2 * j1 + 1) *
    (2 * j2 + 1) *
```

```
        (2 * j3 + 1) *
        (C((j1, j2, j3), (1, 0, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
    ])


def solve_q11(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 36 - 1 / (32 ** 2) * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (C((j1, j2, j3, j4), (0, 0, 0, 1)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
    ])


def solve_q12(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 60 - 1 / (32 ** 2) * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (C((j1, j2, j3, j4), (0, 0, 1, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
```

```
    ])


def solve_q13(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 120 - 1 / (32 ** 2) * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (C((j1, j2, j3, j4), (0, 1, 0, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
    ])


def solve_q14(i):
    """
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
        amount of members
    Returns
    =======
    values : float
        value for iteration of loop that calculates amount of q
    """
    return 1 / 360 - 1 / (32 ** 2) * sum([
        (2 * j1 + 1) *
        (2 * j2 + 1) *
        (2 * j3 + 1) *
        (2 * j4 + 1) *
        (C((j1, j2, j3, j4), (1, 0, 0, 0)) ** 2)
        for j1 in range(i + 1)
        for j2 in range(i + 1)
        for j3 in range(i + 1)
        for j4 in range(i + 1)
    ])


def solve_q15(i):
    """
```

```
    Calculates value for iteration of loop
    Parameters
    ==========
    i : int
      amount of members
    Returns
    =======
    values : float
      value for iteration of loop that calculates amount of q
    """
    return 1 / 720 - 1 / (64 ** 2) * sum([
      (2 * j1 + 1) *
      (2 * j2 + 1) *
      (2 * j3 + 1) *
      (2 * j4 + 1) *
      (2 * j5 + 1) *
      (2 * j6 + 1) *
      (C((j1, j2, j3, j4, j5, j6), (0, 0, 0, 0, 0, 0)) ** 2)
      for j1 in range(i + 1)
      for j2 in range(i + 1)
      for j3 in range(i + 1)
      for j4 in range(i + 1)
      for j5 in range(i + 1)
      for j6 in range(i + 1)
    ])


solvers = [
  solve_q, solve_q1, solve_q2, solve_q3, solve_q8,
  solve_q9, solve_q10, solve_q4, solve_q7, solve_q6,
  solve_q5, solve_q11, solve_q12, solve_q13,
  solve_q14, solve_q15,
]

dt_degrees = [
  [1],
  [2, 1],
  [3, 2, 1, 1],
  [4, 3, 2, 2, 1, 1, 1, 1],
  [5, 4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1],
]

q_ranges = [
  1, 2, 4, 8, 16
]


def loop(dt: float, k: float, degree: int, solver):
  """
  Loop that chooses amount of q that provides necessary accuracy
  Parameters
  ==========
  dt : float
    delta time
  k : float
```

```
        user chosen coefficient of accuracy
    degree : int
        degree of dt depending on q
    solver : function
        function that
    Returns
    =======
    i : int
        amount of q
    """
    i = 0
    while True:
        if solver(i) <= k * dt ** degree:
            break
        i += 1
    return i


def get_q(dt: float, k: float, r: float):
    """
    Iterates solvers and get q values necessary to
    achieve given accuracy
    Parameters
    ==========
    dt: float
        integration step
    k: float
        user chosen coefficient of accuracy
    r: float
        strong numerical scheme order
    Returns
    =======
    qs_result: tuple
        q values
    """
    qs_result = []

    degree = int(r * 2)
    range_id = degree - 2

    for q_id in range(q_ranges[range_id]):
        qs_result.append(loop(dt, k, dt_degrees[range_id][q_id], solvers[q_id]))

    return tuple(qs_result)
```

6.2.6 Source Codes for Strong Taylor–Itô Numerical Schemes with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs

Listing 110: **Euler scheme modeling subprogram**

```
import logging
```

```
from time import time

import numpy as np
from sympy import lambdify, Matrix, symbols, MatrixSymbol, Symbol

from mathematics.sde.nonlinear.symbolic.schemes.euler import Euler


def euler(y0: np.array, a: Matrix, b: Matrix, times: tuple):
    """
    Performs modeling of Euler scheme
    Parameters
    ==========
    y0 : numpy.ndarray
        initial conditions
    a : numpy.ndarray
        vector function a
    b : numpy.ndarray
        matrix function b
    times : tuple
        integration limits and step
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Euler start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)

    # Symbols
    sym_i, sym_t = Symbol("i"), Symbol("t")
    sym_ksi = MatrixSymbol("ksi", 1, m)
    sym_y = Euler(sym_i, Matrix(args), a, b, dt, sym_ksi)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
```

```
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Euler subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(1, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Euler calculations are finished")

    return y, t
```

Listing 111: **Euler scheme**

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.ito.i0 import I0


class Euler(Function):
    """
    Euler scheme
    """
    nargs = 6

    def __new__(cls, *args, **kwargs):
        """
        Creates new Euler object with given args
        Parameters
        ==========
        i : int
            component of stochastic process
        yp : numpy.ndarray
            initial conditions
        a : numpy.ndarray
            algebraic, given in the variables x and t
        b : numpy.ndarray
            algebraic, given in the variables x and t
        dt : float
            integration step
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        Returns
        =======
```

```
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi = sympify(args)
    n, m = b.shape[0], b.shape[1]

    return Add(

      yp[i, 0], a[i, 0] * dt,

      *[b[i, i1] * I0(i1, dt, ksi)
       for i1 in range(m)]

    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    sympy.Expr
    """
    return Euler(*self.args, **hints)
```

Listing 112: **Milstein scheme modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import symbols, MatrixSymbol, Matrix, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.milstein import Milstein


def milstein(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
  """
  Performs modeling of Milstein scheme
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
  a : numpy.ndarray
    vector function a
  b : numpy.ndarray
    matrix function b
  q : tuple
    amount of independent random variables
  times : tuple
    integration limits and step
  Returns
  =======
```

```
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Milstein start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 1)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = Milstein(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Milstein subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Milstein calculations are finished"
        )
```

```

  return y, t
```

Listing 113: **Milstein scheme**

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
from mathematics.sde.nonlinear.symbolic.ito.i00 import I00


class Milstein(Function):
  """
  Milstein scheme
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new Milstein object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    return Add(

      yp[i, 0], a[i, 0] * dt,

      *[b[i, i1] * I0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
```

```
        I00(i1, i2, q[0], dt, ksi)
        for i2 in range(m)
        for i1 in range(m)]

    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    sympy.Expr
    """
    return Milstein(*self.args, **hints)
```

Listing 114: **Strong Taylor–Itô scheme with convergence order 1.5 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, symbols, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_1p5 import
    StrongTaylorIto1p5


def strong_taylor_ito_1p5(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
  """
  Performs modeling of Strong Taylor-Ito scheme with convergence order 1.5
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
  a : numpy.ndarray
    vector function a
  b : numpy.ndarray
    matrix function b
  k : float
    precision constant
  times : tuple
    integration limits and step
  Returns
  =======
  y : numpy.ndarray
    vector of solution
  t : list
    list of time moments
  """
  start_time = time()

  logger = logging.getLogger(__name__)
```

```
    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 1.5 start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 1.5)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = StrongTaylorIto1p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 1.5 subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 1.5 calculations are finished")

    return y, t
```

Listing 115: **Strong Taylor–Itô scheme with convergence order** 1.5

```
from sympy import Function, sympify, Add
```

```
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
from mathematics.sde.nonlinear.symbolic.l import L


class StrongTaylorIto1p5(Function):
  """
  Strong Taylor-Ito scheme with convergence order 1.5
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorIto1p5 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    return Add(

      yp[i, 0], a[i, 0] * dt,

      *[b[i, i1] * I0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
       I00(i1, i2, q[0], dt, ksi)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], a[i, 0], dxs) *
```

```
        (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
        L(a, b, b[i, i1], dxs) *
        I1(i1, dt, ksi)
        for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
        I000(i1, i2, i3, q[1], dt, ksi)
        for i3 in range(m)
        for i2 in range(m)
        for i1 in range(m)],

      dt ** 2 / 2 * L(a, b, a[i, 0], dxs)

    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    sympy.Expr
    """
    return StrongTaylorIto1p5(*self.args, **hints)
```

Listing 116: **Strong Taylor–Itô scheme with convergence order 2.0 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import symbols, Matrix, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_2p0 import
    StrongTaylorIto2p0


def strong_taylor_ito_2p0(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
  """
  Performs modeling of Strong Taylor-Ito scheme with convergence order 2.0
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
  a : numpy.ndarray
    vector function a
  b : numpy.ndarray
    matrix function b
  k : float
    precision constant
  times : tuple
    integration limits and step
```

```
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.0 start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 2)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = StrongTaylorIto2p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.0 subs are
        finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)
```

```
    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.0 calculations
      are finished")

    return y, t
```

Listing 117: **Strong Taylor–Itô scheme with convergence order** 2.0

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
from mathematics.sde.nonlinear.symbolic.l import L


class StrongTaylorIto2p0(Function):
    """
    Strong Taylor-Ito scheme with convergence order 2.0
    """
    nargs = 8

    def __new__(cls, *args, **kwargs):
        """
        Creates new StrongTaylorIto2p0 object with given args
        Parameters
        ==========
        i : int
            component of stochastic process
        yp : numpy.ndarray
            initial conditions
        a : numpy.ndarray
            algebraic, given in the variables x and t
        b : numpy.ndarray
            algebraic, given in the variables x and t
        dt : float
            integration step
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        q : tuple
            amounts of q for stochastic integrals approximations
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i, yp, a, b, dt, ksi, dxs, q = sympify(args)
        n, m = b.shape[0], b.shape[1]
```

```

    return Add(

      yp[i, 0], a[i, 0] * dt,

      *[b[i, i1] * I0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
       I00(i1, i2, q[0], dt, ksi)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], a[i, 0], dxs) *
       (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
       L(a, b, b[i, i1], dxs) *
       I1(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
       I000(i1, i2, i3, q[1], dt, ksi)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)],

      dt ** 2 / 2 * L(a, b, a[i, 0], dxs),

      *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
       (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
       L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
       G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
       (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
       I0000(i1, i2, i3, i4, q[3], dt, ksi)
       for i4 in range(m)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)]

    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    sympy.Expr
    """
    return StrongTaylorIto2p0(*self.args, **hints)
```

Listing 118: **Strong Taylor–Itô scheme with convergence order 2.5 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import symbols, Matrix, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_2p5 import
    StrongTaylorIto2p5


def strong_taylor_ito_2p5(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
    """
    Performs modeling of Strong Taylor-Ito scheme with convergence order 2.5
    Parameters
    ==========
    y0 : numpy.ndarray
        initial conditions
    a : numpy.ndarray
        vector function a
    b : numpy.ndarray
        matrix function b
    k : float
        precision constant
    times : tuple
        integration limits and step
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.5 start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 2.5)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
```

```
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
    sym_y = StrongTaylorIto2p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.5 subs are
      finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
      values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
      for tr in range(n):
        y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 2.5 calculations
      are finished")

    return y, t
```

Listing 119: **Strong Taylor–Itô scheme with convergence order** 2.5

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
from mathematics.sde.nonlinear.symbolic.ito.i00000 import I00000
from mathematics.sde.nonlinear.symbolic.ito.i001 import I001
from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
from mathematics.sde.nonlinear.symbolic.ito.i010 import I010
from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
from mathematics.sde.nonlinear.symbolic.ito.i100 import I100
from mathematics.sde.nonlinear.symbolic.ito.i2 import I2
```

```
from mathematics.sde.nonlinear.symbolic.l import L


class StrongTaylorIto2p5(Function):
  """
  Strong Taylor-Ito scheme with convergence order 2.5
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorIto2p5 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    return Add(

      yp[i, 0], a[i, 0] * dt,

      *[b[i, i1] * I0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
       I00(i1, i2, q[0], dt, ksi)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], a[i, 0], dxs) *
       (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
       L(a, b, b[i, i1], dxs) *
       I1(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
```

```
            I000(i1, i2, i3, q[1], dt, ksi)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

          dt ** 2 / 2 * L(a, b, a[i, 0], dxs),
          *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
            (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
            L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
            G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
            (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
            for i2 in range(m)
            for i1 in range(m)],

          *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
            I0000(i1, i2, i3, i4, q[3], dt, ksi)
            for i4 in range(m)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

          *[G(b[:, i1], L(a, b, a[i, 0], dxs), dxs) *
            (I2(i1, dt, ksi) / 2 + dt * I1(i1, dt, ksi) + dt ** 2 / 2 * I0(i1, dt, ksi)) +
            L(a, b, L(a, b, b[i, i1], dxs), dxs) * I2(i1, dt, ksi) / 2 -
            L(a, b, G(b[:, i1], a[i, 0], dxs), dxs) * (I2(i1, dt, ksi) + dt * I1(i1, dt, ksi))
            for i1 in range(m)],

          *[G(b[:, i1], L(a, b, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
            (I100(i1, i2, i3, q[6], dt, ksi) - I010(i1, i2, i3, q[5], dt, ksi)) +
            G(b[:, i1], G(b[:, i2], L(a, b, b[i, i3], dxs), dxs), dxs) *
            (I010(i1, i2, i3, q[5], dt, ksi) - I001(i1, i2, i3, q[4], dt, ksi)) +
            G(b[:, i1], G(b[:, i2], G(b[:, i3], a[i, 0], dxs), dxs), dxs) *
            (dt * I000(i1, i2, i3, q[1], dt, ksi) + I001(i1, i2, i3, q[4], dt, ksi)) -
            L(a, b, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
            I100(i1, i2, i3, q[6], dt, ksi)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

          *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
            b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
            I00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
            for i5 in range(m)
            for i4 in range(m)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

          dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs)

        )

    def doit(self, **hints):
        """
        Tries to expand or calculate function
```

```
    Returns
    =======
    sympy.Expr
    """
    return StrongTaylorIto2p5(*self.args, **hints)
```

Listing 120: **Strong Taylor–Itô scheme with convergence order 3.0 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, symbols, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_3p0 import
    StrongTaylorIto3p0


def strong_taylor_ito_3p0(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
    """
    Performs modeling of Strong Taylor-Ito scheme with convergence order 3.0
    Parameters
    ==========
    y0 : numpy.ndarray
        initial conditions
    a : numpy.ndarray
        vector function a
    b : numpy.ndarray
        matrix function b
    k : float
        precision constant
    times : tuple
        integration limits and step
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 3.0 start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
```

```
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 3)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
    sym_y = StrongTaylorIto3p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 3.0 subs are
      finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
      values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
      for tr in range(n):
        y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Ito 3.0 calculations
      are finished")

    return y, t
```

Listing 121: **Strong Taylor–Itô scheme with convergence order** 3.0

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
from mathematics.sde.nonlinear.symbolic.ito.i00000 import I00000
```

```
from mathematics.sde.nonlinear.symbolic.ito.i000000 import I000000
from mathematics.sde.nonlinear.symbolic.ito.i0001 import I0001
from mathematics.sde.nonlinear.symbolic.ito.i001 import I001
from mathematics.sde.nonlinear.symbolic.ito.i0010 import I0010
from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
from mathematics.sde.nonlinear.symbolic.ito.i010 import I010
from mathematics.sde.nonlinear.symbolic.ito.i0100 import I0100
from mathematics.sde.nonlinear.symbolic.ito.i02 import I02
from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
from mathematics.sde.nonlinear.symbolic.ito.i100 import I100
from mathematics.sde.nonlinear.symbolic.ito.i1000 import I1000
from mathematics.sde.nonlinear.symbolic.ito.i11 import I11
from mathematics.sde.nonlinear.symbolic.ito.i2 import I2
from mathematics.sde.nonlinear.symbolic.ito.i20 import I20
from mathematics.sde.nonlinear.symbolic.l import L


class StrongTaylorIto3p0(Function):
    """
    Strong Taylor-Ito scheme with convergence order 3.0
    """
    nargs = 8

    def __new__(cls, *args, **kwargs):
        """
        Creates new StrongTaylorIto3p0 object with given args
        Parameters
        ==========
        i : int
            component of stochastic process
        yp : numpy.ndarray
            initial conditions
        a : numpy.ndarray
            algebraic, given in the variables x and t
        b : numpy.ndarray
            algebraic, given in the variables x and t
        dt : float
            integration step
        ksi : numpy.ndarray
            matrix of Gaussian random variables
        q : tuple
            amounts of q for stochastic integrals approximations
        Returns
        =======
        sympy.Expr
            formula to simplify and substitute
        """
        i, yp, a, b, dt, ksi, dxs, q = sympify(args)
        n, m = b.shape[0], b.shape[1]

        return Add(

            yp[i, 0], a[i, 0] * dt,

```

```
        *[b[i, i1] * I0(i1, dt, ksi)
         for i1 in range(m)],

        *[G(b[:, i1], b[i, i2], dxs) *
         I00(i1, i2, q[0], dt, ksi)
         for i2 in range(m)
         for i1 in range(m)],

        *[G(b[:, i1], a[i, 0], dxs) *
         (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
         L(a, b, b[i, i1], dxs) *
         I1(i1, dt, ksi)
         for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
         I000(i1, i2, i3, q[1], dt, ksi)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)],

        dt ** 2 / 2 * L(a, b, a[i, 0], dxs),

        *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
         (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
         L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
         G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
         (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
         for i2 in range(m)
         for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
         I0000(i1, i2, i3, i4, q[3], dt, ksi)
         for i4 in range(m)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)],

        *[G(b[:, i1], L(a, b, a[i, 0], dxs), dxs) *
         (I2(i1, dt, ksi) / 2 + dt * I1(i1, dt, ksi) + dt ** 2 / 2 * I0(i1, dt, ksi)) +
         L(a, b, L(a, b, b[i, i1], dxs), dxs) * I2(i1, dt, ksi) / 2 -
         L(a, b, G(b[:, i1], a[i, 0], dxs), dxs) * (I2(i1, dt, ksi) + dt * I1(i1, dt, ksi))
         for i1 in range(m)],

        *[G(b[:, i1], L(a, b, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
         (I100(i1, i2, i3, q[6], dt, ksi) - I010(i1, i2, i3, q[5], dt, ksi)) +
         G(b[:, i1], G(b[:, i2], L(a, b, b[i, i3], dxs), dxs), dxs) *
         (I010(i1, i2, i3, q[5], dt, ksi) - I001(i1, i2, i3, q[4], dt, ksi)) +
         G(b[:, i1], G(b[:, i2], G(b[:, i3], a[i, 0], dxs), dxs), dxs) *
         (dt * I000(i1, i2, i3, q[1], dt, ksi) + I001(i1, i2, i3, q[4], dt, ksi)) -
         L(a, b, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
         I100(i1, i2, i3, q[6], dt, ksi)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)],

```

```
        *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
         b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
         I00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
         for i5 in range(m)
         for i4 in range(m)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)],

        dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs),

        *[G(b[:, i1], G(b[:, i2], L(a, b, a[i, 0], dxs), dxs), dxs) *
         (I02(i1, i2, q[6], dt, ksi) / 2 + dt * I01(i1, i2, q[2], dt, ksi) +
          dt ** 2 / 2 * I00(i1, i2, q[2], dt, ksi)) +
         L(a, b, L(a, b, G(b[:, i1], b[i, i2], dxs), dxs), dxs) / 2 *
         I20(i1, i2, q[10], dt, ksi) +
         G(b[:, i1], L(a, b, G(b[:, i2], a[i, 0], dxs), dxs), dxs) *
         (I11(i1, i2, q[9], dt, ksi) - I02(i1, i2, q[8], dt, ksi) +
          dt * (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi))) +
         L(a, b, G(b[:, i1], L(a, b, b[i, i2], dxs), dxs), dxs) *
         (I11(i1, i2, q[9], dt, ksi) - I20(i1, i2, q[10], dt, ksi)) +
         G(b[:, i1], L(a, b, L(a, b, b[i, i2], dxs), dxs), dxs) *
         (I02(i1, i2, q[8], dt, ksi) / 2 + I20(i1, i2, q[10], dt, ksi) / 2 -
          I11(i1, i2, q[9], dt, ksi)) -
         L(a, b, G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs), dxs) *
         (dt * I10(i1, i2, q[2], dt, ksi) + I11(i1, i2, q[9], dt, ksi))
         for i2 in range(m)
         for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], a[i, 0], dxs), dxs), dxs), dxs) *
         (dt * I0000(i1, i2, i3, i4, q[3], dt, ksi) +
         I0001(i1, i2, i3, i4, q[11], dt, ksi)) +
         G(b[:, i1], G(b[:, i2], L(a, b, G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
         (I0100(i1, i2, i3, i4, q[13], dt, ksi) - I0010(i1, i2, i3, i4, q[12], dt, ksi)) -
         L(a, b, G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
         I1000(i1, i2, i3, i4, q[14], dt, ksi) +
         G(b[:, i1], L(a, b, G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
         (I1000(i1, i2, i3, i4, q[14], dt, ksi) - I0100(i1, i2, i3, i4, q[13], dt, ksi)) +
         G(b[:, i1], G(b[:, i2], G(b[:, i3], L(a, b, b[i, i4], dxs), dxs), dxs), dxs) *
         (I0010(i1, i2, i3, i4, q[12], dt, ksi) - I0001(i1, i2, i3, i4, q[11], dt, ksi))
         for i4 in range(m)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], G(
         b[:, i5], b[i, i6], dxs), dxs), dxs), dxs), dxs) *
         I000000(i1, i2, i3, i4, i5, i6, q[15], dt, ksi)
         for i6 in range(m)
         for i5 in range(m)
         for i4 in range(m)
         for i3 in range(m)
         for i2 in range(m)
         for i1 in range(m)]

```

```
    )

  def doit(self, **hints):
    """
    Tries to expand or calculate function
    Returns
    =======
    sympy.Expr
    """
    return StrongTaylorIto3p0(*self.args, **hints)
```

6.2.7 Source Codes for Strong Taylor–Stratonovich Numerical Schemes with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs

Listing 122: **Strong Taylor–Stratonovich scheme with convergence order 1.0 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, symbols, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_1p0 import
    StrongTaylorStratonovich1p0


def strong_taylor_stratonovich_1p0(y0: np.array, a: Matrix, b: Matrix, k: float, times:
    tuple):
  """
  Performs modeling of Strong Taylor-Stratonovich scheme with convergence order 1.0
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
  a : numpy.ndarray
    vector function a
  b : numpy.ndarray
    matrix function b
  k : float
    precision constant
  times : tuple
    integration limits and step
  Returns
  =======
  y : numpy.ndarray
    vector of solution
```

```
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Taylor-Stratonovich 1.0 start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 1)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = StrongTaylorStratonovich1p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
                f"Taylor-Stratonovich 1.0 subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
                f"Taylor-Stratonovich 1.0 calculations are finished")

```

```
  return y, t
```

Listing 123: **Strong Taylor–Stratonovich scheme with convergence order** 1.0

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.aj import Aj
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00


class StrongTaylorStratonovich1p0(Function):
  """
  Strong Taylor-Stratonovich scheme with convergence order 1.0
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorStratonovich1p0 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    aj = Aj(i, a, b, dxs)

    return Add(

      yp[i, 0], aj[i, 0] * dt,

      *[b[i, i1] * J0(i1, dt, ksi)
       for i1 in range(m)],
```

```

            *[G(b[:, i1], b[i, i2], dxs) *
             J00(i1, i2, q[0], dt, ksi)
             for i2 in range(m)
             for i1 in range(m)]

        )

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        sympy.Expr
        """
        return StrongTaylorStratonovich1p0(*self.args, **hints)
```

Listing 124: **Strong Taylor–Stratonovich scheme with convergence order** 1.5 **modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, MatrixSymbol, symbols, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_1p5 import
    StrongTaylorStratonovich1p5


def strong_taylor_stratonovich_1p5(y0: np.array, a: Matrix, b: Matrix, k: float, times:
    tuple):
    """
    Performs modeling of Strong Taylor-Stratonovich scheme with convergence order 1.5
    Parameters
    ==========
    y0 : numpy.ndarray
        initial conditions
    a : numpy.ndarray
        vector function a
    b : numpy.ndarray
        matrix function b
    k : float
        precision constant
    times : tuple
        integration limits and step
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
```

```python
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Stratonovich 1.5
      start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 1.5)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = StrongTaylorStratonovich1p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
          f"Taylor-Stratonovich 1.5 subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
      values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
      for tr in range(n):
        y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
          f"Taylor-Stratonovich 1.5 calculations are finished")

    return y, t
```

Listing 125: **Strong Taylor–Stratonovich scheme with convergence order** 1.5

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.aj import Aj
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.l import L
from mathematics.sde.nonlinear.symbolic.lj import Lj
from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1


class StrongTaylorStratonovich1p5(Function):
  """
  Strong Taylor-Stratonovich scheme with convergence order 1.5
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorStratonovich1p5 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    aj = Aj(i, a, b, dxs)

    return Add(

      yp[i, 0], aj[i, 0] * dt,

      *[b[i, i1] * J0(i1, dt, ksi)
       for i1 in range(m)],
```

```

            *[G(b[:, i1], b[i, i2], dxs) *
             J00(i1, i2, q[0], dt, ksi)
             for i2 in range(m)
             for i1 in range(m)],

            *[G(b[:, i1], aj[i, 0], dxs) *
             (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
             Lj(a, b[i, i1], dxs) *
             J1(i1, dt, ksi)
             for i1 in range(m)],

            *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
             J000(i1, i2, i3, q[1], dt, ksi)
             for i3 in range(m)
             for i2 in range(m)
             for i1 in range(m)],

            dt ** 2 / 2 * L(a, b, a[i, 0], dxs)

        )

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        sympy.Expr
        """
        return StrongTaylorStratonovich1p5(*self.args, **hints)
```

Listing 126: **Strong Taylor–Stratonovich scheme with convergence order 2.0 modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import symbols, Matrix, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_2p0 import
    StrongTaylorStratonovich2p0


def strong_taylor_stratonovich_2p0(y0: np.array, a: Matrix, b: Matrix, k: float, times:
    tuple):
  """
  Performs modeling of Strong Taylor-Stratonovich scheme with convergence order 2.0
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
```

```
    a : numpy.ndarray
      vector function a
    b : numpy.ndarray
      matrix function b
    k : float
      precision constant
    times : tuple
      integration limits and step
    Returns
    =======
    y : numpy.ndarray
      vector of solution
    t : list
      list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Stratonovich 2.0
      start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 2)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
    sym_y = StrongTaylorStratonovich2p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
          f"Taylor-Stratonovich 2.0 subs are finished")

    # Substitution values
```

```
  t = [t1 + i * dt for i in range(ticks)]
  y = np.zeros((n, ticks))
  y[:, 0] = y0[:, 0]

  # Dynamic substitutions with integration
  for p in range(ticks - 1):
    values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
    for tr in range(n):
      y[tr, p + 1] = y_compiled[tr](*values)

  logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
        f"Taylor-Stratonovich 2.0 calculations are finished")

  return y, t
```

Listing 127: **Strong Taylor–Stratonovich scheme with convergence order** 2.0

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.aj import Aj
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.lj import Lj
from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10


class StrongTaylorStratonovich2p0(Function):
  """
  Strong Taylor-Stratonovich scheme with convergence order 2.0
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorStratonovich2p0 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
```

```
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    aj = Aj(i, a, b, dxs)

    return Add(

      yp[i, 0], aj[i, 0] * dt,

      *[b[i, i1] * J0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
       J00(i1, i2, q[0], dt, ksi)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], aj[i, 0], dxs) *
       (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
       Lj(a, b[i, i1], dxs) *
       J1(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
       J000(i1, i2, i3, q[1], dt, ksi)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)],

          dt ** 2 / 2 * Lj(a, aj[i, 0], dxs),

      *[G(b[:, i1], Lj(a, b[i, i2], dxs), dxs) *
       (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi)) -
       Lj(a, G(b[:, i1], b[i, i2], dxs), dxs) * J10(i1, i2, q[2], dt, ksi) +
       G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs) *
       (J01(i1, i2, q[2], dt, ksi) + dt * J00(i1, i2, q[0], dt, ksi))
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
       J0000(i1, i2, i3, i4, q[3], dt, ksi)
       for i4 in range(m)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)]

    )
```

```

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        sympy.Expr
        """
        return StrongTaylorStratonovich2p0(*self.args, **hints)
```

Listing 128: **Strong Taylor–Stratonovich scheme with convergence order** 2.5 **modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, symbols, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_2p5 import
    StrongTaylorStratonovich2p5


def strong_taylor_stratonovich_2p5(y0: np.array, a: Matrix, b: Matrix, k: float, times:
    tuple):
    """
    Performs modeling of Strong Taylor-Stratonovich scheme with convergence order 2.5
    Parameters
    ==========
    y0 : numpy.ndarray
        initial conditions
    a : numpy.ndarray
        vector function a
    b : numpy.ndarray
        matrix function b
    k : float
        precision constant
    times : tuple
        integration limits and step
    Returns
    =======
    y : numpy.ndarray
        vector of solution
    t : list
        list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Stratonovich 2.5
        start")
```

```python

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 2.5)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
    sym_y = StrongTaylorStratonovich2p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
        y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
                f"Taylor-Stratonovich 2.5 subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
        values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
        for tr in range(n):
            y[tr, p + 1] = y_compiled[tr](*values)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
                f"Taylor-Stratonovich 2.5 calculations are finished")

    return y, t
```

Listing 129: **Strong Taylor–Stratonovich scheme with convergence order** 2.5

```python
from sympy import Function, sympify, Add

```

```
from mathematics.sde.nonlinear.symbolic.aj import Aj
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.l import L
from mathematics.sde.nonlinear.symbolic.lj import Lj
from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
from mathematics.sde.nonlinear.symbolic.stratonovich.j00000 import J00000
from mathematics.sde.nonlinear.symbolic.stratonovich.j001 import J001
from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
from mathematics.sde.nonlinear.symbolic.stratonovich.j010 import J010
from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10
from mathematics.sde.nonlinear.symbolic.stratonovich.j100 import J100
from mathematics.sde.nonlinear.symbolic.stratonovich.j2 import J2


class StrongTaylorStratonovich2p5(Function):
  """
  Strong Taylor-Stratonovich scheme with convergence order 2.5
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorStratonovich2p5 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
      algebraic, given in the variables x and t
    b : numpy.ndarray
      algebraic, given in the variables x and t
    dt : float
      integration step
    ksi : numpy.ndarray
      matrix of Gaussian random variables
    q : tuple
      amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
      formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    aj = Aj(i, a, b, dxs)

    return Add(

```

```
      yp[i, 0], aj[i, 0] * dt,

      *[b[i, i1] * J0(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], b[i, i2], dxs) *
       J00(i1, i2, q[0], dt, ksi)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], aj[i, 0], dxs) *
       (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
       Lj(a, b[i, i1], dxs) *
       J1(i1, dt, ksi)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
       J000(i1, i2, i3, q[1], dt, ksi)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)],

      dt ** 2 / 2 * Lj(a, aj[i, 0], dxs),

      *[G(b[:, i1], Lj(a, b[i, i2], dxs), dxs) *
       (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi)) -
       Lj(a, G(b[:, i1], b[i, i2], dxs), dxs) * J10(i1, i2, q[2], dt, ksi) +
       G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs) *
       (J01(i1, i2, q[2], dt, ksi) + dt * J00(i1, i2, q[0], dt, ksi))
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
       J0000(i1, i2, i3, i4, q[3], dt, ksi)
       for i4 in range(m)
       for i3 in range(m)
       for i2 in range(m)
       for i1 in range(m)],

      *[G(b[:, i1], Lj(a, aj[i, 0], dxs), dxs) *
       (J2(i1, dt, ksi) / 2 + dt * J1(i1, dt, ksi) + dt ** 2 / 2 * J0(i1, dt, ksi)) +
       Lj(a, Lj(a, b[i, i1], dxs), dxs) * J2(i1, dt, ksi) / 2 -
       Lj(a, G(b[:, i1], aj[i, 0], dxs), dxs) * (J2(i1, dt, ksi) + dt * J1(i1, dt, ksi))
       for i1 in range(m)],

      *[G(b[:, i1], Lj(a, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
       (J100(i1, i2, i3, q[6], dt, ksi) - J010(i1, i2, i3, q[5], dt, ksi)) +
       G(b[:, i1], G(b[:, i2], Lj(a, b[i, i3], dxs), dxs), dxs) *
       (J010(i1, i2, i3, q[5], dt, ksi) - J001(i1, i2, i3, q[4], dt, ksi)) +
       G(b[:, i1], G(b[:, i2], G(b[:, i3], aj[i, 0], dxs), dxs), dxs) *
       (dt * J000(i1, i2, i3, q[1], dt, ksi) + J001(i1, i2, i3, q[4], dt, ksi)) -
       Lj(a, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
       J100(i1, i2, i3, q[6], dt, ksi)
       for i3 in range(m)
       for i2 in range(m)
```

```
            for i1 in range(m)],

          *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
            b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
            J00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
            for i5 in range(m)
            for i4 in range(m)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

          dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs)

        )

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        sympy.Expr
        """
        return StrongTaylorStratonovich2p5(*self.args, **hints)
```

Listing 130: **Strong Taylor–Stratonovich scheme with convergence order** 3.0 **modeling subprogram**

```
import logging
from time import time

import numpy as np
from sympy import Matrix, symbols, MatrixSymbol, lambdify

from mathematics.sde.nonlinear.q import get_q
from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_3p0 import
    StrongTaylorStratonovich3p0


def strong_taylor_stratonovich_3p0(y0: np.array, a: Matrix, b: Matrix, k: float, times:
    tuple):
  """
  Performs modeling of Strong Taylor-Stratonovich scheme with convergence order 3.0
  Parameters
  ==========
  y0 : numpy.ndarray
    initial conditions
  a : numpy.ndarray
    vector function a
  b : numpy.ndarray
    matrix function b
  k : float
    precision constant
  times : tuple
```

```
      integration limits and step
    Returns
    =======
    y : numpy.ndarray
      vector of solution
    t : list
      list of time moments
    """
    start_time = time()

    logger = logging.getLogger(__name__)

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor-Stratonovich 3.0
      start")

    # Ranges
    n = b.shape[0]
    m = b.shape[1]
    t1 = times[0]
    dt = times[1]
    t2 = times[2]

    # Defining context
    args = symbols(f"x1:{n + 1}")
    ticks = int((t2 - t1) / dt)
    q = get_q(dt, k, 3)
    logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
    logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")

    # Symbols
    sym_i, sym_t = symbols("i t")
    sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
    sym_y = StrongTaylorStratonovich3p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)

    args_extended = list()
    args_extended.extend(args)
    args_extended.extend([sym_t, sym_ksi])

    # Compilation of formulas
    y_compiled = list()
    for tr in range(n):
      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))

    logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
          f"Taylor-Stratonovich 3.0 subs are finished")

    # Substitution values
    t = [t1 + i * dt for i in range(ticks)]
    y = np.zeros((n, ticks))
    y[:, 0] = y0[:, 0]

    # Dynamic substitutions with integration
    for p in range(ticks - 1):
      values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
```

```
    for tr in range(n):
      y[tr, p + 1] = y_compiled[tr](*values)

  logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
        f"Taylor-Stratonovich 3.0 calculations are finished")

  return y, t
```

Listing 131: **Strong Taylor–Stratonovich scheme with convergence order** 3.0

```
from sympy import Function, sympify, Add

from mathematics.sde.nonlinear.symbolic.aj import Aj
from mathematics.sde.nonlinear.symbolic.g import G
from mathematics.sde.nonlinear.symbolic.lj import Lj
from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
from mathematics.sde.nonlinear.symbolic.stratonovich.j00000 import J00000
from mathematics.sde.nonlinear.symbolic.stratonovich.j000000 import J000000
from mathematics.sde.nonlinear.symbolic.stratonovich.j0001 import J0001
from mathematics.sde.nonlinear.symbolic.stratonovich.j001 import J001
from mathematics.sde.nonlinear.symbolic.stratonovich.j0010 import J0010
from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
from mathematics.sde.nonlinear.symbolic.stratonovich.j010 import J010
from mathematics.sde.nonlinear.symbolic.stratonovich.j0100 import J0100
from mathematics.sde.nonlinear.symbolic.stratonovich.j02 import J02
from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10
from mathematics.sde.nonlinear.symbolic.stratonovich.j100 import J100
from mathematics.sde.nonlinear.symbolic.stratonovich.j1000 import J1000
from mathematics.sde.nonlinear.symbolic.stratonovich.j11 import J11
from mathematics.sde.nonlinear.symbolic.stratonovich.j2 import J2
from mathematics.sde.nonlinear.symbolic.stratonovich.j20 import J20


class StrongTaylorStratonovich3p0(Function):
  """
  Strong Taylor-Stratonovich scheme with convergence order 3.0
  """
  nargs = 8

  def __new__(cls, *args, **kwargs):
    """
    Creates new StrongTaylorStratonovich3p0 object with given args
    Parameters
    ==========
    i : int
      component of stochastic process
    yp : numpy.ndarray
      initial conditions
    a : numpy.ndarray
```

```
        algebraic, given in the variables x and t
    b : numpy.ndarray
        algebraic, given in the variables x and t
    dt : float
        integration step
    ksi : numpy.ndarray
        matrix of Gaussian random variables
    q : tuple
        amounts of q for stochastic integrals approximations
    Returns
    =======
    sympy.Expr
        formula to simplify and substitute
    """
    i, yp, a, b, dt, ksi, dxs, q = sympify(args)
    n, m = b.shape[0], b.shape[1]

    aj = Aj(i, a, b, dxs)

    return Add(

        yp[i, 0], aj[i, 0] * dt,

        *[b[i, i1] * J0(i1, dt, ksi)
          for i1 in range(m)],

        *[G(b[:, i1], b[i, i2], dxs) *
          J00(i1, i2, q[0], dt, ksi)
          for i2 in range(m)
          for i1 in range(m)],

        *[G(b[:, i1], aj[i, 0], dxs) *
          (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
          Lj(a, b[i, i1], dxs) *
          J1(i1, dt, ksi)
          for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
          J000(i1, i2, i3, q[1], dt, ksi)
          for i3 in range(m)
          for i2 in range(m)
          for i1 in range(m)],

        dt ** 2 / 2 * Lj(a, aj[i, 0], dxs),

        *[G(b[:, i1], Lj(a, b[i, i2], dxs), dxs) *
          (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi)) -
          Lj(a, G(b[:, i1], b[i, i2], dxs), dxs) * J10(i1, i2, q[2], dt, ksi) +
         G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs) *
          (J01(i1, i2, q[2], dt, ksi) + dt * J00(i1, i2, q[0], dt, ksi))
          for i2 in range(m)
          for i1 in range(m)],

        *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
          J0000(i1, i2, i3, i4, q[3], dt, ksi)
```

```
        for i4 in range(m)
        for i3 in range(m)
        for i2 in range(m)
        for i1 in range(m)],

      *[G(b[:, i1], Lj(a, aj[i, 0], dxs), dxs) *
        (J2(i1, dt, ksi) / 2 + dt * J1(i1, dt, ksi) + dt ** 2 / 2 * J0(i1, dt, ksi)) +
        Lj(a, Lj(a, b[i, i1], dxs), dxs) * J2(i1, dt, ksi) / 2 -
        Lj(a, G(b[:, i1], aj[i, 0], dxs), dxs) * (J2(i1, dt, ksi) + dt * J1(i1, dt, ksi))
        for i1 in range(m)],

      *[G(b[:, i1], Lj(a, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
        (J100(i1, i2, i3, q[6], dt, ksi) - J010(i1, i2, i3, q[5], dt, ksi)) +
       G(b[:, i1], G(b[:, i2], Lj(a, b[i, i3], dxs), dxs), dxs) *
        (J010(i1, i2, i3, q[5], dt, ksi) - J001(i1, i2, i3, q[4], dt, ksi)) +
       G(b[:, i1], G(b[:, i2], G(b[:, i3], aj[i, 0], dxs), dxs), dxs) *
        (dt * J000(i1, i2, i3, q[1], dt, ksi) + J001(i1, i2, i3, q[4], dt, ksi)) -
        Lj(a, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
        J100(i1, i2, i3, q[6], dt, ksi)
        for i3 in range(m)
        for i2 in range(m)
        for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
        b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
        J00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
        for i5 in range(m)
        for i4 in range(m)
        for i3 in range(m)
        for i2 in range(m)
        for i1 in range(m)],

      dt ** 3 / 6 * Lj(a, Lj(a, aj[i, 0], dxs), dxs),

      *[G(b[:, i1], G(b[:, i2], Lj(a, aj[i, 0], dxs), dxs), dxs) *
        (J02(i1, i2, q[6], dt, ksi) / 2 + dt * J01(i1, i2, q[2], dt, ksi) +
         dt ** 2 / 2 * J00(i1, i2, q[2], dt, ksi)) +
        Lj(a, Lj(a, G(b[:, i1], b[i, i2], dxs), dxs), dxs) / 2 *
        J20(i1, i2, q[10], dt, ksi) +
       G(b[:, i1], Lj(a, G(b[:, i2], aj[i, 0], dxs), dxs), dxs) *
        (J11(i1, i2, q[9], dt, ksi) - J02(i1, i2, q[8], dt, ksi) +
         dt * (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi))) +
        Lj(a, G(b[:, i1], Lj(a, b[i, i2], dxs), dxs), dxs) *
        (J11(i1, i2, q[9], dt, ksi) - J20(i1, i2, q[10], dt, ksi)) +
       G(b[:, i1], Lj(a, Lj(a, b[i, i2], dxs), dxs), dxs) *
        (J02(i1, i2, q[8], dt, ksi) / 2 + J20(i1, i2, q[10], dt, ksi) / 2 -
        J11(i1, i2, q[9], dt, ksi)) -
        Lj(a, G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs), dxs) *
        (dt * J10(i1, i2, q[2], dt, ksi) + J11(i1, i2, q[9], dt, ksi))
        for i2 in range(m)
        for i1 in range(m)],

      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
        b[:, i4], aj[i, 0], dxs), dxs), dxs), dxs) *
        (dt * J0000(i1, i2, i3, i4, q[3], dt, ksi) +
```

```
            J0001(i1, i2, i3, i4, q[11], dt, ksi)) +
            G(b[:, i1], G(b[:, i2], Lj(a, G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
            (J0100(i1, i2, i3, i4, q[13], dt, ksi) - J0010(i1, i2, i3, i4, q[12], dt, ksi)) -
            Lj(a, G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
            J1000(i1, i2, i3, i4, q[14], dt, ksi) +
            G(b[:, i1], Lj(a, G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
            (J1000(i1, i2, i3, i4, q[14], dt, ksi) - J0100(i1, i2, i3, i4, q[13], dt, ksi)) +
            G(b[:, i1], G(b[:, i2], G(b[:, i3], Lj(a, b[i, i4], dxs), dxs), dxs), dxs) *
            (J0010(i1, i2, i3, i4, q[12], dt, ksi) - J0001(i1, i2, i3, i4, q[11], dt, ksi))
            for i4 in range(m)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)],

           *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], G(
            b[:, i5], b[i, i6], dxs), dxs), dxs), dxs), dxs) *
            J000000(i1, i2, i3, i4, i5, i6, q[15], dt, ksi)
            for i6 in range(m)
            for i5 in range(m)
            for i4 in range(m)
            for i3 in range(m)
            for i2 in range(m)
            for i1 in range(m)]

        )

    def doit(self, **hints):
        """
        Tries to expand or calculate function
        Returns
        =======
        sympy.Expr
        """
        return StrongTaylorStratonovich3p0(*self.args, **hints)
```

6.3 Source Codes for Linear Stationary Systems of Itô SDEs

Listing 132: **Implementation of supplementary functions**

```
import numpy as np


class NotASquareMatrix(Exception):
    pass


def vec_to_eye(vector):
    """
    Converts vector to eye matrix
    Parameters
    ==========
```

```
    vector : iterable
    Returns
    =======
    numpy.ndarray
    """
    n = len(vector)
    matrix = np.zeros((n, n))

    for i in range(len(matrix)):
        matrix[i][i] = vector[i]

    return matrix


def diagonal_to_column(matrix):
    """
    Converts diagonal matrix to column vector
    Parameters
    ==========
    matrix : numpy.ndarray
    Returns
    =======
    column : numpy.ndarray
    """
    height = np.shape(matrix)[0]
    if height != np.shape(matrix)[1]:
        raise NotASquareMatrix()

    column = np.zeros((height, 1))
    for i in range(height):
        column[i][0] = matrix[i][i]

    return column
```

Listing 133: **Implementation of Algorithm 11.2 [62]**

```
import numpy as np
import scipy.linalg as sci


def dindet(n: int, k: int, m_a: np.ndarray, m_b: np.ndarray, dt: float):
    """
    Algorithm 11.2 [61]
    Parameters
    ==========
    n : int
    k : int
    m_a : numpy.array
    m_b : numpy.array
    dt : float
    Returns
    =======
    m_ad : numpy.array
```

```
    m_bd : numpy.array
    """
    m_okn = np.zeros((k, n))
    m_okk = np.zeros((k, k))
    m_idt = np.eye(n + k) * dt
    m_aa = np.vstack((np.hstack((m_a, m_b)),
                np.hstack((m_okn, m_okk))))
    m_ex_aah = sci.expm(m_aa.dot(m_idt))
    m_ad = m_ex_aah[:n, :n]
    m_bd = m_ex_aah[:n, n:(n + k)]
    return m_ad, m_bd
```

Listing 134: **Implementation of Algorithm 11.6 [62]**

```
import numpy as np

from mathematics.sde.linear.dindet import dindet
from mathematics.sde.linear.matrix import vec_to_eye


def stoch(n: int, m_a: np.ndarray, m_f: np.ndarray, dt: float):
    """
    Root function for set of algorithms implemented below
    Parameters
    ==========
    n : int
    m_a : numpy.ndarray
    m_f : numpy.ndarray
    dt : float
    Returns
    =======
    numpy.ndarray
    """
    v_l2, m_s, m_d1 = algorithm_11_2(n, m_a, m_f, dt)
    mat_l = vec_to_eye(np.sqrt(v_l2))
    return m_s.dot(mat_l)


def algorithm_11_2(n: int, m_a: np.ndarray, m_f: np.ndarray, dt: float):
    """
    Parameters
    ==========
    n : int
    m_a : numpy.ndarray
    m_f : numpy.ndarray
    dt : float
    Returns
    =======
    eigenvalues : numpy.ndarray
    eigenvectors : numpy.ndarray
    m_d1 : numpy.ndarray
    """
    m_ac = algorithm_11_5(n, m_a)
```

```
    m_g = m_f.dot(np.transpose(m_f))
    m_gv = algorithm_11_3(n, m_g)
    m_dd, m_dv = dindet(int(n * (n + 1) / 2), 1, m_ac, m_gv, dt)
    m_d1 = algorithm_11_4(n, m_dv)
    eigenvalues, eigenvectors = np.linalg.eig(m_d1)
    return eigenvalues, eigenvectors, m_d1


def algorithm_11_3(n: int, m_g: np.ndarray):
    """
    Algorithm 11.3 [61]
    Parameters
    ----------
    n : int
    m_g : numpy.ndarray
    Returns
    -------
    m_vec : numpy.ndarray
        column vector
    """
    i2 = 0
    v_size = 0
    for i in range(n):
        n2 = n - i
        for j in range(n2):
            if v_size < j + i2:
                v_size = j + i2
        i2 = i2 + n - i

    m_vec = np.ndarray((v_size + 1, 1))

    i2 = 0
    for i in range(n):
        n2 = n - i
        for j in range(n2):
            m_vec[j + i2][0] = m_g[j][j + i]
        i2 = i2 + n - i

    return m_vec


def algorithm_11_4(n: int, m_dv: np.ndarray):
    """
    Algorithm 11.4 [61]
    Parameters
    ----------
    n : int
    m_dv : numpy.ndarray
    Returns
    -------
    m_d1 : numpy.ndarray
    """
    i2 = 0
    size = 0
    for i in range(n):
```

```
    n2 = n - i
    for j in range(n2):
      if size < j + i:
        size = j + i
    i2 = i2 + n - i

  m_d1 = np.ndarray((size + 1, size + 1))

  i2 = 0
  for i in range(n):
    n2 = n - i
    for j in range(n2):
      m_d1[j][j + i] = m_dv[j + i2][0]
      m_d1[j + i][j] = m_dv[j + i2][0]
    i2 = i2 + n - i

  return m_d1


def algorithm_11_5(n: int, m_a: np.ndarray):
  """
  Algorithm 11.5 [61]
  Parameters
  ----------
  n : int
  m_a : numpy.ndarray
  Returns
  -------
  m_ac : numpy.ndarray
  """
  r = 0
  v_size = 0
  h_size = 0

  for i in range(n):
    n2 = n - i
    for j in range(n2):
      o = 0
      for k in range(n):
        n3 = n - k
        for m in range(n3):
          if v_size < m + o:
            v_size = m + o
          if h_size < r:
            h_size = r
        o = o + n - k
      r = r + 1

  m_ones = np.zeros((n, n))
  m_ac = np.ndarray((v_size + 1, h_size + 1))

  r = 0
  for i in range(n):
    n2 = n - i
    for j in range(n2):
```

```python
            i2 = j + i
            m_ones[j][i2] = 1
            m_ones[i2][j] = 1
            m_one_a = m_ones.dot(np.transpose(m_a)) + m_a.dot(m_ones)
            o = 0
            for k in range(n):
                n3 = n - k
                for m in range(n3):
                    m_ac[m + o][r] = m_one_a[m][m + k]
                o = o + n - k
            m_ones = np.zeros((n, n))
            r = r + 1

    return m_ac
```

Listing 135: **Implementation of the vector function $\mathbf{u}(t)$**

```python
import numpy as np
from sympy import lambdify, sympify


class AbstractDistortion:
    def t(self, t: float):
        raise NotImplementedError("Method t is not implemented")


class Symbolic(AbstractDistortion):

    def __init__(self, fn: str):
        from sympy.abc import t
        self._u = lambdify(t, sympify(fn), "numpy")

    def t(self, t):
        return self._u(t)


class ComplexDistortion(AbstractDistortion):
    """
    Vector function u(t)
    """

    def __init__(self, n: int, mat_u: np.ndarray):
        self._mat_u = mat_u
        self._mat_ut = np.ndarray(shape=(n, 1), dtype=float)

    def t(self, t: float):
        """
        Provides vector function u(t) at moment t

        Parameters
        ----------
        t : float
            moment of time
```

```
    Returns
    ———
    numpy.ndarray
      column u(t)
    """
    for i in range(self._mat_u.shape[0]):
      self._mat_ut[i][0] = self._mat_u[i][0].t(t)
    return self._mat_ut
```

Listing 136: **Modeling of linear system of Itô SDEs**

```
import numpy as np
from numpy import transpose

from mathematics.sde.linear.matrix import diagonal_to_column


class Integral:
  """
  Provides numerical integration
  """

  def __init__(self, n: int):
    self.n, self.t0, self.tk, self.dt, self.t = \
      n, 0, 0, 0, 0
    self.m_a, self.m_ad, self.m_bd, self.m_h, self.m_fd, self.distortion = \
      None, None, None, None, None, None
    self.m_x0, self.m_mx0, self.m_dx0, self.m_xt, self.m_mx, self.m_dx = \
      None, None, None, np.ndarray((n, 0)), np.ndarray((n, 0)), np.ndarray((n, 0))
    self.v_yt, self.v_my, self.v_dy, self.v_t = \
      [], [], [], []
    self.v_ry = []

  def integrate(self):
    """
    Performs numerical integration
    """
    higher_limit = self.t + int((self.tk - self.t0) / self.dt + 1)
    lower_limit = self.t

    self.m_xt = np.hstack((self.m_xt, np.ndarray((self.n, higher_limit - lower_limit))))
    self.m_mx = np.hstack((self.m_mx, np.ndarray((self.n, higher_limit - lower_limit))))
    self.m_dx = np.hstack((self.m_dx, np.ndarray((self.n, higher_limit - lower_limit))))

    for self.t in range(lower_limit, higher_limit):
      t = self.t0 + self.t * self.dt
      ft = np.random.randn(self.n, 1)
      mat_ut = self.distortion.t(t)

      # solution of sde
      xt = self.m_ad.dot(self.m_x0) + self.m_bd.dot(mat_ut) + self.m_fd.dot(ft)
      # exit process of stochastic system
      self.m_xt[:, self.t] = xt[:, 0]
```

```
        self.v_yt.append(self.m_h.dot(xt)[0][0])

        # expectation of solution of sde
        mx = self.m_ad.dot(self.m_mx0) + self.m_bd.dot(mat_ut)
        # expectation of exit process
        self.m_mx[:, self.t] = mx[:, 0]
        self.v_my.append(self.m_h.dot(mx)[0][0])

        # dispersion of solution of sde
        dx = self.m_ad.dot(self.m_dx0).dot(np.transpose(self.m_ad)) + self.m_fd.dot(np.
      transpose(self.m_fd))
        # dispersion of exit process
        self.m_dx[:, self.t] = diagonal_to_column(dx)[:, 0]
        self.v_dy.append(self.m_h.dot(dx).dot(transpose(self.m_h))[0][0])

        self.v_t.append(t)

        self.m_x0, self.m_mx0, self.m_dx0 = xt, mx, dx
```

6.4 Source Codes for Utilities and Initialization

Listing 137: **Initialization module**

```
from config import database
from init.database import initdb
from tools.database import connect, disconnect


def initialization():
    """
    Initializes various components of application
    """
    connect(database)
    initdb()
    disconnect()
```

Listing 138: **Module for database initialization**

```
import csv
import logging
import os

import sympy as sp

import config as c
from tools import fsys
from tools.database import execute
from tools.fsys import get_files

logger = logging.getLogger(__name__)
```

```
def initdb():
    """
    Initializes database with necessary table drivers
    """

    if not fsys.is_locked(".db.lock"):
        logger.info("Initializing database...")
        create_files_table()
        create_c_table()
        fsys.lock(".db.lock")
    else:
        logger.info("Updating database...")
        update_coefficients()


def create_files_table():
    """
    Initializes the Fourier-Legendre coefficients table
    """
    execute("DROP TABLE IF EXISTS `files`")
    execute(
        "CREATE TABLE `files` ("
        "  `id`  integer PRIMARY KEY AUTOINCREMENT,"
        "  `name` text unique"
        ")"
    )


def create_c_table():
    """
    Initializes the Fourier-Legendre coefficients table
    """
    execute("DROP TABLE IF EXISTS `C`")
    execute(
        "CREATE TABLE `C` ("
        "  `id`  integer PRIMARY KEY AUTOINCREMENT,"
        "  `index` text unique,"
        "  `value` text,"
        "  `value_f` double"
        ")"
    )

    update_coefficients()


def update_coefficients():
    """
    Updates the Fourier-Legendre coefficients table
    """
    files = get_files(c.csv, r'c_.*\.csv')
    loaded_files = [record[0] for record in execute("SELECT `name` FROM `files`")]
    difference = [f for f in files if f not in loaded_files]

```

```
    rows = []
    for file in difference:

        with open(os.path.join(file)) as f:
            reader = csv.reader(f, delimiter=';', quotechar='"')
            for row in reader:
                if len(rows) > c.read_buffer_size:
                    execute(f"INSERT INTO `C` (`index`, `value`, `value_f`) VALUES {','.join(rows)}
    ")
                    rows.clear()

                rows.append(f"('{row[0]}', '{row[1]}', {float(sp.sympify(row[1]).evalf())})")

        execute(f"INSERT INTO `files` (`name`) VALUES ('{file}')")

    if len(rows) > 0:
        execute(f"INSERT INTO `C` (`index`, `value`, `value_f`) VALUES {','.join(rows)}")
```

Listing 139: **Database module**

```
import logging
import re
import sqlite3

logger = logging.getLogger(__name__)

connection: sqlite3.Connection
cursor: sqlite3.Cursor


def is_connected():
    """
    Checks if application is connected to database
    Returns
    =======
    True or False
    """
    global connection
    if connection is None:
        return False
    else:
        return True


def connect(db: str):
    """
    Connects application to database
    Parameters
    ==========
    db : str
        path to database file
    """
    try:
```

```
        global connection
        global cursor

        connection = sqlite3.connect(db)
        connection.create_function("REGEXP", 2, regex)

        cursor = connection.cursor()
        logger.info(f"SQLite Database is successfully connected")

        query = "select sqlite_version();"
        cursor.execute(query)
        record = cursor.fetchall()
        logger.info(f"SQLite Database Version is: {record[0][0]}")

        query = "PRAGMA foreign_keys = ON;"
        cursor.execute(query)

    except sqlite3.Error as error:
        logger.error(f"Error while connecting to sqlite: {error}")


def disconnect():
    """
    Disconnects application from database
    """
    try:
        global connection
        global cursor

        connection.close()
        logger.info("The SQLite connection is closed")

    except sqlite3.Error as error:
        logger.error(f"Error while connecting to sqlite: {error}")


def execute(query: str):
    """
    Sends query to database and receives data
    Parameters
    ==========
    query : str
        query to database
    Returns
    =======
        list of tuples (rows)
    """
    try:
        global connection
        global cursor

        cursor.execute(query)
        records = cursor.fetchall()
        connection.commit()
        return records
```

```

  except sqlite3.Error as error:
    logger.error(f"Error while connecting to sqlite: {error}")


def regex(value, pattern):
  """
  Regular expression for search in database
  Parameters
  ==========
  value
    column to apply
  pattern
    regular expression
  Returns
  =======
  Search results
  """
  c_pattern = re.compile(r"\b" + pattern.lower() + r"\b")
  return c_pattern.search(value) is not None
```

Listing 140: **File system utilities**

```
import os
import re

import config as c


def get_files(path, pattern):
  """
  Gives list of files containing the Fourier-Legendre coefficients
  Returns
  =======
  list
    list of available files
  """
  return [os.path.join(path, f)
      for f in os.listdir(path) if re.match(pattern, f)]


def is_locked(filename: str):
  """
  Checks if lock is set
  Parameters
  ==========
  filename : str
    name of lock file
  Returns
  =======
  True of False
  """
  if os.path.isfile(os.path.join(c.resources, filename)):
```

```
        return True
    else:
        return False


def lock(filename):
    """
    Performs locking
    Parameters
    ==========
    filename : str
        name of lock file
    """
    f = open(os.path.join(c.resources, filename), "w")
    f.close()


def unlock(filename):
    """
    Performs unlocking
    Parameters
    ==========
    filename : str
        name of lock file
    """
    os.remove(os.path.join(c.resources, filename))
```

7 Future Work

Considering the future work, it is important to say that symbolic algebra gives a wide field for optimizations of modeling process. Symbolic operations is actually operations with strings. Such operations has relatively high complexity and slows down modeling process significantly. One of possible ways to improve modeling performance is to parallelize computations. Since strong numerical schemes for Itô SDEs have massive amount of terms this idea appears justified.

The strong numerical schemes for Itô SDEs seem to be easily optimizable, on the other hand, superpositions of the differential operators (4), (5), and (23) are not. They are called recursively during calculation process which is more difficult to parallelize than strong numerical schemes for Itô SDEs. Differential operators obviously include differentiating which is high cost and optimization of them is a dedicated issue.

In the future, it is possible to improve the SDE-MATH software package in a number of other directions. In particular, high-order strong numerical methods of the Runge-Kutta type [2], [7], [43], [62] (including multistep numerical methods [2], [7], [43], [62]) for Itô SDEs can be implemented. In addition, software for solving the filtering problem and the problem of stochastic optimal control can also be developed. These improvements will lead to changes of the graphical user interface due to new features.

Bibliography

[1] Ito, K. On Stochastic Differential Equations. Memoirs of the American Mathematical Society, 4 (1951), 1-51.

[2] Kloeden, P.E., Platen, E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992, 632 pp.

[3] Kloeden, P.E., Platen, E., Schurz, H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.

[4] Arato, M. Linear Stochastic Systems with Constant Coefficients. A Statistical Approach. Springer-Verlag, Berlin, Heidelberg, N.Y., 1982, 289 pp.

[5] Shiryaev, A.N. Essentials of Stochastic Finance: Facts, Models, Theory. World Scientific Publishing Co United States, 1999, 852 pp.

[6] Karatzas, I., Shreve, S. Methods of Mathematical Finance. Springer-Verlag, New York, 1998, 415 pp.

[7] Platen, E., Bruti-Liberati, N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010, 868 pp.

[8] Han, X., Kloeden, P.E., Random Ordinary Differential Equations and Their Numerical Solution. Springer, Singapore, 2017, 250 pp.

[9] Allen, E. Modeling with Ito Stochastic Differential Equations. Springer, Dordrecht, 2007, 230 pp.

[10] Merton, R.C. Continuous-Time Finance. Blackwell, Oxford, 1992, 754 pp.

[11] Heston, S.L. A closed-form solution for options with stochastic volatility with applications to bond and currency options, Rev. Financial Studies, 6, 2 (1993), 327–343.

[12] Cox, J.C., Ingersoll, J.E., Ross, S.A. A Theory of the term structure of interest rates, Econometrica, 53 (1985), 385-408.

[13] Kimura, M., Ohta, T. Theoretical aspects of Population genetics. Princeton Univ. Press, Boston, 1971, 232 pp.

[14] Iacus, S.M. Simulation and Inference for Stochastic Differential Equations. With R Examples. Springer-Verlag, New York, 2008, 285 pp.

[15] Ricciardi, L.M. Diffusion Processes and Related Topics in Biology, Lecture Notes in Biomathematics. Springer, New York, 1977, 202 pp.

[16] Stratonovich, R.L., Selected Problems of Fluctuations Theory in Radio Engineering. [In Russian]. Sovetskoe Radio, Moscow, 1961, 556 pp.

[17] Liptser, R.Sh., Shirjaev, A.N. Statistics of stochastic processes: nonlinear filtering and related problems. [In Russian]. Nauka, Moscow, 1974, 696 pp.

[18] Nasyrov, F.S. Local times, symmetric integrals and stochastic analysis. [In Russian]. Fizmatlit Publ., Moscow, 2011, 212 pp.

[19] Kagirova, G.R., Nasyrov, F.S. On an optimal filtration problem for one-dimensional diffusion processes. Siberian Adv. Math., 28, 3 (2018), 155-165.

[20] Chugai, K.N., Kosachev, I.M., Rybakov, K.A. Approximate filtering methods in continuous-time stochastic systems. Smart Innovation, Systems and Technologies, vol. 173, Eds. Jain L.C., Favorskaya M.N., Nikitin I.S., Reviznikov D.L. Springer, 2020, pp. 351-371. DOI: http://doi.org/10.1007/978-981-15-2600-8_24

[21] Averina, T.A., Rybakov, K.A. Using maximum cross section method for filtering jump-diffusion random processes. Russian Journal of Numerical Analysis and Mathematical Modelling. 35, 2 (2020), 55-67. DOI: http://doi.org/10.1515/rnam-2020-0005

[22] Kloeden, P.E., Platen, E., Schurz, H., Sorensen, M. On effects of discretization on estimators of drift parameters for diffusion processes. J. Appl. Probab., 33 (1996), 1061-1076.

[23] Clark, J.M.C., Cameron, R.J. The maximum rate of convergence of discrete approximations for stochastic differential equations. Stochastic Differential Systems Filtering and Control. Lecture Notes in Control and

Information Sciences, vol 25. Ed. Grigelionis B. Springer, Berlin, Heidelberg, 1980, pp. 162-171.

[24] Kulchitskiy, O.Yu., Kuznetsov, D.F. The unified Taylor–Ito expansion. Journal of Mathematical Sciences (New York), 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[25] Kuznetsov, D.F. New representations of the Taylor–Stratonovich expansion. Journal of Mathematical Sciences (New York), 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:1026138522239

[26] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[27] Kuznetsov, D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. [In English]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[28] Milstein, G.N. Numerical Integration of Stochastic Differential Equations. [In Russian]. Ural University Press, Sverdlovsk, 1988, 225 pp.

[29] Kloeden, P.E., Platen, E., Wright, I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[30] Averina, T.A., Prigarin, S.M. Calculation of stochastic integrals of Wiener processes. [In Russian]. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.

[31] Prigarin, S.M., Belov, S.M. One application of series expansions of Wiener process. [In Russian]. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.

[32] Wiktorsson, M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. The Annals of Applied Probability, 11, 2 (2001), 470-487.

[33] Ryden, T., Wiktorsson, M. On the simulation of iterated Ito integrals. Stochastic Processes and their Applications, 91, 1 (2001), 151-168.

[34] Gaines, J.G., Lyons, T.J. Random generation of stochastic area integrals. SIAM J. Appl. Math., 54 (1994), 1132-1146.

[35] Milstein, G.N., Tretyakov, M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.

[36] Allen, E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[37] Rybakov, K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2019), 1-31. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html

[38] Tang, X., Xiao, A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Advances in Computational Mathematics, 45 (2019), 813-846.

[39] Zahri, M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. Journal of Taibah University for Science, 8, 2 (2014), 186-198.

[40] Li, C.W., Liu, X.Q. Approximation of multiple stochastic integrals and its application to stochastic differential equations. Nonlinear Anal. Theor. Meth. Appl., 30, 2 (1997), 697-708.

[41] Rybakov, K. Application of Walsh series to represent iterated Stratonovich stochastic integrals. IOP Conference Series: Materials science and engineering. 2020, vol. 927, id 012080. DOI: http://doi.org/10.1088/1757-899X/927/1/012080

[42] Rybakov, K.A. Modeling and analysis of output processes of linear continuous stochastic systems based on orthogonal expansions of random functions. J. of Computer and Systems Sci. Int., 59, 3 (2020), 322-337. DOI: http://doi.org/10.1134/S1064230720030156

[43] Kuznetsov, D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227

[44] Kuznetsov, D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[45] Kuznetsov, D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[46] Kuznetsov, D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[47] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

[48] Jentzen, A., Röckner, M. A Milstein scheme for SPDEs. Foundations Comp. Math., 15, 2 (2015), 313-362.

[49] Becker, S., Jentzen, A., Kloeden, P.E. An exponential Wagner–Platen type scheme for SPDEs. SIAM J. Numer. Anal., 54, 4 (2016), 2389-2426.

[50] Mishura, Y.S., Shevchenko, G.M. Approximation schemes for stochastic differential equations in a Hilbert space. Theor. Prob. Appl., 51, 3 (2007), 442-458.

[51] Bao, J., Reisinger, C., Renz, P., Stockinger, W. First order convergence of Milstein schemes for McKean equations and interacting particle systems arXiv:2004.03325v1 [math.PR], 2020, 27 pp.

[52] Son, L.N., Tuan, A.H., Dung, T.N., Yin G. Milstein-type procedures for numerical solutions of stochastic differential equations with Markovian switching. SIAM J. Numer. Anal., 55, 2 (2017), 953–979.

[53] Sun, Y., Yang, J., Zhao W. Ito–Taylor schemes for solving mean-field stochastic differential equations. Numer. Math. Theor. Meth. Appl., 10, 4 (2017), 798-828.

[54] Higham, D.J. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations. SIAM Rev., 43, 3 (2001), 525–546.

[55] Cyganowski, S., Grune, L., Kloeden, P.E. Maple for stochastic differential equations. Theory and Numerics of Differential Equations. Eds. Blowey, J.F., Coleman, J.P., Craig, A.W. Universitext. Springer, Berlin, Heidelberg, 2001, pp. 127-177.

[56] Higham, D.J., Kloeden, P.E. MAPLE and MATLAB for stochastic differential equations in finance. Programming Languages and Systems in Computational Economics and Finance. Advances in Computational Economics, vol 18, Ed. Nielsen, S.S. Springer, Boston, MA, 2002, pp. 233-269.

[57] Cyganowski, S., Grune, L., Kloeden P.E. MAPLE for jump-diffusion stochastic differential equations in finance. Programming Languages and Systems in Computational Economics and Finance. Advances in Computational Economics, vol. 18, Ed. Nielsen, S.S. Springer, Boston, MA, 2002, pp. 441-460.

[58] Gilsing, H., Shardlow, T. SDELab: A package for solving stochastic differential equations in MATLAB. Journal of Computational and Applied Mathematics, 2, 205 (2007), 1002-1018.

[59] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, 768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230

[60] Kiesewetter, S., Polkinghorne, R., Opanchuk, B., Drummond, P.D. xSPDE: Extensible software for stochastic equations. SoftwareX, 5 (2016), 12-15.

[61] Gevorkyan, M.N., Velieva, T.R., Korolkova, A.V., Kulyabov, D.S., Sevastyanov, L.A. Stochastic Runge–Kutta software package for stochastic differential equations. Dependability Engineering and Complex Systems. DepCoS-RELCOMEX 2016. Advances in Intelligent Systems and Computing, vol. 470, Eds. Zamojski, W., Mazurkiewicz, J., Sugier, J., Walkowiak, T., Kacprzyk, J. Springer, Cham, 2016, pp. 169-179.

[62] Kuznetsov, D.F. Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Edition. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[63] Kulchitskiy, O.Yu., Kuznetsov, D.F. Numerical Simulation of Stochastic Systems of Linear Stationary Differential Equations. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (1998), 41-65. Available at: http://diffjournal.spbu.ru/pdf/j010.pdf

[64] Gihman, I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. [In Russian]. Naukova Dumka, Kiev, 1982, 612 pp.

[65] Kuznetsov, D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 1.5 and 2.0 Orders of Strong Convergence. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[66] Kuznetsov, D.F. On Numerical Modeling of the Multidimentional Dynamic Systems Under Random Perturbations With the 2.5 Order of Strong Convergence. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[67] Kuznetsov, D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

[68] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 912 pp.

[69] Kuznetsov, D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234

[70] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604

[71] Kuznetsov, D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[72] Kuznetsov, D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. Journal of Automation and Information Sciences (Begell House), 32, Issue 12, (2000), 69-86. DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[73] Kuznetsov, D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier–Legendre and trigonometric expansions, approximations, formulas. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (2017), A.1–A.385. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[74] Kuznetsov, M.D., Kuznetsov, D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp.

[75] Kuznetsov, M.D., Kuznetsov, D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. arXiv:2009.14011 [math.PR], 2020, 342 pp.

[76] Kuznetsov, D.F., Kuznetsov, M.D. A software package for implementation of strong numerical methods of convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative multi-dimensional noise.

19th International Conference "Aviation and Cosmonautics" (AviaSpace-2020). Abstracts (Moscow, MAI, 23-27 November, 2020), Publishing house "Pero", 2020, 569-570.

[77] Kuznetsov, D.F., Kuznetsov, M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[78] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf

[79] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, xxxii+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf

[80] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, xxx+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf

[81] Kuznetsov, D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier serieses. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2010), A.1-A.257. Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[82] Kuznetsov, D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, St.-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-232 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf

[83] Kuznetsov, D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, St.-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf

[84] Kuznetsov, D.F. Approximation of Multiple Ito and Stratonovich Stochastic Integrals. Multiple Fourier Series Approach. [In English]. LAP Lambert Academic Publishing: Saarbrucken, 2012 , 409 pp. Available at: http://www.sde-kuznetsov.spb.ru/12a.pdf

[85] Kuznetsov, D.F. Stochastic differential equations: theory and practice of numerical solution. With programs on MATLAB, 5th Edition. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 2 (2017), A.1-A.1000. Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[86] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X97T989878

[87] Kuznetsov, D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. Differencialnie Uravnenia i Protsesy Upravlenia, 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[88] Kuznetsov, D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Prinsheim method [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[89] Kuznetsov, D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [in English].

[90] Kuznetsov, D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR]. 2017, 56 pp. [in English].

[91] Kuznetsov, D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals, based on generalized multiple Fourier series. arXiv:1801.01079 [math.PR]. 2018, 68 pp. [in English].

[92] Kuznetsov, D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions, using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2017, 106 pp. [in English].

[93] Kuznetsov, D.F. Expansions of Iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 203 pp. [in English].

[94] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 77 pp. [in English].

[95] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 65 pp. [in English].

[96] Kuznetsov, D.F. Expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 46 pp. [In English].

[97] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR]. 2018, 20 pp. [In English].

[98] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 126 pp. [in English].

[99] Kuznetsov, D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [in English].

[100] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. Differencialnie Uravnenia i Protsesy Upravlenia, 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[101] Kuznetsov, D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].

[102] Kuznetsov, D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 40 pp. [In English].

[103] Kuznetsov, D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR]. 2018, 28 pp. [In English].

[104] Kuznetsov, D.F, Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor-Stratonovich expansion. arXiv:1806.10705 [math.PR]. 2018, 28 pp. [In English].

[105] Kuznetsov, D.F, Numerical simulation of 2.5-set of iterated Ito stochastic integrals of multiplicities 1 to 5 from the Taylor-Ito expansion. arXiv:1805.12527 [math.PR]. 2018, 28 pp. [In English].

[106] Kuznetsov, D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv:1802.04844 [math.PR]. 2018, 36 pp. [in English].

[107] Kuznetsov, D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR], 2018, 44 pp. [in English].

[108] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method arXiv:1801.01962 [math.PR]. 2018, 49 pp. [in English].

[109] Kuznetsov, D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM], 2019, 41 pp.

[110] Kuznetsov, D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[111] Kuznetsov, D.F. Expansions of iterated Stratonovich stochastic integrals from the Taylor-Stratonovich expansion based on multiple trigonometric Fourier series. Comparison with the Milstein expansion. arXiv:1801.08862 [math.PR], 2018, 36 p. [In English].

[112] Kuznetsov, D.F. New simple method for obtainment an expansion of double stochastic Ito integrals based on the expansion of Brownian motion using Legendre polynomials and trigonometric functions. arXiv:1807.00409 [math.PR], 2019, 23 pp. [In English].

[113] Kuznetsov, D.F. Four new forms of the Taylor-Ito and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Ito stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 90 pp. [In English].

[114] Kuznetsov, M.D., Kuznetsov, D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Itô SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/collection.html

[115] Kuznetsov, D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[116] Kuznetsov, D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Differencialnie Uravnenia i Protsesy Up-

ravlenia, 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html

[117] Kuznetsov, D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: http://www.sde-kuznetsov.spb.ru/01b.pdf

[118] Kuznetsov, D.F. Methods of numerical simulation of stochastic differential Ito equations solutions in problems of mechanics. Ph. D. Thesis, St.-Petersburg, 1996, 260 pp.

[119] Kuznetsov, D.F. Theorems about integration order replacement in multiple Ito stochastic integrals. [In Russian]. VINITI. 3607-V97 (1997), 31 pp.

[120] Kuznetsov, D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [In English]. arXiv:1801.04634 [math.PR], 2018, 27 pp.

[121] Kuznetsov, D.F. Two new representations of the Taylor–Stratonovich expansion. Preprint. [In Russian]. SPbGTU Publishing House, 1999, 13 pp. DOI: http://doi.org/10.13140/RG.2.2.18258.86729 Available at: http://www.sde-kuznetsov.spb.ru/99b.pdf

[122] Kuznetsov, D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2019), 32-52. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[123] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html