

DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 1, 2021
Electronic Journal,
reg. N ΦC77-39410 at 15.04.2010
ISSN 1817-2172

<http://diffjournal.spbu.ru/>
e-mail: jodiff@mail.ru

Computer software for the investigation of differential
equations, dynamical systems, and control processes
Stochastic differential equations
Numerical methods
Computer modeling in dynamical and control systems

SDE-MATH: A software package for the implementation of strong high-order numerical methods for Itô SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series

Mikhail D. Kuznetsov¹, Dmitriy F. Kuznetsov²

¹Faculty of Computer Technologies and Informatics, St. Petersburg Electrotechnical University, Saint-Petersburg, Russia

²Institute of Applied Mathematics and Mechanics, Peter the Great St. Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to the implementation of strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô stochastic differential equations with multidimensional non-commutative noise based on multiple Fourier–Legendre series and unified Taylor–Itô and Taylor–Stratonovich expansions. Algorithms for the implementation of these methods are constructed and a package of programs in the Python programming language is presented. An important part of this software package concerning the

mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 with respect to components of the multidimensional Wiener process is based on the method of generalized multiple Fourier series. More precisely, we used multiple Fourier–Legendre series converging in the sense of norm in Hilbert space for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals.

Key words: Software package, Python programming language, numerical method, strong convergence, Itô stochastic differential equation, multidimensional Wiener process, non-commutative noise, unified Taylor–Itô expansion, unified Taylor–Stratonovich expansion, Milstein scheme, high-order strong numerical scheme, iterated Itô stochastic integral, iterated Stratonovich stochastic integral, mean-square approximation, generalized multiple Fourier series, multiple Fourier–Legendre series, Legendre polynomial.

Contents

1	Introduction	94
2	Theoretical Results Underlying the SDE-MATH Software Package	98
2.1	Strong Numerical Methods with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Itô Expansion	98
2.2	Strong Numerical Methods with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Stratonovich Expansion	106
2.3	Method of Expansion and Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series	113
2.4	Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (11)–(16) Using Legendre Polynomials	123
2.5	Optimization of Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (12)–(16)	132
2.6	Approximations of Iterated Stratonovich Stochastic Integrals from the Numerical Schemes (24)–(28) Using Legendre Polynomials	141
2.7	Numerical Algorithm for Linear Stationary Systems of Itô SDEs Based on Spectral Decomposition	147

3	The Structure of the SDE-MATH Software Package	148
3.1	Development Tools	148
3.2	Dependency Libraries	149
3.3	Architecture	149
3.3.1	Integration with SymPy	151
3.3.2	Purpose of NumPy	151
3.3.3	Purpose of SQLite Database	152
3.3.4	Purpose of Matplotlib	153
3.4	Implementation Plan	153
3.4.1	Calculation of the Fourier–Legendre Coefficients	153
3.4.2	Differential Operators $L, \bar{L}, G_0^{(i)}, i = 1, \dots, m$	154
3.4.3	Approximations of Iterated Stochastic Integrals	154
3.4.4	Strong Numerical Schemes for Itô SDEs	154
3.4.5	Graphical User Interface	154
4	Software Package Graphical User Interface	154
4.1	Information Model of The Graphical User Interface	154
4.1.1	Processing Screens	156
4.1.2	Greetings Dialog	156
4.1.3	Main Menu Dialog	156
4.1.4	Visualization Tool	156
4.1.5	Data Input Dialogs	156
4.2	The User Experience and Implementation Results	157
5	The Results Obtained Using the SDE-MATH Software Package	175
5.1	The Calculated Fourier–Legendre Coefficients	175
5.2	Accuracy Settings	176
5.3	Testing Example (Nonlinear System of Itô SDEs)	179
5.4	Visualization and Numerical Results for Nonlinear System of Itô SDEs Obtained via the SDE-MATH Software Package	180
5.5	Example of Linear System of Itô SDEs (Solar Activity)	211

5.6	Visualization and Numerical Results for Solar Activity Model	211
5.7	Example of Abstract Linear System of Itô SDEs	214
5.8	Visualization and Numerical Results for Abstract Linear System of Itô SDEs Obtained via the SDE-MATH Software Package	215
6	Source Codes of the SDE-MATH Software Package in the Python Programming Language	217
6.1	Source Codes of Graphical User Interface	217
6.1.1	Source Codes of Main Menu	217
6.1.2	Source Codes of Charts Window	230
6.1.3	Source Codes of Input for Nonlinear Systems of Itô SDEs	240
6.1.4	Source Codes of Input for Linear Systems of Itô SDEs	255
6.2	Source Codes for Nonlinear Systems of Itô SDEs	275
6.2.1	Source Codes for Calculation of the Fourier–Legendre Coefficients	275
6.2.2	Source Codes for Supplementary Differential Operators and Functions	299
6.2.3	Source Codes for Iterated Itô Stochastic Integrals Approximations Subprograms	303
6.2.4	Source Codes for Iterated Stratonovich Stochastic Integrals Approximations Subprograms	329
6.2.5	Source Codes for Calculation of the Numbers q, q_1, \dots, q_{15}	351
6.2.6	Source Codes for Strong Taylor–Itô Numerical Schemes with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs	359
6.2.7	Source Codes for Strong Taylor–Stratonovich Numerical Schemes with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs	381
6.3	Source Codes for Linear Stationary Systems of Itô SDEs	400
6.4	Source Codes for Utilities and Initialization	407
7	Future Work	412
	References	413

1 Introduction

As known, Itô stochastic differential equations (SDEs) have appeared in the theory of random processes relatively recently [1] (1951). Nevertheless, to date, a large number of mathematical models for dynamical systems of different physical nature under the influence of random perturbations have been built on the basis of such equations [2]-[16]. Among them we note mathematical models in stochastic financial mathematics [5]-[7], [10]-[12], geophysics [2], [4], genetics [13], hydrology [2], epidemiology [9], chemical kinetics [2], [9], biology [8], [15], seismology [2], electrodynamics [16] and many other fields [2], [9], [14]. In addition, Itô SDEs arise when solving a number of mathematical problems, such as filtration [2], [3], [17]-[21], stochastic control [2], [17], stochastic stability [2], parameter estimation of stochastic systems [2], [3], [22].

Exact solutions of Itô SDEs are known in rare cases. For this reason, it becomes necessary to construct numerical methods for Itô SDEs. Moreover, the problem of numerical solution of Itô SDEs often occurs even in cases when the exact solution of Itô SDE is known. This means that in some cases, knowing the exact solution of the Itô SDE does not allow us to simulate it numerically in a simple way.

This article is devoted to the implementation of high-order strong numerical methods for systems of Itô SDEs with multidimensional non-commutative noise. More precisely, we consider strong numerical methods with convergence orders 1.0, 1.5, 2.0, 2.5, and 3.0. The article also considers the Euler method, which under suitable conditions [2] has the order 0.5 of strong convergence. To construct the mentioned numerical methods in this article, we use the so-called unified Taylor–Itô and Taylor–Stratonovich expansions [24], [25] (also see [26], Chapter 4). The important components of these expansions are the iterated Itô and Stratonovich stochastic integrals, which are functionals of a complex structure with respect to the components of a multidimensional Wiener process.

It should be noted that it is impossible to construct a numerical method for Itô SDE in a general case (multidimensional non-commutative noise) that includes only increments of the multidimensional Wiener processes, but has a higher order of convergence (in the mean-square sense) than the Euler method (simplest numerical method for Itô SDEs). This result is known as the "Clark–Cameron paradox" [23] (1980) and well explains the need to use high-order numerical methods for Itô SDEs, since the accuracy of the Euler method is insufficient for solving a number of practical problems related to Itô SDEs [2].

According to the "Clark–Cameron paradox" [23], avoidance of the problem of mean-square approximation of the mentioned iterated stochastic integrals is impossible in the general case when constructing high-order strong numerical methods for Itô SDEs.

The problem of mean-square approximation of iterated Itô and Stratonovich stochastic integrals in the context of the numerical integration of Itô SDEs was considered in a number of works [2], [3], [7], [8], [27]–[39].

It should be explained why the results of these works are insufficient for constructing effective procedures for the implementation of strong numerical methods of order 1.5 and higher for Itô SDEs.

There exists an approach to the mean-square approximation of iterated stochastic integrals based on integral sums [27], [34], [35]. Note that one of the variants of this method is based on reducing the problem of mean-square approximation of iterated stochastic integrals to the numerical integration of systems of linear Itô SDEs by the Euler method [39]. However, this approach [27], [34], [35], [39] implies the partitioning of the interval of integration for iterated stochastic integrals. It should be noted that the length of this interval is an integration step for numerical methods for Itô SDEs, which is already a fairly small value even without additional partitioning. Computational experiments show that the numerical modeling of iterated stochastic integrals by the method of integral sums [27], [34], [35], [39] leads to unacceptably high computational cost and accumulation of computation errors [42].

More efficient approach of the mean-square approximation of iterated stochastic integrals is based on the expansion of the so-called Brownian bridge process into the trigonometric Fourier series with random terms (version of the so-called Karhunen–Loève expansion) [2], [3], [7], [27], [28], [33], [34], [37], [38]. However, in [27], [33], [34], [38], this approach was used to approximate only iterated stochastic integrals of multiplicities 1 and 2, which makes it possible to implement numerical method with order 1.0 of strong convergence for Itô SDEs (Milstein method [27]). In papers [2], [3], [7], [28], the approximation of iterated stochastic integrals of multiplicities 1 to 3 was considered by the above approach, which makes it possible to implement numerical methods with orders 1.0 and 1.5 of strong convergence for Itô SDEs. However, formulas concerning integrals of multiplicity 3 turned out to be too complicated and did not find wide application in practice. Moreover, these formulas (for iterated stochastic integrals of multiplicity 3) were obtained without strict theoretical justification and exclude the possibility of effective estimation of the mean-square error of

approximation (see discussion in [26] (Sections 2.6.2, 6.2) for details).

It should be noted that in papers [29], [30], a similar approach was used to approximate iterated stochastic integrals of multiplicities 1 to 3 based on the series expansion of the Wiener process using trigonometric functions and Haar functions. In [40] orthonormal expansions of functions in terms of Walsh series were used to represent the iterated stochastic integrals.

Note that the iterated stochastic integrals under consideration are the random variables with unknown density functions. The only exception is the iterated Itô stochastic integral with multiplicity 2 [31]. However, the knowledge of density function of the mentioned stochastic integral gives no simple way of its approximation [31].

In this work, we use a more efficient method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals than the methods considered above. This method (the so-called method of generalized multiple Fourier series) is based on the theory constructed in Chapters 1, 2, and 5 of monograph [26] (also see bibliography therein). The method of generalized multiple Fourier series made it possible in this work to successfully implement the procedures for the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6. In this case, we use multiple Fourier–Legendre series, that is, we have chosen Legendre polynomials as a basis system of functions for approximating iterated stochastic integrals. It is important to note that the Legendre polynomials were first applied in the context of this problem in [43] (1997), while in the works of other authors Legendre polynomials were not considered as a system of basis functions for approximating iterated stochastic integrals (an exception is work [36]). As shown in [44], the Legendre polynomials are an optimal system of basis functions for approximating iterated Itô and Stratonovich stochastic integrals.

In this article, to build the SDE-MATH software package in the Python programming language, we use a database with 270,000 exactly calculated Fourier–Legendre coefficients to approximate iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6. It should be noted that the procedures for the mean-square approximation of iterated stochastic integrals of multiplicities 4, 5, and 6 constructed in this work have no analogues in the literature. At the same time, we propose a much more convenient procedure for the mean-square approximation of iterated stochastic integrals of multiplicity 3 than in works [2], [3], [7], [28]. This procedure provides an accurate calculation of the mean-square error of approximation of the mentioned stochastic integrals.

Another important feature of the presented software package is the use of unified Taylor–Itô and Taylor–Stratonovich expansions [24], [25] (also see [26], Chapter 4) for constructing strong numerical methods with convergence orders 1.5, 2.0, 2.5, and 3.0 for Itô SDEs. Unified Taylor–Itô and Taylor–Stratonovich expansions make it possible (in contrast with its classical analogues [2]) to use the minimal sets of iterated Itô and Stratonovich stochastic integrals. This property well explains the motive for using the mentioned unified expansions.

The results of this work on the approximation of iterated stochastic integrals can be used to numerically solve various types of SDEs. For example, for semilinear SPDEs with multiplicative trace class noise [26] (Chapter 7), [45], [46]. This is due to the fact that iterated stochastic integrals are a universal tool for constructing high-order strong numerical methods for various types of SDEs. In recent years, the mentioned numerical methods have been constructed for SDEs with jumps [7], SPDEs with multiplicative trace class noise [47]–[49], McKean SDEs [50], SDEs with switchings [51], mean-field SDEs [52], Itô–Volterra stochastic integral equations [49], etc.

There are many publications in which codes of programs in various programming languages are given for the numerical solution of SDEs [3], [9], [14], [53]–[61]. Among them, we note the software described in [3], [54], [56], [60]. Some of the mentioned works [3], [54], [56], [57], [60] are based on the results of monograph [2] on the approximation of iterated stochastic integrals (see above discussion on the disadvantages of approach [2]). Other publications [9], [14], [53], [55] do not use the modeling of iterated stochastic integrals for the case of multidimensional non-commutative noise at all.

In this article, we develop software for the numerical integration of Itô SDEs based on theoretical results and MATLAB codes from monographs [58], [61] for modeling iterated stochastic integrals of multiplicities 1 to 6 (the case of multidimensional non-commutative noise). In addition, we provide software (as a part of the SDE-MATH software package) for the numerical integration of linear stationary systems of Itô SDEs based on the results of article [62] and MATLAB codes from monographs [58], [61].

In Section 7 we discuss possible directions for the development of the SDE-MATH software package. In particular, the parallelization of computations, the implementation of methods of the Runge–Kutta type [2], [7], [42], [61] and multistep numerical methods for Itô SDEs [2], [7], [42], [61], the development of a part of the software package for solving filtering problem and stochastic optimal control problem [2], as well as improvement of the graphical user interface.

2 Theoretical Results Underlying the SDE-MATH Software Package

2.1 Strong Numerical Methods with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Itô Expansion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\{\mathcal{F}_t, t \in [0, T]\}$ be a nondecreasing right-continuous family of σ -algebras of \mathcal{F} . Let \mathbf{w}_t be a standard m -dimensional Wiener stochastic process with independent components $\mathbf{w}_t^{(i)}$ ($i = 1, \dots, m$), which is \mathcal{F}_t -measurable for any $t \in [0, T]$. Consider an Itô SDE in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \sum_{i=1}^m \int_0^t B_i(\mathbf{x}_\tau, \tau) d\mathbf{w}_\tau^{(i)}, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad (1)$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is a solution of the Itô SDE (1), the nonrandom functions $\mathbf{a}(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$, $B(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1) [63], the second integral on the right-hand side of (1) is interpreted as an Itô stochastic integral, $B_i(\mathbf{x}, t)$ is the i th column of the matrix function $B(\mathbf{x}, t)$, \mathbf{x}_0 is an n -dimensional and \mathcal{F}_0 -measurable random variable, $\mathbb{M}\{|\mathbf{x}_0|^2\} < \infty$ (\mathbb{M} is an expectation operator). We assume that \mathbf{x}_0 and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

It is well known that one of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions [2], [7], [42]. The essential feature of such expansions are the so-called iterated Itô and Stratonovich stochastic integrals, which have the form

$$J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \dots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (2)$$

$$J^*[\psi^{(k)}]_{T,t} = \int_t^{*T} \psi_k(t_k) \dots \int_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (3)$$

where every $\psi_l(\tau)$ ($l = 1, \dots, k$) is a continuous nonrandom function on $[t, T]$,

$\mathbf{w}_\tau^{(i)}$ ($i = 1, \dots, m$) are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} \stackrel{\text{def}}{=} \tau$,

$$\int \text{ and } \int^*$$

denote Itô and Stratonovich stochastic integrals, respectively; $i_1, \dots, i_k = 0, 1, \dots, m$.

Note that $\psi_l(\tau) \equiv 1$ ($l = 1, \dots, k$) and $i_1, \dots, i_k = 0, 1, \dots, m$ in the classical Taylor–Itô and Taylor–Stratonovich expansions [2]. At the same time $\psi_l(\tau) \equiv (t - \tau)^{q_l}$ ($l = 1, \dots, k$; $q_1, \dots, q_k = 0, 1, 2, \dots$) and $i_1, \dots, i_k = 1, \dots, m$ in the unified Taylor–Itô and Taylor–Stratonovich expansions [24], [25] (also see [26], Chapter 4).

Let $C^{2,1}(\mathbb{R}^n \times [0, T])$ be the space of functions $R(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^1$ with the following property: these functions are twice continuously differentiable in \mathbf{x} and have one continuous derivative in t . Let us consider the following differential operators on the space $C^{2,1}(\mathbb{R}^n \times [0, T])$

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n a^{(i)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(i)}} + \frac{1}{2} \sum_{j=1}^m \sum_{l,i=1}^n B^{(lj)}(\mathbf{x}, t) B^{(ij)}(\mathbf{x}, t) \frac{\partial^2}{\partial \mathbf{x}^{(l)} \partial \mathbf{x}^{(i)}}, \quad (4)$$

$$G_0^{(i)} = \sum_{j=1}^n B^{(ji)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(j)}}, \quad i = 1, \dots, m, \quad (5)$$

where $a^{(i)}(\mathbf{x}, t)$ is the i th component of the vector function $a(\mathbf{x}, t)$ and $B^{(ij)}(\mathbf{x}, t)$ is the ij th component of the matrix function $B(\mathbf{x}, t)$.

Consider the following sequence of differential operators

$$G_p^{(i)} = \frac{1}{p} \left(G_{p-1}^{(i)} L - L G_{p-1}^{(i)} \right), \quad p = 1, 2, \dots, \quad i = 1, \dots, m,$$

where L and $G_0^{(i)}$, $i = 1, \dots, m$ are defined by (4), (5).

For the further consideration, we need to introduce the following set of iterated Itô stochastic integrals

$$I_{(l_1 \dots l_k) s, t}^{(i_1 \dots i_k)} = \int_t^s (t - t_k)^{l_k} \dots \int_t^{t_2} (t - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (6)$$

where $l_1, \dots, l_k = 0, 1, \dots$ and $i_1, \dots, i_k = 1, \dots, m$.

Assume that $R(\mathbf{x}, t)$, $\mathbf{a}(\mathbf{x}, t)$, and $B_i(\mathbf{x}, t)$, $i = 1, \dots, m$ are enough smooth functions with respect to the variables \mathbf{x} and t . Then for all $s, t \in [0, T]$ such that $s > t$ we can write the following unified Taylor–Itô expansion [24] (also see [26], Chapter 4)

$$\begin{aligned} & R(\mathbf{x}_s, s) = \\ & = R(\mathbf{x}_t, t) + \sum_{q=1}^r \sum_{(k,j,l_1,\dots,l_k) \in D_q} \frac{(s-t)^j}{j!} \sum_{i_1,\dots,i_k=1}^m G_{l_1}^{(i_1)} \dots G_{l_k}^{(i_k)} L^j R(\mathbf{x}_t, t) I_{(l_1 \dots l_k)_{s,t}}^{(i_1 \dots i_k)} + \\ & \quad + (H_{r+1})_{s,t} \quad \text{w. p. 1,} \end{aligned} \quad (7)$$

where

$$L^j R(\mathbf{x}, t) \stackrel{\text{def}}{=} \begin{cases} \underbrace{L \dots L}_j R(\mathbf{x}, t) & \text{for } j \geq 1 \\ R(\mathbf{x}, t) & \text{for } j = 0 \end{cases},$$

$$D_q = \left\{ (k, j, l_1, \dots, l_k) : k + 2 \left(j + \sum_{p=1}^k l_p \right) = q; k, j, l_1, \dots, l_k = 0, 1, \dots \right\}, \quad (8)$$

and $(H_{r+1})_{s,t}$ is the remainder term in integral form [26].

Consider the partition $\{\tau_p\}_{p=0}^N$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \tau_1 < \dots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} |\tau_{j+1} - \tau_j|. \quad (9)$$

Let $\mathbf{y}_{\tau_j} \stackrel{\text{def}}{=} \mathbf{y}_j$, $j = 0, 1, \dots, N$ be a time discrete approximation of the process \mathbf{x}_t , $t \in [0, T]$, which is a solution of the Itô SDE (I).

Definiton 1 [2]. We will say that a time discrete approximation \mathbf{y}_j , $j = 0, 1, \dots, N$, corresponding to the maximal step of discretization Δ_N , converges strongly with order $\gamma > 0$ at time moment T to the process \mathbf{x}_t , $t \in [0, T]$, if there exists a constant $C > 0$, which does not depend on Δ_N , and a $\delta > 0$ such that $\mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T|\} \leq C(\Delta_N)^\gamma$ for each $\Delta_N \in (0, \delta)$.

From (7) for $s = \tau_{p+1}$ and $t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (I)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^r \sum_{(k,j,l_1,\dots,l_k) \in D_q} \frac{(\tau_{p+1} - \tau_p)^j}{j!} \sum_{i_1,\dots,i_k=1}^m G_{l_1}^{(i_1)} \dots G_{l_k}^{(i_k)} L^j \mathbf{y}_p \hat{I}_{(l_1 \dots l_k)_{\tau_{p+1}, \tau_p}}^{(i_1 \dots i_k)} +$$

$$+ \mathbf{1}_{\{r=2d-1, d \in \mathbb{N}\}} \frac{(\tau_{p+1} - \tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \quad (10)$$

where $\hat{I}_{(l_1 \dots l_k) \tau_{p+1}, \tau_p}^{(i_1 \dots i_k)}$ is an approximation of the iterated Itô stochastic integral (6) and $\mathbf{1}_A$ is the indicator of the set A . Note that we understand the equality (10) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column \mathbf{y}_p . Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $p = 0, 1, \dots, N$.

Under the appropriate conditions [2] the numerical scheme (10) has strong order $r/2$ ($r \in \mathbb{N}$) of convergence.

Below we consider particular cases of the numerical scheme (10) for $r = 1, 2, 3, 4, 5$, and 6, i.e. explicit one-step strong numerical schemes with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for the Itô SDE (1) [26], [64], [65]. At that for simplicity we will write \mathbf{a} , $L\mathbf{a}$, B_i , $G_0^{(i)} B_j$ etc. instead of $\mathbf{a}(\mathbf{y}_p, \tau_p)$, $L\mathbf{a}(\mathbf{y}_p, \tau_p)$, $B_i(\mathbf{y}_p, \tau_p)$, $G_0^{(i)} B_j(\mathbf{y}_p, \tau_p)$ etc. correspondingly. Moreover, the operators L and $G_0^{(i)}$, $i = 1, \dots, m$ are defined by (4), (5).

Scheme with strong order 0.5 (Euler scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0) \tau_{p+1}, \tau_p}^{(i_1)} + \Delta \mathbf{a}. \quad (11)$$

Scheme with strong order 1.0 (Milstein scheme)

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0) \tau_{p+1}, \tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00) \tau_{p+1}, \tau_p}^{(i_1 i_2)}. \quad (12)$$

Scheme with strong order 1.5

$$\begin{aligned} \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0) \tau_{p+1}, \tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00) \tau_{p+1}, \tau_p}^{(i_1 i_2)} + \\ & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{(0) \tau_{p+1}, \tau_p}^{(i_1)} + \hat{I}_{(1) \tau_{p+1}, \tau_p}^{(i_1)} \right) - L B_{i_1} \hat{I}_{(1) \tau_{p+1}, \tau_p}^{(i_1)} \right] + \\ & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000) \tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2} L \mathbf{a}. \end{aligned} \quad (13)$$

Scheme with strong order 2.0

$$\begin{aligned}
 \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} \right) - L B_{i_1} \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2} L \mathbf{a} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} L B_{i_2} \left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) - L G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \right. \\
 & \left. + G_0^{(i_1)} G_0^{(i_2)} \mathbf{a} \left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)}. \tag{14}
 \end{aligned}$$

Scheme with strong order 2.5

$$\begin{aligned}
 \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} \right) - L B_{i_1} \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2} L \mathbf{a} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} L B_{i_2} \left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) - L G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \right. \\
 & \left. + G_0^{(i_1)} G_0^{(i_2)} \mathbf{a} \left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4)} +
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \mathbf{La} \left(\frac{1}{2} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{(i_1)} + \frac{\Delta^2}{2} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{(i_1)} \right) + \right. \\
 & \left. + \frac{1}{2} L L B_{i_1} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{(i_1)} - L G_0^{(i_1)} \mathbf{a} \left(\hat{I}_{(2)\tau_{p+1}, \tau_p}^{(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{(i_1)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m \left[G_0^{(i_1)} L G_0^{(i_2)} B_{i_3} \left(\hat{I}_{(100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} L B_{i_3} \left(\hat{I}_{(010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} \right) + \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} \mathbf{a} \left(\Delta \hat{I}_{(000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} \right) - \\
 & \quad \left. - L G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4, i_5=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} B_{i_5} \hat{I}_{(00000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4 i_5)} + \\
 & \quad + \frac{\Delta^3}{6} L L \mathbf{a}. \tag{15}
 \end{aligned}$$

Scheme with strong order 3.0

$$\begin{aligned}
 \mathbf{y}_{p+1} & = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{(i_1)} + \Delta \mathbf{a} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1}, \tau_p}^{(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \mathbf{a} \left(\Delta \hat{I}_{(0)\tau_{p+1}, \tau_p}^{(i_1)} + \hat{I}_{(1)\tau_{p+1}, \tau_p}^{(i_1)} \right) - L B_{i_1} \hat{I}_{(1)\tau_{p+1}, \tau_p}^{(i_1)} \right] + \\
 & \quad + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)} + \frac{\Delta^2}{2} L \mathbf{a} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} L B_{i_2} \left(\hat{I}_{(10)\tau_{p+1}, \tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1}, \tau_p}^{(i_1 i_2)} \right) - L G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1}, \tau_p}^{(i_1 i_2)} \right. \\
 & \quad \left. + G_0^{(i_1)} G_0^{(i_2)} \mathbf{a} \left(\hat{I}_{(01)\tau_{p+1}, \tau_p}^{(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1}, \tau_p}^{(i_1 i_2)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} + \mathbf{q}_{p+1, p} + \mathbf{r}_{p+1, p}, \tag{16}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{q}_{p+1,p} = & \sum_{i_1=1}^m \left[G_0^{(i_1)} L \mathbf{a} \left(\frac{1}{2} \hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} + \frac{\Delta^2}{2} \hat{I}_{(0)\tau_{p+1},\tau_p}^{(i_1)} \right) + \right. \\
 & \left. + \frac{1}{2} L L B_{i_1} \hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} - L G_0^{(i_1)} \mathbf{a} \left(\hat{I}_{(2)\tau_{p+1},\tau_p}^{(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1},\tau_p}^{(i_1)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m \left[G_0^{(i_1)} L G_0^{(i_2)} B_{i_3} \left(\hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} L B_{i_3} \left(\hat{I}_{(010)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} \right) + \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} \mathbf{a} \left(\Delta \hat{I}_{(000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} \right) - \\
 & \quad \left. - L G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(100)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3)} \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4, i_5=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} B_{i_5} \hat{I}_{(00000)\tau_{p+1},\tau_p}^{(i_1 i_2 i_3 i_4 i_5)} + \\
 & \quad + \frac{\Delta^3}{6} L L \mathbf{a},
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbf{r}_{p+1,p} = & \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} G_0^{(i_2)} L \mathbf{a} \left(\frac{1}{2} \hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \frac{\Delta^2}{2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) + \right. \\
 & \quad + \frac{1}{2} L L G_0^{(i_1)} B_{i_2} \hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \\
 & \quad + G_0^{(i_1)} L G_0^{(i_2)} \mathbf{a} \left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \Delta \left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) \right) + \\
 & \quad + L G_0^{(i_1)} L B_{i_2} \left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) + \\
 & \quad + G_0^{(i_1)} L L B_{i_2} \left(\frac{1}{2} \hat{I}_{(02)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \frac{1}{2} \hat{I}_{(20)\tau_{p+1},\tau_p}^{(i_1 i_2)} - \hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) - \\
 & \quad \left. - L G_0^{(i_1)} G_0^{(i_2)} \mathbf{a} \left(\Delta \hat{I}_{(10)\tau_{p+1},\tau_p}^{(i_1 i_2)} + \hat{I}_{(11)\tau_{p+1},\tau_p}^{(i_1 i_2)} \right) \right] +
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i_1, i_2, i_3, i_4=1}^m \left[G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} \mathbf{a} \left(\Delta \hat{I}_{(0000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} + \hat{I}_{(0001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} L G_0^{(i_3)} B_{i_4} \left(\hat{I}_{(0100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} - \hat{I}_{(0010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} \right) - \\
 & \quad - L G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(1000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} + \\
 & \quad + G_0^{(i_1)} L G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \left(\hat{I}_{(1000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} - \hat{I}_{(0100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} \right) + \\
 & \quad \left. + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} L B_{i_4} \left(\hat{I}_{(0010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} - \hat{I}_{(0001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4, i_5, i_6=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} G_0^{(i_5)} B_{i_6} \hat{I}_{(000000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4 i_5 i_6)}.
 \end{aligned}$$

Under the suitable conditions [2] the numerical schemes (I12)–(I16) have strong orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence correspondingly. Among these conditions we consider only the condition for approximations of iterated Itô stochastic integrals from (I12)–(I16) [2] (also see [42])

$$\mathbb{M} \left\{ \left(I_{(l_1 \dots l_k)\tau_{p+1}, \tau_p}^{(i_1 \dots i_k)} - \hat{I}_{(l_1 \dots l_k)\tau_{p+1}, \tau_p}^{(i_1 \dots i_k)} \right)^2 \right\} \leq C \Delta^{r+1}, \tag{17}$$

where constant C is independent of Δ and $r/2$ are the strong convergence orders for the numerical schemes (I12)–(I16), i.e. $r/2 = 1.0, 1.5, 2.0, 2.5,$ and 3.0 .

Note that the numerical schemes (I12)–(I16) are unrealizable in practice without procedures for the numerical simulation of iterated Itô stochastic integrals from (I10). In Section 2.3 we give a brief overview of the effective method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$).

2.2 Strong Numerical Methods with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs Based on the Unified Taylor–Stratonovich Expansion

Let us consider the following differential operator on the space $C^{2,1}(\mathbb{R}^n \times [0, T])$

$$\bar{L} = L - \frac{1}{2} \sum_{i=1}^m G_0^{(i)} G_0^{(i)}, \quad (18)$$

where operators L and $G_0^{(i)}$, $i = 1, \dots, m$ are defined by (4), (5).

Define the following sequence of differential operators

$$\bar{G}_p^{(i)} \stackrel{\text{def}}{=} \frac{1}{p} \left(\bar{G}_{p-1}^{(i)} \bar{L} - \bar{L} \bar{G}_{p-1}^{(i)} \right), \quad p = 1, 2, \dots, \quad i = 1, \dots, m, \quad (19)$$

where $\bar{G}_0^{(i)} \stackrel{\text{def}}{=} G_0^{(i)}$, $i = 1, \dots, m$. The operators \bar{L} and $G_0^{(i)}$, $i = 1, \dots, m$ are defined by (18) and (5) correspondingly.

For the further consideration, we need to introduce the following set of iterated Stratonovich stochastic integrals

$$I_{(l_1 \dots l_k) s, t}^{*(i_1 \dots i_k)} = \int_t^{*s} (t - t_k)^{l_k} \dots \int_t^{*t_2} (t - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (20)$$

where $l_1, \dots, l_k = 0, 1, \dots$ and $i_1, \dots, i_k = 1, \dots, m$.

Assume that $R(\mathbf{x}, t)$, $\mathbf{a}(\mathbf{x}, t)$, and $B_i(\mathbf{x}, t)$, $i = 1, \dots, m$ are enough smooth functions with respect to the variables \mathbf{x} and t . Then for all $s, t \in [0, T]$ such that $s > t$ we can write the following unified Taylor–Stratonovich expansion [25] (also see [26], Chapter 4)

$$\begin{aligned} R(\mathbf{x}_s, s) &= \\ &= R(\mathbf{x}_t, t) + \sum_{q=1}^r \sum_{(k, j, l_1, \dots, l_k) \in D_q} \frac{(s-t)^j}{j!} \sum_{i_1, \dots, i_k=1}^m \bar{G}_{l_1}^{(i_1)} \dots \bar{G}_{l_k}^{(i_k)} \bar{L}^j R(\mathbf{x}_t, t) I_{(l_1 \dots l_k) s, t}^{*(i_1 \dots i_k)} + \\ &\quad + (\bar{H}_{r+1})_{s, t} \quad \text{w. p. 1,} \end{aligned} \quad (21)$$

where

$$\bar{L}^j R(\mathbf{x}, t) \stackrel{\text{def}}{=} \begin{cases} \underbrace{\bar{L} \dots \bar{L}}_j R(\mathbf{x}, t) & \text{for } j \geq 1 \\ R(\mathbf{x}, t) & \text{for } j = 0 \end{cases},$$

the set D_q is defined by the equality (8) and $(\bar{H}_{r+1})_{s, t}$ is the remainder term in integral form [25] (also see [26], Chapter 4).

Consider the partition (9) of the interval $[0, T]$. From (21) for $s = \tau_{p+1}$ and $t = \tau_p$ we obtain the following representation of explicit one-step strong numerical scheme for the Itô SDE (1)

$$\begin{aligned} \mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{q=1}^r \sum_{(k,j,l_1,\dots,l_k) \in \mathcal{D}_q} \frac{(\tau_{p+1} - \tau_p)^j}{j!} \sum_{i_1, \dots, i_k=1}^m \bar{G}_{l_1}^{(i_1)} \dots \bar{G}_{l_k}^{(i_k)} \bar{L}^j \mathbf{y}_p \hat{I}_{(l_1 \dots l_k)\tau_{p+1}, \tau_p}^{*(i_1 \dots i_k)} + \\ + \mathbf{1}_{\{r=2d-1, d \in \mathbb{N}\}} \frac{(\tau_{p+1} - \tau_p)^{(r+1)/2}}{((r+1)/2)!} L^{(r+1)/2} \mathbf{y}_p, \end{aligned} \quad (22)$$

where $\hat{I}_{(l_1 \dots l_k)\tau_{p+1}, \tau_p}^{*(i_1 \dots i_k)}$ is an approximation of the iterated Stratonovich stochastic integral (20) and $\mathbf{1}_A$ is the indicator of the set A . Note that we understand the equality (22) componentwise with respect to the components $\mathbf{y}_p^{(i)}$ of the column \mathbf{y}_p . Also for simplicity we put $\tau_p = p\Delta$, $\Delta = T/N$, $p = 0, 1, \dots, N$.

Under the appropriate conditions [2] the numerical scheme (22) has strong order $r/2$ ($r \in \mathbb{N}$) of convergence.

Denote

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t) - \frac{1}{2} \sum_{j=1}^m G_0^{(j)} B_j(\mathbf{x}, t),$$

where $B_j(\mathbf{x}, t)$ is the j th column of the matrix function $B(\mathbf{x}, t)$.

It is not difficult to show that (see (18))

$$\bar{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n \bar{\mathbf{a}}^{(i)}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}^{(i)}}, \quad (23)$$

where $\bar{\mathbf{a}}^{(i)}(\mathbf{x}, t)$ is the i th component of the vector function $\bar{\mathbf{a}}(\mathbf{x}, t)$.

Below we consider particular cases of the numerical scheme (22) for $r = 2, 3, 4, 5$, and 6 , i.e. explicit one-step strong numerical schemes with convergence orders $1.0, 1.5, 2.0, 2.5$, and 3.0 for the Itô SDE (1) [26], [66], [67]. At that for simplicity we will write $\bar{\mathbf{a}}, \bar{L}\bar{\mathbf{a}}, L\mathbf{a}, B_i, G_0^{(i)} B_j$ etc. instead of $\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p), \bar{L}\bar{\mathbf{a}}(\mathbf{y}_p, \tau_p), L\mathbf{a}(\mathbf{y}_p, \tau_p), B_i(\mathbf{y}_p, \tau_p), G_0^{(i)} B_j(\mathbf{y}_p, \tau_p)$ etc. correspondingly.

Scheme with strong order 1.0

$$\mathbf{y}_{p+1} = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \bar{\mathbf{a}} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1}, \tau_p}^{*(i_1 i_2)}. \quad (24)$$

Scheme with strong order 1.5

$$\begin{aligned}
 \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \Delta \bar{\mathbf{a}} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right) - \bar{L} B_{i_1} \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)} + \frac{\Delta^2}{2} L \mathbf{a}. \tag{25}
 \end{aligned}$$

Scheme with strong order 2.0

$$\begin{aligned}
 \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \Delta \bar{\mathbf{a}} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right) - \bar{L} B_{i_1} \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)} + \frac{\Delta^2}{2} \bar{L} \bar{\mathbf{a}} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} \bar{L} B_{i_2} \left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{*(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) - \bar{L} G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \right. \\
 & \left. + G_0^{(i_1)} G_0^{(i_2)} \bar{\mathbf{a}} \left(\hat{I}_{(01)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)}. \tag{26}
 \end{aligned}$$

Scheme with strong order 2.5

$$\begin{aligned}
 \mathbf{y}_{p+1} = & \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \Delta \bar{\mathbf{a}} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(0)\tau_{p+1},\tau_p}^{*(i_1)} + \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right) - \bar{L} B_{i_1} \hat{I}_{(1)\tau_{p+1},\tau_p}^{*(i_1)} \right] +
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} + \frac{\Delta^2}{2} \bar{L} \bar{\mathbf{a}} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} \bar{L} B_{i_2} \left(\hat{I}_{(10)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} \right) - \bar{L} G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \right. \\
 & \quad \left. + G_0^{(i_1)} G_0^{(i_2)} \bar{\mathbf{a}} \left(\hat{I}_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} \right) \right] + \\
 & \quad + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{L} \bar{\mathbf{a}} \left(\frac{1}{2} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} + \frac{\Delta^2}{2} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{*(i_1)} \right) + \right. \\
 & \quad \left. + \frac{1}{2} \bar{L} \bar{L} B_{i_1} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} - \bar{L} G_0^{(i_1)} \bar{\mathbf{a}} \left(\hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m \left[G_0^{(i_1)} \bar{L} G_0^{(i_2)} B_{i_3} \left(\hat{I}_{(100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} \bar{L} B_{i_3} \left(\hat{I}_{(010)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) + \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) - \\
 & \quad \left. - \bar{L} G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4, i_5=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} B_{i_5} \hat{I}_{(00000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4 i_5)} + \\
 & \quad + \frac{\Delta^3}{6} L L \mathbf{a}. \tag{27}
 \end{aligned}$$

Scheme with strong order 3.0

$$\begin{aligned}
 \mathbf{y}_{p+1} & = \mathbf{y}_p + \sum_{i_1=1}^m B_{i_1} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \bar{\mathbf{a}} + \sum_{i_1, i_2=1}^m G_0^{(i_1)} B_{i_2} \hat{I}_{(00)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \\
 & + \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(0)\tau_{p+1}, \tau_p}^{*(i_1)} + \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} \right) - \bar{L} B_{i_1} \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} \right] +
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i_1, i_2, i_3=1}^m G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} + \frac{\Delta^2}{2} \bar{L} \bar{\mathbf{a}} + \\
 & + \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} \bar{L} B_{i_2} \left(\hat{I}_{(10)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} \right) - \bar{L} G_0^{(i_1)} B_{i_2} \hat{I}_{(10)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \right. \\
 & \quad \left. + G_0^{(i_1)} G_0^{(i_2)} \bar{\mathbf{a}} \left(\hat{I}_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \Delta \hat{I}_{(00)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(0000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)} + \mathbf{q}_{p+1, p} + \mathbf{r}_{p+1, p}, \tag{28}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{q}_{p+1, p} = & \sum_{i_1=1}^m \left[G_0^{(i_1)} \bar{L} \bar{\mathbf{a}} \left(\frac{1}{2} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} + \frac{\Delta^2}{2} \hat{I}_{(0)\tau_{p+1}, \tau_p}^{*(i_1)} \right) + \right. \\
 & \left. + \frac{1}{2} \bar{L} \bar{L} B_{i_1} \hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} - \bar{L} G_0^{(i_1)} \bar{\mathbf{a}} \left(\hat{I}_{(2)\tau_{p+1}, \tau_p}^{*(i_1)} + \Delta \hat{I}_{(1)\tau_{p+1}, \tau_p}^{*(i_1)} \right) \right] + \\
 & + \sum_{i_1, i_2, i_3=1}^m \left[G_0^{(i_1)} \bar{L} G_0^{(i_2)} B_{i_3} \left(\hat{I}_{(100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} - \hat{I}_{(010)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} \bar{L} B_{i_3} \left(\hat{I}_{(010)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} - \hat{I}_{(001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) + \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} + \hat{I}_{(001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right) - \\
 & \quad \left. - \bar{L} G_0^{(i_1)} G_0^{(i_2)} B_{i_3} \hat{I}_{(100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)} \right] + \\
 & + \sum_{i_1, i_2, i_3, i_4, i_5=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} B_{i_5} \hat{I}_{(00000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4 i_5)} + \\
 & \quad + \frac{\Delta^3}{6} \bar{L} \bar{L} \bar{\mathbf{a}},
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbf{r}_{p+1, p} = & \sum_{i_1, i_2=1}^m \left[G_0^{(i_1)} G_0^{(i_2)} \bar{L} \bar{\mathbf{a}} \left(\frac{1}{2} \hat{I}_{(02)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \Delta \hat{I}_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \frac{\Delta^2}{2} \hat{I}_{(00)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} \right) + \right. \\
 & \left. + \frac{1}{2} \bar{L} \bar{L} G_0^{(i_1)} B_{i_2} \hat{I}_{(20)\tau_{p+1}, \tau_p}^{*(i_1 i_2)} + \right.
 \end{aligned}$$

$$\begin{aligned}
 & +G_0^{(i_1)} \bar{L}G_0^{(i_2)} \bar{\mathbf{a}} \left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{*(i_1 i_2)} - \hat{I}_{(02)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \Delta \left(\hat{I}_{(10)\tau_{p+1},\tau_p}^{*(i_1 i_2)} - \hat{I}_{(01)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) \right) + \\
 & \quad + \bar{L}G_0^{(i_1)} \bar{L}B_{i_2} \left(\hat{I}_{(11)\tau_{p+1},\tau_p}^{*(i_1 i_2)} - \hat{I}_{(20)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) + \\
 & \quad + G_0^{(i_1)} \bar{L}\bar{L}B_{i_2} \left(\frac{1}{2}\hat{I}_{(02)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \frac{1}{2}\hat{I}_{(20)\tau_{p+1},\tau_p}^{*(i_1 i_2)} - \hat{I}_{(11)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) - \\
 & \quad \left. - \bar{L}G_0^{(i_1)} G_0^{(i_2)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(10)\tau_{p+1},\tau_p}^{*(i_1 i_2)} + \hat{I}_{(11)\tau_{p+1},\tau_p}^{*(i_1 i_2)} \right) \right] + \\
 & \quad + \sum_{i_1, i_2, i_3, i_4=1}^m \left[G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} \bar{\mathbf{a}} \left(\Delta \hat{I}_{(0000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} + \hat{I}_{(0001)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} \right) + \right. \\
 & \quad + G_0^{(i_1)} G_0^{(i_2)} \bar{L}G_0^{(i_3)} B_{i_4} \left(\hat{I}_{(0100)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} - \hat{I}_{(0010)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} \right) - \\
 & \quad \left. - \bar{L}G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \hat{I}_{(1000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} + \right. \\
 & \quad + G_0^{(i_1)} \bar{L}G_0^{(i_2)} G_0^{(i_3)} B_{i_4} \left(\hat{I}_{(1000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} - \hat{I}_{(0100)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} \right) + \\
 & \quad \left. + G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} \bar{L}B_{i_4} \left(\hat{I}_{(0010)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} - \hat{I}_{(0001)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4)} \right) \right] + \\
 & \quad + \sum_{i_1, i_2, i_3, i_4, i_5, i_6=1}^m G_0^{(i_1)} G_0^{(i_2)} G_0^{(i_3)} G_0^{(i_4)} G_0^{(i_5)} B_{i_6} \hat{I}_{(000000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3 i_4 i_5 i_6)}.
 \end{aligned}$$

Under the suitable conditions [2] the numerical schemes (24)–(28) have strong orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence correspondingly. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from (24)–(28) [2] (also see [42])

$$\mathbf{M} \left\{ \left(I_{(l_1 \dots l_k)\tau_{p+1},\tau_p}^{*(i_1 \dots i_k)} - \hat{I}_{(l_1 \dots l_k)\tau_{p+1},\tau_p}^{*(i_1 \dots i_k)} \right)^2 \right\} \leq C \Delta^{r+1}, \quad (29)$$

where constant C is independent of Δ and $r/2$ are the strong convergence orders for the numerical schemes (24)–(28), i.e. $r/2 = 1.0, 1.5, 2.0, 2.5,$ and 3.0 .

Note that the numerical schemes (24)–(28) are unrealizable in practice without procedures for the numerical simulation of iterated Stratonovich stochastic

integrals from [22]. The next section is devoted to the effective method of the mean-square approximation of iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k ($k \in \mathbb{N}$).

2.3 Method of Expansion and Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series

Let us consider the effective approach to expansion of iterated Itô stochastic integrals [42] (2006) (also see [26], [44]-[46], [58], [61], [67], [68]). This method is referred to as the method of generalized multiple Fourier series.

Suppose that every $\psi_l(\tau)$ ($l = 1, \dots, k$) is a continuous nonrandom function on $[t, T]$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \dots, t_k) = \begin{cases} \psi_1(t_1) \dots \psi_k(t_k) & \text{for } t_1 < \dots < t_k \\ 0 & \text{otherwise} \end{cases}, \quad t_1, \dots, t_k \in [t, T], \tag{30}$$

where $k \geq 2$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t, T])$.

The function $K(t_1, \dots, t_k)$ is piecewise continuous in the hypercube $[t, T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \dots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \dots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \dots, p_k \rightarrow \infty} \left\| K(t_1, \dots, t_k) - \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right\|_{L_2([t, T]^k)} = 0,$$

where

$$C_{j_k \dots j_1} = \int_{[t, T]^k} K(t_1, \dots, t_k) \prod_{l=1}^k \phi_{j_l}(t_l) dt_1 \dots dt_k \tag{31}$$

is the Fourier coefficient and

$$\|f\|_{L_2([t, T]^k)} = \left(\int_{[t, T]^k} f^2(t_1, \dots, t_k) dt_1 \dots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t, T]$ such that

$$t = \tau_0 < \dots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta\tau_j \rightarrow 0 \text{ if } N \rightarrow \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \quad (32)$$

Theorem 1 [42] (2006) (also see [26], [44]–[46], [58], [61], [67], [68]). *Suppose that every $\psi_l(\tau)$ ($l = 1, \dots, k$) is a continuous nonrandom function on $[t, T]$ and $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of continuous functions in the space $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_k \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \left(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} - \text{l.i.m.}_{N \rightarrow \infty} \sum_{(l_1, \dots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \dots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right), \quad (33)$$

where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \dots, l_k) : l_1, \dots, l_k = 0, 1, \dots, N-1\},$$

$$L_k = \{(l_1, \dots, l_k) : l_1, \dots, l_k = 0, 1, \dots, N-1; l_g \neq l_r \ (g \neq r); g, r = 1, \dots, k\},$$

l.i.m. is a limit in the mean-square sense, $i_1, \dots, i_k = 0, 1, \dots, m$,

$$\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \quad (34)$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $C_{j_k \dots j_1}$ is the Fourier coefficient (31), $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$ ($i = 0, 1, \dots, m$), $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition (32).

Note that a number of modifications and generalizations of Theorem 1 can be found in [26], [67].

Consider transformed particular cases of (33) for $k = 1, \dots, 6$ [26], [58], [61], [67], [68]

$$J[\psi^{(1)}]_{T,t} = \text{l.i.m.}_{p_1 \rightarrow \infty} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \quad (35)$$

$$J[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1, p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \right), \quad (36)$$

$$J[\psi^{(3)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_3 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \quad (37)$$

$$J[\psi^{(4)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_4 \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_4=0}^{p_4} C_{j_4 \dots j_1} \left(\prod_{l=1}^4 \zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \right), \quad (38)$$

$$J[\psi^{(5)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_5 \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_5=0}^{p_5} C_{j_5 \dots j_1} \left(\prod_{l=1}^5 \zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_3}^{(i_3)} +$$

$$\begin{aligned}
 & + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} + \\
 & \left. + \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \right), \quad (39)
 \end{aligned}$$

$$\begin{aligned}
 J[\psi^{(6)}]_{T,t} = & \text{l.i.m.}_{p_1, \dots, p_6 \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_6=0}^{p_6} C_{j_6 \dots j_1} \left(\prod_{l=1}^6 \zeta_{j_l}^{(i_l)} - \right. \\
 & - \mathbf{1}_{\{i_1=i_6 \neq 0\}} \mathbf{1}_{\{j_1=j_6\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_6 \neq 0\}} \mathbf{1}_{\{j_2=j_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \\
 & - \mathbf{1}_{\{i_3=i_6 \neq 0\}} \mathbf{1}_{\{j_3=j_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_4=i_6 \neq 0\}} \mathbf{1}_{\{j_4=j_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} - \\
 & - \mathbf{1}_{\{i_5=i_6 \neq 0\}} \mathbf{1}_{\{j_5=j_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_3}^{(i_3)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_3}^{(i_3)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_6}^{(i_6)} +
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_3}^{(i_3)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_6}^{(i_6)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} + \\
 & + \mathbf{1}_{\{i_6=i_1 \neq 0\}} \mathbf{1}_{\{j_6=j_1\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_2 \neq 0\}} \mathbf{1}_{\{j_6=j_2\}} \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_2=i_5 \neq 0\}} \mathbf{1}_{\{j_2=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_4=i_5 \neq 0\}} \mathbf{1}_{\{j_4=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_1=i_5 \neq 0\}} \mathbf{1}_{\{j_1=j_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_3 \neq 0\}} \mathbf{1}_{\{j_6=j_3\}} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} + \\
 & + \mathbf{1}_{\{i_6=i_4 \neq 0\}} \mathbf{1}_{\{j_6=j_4\}} \mathbf{1}_{\{i_3=i_5 \neq 0\}} \mathbf{1}_{\{j_3=j_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} +
 \end{aligned}$$

where $\mathbf{1}_A$ is the indicator of the set A .

A detailed discussion of advantages of the method based on Theorem 1 over the approximation methods from works [2], [3], [7], [8], [27]-[35], [37]-[39] can be found in [26] (Section 1.1.10) or in [67].

As it turned out, Theorem 1 can be adapted for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 5 [26], [67]-[69] (also see bibliography therein). Let us collect some of these results in the following theorem.

Theorem 2 [26], [67]-[69]. *Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. At the same time $\psi_2(s)$ is a continuously differentiable function on $[t, T]$ and $\psi_1(s)$, $\psi_3(s)$ are twice continuously differentiable functions on $[t, T]$. Then*

$$J^*[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1, p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \quad (i_1, i_2 = 1, \dots, m), \quad (41)$$

$$J^*[\psi^{(3)}]_{T,t} = \text{l.i.m.}_{p_1, p_2, p_3 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \quad (i_1, i_2, i_3 = 0, 1, \dots, m), \quad (42)$$

$$J^*[\psi^{(3)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2, j_3=0}^p C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \dots, m), \quad (43)$$

$$J^*[\psi^{(4)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, \dots, j_4=0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \quad (i_1, \dots, i_4 = 0, 1, \dots, m), \quad (44)$$

$$J^*[\psi^{(5)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, \dots, j_5=0}^p C_{j_5 j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \quad (i_1, \dots, i_5 = 0, 1, \dots, m), \quad (45)$$

where $J^*[\psi^{(k)}]_{T,t}$ is defined by (3), and $\psi_l(s) \equiv 1$ ($l = 1, \dots, 5$) in (42), (44), (45); another notations are the same as in Theorem 1.

Consider the following Hypothesis on expansion of the iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k ($k \in \mathbb{N}$).

Hypothesis 1 [26], [67], [68]. *Assume that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the*

space $L_2([t, T])$. Moreover, every $\psi_l(\tau)$ ($l = 1, \dots, k$) is an enough smooth non-random function on $[t, T]$. Then, for the iterated Stratonovich stochastic integral (3) of multiplicity k the following expansion

$$J^*[\psi^{(k)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1} \prod_{l=1}^k \zeta_{j_l}^{(i_l)} \tag{46}$$

that converges in the mean-square sense is valid, where notations are the same as in Theorem 1.

Hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral $J^*[\psi^{(k)}]_{T,t}$ by the sum

$$J^*[\psi^{(k)}]_{T,t}^p = \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1} \prod_{l=1}^k \zeta_{j_l}^{(i_l)}, \tag{47}$$

where

$$\lim_{p \rightarrow \infty} \mathbf{M} \left\{ \left(J^*[\psi^{(k)}]_{T,t} - J^*[\psi^{(k)}]_{T,t}^p \right)^2 \right\} = 0.$$

Assume that $J[\psi^{(k)}]_{T,t}^p$ is the approximation of (2), which is the expression in (33) before passing to the limit for the case $p_1 = \dots = p_k = p$, i.e.

$$J[\psi^{(k)}]_{T,t}^p = \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1} \left(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} - \text{l.i.m.}_{N \rightarrow \infty} \sum_{(l_1, \dots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \dots \phi_{j_k}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right). \tag{48}$$

Let us denote

$$\mathbf{M} \left\{ \left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^p \right)^2 \right\} \stackrel{\text{def}}{=} E_k^p,$$

$$\|K\|_{L_2([t, T]^k)}^2 = \int_{[t, T]^k} K^2(t_1, \dots, t_k) dt_1 \dots dt_k \stackrel{\text{def}}{=} I_k.$$

For the further consideration, we need the following useful estimate [26], [67]

$$E_k^p \leq k! \left(I_k - \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1}^2 \right), \tag{49}$$

where $i_1, \dots, i_k = 1, \dots, m$ for $T - t \in (0, \infty)$ and $i_1, \dots, i_k = 0, 1, \dots, m$ for $T - t \in (0, 1)$; another notations are the same as in Theorem 1.

The value E_k^p can be calculated exactly.

Theorem 3 [26], [67], [70]. *Suppose that the conditions of Theorem 1 are satisfied. Then*

$$E_k^p = I_k - \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1} \times \mathbb{M} \left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_1, \dots, j_k)} \int_t^T \phi_{j_k}(t_k) \dots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)} \right\}, \quad (50)$$

where $i_1, \dots, i_k = 1, \dots, m$; expression

$$\sum_{(j_1, \dots, j_k)}$$

means the sum with respect to all possible permutations (j_1, \dots, j_k) . At the same time if j_r swapped with j_q in the permutation (j_1, \dots, j_k) , then i_r swapped with i_q in the permutation (i_1, \dots, i_k) ; another notations are the same as in Theorem 1.

Note that

$$\mathbb{M} \left\{ J[\psi^{(k)}]_{T,t} \int_t^T \phi_{j_k}(t_k) \dots \int_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)} \right\} = C_{j_k \dots j_1}.$$

Then from Theorem 3 we obtain

$$E_k^p = I_k - \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1}^2 \quad (i_1, \dots, i_k \text{ are pairwise different}), \quad (51)$$

$$E_k^p = I_k - \sum_{j_1, \dots, j_k=0}^p C_{j_k \dots j_1} \left(\sum_{(j_1, \dots, j_k)} C_{j_k \dots j_1} \right) \quad (i_1 = \dots = i_k). \quad (52)$$

Consider some examples of the application of Theorem 3 ($i_1, \dots, i_5 = 1, \dots, m$):

$$E_2^p = I_2 - \sum_{j_1, j_2=0}^p C_{j_2 j_1}^2 - \sum_{j_1, j_2=0}^p C_{j_2 j_1} C_{j_1 j_2} \quad (i_1 = i_2), \quad (53)$$

$$E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3), \quad (54)$$

$$E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3), \quad (55)$$

$$E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1} C_{j_1 j_2 j_3} \quad (i_1 = i_3 \neq i_2), \quad (56)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_1, j_2)} C_{j_4 \dots j_1} \right) \quad (i_1 = i_2 \neq i_3, i_4; i_3 \neq i_4), \quad (57)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_1, j_3)} C_{j_4 \dots j_1} \right) \quad (i_1 = i_3 \neq i_2, i_4; i_2 \neq i_4), \quad (58)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_2, j_3)} C_{j_4 \dots j_1} \right) \quad (i_2 = i_3 \neq i_1, i_4; i_1 \neq i_4), \quad (59)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_1, j_4)} C_{j_4 \dots j_1} \right) \quad (i_1 = i_4 \neq i_2, i_3; i_2 \neq i_3), \quad (60)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_1, j_4)} \left(\sum_{(j_2, j_3)} C_{j_4 \dots j_1} \right) \right) \quad (i_1 = i_4 \neq i_2 = i_3), \quad (61)$$

$$E_4^p = I_4 - \sum_{j_1, \dots, j_4=0}^p C_{j_4 \dots j_1} \left(\sum_{(j_1, j_2, j_3)} C_{j_4 \dots j_1} \right) \quad (i_1 = i_2 = i_3 \neq i_4), \quad (62)$$

$$E_5^p = I_5 - \sum_{j_1, \dots, j_5=0}^p C_{j_5 \dots j_1} \left(\sum_{(j_1, j_2)} C_{j_5 \dots j_1} \right), \quad (63)$$

where $i_1 = i_2 \neq i_3, i_4, i_5$ and i_3, i_4, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1, \dots, j_5=0}^p C_{j_5 \dots j_1} \left(\sum_{(j_2, j_3)} C_{j_5 \dots j_1} \right), \quad (64)$$

where $i_2 = i_3 \neq i_1, i_4, i_5$ and i_1, i_4, i_5 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1, \dots, j_5=0}^p C_{j_5 \dots j_1} \left(\sum_{(j_4, j_5)} C_{j_5 \dots j_1} \right), \tag{65}$$

where $i_4 = i_5 \neq i_1, i_2, i_3$ and i_1, i_2, i_3 are pairwise different,

$$E_5^p = I_5 - \sum_{j_1, \dots, j_5=0}^p C_{j_5 \dots j_1} \left(\sum_{(j_2, j_4)} \left(\sum_{(j_3, j_5)} C_{j_5 \dots j_1} \right) \right) \quad (i_1 \neq i_2 = i_4 \neq i_3 = i_5 \neq i_1). \tag{66}$$

2.4 Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (11)–(16) Using Legendre Polynomials

This section is devoted to approximation of the iterated Itô stochastic integrals (6) of multiplicities 1 to 6 based on Theorem 1. At that we will use multiple Fourier–Legendre series for approximation of the mentioned stochastic integrals.

The numerical schemes (11)–(16) contain the following set (see (6)) of iterated Itô stochastic integrals

$$I_{(0)T,t}^{(i_1)}, \quad I_{(1)T,t}^{(i_1)}, \quad I_{(2)T,t}^{(i_1)}, \quad I_{(00)T,t}^{(i_1 i_2)}, \quad I_{(10)T,t}^{(i_1 i_2)}, \quad I_{(01)T,t}^{(i_1 i_2)}, \quad I_{(000)T,t}^{(i_1 i_2 i_3)}, \quad I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)} \tag{67}$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)}, \quad I_{(02)T,t}^{(i_1 i_2)}, \quad I_{(20)T,t}^{(i_1 i_2)}, \quad I_{(11)T,t}^{(i_1 i_2)}, \quad I_{(100)T,t}^{(i_1 i_2 i_3)}, \quad I_{(010)T,t}^{(i_1 i_2 i_3)}, \quad I_{(001)T,t}^{(i_1 i_2 i_3)} \tag{68}$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)}, \quad I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)}, \quad I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)}, \quad I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)}, \quad I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)}. \tag{69}$$

Let us consider the complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j \left(\left(x - \frac{T+t}{2} \right) \frac{2}{T-t} \right), \quad j = 0, 1, 2, \dots, \tag{70}$$

where $P_j(x)$ is the Legendre polynomial

$$P_j(x) = \frac{1}{2^j j!} \frac{d^j}{dx^j} (x^2 - 1)^j.$$

Using Theorem 1 and well known properties of the Legendre polynomials, we obtain the following formulas for numerical modeling of the stochastic integrals (67)–(69) [26], [42]–[46], [58], [61], [67], [68], [70]–[72]

$$I_{(0)T,t}^{(i_1)} = \sqrt{T-t} \zeta_0^{(i_1)}, \tag{71}$$

$$I_{(1)T,t}^{(i_1)} = -\frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right), \quad (72)$$

$$I_{(2)T,t}^{(i_1)} = \frac{(T-t)^{5/2}}{3} \left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2} \zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}} \zeta_2^{(i_1)} \right), \quad (73)$$

$$I_{(00)T,t}^{(i_1 i_2)q} = \frac{T-t}{2} \left(\zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{i=1}^q \frac{1}{\sqrt{4i^2-1}} \left(\zeta_{i-1}^{(i_1)} \zeta_i^{(i_2)} - \zeta_i^{(i_1)} \zeta_{i-1}^{(i_2)} \right) - \mathbf{1}_{\{i_1=i_2\}} \right), \quad (74)$$

$$I_{(000)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1, j_2, j_3=0}^{q_1} C_{j_3 j_2 j_1}^{000} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \right. \\ \left. - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \quad (75)$$

$$I_{(10)T,t}^{(i_1 i_2)q_2} = \sum_{j_1, j_2=0}^{q_2} C_{j_2 j_1}^{10} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right), \quad (76)$$

$$I_{(01)T,t}^{(i_1 i_2)q_2} = \sum_{j_1, j_2=0}^{q_2} C_{j_2 j_1}^{01} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right), \quad (77)$$

$$I_{(0000)T,t}^{(i_1 i_2 i_3 i_4)q_3} = \sum_{j_1, j_2, j_3, j_4=0}^{q_3} C_{j_4 j_3 j_2 j_1}^{0000} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \right. \\ - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\ - \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\ - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\ + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\ \left. + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \right), \quad (78)$$

$$I_{(00000)T,t}^{(i_1 i_2 i_3 i_4 i_5)q_4} = \sum_{j_1, j_2, j_3, j_4, j_5=0}^{q_4} C_{j_5 j_4 j_3 j_2 j_1}^{00000} \left(\prod_{l=1}^5 \zeta_{j_l}^{(i_l)} - \right. \\ \left. - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \right.$$

$$\begin{aligned}
 & -\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \\
 & -\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \\
 & -\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)} - \\
 & -\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} + \\
 & +\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)} + \\
 & +\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)} + \\
 & +\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)} + \\
 & +\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)} + \\
 & +\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)} + \\
 & +\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)} + \\
 & +\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)} + \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)} + \\
 & + \mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)} \Big), \tag{79}
 \end{aligned}$$

$$I_{(20)T,t}^{(i_1 i_2)q_5} = \sum_{j_1, j_2=0}^{q_5} C_{j_2 j_1}^{20} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right), \tag{80}$$

$$I_{(11)T,t}^{(i_1 i_2)q_6} = \sum_{j_1, j_2=0}^{q_6} C_{j_2 j_1}^{11} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right), \tag{81}$$

$$I_{(02)T,t}^{(i_1 i_2)q_7} = \sum_{j_1, j_2=0}^{q_7} C_{j_2 j_1}^{02} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \right), \tag{82}$$

$$\begin{aligned}
 I_{(001)T,t}^{(i_1 i_2 i_3)q_8} = & \sum_{j_1, j_2, j_3=0}^{q_8} C_{j_3 j_2 j_1}^{001} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \right. \\
 & \left. - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \tag{83}
 \end{aligned}$$

$$\begin{aligned}
 I_{(010)T,t}^{(i_1 i_2 i_3)q_9} = & \sum_{j_1, j_2, j_3=0}^{q_9} C_{j_3 j_2 j_1}^{010} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \right. \\
 & \left. - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \quad (84)
 \end{aligned}$$

$$\begin{aligned}
 I_{(100)T,t}^{(i_1 i_2 i_3)q_{10}} = & \sum_{j_1, j_2, j_3=0}^{q_{10}} C_{j_3 j_2 j_1}^{100} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \right. \\
 & \left. - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \quad (85)
 \end{aligned}$$

$$\begin{aligned}
 I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)q_{11}} = & \sum_{j_1, j_2, j_3, j_4=0}^{q_{11}} C_{j_4 j_3 j_2 j_1}^{0001} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \right. \\
 & - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\
 & \left. + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \right), \quad (86)
 \end{aligned}$$

$$\begin{aligned}
 I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)q_{12}} = & \sum_{j_1, j_2, j_3, j_4=0}^{q_{12}} C_{j_4 j_3 j_2 j_1}^{0010} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \right. \\
 & - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\
 & \left. + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \right), \quad (87)
 \end{aligned}$$

$$\begin{aligned}
 I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)q_{13}} = & \sum_{j_1, j_2, j_3, j_4=0}^{q_{13}} C_{j_4 j_3 j_2 j_1}^{0100} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \right. \\
 & - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\
 & \left. + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \right), \tag{88}
 \end{aligned}$$

$$\begin{aligned}
 I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)q_{14}} = & \sum_{j_1, j_2, j_3, j_4=0}^{q_{14}} C_{j_4 j_3 j_2 j_1}^{1000} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \right. \\
 & - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \\
 & - \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \\
 & + \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4\}} \mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4\}} \mathbf{1}_{\{j_2=j_4\}} + \\
 & \left. + \mathbf{1}_{\{i_1=i_4\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \right), \tag{89}
 \end{aligned}$$

$$\begin{aligned}
 I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)q_{15}} = & \sum_{j_1, j_2, j_3, j_4, j_5, j_6=0}^{q_{15}} C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000} \left(\prod_{l=1}^6 \zeta_{j_l}^{(i_l)} - \right. \\
 & - \mathbf{1}_{\{j_1=j_6\}} \mathbf{1}_{\{i_1=i_6\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_2=j_6\}} \mathbf{1}_{\{i_2=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \\
 & - \mathbf{1}_{\{j_3=j_6\}} \mathbf{1}_{\{i_3=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{j_4=j_6\}} \mathbf{1}_{\{i_4=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} - \\
 & - \mathbf{1}_{\{j_5=j_6\}} \mathbf{1}_{\{i_5=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_1=i_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_1=i_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_1=i_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{j_1=j_5\}} \mathbf{1}_{\{i_1=i_5\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_2=i_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \\
 & - \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{i_2=i_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)} - \mathbf{1}_{\{j_2=j_5\}} \mathbf{1}_{\{i_2=i_5\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_6}^{(i_6)} - \\
 & \left. - \mathbf{1}_{\{j_2=j_6\}} \mathbf{1}_{\{i_2=i_6\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & -\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}} - \\
 & -\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_6=i_1\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}} - \\
 & -\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_6=i_2\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}} - \\
 & -\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_6=i_3\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}} - \\
 & -\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_3=i_6\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_4=j_5\}}\mathbf{1}_{\{i_4=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}} - \\
 & -\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_2=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_6=i_4\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_5\}}\mathbf{1}_{\{i_3=i_5\}} - \\
 & -\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_2=i_3\}} - \\
 & -\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_3=j_4\}}\mathbf{1}_{\{i_3=i_4\}} - \\
 & -\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_6=i_5\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_2=i_4\}}
 \end{aligned}
 \tag{90}$$

where $\mathbf{1}_A$ is the indicator of the set A and

$$C_{j_3 j_2 j_1}^{000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8} (T-t)^{3/2} \bar{C}_{j_3 j_2 j_1}^{000}, \tag{91}$$

$$C_{j_2 j_1}^{01} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{01}, \tag{92}$$

$$C_{j_2 j_1}^{10} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{8} (T-t)^2 \bar{C}_{j_2 j_1}^{10}, \tag{93}$$

$$C_{j_4 j_3 j_2 j_1}^{0000} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}{16} (T-t)^2 \bar{C}_{j_4 j_3 j_2 j_1}^{0000}, \tag{94}$$

$$C_{j_2 j_1}^{02} = \frac{\sqrt{(2j_1+1)(2j_2+1)}}{16} (T-t)^3 \bar{C}_{j_2 j_1}^{02}, \tag{95}$$

$$C_{j_2 j_1}^{20} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)}}{16} (T - t)^3 \bar{C}_{j_2 j_1}^{20}, \quad (96)$$

$$C_{j_2 j_1}^{11} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)}}{16} (T - t)^3 \bar{C}_{j_2 j_1}^{11}, \quad (97)$$

$$C_{j_3 j_2 j_1}^{001} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} (T - t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{001}, \quad (98)$$

$$C_{j_3 j_2 j_1}^{010} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} (T - t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{010}, \quad (99)$$

$$C_{j_3 j_2 j_1}^{100} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} (T - t)^{5/2} \bar{C}_{j_3 j_2 j_1}^{100}, \quad (100)$$

$$C_{j_5 j_4 j_3 j_2 j_1}^{00000} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)}}{32} (T - t)^{5/2} \bar{C}_{j_5 j_4 j_3 j_2 j_1}^{00000}, \quad (101)$$

$$C_{j_4 j_3 j_2 j_1}^{0001} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} (T - t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0001}, \quad (102)$$

$$C_{j_3 j_2 j_1}^{0010} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} (T - t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{0010}, \quad (103)$$

$$C_{j_4 j_3 j_2 j_1}^{0100} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} (T - t)^3 \bar{C}_{j_3 j_2 j_1}^{0100}, \quad (104)$$

$$C_{j_4 j_3 j_2 j_1}^{1000} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} (T - t)^3 \bar{C}_{j_4 j_3 j_2 j_1}^{1000}, \quad (105)$$

$$\begin{aligned} & C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000} = \\ & = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)(2j_6 + 1)}}{64} (T - t)^3 \bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000}, \end{aligned} \quad (106)$$

where

$$\bar{C}_{j_3 j_2 j_1}^{000} = \int_{-1}^1 P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (107)$$

$$\bar{C}_{j_2 j_1}^{01} = - \int_{-1}^1 (1+y) P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy, \quad (108)$$

$$\bar{C}_{j_2 j_1}^{10} = - \int_{-1}^1 P_{j_2}(y) \int_{-1}^y (1+x) P_{j_1}(x) dx dy, \quad (109)$$

$$\bar{C}_{j_4 j_3 j_2 j_1}^{0000} = \int_{-1}^1 P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz du, \quad (110)$$

$$\bar{C}_{j_2 j_1}^{02} = \int_{-1}^1 P_{j_2}(y) (y+1)^2 \int_{-1}^y P_{j_1}(x) dx dy, \quad (111)$$

$$\bar{C}_{j_2 j_1}^{20} = \int_{-1}^1 P_{j_2}(y) \int_{-1}^y P_{j_1}(x) (x+1)^2 dx dy, \quad (112)$$

$$\bar{C}_{j_2 j_1}^{11} = \int_{-1}^1 P_{j_2}(y) (y+1) \int_{-1}^y P_{j_1}(x) (x+1) dx dy, \quad (113)$$

$$\bar{C}_{j_3 j_2 j_1}^{001} = - \int_{-1}^1 P_{j_3}(z) (z+1) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (114)$$

$$\bar{C}_{j_3 j_2 j_1}^{010} = - \int_{-1}^1 P_{j_3}(z) \int_{-1}^z P_{j_2}(y) (y+1) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (115)$$

$$\bar{C}_{j_3 j_2 j_1}^{100} = - \int_{-1}^1 P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) (x+1) dx dy dz, \quad (116)$$

$$\bar{C}_{j_5 j_4 j_3 j_2 j_1}^{00000} = \int_{-1}^1 P_{j_5}(v) \int_{-1}^v P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz du dv, \quad (117)$$

$$\bar{C}_{j_4 j_3 j_2 j_1}^{1000} = - \int_{-1}^1 P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x)(x+1) dx dy dz, \quad (118)$$

$$\bar{C}_{j_4 j_3 j_2 j_1}^{0100} = - \int_{-1}^1 P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y)(y+1) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (119)$$

$$\bar{C}_{j_4 j_3 j_2 j_1}^{0010} = - \int_{-1}^1 P_{j_4}(u) \int_{-1}^u P_{j_3}(z)(z+1) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (120)$$

$$\bar{C}_{j_4 j_3 j_2 j_1}^{0001} = - \int_{-1}^1 P_{j_4}(u)(u+1) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz, \quad (121)$$

$$\begin{aligned} & \bar{C}_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000} = \\ & = \int_{-1}^1 P_{j_6}(w) \int_{-1}^w P_{j_5}(v) \int_{-1}^v P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz dudv dw; \end{aligned} \quad (122)$$

another notations are the same as in Theorem 1.

2.5 Optimization of Approximations of Iterated Itô Stochastic Integrals from the Numerical Schemes (12)–(16)

This section is devoted to the optimization of approximations of iterated Itô stochastic integrals from the numerical schemes (12)–(16). More precisely, we discuss how to minimize the numbers $q, q_1, q_2, \dots, q_{15}$ from Section 2.4.

Suppose that $\varepsilon > 0$ is the mean-square accuracy of approximation of the iterated Itô stochastic integrals (6), i.e.

$$E_{(l_1 \dots l_k)T,t}^{(i_1 \dots i_k)p} \stackrel{\text{def}}{=} \mathbf{M} \left\{ \left(I_{(l_1 \dots l_k)T,t}^{(i_1 \dots i_k)} - I_{(l_1 \dots l_k)T,t}^{(i_1 \dots i_k)p} \right)^2 \right\} \leq \varepsilon,$$

where $I_{(l_1 \dots l_k)T,t}^{(i_1 \dots i_k)p}$, $p \in \mathbb{N}$ is the approximation of the iterated Itô stochastic integral $I_{(l_1 \dots l_k)T,t}^{(i_1 \dots i_k)}$. Then from (74) and (49) we obtain the following conditions

for choosing the numbers $q, q_1, q_2, \dots, q_{15}$ for approximations of the iterated Itô stochastic integrals (67)–(69) [26], [67]

$$E_{(00)T,t}^{(i_1 i_2)q} = \frac{(T-t)^2}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq \varepsilon, \quad (123)$$

$$E_{(000)T,t}^{(i_1 i_2 i_3)q_1} \leq 6 \left(\frac{(T-t)^3}{6} - \sum_{j_1, j_2, j_3=0}^{q_1} (C_{j_3 j_2 j_1}^{000})^2 \right) \leq \varepsilon, \quad (124)$$

$$E_{(01)T,t}^{(i_1 i_2)q_2} \leq 2 \left(\frac{(T-t)^4}{4} - \sum_{j_1, j_2=0}^{q_2} (C_{j_2 j_1}^{01})^2 \right) \leq \varepsilon, \quad (125)$$

$$E_{(10)T,t}^{(i_1 i_2)q_2} \leq 2 \left(\frac{(T-t)^4}{12} - \sum_{j_1, j_2=0}^{q_2} (C_{j_2 j_1}^{10})^2 \right) \leq \varepsilon, \quad (126)$$

$$E_{(0000)T,t}^{(i_1 \dots i_4)q_3} \leq 24 \left(\frac{(T-t)^4}{24} - \sum_{j_1, j_2, j_3, j_4=0}^{q_3} (C_{j_4 j_3 j_2 j_1}^{0000})^2 \right) \leq \varepsilon, \quad (127)$$

$$E_{(00000)T,t}^{(i_1 \dots i_5)q_4} \leq 120 \left(\frac{(T-t)^5}{120} - \sum_{j_1, j_2, j_3, j_4, j_5=0}^{q_4} (C_{j_5 j_4 j_3 j_2 j_1}^{00000})^2 \right) \leq \varepsilon, \quad (128)$$

$$E_{(20)T,t}^{(i_1 i_2)q_5} \leq 2 \left(\frac{(T-t)^6}{30} - \sum_{j_2, j_1=0}^{q_5} (C_{j_2 j_1}^{20})^2 \right) \leq \varepsilon, \quad (129)$$

$$E_{(11)T,t}^{(i_1 i_2)q_6} \leq 2 \left(\frac{(T-t)^6}{18} - \sum_{j_2, j_1=0}^{q_6} (C_{j_2 j_1}^{11})^2 \right) \leq \varepsilon, \quad (130)$$

$$E_{(02)T,t}^{(i_1 i_2)q_7} \leq 2 \left(\frac{(T-t)^6}{6} - \sum_{j_2, j_1=0}^{q_7} (C_{j_2 j_1}^{02})^2 \right) \leq \varepsilon, \quad (131)$$

$$E_{(001)T,t}^{(i_1 i_2 i_3)q_8} \leq 6 \left(\frac{(T-t)^5}{10} - \sum_{j_1, j_2, j_3=0}^{q_8} (C_{j_3 j_2 j_1}^{001})^2 \right) \leq \varepsilon, \quad (132)$$

$$E_{(010)T,t}^{(i_1 i_2 i_3)q_9} \leq 6 \left(\frac{(T-t)^5}{20} - \sum_{j_1, j_2, j_3=0}^{q_9} (C_{j_3 j_2 j_1}^{010})^2 \right) \leq \varepsilon, \quad (133)$$

$$E_{(100)T,t}^{(i_1 i_2 i_3)q_{10}} \leq 6 \left(\frac{(T-t)^5}{60} - \sum_{j_1, j_2, j_3=0}^{q_{10}} (C_{j_3 j_2 j_1}^{100})^2 \right) \leq \varepsilon, \quad (134)$$

$$E_{(0001)T,t}^{(i_1 \dots i_4)q_{11}} \leq 24 \left(\frac{(T-t)^6}{36} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{11}} (C_{j_4 j_3 j_2 j_1}^{0001})^2 \right) \leq \varepsilon, \quad (135)$$

$$E_{(0010)T,t}^{(i_1 \dots i_4)q_{12}} \leq 24 \left(\frac{(T-t)^6}{60} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{12}} (C_{j_4 j_3 j_2 j_1}^{0010})^2 \right) \leq \varepsilon, \quad (136)$$

$$E_{(0100)T,t}^{(i_1 \dots i_4)q_{13}} \leq 24 \left(\frac{(T-t)^6}{120} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{13}} (C_{j_4 j_3 j_2 j_1}^{0100})^2 \right) \leq \varepsilon, \quad (137)$$

$$E_{(1000)T,t}^{(i_1 \dots i_4)q_{14}} \leq 24 \left(\frac{(T-t)^6}{360} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{14}} (C_{j_4 j_3 j_2 j_1}^{1000})^2 \right) \leq \varepsilon, \quad (138)$$

$$E_{(000000)T,t}^{(i_1 \dots i_6)q_{15}} \leq 720 \left(\frac{(T-t)^6}{720} - \sum_{j_1, j_2, j_3, j_4, j_5, j_6=0}^{q_{15}} (C_{j_6 j_5 j_4 j_3 j_2 j_1})^2 \right) \leq \varepsilon. \quad (139)$$

Taking into account (17) and (91)–(122), (123)–(139), we obtain the following conditions for choosing the numbers $q, q_1, q_2, \dots, q_{15}$ for the numerical schemes (12)–(16) (constant C is independent of $T - t$ (see below)).

Milstein scheme (12)

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t).$$

Strong Taylor–Itô scheme (13) with convergence order 1.5

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^2,$$

$$6 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \right) \leq C(T - t). \quad (140)$$

Strong Taylor–Itô scheme (14) with convergence order 2.0

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^3,$$

$$6 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \right) \leq C(T-t)^2, \quad (141)$$

$$2 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \right) \leq C(T-t), \quad (142)$$

$$2 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \right) \leq C(T-t), \quad (143)$$

$$24 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \right) \leq \\ \leq C(T-t). \quad (144)$$

Strong Taylor–Itô scheme (15) with convergence order 2.5

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T-t)^4,$$

$$6 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \right) \leq C(T-t)^3, \quad (145)$$

$$2 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \right) \leq C(T-t)^2, \quad (146)$$

$$2 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \right) \leq C(T-t)^2, \quad (147)$$

$$24 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \right) \leq \\ \leq C(T-t)^2, \quad (148)$$

$$6 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{001})^2 \right) \leq \\ \leq C(T-t), \quad (149)$$

$$6 \left(\frac{1}{20} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_9} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{010})^2 \right) \leq C(T - t), \tag{150}$$

$$6 \left(\frac{1}{60} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_{10}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{100})^2 \right) \leq C(T - t), \tag{151}$$

$$120 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_5=0}^{q_4} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1) \times (\bar{C}_{j_5 \dots j_1}^{00000})^2 \right) \leq C(T - t). \tag{152}$$

Strong Taylor–Itô scheme (16) with convergence order 3.0

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^5,$$

$$6 \left(\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \right) \leq C(T - t)^4, \tag{153}$$

$$2 \left(\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \right) \leq C(T - t)^3, \tag{154}$$

$$2 \left(\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \right) \leq C(T - t)^3, \tag{155}$$

$$24 \left(\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \right) \leq C(T - t)^3, \tag{156}$$

$$6 \left(\frac{1}{10} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_8} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{001})^2 \right) \leq C(T - t)^2, \tag{157}$$

$$6 \left(\frac{1}{20} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_9} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{010})^2 \right) \leq \\ \leq C(T - t)^2, \quad (158)$$

$$6 \left(\frac{1}{60} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_{10}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{100})^2 \right) \leq \\ \leq C(T - t)^2, \quad (159)$$

$$120 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_5=0}^{q_4} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1) \times \right. \\ \left. \times (\bar{C}_{j_5 \dots j_1}^{00000})^2 \right) \leq C(T - t)^2, \quad (160)$$

$$2 \left(\frac{1}{30} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_5} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{20})^2 \right) \leq C(T - t), \quad (161)$$

$$2 \left(\frac{1}{18} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_6} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{11})^2 \right) \leq C(T - t), \quad (162)$$

$$2 \left(\frac{1}{6} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_7} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{02})^2 \right) \leq C(T - t), \quad (163)$$

$$24 \left(\frac{1}{36} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{11}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0001})^2 \right) \leq \\ \leq C(T - t), \quad (164)$$

$$24 \left(\frac{1}{60} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{12}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0010})^2 \right) \leq \\ \leq C(T - t), \quad (165)$$

$$24 \left(\frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{13}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0100})^2 \right) \leq \\ \leq C(T - t), \quad (166)$$

$$24 \left(\frac{1}{360} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{14}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{1000})^2 \right) \leq \leq C(T - t), \quad (167)$$

$$720 \left(\frac{1}{720} - \frac{1}{64^2} \sum_{j_1, \dots, j_6=0}^{q_{15}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)(2j_6 + 1) \times \times (\bar{C}_{j_6 \dots j_1}^{000000})^2 \right) \leq C(T - t). \quad (168)$$

Taking into account Theorem 3 and the results of Listings 5 and 6 (see Section 5) we decided to exclude the multiplier factors $k!$ from the left-hand sides of (140), (141)–(144), (145)–(152), (153)–(168). The detailed numerical confirmation of the mentioned possibility can be found in [73]. This means that we will use the following conditions for choosing the numbers $q, q_1, q_2, \dots, q_{15}$ for the numerical schemes (12)–(16) (constant C is independent of $T - t$ (see below)).

Milstein scheme (12)

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t). \quad (169)$$

Strong Taylor–Itô scheme (13) with convergence order 1.5

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^2, \quad (170)$$

$$\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \leq C(T - t). \quad (171)$$

Strong Taylor–Itô scheme (14) with convergence order 2.0

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^3, \quad (172)$$

$$\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \leq C(T - t)^2, \quad (173)$$

$$\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \leq C(T - t), \quad (174)$$

$$\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \leq C(T - t), \quad (175)$$

$$\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \leq C(T - t). \quad (176)$$

Strong Taylor–Itô scheme (15) with convergence order 2.5

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^4, \quad (177)$$

$$\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \leq C(T - t)^3, \quad (178)$$

$$\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \leq C(T - t)^2, \quad (179)$$

$$\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \leq C(T - t)^2, \quad (180)$$

$$\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \leq C(T - t)^2, \quad (181)$$

$$\frac{1}{10} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_8} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{001})^2 \leq C(T - t), \quad (182)$$

$$\frac{1}{20} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_9} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{010})^2 \leq C(T - t), \quad (183)$$

$$\frac{1}{60} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_{10}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{100})^2 \leq C(T - t), \quad (184)$$

$$\begin{aligned} \frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_5=0}^{q_4} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1) (\bar{C}_{j_5 \dots j_1}^{00000})^2 \leq \\ \leq C(T - t), \end{aligned} \quad (185)$$

Strong Taylor–Itô scheme (16) with convergence order 3.0

$$\frac{1}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right) \leq C(T - t)^5, \quad (186)$$

$$\frac{1}{6} - \frac{1}{64} \sum_{j_1, j_2, j_3=0}^{q_1} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{000})^2 \leq C(T - t)^4, \quad (187)$$

$$\frac{1}{4} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{01})^2 \leq C(T - t)^3, \quad (188)$$

$$\frac{1}{12} - \frac{1}{64} \sum_{j_1, j_2=0}^{q_2} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{10})^2 \leq C(T - t)^3, \quad (189)$$

$$\frac{1}{24} - \frac{1}{256} \sum_{j_1, \dots, j_4=0}^{q_3} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0000})^2 \leq C(T - t)^3, \quad (190)$$

$$\frac{1}{10} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_8} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{001})^2 \leq C(T - t)^2, \quad (191)$$

$$\frac{1}{20} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_9} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{010})^2 \leq C(T - t)^2, \quad (192)$$

$$\frac{1}{60} - \frac{1}{256} \sum_{j_1, j_2, j_3=0}^{q_{10}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) (\bar{C}_{j_3 j_2 j_1}^{100})^2 \leq C(T - t)^2, \quad (193)$$

$$\begin{aligned} \frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_5=0}^{q_4} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1) (\bar{C}_{j_5 \dots j_1}^{00000})^2 \leq \\ \leq C(T - t)^2, \end{aligned} \quad (194)$$

$$\frac{1}{30} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_5} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{20})^2 \leq C(T - t), \quad (195)$$

$$\frac{1}{18} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_6} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{11})^2 \leq C(T - t), \quad (196)$$

$$\frac{1}{6} - \frac{1}{256} \sum_{j_1, j_2=0}^{q_7} (2j_1 + 1)(2j_2 + 1) (\bar{C}_{j_2 j_1}^{02})^2 \leq C(T - t), \quad (197)$$

$$\frac{1}{36} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{11}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0001})^2 \leq C(T - t), \quad (198)$$

$$\frac{1}{60} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{12}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0010})^2 \leq C(T - t), \quad (199)$$

$$\frac{1}{120} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{13}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{0100})^2 \leq C(T - t), \quad (200)$$

$$\frac{1}{360} - \frac{1}{32^2} \sum_{j_1, \dots, j_4=0}^{q_{14}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) (\bar{C}_{j_4 \dots j_1}^{1000})^2 \leq C(T - t), \quad (201)$$

$$\begin{aligned} \frac{1}{720} - \frac{1}{64^2} \sum_{j_1, \dots, j_6=0}^{q_{15}} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)(2j_6 + 1) \times \\ \times (\bar{C}_{j_6 \dots j_1}^{000000})^2 \leq C(T - t). \end{aligned} \quad (202)$$

2.6 Approximations of Iterated Stratonovich Stochastic Integrals from the Numerical Schemes (24)–(28) Using Legendre Polynomials

This section is devoted to approximation of the Stratonovich stochastic integrals (20) of multiplicities 1 to 6 based on Theorem 2 and Hypothesis 1. At that we will use multiple Fourier–Legendre series for approximation of the mentioned stochastic integrals.

The numerical schemes (24)–(28) contain the following set (see (20)) of iterated Stratonovich stochastic integrals

$$I_{(0)T,t}^{*(i_1)}, \quad I_{(1)T,t}^{*(i_1)}, \quad I_{(2)T,t}^{*(i_1)}, \quad I_{(00)T,t}^{*(i_1 i_2)}, \quad I_{(10)T,t}^{*(i_1 i_2)}, \quad I_{(01)T,t}^{*(i_1 i_2)}, \quad I_{(000)T,t}^{*(i_1 i_2 i_3)}, \quad I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)}, \quad (203)$$

$$I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)}, \quad I_{(02)T,t}^{*(i_1 i_2)}, \quad I_{(20)T,t}^{*(i_1 i_2)}, \quad I_{(11)T,t}^{*(i_1 i_2)}, \quad I_{(100)T,t}^{*(i_1 i_2 i_3)}, \quad I_{(010)T,t}^{*(i_1 i_2 i_3)}, \quad I_{(001)T,t}^{*(i_1 i_2 i_3)}, \quad (204)$$

$$I_{(0001)T,t}^{*(i_1 i_2 i_3 i_4)}, \quad I_{(0010)T,t}^{*(i_1 i_2 i_3 i_4)}, \quad I_{(0100)T,t}^{*(i_1 i_2 i_3 i_4)}, \quad I_{(1000)T,t}^{*(i_1 i_2 i_3 i_4)}, \quad I_{(000000)T,t}^{*(i_1 i_2 i_3 i_4 i_5 i_6)}. \quad (205)$$

Using Theorem 2, Hypothesis 1, and well known properties of the Legendre polynomials, we obtain the following formulas for numerical modeling of the stochastic integrals (203)–(205) [26], [42]–[46], [58], [61], [67], [68], [70]–[72]

$$I_{(0)T,t}^{*(i_1)} = \sqrt{T - t} \zeta_0^{(i_1)}, \quad (206)$$

$$I_{(1)T,t}^{*(i_1)} = -\frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right), \quad (207)$$

$$I_{(2)T,t}^{*(i_1)} = \frac{(T-t)^{5/2}}{3} \left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2} \zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}} \zeta_2^{(i_1)} \right), \quad (208)$$

$$I_{(00)T,t}^{*(i_1 i_2)q} = \frac{T-t}{2} \left(\zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{i=1}^q \frac{1}{\sqrt{4i^2-1}} \left(\zeta_{i-1}^{(i_1)} \zeta_i^{(i_2)} - \zeta_i^{(i_1)} \zeta_{i-1}^{(i_2)} \right) \right), \quad (209)$$

$$I_{(000)T,t}^{*(i_1 i_2 i_3)q_1} = \sum_{j_1, j_2, j_3=0}^{q_1} C_{j_3 j_2 j_1}^{000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad (210)$$

$$I_{(10)T,t}^{*(i_1 i_2)q_2} = \sum_{j_1, j_2=0}^{q_2} C_{j_2 j_1}^{10} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad (211)$$

$$I_{(01)T,t}^{*(i_1 i_2)q_2} = \sum_{j_1, j_2=0}^{q_2} C_{j_2 j_1}^{01} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad (212)$$

$$I_{(0000)T,t}^{*(i_1 i_2 i_3 i_4)q_3} = \sum_{j_1, j_2, j_3, j_4=0}^{q_3} C_{j_4 j_3 j_2 j_1}^{0000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (213)$$

$$I_{(00000)T,t}^{*(i_1 i_2 i_3 i_4 i_5)q_4} = \sum_{j_1, j_2, j_3, j_4, j_5=0}^{q_4} C_{j_5 j_4 j_3 j_2 j_1}^{00000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)}, \quad (214)$$

$$I_{(20)T,t}^{*(i_1 i_2)q_5} = \sum_{j_1, j_2=0}^{q_5} C_{j_2 j_1}^{20} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad (215)$$

$$I_{(11)T,t}^{*(i_1 i_2)q_6} = \sum_{j_1, j_2=0}^{q_6} C_{j_2 j_1}^{11} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad (216)$$

$$I_{(02)T,t}^{*(i_1 i_2)q_7} = \sum_{j_1, j_2=0}^{q_7} C_{j_2 j_1}^{02} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad (217)$$

$$I_{(001)T,t}^{*(i_1 i_2 i_3)q_8} = \sum_{j_1, j_2, j_3=0}^{q_8} C_{j_3 j_2 j_1}^{001} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad (218)$$

$$I_{(010)T,t}^{*(i_1 i_2 i_3)q_9} = \sum_{j_1, j_2, j_3=0}^{q_9} C_{j_3 j_2 j_1}^{010} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad (219)$$

$$I_{(100)T,t}^{(i_1 i_2 i_3)q_{10}} = \sum_{j_1, j_2, j_3=0}^{q_{10}} C_{j_3 j_2 j_1}^{100} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad (220)$$

$$I_{(0001)T,t}^{(i_1 i_2 i_3 i_4)q_{11}} = \sum_{j_1, j_2, j_3, j_4=0}^{q_{11}} C_{j_4 j_3 j_2 j_1}^{0001} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (221)$$

$$I_{(0010)T,t}^{(i_1 i_2 i_3 i_4)q_{12}} = \sum_{j_1, j_2, j_3, j_4=0}^{q_{12}} C_{j_4 j_3 j_2 j_1}^{0010} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (222)$$

$$I_{(0100)T,t}^{(i_1 i_2 i_3 i_4)q_{13}} = \sum_{j_1, j_2, j_3, j_4=0}^{q_{13}} C_{j_4 j_3 j_2 j_1}^{0100} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (223)$$

$$I_{(1000)T,t}^{(i_1 i_2 i_3 i_4)q_{14}} = \sum_{j_1, j_2, j_3, j_4=0}^{q_{14}} C_{j_4 j_3 j_2 j_1}^{1000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (224)$$

$$I_{(000000)T,t}^{(i_1 i_2 i_3 i_4 i_5 i_6)q_{15}} = \sum_{j_1, j_2, j_3, j_4, j_5, j_6=0}^{q_{15}} C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \zeta_{j_5}^{(i_5)} \zeta_{j_6}^{(i_6)}, \quad (225)$$

where $\mathbf{1}_A$ is the indicator of the set A ; another notations are the same as in Section 2.4.

The question on choosing the numbers q_1, q_2, \dots, q_{15} in (210)–(225) turned out to be nontrivial [26] (Chapter 5). The expansions (210)–(225) for iterated Stratonovich stochastic integrals are simpler than their analogues (75)–(90) for iterated Itô stochastic integrals. However, the calculation of the mean-square approximation error for iterated Stratonovich stochastic integrals turns out to be much more difficult than for iterated Itô stochastic integrals [26] (Chapter 5). Below we give some reasoning regarding this problem.

Denote

$$E_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)p} \stackrel{\text{def}}{=} \mathbb{M} \left\{ \left(I_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)} - I_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)p} \right)^2 \right\},$$

where $I_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)p}$, $p \in \mathbb{N}$ is the approximation of the iterated Stratonovich stochastic integral $I_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)}$.

From (209) for $i_1 \neq i_2$ we obtain [26], [67]

$$E_{(00)T,t}^{*(i_1 i_2)q} = \frac{(T-t)^2}{2} \sum_{i=q+1}^{\infty} \frac{1}{4i^2-1} \leq \frac{(T-t)^2}{2} \int_q^{\infty} \frac{1}{4x^2-1} dx =$$

$$= -\frac{(T-t)^2}{8} \ln \left| 1 - \frac{2}{2q+1} \right| \leq C_1 \frac{(T-t)^2}{q}, \tag{226}$$

where constant C_1 is independent of q .

It is easy to notice that for a sufficiently small $T-t$ (recall that $T-t \ll 1$ since it is a step of integration for numerical schemes for Itô SDEs) there exists a constant C_2 such that

$$E_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)q} \leq C_2 E_{(00)T,t}^{*(i_1 i_2)q}. \tag{227}$$

From (226) and (227) we finally obtain

$$E_{(l_1 \dots l_k)T,t}^{*(i_1 \dots i_k)q} \leq C \frac{(T-t)^2}{q}, \tag{228}$$

where constant C does not depend on $T-t$. The same idea can be found in [2] in the framework of the method of approximation of iterated Stratonovich stochastic integrals based on the trigonometric expansion of the Brownian bridge process.

Obviously, we can get more information about the numbers q_1, q_2, \dots, q_{15} (these numbers are different for different iterated Stratonovich stochastic integrals) using the another approach. Since

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} \quad \text{w. p. 1}$$

for pairwise different $i_1, \dots, i_k = 1, \dots, m$, where $J[\psi^{(k)}]_{T,t}$, $J^*[\psi^{(k)}]_{T,t}$ are defined by (2) and (3) correspondingly, then for pairwise different $i_1, \dots, i_6 = 1, \dots, m$ from (51) we obtain [26], [67]

$$E_{(00)T,t}^{*(i_1 i_2)q} = \frac{(T-t)^2}{2} \left(\frac{1}{2} - \sum_{i=1}^q \frac{1}{4i^2 - 1} \right), \tag{229}$$

$$E_{(000)T,t}^{*(i_1 i_2 i_3)q_1} = \frac{(T-t)^3}{6} - \sum_{j_3, j_2, j_1=0}^{q_1} (C_{j_3 j_2 j_1}^{000})^2, \tag{230}$$

$$E_{(01)T,t}^{*(i_1 i_2)q_2} = \frac{(T-t)^4}{4} - \sum_{j_1, j_2=0}^{q_2} (C_{j_2 j_1}^{01})^2, \tag{231}$$

$$E_{(10)T,t}^{*(i_1 i_2)q_2} = \frac{(T-t)^4}{12} - \sum_{j_1, j_2=0}^{q_2} (C_{j_2 j_1}^{10})^2, \tag{232}$$

$$E_{(0000)T,t}^{*(i_1 \dots i_4)q_3} = \frac{(T-t)^4}{24} - \sum_{j_1, j_2, j_3, j_4=0}^{q_3} (C_{j_4 j_3 j_2 j_1}^{0000})^2, \quad (233)$$

$$E_{(00000)T,t}^{*(i_1 \dots i_5)q_4} = \frac{(T-t)^5}{120} - \sum_{j_1, j_2, j_3, j_4, j_5=0}^{q_4} (C_{j_5 i_4 i_3 i_2 j_1}^{00000})^2, \quad (234)$$

$$E_{(20)T,t}^{*(i_1 i_2)q_5} = \frac{(T-t)^6}{30} - \sum_{j_2, j_1=0}^{q_5} (C_{j_2 j_1}^{20})^2, \quad (235)$$

$$E_{(11)T,t}^{*(i_1 i_2)q_6} = \frac{(T-t)^6}{18} - \sum_{j_2, j_1=0}^{q_6} (C_{j_2 j_1}^{11})^2, \quad (236)$$

$$E_{(02)T,t}^{*(i_1 i_2)q_7} = \frac{(T-t)^6}{6} - \sum_{j_2, j_1=0}^{q_7} (C_{j_2 j_1}^{02})^2, \quad (237)$$

$$E_{(001)T,t}^{*(i_1 i_2 i_3)q_8} = \frac{(T-t)^5}{10} - \sum_{j_1, j_2, j_3=0}^{q_8} (C_{j_3 j_2 j_1}^{001})^2, \quad (238)$$

$$E_{(010)T,t}^{*(i_1 i_2 i_3)q_9} = \frac{(T-t)^5}{20} - \sum_{j_1, j_2, j_3=0}^{q_9} (C_{j_3 j_2 j_1}^{010})^2, \quad (239)$$

$$E_{(100)T,t}^{*(i_1 i_2 i_3)q_{10}} = \frac{(T-t)^5}{60} - \sum_{j_1, j_2, j_3=0}^{q_{10}} (C_{j_3 j_2 j_1}^{100})^2, \quad (240)$$

$$E_{(0001)T,t}^{*(i_1 \dots i_4)q_{11}} = \frac{(T-t)^6}{36} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{11}} (C_{j_4 j_3 j_2 j_1}^{0001})^2, \quad (241)$$

$$E_{(0010)T,t}^{*(i_1 \dots i_4)q_{12}} = \frac{(T-t)^6}{60} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{12}} (C_{j_4 j_3 j_2 j_1}^{0010})^2, \quad (242)$$

$$E_{(0100)T,t}^{*(i_1 \dots i_4)q_{13}} = \frac{(T-t)^6}{120} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{13}} (C_{j_4 j_3 j_2 j_1}^{0100})^2, \quad (243)$$

$$E_{(1000)T,t}^{*(i_1 \dots i_4)q_{14}} = \frac{(T-t)^6}{360} - \sum_{j_1, j_2, j_3, j_4=0}^{q_{14}} (C_{j_4 j_3 j_2 j_1}^{1000})^2, \quad (244)$$

$$E_{(000000)T,t}^{*(i_1 \dots i_6)q_{15}} = \frac{(T-t)^6}{720} - \sum_{j_1, j_2, j_3, j_4, j_5, j_6=0}^{q_{15}} (C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000})^2. \quad (245)$$

Table 1. High-order strong Taylor–Stratonovich schemes.

Order of convergence	Scheme	Conditions for choosing the numbers q, q_1, \dots, q_{15}
1.0	(24)	(169)
1.5	(25)	(170), (171)
2.0	(26)	(172)–(176)
2.5	(27)	(177)–(185)
3.0	(28)	(186)–(202)

Taking into account (229)–(245) and the results of paper [73], we use in the SDE-MATH software package the conditions from Table 1 for choosing the numbers $q, q_1, q_2, \dots, q_{15}$ for the numerical schemes (24)–(28).

Note that in the SDE-MATH software package, which is presented in the following sections, we use the following upper bounds b on the numbers q_1, \dots, q_{15}

$$b = 56 \quad \text{for } q_1, \quad b = 15 \quad \text{for } q_2, q_3, \quad b = 6 \quad \text{for } q_4, q_8, q_9, q_{10},$$

$$b = 2 \quad \text{for } q_5, q_6, q_7, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}.$$

This means that for the implementing of the numerical methods (13)–(16) and (25)–(28) we use in the SDE-MATH software package the following quantities of the exactly calculated Fourier–Legendre coefficients

$$57^3 = 185,193 \quad \text{for } C_{j_3 j_2 j_1}^{000},$$

$$16^3 = 4,096 \quad \text{for each of } C_{j_2 j_1}^{10}, C_{j_2 j_1}^{01},$$

$$16^4 = 65,536 \quad \text{for } C_{j_4 j_3 j_2 j_1}^{0000},$$

$$7^3 = 343 \quad \text{for each of } C_{j_3 j_2 j_1}^{100}, C_{j_3 j_2 j_1}^{010}, C_{j_3 j_2 j_1}^{001},$$

$$7^5 = 16,807 \quad \text{for } C_{j_5 j_4 j_3 j_2 j_1}^{00000},$$

$$3^2 = 9 \quad \text{for each of } C_{j_2 j_1}^{20}, C_{j_2 j_1}^{02}, C_{j_2 j_1}^{11},$$

$$3^4 = 81 \quad \text{for each of } C_{j_4 j_3 j_2 j_1}^{1000}, C_{j_4 j_3 j_2 j_1}^{0100}, C_{j_4 j_3 j_2 j_1}^{0010}, C_{j_4 j_3 j_2 j_1}^{0001},$$

$$3^6 = 729 \quad \text{for } C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000}.$$

It should be noted that unlike the method based on Theorems 1–3, existing and well-known approaches to the mean-square approximation of iterated stochastic integrals based on the trigonometric basis functions [2], [3], [7], [27],

[28], [34], [37] do not allow choosing theoretically different numbers q for approximations of different iterated stochastic integrals (starting from the multiplicity 2 of stochastic integrals). Moreover, the noted approaches [2], [3], [7], [27], [28], [34], [37] exclude the possibility for obtaining of approximate and exact expressions for the mean-square approximation error similar to the formulas (49), (50).

2.7 Numerical Algorithm for Linear Stationary Systems of Itô SDEs Based on Spectral Decomposition

Consider the following linear stationary system of Itô SDEs

$$d\mathbf{x}_t = (A\mathbf{x}_t + B\mathbf{u}(t)) dt + Fd\mathbf{w}_t, \quad \mathbf{x}_0 = \mathbf{x}(0), \quad t \in [0, T], \quad (246)$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is a solution of the system (246), $\mathbf{u}(t) : [0, T] \rightarrow \mathbb{R}^k$ is a non-random function, $A \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times k}$, and \mathbf{w}_t is a standard m -dimensional Wiener process with independent components $\mathbf{w}_t^{(i)}$, $i = 1, \dots, m$. Also we suppose that $n, m, k \geq 1$. The process $\mathbf{y}_t = H\mathbf{x}_t \in \mathbb{R}^1$ is interpreted as an output process of the system (246), where $H \in \mathbb{R}^{1 \times n}$.

It is well-known that the solution of (246) has the form [4]

$$\mathbf{x}_t = e^{A(t-t_0)}\mathbf{x}_{t_0} + \int_{t_0}^t e^{A(t-s)}B\mathbf{u}(s)ds + \int_{t_0}^t e^{A(t-s)}Fd\mathbf{w}_s, \quad 0 \leq t_0 \leq t \leq T, \quad (247)$$

where e^C is a matrix exponent

$$\sum_{j=0}^{\infty} \frac{C^j}{j!} \stackrel{\text{def}}{=} e^C,$$

C is a square matrix, and $C^0 \stackrel{\text{def}}{=} I$ is a unity matrix.

Consider the partition $\{\tau_p\}_{p=0}^N$ of $[0, T]$ such that $\tau_p = p\Delta$, $\Delta > 0$. For simplicity, we will suppose that $u(s)$, $s \in [0, T]$ can be approximated by the step function, i.e. $\mathbf{u}(s) \approx \hat{\mathbf{u}}(s)$, $s \in [0, T]$, where $\hat{\mathbf{u}}(s) = \mathbf{u}(\tau_p)$ for $s \in [\tau_p, \tau_{p+1})$, $p = 0, 1, \dots, N - 1$ (more accurate approximations of $u(s)$ are discussed in [62] (also see [58], [61])). Substituting $t = \tau_{p+1}$, $t_0 = \tau_p$, and $\hat{\mathbf{u}}(s)$ instead of $\mathbf{u}(s)$ into (247), we obtain

$$\hat{\mathbf{x}}_{p+1} = e^{A\Delta}\hat{\mathbf{x}}_p + A^{-1}(e^{A\Delta} - I)B\mathbf{u}(p\Delta) + \tilde{\mathbf{w}}_{p+1}(\Delta), \quad \mathbf{x}_0 = \mathbf{x}(0), \quad (248)$$

where $\hat{\mathbf{x}}_p$ is the approximation of \mathbf{x}_{τ_p} and

$$\int_0^{\Delta} e^{A(\Delta-s)} F d\mathbf{w}_{s+p\Delta} \stackrel{\text{def}}{=} \tilde{\mathbf{w}}_{p+1}(\Delta).$$

Also we assume that $\hat{\mathbf{y}}_p = H\hat{\mathbf{x}}_p$, where $\hat{\mathbf{y}}_p$ is the approximation of \mathbf{y}_{τ_p} . The random column $\tilde{\mathbf{w}}_{p+1}(\Delta)$ admits the following representation [4]

$$\tilde{\mathbf{w}}_{p+1}(\Delta) = S_D(\Delta)\Lambda_D(\Delta)\bar{\mathbf{w}}_{p+1}, \quad (249)$$

where $\bar{\mathbf{w}}_p \in \mathbb{R}^n$ is a column of independent standard Gaussian random variables such that $\mathbf{M}\{\bar{\mathbf{w}}_p\bar{\mathbf{w}}_q^T\} = \mathcal{O}$ for $p \neq q$, \mathcal{O} is a zero matrix of size $n \times n$, $S_D(\Delta)$ is a matrix of orthonormal eigenvectors of the matrix $D_f(\Delta)$ and $\Lambda_D^2(\Delta)$ is a diagonal matrix on the main diagonal of which are the eigenvalues of the matrix $D_f(\Delta)$, the matrix $D_f(\Delta)$ is defined by

$$D_f(\Delta) = \mathbf{M}\{\tilde{\mathbf{w}}_{p+1}(\Delta)\tilde{\mathbf{w}}_{p+1}^T(\Delta)\} = \int_0^{\Delta} \exp(A(\Delta-s))FF^T \exp(A^T(\Delta-s))ds,$$

where C^T is a transposed matrix C . Moreover, $D_f(\Delta) = D_f(t)|_{t=\Delta}$, where $D_f(t)$ is a solution of the following Cauchy problem [4]

$$\frac{dD_f}{dt}(t) = AD_f(t) + D_f(t)A^T + FF^T, \quad D_f(0) = \mathcal{O}.$$

In the SDE-MATH software package, we implement the numerical modeling of the system (246) by the formulas (248), (249). At that we use Algorithms 2.3–2.6 from [62] (also see [61], Chapter 11) for the implementation of (249).

3 The Structure of the SDE-MATH Software Package

3.1 Development Tools

The software package was implemented with Python programming language. The main reason to use it is a huge community and significant amount of helpful libraries for calculations and mathematics. The development was performed in free to use Atom text editor¹.

¹All programs in Python programming language from this paper were written by the first author

3.2 Dependency Libraries

In the development of the SDE-MATH software package such libraries as SymPy, NumPy, PyQt5, and Matplotlib were involved. All these libraries and tools are free and open source.

- SymPy is a Python library able to perform symbolic algebra calculations.
- NumPy is a library which specialization is efficient mathematical calculations. Most part of this library is written in C programming language that guarantees high calculation performance.
- The database is SQLite3. This is a tiny database for a local usage on one machine.
- Matplotlib library is a piece of software used to present obtained results in a best way.
- PyQt5 is a library used to build graphical user interface for the SDE-MATH software package.

3.3 Architecture

Taking into account, that the SDE-MATH software package is oriented on a numerical modeling its architecture is clear. There are two main statements. The first is that mathematical formulas are strongly integrated with SymPy library. By that we mean that they completely rely on SymPy. And the second is usage of database to make some calculations able for caching. The architecture itself is provided on Figure [1](#). Here all parts of the software package can be seen.

The main package is responsible for startup, so it decides which part of the software package must be started. The software package has several modes of operation. The objectives now are

- Run program to calculate and store the Fourier–Legendre coefficients in few text files with further loading in database.
- Run program with graphical user interface. This is the main program entry for the SDE-MATH software package.

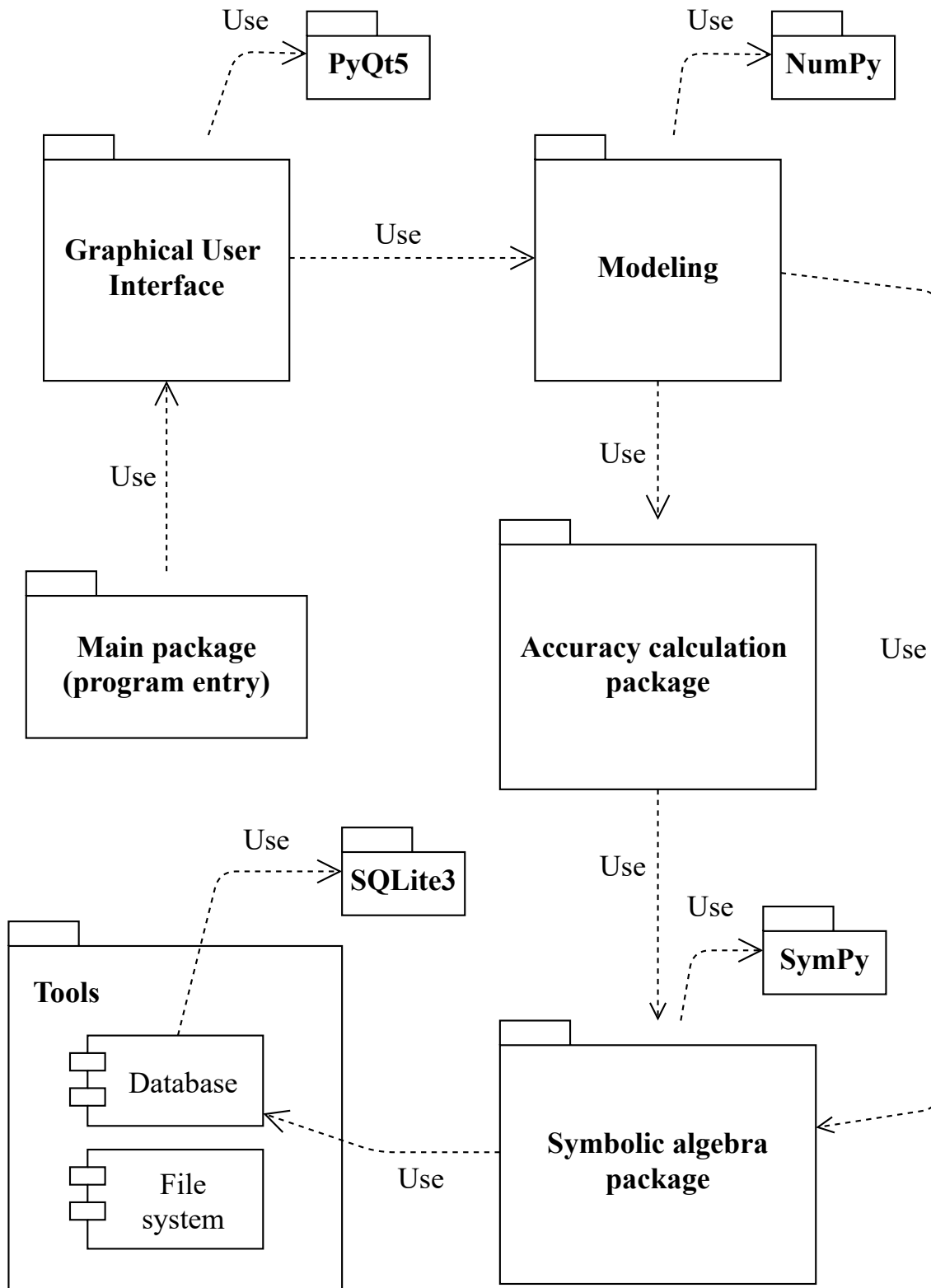


Figure 1: The SDE-MATH software package architecture

On the current state of development the main entry package booting up PyQt5 library with all necessary widgets. More detailed description of this process will be provided later.

Moving further, the modeling package comes up. This package responsible for all work referenced to modeling including initialization of modeling environment, calculations loops and more. Also, it depends on accuracy calculation module deciding which amount of members in each approximation of iterated stochastic integral should be used in modeling of the Itô SDE (I) solution.

Accuracy calculation module accepts the order of strong numerical scheme for the Itô SDE (I) and its integration step and then calculates necessary amount of members in approximations of iterated Itô and Stratonovich stochastic integrals.

Symbolic algebra module is the construction part which combines many supplementary differential operators with strong numerical schemes for the Itô SDE (I). Having these components combined this module performs simplification of resulting formula so the modeling package can do its modeling work.

Tools module provides some functionality related to bootstrap of runtime environment and external instruments such as database and file system.

3.3.1 Integration with SymPy

Class inheritance tree was extended to implement strong numerical schemes for Itô SDEs. While numerical schemes for Itô SDEs were being implemented it was also necessary to implement supplementary subprograms. SymPy is a Python library able to perform symbolic algebra calculations. This is a core part of the project since it differentiates input functions, builds and simplifies strong numerical schemes for Itô SDEs to model the Itô SDE (I) solution. Without this part the program package cannot be able to provide such flexible input of data.

3.3.2 Purpose of NumPy

NumPy is a library that helps with calculation optimizations in this project. The library specialization is efficient mathematical calculations. The main usage case is to calculate compiled symbolic formulas with it. It has integration with SymPy to replace symbolic functions with high performance numerical functions.

3.3.3 Purpose of SQLite Database

The database was used to store the precalculated Fourier–Legendre coefficients, so getting them from there made numerical modeling much faster, because calculation process for these Fourier–Legendre coefficients involve high-cost symbolic operations. The database contains only one table, and might be thought redundant, but modeling needs hundreds (or even thousands) of precalculated coefficients. Obviously, calculation of them at runtime is terribly inefficient, but text files also not the best choice. Text files provide a sequential access memory and combining different accuracy values q_1, \dots, q_{15} it causes sequential search which extends time to give the result. That is where database comes up. The random access allows to get any Fourier–Legendre coefficient or any quantity of them which makes solution as flexible as it possible.

The download of precalculated Fourier–Legendre coefficients is built in supplemental subprograms to provide fluent calculation pipeline. Having the precalculated Fourier–Legendre coefficient not found, subprogram initiates calculation for it with following store in the database.

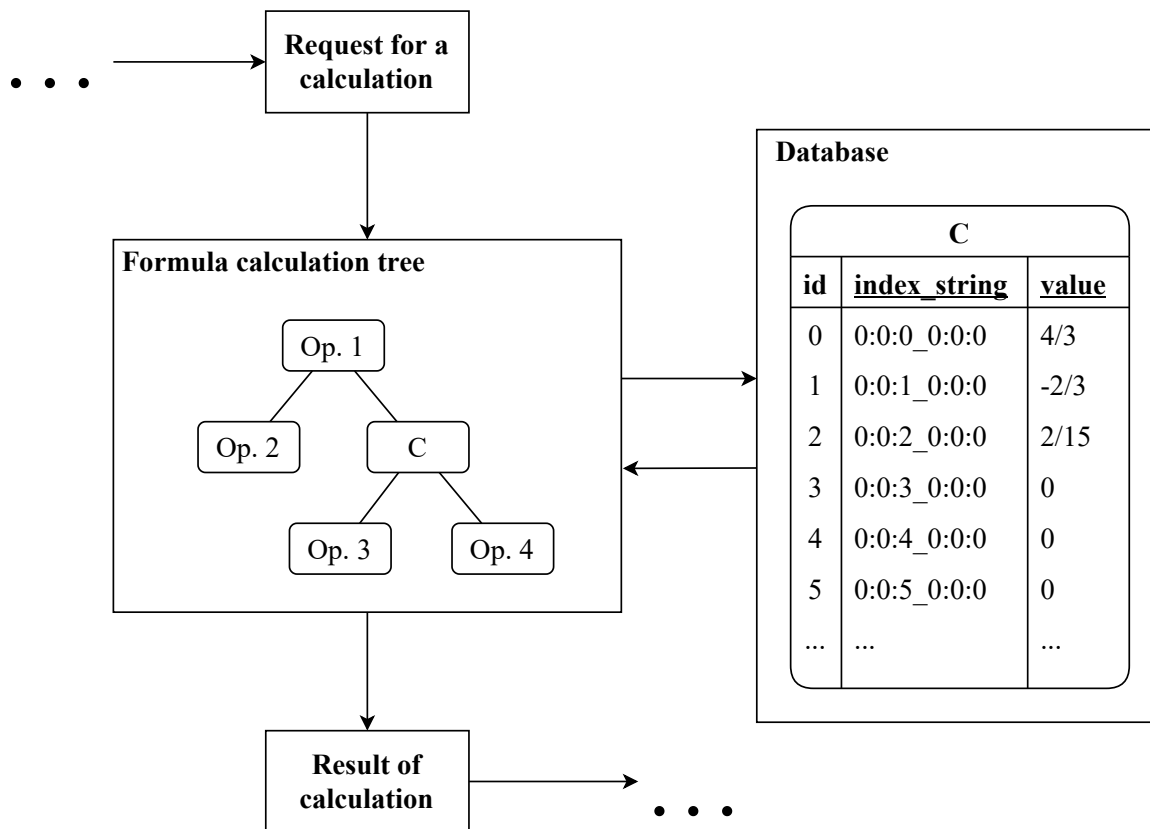


Figure 2: Fourier–Legendre coefficients calculations explanation

It is an interesting note that having mentioned earlier optimization done, the calculations performance were increased in several times. Now the most heavy operation is symbolic simplifications before modeling. The actual modeling takes seconds, for thousands of iterations on m components of stochastic process, so it is not such important how long modeling period of time, as the accuracy that needs to be accomplished.

The scheme of calculation process is presented on Figure [2](#).

3.3.4 Purpose of Matplotlib

Matplotlib library is a piece of software used to present obtained results in a best way. This library has many features, but feature that needed in this project is to print charts with modeling results in a PyQt5 widget. Thus the data visualization is integrated in graphical user interface.

3.4 Implementation Plan

The implementation of SDE-MATH software package was performed sequentially. The components of SDE-MATH software package were implemented in order of their necessity for calculation pipeline completion.

3.4.1 Calculation of the Fourier–Legendre Coefficients

The Fourier–Legendre coefficients for the approximations of iterated Itô and Stratonovich stochastic integrals were implemented and placed in Listings [43](#)–[61](#). This was the first step since the Fourier–Legendre coefficients involved almost in every strong numerical scheme for the Itô SDE [\(1\)](#).

Also it is important to note that the SDE-MATH software package contains a Python script intended for generating of Fourier–Legendre coefficients using multiprocessing. This script placed in Listing [62](#) and already contains tasks that were performed to generate about 300,000 Fourier–Legendre coefficients. Similarly, user can run and calculate additional Fourier–Legendre coefficients if they are needed. To determine which Fourier–Legendre coefficients will be calculated user must specify pairs of starting and ending values of components in lower multi-index and specify upper multi-index of the Fourier–Legendre coefficient. For example $((0, 15), (0, 15), (0, 15)), (0, 1, 0)$. This means that

program calculates the Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{010}$, where $j_1, j_2, j_3 \in \{0, 1, \dots, 14\}$.

3.4.2 Differential Operators $L, \bar{L}, G_0^{(i)}, i = 1, \dots, m$

Moving further, strong numerical schemes for Itô SDEs rely on the differential operators (4), (5), and (23). They were implemented and placed in Listings 64–67.

3.4.3 Approximations of Iterated Stochastic Integrals

The next step is implementation of approximations of iterated Itô and Stratonovich stochastic integrals for the numerical schemes (12)–(16), (24)–(28). They are implemented and definition of their classes are placed in Listings 69–108.

3.4.4 Strong Numerical Schemes for Itô SDEs

The strong numerical schemes (12)–(16), (24)–(28) for Itô SDEs were implemented. They are placed in Listings 110–131.

3.4.5 Graphical User Interface

Finally, the graphical user interface was implemented. The source codes referenced to graphical user interface are placed in Listings 7–42.

4 Software Package Graphical User Interface

For the SDE-MATH software package mentioned above the graphical user interface was developed. The graphical user interface is important and massive part of SDE-MATH software package because it allows user to perform modeling experiments without programming skills and understanding of program package architecture and principles of work.

4.1 Information Model of The Graphical User Interface

The development of graphical user interface was started from consideration of

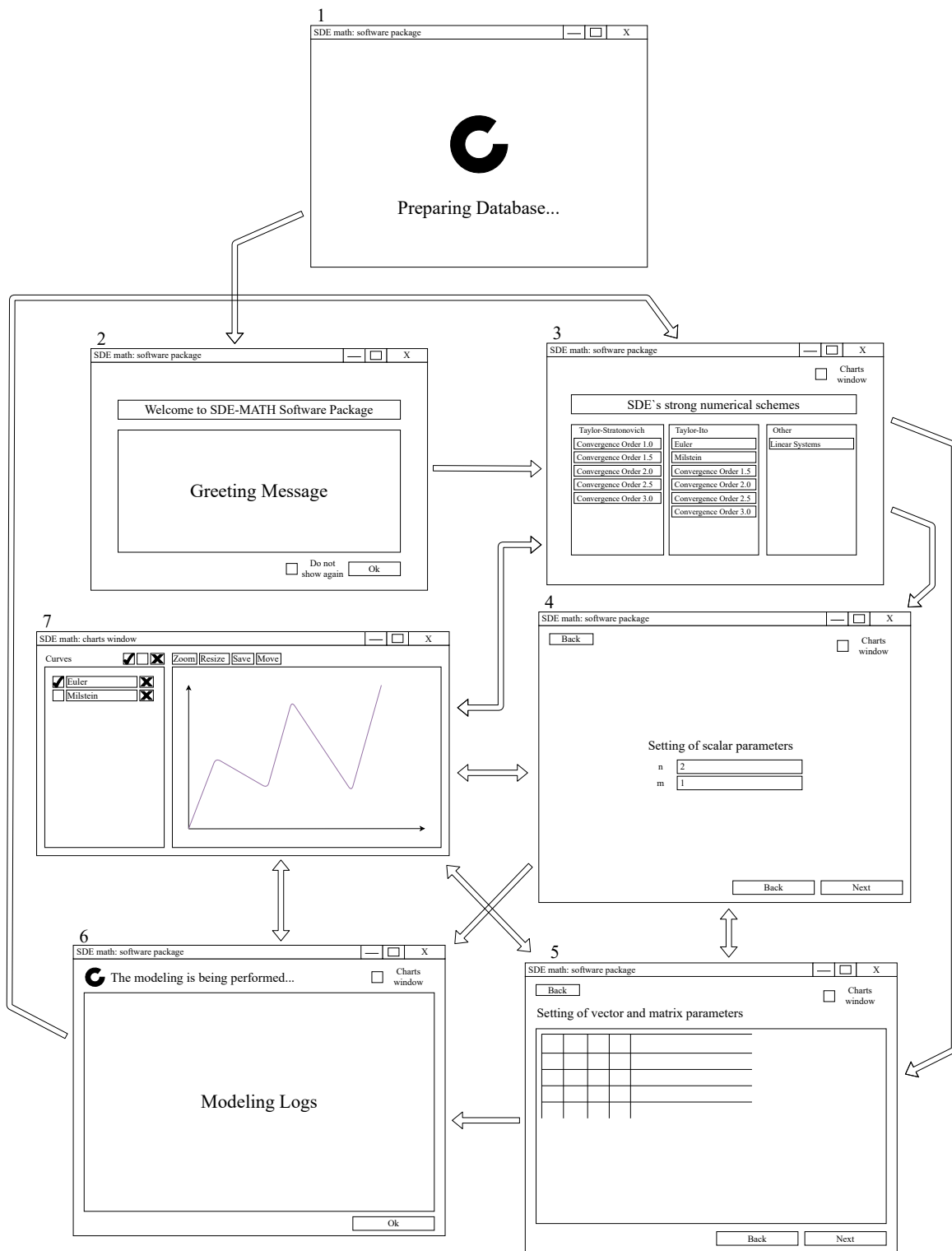


Figure 3: Information model of graphical user interface

experiments and routines which can be performed with the SDE-MATH software package. The graphical user interface is aimed on provision of user capabilities to perform nonlinear and linear systems of Itô SDEs modeling experi-

ments. The information model which schematically describes the graphical user interface structure is presented on Figure 3.

4.1.1 Processing Screens

To represent long duration processes the graphical user interface has two dialogs which can be seen on Figure 3, in Windows 1 and 6. The first one represents database preparing process on application very first run. During this process the Fourier–Legendre coefficients are being loaded into the SQLite database. This screen appears also when user calculates new Fourier–Legendre coefficients. The other screen shows logs during modeling experiment.

4.1.2 Greetings Dialog

After the SDE-MATH software package has completed the database preparation, it shows greeting dialog which represents short information about its purposes. The greeting dialog can be seen on Figure 3 in Window 2.

4.1.3 Main Menu Dialog

In the main menu of the SDE-MATH software package user can choose one of strong numerical schemes for Itô SDEs to perform modeling experiments. The main menu dialog can be seen on Figure 3 in Window 3.

4.1.4 Visualization Tool

It is important to note that the main SDE-MATH software package window has a checkbox in right upper corner which do switching on and off charts window. In any time user can call this window or hide it if it is not needed. The charts window is universal utility for modeling experiments results visualization. This window has few instruments on it. The left side bar contains all curves labels, and control elements for hiding, showing, and deleting curves. On the right side of the window there are plot which draws the curves. The charts window can be seen on Figure 3, it is Window 3.

4.1.5 Data Input Dialogs

Since the software package has options to perform linear Itô SDEs modeling experiments it is necessary to provide user with input fields for numerical data

both scalar and matrix. On the other side, for nonlinear Itô SDEs it is necessary to provide symbolic input. The choice of control elements is conditioned by the above obstacles. On Figure 3, and especially in Windows 4 and 5, these input controls can be seen. There are "LineEditWidget" and "TableWidget" which are sufficient to provide input abilities. The topic of input data validation is also important but to be more accurate referenced to user experience rather than to information model, so it will be described further.

4.2 The User Experience and Implementation Results

The above part represents the structure of software package but not the dynamics and user experience of it. Let us discuss the SDE-MATH software package user experience on few examples provided further on Figures 4-36. This examples represent two scenarios of the SDE-MATH software package use.

The database preparation screen is presented on Figure 4. During the database preparation this screen displays informational message and spinning visualizer of process continuation.

The screen that presented on Figure 5 appears every time when software package runs unless user presses "Ok" button with marked checkbox. In such case this message screen will not be shown again.

On Figure 6 the main menu dialog is presented. In this dialog user can choose any strong numerical scheme for Itô SDEs to perform modeling experiment.

The tooltip example can be seen on Figure 7. Such tooltips displayed with characteristic icon are placed all over software package interface to help user with explanations.

As noted earlier, the dedicated charts window is universal tool for visualization. The specific examples of such visualization are presented on Figure 8, 19, 34-36.

The initial state of input dialogs for nonlinear and linear Itô SDEs are displayed on Figures 9 and 20. At that moment user can start to input the data.

The example of wrong scalar data input is presented on Figures 10, 15, 21, and 31. When user input wrong data the error message appears and "Next" or "Perform modeling" button is blocked. The input field is being checked all the user data input process, and as soon as wrong character is entered notification

pops up.

If scalar data is correct the "Next" button is automatically unblocked. On Figures 11, 16, 22, and 32 the examples of scenario are displayed.

On Figures 12, 13, and 28 the example of correct matrix data input is presented. In this particular case the input is symbolic. Symbolic algebra input errors are much harder to determine so this is done on further stages, in modeling runtime.

In the other case when matrix input data are numerical, the validation is performed right after user has finished input. The examples of incorrect matrix numerical input can be found on Figure 24.

When user finishes input with a success the "Next" or "Perform modeling" button is automatically unblocked. On Figures 23, 25–27, 29, and 30 that can be clearly seen.

The Figures 17, 18, and 33 displays sequence of log messages emerged during the modeling process.

After modeling has been done the focus moves to the charts window where obtained modeling results can be seen. The results of modeling is displayed on Figures 19, 34–36. On Figures 35 and 36 the expectations and variances of obtained components of solution are displayed.

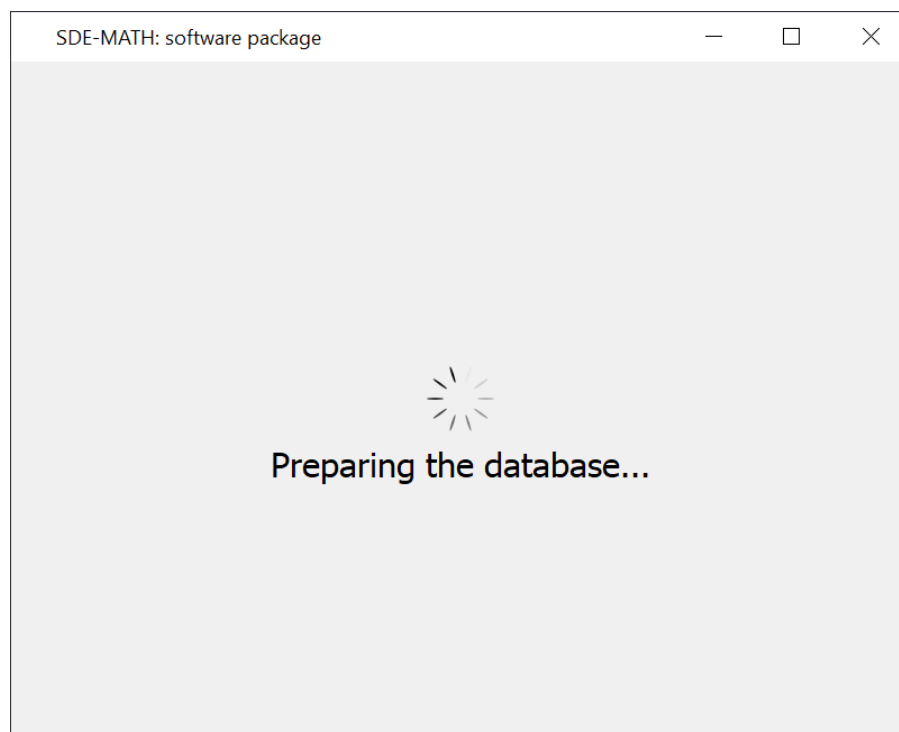


Figure 4: Fourier–Legendre coefficients database preparation screen

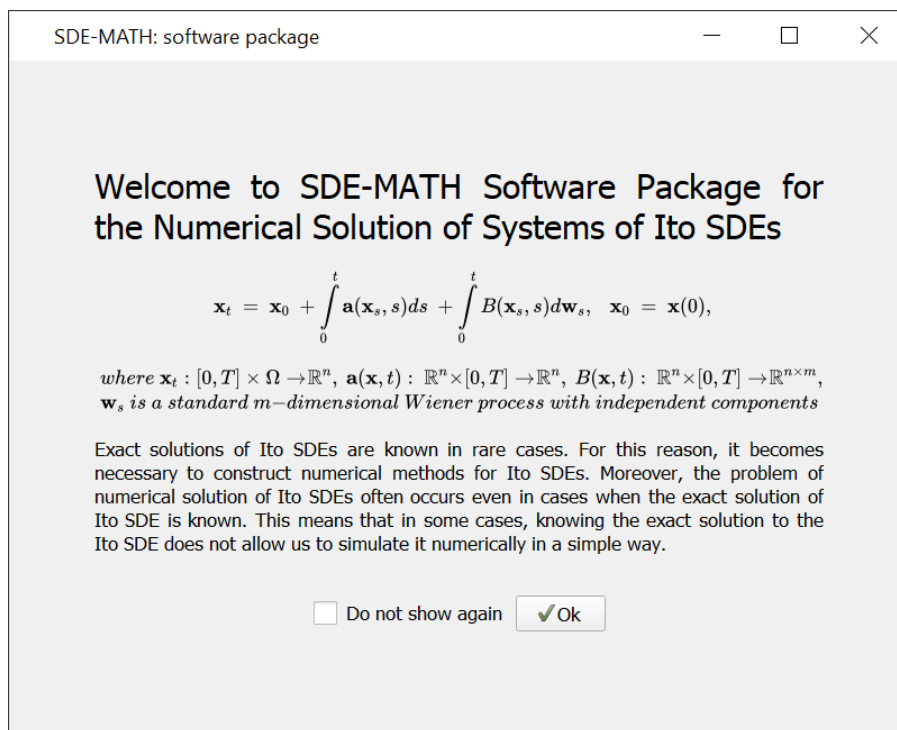


Figure 5: Greetings screen

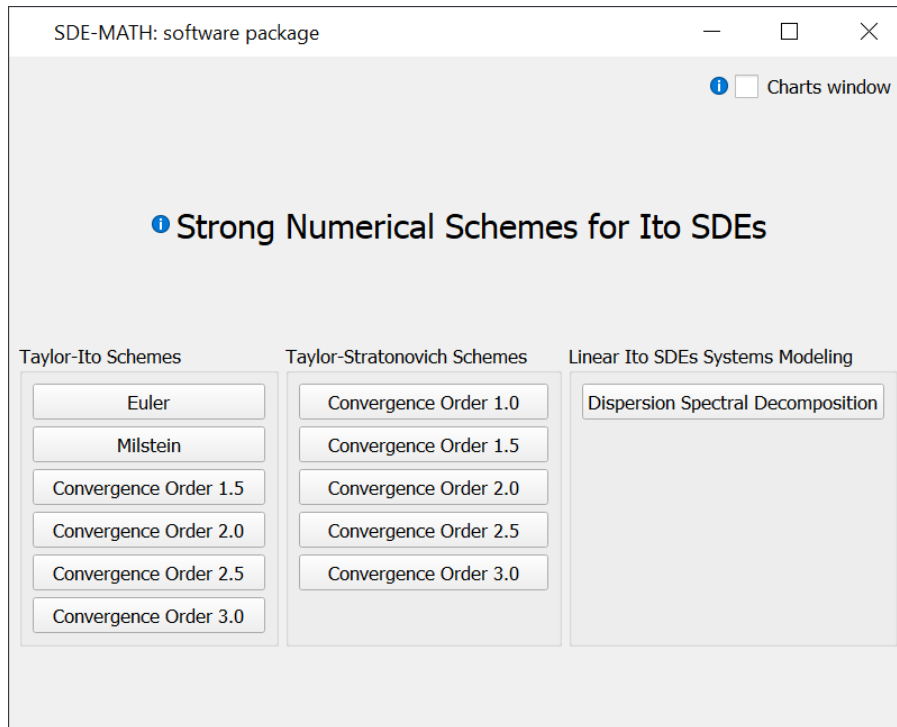


Figure 6: Main menu dialog

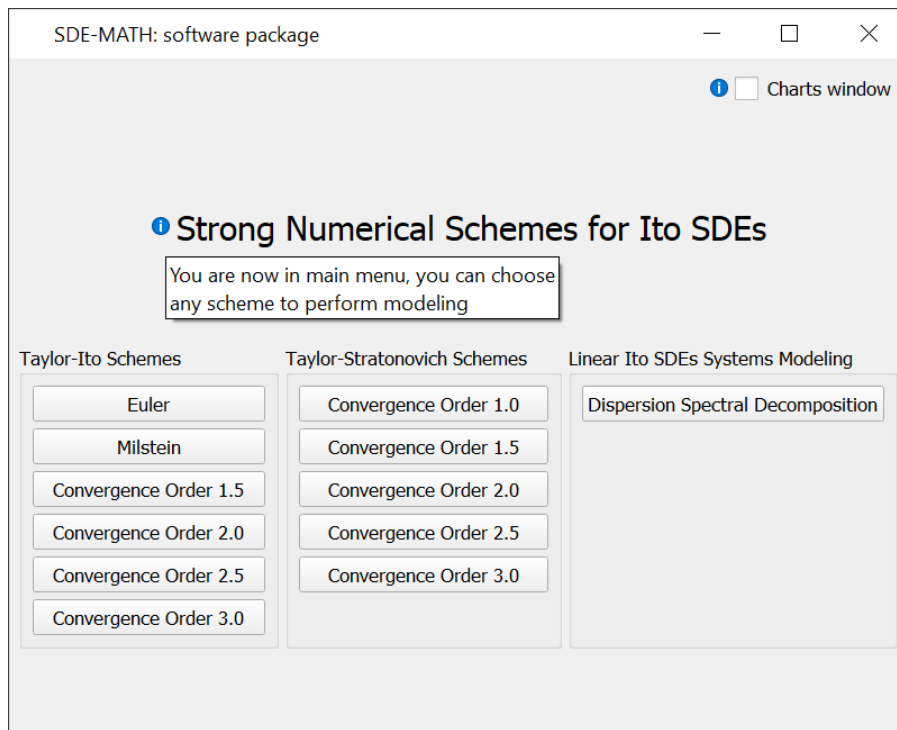


Figure 7: Tooltip

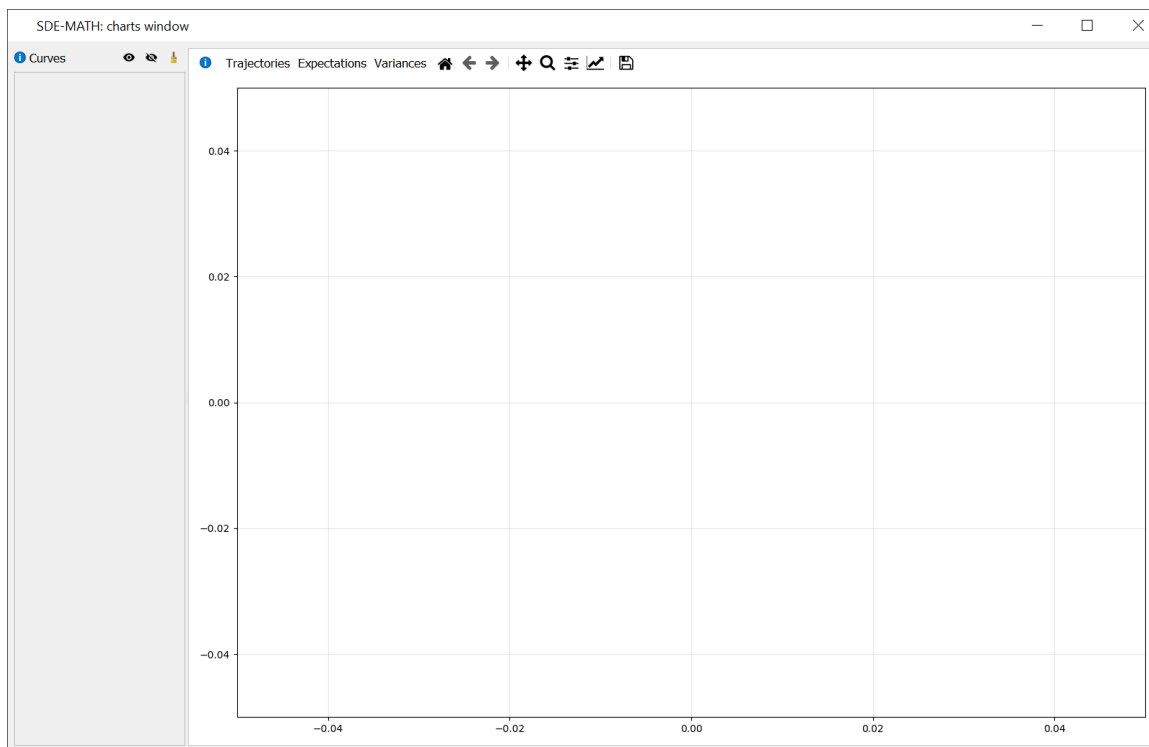


Figure 8: Charts window

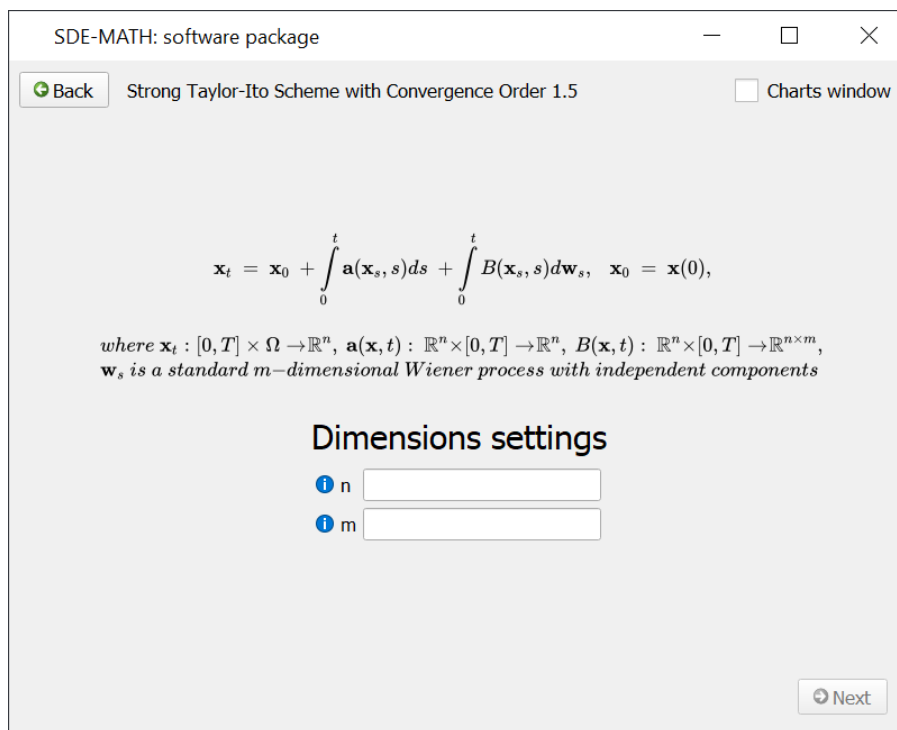


Figure 9: Nonlinear system of Itô SDEs data input

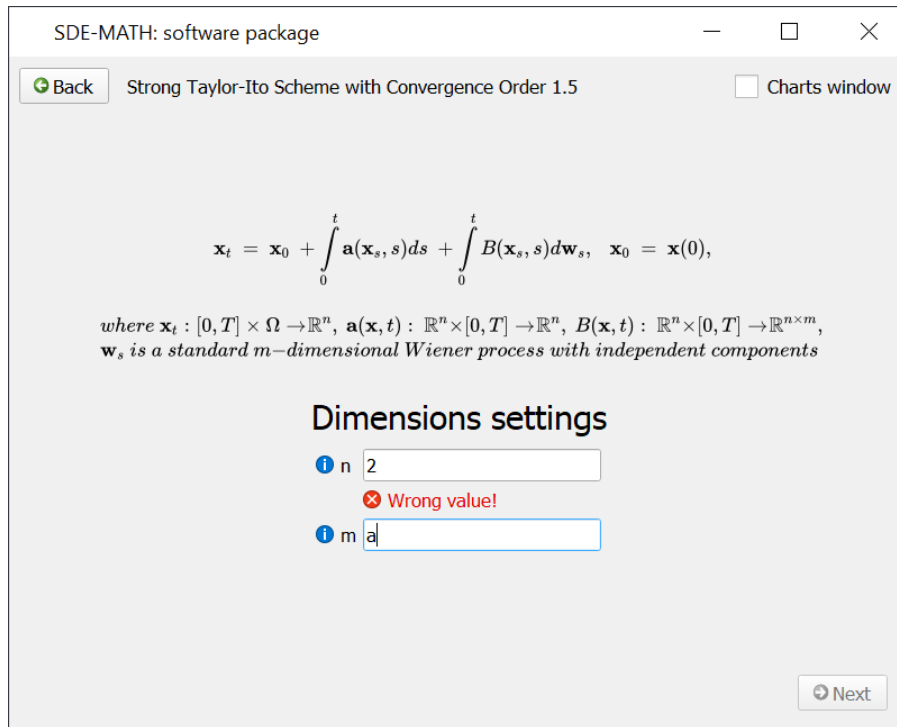


Figure 10: Wrong data input

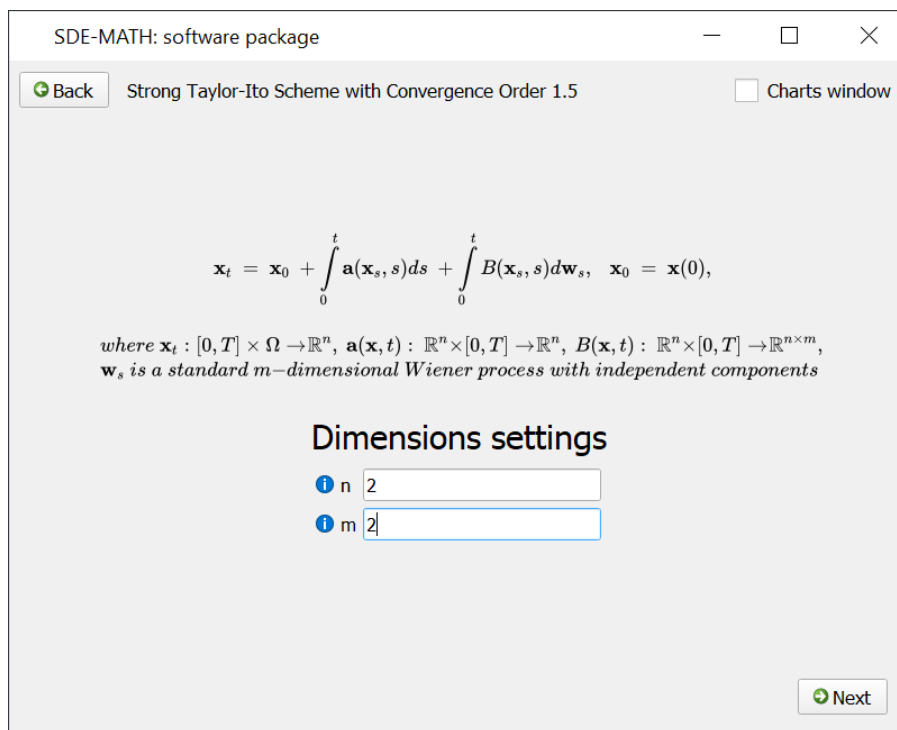


Figure 11: Correct data input

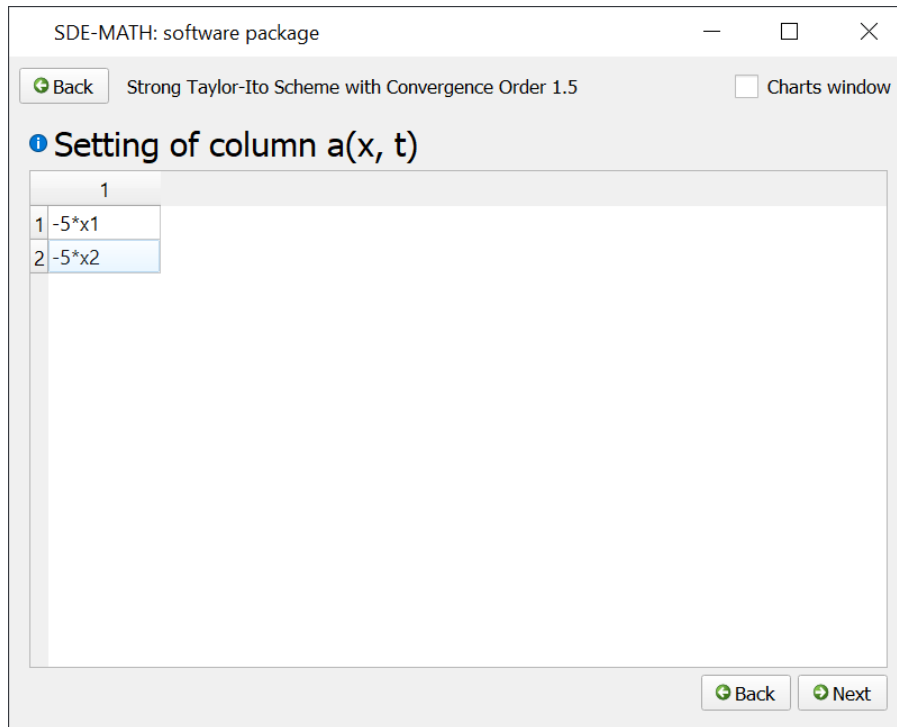


Figure 12: Vector function $\mathbf{a}(\mathbf{x}, t)$ input

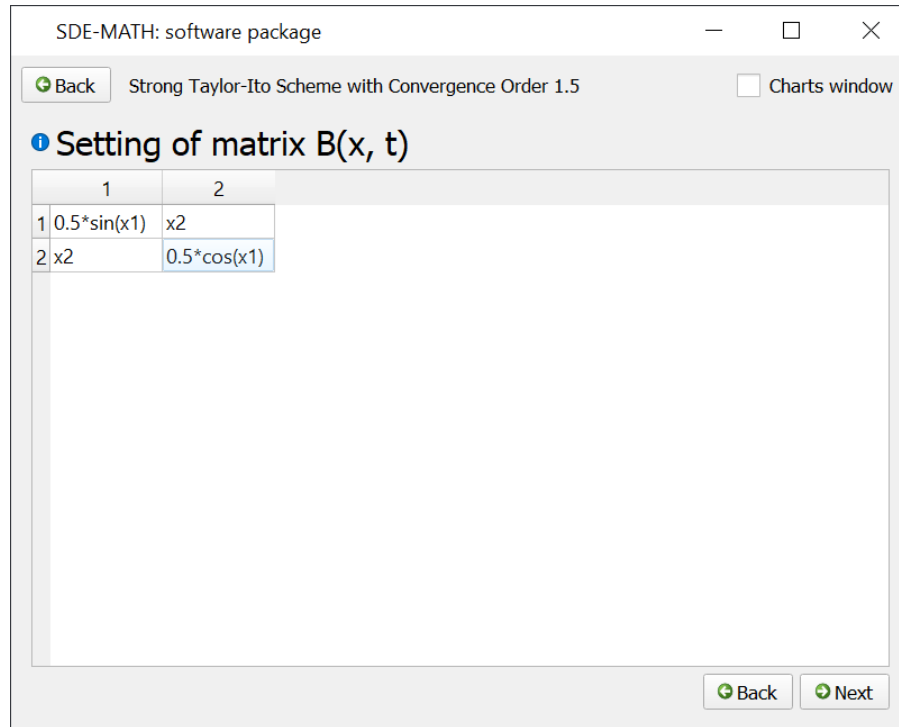


Figure 13: Matrix function $B(x, t)$ input

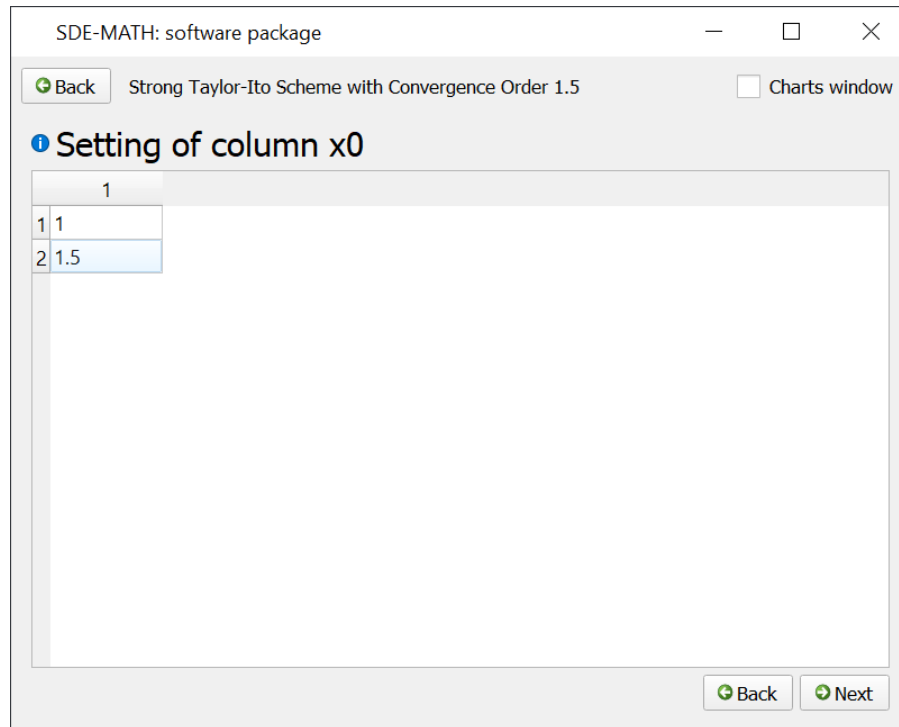


Figure 14: Initial data input

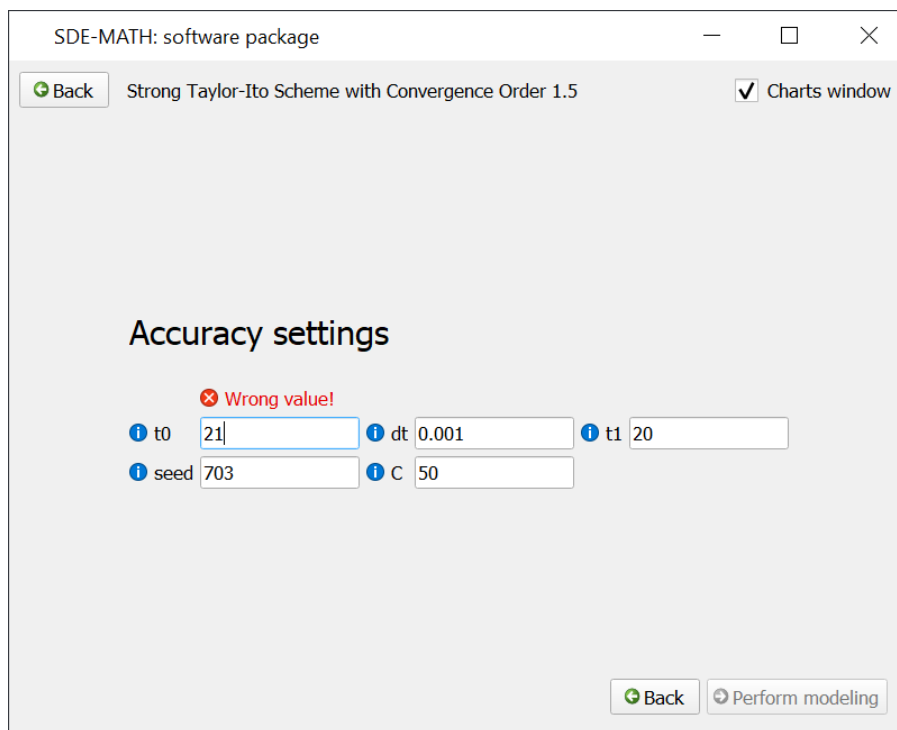


Figure 15: Wrong data input

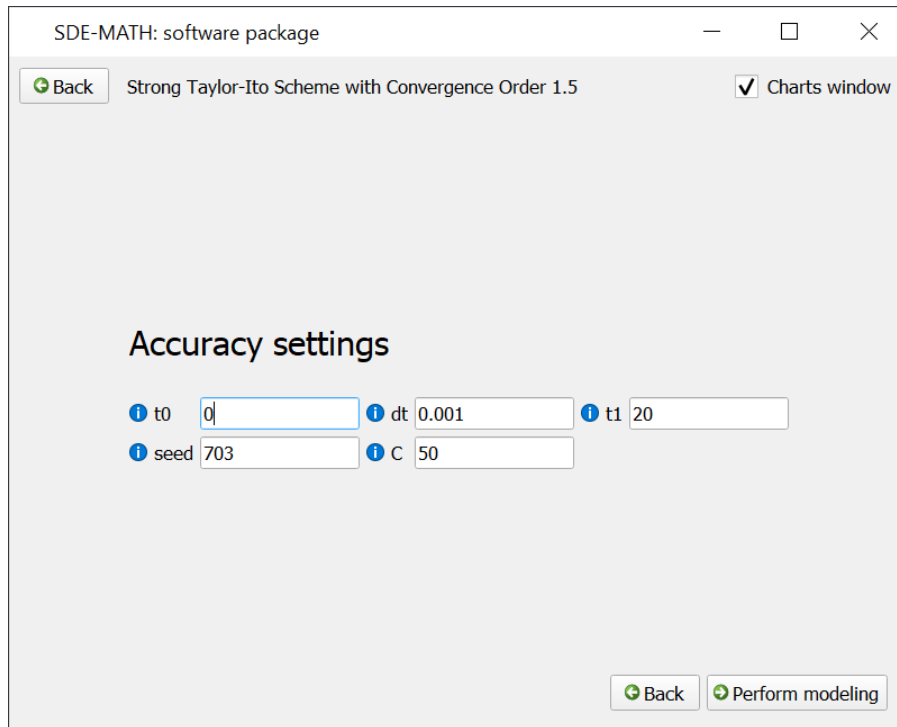


Figure 16: Correct data input

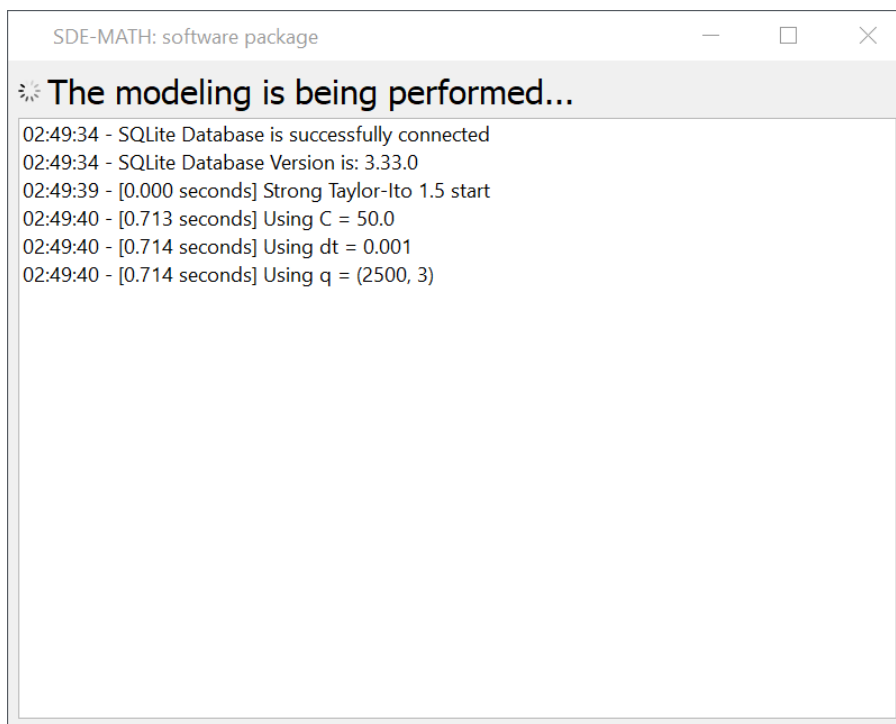


Figure 17: Modeling logs

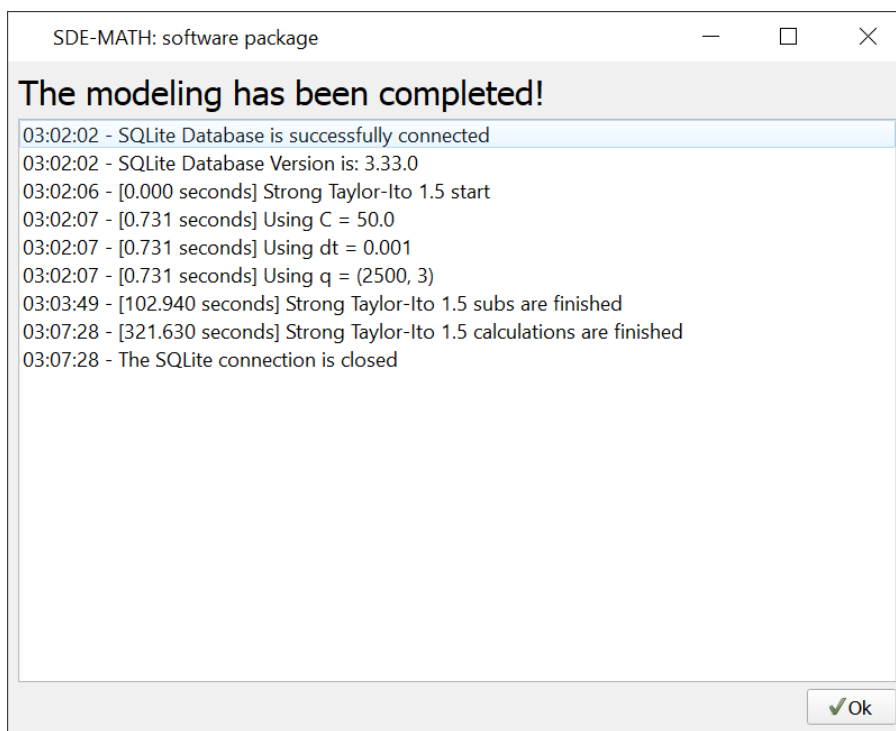


Figure 18: Modeling logs

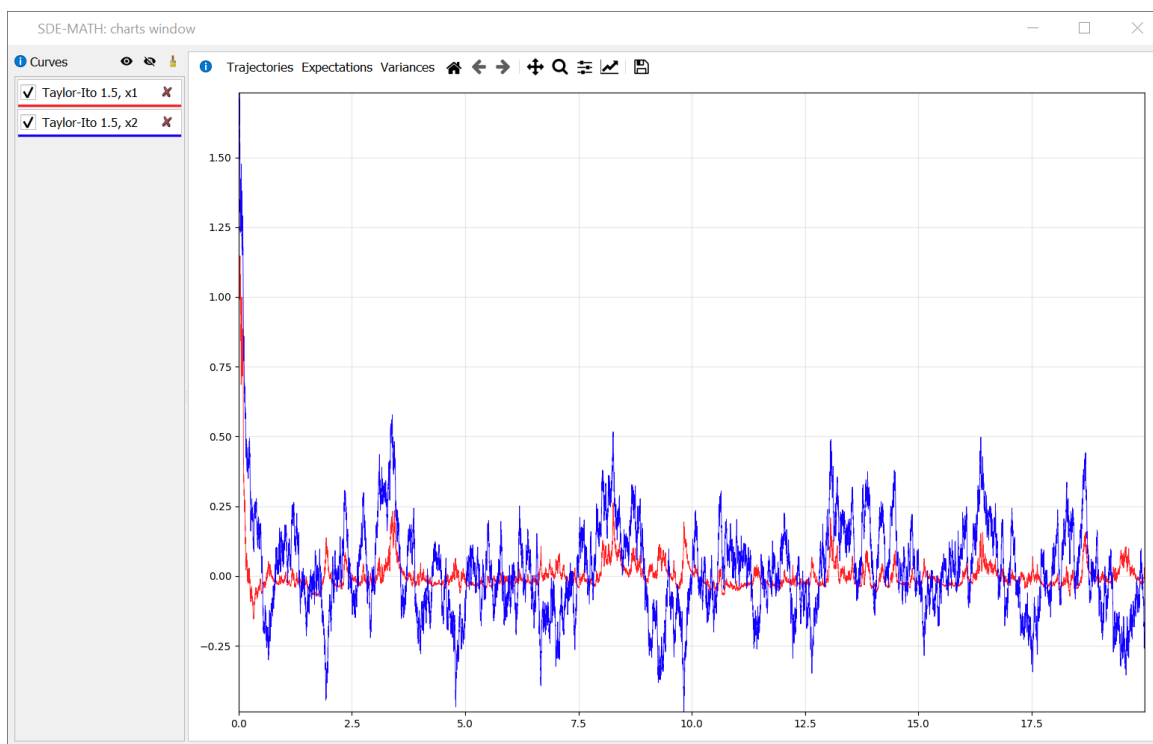


Figure 19: Modeling results

SDE-MATH: software package - □ ×

← Back Linear Systems of Ito SDEs Charts window

$$\begin{cases} dx_t = (Ax_t + Bu(t))dt + Fdw_t \\ y_t = Hx_t \end{cases}, \quad x_0 = x(0),$$

where $x_t: [0, T] \times \Omega \rightarrow \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $u(t): [0, T] \rightarrow \mathbb{R}^k$, $F \in \mathbb{R}^{n \times m}$, $H \in \mathbb{R}^{1 \times n}$, w_t is a standard m -dimensional Wiener process with independent components

Dimensions settings

i n

i m

i k

→ Next

Figure 20: Linear system of Itô SDEs data input

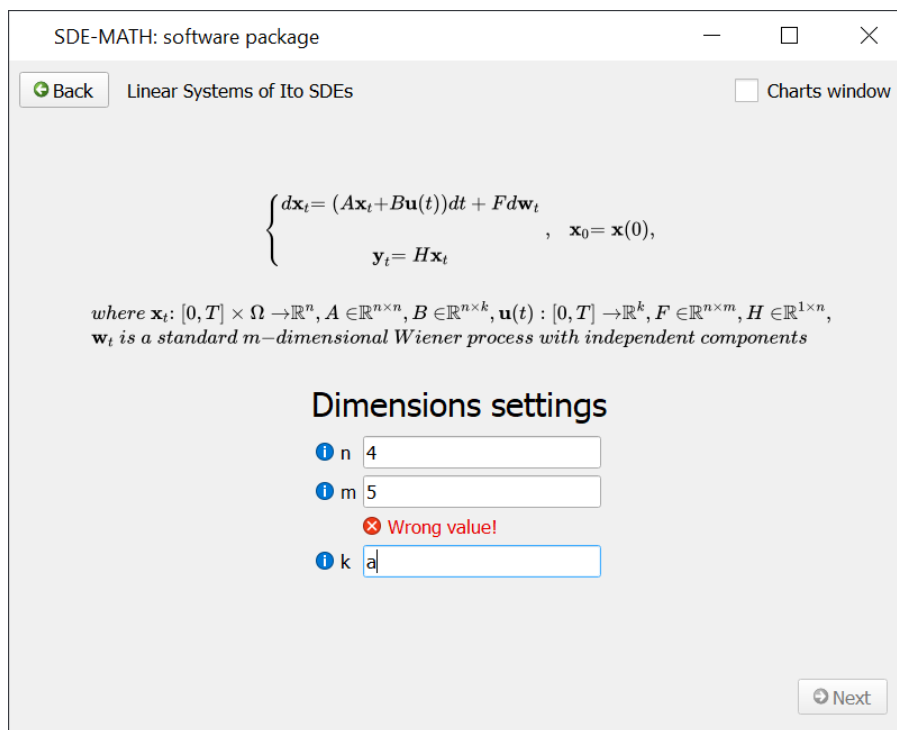


Figure 21: Wrong data input

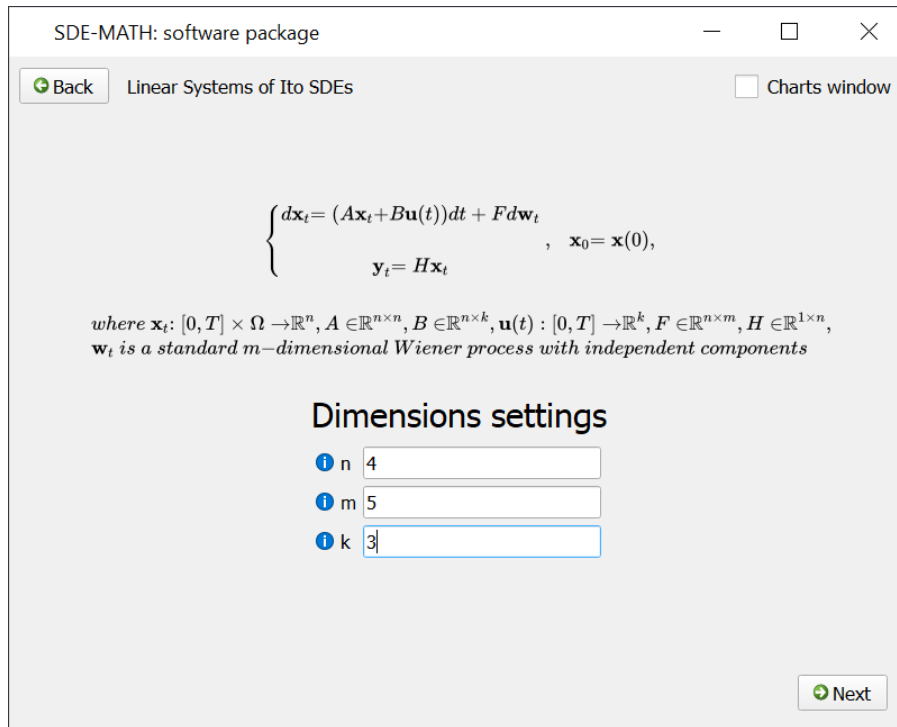


Figure 22: Correct data input

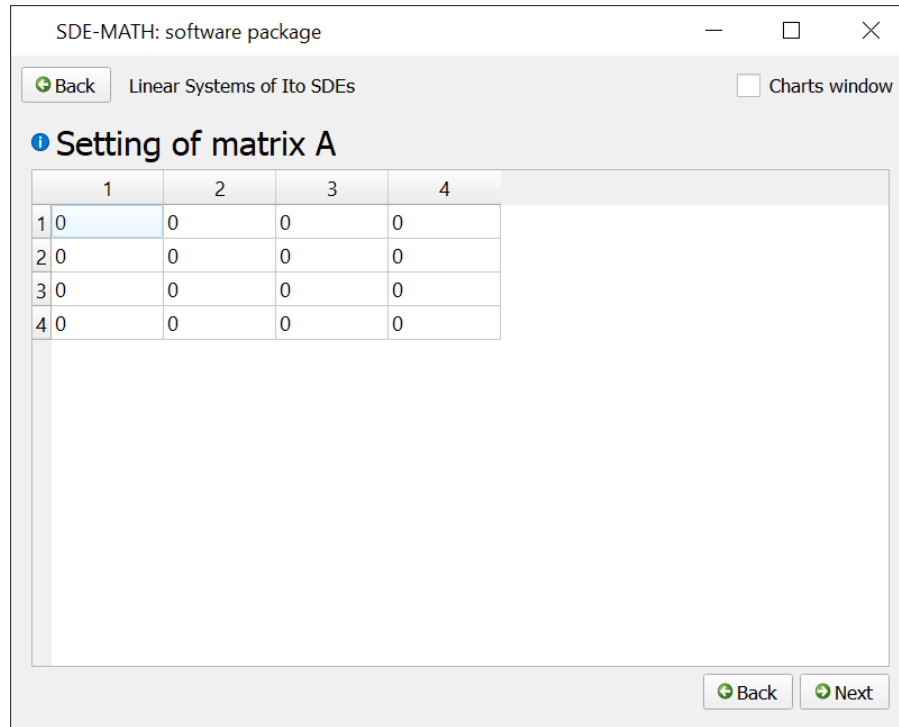


Figure 23: Matrix A input

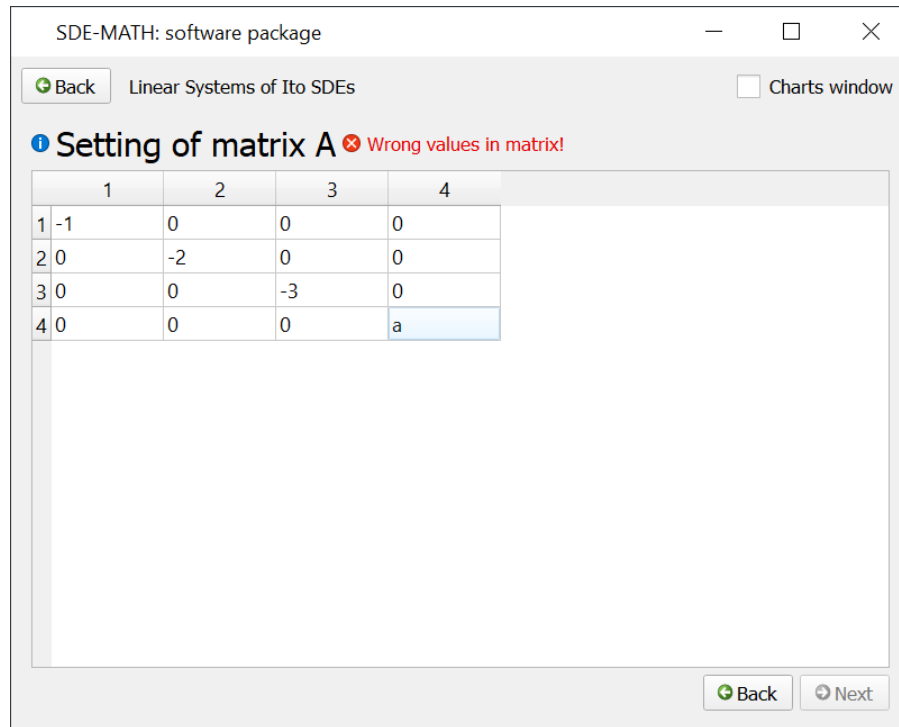


Figure 24: Wrong matrix A input

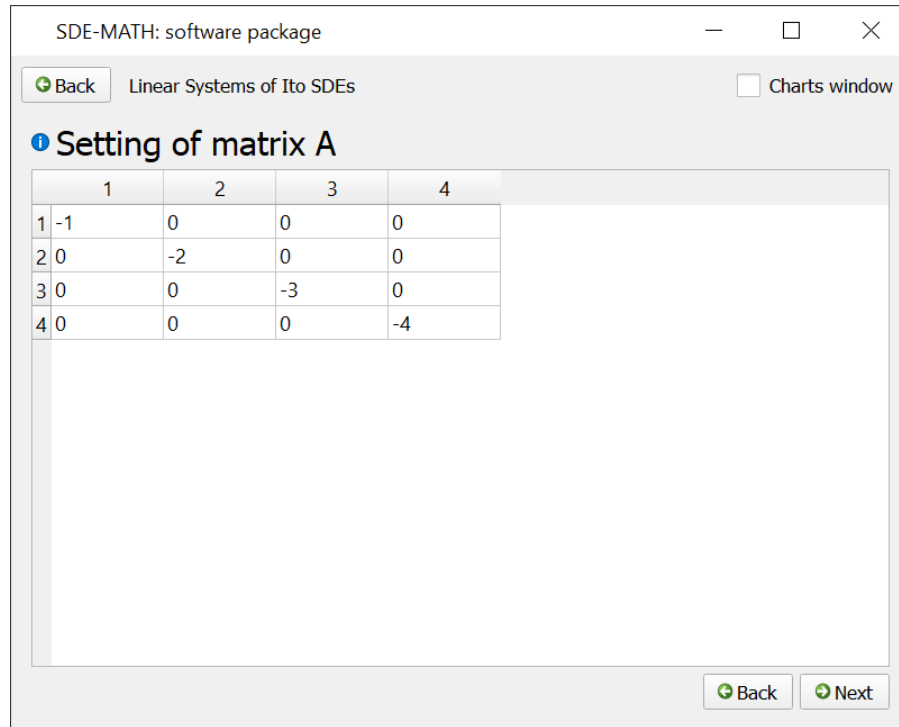


Figure 25: Correct matrix A input

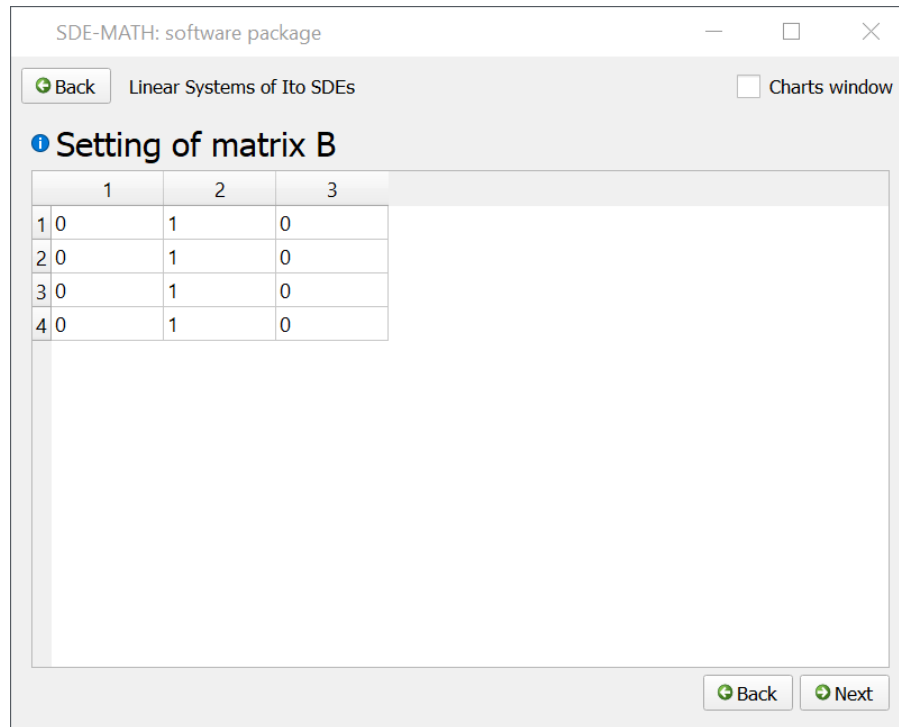


Figure 26: Matrix B input

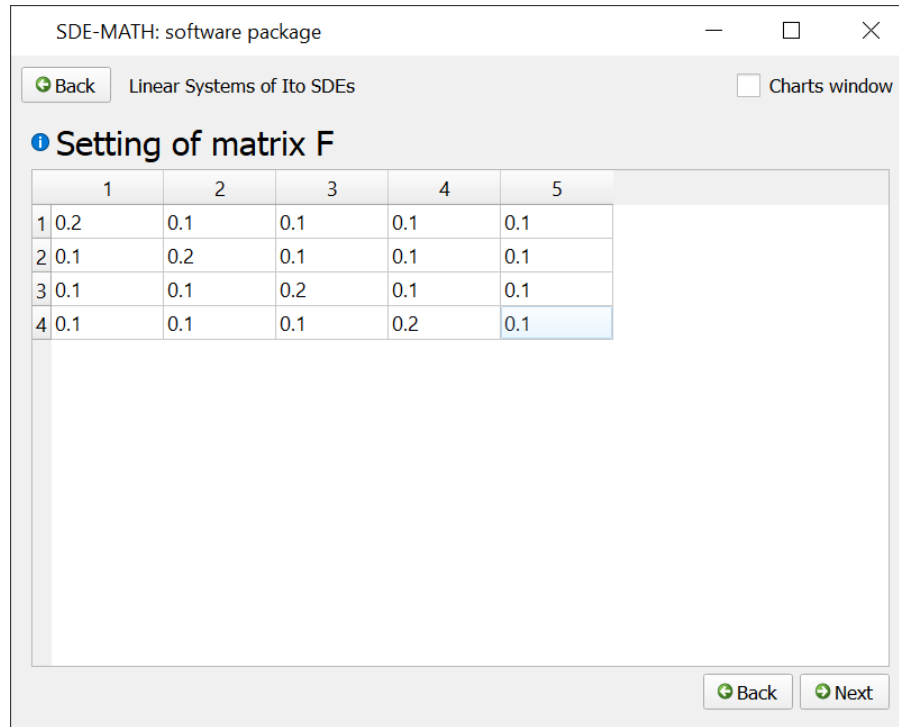


Figure 27: Matrix F input

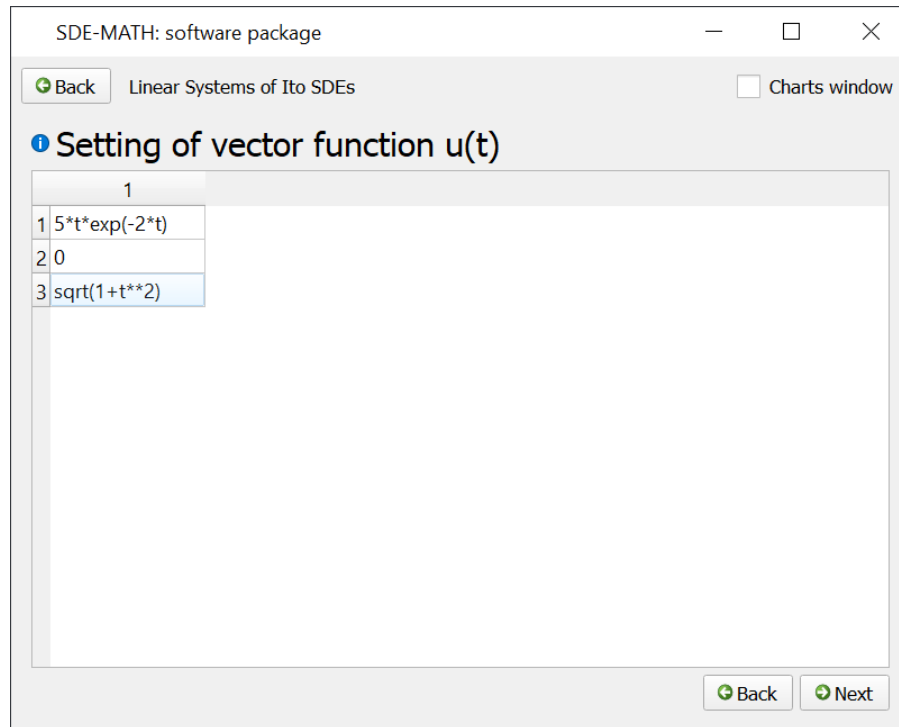


Figure 28: Vector function $u(t)$ input

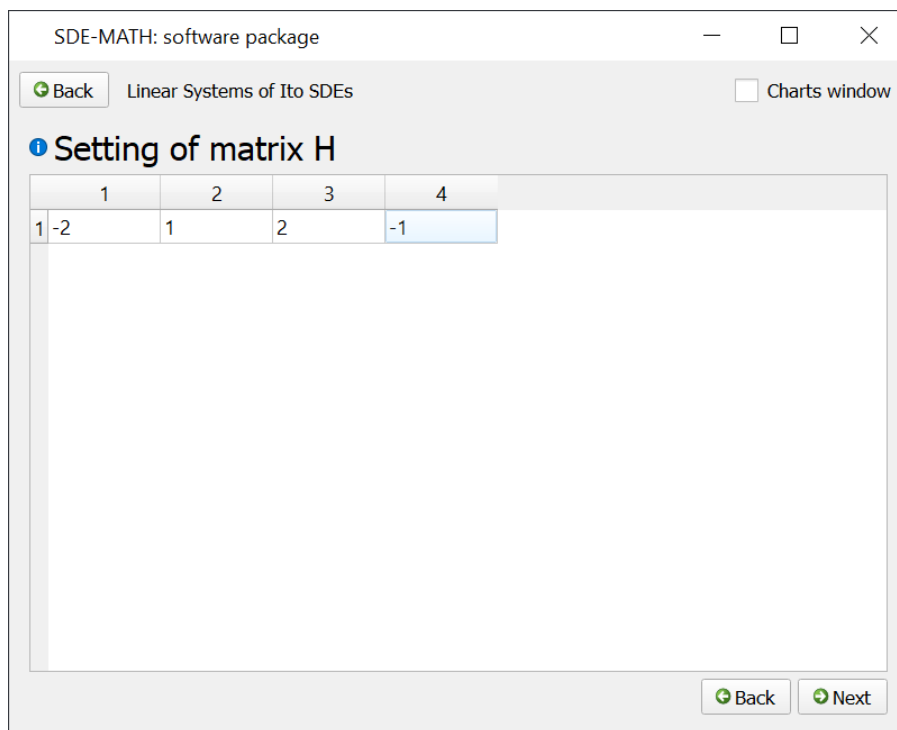


Figure 29: Matrix H input

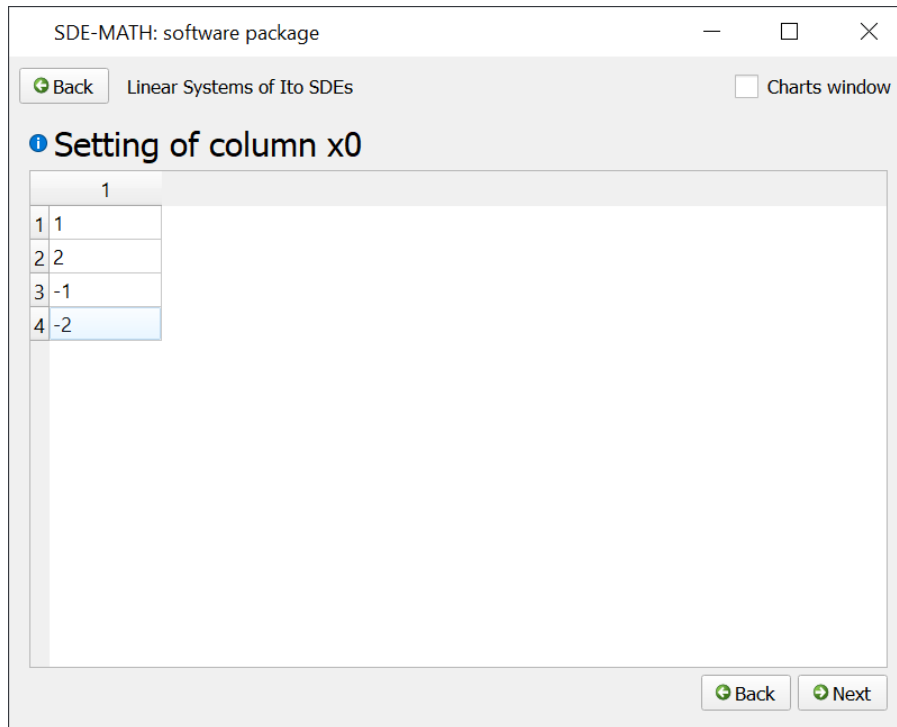


Figure 30: Initial data input

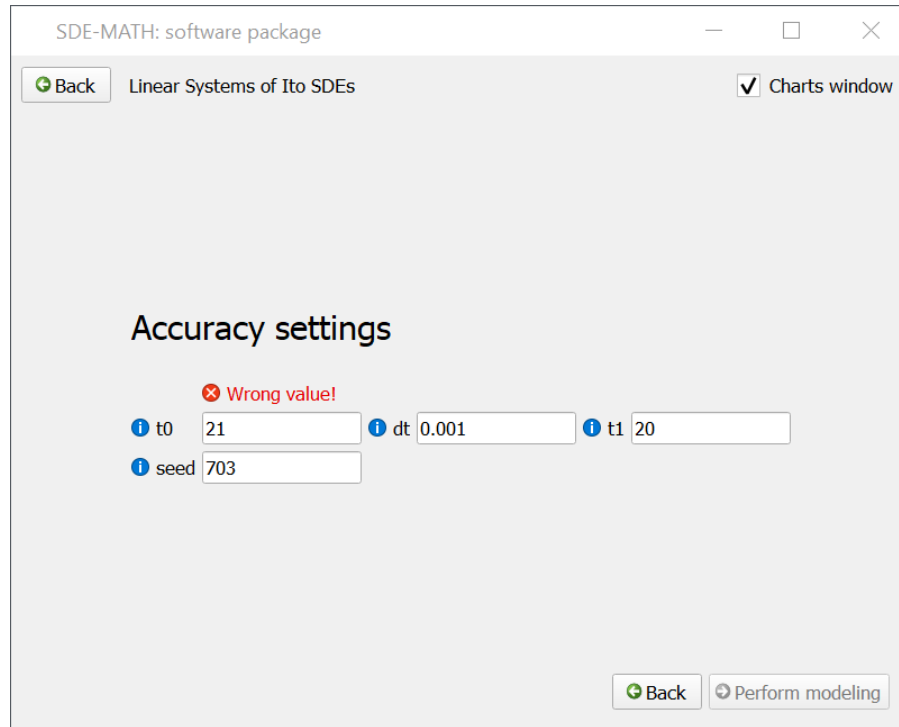


Figure 31: Wrong data input

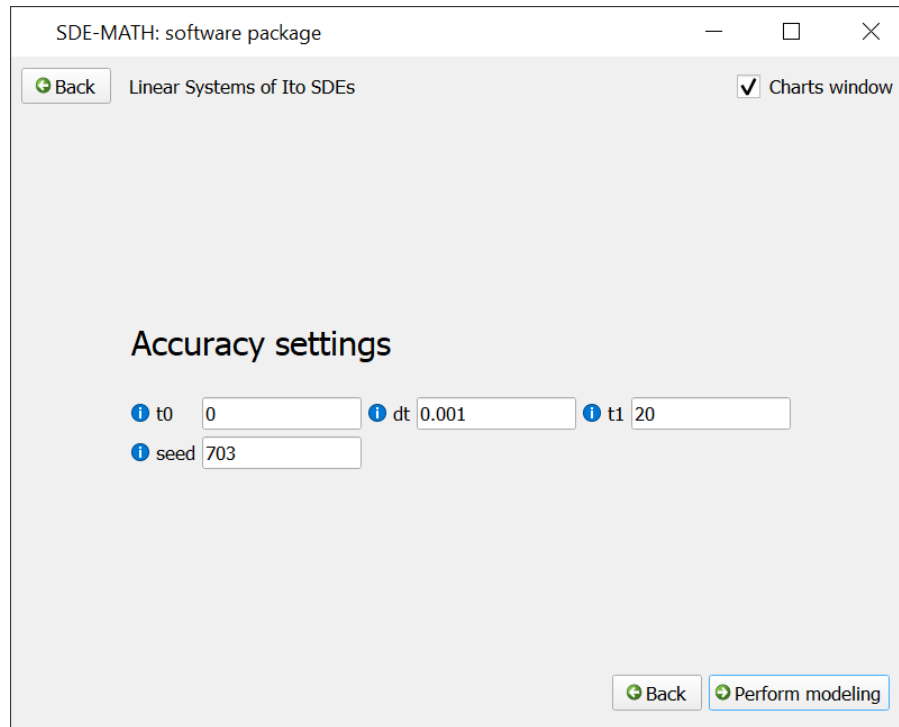


Figure 32: Correct data input

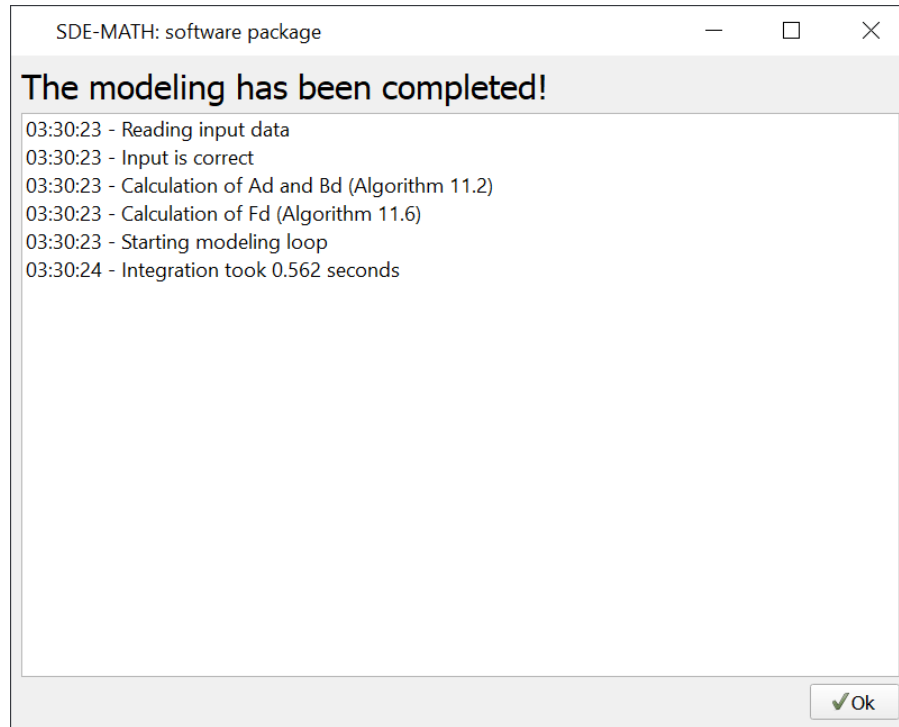


Figure 33: Modeling logs

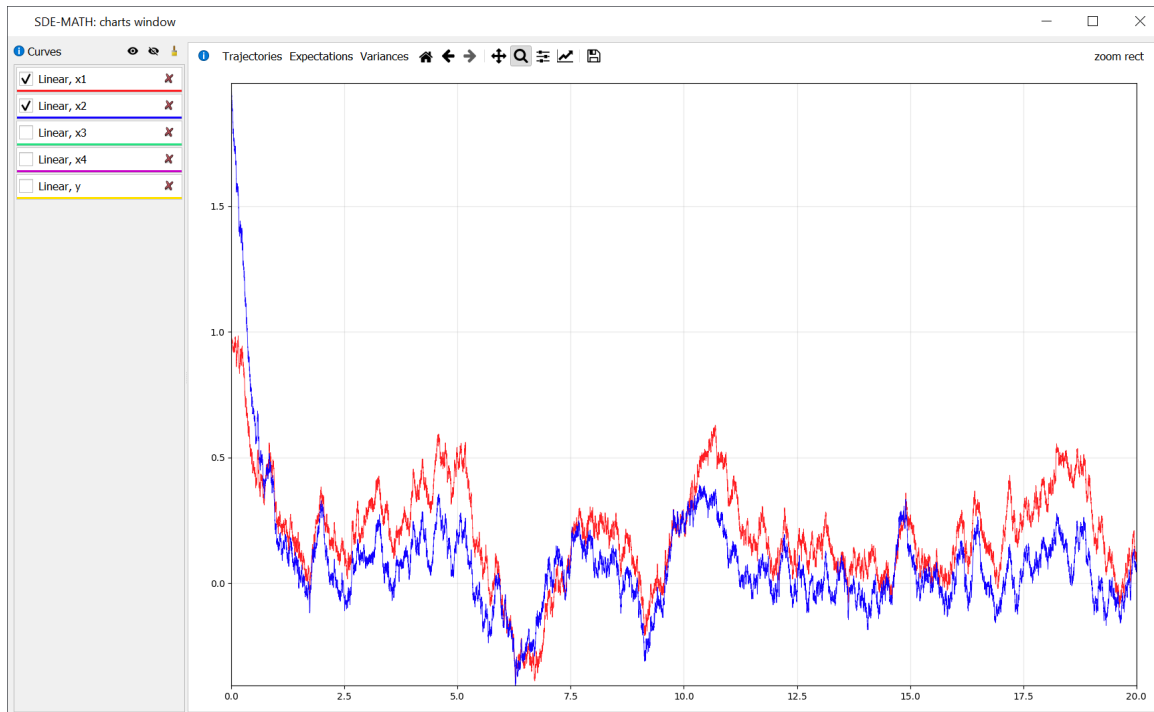


Figure 34: Modeling results (components of solution)

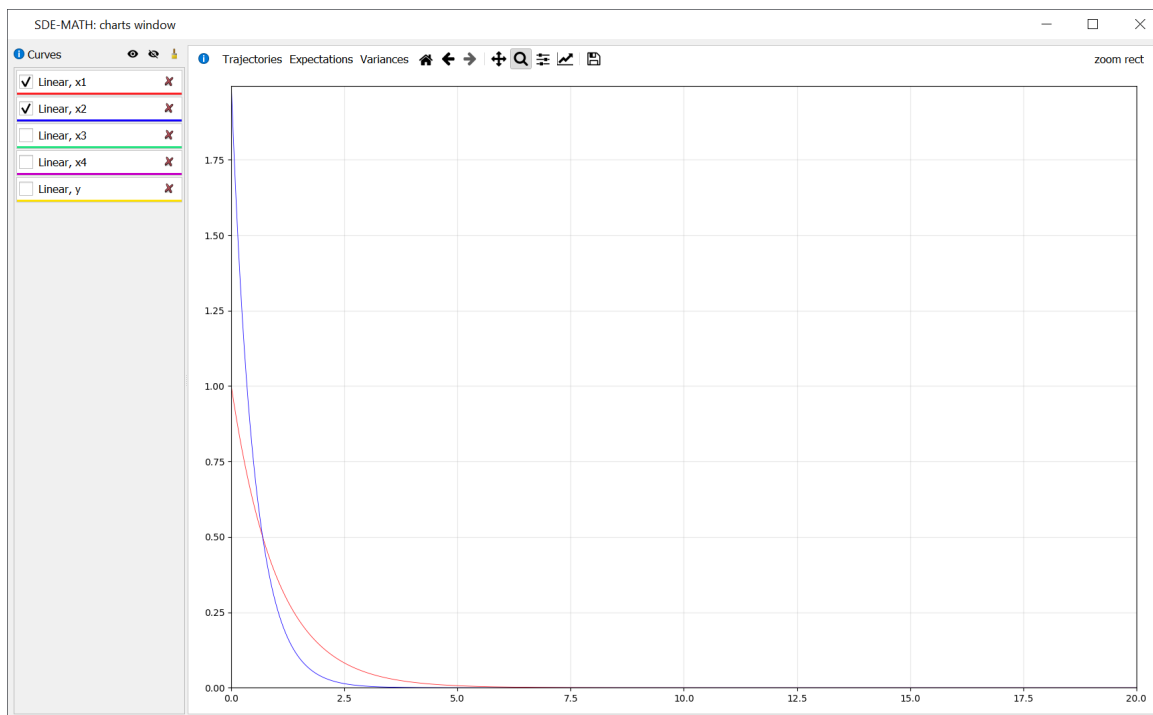


Figure 35: Modeling results (expectations)

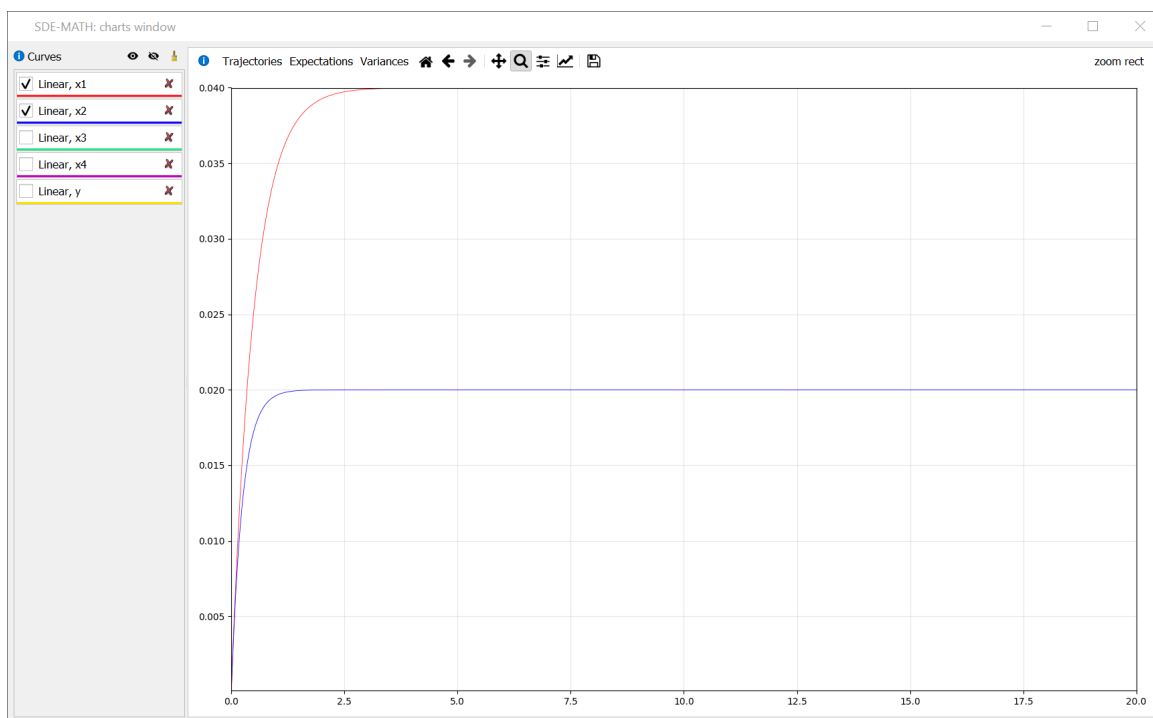


Figure 36: Modeling results (variances)

5 The Results Obtained Using the SDE-MATH Software Package

This section represents the results that were obtained with the SDE-MATH software package at the current stage of the development.

5.1 The Calculated Fourier–Legendre Coefficients

When application runs first time it performs loading of Fourier–Legendre coefficients basic pack in the database from the files. Further, in Listings [1–4](#) few examples of them can be seen.

Listing 1: The Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{000}$ examples

```

1 C_0:0:0 = 4/3
2 C_0:0:1 = -2/3
3 C_0:0:2 = 2/15
4 C_0:0:3 = 0
5 ...
6 C_0:6:4 = -4/429
7 C_0:6:5 = 2/143
8 C_0:6:6 = 2/2145
9 C_1:0:0 = 2/3
10 ...
11 C_47:33:44 = 3874457388633368/31334948307735906710660485
12 C_47:33:45 = 0
13 C_47:33:46 = 52892292737827468/2224781329849249376456894435
14 C_47:34:0 = 0
15 C_47:34:1 = 0

```

Listing 2: The Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{0000}$ examples

```

1 C_0:0:0:0 = 2/3
2 C_0:0:0:1 = -2/5
3 C_0:0:0:2 = 2/15
4 C_0:0:1:0 = -2/15
5 C_0:0:1:1 = 2/15
6 ...
7 C_1:1:0:1 = -2/35
8 C_1:1:0:2 = 0
9 C_1:1:1:0 = 2/105
10 C_1:1:1:1 = 0
11 ...
12 C_20:20:20:1 = -2401828/165607444685315115
13 C_20:20:20:2 = 0
14 C_20:20:20:3 = -1241929832/77669891557412788935
15 C_20:20:20:4 = 0

```

Listing 3: The Fourier–Legendre coefficients $C_{j_5 j_4 j_3 j_2 j_1}^{00000}$ examples

```

1 C_0:0:0:0:0 = 4/15
2 C_0:0:0:0:1 = -8/45
3 C_0:0:0:1:0 = -4/45
4 C_0:0:0:1:1 = 8/105
5 C_0:0:1:0:0 = 0
6 C_0:0:1:0:1 = 4/315
7 ...
8 C_1:1:0:1:0 = -4/315
9 C_1:1:0:1:1 = 4/315
10 C_1:1:1:0:0 = 2/105
11 C_1:1:1:0:1 = -8/945
12 C_1:1:1:1:0 = 2/945
13 C_1:1:1:1:1 = 0
14 ...
15 C_20:20:20:20:1 = 0
16 C_20:20:20:20:2 = -249877207023610010/10969028984480026752856704371
17 C_20:20:20:20:3 = 0
18 C_20:20:20:20:4 = -307937246954575016/571102494076952592887484313076115

```

Listing 4: The Fourier–Legendre coefficients $C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000}$ examples

```

1 C_0:0:0:0:0:0 = 4/45
2 C_0:0:0:0:0:1 = -4/63
3 C_0:0:0:0:0:2 = 2/63
4 C_0:0:0:0:1:0 = -4/105
5 ...
6 C_2:1:0:1:0:2 = -2/1575
7 C_2:1:0:1:1:0 = 38/22275
8 C_2:1:0:1:1:1 = -2/1575
9 C_2:1:0:1:1:2 = 68/81081
10 ...
11 C_15:15:15:15:15:15 = 0
12 C_15:15:15:15:15:16 = -798538765964/243076352242280511713913783475
13 C_15:15:15:15:15:17 = 0
14 C_15:15:15:15:15:18 = -59075427603328/17302616709609603697454044769175

```

5.2 Accuracy Settings

From Theorem 3 (see formulas (51)–(66)) it follows that the number p in the formula (48) should be chosen individually for various combinations of indices $i_1, \dots, i_k \in \{1, \dots, m\}$. As follows from Listing 5 (see below) and the results of work [73], these numbers p in the overwhelming majority of cases do not exceed the number p from the formula (51). Moreover, all the mentioned numbers p are many times less than the number p selected using the formula (49) (due to the presence of the multiplier factor $k!$ on the left-hand side of (49)).

In this work, we have replaced the mentioned numbers p for all possible

combinations of indices $i_1, \dots, i_k \in \{1, \dots, m\}$ with the number p according to the formula (51). This is possible due to the results of Listing 6. This listing shows that the above replacement does not lead to noticeable accuracy loss of the mean square approximation of iterated Itô stochastic integrals (for more details see [73]).

Thus, in this paper we decided to exclude the multiplier factor $k!$ in the conditions for choosing the numbers q_1, \dots, q_{15} (see (169)–(202)). Recall that these numbers are used to construct the approximations of iterated Itô and Stratonovich stochastic integrals from the numerical schemes (13)–(16), (25)–(28). The test script was written. The results of its work are presented in Listings 5 and 6, where

1. dt is the integration step;
2. $q1(1,2)$ means p from (54), $q1(2,3)$ means p from (55), $q1(1,3)$ means p from (56), $q1$ means p from (51) for $k = 3$;
3. $C = 1$ (see (17) and (29));
4. error 1 means the left-hand side of (171);
5. error 2 means the left-hand side of (54) divided by $(T - t)^3$;
6. error 3 means the left-hand side of (56) divided by $(T - t)^3$;
7. error 4 means the left-hand side of (55) divided by $(T - t)^3$.

The above idea of calculation of the numbers q_1, \dots, q_{15} is described in Listing 109.

Listing 5: Accuracy calculation module

```

1
2 dt = 0.011
3   q1   = 12
4   q1 (1, 2) = 6
5   q1 (1, 3) = 12
6   q1 (2, 3) = 6
7
8 dt = 0.008
9   q1   = 16
10  q1 (1, 2) = 8
11  q1 (1, 3) = 16
12  q1 (2, 3) = 8

```

```
13
14 dt = 0.0045
15   q1    = 28
16   q1 (1, 2) = 14
17   q1 (1, 3) = 28
18   q1 (2, 3) = 14
19
20 dt = 0.0035
21   q1    = 36
22   q1 (1, 2) = 18
23   q1 (1, 3) = 36
24   q1 (2, 3) = 18
25
26 dt = 0.0027
27   q1    = 47
28   q1 (1, 2) = 23
29   q1 (1, 3) = 47
30   q1 (2, 3) = 23
31
32 dt = 0.0025
33   q1    = 50
34   q1 (1, 2) = 25
35   q1 (1, 3) = 51
36   q1 (2, 3) = 25
37
38
39 Process finished with exit code 0
```

Listing 6: Accuracy calculation module

```
1
2 dt = 0.011
3   error 1 = 0.010153888451696458
4   q1     = 12
5   error 2 = 0.005076944225848201
6   q1 (1, 2) = 12
7   error 3 = 0.010307776903394072
8   q1 (1, 3) = 12
9   error 4 = 0.005076944225848284
10  q1 (2, 3) = 12
11
12 dt = 0.008
13   error 1 = 0.007681193827577537
14   q1     = 16
15   error 2 = 0.003840596913789046
16   q1 (1, 2) = 16
17   error 3 = 0.0077866300793989485
18   q1 (1, 3) = 16
19   error 4 = 0.003840596913789157
20   q1 (2, 3) = 16
21
22 dt = 0.0045
```

```

23 error 1 = 0.004432832059862973
24 q1 = 28
25 error 2 = 0.0022164160299319446
26 q1 (1, 2) = 28
27 error 3 = 0.004479699207443705
28 q1 (1, 3) = 28
29 error 4 = 0.002216416029932139
30 q1 (2, 3) = 28
31
32 dt = 0.0035
33 error 1 = 0.0034564405520411956
34 q1 = 36
35 error 2 = 0.0017282202760207088
36 q1 (1, 2) = 36
37 error 3 = 0.003488223569838411
38 q1 (1, 3) = 36
39 error 4 = 0.0017282202760210141
40 q1 (2, 3) = 36
41
42 dt = 0.0027
43 error 1 = 0.0026523659377455377
44 q1 = 47
45 error 2 = 0.00132618296887127
46 q1 (1, 2) = 47
47 error 3 = 0.0026731529281250332
48 q1 (1, 3) = 47
49 error 4 = 0.0013261829688714366
50 q1 (2, 3) = 47
51
52 dt = 0.0025
53 error 1 = 0.002494053620431952
54 q1 = 50
55 error 2 = 0.0012470268102122428
56 q1 (1, 2) = 50
57 error 3 = 0.0025128597161119537
58 q1 (1, 3) = 50
59 error 4 = 0.0012470268102123538
60 q1 (2, 3) = 50
61
62
63 Process finished with exit code 0

```

5.3 Testing Example (Nonlinear System of Itô SDEs)

The input data for testing of the SDE-MATH software package correspond to the autonomous variant of nonlinear system of Itô SDE (II) with multidimensional non-commutative noise. More precisely, we choose $n = 2$, $m = 2$, $\mathbf{x}_0^{(1)} = 1$, $\mathbf{x}_0^{(2)} = 1.5$,

$$\mathbf{a}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \begin{pmatrix} -5\mathbf{x}^{(1)} \\ -5\mathbf{x}^{(2)} \end{pmatrix},$$

$$B(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \begin{pmatrix} 0.5 \cdot \sin(\mathbf{x}^{(1)}) & \mathbf{x}^{(2)} \\ \mathbf{x}^{(2)} & 0.5 \cdot \cos(\mathbf{x}^{(1)}) \end{pmatrix}.$$

Figures 37–92 related to the strong high-order Taylor–Itô and Taylor–Stratonovich schemes (12)–(16), (24)–(28) for the Itô SDE (1) represent modeling results.

Test machine specifications are CPU with maximum core frequency 4.2 GHz and 16GB of RAM.

5.4 Visualization and Numerical Results for Nonlinear System of Itô SDEs Obtained via the SDE-MATH Software Package

This subsection is fully devoted to modeling logs and results visualization. They are presented on Figures 37–91

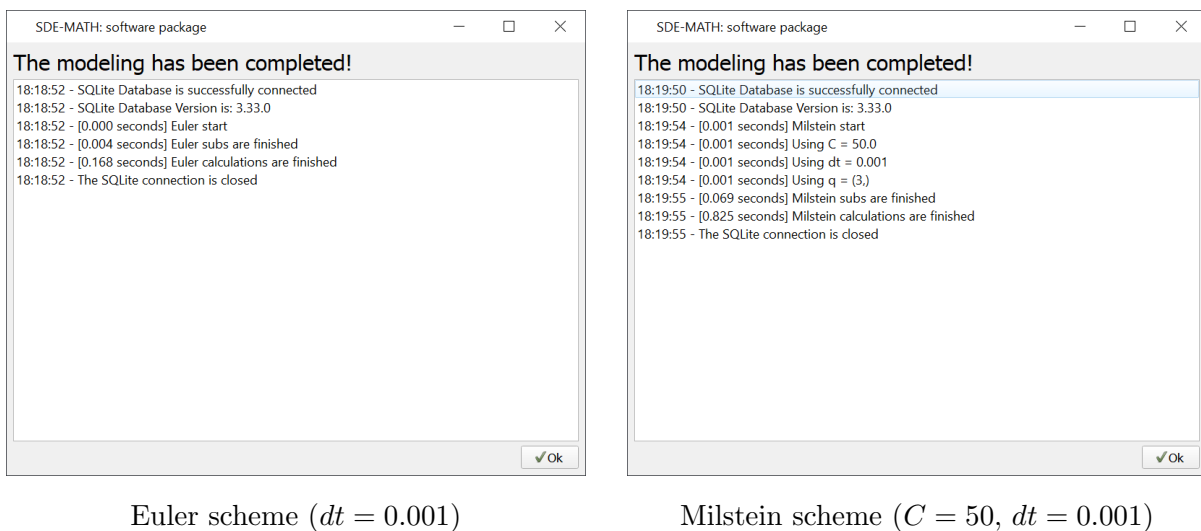


Figure 37: Modeling logs

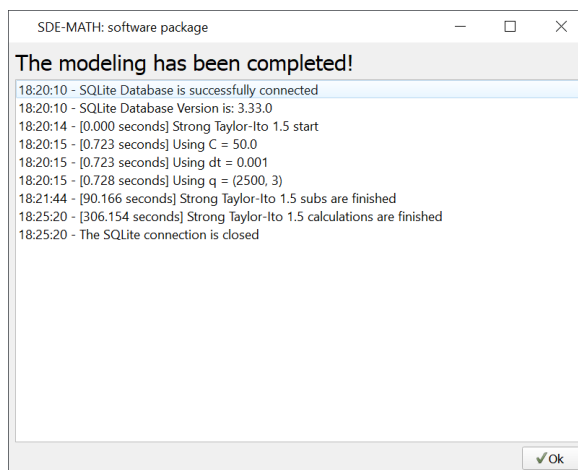


Figure 38: Strong Taylor–Itô scheme of order 1.5 ($C = 50$, $dt = 0.001$)

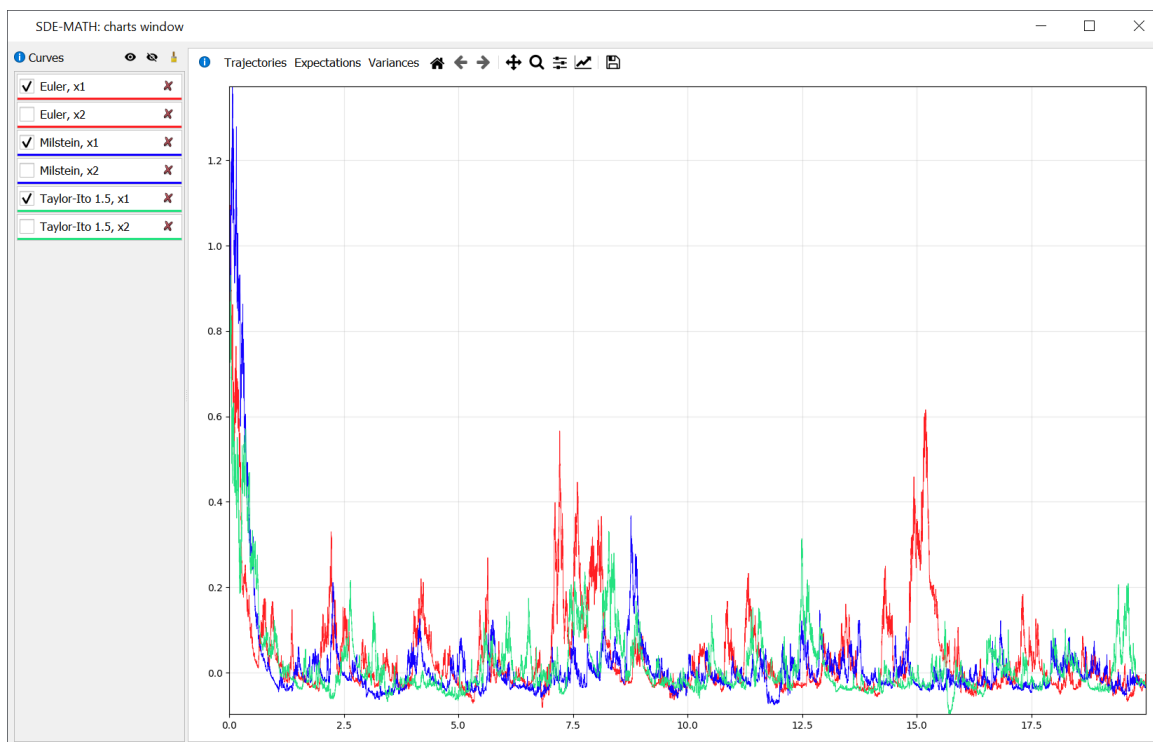


Figure 39: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 50$, $dt = 0.001$)

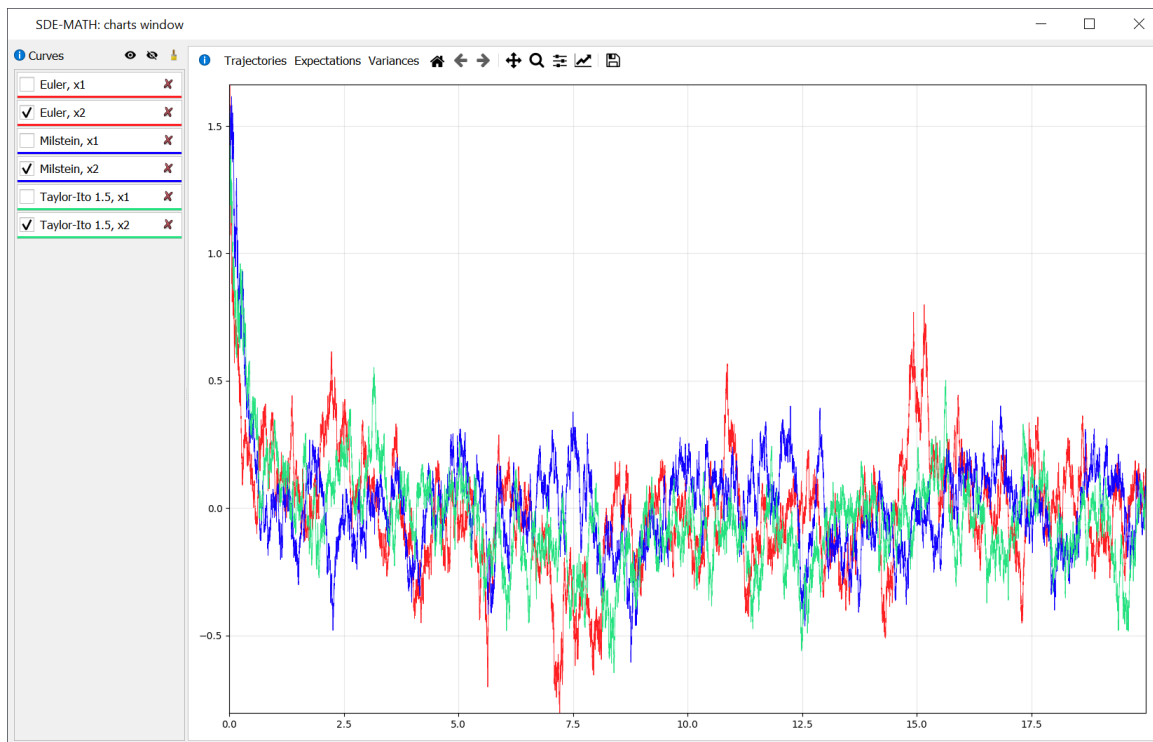
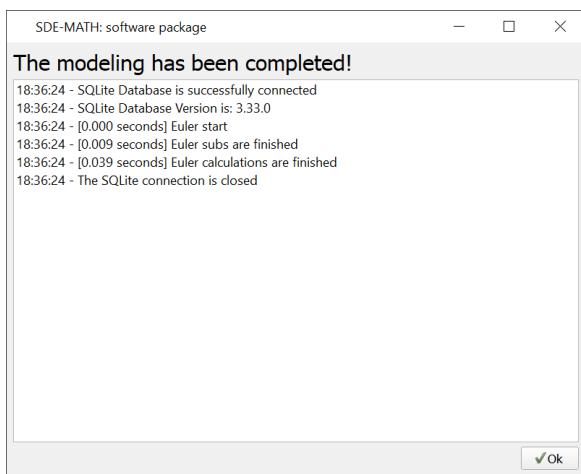
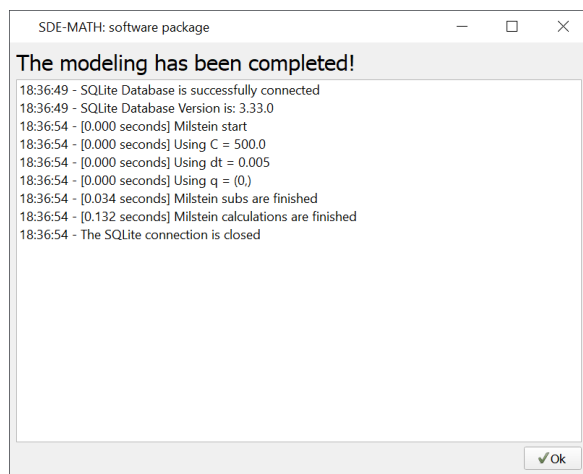


Figure 40: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 50$, $dt = 0.001$)

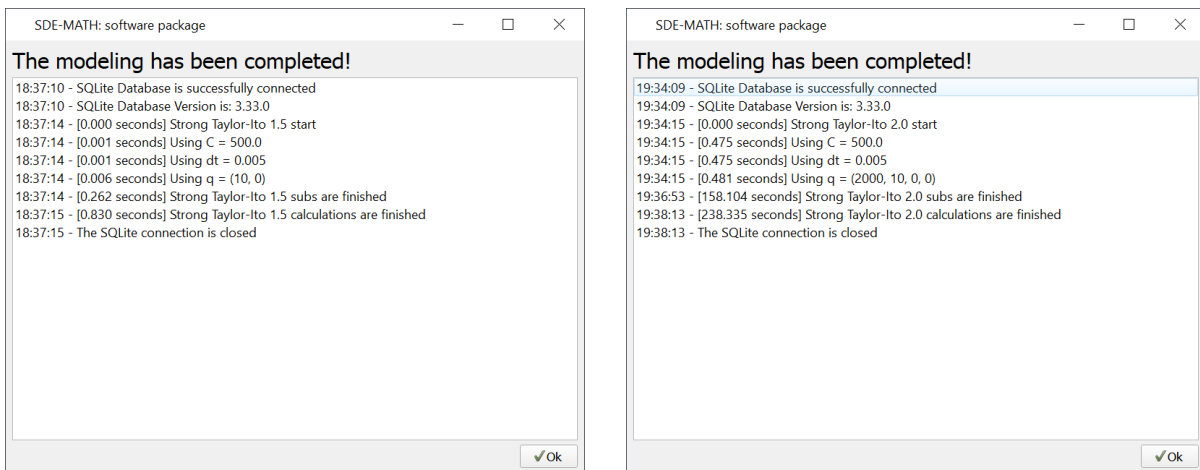


Euler scheme ($dt = 0.005$)



Milstein scheme ($C = 500$, $dt = 0.005$)

Figure 41: Modeling logs



Strong Taylor-Itô scheme of order 1.5 ($C = 500, dt = 0.005$)

Strong Taylor-Itô scheme of order 2.0 ($C = 500, dt = 0.005$)

Figure 42: Modeling logs

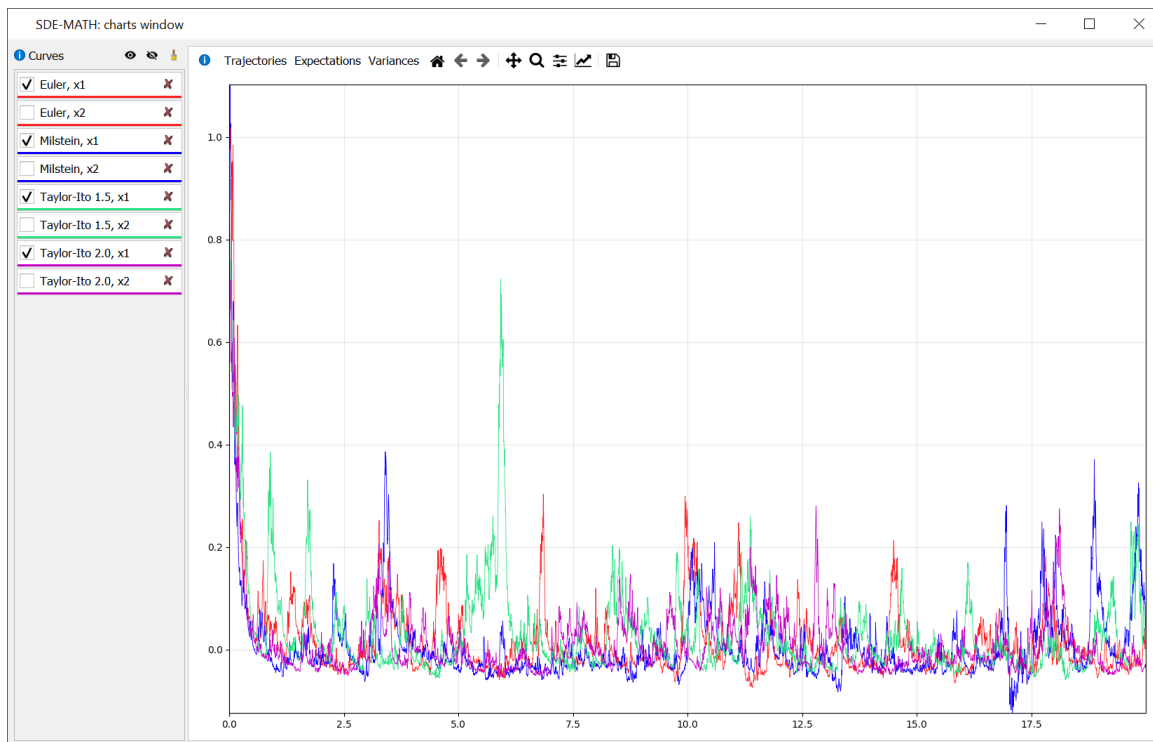


Figure 43: Strong Taylor-Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 500, dt = 0.005$)

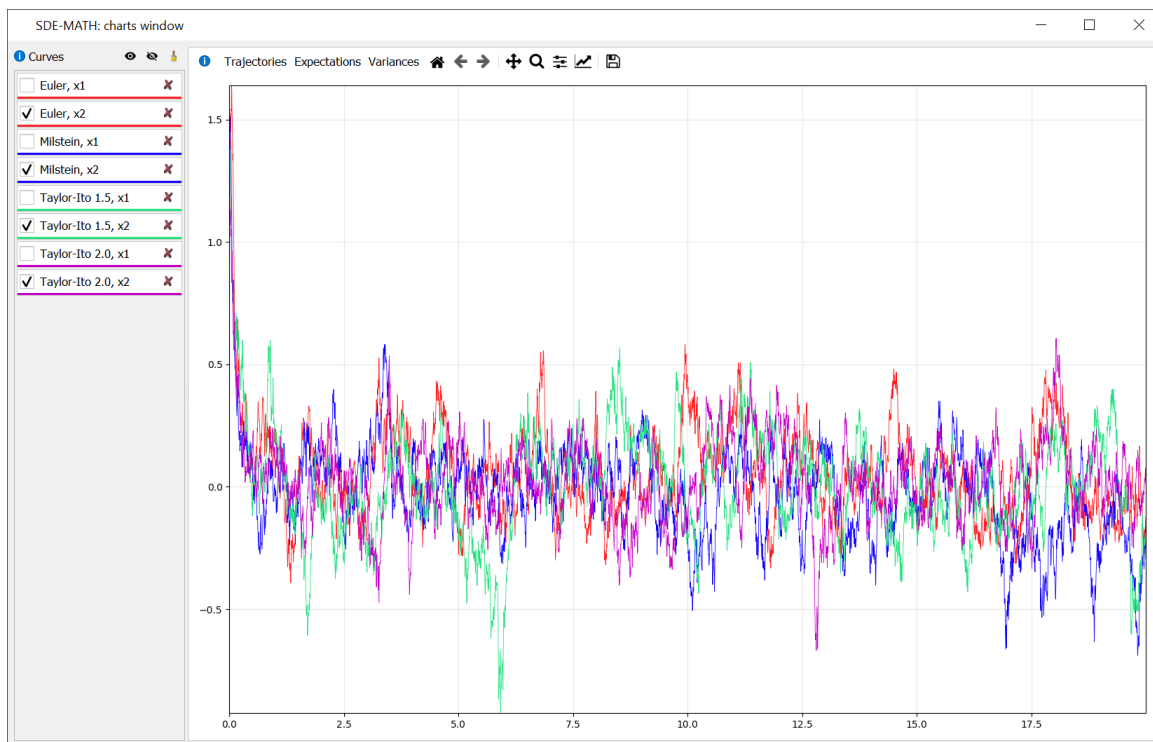


Figure 44: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 500$, $dt = 0.005$)

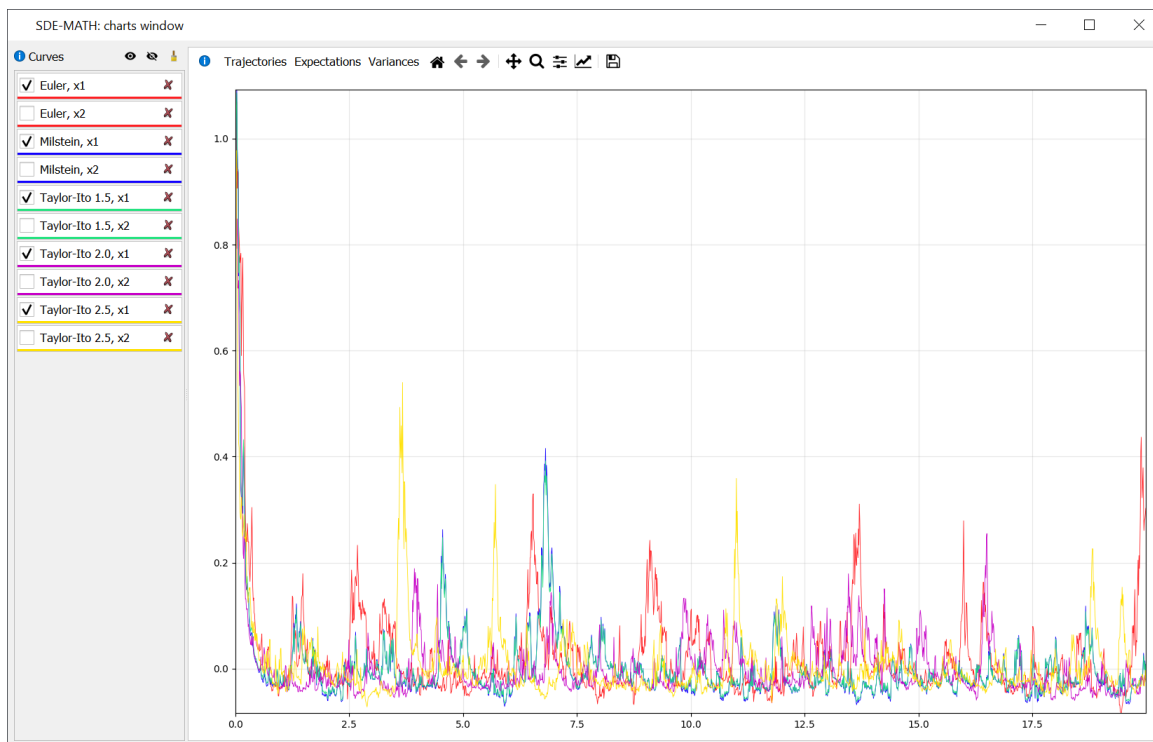
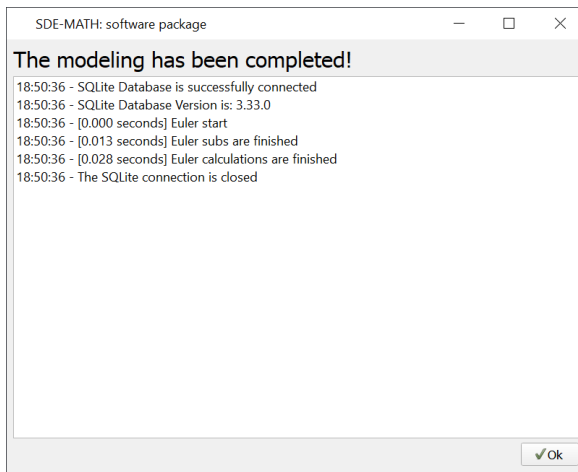
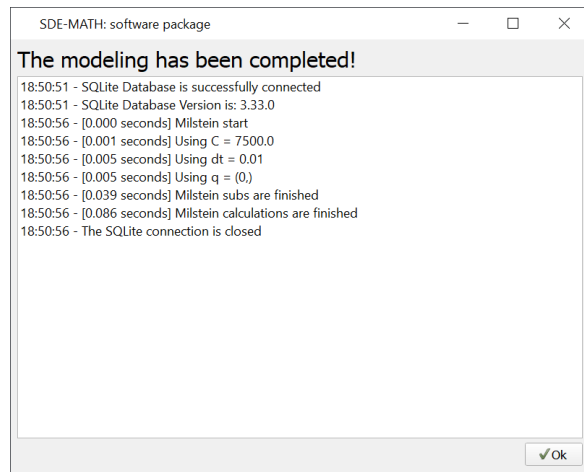


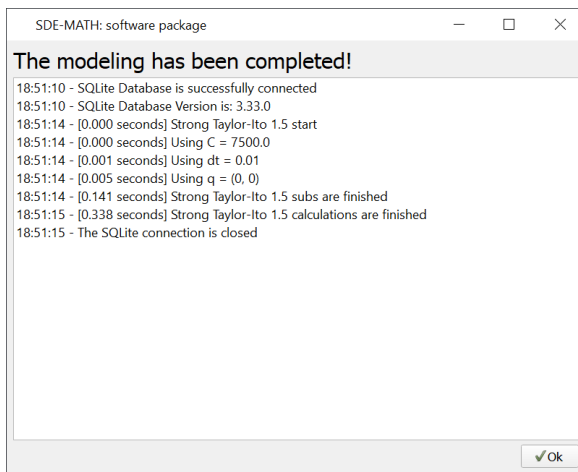
Figure 45: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 7500$, $dt = 0.01$)



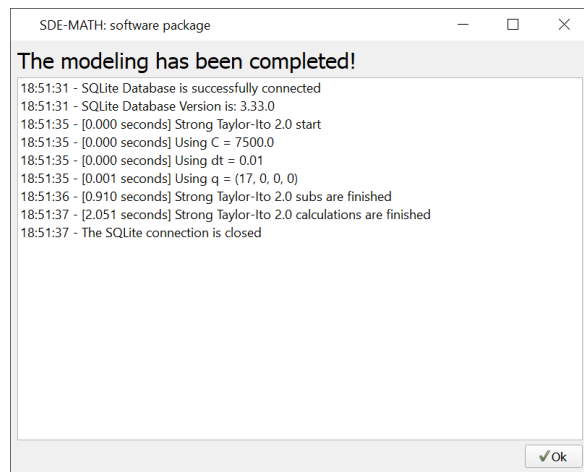
Euler scheme ($dt = 0.01$)



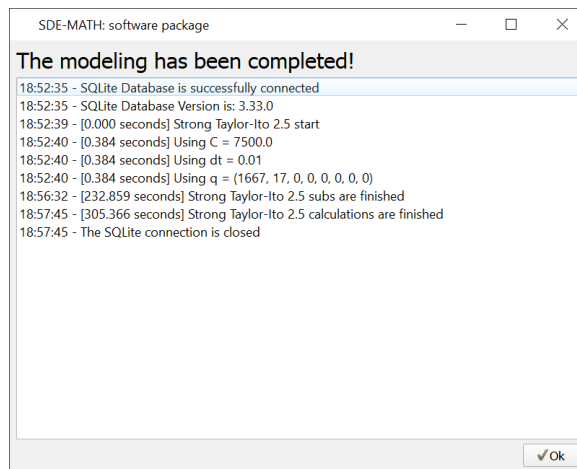
Milstein scheme ($C = 7500, dt = 0.01$)



Strong Taylor-Itô scheme of order 1.5 ($C = 7500, dt = 0.01$)



Strong Taylor-Itô scheme of order 2.0 ($C = 7500, dt = 0.01$)



Strong Taylor-Itô scheme of order 2.5 ($C = 7500, dt = 0.01$)

Figure 46: Modeling logs

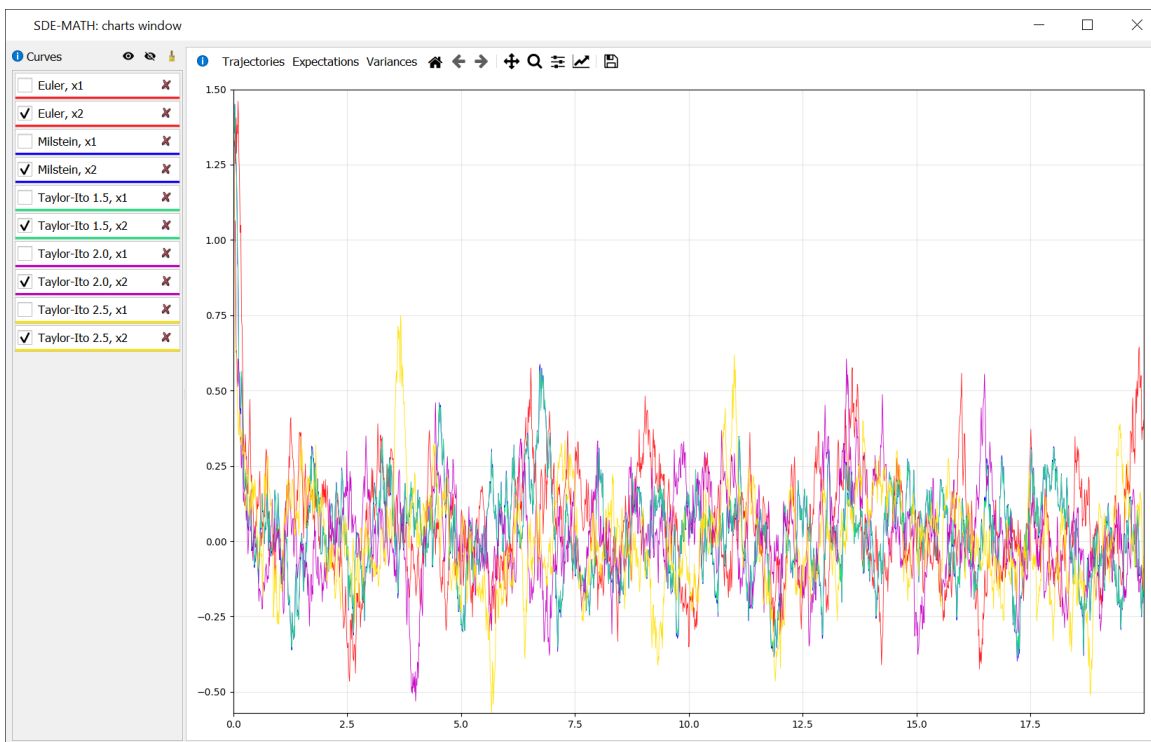


Figure 47: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 7500$, $dt = 0.01$)

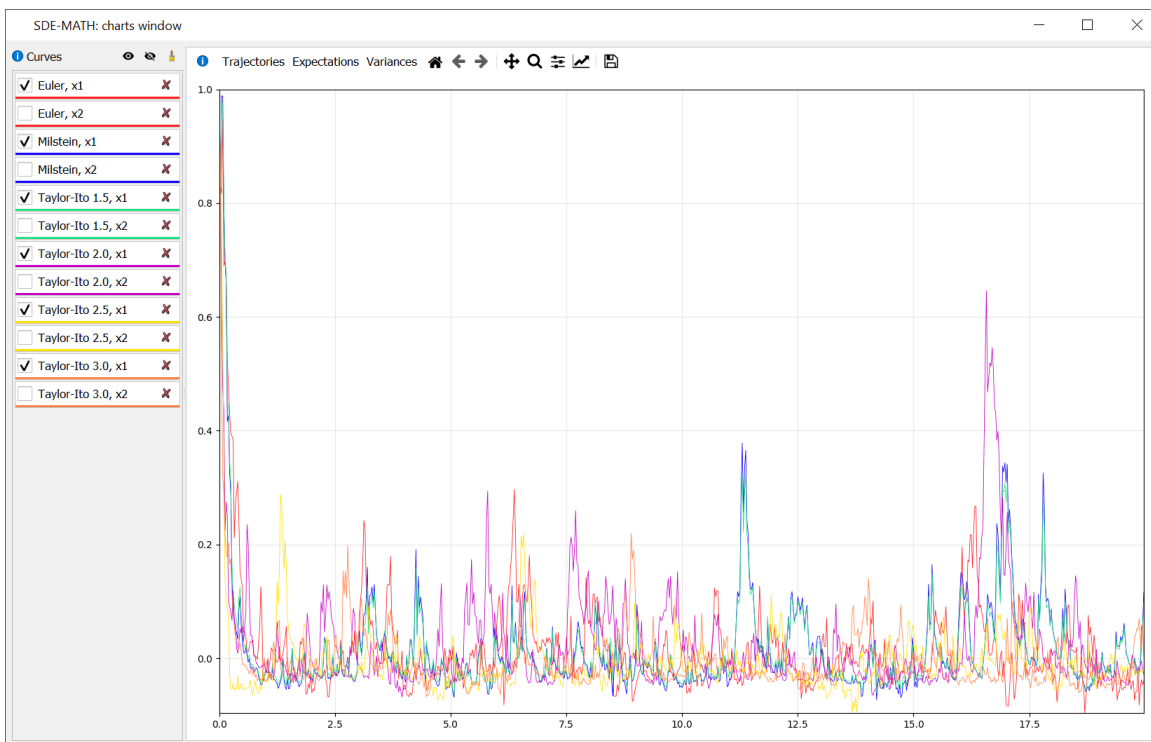
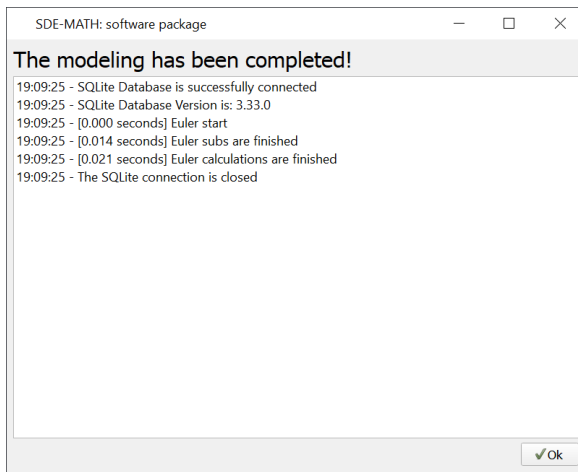
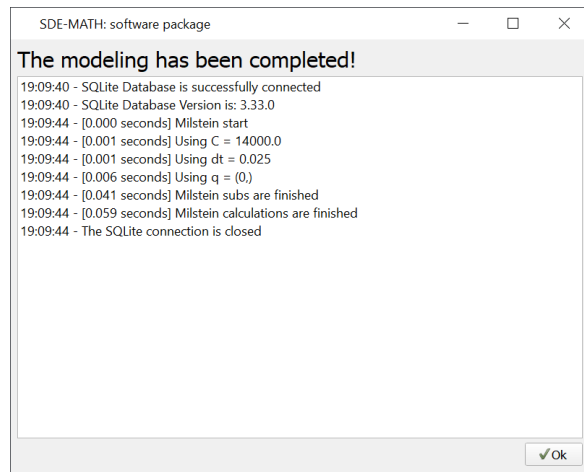


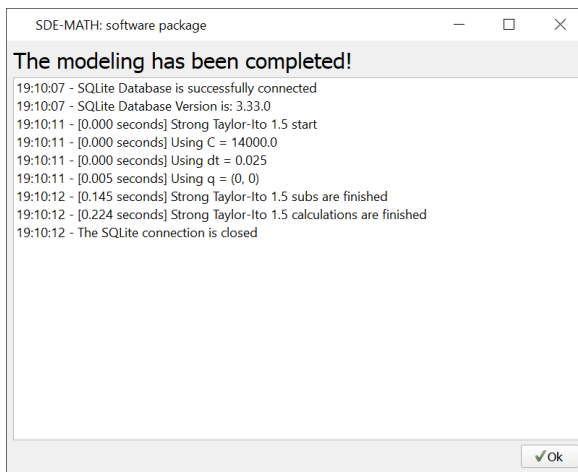
Figure 48: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 14000$, $dt = 0.025$)



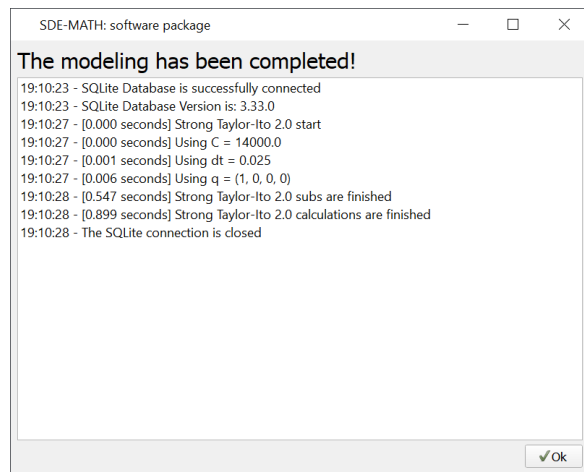
Euler scheme ($dt = 0.025$)



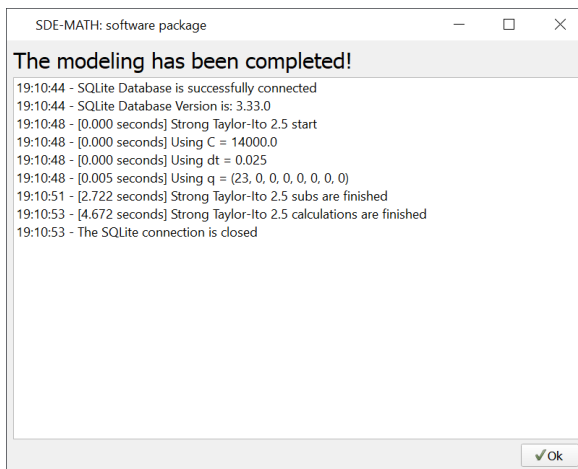
Milstein scheme ($C = 14000, dt = 0.025$)



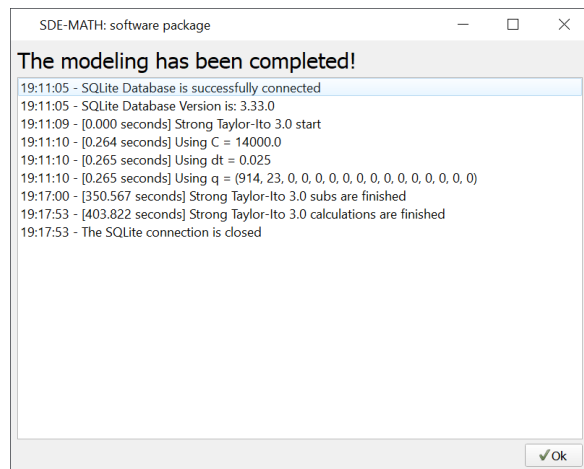
Strong Taylor-Itô scheme of order 1.5 ($C = 14000, dt = 0.025$)



Strong Taylor-Itô scheme of order 2.0 ($C = 14000, dt = 0.025$)



Strong Taylor-Itô scheme of order 2.5 ($C = 14000, dt = 0.025$)



Strong Taylor-Itô scheme of order 3.0 ($C = 14000, dt = 0.025$)

Figure 49: Modeling logs

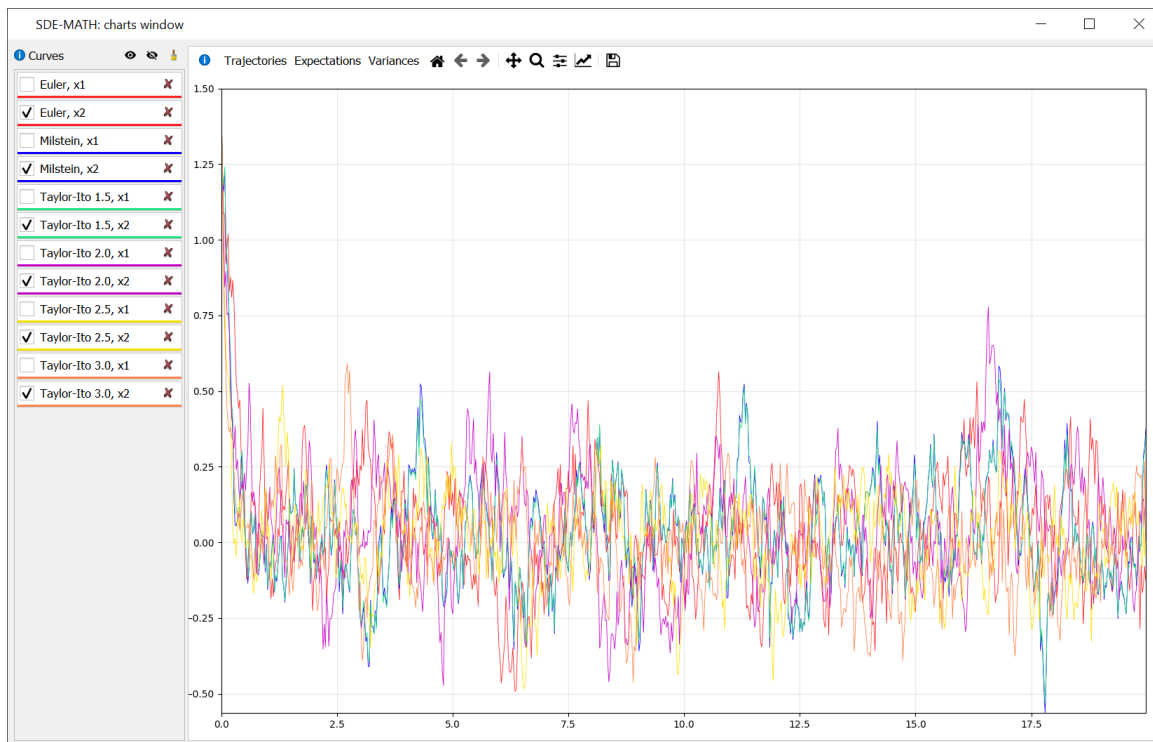
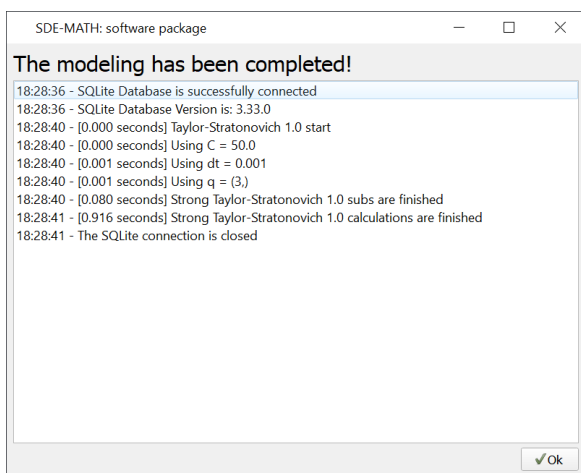
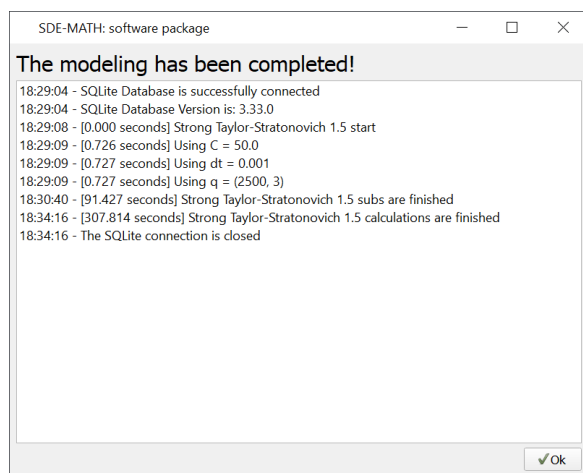


Figure 50: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 14000$, $dt = 0.025$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 50$, $dt = 0.001$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 50$, $dt = 0.001$)

Figure 51: Modeling logs

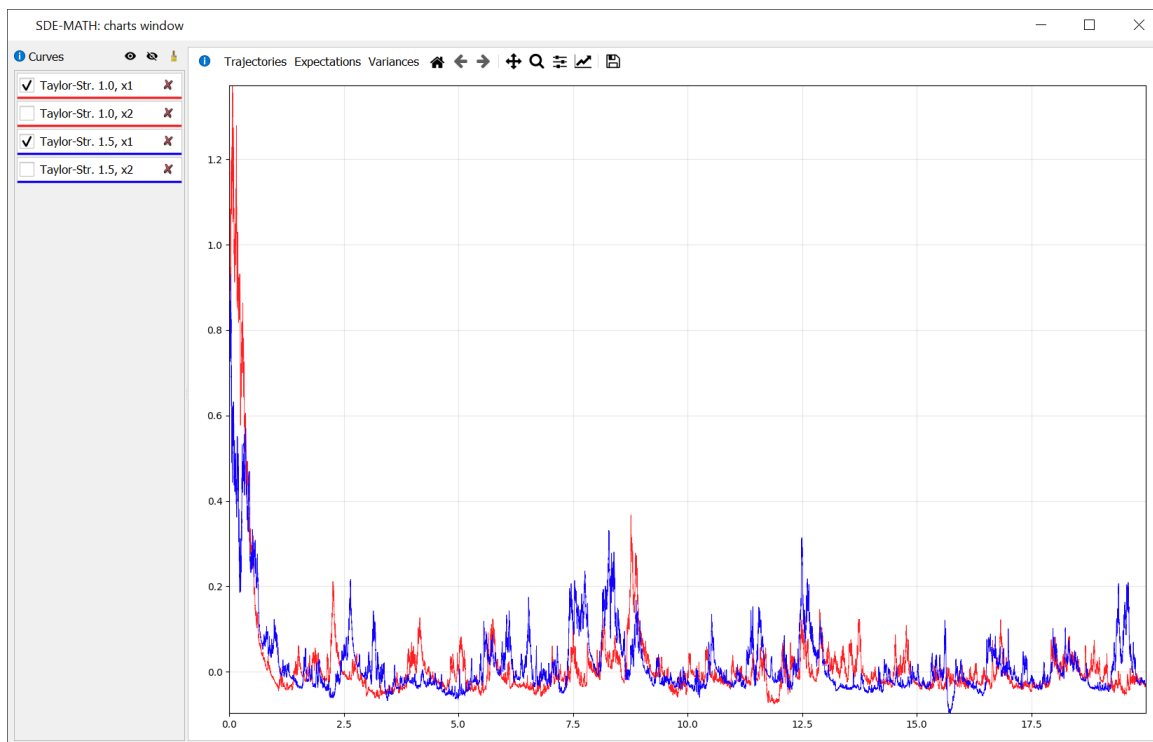


Figure 52: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 50$, $dt = 0.001$)

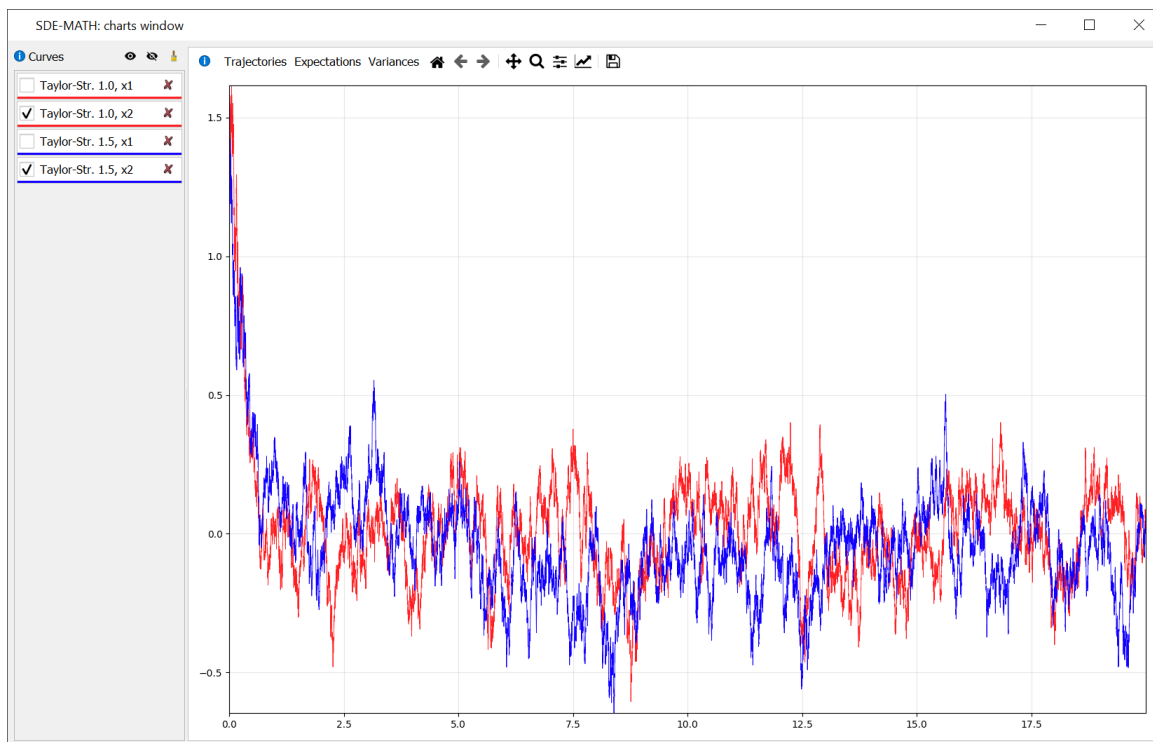


Figure 53: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 50$, $dt = 0.001$)

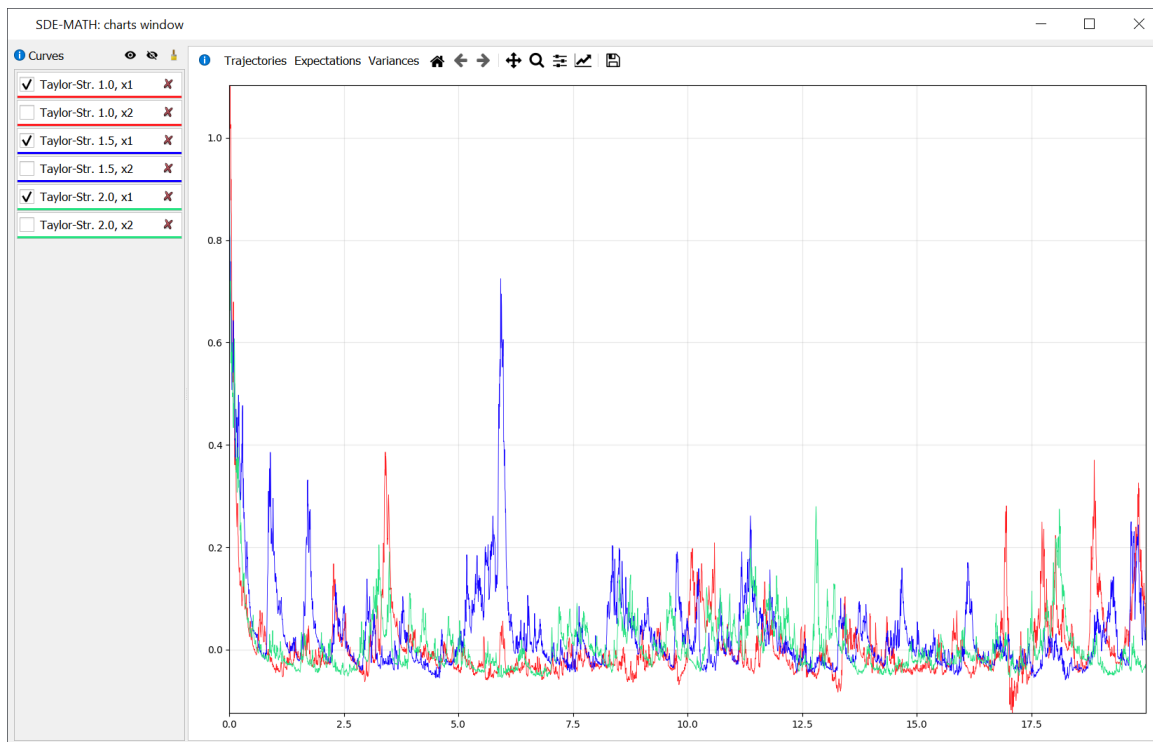
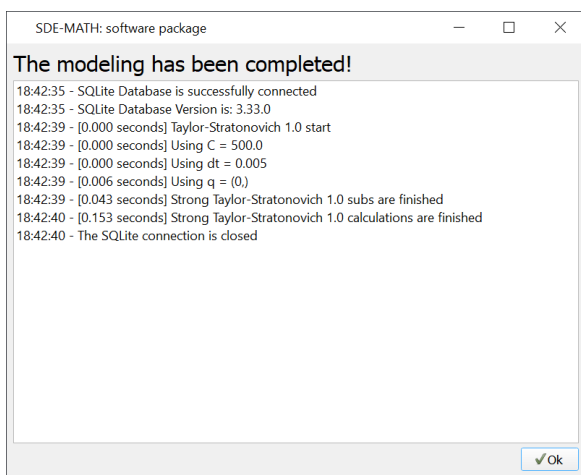
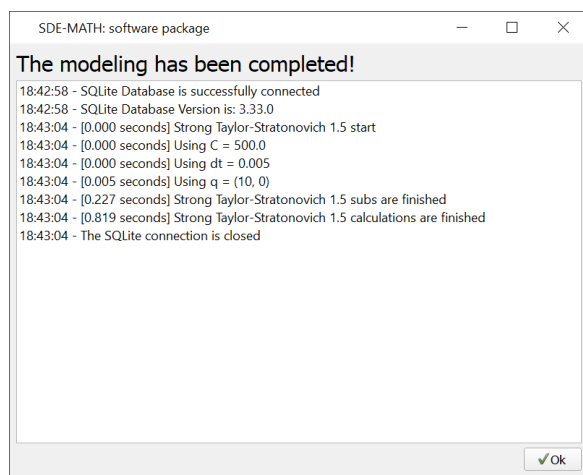


Figure 54: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 500$, $dt = 0.005$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 500$, $dt = 0.005$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 500$, $dt = 0.005$)

Figure 55: Modeling logs

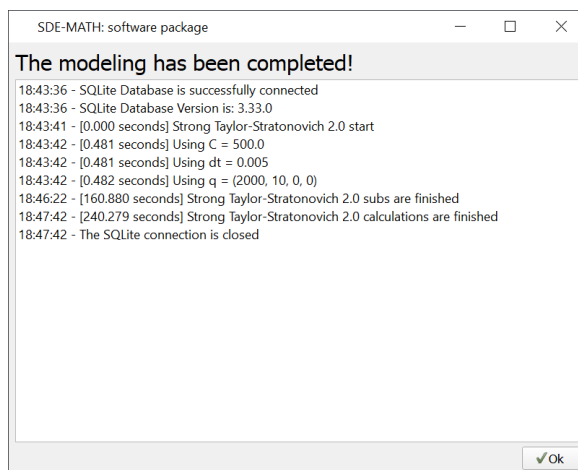


Figure 56: Strong Taylor–Stratonovich scheme of order 2.0 ($C = 500$, $dt = 0.005$)

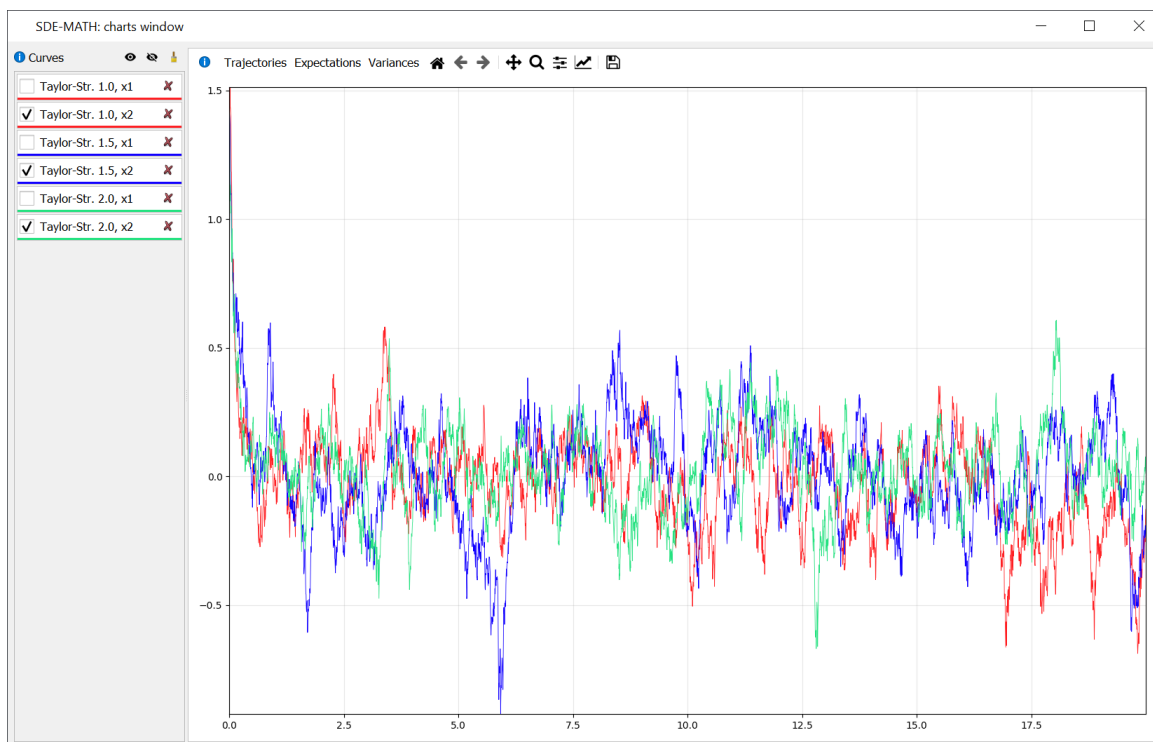


Figure 57: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 500$, $dt = 0.005$)

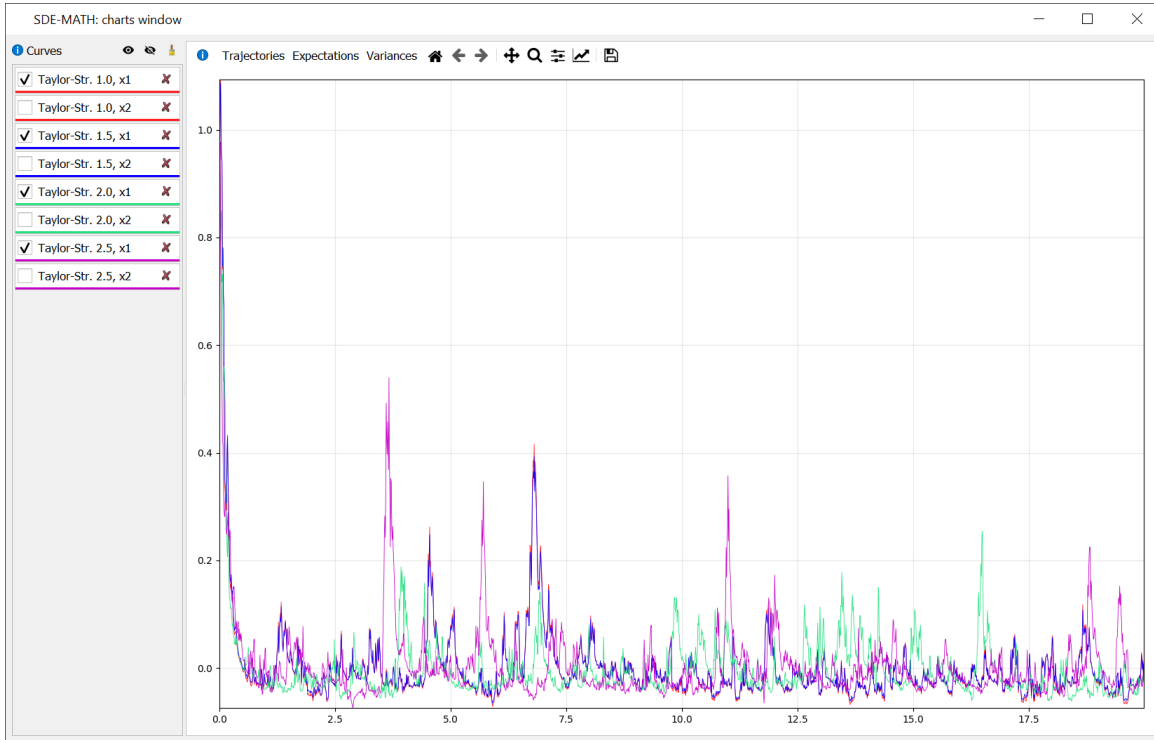
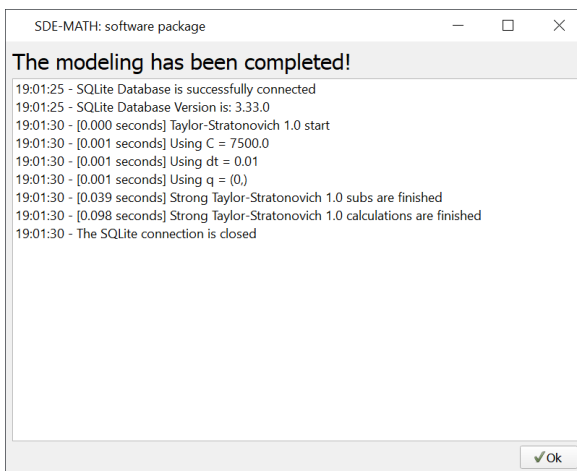
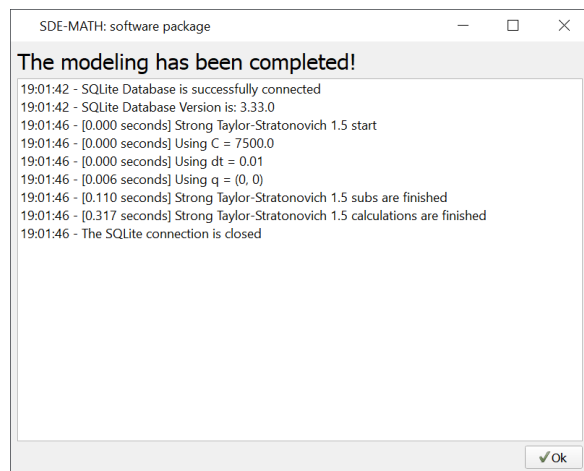


Figure 58: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 7500$, $dt = 0.01$)

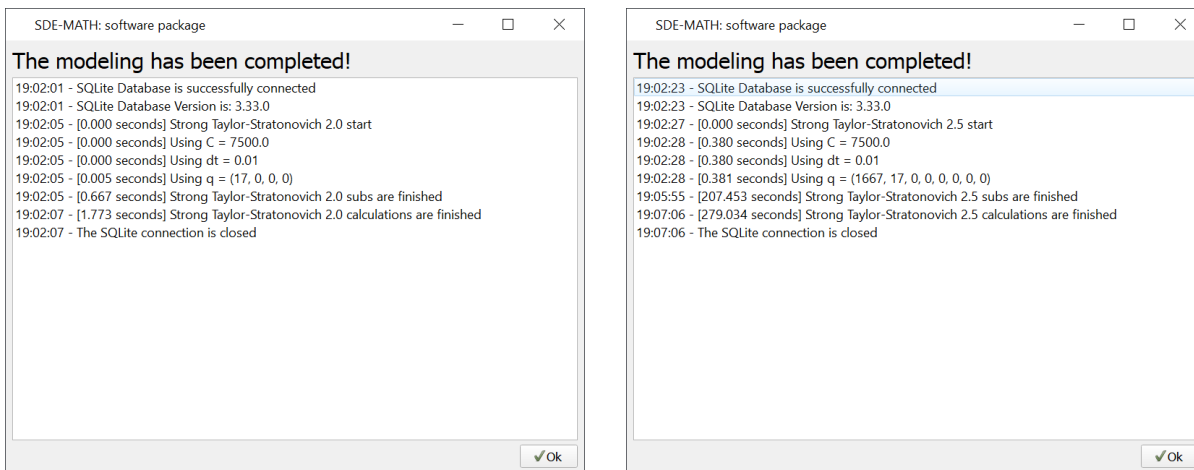


Strong Taylor–Stratonovich scheme of order 1.0 ($C = 7500$, $dt = 0.01$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 7500$, $dt = 0.01$)

Figure 59: Modeling logs



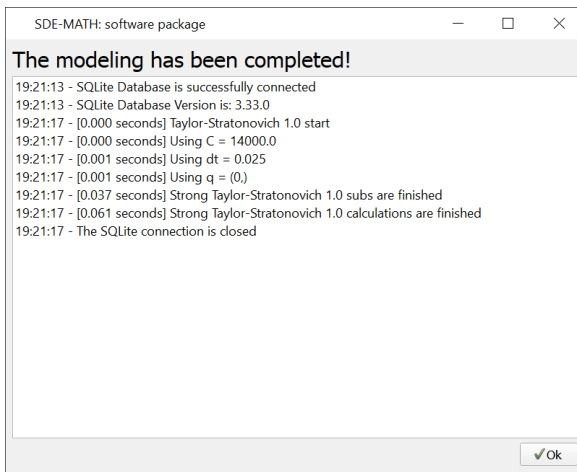
Strong Taylor–Stratonovich scheme of order 2.0 ($C = 7500$, $dt = 0.01$)

Strong Taylor–Stratonovich scheme of order 2.5 ($C = 7500$, $dt = 0.01$)

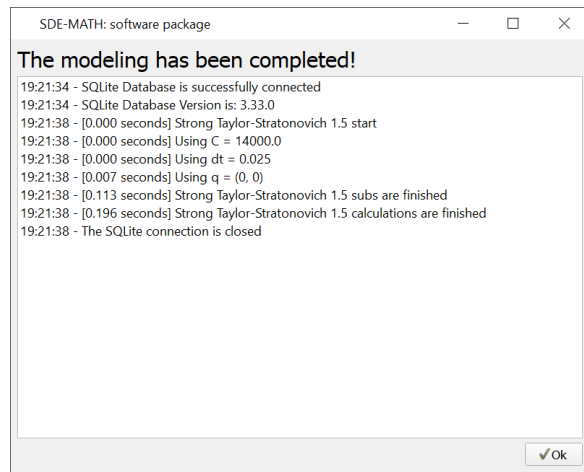
Figure 60: Modeling logs



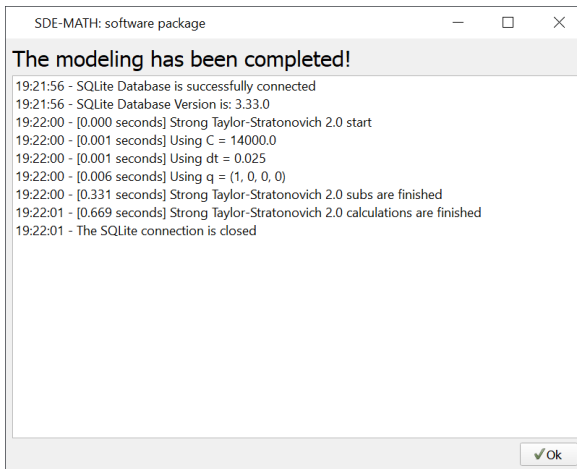
Figure 61: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 7500$, $dt = 0.01$)



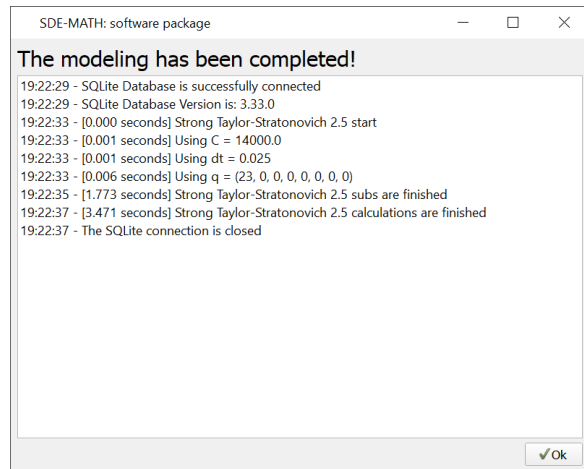
Strong Taylor-Stratonovich scheme of order 1.0 ($C = 14000$, $dt = 0.025$)



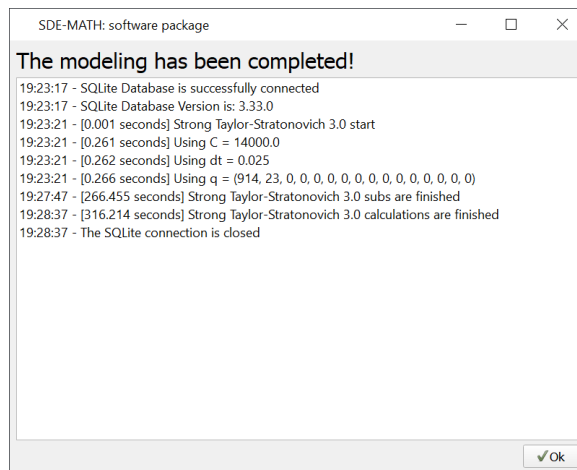
Strong Taylor-Stratonovich scheme of order 1.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor-Stratonovich scheme of order 2.0 ($C = 14000$, $dt = 0.025$)



Strong Taylor-Stratonovich scheme of order 2.5 ($C = 14000$, $dt = 0.025$)



Strong Taylor-Stratonovich scheme of order 3.0 ($C = 14000$, $dt = 0.025$)

Figure 62: Modeling logs

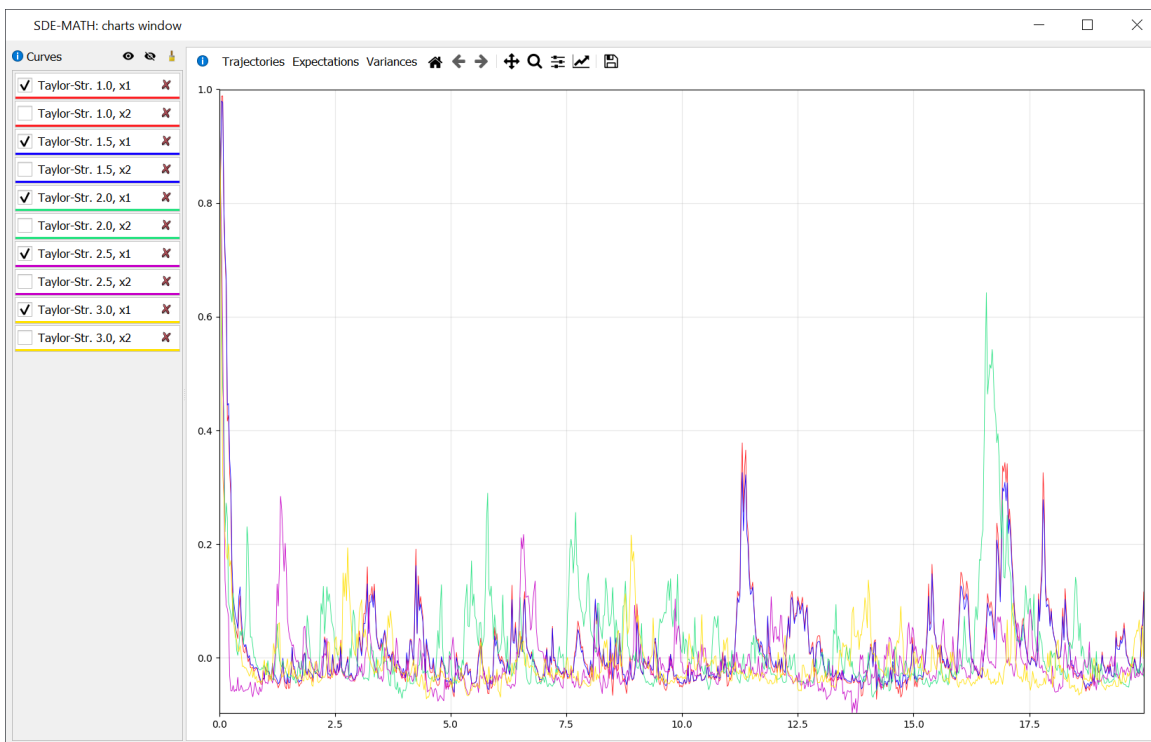


Figure 63: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 14000$, $dt = 0.025$)

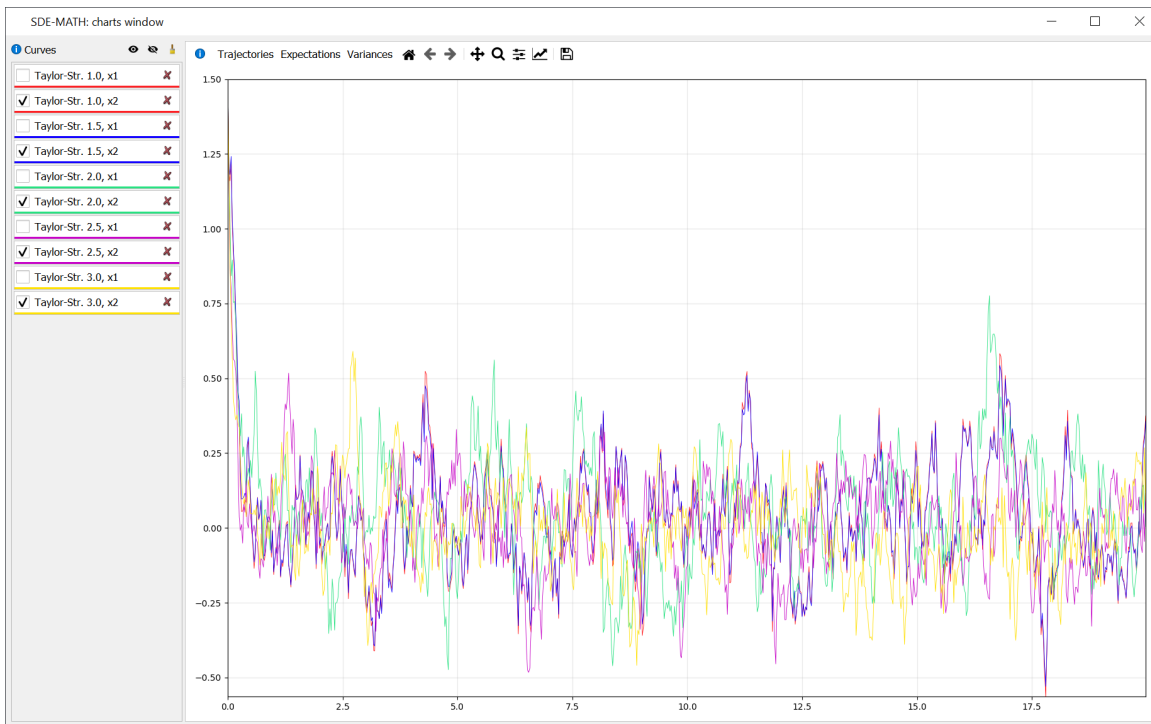


Figure 64: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 14000$, $dt = 0.025$)

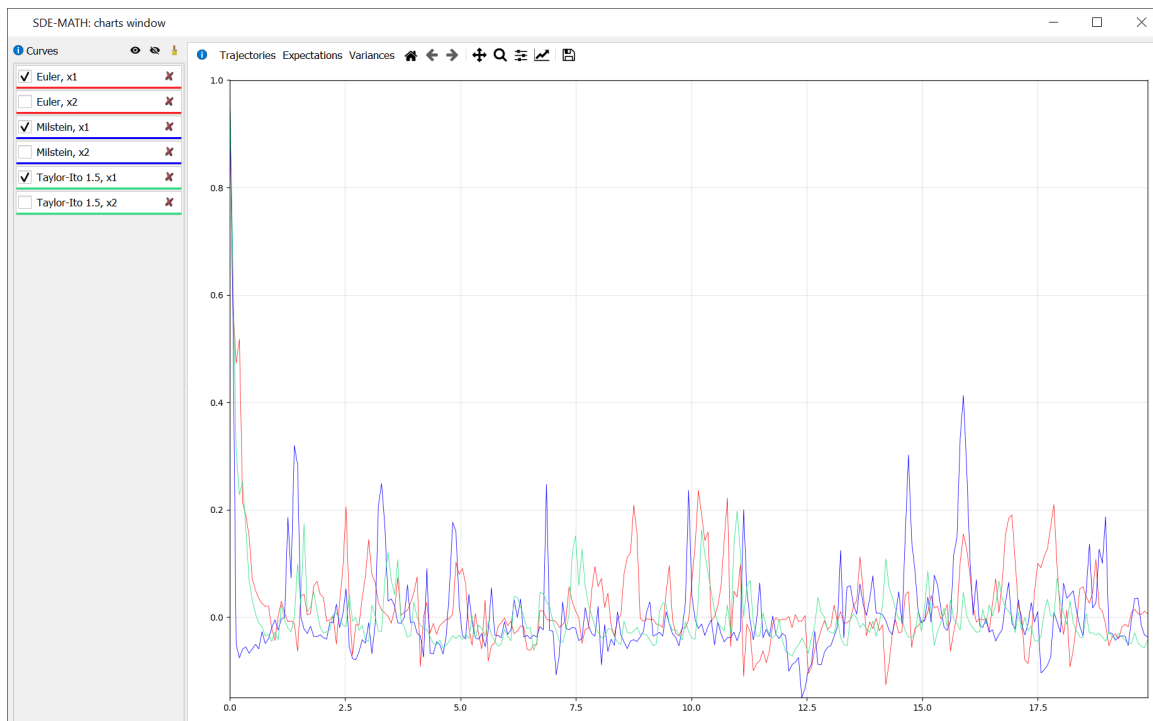
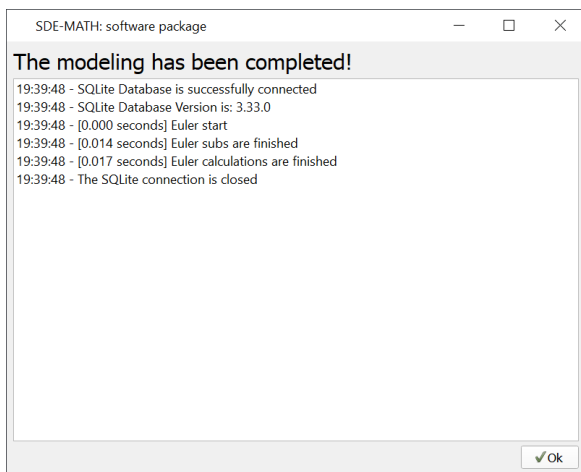
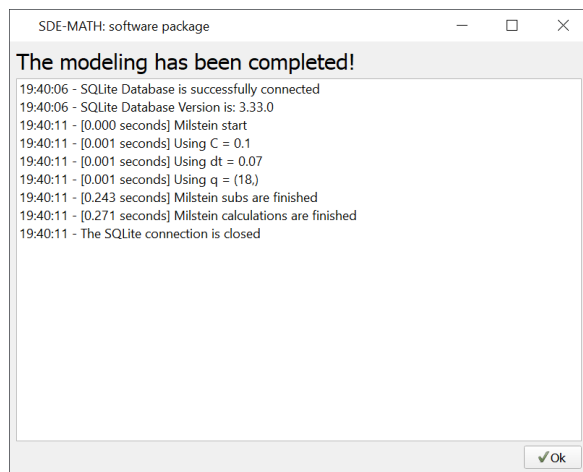


Figure 65: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.1$, $dt = 0.07$)



Euler scheme ($dt = 0.07$)



Milstein scheme ($C = 0.1$, $dt = 0.07$)

Figure 66: Modeling logs

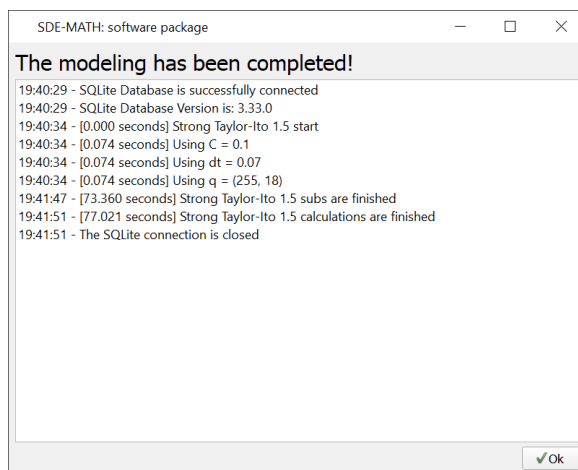


Figure 67: Strong Taylor–Itô scheme of order 1.5 ($C = 0.1$, $dt = 0.07$)

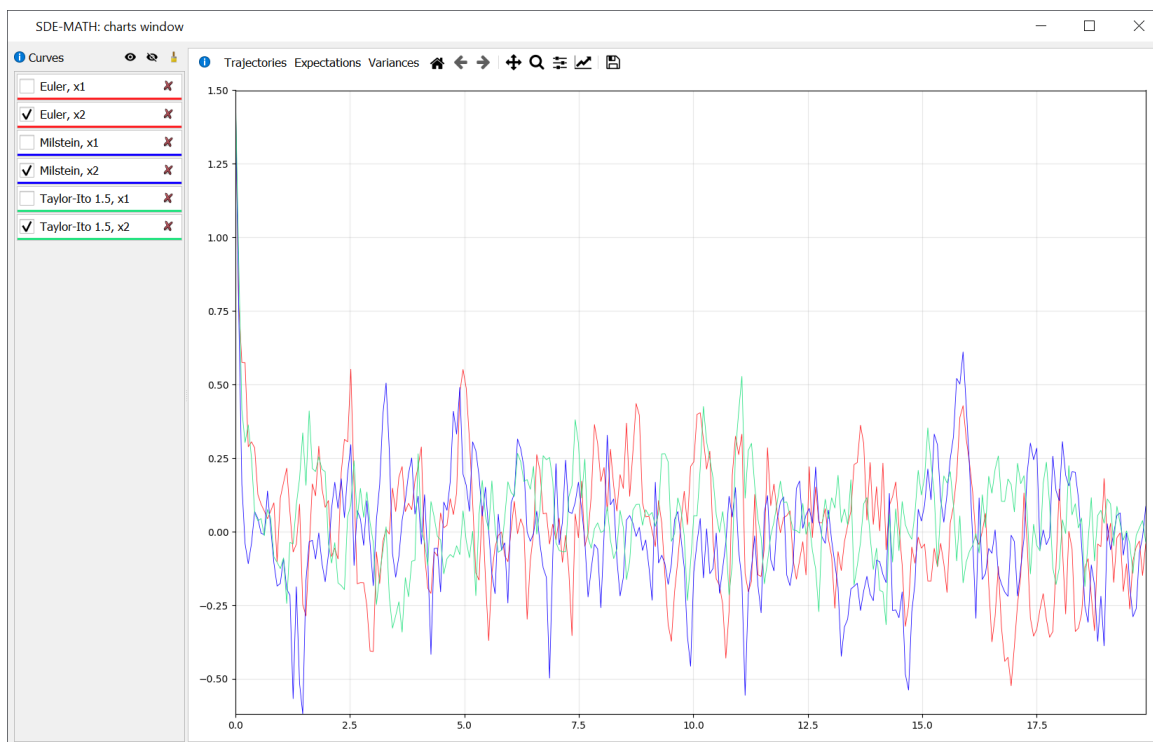


Figure 68: Strong Taylor–Itô schemes of orders 0.5, 1.0, and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.1$, $dt = 0.07$)

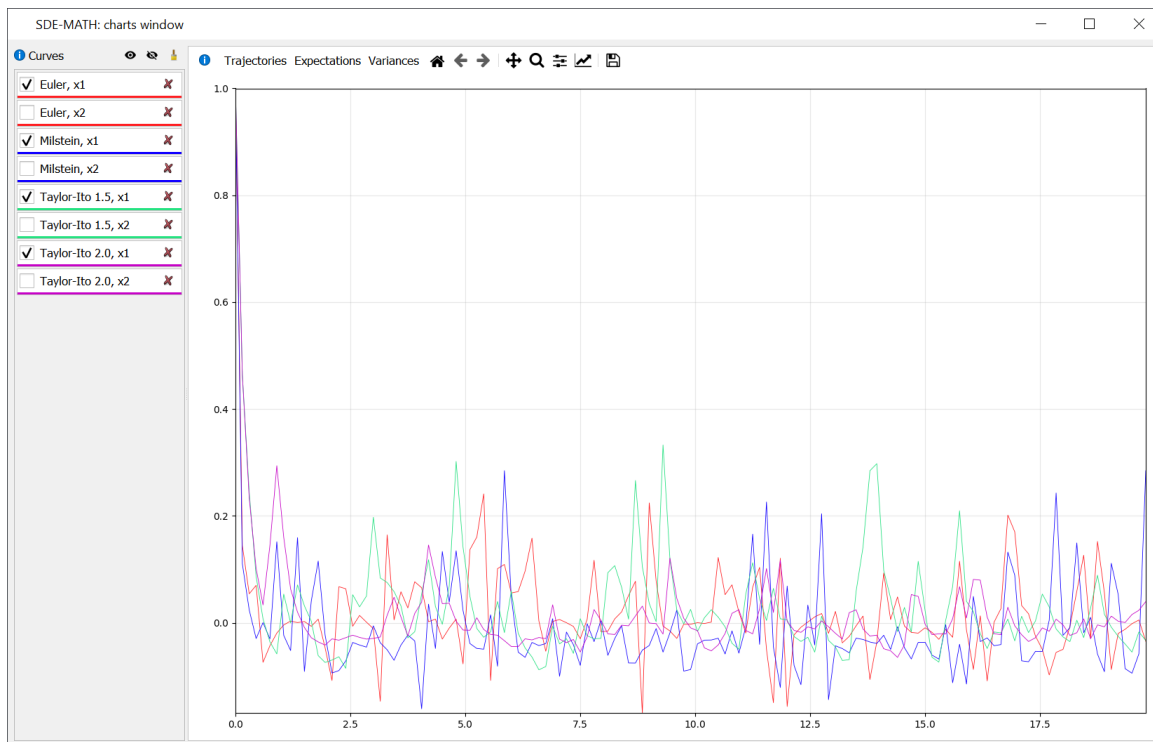
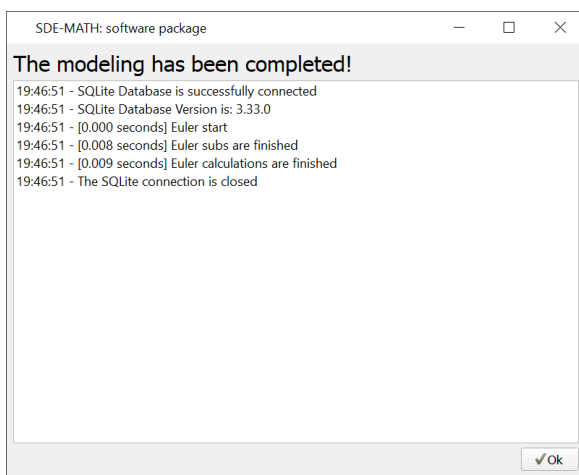
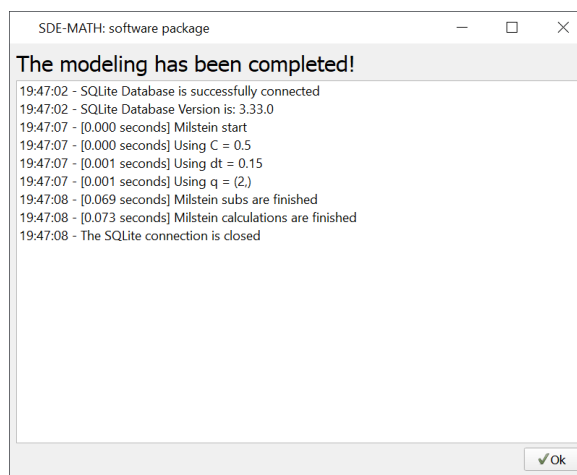


Figure 69: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 0.5$, $dt = 0.15$)

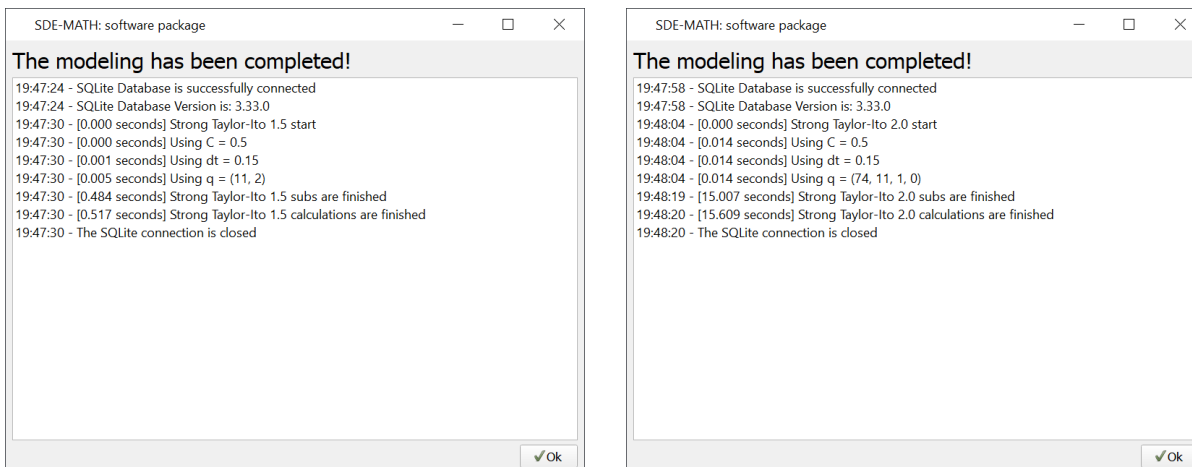


Euler scheme ($dt = 0.15$)



Milstein scheme ($C = 0.5$, $dt = 0.15$)

Figure 70: Modeling logs



Strong Taylor-Itô scheme of order 1.5 ($C = 0.5$, $dt = 0.15$)

Strong Taylor-Itô scheme of order 2.0 ($C = 0.5$, $dt = 0.15$)

Figure 71: Modeling logs

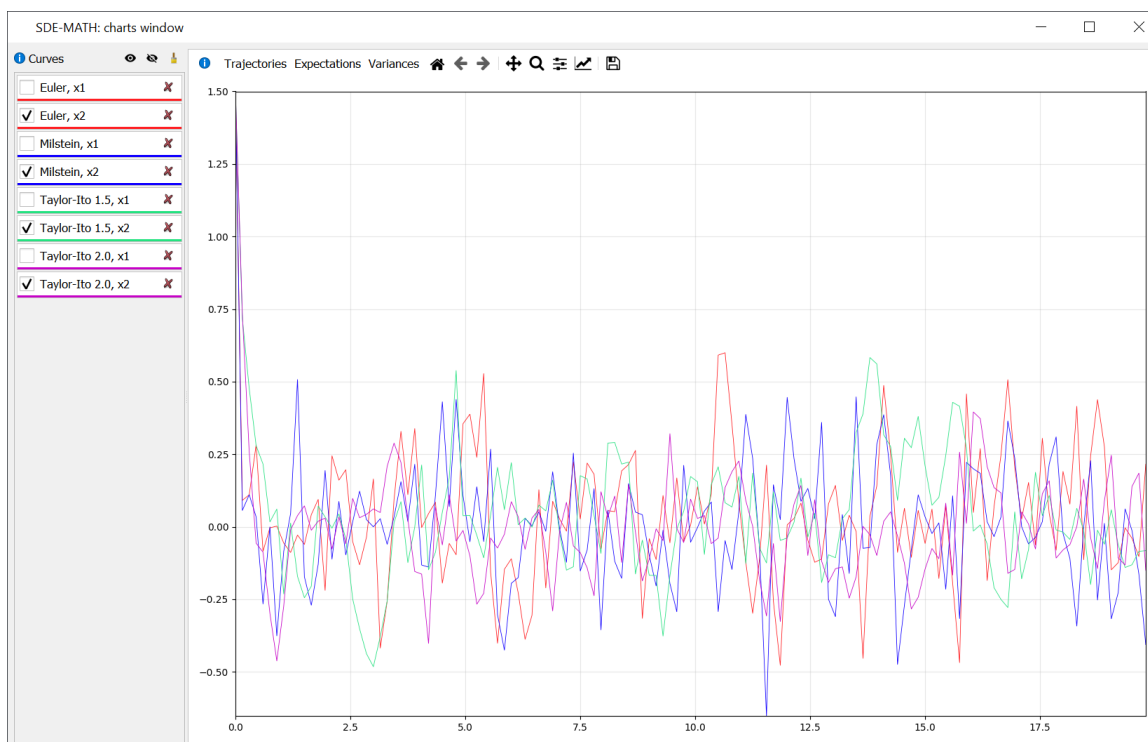
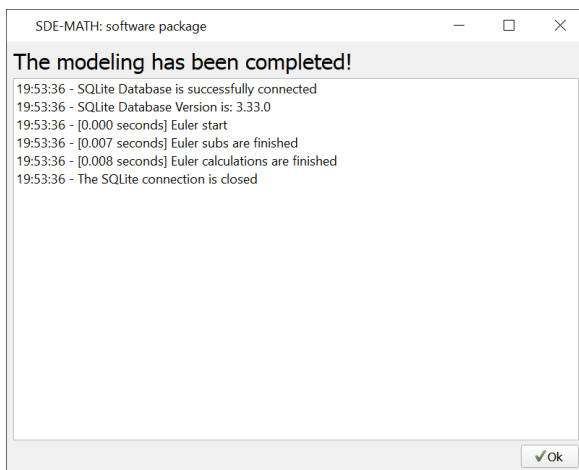
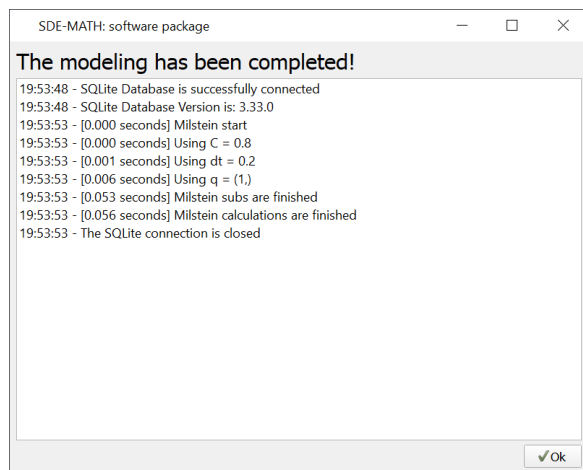


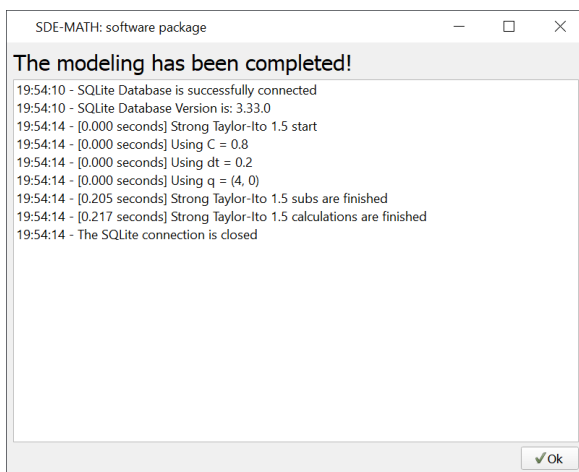
Figure 72: Strong Taylor-Itô schemes of orders 0.5, 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 0.5$, $dt = 0.15$)



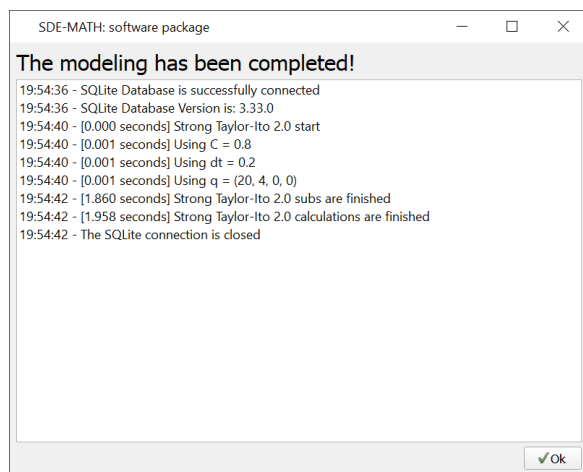
Euler scheme ($dt = 0.2$)



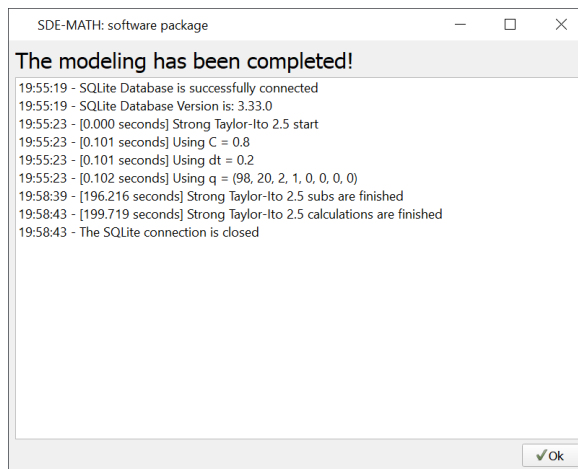
Milstein scheme ($C = 0.8, dt = 0.2$)



Strong Taylor-Itô scheme of order 1.5 ($C = 0.8, dt = 0.2$)



Strong Taylor-Itô scheme of order 2.0 ($C = 0.8, dt = 0.2$)



Strong Taylor-Itô scheme of order 2.5 ($C = 0.8, dt = 0.2$)

Figure 73: Modeling logs



Figure 74: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.8$, $dt = 0.2$)

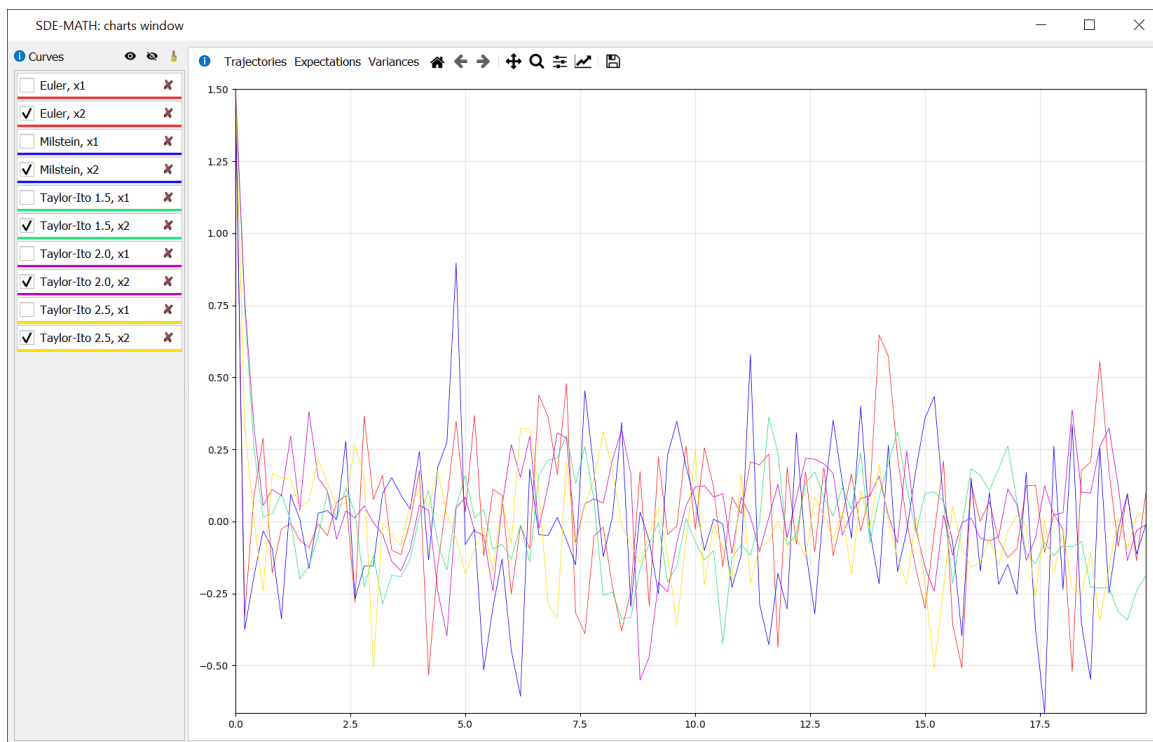
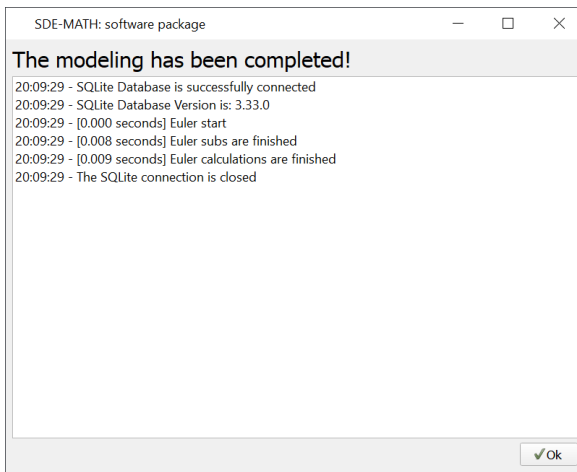
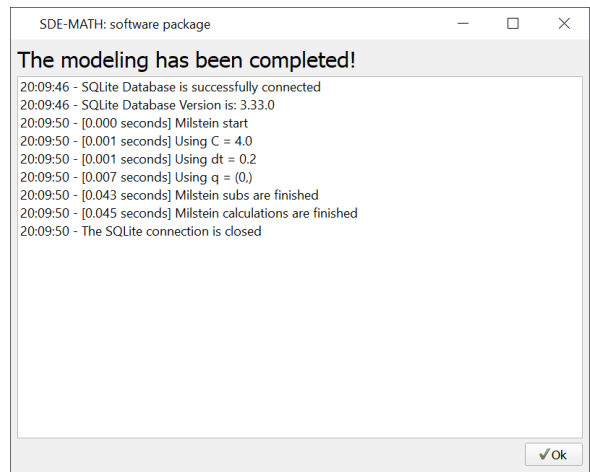


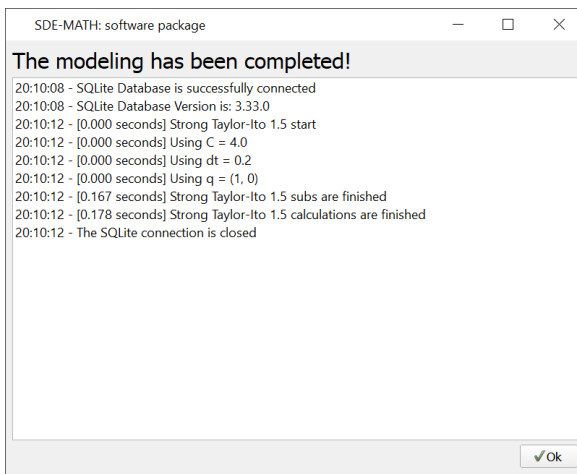
Figure 75: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.8$, $dt = 0.2$)



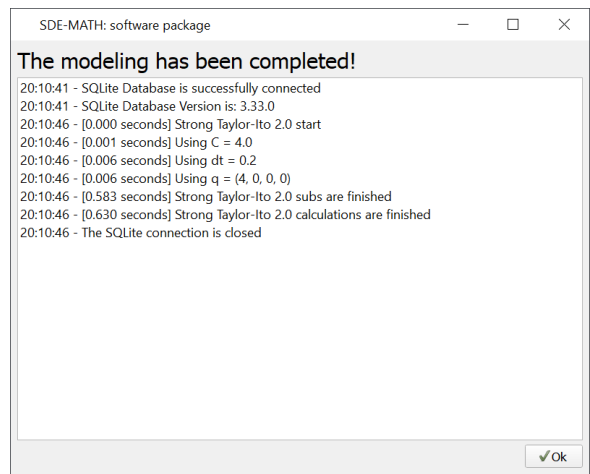
Euler scheme ($dt = 0.2$)



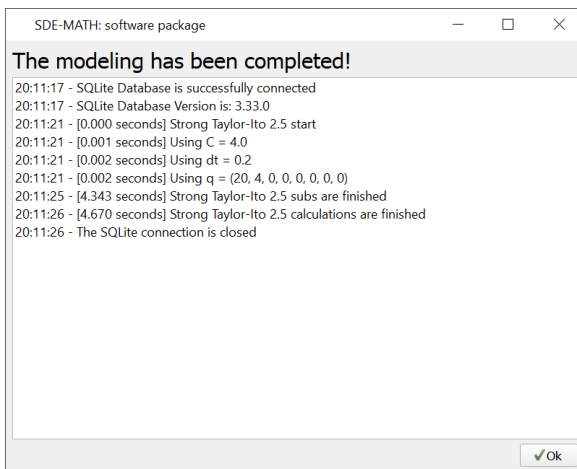
Milstein scheme ($C = 4, dt = 0.2$)



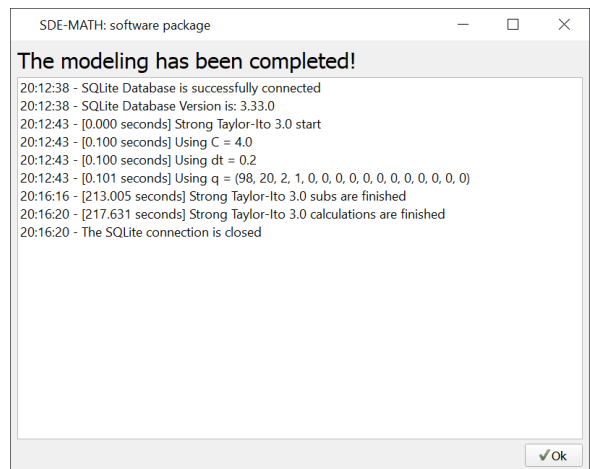
Strong Taylor-Itô scheme of order 1.5 ($C = 4, dt = 0.2$)



Strong Taylor-Itô scheme of order 2.0 ($C = 4, dt = 0.2$)



Strong Taylor-Itô scheme of order 2.5 ($C = 4, dt = 0.2$)



Strong Taylor-Itô scheme of order 3.0 ($C = 4, dt = 0.2$)

Figure 76: Modeling logs

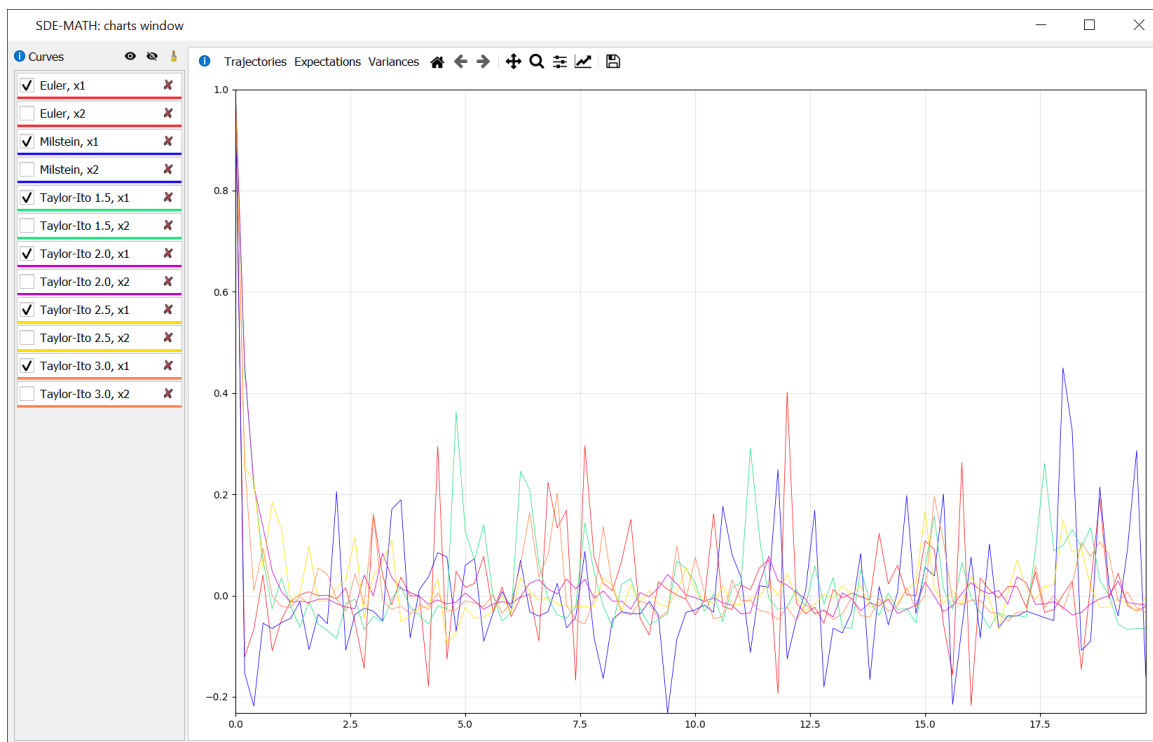


Figure 77: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 4$, $dt = 0.2$)

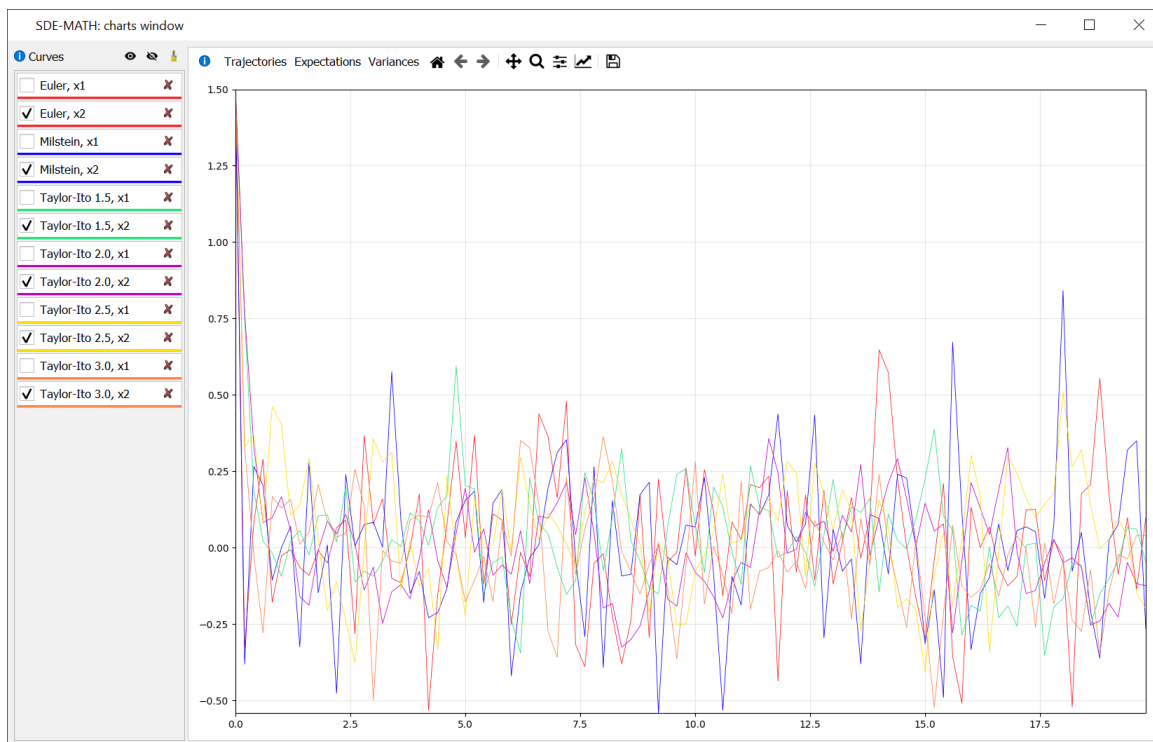
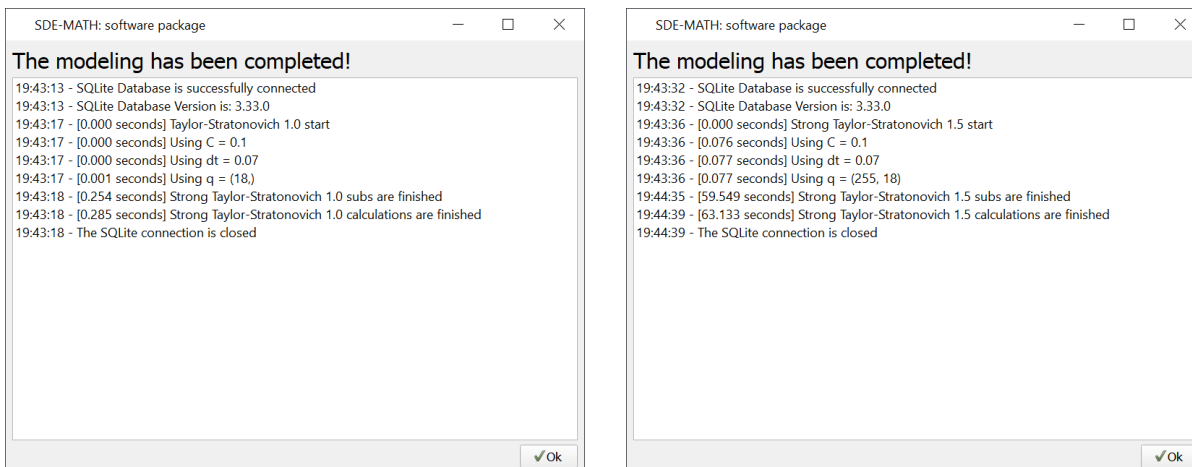


Figure 78: Strong Taylor–Itô schemes of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 4$, $dt = 0.2$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.1, dt = 0.07$)

Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.1, dt = 0.07$)

Figure 79: Modeling logs

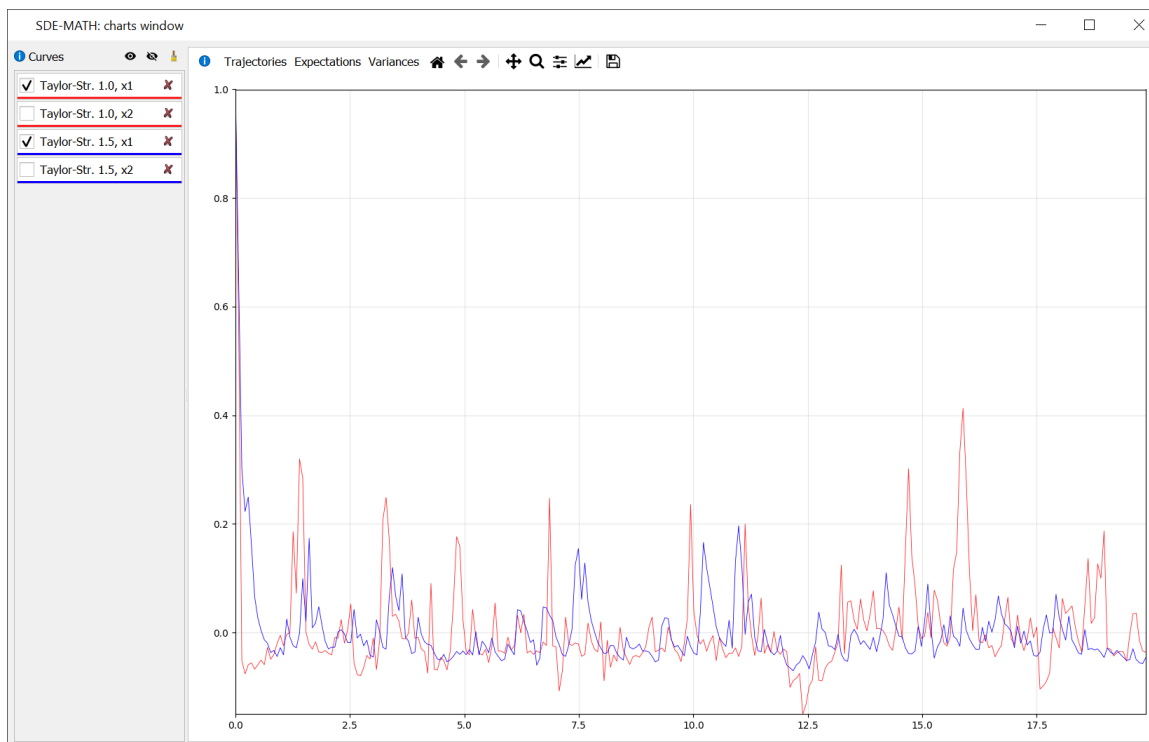


Figure 80: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.1, dt = 0.07$)

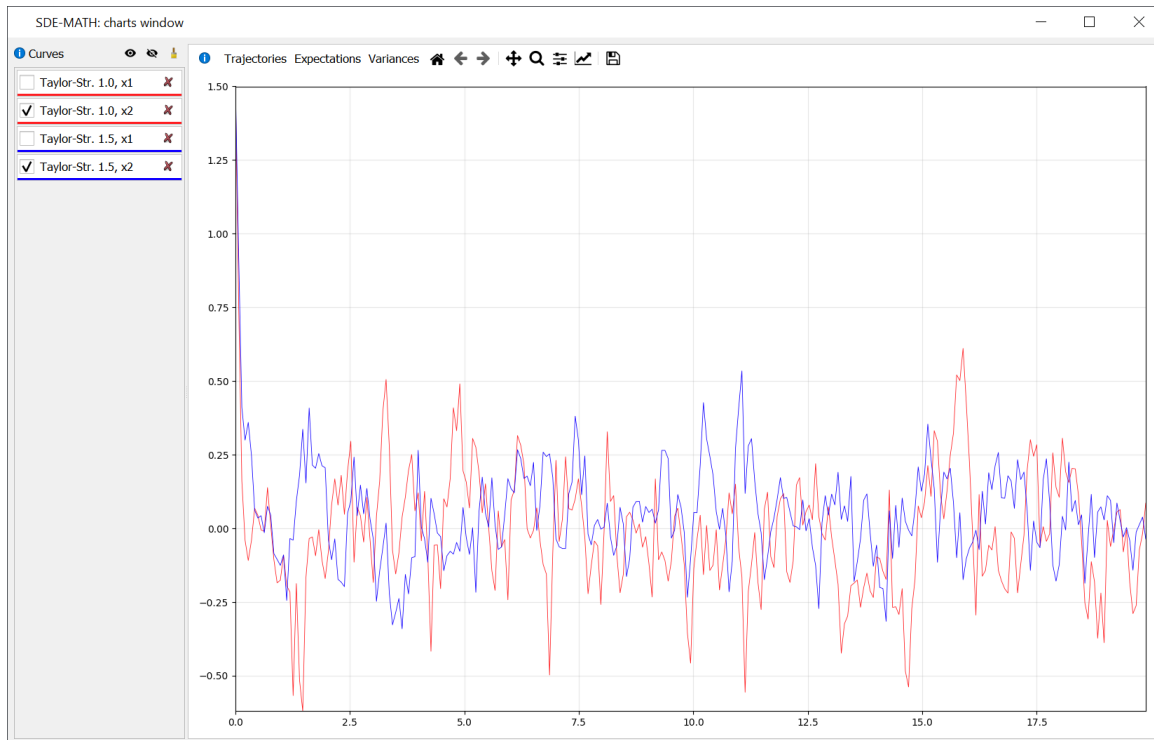
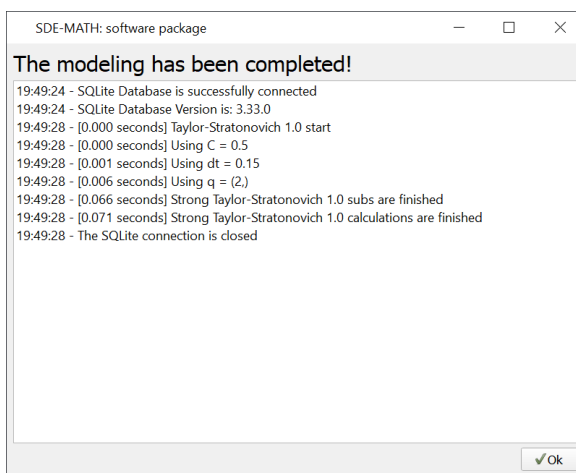
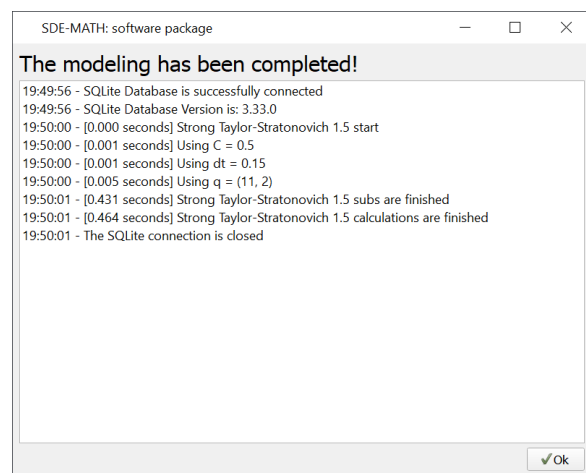


Figure 81: Strong Taylor–Stratonovich schemes of orders 1.0 and 1.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.1$, $dt = 0.07$)



Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.5$, $dt = 0.15$)



Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.5$, $dt = 0.15$)

Figure 82: Modeling logs

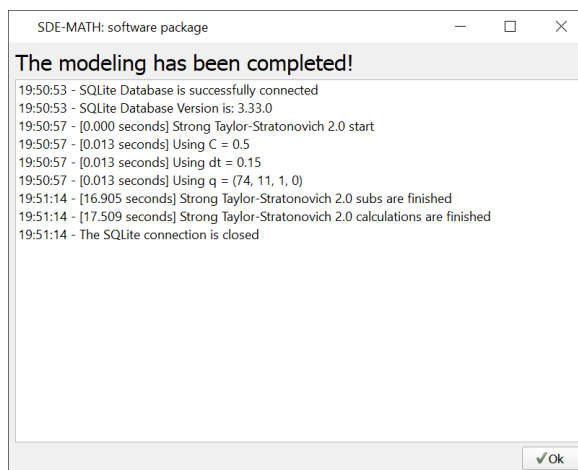


Figure 83: Strong Taylor–Stratonovich scheme of order 2.0 ($C = 0.5$, $dt = 0.15$)

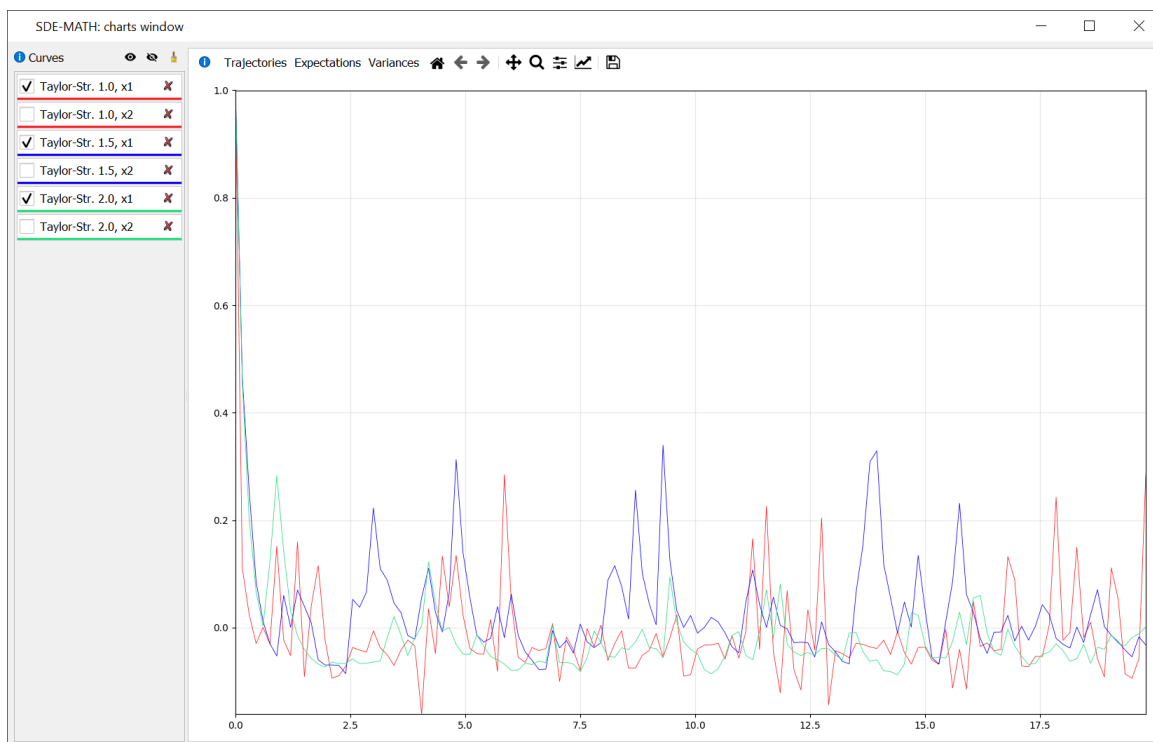


Figure 84: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(1)}$ component, $C = 0.5$, $dt = 0.15$)

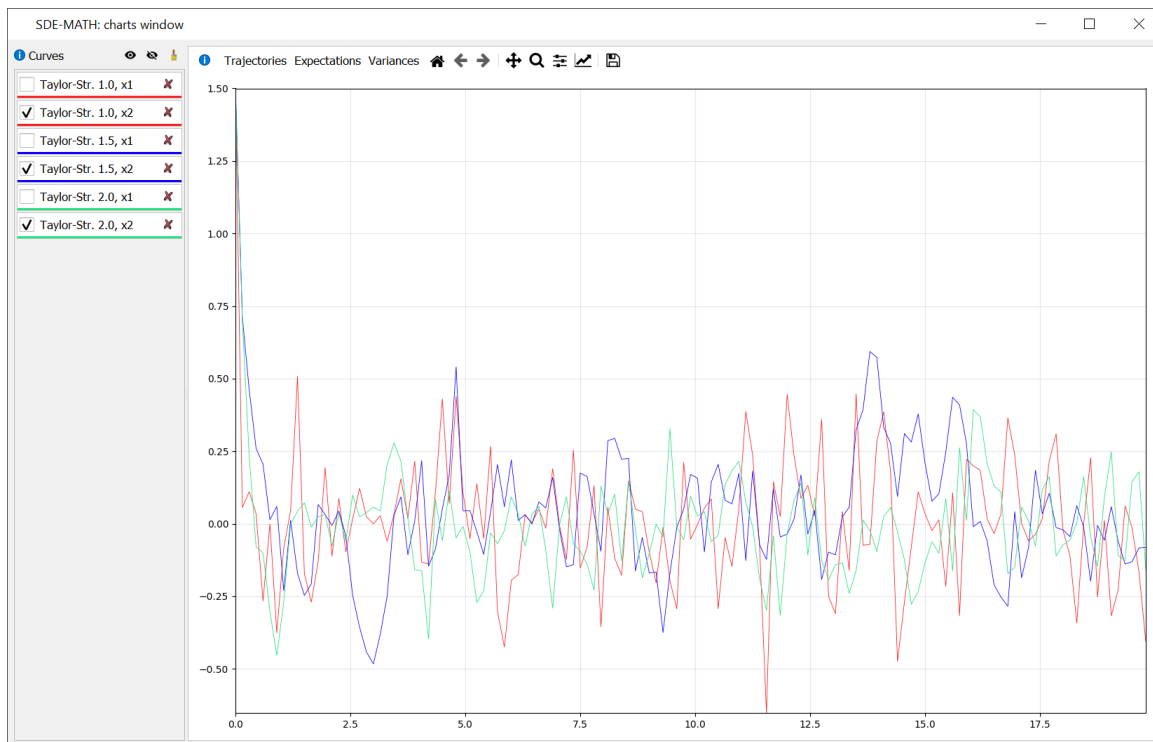
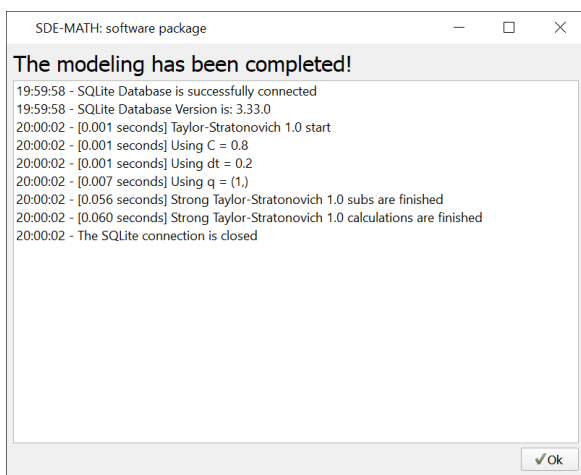
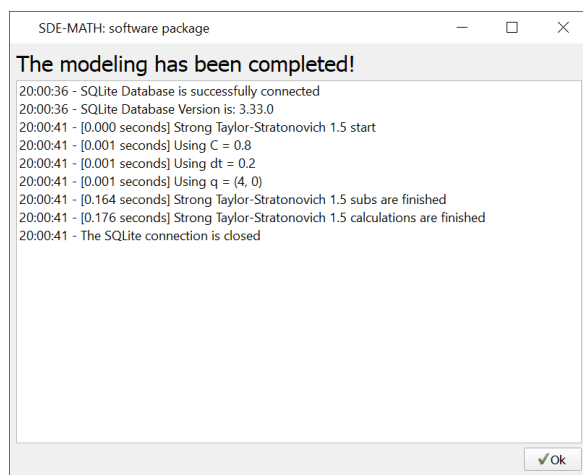


Figure 85: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, and 2.0 ($\mathbf{x}_t^{(2)}$ component, $C = 0.5$, $dt = 0.15$)

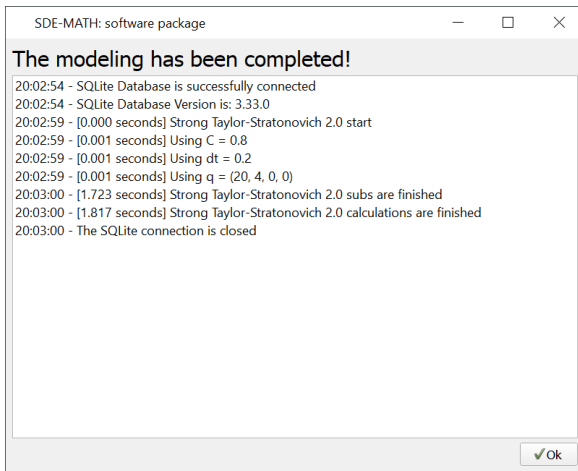


Strong Taylor–Stratonovich scheme of order 1.0 ($C = 0.8$, $dt = 0.2$)

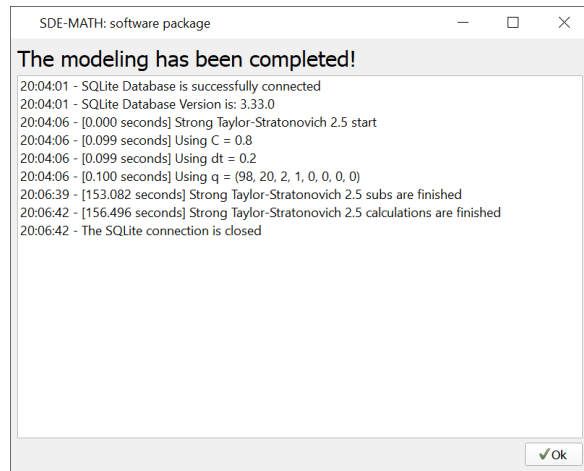


Strong Taylor–Stratonovich scheme of order 1.5 ($C = 0.8$, $dt = 0.2$)

Figure 86: Modeling logs



Strong Taylor-Stratonovich scheme of order 2.0 ($C = 0.8$, $dt = 0.2$)



Strong Taylor-Stratonovich scheme of order 2.5 ($C = 0.8$, $dt = 0.2$)

Figure 87: Modeling logs

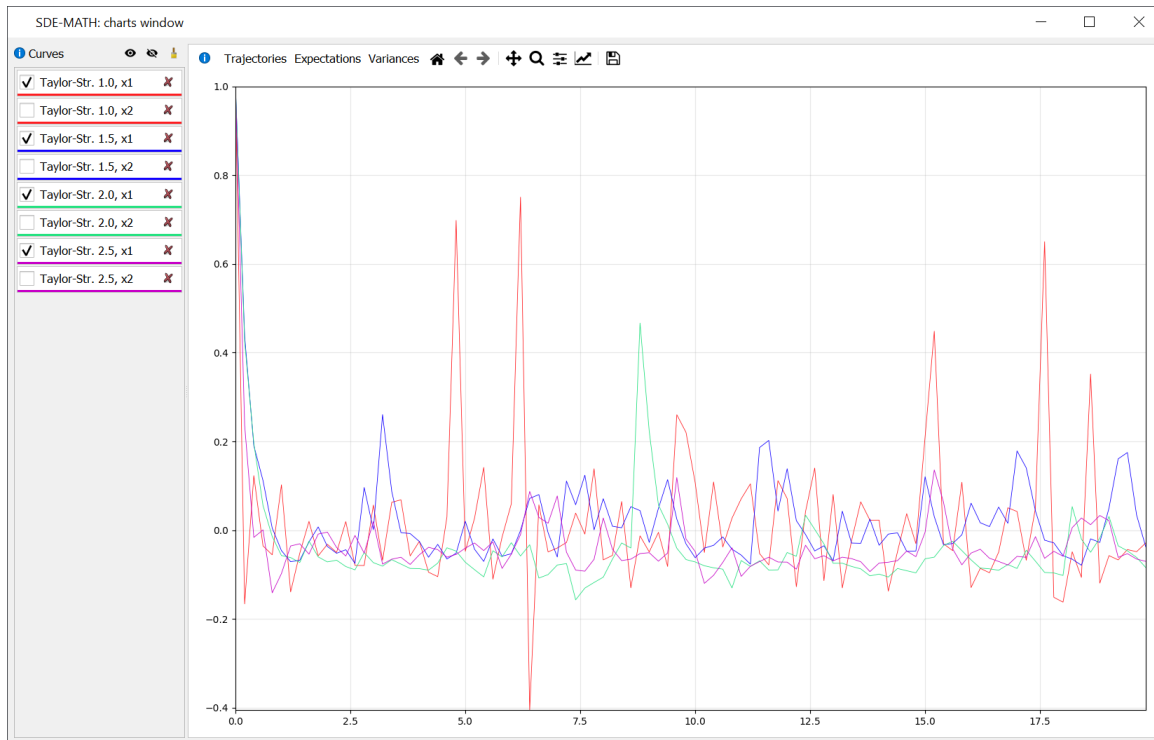


Figure 88: Strong Taylor-Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(1)}$ component, $C = 0.8$, $dt = 0.2$)



Figure 89: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, and 2.5 ($\mathbf{x}_t^{(2)}$ component, $C = 0.8$, $dt = 0.2$)

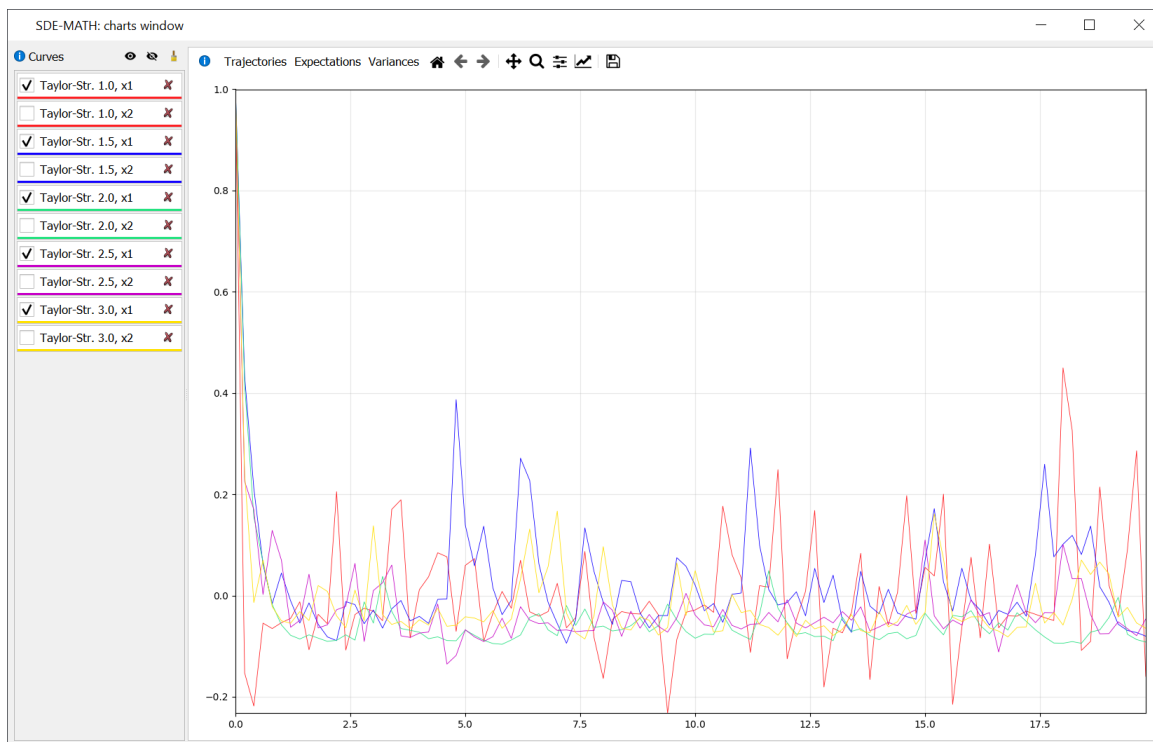
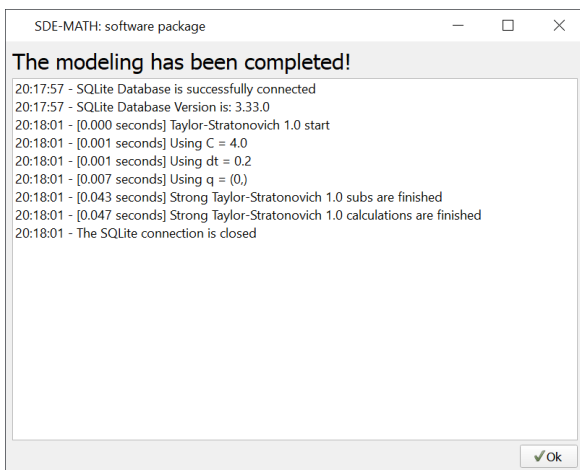
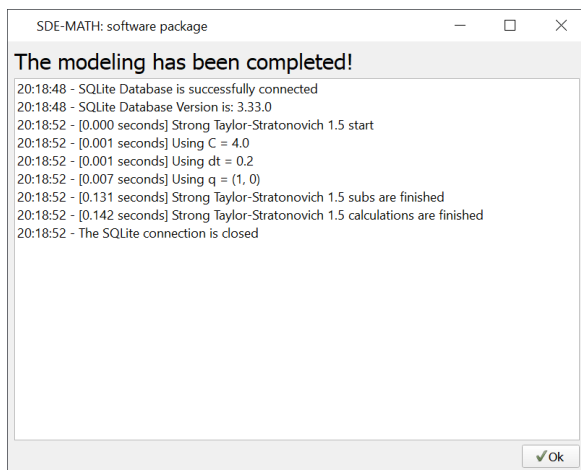


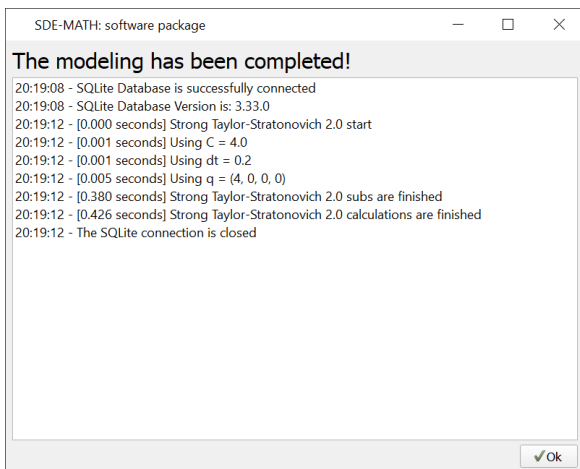
Figure 90: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(1)}$ component, $C = 4$, $dt = 0.2$)



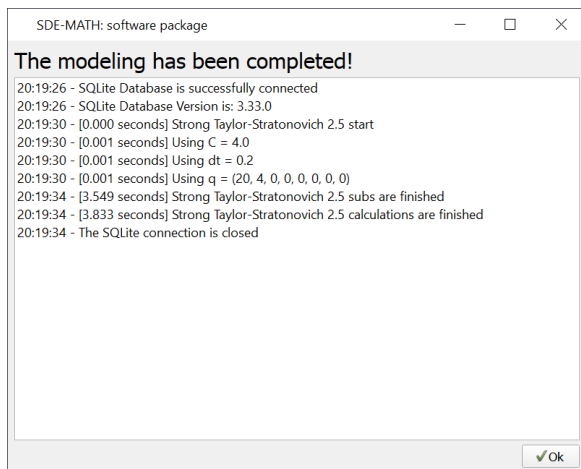
Strong Taylor-Stratonovich scheme of order 1.0 ($C = 4$, $dt = 0.2$)



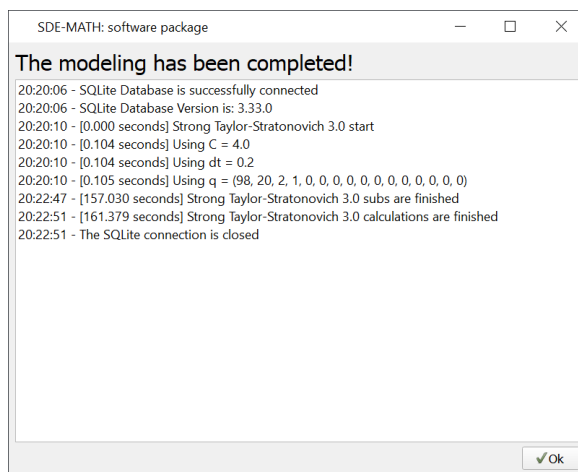
Strong Taylor-Stratonovich scheme of order 1.5 ($C = 4$, $dt = 0.2$)



Strong Taylor-Stratonovich scheme of order 2.0 ($C = 4$, $dt = 0.2$)



Strong Taylor-Stratonovich scheme of order 2.5 ($C = 4$, $dt = 0.2$)



Strong Taylor-Stratonovich scheme of order 3.0 ($C = 4$, $dt = 0.2$)

Figure 91: Modeling logs

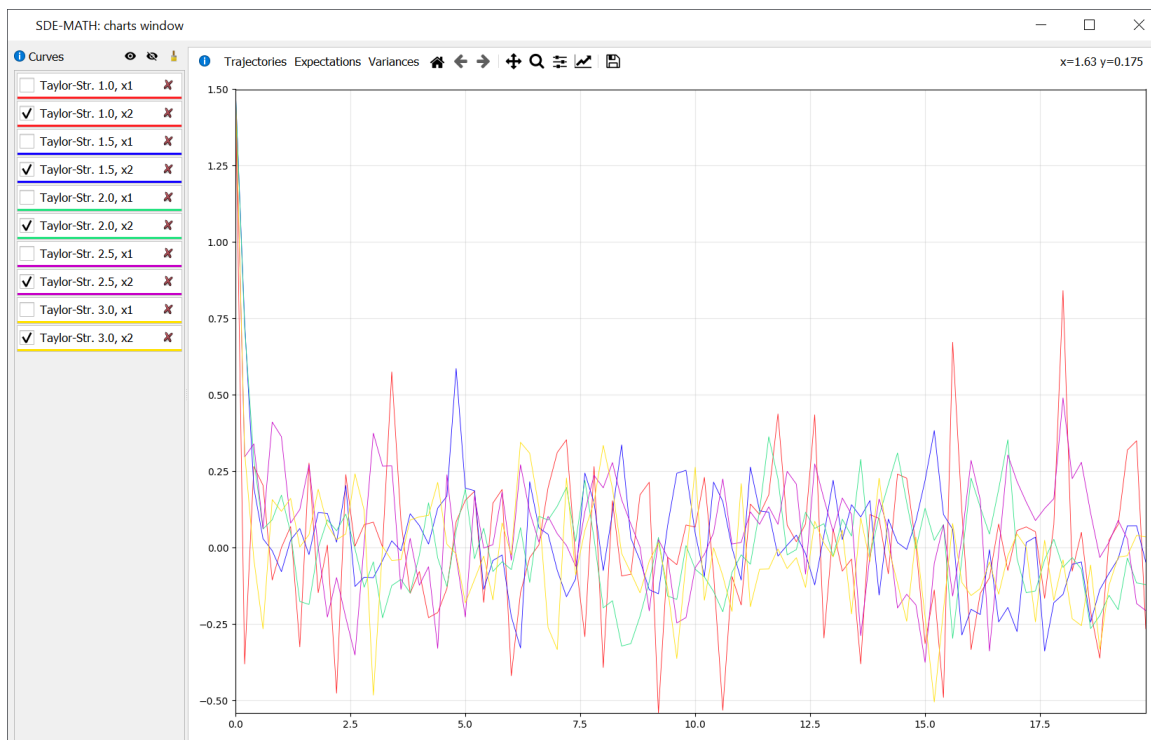


Figure 92: Strong Taylor–Stratonovich schemes of orders 1.0, 1.5, 2.0, 2.5, and 3.0 ($\mathbf{x}_t^{(2)}$ component, $C = 4$, $dt = 0.2$)

5.5 Example of Linear System of Itô SDEs (Solar Activity)

Consider a mathematical model of the solar activity without its average value in a form of the system of linear Itô SDEs (246) [4]. In (246) we choose [4] $n = 2$, $m = 1$, $k = 2$, $\mathbf{x}_0^{(1)} = 7$, $\mathbf{x}_0^{(2)} = -0.25$,

$$A = \begin{pmatrix} 0 & 1 \\ -0.3205 & -0.14 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (250)$$

$$\mathbf{u}(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 5.08 \end{pmatrix}. \quad (251)$$

5.6 Visualization and Numerical Results for Solar Activity Model

This subsection is devoted to the visualization and numerical results for the model of solar activity (246), (250), (251).

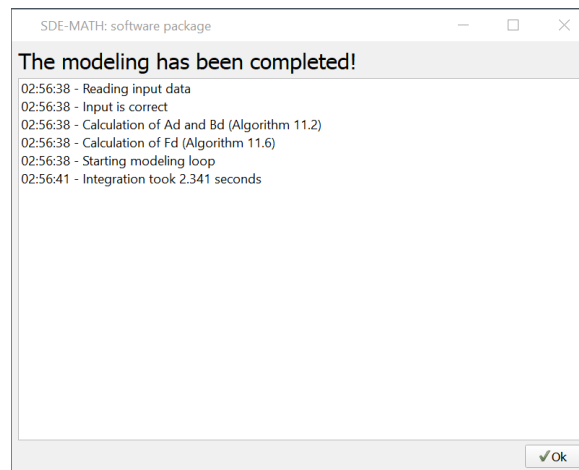


Figure 93: Modeling logs for solar activity model

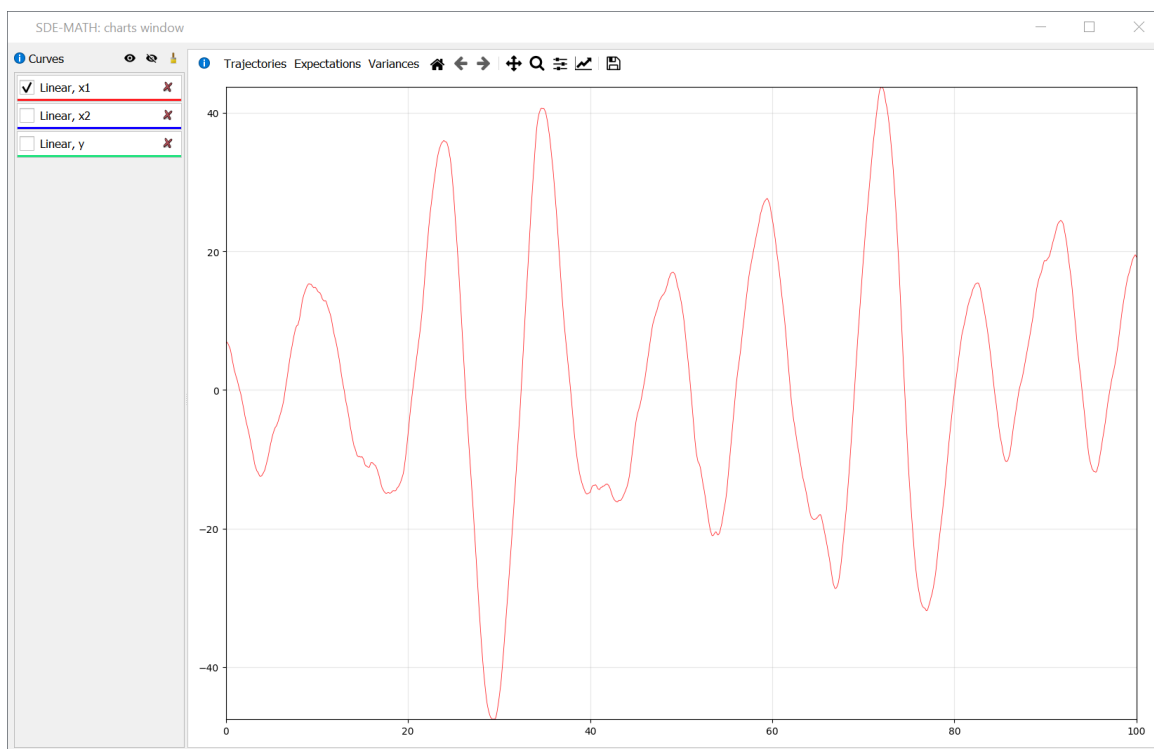


Figure 94: Solar activity model ($x_t^{(1)}$ component)

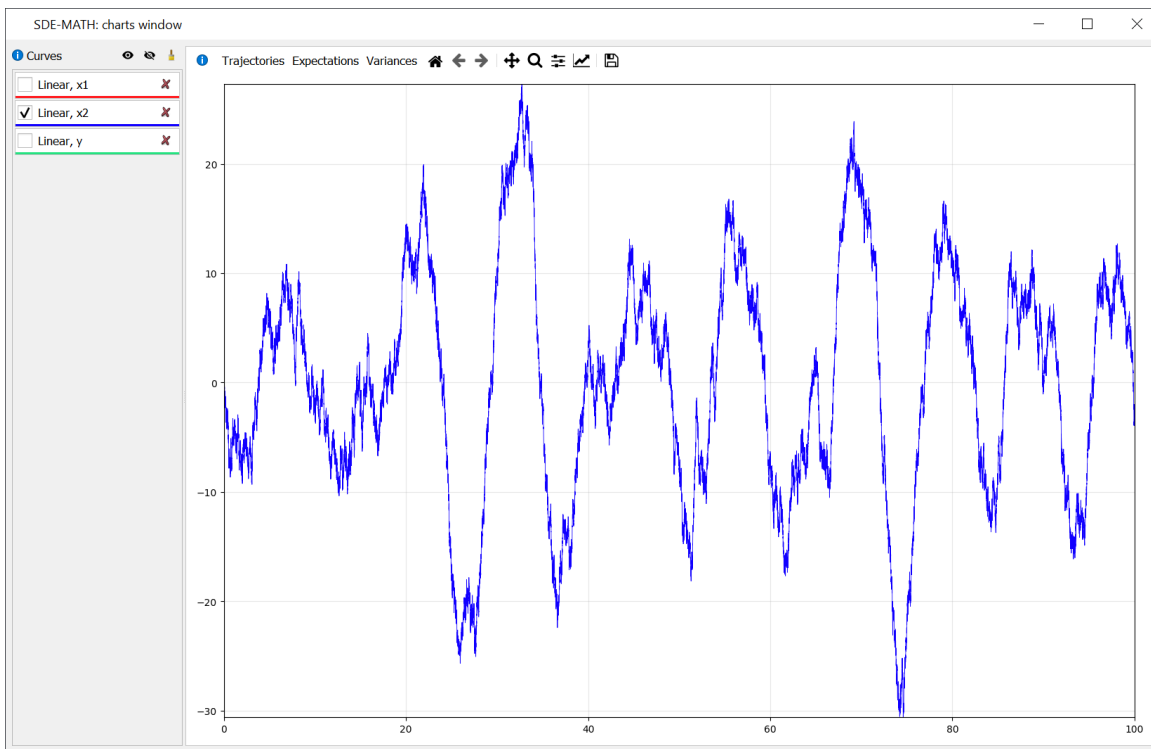


Figure 95: Solar activity model ($x_t^{(2)}$ component)

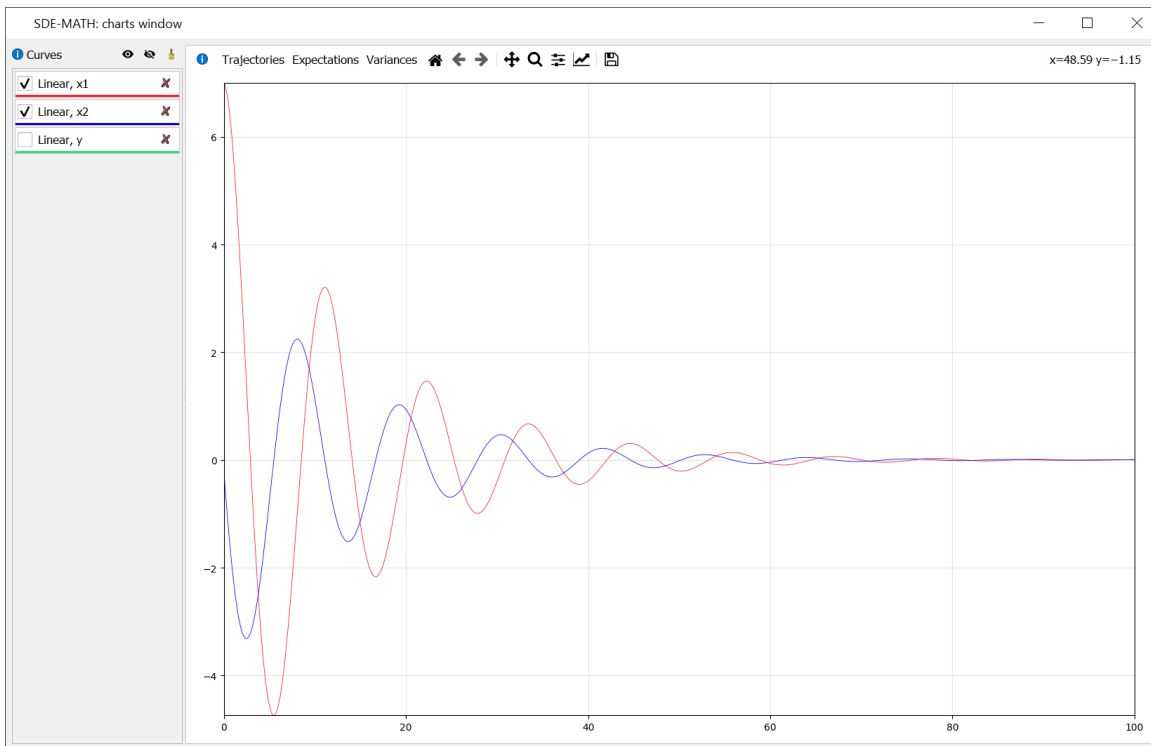


Figure 96: Solar activity model (expectations)

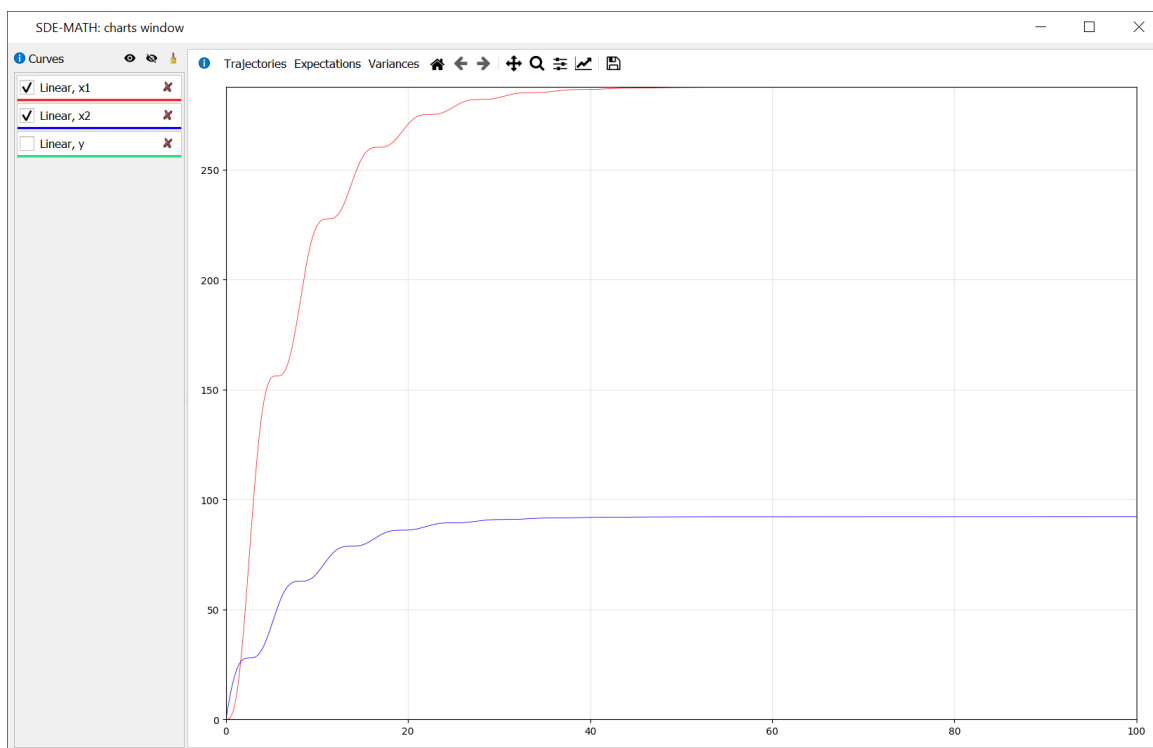


Figure 97: Solar activity model (variances)

5.7 Example of Abstract Linear System of Itô SDEs

Now consider the system of linear Itô SDEs (246) with the following data

$$n = 4, \quad m = 5, \quad k = 3, \quad \mathbf{x}_0^{(1)} = 1, \quad \mathbf{x}_0^{(2)} = 2, \quad \mathbf{x}_0^{(3)} = -1, \quad \mathbf{x}_0^{(4)} = -2, \quad (252)$$

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.2 & 0.1 \end{pmatrix}, \quad (253)$$

$$\mathbf{u}(t) \equiv (0 \ 0 \ 0)^T, \quad H = (0.1 \ 0.1 \ 0.1 \ 0.1). \quad (254)$$

5.8 Visualization and Numerical Results for Abstract Linear System of Itô SDEs Obtained via the SDE-MATH Software Package

This subsection is devoted to the visualization and numerical results for the model (246), (252)–(254).

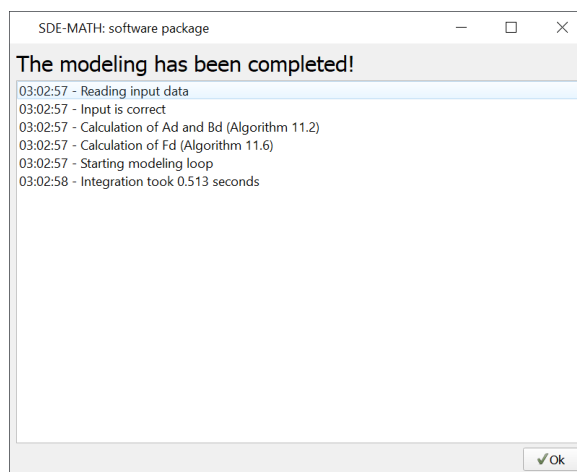


Figure 98: Modeling logs (linear system of Itô SDEs (246), (252)–(254))

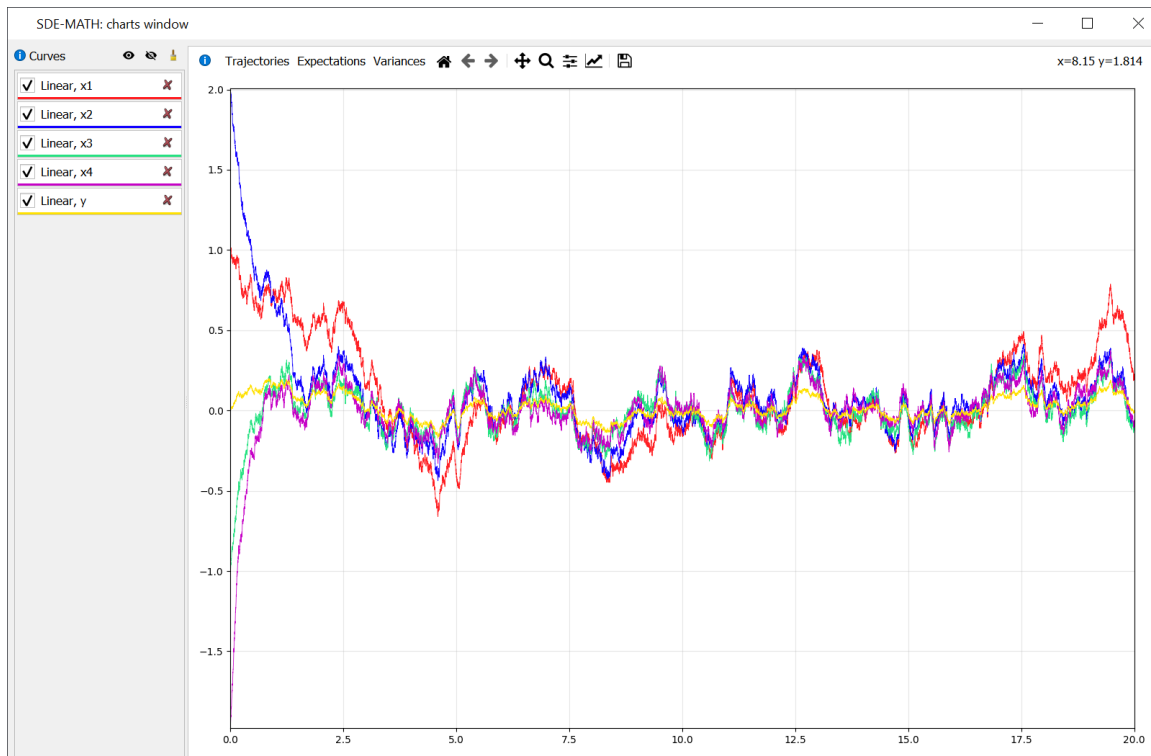


Figure 99: Linear system of Itô SDEs (246), (252)–(254) (components of solution)

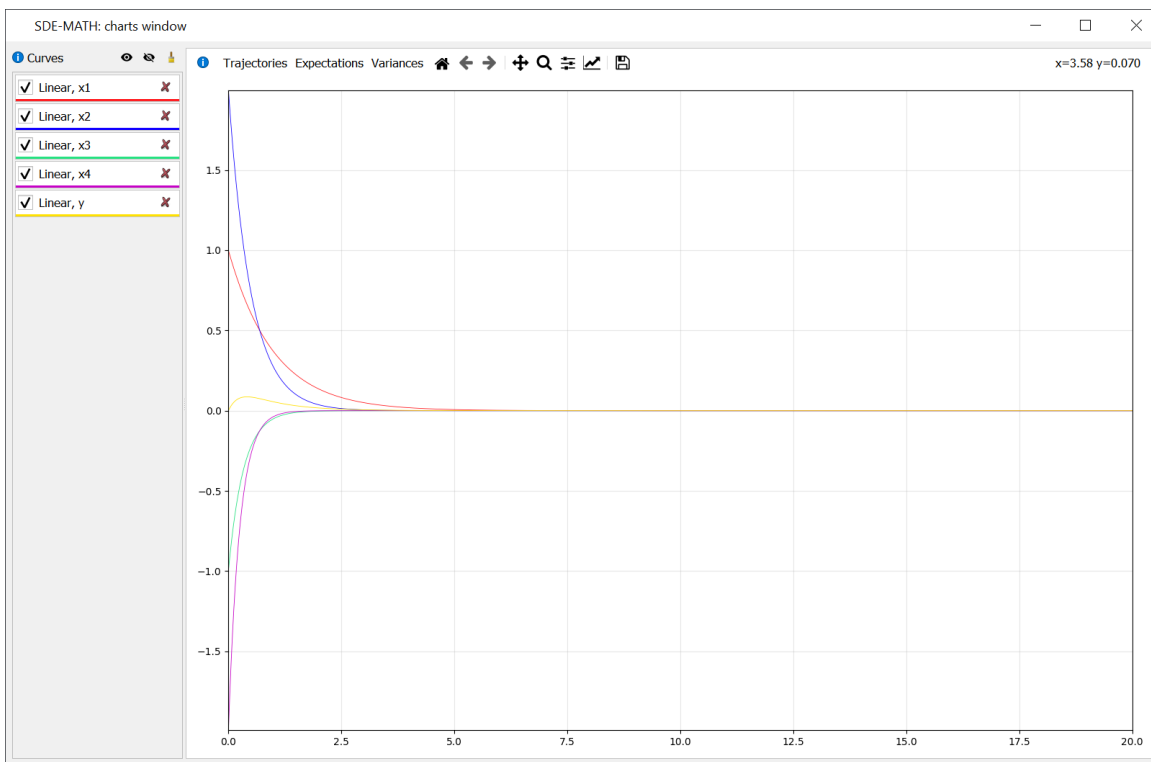


Figure 100: Linear system of Itô SDEs (246), (252)–(254) (expectations)

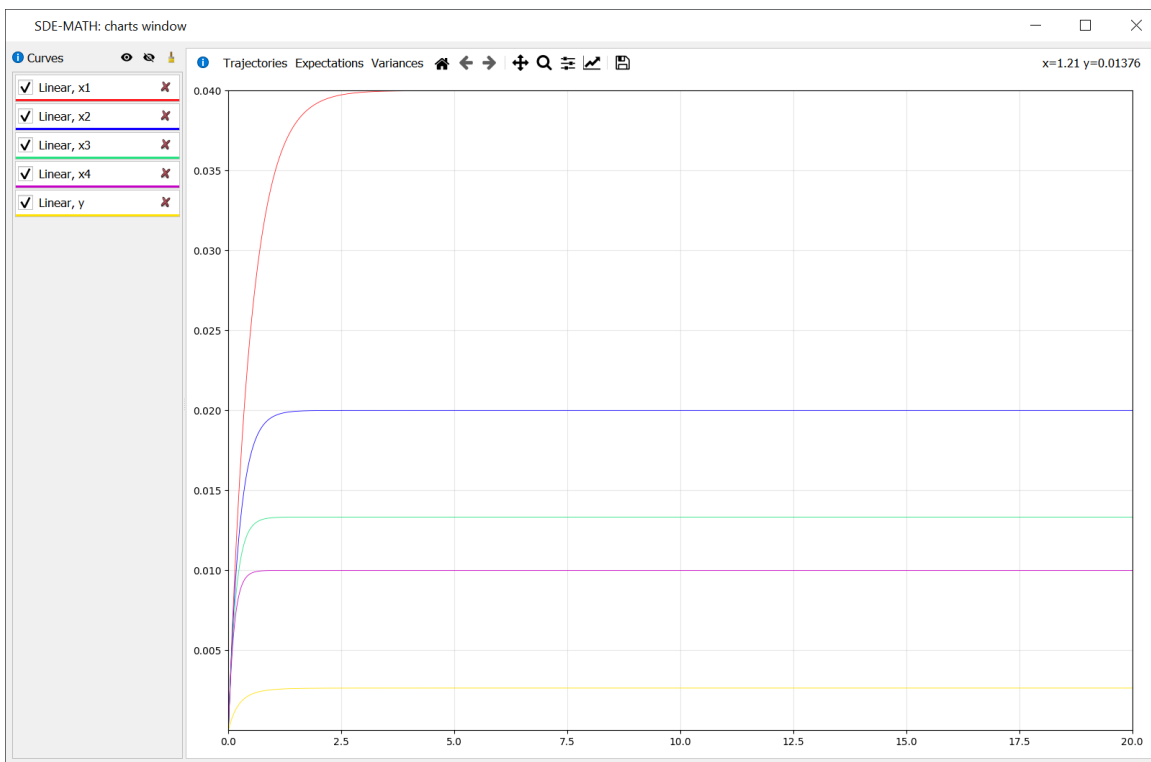


Figure 101: Linear system of Itô SDEs (246), (252)–(254) (variances)

6 Source Codes of the SDE-MATH Software Package in the Python Programming Language

6.1 Source Codes of Graphical User Interface

6.1.1 Source Codes of Main Menu

Listing 7: Configuration file example

```

1  """
2  Configuration file example, change this paths to yours
3  """
4
5  # Paths to resources
6  resources = "./resources/"
7
8  # Path to database
9  database = "./resources/database.db"
10
11 # Size of read buffer for the Fourier–Legendre coefficients
12 read_buffer_size = 8192
13
14 # Recursion limit for the Fourier–Legendre calculations
15 recursion_limit = 10 ** 8

```

Listing 8: Program entry

```

1  #!/usr/bin/env python
2  import logging
3  import os
4  import sys
5
6  from PyQt5 import QtGui, QtWidgets
7  from PyQt5.QtWidgets import QApplication
8  from PyQt5.QtWinExtras import QWinTaskbarButton
9
10 from config import database, images
11 from mathematics.sde.nonlinear.symbolic.coefficients.c import C
12 from tools.database import connect, disconnect
13 from ui.main.main_window import MainWindow
14
15
16 def main():
17
18     logging.basicConfig(
19         level=logging.INFO,
20         format="%(asctime)s - %(levelname)s - %(message)s",
21         datefmt="%H:%M:%S"

```

```

22     )
23
24     app = QApplication(sys.argv)
25     app.setWindowIcon(QtGui.QIcon(os.path.join(images, "function.png")))
26     app.setStyle(QtGui.QStyleFactory.create('Fusion'))
27
28     main_window = MainWindow()
29
30     main_window.taskbar_button = QWinTaskbarButton()
31     main_window.taskbar_button.setOverlayIcon(QtGui.QIcon("resources/function.svg"))
32
33     exit(app.exec())
34
35
36 if __name__ == "__main__":
37     main()

```

Listing 9: Main window

```

1  from PyQt5.QtCore import QThreadPool, pyqtSignal
2  from PyQt5.QtWidgets import QStackedWidget, QMainWindow
3  from sympy.physics.mechanics.tests.test_system import lam
4
5  from init.initialization import initialization
6  from tools.fsys import is_locked
7  from ui.async_calls.worker import Worker
8  from ui.charts.charts_window import PlotWindow
9  from ui.main.greetings import GreetingsWidget
10 from ui.main.menu.base import MainMenuWidget
11 from ui.main.modeling.linear.base import LinearModelingWidget
12 from ui.main.modeling.nonlinear.base import NonlinearModelingWidget
13 from ui.main.progress.complex_progress import ComplexProgressWidget
14 from ui.main.progress.simple_progress import SimpleProgressWidget
15
16
17 class MainWindow(QMainWindow):
18
19     main_window_close = pyqtSignal()
20     start_simple_progress = pyqtSignal(str)
21     stop_simple_progress = pyqtSignal(str)
22
23     def __init__(self):
24         super(QMainWindow, self).__init__()
25
26         self.plot_window = PlotWindow()
27
28         self.stack_widget = QStackedWidget(self)
29
30         self.main_menu = MainMenuWidget(self.stack_widget)
31         self.complex_progress = ComplexProgressWidget(self.stack_widget)
32         self.simple_progress = SimpleProgressWidget(self.stack_widget)
33         self.greetings = GreetingsWidget(self.stack_widget)
34         self.linear_modeling = LinearModelingWidget(self.stack_widget)

```

```

35 self.nonlinear_modeling = NonlinearModelingWidget(self.stack_widget)
36
37 self.stack_widget.addWidget(self.main_menu)
38 self.stack_widget.addWidget(self.nonlinear_modeling)
39 self.stack_widget.addWidget(self.linear_modeling)
40 self.stack_widget.addWidget(self.greetings)
41 self.stack_widget.addWidget(self.simple_progress)
42 self.stack_widget.addWidget(self.complex_progress)
43
44 self.setCentralWidget(self.stack_widget)
45
46 self.exec_init()
47
48 self.setWindowTitle("SDE-MATH: software package")
49 self.setMinimumSize(640, 480)
50 self.resize(800, 600)
51 self.show()
52
53 self.main_menu.group1.show_nonlinear_dialog.connect(self.show_nonlinear)
54 self.main_menu.group2.show_nonlinear_dialog.connect(self.show_nonlinear)
55 self.main_menu.group3.show_linear_dialog.connect(
56     lambda: self.stack_widget.setCurrentWidget(self.linear_modeling))
57
58 self.nonlinear_modeling.show_main_menu.connect(
59     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
60 self.nonlinear_modeling.start_progress.connect(
61     lambda: self.stack_widget.setCurrentWidget(self.complex_progress))
62
63 self.linear_modeling.show_main_menu.connect(
64     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
65 self.linear_modeling.start_progress.connect(
66     lambda: self.stack_widget.setCurrentWidget(self.complex_progress))
67
68 self.greetings.show_main_menu.connect(
69     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
70
71 self.greetings.show_main_menu.connect(
72     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
73
74 self.greetings.show_main_menu.connect(
75     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
76
77 self.complex_progress.back_btn.clicked.connect(
78     lambda: self.stack_widget.setCurrentWidget(self.main_menu))
79
80 # plot events
81
82 self.main_window_close.connect(self.plot_window.close)
83
84 self.main_menu.charts_check.clicked.connect(self.plot_window.setVisible)
85 self.plot_window.charts_show.connect(
86     lambda: self.main_menu.charts_check.setChecked(True))
87 self.plot_window.charts_hide.connect(
88     lambda: self.main_menu.charts_check.setChecked(False))
89

```

```

90     self.nonlinear_modeling.draw_chart.connect(self.plot_window.charts_list.new_items)
91     self.nonlinear_modeling.draw_chart.connect(self.plot_window.plot_widget.new_items)
92     self.nonlinear_modeling.draw_chart.connect(self.plot_window.show)
93
94     self.nonlinear_modeling.charts_check.stateChanged.connect(self.plot_window.setVisible
95     )
96     self.plot_window.charts_show.connect(
97         lambda: self.nonlinear_modeling.charts_check.setChecked(True))
98     self.plot_window.charts_hide.connect(
99         lambda: self.nonlinear_modeling.charts_check.setChecked(False))
100
101     self.linear_modeling.draw_chart.connect(self.plot_window.charts_list.new_items)
102     self.linear_modeling.draw_chart.connect(self.plot_window.plot_widget.new_items)
103     self.linear_modeling.draw_chart.connect(self.plot_window.show)
104
105     self.linear_modeling.charts_check.clicked.connect(self.plot_window.setVisible)
106     self.plot_window.charts_show.connect(
107         lambda: self.linear_modeling.charts_check.setChecked(True))
108     self.plot_window.charts_hide.connect(
109         lambda: self.linear_modeling.charts_check.setChecked(False))
110
111     self.linear_modeling.start_progress.connect(self.complex_progress.spin)
112     self.nonlinear_modeling.start_progress.connect(self.complex_progress.spin)
113     self.linear_modeling.stop_progress.connect(self.complex_progress.stop)
114     self.nonlinear_modeling.stop_progress.connect(self.complex_progress.stop)
115
116     def closeEvent(self, event):
117         self.main_window_close.emit()
118
119     def exec_init(self):
120         self.simple_progress.spin("Preparing the database...")
121         self.stack_widget.setCurrentWidget(self.simple_progress)
122
123         worker = Worker(initialization)
124         worker.signals.finished.connect(self.init_done)
125
126         QThreadPool.globalInstance().start(worker)
127
128     def init_done(self):
129         if not is_locked(".welcome.lock"):
130             self.stack_widget.setCurrentWidget(self.greetings)
131         else:
132             self.stack_widget.setCurrentWidget(self.main_menu)
133         self.simple_progress.stop()
134
135     def show_nonlinear(self, scheme_id):
136         self.nonlinear_modeling.set_scheme(scheme_id)
137         self.stack_widget.setCurrentWidget(self.nonlinear_modeling)

```

Listing 10: Greetings window

```

1 from PyQt5.QtCore import pyqtSignal, Qt
2 from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QWidget, QSizePolicy, \

```

```

3   QSpacerItem, QHBoxLayout, QLabel, QCheckBox, QApplication, QStyle
4
5   from tools.fsystools import lock, unlock
6   from ui.main.svg import SVG
7
8
9   class GreetingsWidget(QWidget):
10
11       show_main_menu = pyqtSignal()
12
13       def __init__(self, parent=None):
14           super(QWidget, self).__init__(parent)
15
16           header = QLabel("Welcome to SDE-MATH Software Package for "
17                           "the Numerical Solution of Systems of Ito SDEs")
18           font = header.font()
19           font.setPointSize(15)
20           header.setAlignment(Qt.AlignJustify)
21           header.setWordWrap(True)
22           header.setFont(font)
23
24           welcome = QLabel(
25               "Exact solutions of Ito SDEs are known in rare cases. For this "
26               "reason, it becomes necessary to construct numerical methods for "
27               "Ito SDEs. Moreover, the problem of numerical solution of Ito SDEs "
28               "often occurs even in cases when the exact solution of Ito SDE is known. "
29               "This means that in some cases, knowing the exact solution to the Ito "
30               "SDE does not allow us to simulate it numerically in a simple way.", self
31           )
32
33           font = welcome.font()
34           welcome.setFont(font)
35
36           welcome.setAlignment(Qt.AlignJustify)
37           welcome.setWordWrap(True)
38           welcome.setSizePolicy(QSizePolicy.Expanding, QSizePolicy.Minimum)
39
40           check_again = QCheckBox("Do not show again", self)
41           next_btn = QPushButton("Ok", self)
42           next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogApplyButton))
43
44           check_again.clicked.connect(self.check_lock)
45           next_btn.clicked.connect(lambda: self.show_main_menu.emit())
46
47           controls = QHBoxLayout()
48           controls.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
49           controls.addWidget(check_again)
50           controls.addWidget(next_btn)
51           controls.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
52
53           eq1 = QHBoxLayout()
54           eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
55           eq1.addWidget(SVG("equation1.svg", scale_factor=1.))
56           eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
57

```

```

58 eq2 = QHBoxLayout()
59 eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
60 eq2.addWidget(SVG("equation2.svg", scale_factor=1.))
61 eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
62
63 column = QVBoxLayout()
64 column.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
65 column.addWidget(header)
66 column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
67 column.addLayout(eq1)
68 column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
69 column.addLayout(eq2)
70 column.addItem(QSpacerItem(0, 15, QSizePolicy.Expanding, QSizePolicy.Minimum))
71 column.addWidget(welcome)
72 column.addItem(QSpacerItem(0, 30, QSizePolicy.Expanding, QSizePolicy.Minimum))
73 column.addLayout(controls)
74 column.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
75
76 layout = QHBoxLayout()
77 layout.addItem(QSpacerItem(50, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
78 layout.addLayout(column)
79 layout.addItem(QSpacerItem(50, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
80
81 self.setLayout(layout)
82
83 def check_lock(self):
84     if self.sender().isChecked():
85         lock(".welcome.lock")
86     else:
87         unlock(".welcome.lock")

```

Listing 11: Info icon

```

1 from PyQt5.QtCore import QSize
2 from PyQt5.QtWidgets import QWidget, QApplication, QStyle, QLabel
3
4
5 class InfoIcon(QLabel):
6
7     def __init__(self, text: str, parent=None):
8         super(QWidget, self).__init__(parent)
9
10        self.setToolTip(text)
11        self.setStyleSheet("QToolTip {background: white;}")
12        self.setPixmap(QApplication.style().standardIcon(
13            QStyle.SP_MessageBoxInformation).pixmap(QSize(16, 16)))

```

Listing 12: Error widget

```

1 from PyQt5.QtCore import QSize
2 from PyQt5.QtWidgets import QWidget, QApplication, QStyle, QLabel, QHBoxLayout,
   QSpacerItem, QSizePolicy

```



```

3
4
5 class ErrorWidget(QWidget):
6
7     def __init__(self, text: str, parent=None):
8         super(QWidget, self).__init__(parent)
9
10        msg_m = QLabel(text)
11        msg_m.setStyleSheet("QLabel { color: rgb(230, 0, 0); }")
12
13        msg_i = QLabel()
14        msg_i.setStyleSheet("QToolTip { background: white; }")
15        msg_i.setPixmap(QApplication.style().standardIcon(
16            QStyle.SP_MessageBoxCritical).pixmap(QSize(16, 16)))
17
18        layout = QHBoxLayout()
19        layout.setContentsMargins(0, 0, 0, 0)
20        layout.addWidget(msg_i)
21        layout.addWidget(msg_m)
22        layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
23
24        self.setLayout(layout)

```

Listing 13: Svg picture

```

1 import os
2
3 from PyQt5.QtCore import QSize
4 from PyQt5.QtSvg import QSvgWidget
5 from PyQt5.QtWidgets import QSizePolicy
6
7 from config import images
8
9
10 class SVG(QSvgWidget):
11
12     def __init__(self, name: str, scale_factor=1.):
13         super(QSvgWidget, self).__init__()
14
15         self.load(os.path.join(images, name))
16
17         self.scale_factor = scale_factor
18         self.setSizePolicy(QSizePolicy.Fixed, QSizePolicy.Fixed)
19
20     def sizeHint(self):
21         size = self.renderer().defaultSize()
22         return QSize(size.width() * self.scale_factor,
23                     size.height() * self.scale_factor)

```

Listing 14: Main menu (base part)

```

1 from PyQt5.QtWidgets import QWidget, QSizePolicy, QSpacerItem, QHBoxLayout, QVBoxLayout,

```

```

    QCheckBox, QLabel
2
3 from ui.main.info import InfoIcon
4 from ui.main.menu.linear import LinearGroupWidget
5 from ui.main.menu.taylor_ito import ItoGroupWidget
6 from ui.main.menu.taylor_stratonovich import StratonovichGroupWidget
7
8
9 class MainMenuWidget(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.charts_check = QCheckBox("Charts window", self)
15
16         icon = InfoIcon("This is charts window checkbox, it will\n"
17             "follow you on every application dialog, so\n"
18             "you can easily open or close window with available charts")
19
20         bar_layout = QHBoxLayout()
21         bar_layout.addItem(QSpacerItem(0, 35, QSizePolicy.Expanding, QSizePolicy.Minimum))
22         bar_layout.addWidget(icon)
23         bar_layout.addWidget(self.charts_check)
24
25         header = QLabel("Strong Numerical Schemes for Ito SDEs", parent=self)
26         font = header.font()
27         font.setPointSize(15)
28         header.setFont(font)
29
30         icon = InfoIcon("You are now in main menu, you can choose\n"
31             "any scheme to perform modeling")
32
33         header_layout = QHBoxLayout()
34         header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
35             )
36         header_layout.addWidget(icon)
37         header_layout.addWidget(header)
38         header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
39             )
40
41         self.group3 = LinearGroupWidget(self)
42         self.group1 = ItoGroupWidget(self)
43         self.group2 = StratonovichGroupWidget(self)
44
45         menu_layout = QHBoxLayout()
46         menu_layout.addWidget(self.group1)
47         menu_layout.addWidget(self.group2)
48         menu_layout.addWidget(self.group3)
49
50         layout = QVBoxLayout()
51         layout.addLayout(bar_layout)
52         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
53         layout.addLayout(header_layout)
54         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
55         layout.addLayout(menu_layout)

```

```

54 layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
55
56 self.setLayout(layout)

```

Listing 15: Main menu (linear part)

```

1 from PyQt5.QtCore import pyqtSignal
2 from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox
3
4
5 class LinearGroupWidget(QGroupBox):
6
7     show_linear_dialog = pyqtSignal()
8
9     def __init__(self, parent=None):
10         super(QGroupBox, self).__init__(parent)
11
12         linear_btn = QPushButton("Dispersion Spectral Decomposition")
13
14         layout = QVBoxLayout()
15         layout.addWidget(linear_btn)
16         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
17
18         self.setLayout(layout)
19
20         self.setTitle("Linear Ito SDEs Systems Modeling")
21         self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))
22
23         linear_btn.clicked.connect(lambda: self.show_linear_dialog.emit())

```

Listing 16: Main menu (Taylor-Itô part)

```

1 from PyQt5.QtCore import pyqtSignal
2 from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox
3
4
5 class ItoGroupWidget(QGroupBox):
6
7     show_nonlinear_dialog = pyqtSignal(int)
8
9     def __init__(self, parent=None):
10         super(QGroupBox, self).__init__(parent)
11
12         btn1 = QPushButton("Euler")
13         btn2 = QPushButton("Milstein")
14         btn3 = QPushButton("Convergence Order 1.5")
15         btn4 = QPushButton("Convergence Order 2.0")
16         btn5 = QPushButton("Convergence Order 2.5")
17         btn6 = QPushButton("Convergence Order 3.0")
18
19         layout = QVBoxLayout()
20         layout.addWidget(btn1)

```

```

21 layout.addWidget(btn2)
22 layout.addWidget(btn3)
23 layout.addWidget(btn4)
24 layout.addWidget(btn5)
25 layout.addWidget(btn6)
26 layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
27
28 self.setLayout(layout)
29
30 self.setTitle("Taylor–Ito Schemes")
31 self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))
32
33 btn1.clicked.connect(lambda: self.show_nonlinear_dialog.emit(0))
34 btn2.clicked.connect(lambda: self.show_nonlinear_dialog.emit(1))
35 btn3.clicked.connect(lambda: self.show_nonlinear_dialog.emit(2))
36 btn4.clicked.connect(lambda: self.show_nonlinear_dialog.emit(3))
37 btn5.clicked.connect(lambda: self.show_nonlinear_dialog.emit(4))
38 btn6.clicked.connect(lambda: self.show_nonlinear_dialog.emit(5))

```

Listing 17: Main menu (Taylor–Stratonovich part)

```

1 from PyQt5.QtCore import pyqtSignal
2 from PyQt5.QtWidgets import QPushButton, QVBoxLayout, QSizePolicy, QSpacerItem, QGroupBox
3
4
5 class StratonovichGroupWidget(QGroupBox):
6
7     show_nonlinear_dialog = pyqtSignal(int)
8
9     def __init__(self, parent=None):
10         super(QGroupBox, self).__init__(parent)
11
12         btn1 = QPushButton("Convergence Order 1.0")
13         btn2 = QPushButton("Convergence Order 1.5")
14         btn3 = QPushButton("Convergence Order 2.0")
15         btn4 = QPushButton("Convergence Order 2.5")
16         btn5 = QPushButton("Convergence Order 3.0")
17
18         layout = QVBoxLayout()
19         layout.addWidget(btn1)
20         layout.addWidget(btn2)
21         layout.addWidget(btn3)
22         layout.addWidget(btn4)
23         layout.addWidget(btn5)
24         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
25
26         self.setLayout(layout)
27
28         self.setTitle("Taylor–Stratonovich Schemes")
29         self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))
30
31         btn1.clicked.connect(lambda: self.show_nonlinear_dialog.emit(6))
32         btn2.clicked.connect(lambda: self.show_nonlinear_dialog.emit(7))

```

```

33 btn3.clicked.connect(lambda: self.show_nonlinear_dialog.emit(8))
34 btn4.clicked.connect(lambda: self.show_nonlinear_dialog.emit(9))
35 btn5.clicked.connect(lambda: self.show_nonlinear_dialog.emit(10))

```

Listing 18: Complex progress view

```

1  import logging
2
3  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QSpacerItem, QSizePolicy,
4     \
5     QListWidget, QLabel, QPushButton, QApplication, QStyle
6  from pyqtspinner.spinner import WaitingSpinner
7
8  from ui.main.progress.log_handler import LogHandler
9
10 class ComplexProgressWidget(QWidget):
11
12     def __init__(self, parent=None):
13         super(QWidget, self).__init__(parent)
14
15         self.spinner = WaitingSpinner(self,
16                                     radius=5.0,
17                                     lines=10,
18                                     line_length=5.0,
19                                     centerOnParent=False)
20         self.list_widget = QListWidget(self)
21         self.handler = LogHandler(self.handle_message)
22
23         self.label = QLabel(self)
24         font = self.label.font()
25         font.setPointSize(15)
26         self.label.setFont(font)
27
28         self.back_btn = QPushButton("Ok")
29         self.back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogApplyButton))
30         self.back_btn.hide()
31
32         spinner_layout = QHBoxLayout()
33         spinner_layout.addWidget(self.spinner)
34         spinner_layout.addWidget(self.label)
35         spinner_layout.addSpacerItem(QSpacerItem(0, 0, QSizePolicy.Expanding,
36                                               QSizePolicy.Minimum))
37
38         bottom_bar = QHBoxLayout()
39         bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
40         bottom_bar.addWidget(self.back_btn)
41
42         layout = QVBoxLayout()
43         layout.addLayout(spinner_layout)
44         layout.addWidget(self.list_widget)
45         layout.addLayout(bottom_bar)
46

```

```

47     self.setLayout(layout)
48
49     def handle_message(self, text):
50         self.list_widget.addItem(text)
51         self.list_widget.scrollToBottom()
52
53     def spin(self, text):
54         self.list_widget.clear()
55         logging.getLogger().addHandler(self.handler)
56         self.back_btn.hide()
57         self.spinner.start()
58         self.label.setText(text)
59
60     def stop(self, text):
61         self.back_btn.show()
62         self.list_widget.scrollToBottom()
63         self.label.setText(text)
64         self.spinner.stop()
65         logging.getLogger().removeHandler(self.handler)

```

Listing 19: Simple progress view

```

1  from PyQt5.QtCore import Qt
2  from PyQt5.QtWidgets import QVBoxLayout, QWidget, QLabel, QSpacerItem, QSizePolicy
3  from pyqtspinner.spinner import WaitingSpinner
4
5
6  class SimpleProgressWidget(QWidget):
7
8      def __init__(self, parent=None):
9          super(QWidget, self).__init__(parent)
10
11         self.spinner = WaitingSpinner(self, radius=15.0, lines=10, line_length=15.0)
12
13         self.label = QLabel()
14         self.label.setAlignment(Qt.AlignCenter)
15         font = self.label.font()
16         font.setPointSize(15)
17         self.label.setFont(font)
18
19         layout = QVBoxLayout()
20         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
21         layout.addWidget(self.spinner)
22         layout.addItem(QSpacerItem(0, 50, QSizePolicy.Minimum, QSizePolicy.Minimum))
23         layout.addWidget(self.label)
24         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding))
25
26         self.setLayout(layout)
27
28     def spin(self, text):
29         self.spinner.start()
30         self.label.setText(text)
31

```

```

32 def stop(self):
33     self.spinner.stop()

```

Listing 20: Log handler for application

```

1 import logging
2
3
4 class LogHandler(logging.Handler):
5
6     def __init__(self, callback):
7         super().__init__()
8         self.callback = callback
9         self.setFormatter(logging.Formatter("%(asctime)s - %(message)s", "%H:%M:%S"))
10
11     def handle(self, record):
12         self.callback(self.format(record))

```

Listing 21: Matrix widget

```

1 from PyQt5.QtWidgets import QTableWidgetItem, QTableWidget, QSizePolicy
2
3
4 class MatrixWidget(QTableWidget):
5
6     def __init__(self, parent=None):
7         super(QTableWidget, self).__init__(parent)
8
9         self.itemChanged.connect(self.item_changed)
10        self.m = [["0"]]
11
12        self.setRowCount(1)
13        self.setColumnCount(1)
14        self.setItem(0, 0, QTableWidgetItem("0"))
15
16        self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))
17
18     def resize_w(self, w: int):
19
20        self.blockSignals(True)
21
22        old_w = self.columnCount()
23        self.setColumnCount(w)
24        h = self.rowCount()
25
26        self.m = [[self.m[i][j] if i < h and j < old_w else "0"
27                  for j in range(w)]
28                  for i in range(h)]
29
30     for i in range(h):
31         for j in range(w):
32             item = self.item(i, j)

```

```

33     if item is not None:
34         item.setText(self.m[i][j])
35     else:
36         self.setItem(i, j, CustomItem(self.m[i][j]))
37
38     self.blockSignals(False)
39
40     def resize_h(self, h: int):
41
42         self.blockSignals(True)
43
44         old_h = self.rowCount()
45         w = self.columnCount()
46         self.setRowCount(h)
47
48         self.m = [[self.m[i][j] if i < old_h and j < w else "0"
49                   for j in range(w)]
50                  for i in range(h)]
51
52         for i in range(h):
53             for j in range(w):
54                 item = self.item(i, j)
55                 if item is not None:
56                     item.setText(self.m[i][j])
57                 else:
58                     self.setItem(i, j, CustomItem(self.m[i][j]))
59
60         self.blockSignals(False)
61
62     def item_changed(self, item):
63         self.m[item.row()][item.column()] = item.text()
64
65
66 class CustomItem(QTableWidgetItem):
67
68     def __init__(self, value: str):
69         super(QTableWidgetItem, self).__init__(value)
70
71         self.valid = True

```

6.1.2 Source Codes of Charts Window

Listing 22: Charts window

```

1 from PyQt5.QtCore import pyqtSignal
2 from PyQt5.QtWidgets import QWidget, QHBoxLayout, QMainWindow, QSplitter
3
4 from ui.charts.side.available_charts_widget import AvailableChartsWidget
5 from ui.charts.visuals.charts_widget import ChartsWidget
6
7

```



```

8  class PlotWindow(QMainWindow):
9
10     charts_show = pyqtSignal()
11     charts_hide = pyqtSignal()
12
13     def __init__(self):
14         super(QMainWindow, self).__init__()
15
16         self.plot_widget = ChartsWidget(self)
17         self.charts_list = AvailableChartsWidget(self)
18
19         splitter = QSplitter()
20         splitter.addWidget(self.charts_list)
21         splitter.addWidget(self.plot_widget)
22         splitter.setSizes([splitter.width() / 0.85,
23                           splitter.width() / 0.15])
24
25         layout = QHBoxLayout(self)
26         layout.addWidget(splitter)
27
28         central_widget = QWidget()
29         central_widget.setLayout(layout)
30         self.setCentralWidget(central_widget)
31
32         self.setWindowTitle("SDE-MATH: charts window")
33         self.resize(1200, 800)
34
35         self.charts_list.on_show_all.connect(self.plot_widget.show_all)
36         self.charts_list.on_hide_all.connect(self.plot_widget.hide_all)
37         self.charts_list.on_remove_all.connect(self.plot_widget.delete_all)
38
39     def showEvent(self, event):
40         self.charts_show.emit()
41
42     def closeEvent(self, event):
43         self.charts_hide.emit()

```

Listing 23: Curves list

```

1  import os
2
3  from PyQt5 import QtGui
4  from PyQt5.QtCore import pyqtSignal
5  from PyQt5.QtWidgets import QVBoxLayout, QWidget, QSizePolicy, QSpacerItem, \
6     QScrollArea, QLabel, QHBoxLayout, QPushButton, QApplication, QStyle
7
8  from config import images
9  from ui.charts.side.item_widget import ItemWidget
10 from ui.main.info import InfoIcon
11
12
13 class AvailableChartsWidget(QWidget):
14

```

```

15 on_hide_all = pyqtSignal()
16 on_show_all = pyqtSignal()
17 on_remove_all = pyqtSignal()
18
19 def __init__(self, parent=None):
20     super(QWidget, self).__init__(parent)
21
22     self.items = dict()
23
24     self.spacer = QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding)
25     self.plot_widget = self.parent().plot_widget
26
27     remove_all = QPushButton()
28     remove_all.setFlat(True)
29     remove_all.setIcon(
30         QApplication.style().standardIcon(QStyle.SP_DialogResetButton))
31
32     hide_all = QPushButton()
33     hide_all.setFlat(True)
34     hide_all.setIcon(QtGui.QIcon(os.path.join(images, "crossed.png")))
35
36     show_all = QPushButton()
37     show_all.setFlat(True)
38     show_all.setIcon(QtGui.QIcon(os.path.join(images, "eye.png")))
39
40     header_layout = QHBoxLayout()
41     header_layout.addWidget(
42         InfoIcon("Here You will see all modeling series\n"
43                "You can hide them or delete, if you need to"))
44     header_layout.addItem(
45         QSpacerItem(5, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
46     header_layout.addWidget(QLabel("Curves"))
47     header_layout.addItem(
48         QSpacerItem(5, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
49     header_layout.setContentsMargins(0, 0, 0, 0)
50     header_layout.setSpacing(0)
51     header_layout.addItem(
52         QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
53     header_layout.addWidget(show_all)
54     header_layout.addWidget(hide_all)
55     header_layout.addWidget(remove_all)
56
57     self.layout = QVBoxLayout()
58     self.layout.setContentsMargins(3, 3, 3, 3)
59     self.layout.setSpacing(2)
60
61     scroll_widget = QWidget(self)
62     scroll_widget.setLayout(self.layout)
63
64     scroll_area = QScrollArea(self)
65     scroll_area.setWidgetResizable(True)
66     scroll_area.setWidget(scroll_widget)
67
68     self.layout.addItem(self.spacer)
69

```

```
70 layout = QVBoxLayout()
71 layout.setContentsMargins(0, 0, 0, 0)
72 layout.addLayout(header_layout)
73 layout.addWidget(scroll_area)
74
75 self.setLayout(layout)
76
77 self.setSizePolicy(
78     QSizePolicy(QSizePolicy.MinimumExpanding,
79                 QSizePolicy.MinimumExpanding))
80
81 show_all.clicked.connect(self.show_all)
82 hide_all.clicked.connect(self.hide_all)
83 remove_all.clicked.connect(self.delete_all)
84
85 def new_items(self, lines: list):
86     self.layout.removeItem(self.spacer)
87
88     for i in range(len(lines)):
89         item_widget = ItemWidget(lines[i].name, lines[i].color, parent=self)
90         item_widget.uid = lines[i].uid
91         item_widget.on_show.connect(self.plot_widget.show_item)
92         item_widget.on_hide.connect(self.plot_widget.hide_item)
93         item_widget.on_delete.connect(self.plot_widget.delete_item)
94         item_widget.on_delete.connect(self.delete_item)
95         self.items[lines[i].uid] = item_widget
96
97         self.plot_widget.hide_label.connect(lambda uid: self.items[uid].hide())
98         self.plot_widget.show_label.connect(lambda uid: self.items[uid].show())
99
100        self.layout.addWidget(item_widget)
101
102        self.layout.addItem(self.spacer)
103
104 def delete_item(self):
105     s = self.sender()
106     s.setParent(None)
107     self.items.pop(s.uid)
108     self.layout.removeWidget(s)
109
110 def delete_all(self):
111     for item in self.items.values():
112         item.setParent(None)
113         self.layout.removeWidget(item)
114     self.items.clear()
115     self.on_remove_all.emit()
116
117 def hide_all(self):
118     for item in self.items.values():
119         item.checkbox.blockSignals(True)
120         item.checkbox.setChecked(False)
121         item.checkbox.blockSignals(False)
122     self.on_hide_all.emit()
123
124 def show_all(self):
```

```

125     for item in self.items.values():
126         item.checkbox.blockSignals(True)
127         item.checkbox.setChecked(True)
128         item.checkbox.blockSignals(False)
129     self.on_show_all.emit()

```

Listing 24: Curves list item

```

1  from PyQt5.QtCore import pyqtSignal
2  from PyQt5.QtWidgets import QPushButton, QSizePolicy, QHBoxLayout, QStyle, \
3      QApplication, QCheckBox, QVBoxLayout, QLabel, QSpacerItem, QFrame
4
5
6  class ItemWidget(QFrame):
7
8      on_delete = pyqtSignal(object)
9      on_hide = pyqtSignal(int)
10     on_show = pyqtSignal(int)
11
12     def __init__(self, name, color, parent=None):
13         super(QFrame, self).__init__(parent)
14
15         self.uid = 0
16
17         self setFrameShape(QFrame.StyledPanel)
18         self.setStyleSheet("QFrame { background: white; }")
19
20         self.checkbox = QCheckBox(name)
21         self.checkbox.setChecked(True)
22
23         btn = QPushButton()
24         btn.setFlat(True)
25         btn.setIcon(QApplication.style().standardIcon(QStyle.SP_DialogCancelButton))
26
27         underline = QLabel()
28         underline.setSizePolicy(QSizePolicy.Expanding, QSizePolicy.Maximum)
29         underline.setMaximumHeight(3)
30         underline.setStyleSheet(f"QLabel {{ background: {color}; }}")
31
32         layout = QHBoxLayout()
33         layout.setContentsMargins(3, 3, 3, 3)
34         layout.setSpacing(0)
35         layout.addWidget(self.checkbox)
36         layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
37         layout.addWidget(btn)
38
39         layout_underlined = QVBoxLayout()
40         layout_underlined.setContentsMargins(0, 0, 0, 0)
41         layout_underlined.setSpacing(0)
42         layout_underlined.addLayout(layout)
43         layout_underlined.addWidget(underline)
44
45         self.setLayout(layout_underlined)

```

```

46
47     self.checkbox.stateChanged.connect(self.checkbox_changed)
48     btn.clicked.connect(lambda: self.on_delete.emit(self.uid))
49
50     def checkbox_changed(self, v):
51         if v > 0:
52             self.on_show.emit(self.uid)
53         else:
54             self.on_hide.emit(self.uid)

```

Listing 25: Charts area

```

1  import matplotlib.pyplot as plt
2  from PyQt5.QtCore import pyqtSignal
3  from PyQt5.QtWidgets import QSizePolicy, QFrame, QHBoxLayout, QPushButton, \
4      QSpacerItem
5  from PyQt5.QtWidgets import QVBoxLayout
6  from matplotlib.backends.backend_qt5agg import FigureCanvasQTAgg as FigureCanvas
7
8  from ui.charts.visuals.color import Color
9  from ui.charts.visuals.toolbar import ToolBar
10 from ui.main.info import InfoIcon
11
12
13 class ChartsWidget(QFrame):
14
15     hide_label = pyqtSignal(int)
16     show_label = pyqtSignal(int)
17
18     def __init__(self, parent=None):
19         super(QFrame, self).__init__(parent)
20
21         self.plots = dict()
22         self.mode = 0
23
24         self setFrameStyle(QFrame.StyledPanel)
25         self.setStyleSheet("QFrame { background: white; }")
26
27         self.figure = plt.figure()
28         self.canvas = FigureCanvas(self.figure)
29         self.canvas.mpl_connect('resize_event', self.on_resize)
30         self.toolbar = ToolBar(self.canvas, self)
31
32         self.ax = self.figure.add_subplot(111)
33         self.ax.margins(0)
34         self.ax.grid(axis='both', alpha=.3)
35         self.ax.relim(visible_only=True)
36         self.ax.autoscale()
37
38         self.figure.tight_layout()
39         self.canvas.draw()
40
41         self.btn_to_fn = QPushButton("Trajectories")

```

```

42     self.btn_to_fn.setFlat(True)
43     self.btn_to_mx = QPushButton("Expectations")
44     self.btn_to_mx.setFlat(True)
45     self.btn_to_dx = QPushButton("Variances")
46     self.btn_to_dx.setFlat(True)
47
48     toolbar_layout = QHBoxLayout()
49     toolbar_layout.setContentsMargins(0, 0, 0, 0)
50     toolbar_layout.setSpacing(0)
51     toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
52     toolbar_layout.addWidget(InfoIcon("Click this buttons to switch plot modes\n"
53         "between trajectories, expectations and variances"))
54     toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
55     toolbar_layout.addWidget(self.btn_to_fn)
56     toolbar_layout.addWidget(self.btn_to_mx)
57     toolbar_layout.addWidget(self.btn_to_dx)
58     toolbar_layout.addWidget(self.toolbar)
59     toolbar_layout.addItem(QSpacerItem(15, 0, QSizePolicy.Minimum, QSizePolicy.Minimum))
60
61     layout = QVBoxLayout()
62     layout.setContentsMargins(0, 0, 0, 0)
63     layout.setSpacing(0)
64     layout.addLayout(toolbar_layout)
65     layout.addWidget(self.canvas)
66
67     self.setLayout(layout)
68
69     self.setSizePolicy(QSizePolicy(QSizePolicy.Expanding, QSizePolicy.Expanding))
70
71     self.btn_to_fn.pressed.connect(self.fn_mode)
72     self.btn_to_mx.pressed.connect(self.mx_mode)
73     self.btn_to_dx.pressed.connect(self.dx_mode)
74
75     def rescale(self):
76         self.ax.relim(visible_only=True)
77         self.ax.autoscale()
78         self.canvas.draw()
79
80     def clear(self):
81         for f in self.plots.values():
82             self.hide_label.emit(f.uid)
83             if f.line_fn is not None:
84                 f.line_fn.set_visible(False)
85             if f.line_mx is not None:
86                 f.line_mx.set_visible(False)
87             if f.line_dx is not None:
88                 f.line_dx.set_visible(False)
89
90     def fn_mode(self):
91
92         self.mode = 0
93
94         self.clear()
95
96         for f in self.plots.values():

```

```
97     if f.line_fn is not None:
98         self.show_label.emit(f.uid)
99         if f.visible:
100             f.line_fn.set_visible(True)
101
102     self.rescale()
103
104     def mx_mode(self):
105
106         self.mode = 1
107
108         self.clear()
109
110         for f in self.plots.values():
111             if f.line_mx is not None:
112                 self.show_label.emit(f.uid)
113                 if f.visible:
114                     f.line_mx.set_visible(True)
115
116         self.rescale()
117
118     def dx_mode(self):
119
120         self.mode = 2
121
122         self.clear()
123
124         for f in self.plots.values():
125             if f.line_dx is not None:
126                 self.show_label.emit(f.uid)
127                 if f.visible:
128                     f.line_dx.set_visible(True)
129
130         self.rescale()
131
132     def new_items(self, lines: list):
133
134         for line in lines:
135             self.plots[line.uid] = line
136
137         for f in self.plots.values():
138             if f.line_fn is None and f.fn is not None:
139                 f.line_fn = self.ax.plot(f.t, f.fn, linewidth=1, color=f.color)[0]
140             if f.line_mx is None and f.mx is not None:
141                 f.line_mx = self.ax.plot(f.t, f.mx, linewidth=1, color=f.color)[0]
142             if f.line_dx is None and f.dx is not None:
143                 f.line_dx = self.ax.plot(f.t, f.dx, linewidth=1, color=f.color)[0]
144
145         if self.mode == 0:
146             self.fn_mode()
147
148         if self.mode == 1:
149             self.mx_mode()
150
151         if self.mode == 2:
```

```
152     self.dx_mode()
153
154     def delete_item(self, uid: int):
155         item = self.plots.pop(uid)
156         Color.free(item.color)
157         if item.line_fn is not None:
158             item.line_fn.remove()
159         if item.line_mx is not None:
160             item.line_mx.remove()
161         if item.line_dx is not None:
162             item.line_dx.remove()
163
164         self.rescale()
165
166     def hide_item(self, uid: int):
167         item = self.plots[uid]
168         if item.line_fn is not None and self.mode == 0:
169             item.line_fn.set_visible(False)
170         if item.line_mx is not None and self.mode == 1:
171             item.line_mx.set_visible(False)
172         if item.line_dx is not None and self.mode == 2:
173             item.line_dx.set_visible(False)
174         item.visible = False
175
176         self.rescale()
177
178     def show_item(self, uid: int):
179         item = self.plots[uid]
180         if item.line_fn is not None and self.mode == 0:
181             item.line_fn.set_visible(True)
182         if item.line_mx is not None and self.mode == 1:
183             item.line_mx.set_visible(True)
184         if item.line_dx is not None and self.mode == 2:
185             item.line_dx.set_visible(True)
186         item.visible = True
187
188         self.rescale()
189
190     def show_all(self):
191         for item in self.plots.values():
192             if item.line_fn is not None and self.mode == 0:
193                 item.line_fn.set_visible(True)
194             if item.line_mx is not None and self.mode == 1:
195                 item.line_mx.set_visible(True)
196             if item.line_dx is not None and self.mode == 2:
197                 item.line_dx.set_visible(True)
198             item.visible = True
199
200         self.rescale()
201
202     def hide_all(self):
203         for item in self.plots.values():
204             if item.line_fn is not None and self.mode == 0:
205                 item.line_fn.set_visible(False)
206             if item.line_mx is not None and self.mode == 1:
```



```

207     item.line_mx.set_visible(False)
208     if item.line_dx is not None and self.mode == 2:
209         item.line_dx.set_visible(False)
210         item.visible = False
211
212     self.rescale()
213
214     def delete_all(self):
215         for item in reversed(self.plots.values()):
216             Color.free(item.color)
217             if item.line_fn is not None:
218                 item.line_fn.remove()
219             if item.line_mx is not None:
220                 item.line_mx.remove()
221             if item.line_dx is not None:
222                 item.line_dx.remove()
223
224         self.plots.clear()
225
226         self.rescale()
227
228     def on_resize(self, event):
229         self.figure.tight_layout()
230         self.canvas.draw()

```

Listing 26: Curves color

```

1  from random import choice
2
3
4  class Color:
5
6      reserved_colors = [
7          "#ff834a",
8          "#ffe100",
9          "#c700c7",
10         "#24e280",
11         "#1100ff",
12         "#ff1e22",
13     ]
14     available_colors = [
15         "#ff834a",
16         "#ffe100",
17         "#c700c7",
18         "#24e280",
19         "#1100ff",
20         "#ff1e22",
21     ]
22
23     def __new__(cls, *args, **kwargs):
24
25         try:
26             return cls.available_colors.pop()

```

```

27
28     except IndexError:
29         return f"#{''.join([choice('0123456789ABCDEF') for j in range(6)])}"
30
31     @classmethod
32     def free(cls, code: str):
33
34         if code in cls.reserved_colors:
35             cls.available_colors.append(code)

```

Listing 27: Curve

```

1 from ui.charts.visuals.color import Color
2
3
4 class Line:
5     count = 0
6
7     def __init__(self, name, t, fn, mx=None, dx=None):
8         self.name = name
9         self.t = t
10        self.fn = fn
11        self.mx = mx
12        self.dx = dx
13        self.line_fn = None
14        self.line_mx = None
15        self.line_dx = None
16        self.visible = True
17        self.color = Color()
18
19        self.uid = Line.count
20        Line.count += 1

```

6.1.3 Source Codes of Input for Nonlinear Systems of Itô SDEs

Listing 28: Base part of data input for nonlinear systems

```

1 import logging
2
3 import numpy as np
4 from PyQt5.QtCore import QThreadPool, pyqtSignal
5 from PyQt5.QtWidgets import QCheckBox, QPushButton, QStyle, QApplication, \
6     QSizePolicy, QHBoxLayout, QSpacerItem, QVBoxLayout, QStackedWidget, \
7     QWidget, QLabel
8 from sympy import Matrix
9
10 import config
11 from mathematics.sde.nonlinear.drivers.euler import euler
12 from mathematics.sde.nonlinear.drivers.milstein import milstein
13 from mathematics.sde.nonlinear.drivers.strong_taylor_ito_1p5 import strong_taylor_ito_1p5

```

```

14 from mathematics.sde.nonlinear.drivers.strong_taylor_ito_2p0 import strong_taylor_ito_2p0
15 from mathematics.sde.nonlinear.drivers.strong_taylor_ito_2p5 import strong_taylor_ito_2p5
16 from mathematics.sde.nonlinear.drivers.strong_taylor_ito_3p0 import strong_taylor_ito_3p0
17 from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_1p0 import
    strong_taylor_stratonovich_1p0
18 from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_1p5 import
    strong_taylor_stratonovich_1p5
19 from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_2p0 import
    strong_taylor_stratonovich_2p0
20 from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_2p5 import
    strong_taylor_stratonovich_2p5
21 from mathematics.sde.nonlinear.drivers.strong_taylor_stratonovich_3p0 import
    strong_taylor_stratonovich_3p0
22 from mathematics.sde.nonlinear.symbolic.coefficients.c import C
23 from tools import database
24 from ui.async_calls.worker import Worker
25 from ui.charts.visuals.line import Line
26 from ui.main.modeling.nonlinear.step1 import Step1
27 from ui.main.modeling.nonlinear.step2 import Step2
28 from ui.main.modeling.nonlinear.step3 import Step3
29 from ui.main.modeling.nonlinear.step4 import Step4
30 from ui.main.modeling.nonlinear.step5 import Step5
31
32
33 class NonlinearModelingWidget(QWidget):
34
35     show_main_menu = pyqtSignal()
36     start_progress = pyqtSignal(str)
37     stop_progress = pyqtSignal(str)
38     draw_chart = pyqtSignal(list)
39
40     def __init__(self, parent=None):
41         super(QWidget, self).__init__(parent)
42
43         self.logger = logging.getLogger(__name__)
44         self.scheme_id = 0
45         self.schemes = [
46             (euler, "Euler", "Euler Scheme"),
47             (milstein, "Milstein", "Milstein Scheme"),
48             (strong_taylor_ito_1p5, "Taylor–Ito 1.5",
49              "Strong Taylor–Ito Scheme with Convergence Order 1.5"),
50             (strong_taylor_ito_2p0, "Taylor–Ito 2.0",
51              "Strong Taylor–Ito Scheme with Convergence Order 2.0"),
52             (strong_taylor_ito_2p5, "Taylor–Ito 2.5",
53              "Strong Taylor–Ito Scheme with Convergence Order 2.5"),
54             (strong_taylor_ito_3p0, "Taylor–Ito 3.0",
55              "Strong Taylor–Ito Scheme with Convergence Order 3.0"),
56             (strong_taylor_stratonovich_1p0, "Taylor–Str. 1.0",
57              "Strong Taylor–Stratonovich Scheme with Convergence Order 1.0"),
58             (strong_taylor_stratonovich_1p5, "Taylor–Str. 1.5",
59              "Strong Taylor–Stratonovich Scheme with Convergence Order 1.5"),
60             (strong_taylor_stratonovich_2p0, "Taylor–Str. 2.0",
61              "Strong Taylor–Stratonovich Scheme with Convergence Order 2.0"),
62             (strong_taylor_stratonovich_2p5, "Taylor–Str. 2.5",
63              "Strong Taylor–Stratonovich Scheme with Convergence Order 2.5"),

```

```
64     (strong_taylor_stratonovich_3p0 , "Taylor–Str. 3.0" ,
65         "Strong Taylor–Stratonovich Scheme with Convergence Order 3.0") ,
66     ]
67
68     self.stack_widget = QStackedWidget(self)
69
70     self.step1 = Step1()
71     self.step2 = Step2()
72     self.step3 = Step3()
73     self.step4 = Step4()
74     self.step5 = Step5()
75
76     back_btn = QPushButton("Back" , self)
77     back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
78
79     self.charts_check = QCheckBox("Charts window" , self)
80
81     self.scheme_name = QLabel()
82     self.scheme_name.setSizePolicy(QSizePolicy.Expanding , QSizePolicy.Minimum)
83
84     bar_layout = QHBoxLayout()
85     bar_layout.addWidget(back_btn)
86     bar_layout.addItem(QSpacerItem(10 , 35 , QSizePolicy.Minimum , QSizePolicy.Minimum))
87     bar_layout.addWidget(self.scheme_name)
88     bar_layout.addItem(QSpacerItem(0 , 0 , QSizePolicy.Expanding , QSizePolicy.Minimum))
89     bar_layout.addWidget(self.charts_check)
90
91     self.stack_widget.addWidget(self.step1)
92     self.stack_widget.addWidget(self.step2)
93     self.stack_widget.addWidget(self.step3)
94     self.stack_widget.addWidget(self.step4)
95     self.stack_widget.addWidget(self.step5)
96
97     layout = QVBoxLayout()
98     layout.addLayout(bar_layout)
99     layout.addWidget(self.stack_widget)
100
101     self.setLayout(layout)
102
103     back_btn.clicked.connect(self.show_main_menu.emit)
104     back_btn.clicked.connect(
105         lambda: self.stack_widget.setCurrentWidget(self.step1))
106
107     self.step1.next_btn.clicked.connect(
108         lambda: self.stack_widget.setCurrentWidget(self.step2))
109     self.step2.prev_btn.clicked.connect(
110         lambda: self.stack_widget.setCurrentWidget(self.step1))
111     self.step2.next_btn.clicked.connect(
112         lambda: self.stack_widget.setCurrentWidget(self.step3))
113     self.step3.prev_btn.clicked.connect(
114         lambda: self.stack_widget.setCurrentWidget(self.step2))
115     self.step3.next_btn.clicked.connect(
116         lambda: self.stack_widget.setCurrentWidget(self.step4))
117     self.step4.prev_btn.clicked.connect(
118         lambda: self.stack_widget.setCurrentWidget(self.step3))
```

```

119     self.step4.next_btn.clicked.connect(
120         lambda: self.stack_widget.setCurrentWidget(self.step5))
121     self.step5.prev_btn.clicked.connect(
122         lambda: self.stack_widget.setCurrentWidget(self.step4))
123
124     self.step5.run_btn.clicked.connect(
125         lambda: self.run_modeling())
126     self.step5.run_btn.clicked.connect(
127         lambda: self.stack_widget.setCurrentWidget(self.step1))
128
129     self.step1.n_valid.connect(self.step2.matrix.resize_h)
130     self.step1.n_valid.connect(self.step3.matrix.resize_h)
131     self.step1.n_valid.connect(self.step4.matrix.resize_h)
132
133     self.step1.m_valid.connect(self.step3.matrix.resize_w)
134
135     def set_scheme(self, scheme_id):
136         self.scheme_id = scheme_id
137         self.scheme_name.setText(self.schemes[scheme_id][2])
138         if scheme_id == 0:
139             self.step5.count_c(False)
140         else:
141             self.step5.count_c(True)
142
143     def run_modeling(self):
144         self.start_progress.emit("The modeling is being performed...")
145
146         worker = Worker(self.routine)
147         worker.signals.result.connect(self.on_modeling_finish)
148         worker.signals.error.connect(self.on_modeling_corrupted)
149         QThreadPool.globalInstance().start(worker)
150
151     def routine(self):
152
153         scheme = self.schemes[self.scheme_id]
154
155         a = Matrix(self.step2.matrix.m)
156         b = Matrix(self.step3.matrix.m)
157
158         x0 = np.ndarray(shape=(self.step4.matrix.rowCount(),
159                               self.step4.matrix.columnCount()), dtype=float)
160         for i in range(self.step4.matrix.rowCount()):
161             for j in range(self.step4.matrix.columnCount()):
162                 x0[i][j] = float(self.step4.matrix.m[i][j])
163
164         if self.step5.s != 0:
165             np.random.seed(self.step5.s)
166
167         database.connect(config.database)
168
169         if self.scheme_id == 0:
170             result = scheme[0](
171                 x0, a, b,
172                 (self.step5.t0,
173                  self.step5.dt,

```

```

174         self.step5.t1)
175     )
176     else:
177         C.preload(56, 56, 56, 56, 56)
178         result = scheme[0](
179             x0, a, b, self.step5.c,
180             (self.step5.t0,
181              self.step5.dt,
182              self.step5.t1)
183         )
184
185     database.disconnect()
186
187     lines = [Line(f"{scheme[1]}, x{i + 1}",
188                 np.array(result[1]).astype(float),
189                 np.array(result[0][i, :]).astype(float))
190             for i in range(len(result[0]))]
191
192     return lines
193
194     def on_modeling_finish(self, result):
195         self.stop_progress.emit("The modeling has been completed!")
196         self.draw_chart.emit(result)
197
198     def on_modeling_corrupted(self, result):
199
200         self.logger.error(result[0])
201         self.logger.error(result[1])
202         self.logger.error(result[2])
203
204         self.stop_progress.emit("The modeling failed!")

```

Listing 29: Step 1 of data input for nonlinear systems

```

1  from PyQt5.QtCore import pyqtSignal
2  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QGridLayout, QLineEdit, QLabel, \
3      QVBoxLayout, QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle
4
5  from ui.main.error import ErrorWidget
6  from ui.main.info import InfoIcon
7  from ui.main.svg import SVG
8
9
10 class Step1(QWidget):
11
12     n_valid = pyqtSignal(int)
13     m_valid = pyqtSignal(int)
14
15     def __init__(self, parent=None):
16         super(QWidget, self).__init__(parent)
17
18         self.n_is_valid = False
19         self.m_is_valid = False

```

```

20
21     self.input_stack = self.parent()
22
23     info_n = QIcon("Dimension of linear system of Ito SDEs")
24     info_m = QIcon("Dimension of vector Wiener process")
25
26     label_n = QLabel("n")
27     label_m = QLabel("m")
28
29     self.lineedit_n = QLineEdit()
30     self.lineedit_m = QLineEdit()
31
32     self.msg_n = ErrorWidget("Wrong value!")
33     self.msg_n.hide()
34
35     self.msg_m = ErrorWidget("Wrong value!")
36     self.msg_m.hide()
37
38     grid_layout = QGridLayout()
39     grid_layout.addWidget(self.msg_n, 0, 2)
40     grid_layout.addWidget(self.msg_m, 2, 2)
41     grid_layout.addWidget(info_n, 1, 0)
42     grid_layout.addWidget(info_m, 3, 0)
43     grid_layout.addWidget(label_n, 1, 1)
44     grid_layout.addWidget(label_m, 3, 1)
45     grid_layout.addWidget(self.lineedit_n, 1, 2)
46     grid_layout.addWidget(self.lineedit_m, 3, 2)
47
48     header = QLabel("Dimensions settings", parent=self)
49     font = header.font()
50     font.setPointSize(15)
51     header.setFont(font)
52
53     self.next_btn = QPushButton("Next", self)
54     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
55     self.next_btn.setEnabled(False)
56
57     header_layout = QHBoxLayout()
58     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
59     header_layout.addWidget(header)
60     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
61
62     bottom_bar = QHBoxLayout()
63     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
64     bottom_bar.addWidget(self.next_btn)
65
66     eq1 = QHBoxLayout()
67     eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
68     eq1.addWidget(SVG("equation1.svg", scale_factor=1.))
69     eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
70
71     eq2 = QHBoxLayout()
72     eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
73     eq2.addWidget(SVG("equation2.svg", scale_factor=1.))
74     eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))

```

```

75
76 equalities_layout = QVBoxLayout()
77 equalities_layout.addLayout(eq1)
78 equalities_layout.addItem(QSpacerItem(0, 20, QSizePolicy.Minimum, QSizePolicy.Minimum
))
79 equalities_layout.addLayout(eq2)
80
81 equalities_wrap = QHBoxLayout()
82 equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum)
)
83 equalities_wrap.addLayout(equalities_layout)
84 equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum)
)
85
86 grid_wrap = QHBoxLayout()
87 grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
88 grid_wrap.addLayout(grid_layout)
89 grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
90
91 layout = QVBoxLayout()
92 layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
93 layout.addLayout(equalities_wrap)
94 layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
95 layout.addLayout(header_layout)
96 layout.addItem(QSpacerItem(0, 5, QSizePolicy.Minimum, QSizePolicy.Minimum))
97 layout.addLayout(grid_wrap)
98 layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
99 layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
100 layout.addLayout(bottom_bar)
101
102 self.setLayout(layout)
103
104 self.lineedit_n.textChanged.connect(self.validate_n)
105 self.lineedit_m.textChanged.connect(self.validate_m)
106
107 def validate_form(self):
108     if self.n_is_valid and self.m_is_valid:
109         self.next_btn.setEnabled(True)
110     else:
111         self.next_btn.setEnabled(False)
112
113 def validate_n(self, value):
114     try:
115         typed_value = int(value)
116         if typed_value <= 0:
117             raise ValueError()
118
119         self.n_is_valid = True
120         self.n_valid.emit(typed_value)
121         self.msg_n.hide()
122
123     except ValueError:
124         self.n_is_valid = False
125         self.msg_n.show()
126

```



```

127     finally :
128         self.validate_form()
129
130     def validate_m(self, value):
131         try:
132             typed_value = int(value)
133             if typed_value <= 0:
134                 raise ValueError()
135
136             self.m_is_valid = True
137             self.m_valid.emit(typed_value)
138             self.msg_m.hide()
139
140         except ValueError:
141             self.m_is_valid = False
142             self.msg_m.show()
143
144     finally :
145         self.validate_form()

```

Listing 30: Step 2 of data input for nonlinear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
2      QVBoxLayout, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step2(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of column a(x, t)", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of column a(x, t) are expected to be functions\n"
22             "Size: n x 1\n"
23             "Functions must be set in python and SymPy notation")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()
27
28         self.matrix = MatrixWidget(self)
29
30         self.next_btn = QPushButton("Next", self)
31         self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))

```

```

32
33     self.prev_btn = QPushButton(" Back", self)
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36     header_layout = QHBoxLayout()
37     header_layout.addWidget(info)
38     header_layout.addWidget(header)
39     header_layout.addWidget(self.msg)
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42     bottom_bar = QHBoxLayout()
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44     bottom_bar.addWidget(self.prev_btn)
45     bottom_bar.addWidget(self.next_btn)
46
47     layout = QVBoxLayout()
48     layout.addLayout(header_layout)
49     layout.addWidget(self.matrix)
50     layout.addLayout(bottom_bar)
51
52     self.setLayout(layout)

```

Listing 31: Step 3 of data input for nonlinear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
2      QVBoxLayout, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step3(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of matrix B(x, t)", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of matrix B(x, t) are expected to be functions\n"
22             "Size: n x m\n"
23             "Functions must be set in python and SymPy notation")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()
27
28         self.matrix = MatrixWidget(self)
29

```

```

30     self.next_btn = QPushButton("Next", self)
31     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33     self.prev_btn = QPushButton("Back", self)
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36     header_layout = QHBoxLayout()
37     header_layout.addWidget(info)
38     header_layout.addWidget(header)
39     header_layout.addWidget(self.msg)
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42     bottom_bar = QHBoxLayout()
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44     bottom_bar.addWidget(self.prev_btn)
45     bottom_bar.addWidget(self.next_btn)
46
47     layout = QVBoxLayout()
48     layout.addLayout(header_layout)
49     layout.addWidget(self.matrix)
50     layout.addLayout(bottom_bar)
51
52     self.setLayout(layout)

```

Listing 32: Step 4 of data input for nonlinear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, \
2      QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step4(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of column x0", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of column x0 are expected to be functions\n"
22             "Size: n x 1\n"
23             "Functions must be set in python and SymPy notation")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()
27

```

```

28     self.matrix = MatrixWidget(self)
29
30     self.next_btn = QPushButton("Next", self)
31     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33     self.prev_btn = QPushButton("Back", self)
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36     header_layout = QHBoxLayout()
37     header_layout.addWidget(info)
38     header_layout.addWidget(header)
39     header_layout.addWidget(self.msg)
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42     bottom_bar = QHBoxLayout()
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44     bottom_bar.addWidget(self.prev_btn)
45     bottom_bar.addWidget(self.next_btn)
46
47     layout = QVBoxLayout()
48     layout.addLayout(header_layout)
49     layout.addWidget(self.matrix)
50     layout.addLayout(bottom_bar)
51
52     self.setLayout(layout)

```

Listing 33: Step 5 of data input for nonlinear systems

```

1  import sys
2
3  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QLineEdit, \
4      QGridLayout, QVBoxLayout, QSizePolicy, QSpacerItem, QPushButton, \
5      QApplication, QStyle
6
7  from ui.main.error import ErrorWidget
8  from ui.main.info import InfoIcon
9
10
11  class Step5(QWidget):
12
13      def __init__(self, parent=None):
14          super(QWidget, self).__init__(parent)
15
16          self.t0 = sys.float_info.min
17          self.dt = 0
18          self.t1 = sys.float_info.max
19          self.s = 0
20          self.c = 0
21
22          self.t0_is_valid = False
23          self.dt_is_valid = False
24          self.t1_is_valid = False
25          self.s_is_valid = True

```

```

26     self.c_is_valid = False
27
28     self.input_stack = self.parent()
29
30     info_t0 = InfoIcon("Start point of integration interval\n"
31                       "Must be in [0, t1) range")
32     info_dt = InfoIcon("Integration step\n"
33                       "Must be set in (0, 1) interval")
34     info_t1 = InfoIcon("Final point of integration interval\n"
35                       "Must be more then t0")
36     info_s = InfoIcon("This is random generator seed\n"
37                      "If You do not want to use specific\n"
38                      "seed just leave this field empty")
39     self.info_c = InfoIcon("The constant which defines approximation accuracy")
40
41     label_t0 = QLabel("t0")
42     label_dt = QLabel("dt")
43     label_t1 = QLabel("t1")
44     label_s = QLabel("seed")
45     self.label_c = QLabel("C")
46
47     self.lineedit_t0 = QLineEdit()
48     self.lineedit_dt = QLineEdit()
49     self.lineedit_t1 = QLineEdit()
50     self.lineedit_s = QLineEdit()
51     self.lineedit_c = QLineEdit()
52
53     self.msg_t0 = ErrorWidget("Wrong value!")
54     self.msg_t0.hide()
55
56     self.msg_dt = ErrorWidget("Wrong value!")
57     self.msg_dt.hide()
58
59     self.msg_t1 = ErrorWidget("Wrong value!")
60     self.msg_t1.hide()
61
62     self.msg_s = ErrorWidget("Wrong value!")
63     self.msg_s.hide()
64
65     self.msg_c = ErrorWidget("Wrong value!")
66     self.msg_c.hide()
67
68     header = QLabel("Accuracy settings", parent=self)
69     font = header.font()
70     font.setPointSize(15)
71     header.setFont(font)
72
73     self.prev_btn = QPushButton("Back", self)
74     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
75
76     self.run_btn = QPushButton("Perform modeling", self)
77     self.run_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
78     self.run_btn.setEnabled(False)
79
80     header_layout = QHBoxLayout()

```

```
81     header_layout.addWidget(header)
82     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
83
84     bottom_bar = QHBoxLayout()
85     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
86     bottom_bar.addWidget(self.prev_btn)
87     bottom_bar.addWidget(self.run_btn)
88
89     grid_layout = QGridLayout()
90
91     grid_layout.addWidget(self.msg_t0, 0, 2)
92
93     grid_layout.addWidget(info_t0, 1, 0)
94     grid_layout.addWidget(label_t0, 1, 1)
95     grid_layout.addWidget(self.lineedit_t0, 1, 2)
96
97     grid_layout.addWidget(self.msg_dt, 0, 5)
98
99     grid_layout.addWidget(info_dt, 1, 3)
100    grid_layout.addWidget(label_dt, 1, 4)
101    grid_layout.addWidget(self.lineedit_dt, 1, 5)
102
103    grid_layout.addWidget(self.msg_t1, 0, 8)
104
105    grid_layout.addWidget(info_t1, 1, 6)
106    grid_layout.addWidget(label_t1, 1, 7)
107    grid_layout.addWidget(self.lineedit_t1, 1, 8)
108
109    grid_layout.addWidget(self.msg_s, 2, 2)
110
111    grid_layout.addWidget(info_s, 3, 0)
112    grid_layout.addWidget(label_s, 3, 1)
113    grid_layout.addWidget(self.lineedit_s, 3, 2)
114
115    grid_layout.addWidget(self.msg_c, 2, 5)
116
117    grid_layout.addWidget(self.info_c, 3, 3)
118    grid_layout.addWidget(self.label_c, 3, 4)
119    grid_layout.addWidget(self.lineedit_c, 3, 5)
120
121    column_layout = QVBoxLayout()
122    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
123    )
124    column_layout.addLayout(header_layout)
125    column_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
126    column_layout.addLayout(grid_layout)
127    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
128    )
129
130    control_layout = QHBoxLayout()
131    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
132    ))
133    control_layout.addLayout(column_layout)
134    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
135    ))
```

```
132
133 layout = QVBoxLayout()
134 layout.addLayout(control_layout)
135 layout.addLayout(bottom_bar)
136
137 self.setLayout(layout)
138
139 self.lineEdit_t0.textChanged.connect(self.validate_t0)
140 self.lineEdit_t0.textChanged.connect(self.validate_t1)
141 self.lineEdit_t0.textChanged.connect(self.validate_dt)
142 self.lineEdit_dt.textChanged.connect(self.validate_dt)
143 self.lineEdit_t1.textChanged.connect(self.validate_t0)
144 self.lineEdit_t1.textChanged.connect(self.validate_t1)
145 self.lineEdit_t1.textChanged.connect(self.validate_dt)
146 self.lineEdit_s.textChanged.connect(self.validate_s)
147 self.lineEdit_c.textChanged.connect(self.validate_c)
148
149 def count_c(self, flag: bool):
150     if flag:
151         self.info_c.show()
152         self.label_c.show()
153         self.lineEdit_c.show()
154         self.validate_c()
155     else:
156         self.info_c.hide()
157         self.label_c.hide()
158         self.lineEdit_c.hide()
159         self.c_is_valid = True
160
161     self.msg_c.hide()
162
163 def validate_form(self):
164     if self.t0_is_valid \
165         and self.dt_is_valid \
166         and self.t1_is_valid \
167         and self.s_is_valid \
168         and self.c_is_valid:
169         self.run_btn.setEnabled(True)
170     else:
171         self.run_btn.setEnabled(False)
172
173 def validate_t0(self):
174     try:
175         typed_value = float(self.lineEdit_t0.text())
176         if typed_value >= self.t1:
177             raise ValueError()
178
179         self.t0_is_valid = True
180         self.t0 = typed_value
181         self.msg_t0.hide()
182
183     except ValueError:
184         self.t0_is_valid = False
185         self.msg_t0.show()
186
```

```
187 def validate_dt(self):
188     try:
189         typed_value = float(self.lineedit_dt.text())
190         if typed_value <= 0 \
191             or (self.t1 - self.t0) / typed_value < 1 \
192             or typed_value >= 1:
193             raise ValueError()
194
195         self.dt_is_valid = True
196         self.dt = typed_value
197         self.msg_dt.hide()
198         self.validate_form()
199
200     except ValueError:
201         self.dt_is_valid = False
202         self.msg_dt.show()
203
204 def validate_t1(self):
205     try:
206         typed_value = float(self.lineedit_t1.text())
207         if typed_value <= self.t0:
208             raise ValueError()
209
210         self.t1_is_valid = True
211         self.t1 = typed_value
212         self.msg_t1.hide()
213
214     except ValueError:
215         self.t1_is_valid = False
216         self.msg_t1.show()
217
218 def validate_s(self):
219     try:
220         if self.lineedit_s.text() == "":
221             self.s_is_valid = True
222             self.s = 0
223         else:
224             typed_value = int(self.lineedit_s.text())
225             if typed_value <= 0:
226                 raise ValueError()
227             self.s = typed_value
228
229         self.s_is_valid = True
230         self.msg_s.hide()
231
232     except ValueError:
233         self.s_is_valid = False
234         self.msg_s.show()
235
236     finally:
237         self.validate_form()
238
239 def validate_c(self):
240     try:
241         typed_value = float(self.lineedit_c.text())
```



```

242     if typed_value <= 0:
243         raise ValueError()
244
245     self.c_is_valid = True
246     self.c = typed_value
247     self.msg_c.hide()
248
249     except ValueError:
250         self.c_is_valid = False
251         self.msg_c.show()
252
253     finally:
254         self.validate_form()

```

6.1.4 Source Codes of Input for Linear Systems of Itô SDEs

Listing 34: Base part of data input for linear systems

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from PyQt5.QtCore import QThreadPool, pyqtSignal
6  from PyQt5.QtWidgets import QCheckBox, QPushButton, QStyle, QApplication, \
7      QSizePolicy, QHBoxLayout, QSpacerItem, QVBoxLayout, QStackedWidget, \
8      QWidget, QLabel
9
10 from mathematics.sde.linear.dindet import dindet
11 from mathematics.sde.linear.distortions import Symbolic, ComplexDistortion
12 from mathematics.sde.linear.integration import Integral
13 from mathematics.sde.linear.stoch import stoch
14 from ui.async_calls.worker import Worker
15 from ui.charts.visuals.line import Line
16 from ui.main.modeling.linear.step1 import Step1
17 from ui.main.modeling.linear.step2 import Step2
18 from ui.main.modeling.linear.step3 import Step3
19 from ui.main.modeling.linear.step4 import Step4
20 from ui.main.modeling.linear.step5 import Step5
21 from ui.main.modeling.linear.step6 import Step6
22 from ui.main.modeling.linear.step7 import Step7
23 from ui.main.modeling.linear.step8 import Step8
24
25
26 class LinearModelingWidget(QWidget):
27
28     show_main_menu = pyqtSignal()
29     start_progress = pyqtSignal(str)
30     stop_progress = pyqtSignal(str)
31     draw_chart = pyqtSignal(list)
32
33     def __init__(self, parent=None):

```

```

34     super(QWidget, self).__init__(parent)
35
36     self.logger = logging.getLogger(__name__)
37
38     self.stack_widget = QStackedWidget(self)
39
40     self.step1 = Step1()
41     self.step2 = Step2()
42     self.step3 = Step3()
43     self.step4 = Step4()
44     self.step5 = Step5()
45     self.step6 = Step6()
46     self.step7 = Step7()
47     self.step8 = Step8()
48
49     back_btn = QPushButton("Back", self)
50     back_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
51
52     self.charts_check = QCheckBox("Charts window", self)
53
54     self.scheme_name = QLabel("Linear Systems of Ito SDEs")
55
56     bar_layout = QHBoxLayout()
57     bar_layout.addWidget(back_btn)
58     bar_layout.addItem(QSpacerItem(10, 35, QSizePolicy.Minimum, QSizePolicy.Minimum))
59     bar_layout.addWidget(self.scheme_name)
60     bar_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
61     bar_layout.addWidget(self.charts_check)
62
63     self.stack_widget.addWidget(self.step1)
64     self.stack_widget.addWidget(self.step2)
65     self.stack_widget.addWidget(self.step3)
66     self.stack_widget.addWidget(self.step4)
67     self.stack_widget.addWidget(self.step5)
68     self.stack_widget.addWidget(self.step6)
69     self.stack_widget.addWidget(self.step7)
70     self.stack_widget.addWidget(self.step8)
71
72     layout = QVBoxLayout()
73     layout.addLayout(bar_layout)
74     layout.addWidget(self.stack_widget)
75
76     self.setLayout(layout)
77
78     back_btn.clicked.connect(self.show_main_menu.emit)
79     back_btn.clicked.connect(
80         lambda: self.stack_widget.setCurrentWidget(self.step1))
81
82     self.step1.next_btn.clicked.connect(
83         lambda: self.stack_widget.setCurrentWidget(self.step2))
84     self.step2.prev_btn.clicked.connect(
85         lambda: self.stack_widget.setCurrentWidget(self.step1))
86     self.step2.next_btn.clicked.connect(
87         lambda: self.stack_widget.setCurrentWidget(self.step3))
88     self.step3.prev_btn.clicked.connect(

```

```

89     lambda: self.stack_widget.setCurrentWidget(self.step2))
90 self.step3.next_btn.clicked.connect(
91     lambda: self.stack_widget.setCurrentWidget(self.step4))
92 self.step4.prev_btn.clicked.connect(
93     lambda: self.stack_widget.setCurrentWidget(self.step3))
94 self.step4.next_btn.clicked.connect(
95     lambda: self.stack_widget.setCurrentWidget(self.step5))
96 self.step5.prev_btn.clicked.connect(
97     lambda: self.stack_widget.setCurrentWidget(self.step4))
98 self.step5.next_btn.clicked.connect(
99     lambda: self.stack_widget.setCurrentWidget(self.step6))
100 self.step6.prev_btn.clicked.connect(
101     lambda: self.stack_widget.setCurrentWidget(self.step5))
102 self.step6.next_btn.clicked.connect(
103     lambda: self.stack_widget.setCurrentWidget(self.step7))
104 self.step7.prev_btn.clicked.connect(
105     lambda: self.stack_widget.setCurrentWidget(self.step6))
106 self.step7.next_btn.clicked.connect(
107     lambda: self.stack_widget.setCurrentWidget(self.step8))
108 self.step8.prev_btn.clicked.connect(
109     lambda: self.stack_widget.setCurrentWidget(self.step7))
110
111 self.step8.run_btn.clicked.connect(
112     lambda: self.run_modeling())
113 self.step8.run_btn.clicked.connect(
114     lambda: self.stack_widget.setCurrentWidget(self.step1))
115
116 self.step1.n_valid.connect(self.step2.matrix.resize_h)
117 self.step1.n_valid.connect(self.step2.matrix.resize_w)
118
119 self.step1.n_valid.connect(self.step3.matrix.resize_h)
120 self.step1.k_valid.connect(self.step3.matrix.resize_w)
121
122 self.step1.n_valid.connect(self.step4.matrix.resize_h)
123 self.step1.m_valid.connect(self.step4.matrix.resize_w)
124
125 self.step1.k_valid.connect(self.step5.matrix.resize_h)
126
127 self.step1.n_valid.connect(self.step6.matrix.resize_w)
128
129 self.step1.n_valid.connect(self.step7.matrix.resize_h)
130
131 def run_modeling(self):
132     self.start_progress.emit("The modeling is being performed...")
133
134     worker = Worker(self.routine)
135     worker.signals.result.connect(self.on_modeling_finish)
136     worker.signals.error.connect(self.on_modeling_corrupted)
137     QThreadPool.globalInstance().start(worker)
138
139 def routine(self):
140     n = int(self.step1.lineedit_n.text())
141     m = int(self.step1.lineedit_m.text())
142     k = int(self.step1.lineedit_k.text())
143     t0 = float(self.step8.lineedit_t0.text())

```

```

144 dt = float(self.step8.lineedit_dt.text())
145 t1 = float(self.step8.lineedit_t1.text())
146
147 self.logger.info("Reading input data")
148
149 integral = Integral(n)
150
151 integral.k, integral.m, integral.dt, integral.t0, integral.tk = \
152     k, m, dt, t0, t1
153
154 integral.m_a = np.array([[float(self.step2.matrix.m[i][j])
155                          for j in range(self.step2.matrix.columnCount())
156                          for i in range(self.step2.matrix.rowCount())])
157
158 integral.mat_b = np.array([[float(self.step3.matrix.m[i][j])
159                             for j in range(self.step3.matrix.columnCount())
160                             for i in range(self.step3.matrix.rowCount())])
161
162 integral.mat_f = np.array([[float(self.step4.matrix.m[i][j])
163                             for j in range(self.step4.matrix.columnCount())
164                             for i in range(self.step4.matrix.rowCount())])
165
166 integral.m_h = np.array([[float(self.step6.matrix.m[i][j])
167                             for j in range(self.step6.matrix.columnCount())
168                             for i in range(self.step6.matrix.rowCount())])
169
170 integral.m_x0 = np.array([[float(self.step7.matrix.m[i][j])
171                             for j in range(self.step7.matrix.columnCount())
172                             for i in range(self.step7.matrix.rowCount())])
173
174 integral.m_mx0 = np.array([[float(self.step7.matrix.m[i][j])
175                             for j in range(self.step7.matrix.columnCount())
176                             for i in range(self.step7.matrix.rowCount())])
177
178 integral.m_dx0 = np.zeros((integral.n, integral.n))
179
180 self.logger.info("Input is correct")
181 self.logger.info("Calculation of Ad and Bd (Algorithm 11.2)")
182
183 integral.m_ad, integral.m_bd = dindet(
184     integral.n, integral.k, integral.m_a, integral.mat_b, integral.dt)
185
186 self.logger.info("Calculation of Fd (Algorithm 11.6)")
187
188 integral.m_fd = stoch(integral.n, integral.m_a, integral.mat_f, integral.dt)
189
190 mat_u = np.array([[object]] * self.step5.matrix.rowCount())
191 for i in range(self.step5.matrix.rowCount()):
192     mat_u[i][0] = Symbolic(self.step5.matrix.m[i][0])
193
194 integral.distortion = ComplexDistortion(self.step5.matrix.rowCount(), mat_u)
195
196 self.logger.info("Starting modeling loop")
197
198 start_time = time()

```

```

199     integral.integrate()
200
201     self.logger.info(f"Integration took {(time() - start_time):.3f} seconds")
202
203     lines = [Line(f"Linear, x{i + 1}",
204                 np.array(integral.v_t).astype(float),
205                 np.array(integral.m_xt[i, :]).astype(float),
206                 mx=np.array(integral.m_mx[i, :]).astype(float),
207                 dx=np.array(integral.m_dx[i, :]).astype(float))
208             for i in range(integral.m_xt.shape[0])]
209
210     name = f"Linear"
211
212     lines.append(Line(f"Linear, y",
213                     np.array(integral.v_t).astype(float),
214                     np.array(integral.v_yt).astype(float),
215                     mx=np.array(integral.v_my).astype(float),
216                     dx=np.array(integral.v_dy).astype(float)))
217
218     return lines
219
220     def on_modeling_finish(self, result):
221         self.stop_progress.emit("The modeling has been completed!")
222         self.draw_chart.emit(result)
223
224     def on_modeling_corrupted(self, result):
225         self.logger.error(result[0])
226         self.logger.error(result[1])
227         self.logger.error(result[2])
228
229         self.stop_progress.emit("The modeling failed!")

```

Listing 35: Step 1 of data input for linear systems

```

1  from PyQt5.QtCore import pyqtSignal
2  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QGridLayout, QLineEdit, QLabel, \
3      QVBoxLayout, QSpacerItem, QSizePolicy, QPushButton, QApplication, QStyle
4
5  from ui.main.error import ErrorWidget
6  from ui.main.info import InfoIcon
7  from ui.main.svg import SVG
8
9
10 class Step1(QWidget):
11
12     n_valid = pyqtSignal(int)
13     m_valid = pyqtSignal(int)
14     k_valid = pyqtSignal(int)
15
16     def __init__(self, parent=None):
17         super(QWidget, self).__init__(parent)
18
19         self.n_is_valid = False

```

```

20     self.m_is_valid = False
21     self.k_is_valid = False
22
23     self.input_stack = self.parent()
24
25     info_n = InfoIcon("Dimension of linear system of Ito SDEs")
26     info_m = InfoIcon("Dimension of vector Wiener process")
27     info_k = InfoIcon("Dimension of vector function u(t)")
28
29     label_n = QLabel("n")
30     label_m = QLabel("m")
31     label_k = QLabel("k")
32
33     self.lineedit_n = QLineEdit()
34     self.lineedit_m = QLineEdit()
35     self.lineedit_k = QLineEdit()
36
37     self.msg_n = ErrorWidget("Wrong value!")
38     self.msg_n.hide()
39
40     self.msg_m = ErrorWidget("Wrong value!")
41     self.msg_m.hide()
42
43     self.msg_k = ErrorWidget("Wrong value!")
44     self.msg_k.hide()
45
46     grid_layout = QGridLayout()
47     grid_layout.addWidget(self.msg_n, 0, 2)
48     grid_layout.addWidget(self.msg_m, 2, 2)
49     grid_layout.addWidget(self.msg_k, 4, 2)
50     grid_layout.addWidget(info_n, 1, 0)
51     grid_layout.addWidget(info_m, 3, 0)
52     grid_layout.addWidget(info_k, 5, 0)
53     grid_layout.addWidget(label_n, 1, 1)
54     grid_layout.addWidget(label_m, 3, 1)
55     grid_layout.addWidget(label_k, 5, 1)
56     grid_layout.addWidget(self.lineedit_n, 1, 2)
57     grid_layout.addWidget(self.lineedit_m, 3, 2)
58     grid_layout.addWidget(self.lineedit_k, 5, 2)
59
60     header = QLabel("Dimensions settings", parent=self)
61     font = header.font()
62     font.setPointSize(15)
63     header.setFont(font)
64
65     self.next_btn = QPushButton("Next", self)
66     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
67     self.next_btn.setEnabled(False)
68
69     header_layout = QHBoxLayout()
70     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
71     header_layout.addWidget(header)
72     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
73
74     bottom_bar = QHBoxLayout()

```

```

75     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
76     bottom_bar.addWidget(self.next_btn)
77
78     eq1 = QHBoxLayout()
79     eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
80     eq1.addWidget(SVG("equation3.svg", scale_factor=1.))
81     eq1.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
82
83     eq2 = QHBoxLayout()
84     eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
85     eq2.addWidget(SVG("equation4.svg", scale_factor=1.))
86     eq2.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
87
88     equalities_layout = QVBoxLayout()
89     equalities_layout.addLayout(eq1)
90     equalities_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum
91 ))
92     equalities_layout.addLayout(eq2)
93
94     equalities_wrap = QHBoxLayout()
95     equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum
96 ))
97     equalities_wrap.addLayout(equalities_layout)
98     equalities_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum
99 ))
100    grid_wrap = QHBoxLayout()
101    grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
102    grid_wrap.addLayout(grid_layout)
103    grid_wrap.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
104
105    layout = QVBoxLayout()
106    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
107    layout.addLayout(equalities_wrap)
108    layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
109    layout.addLayout(header_layout)
110    layout.addItem(QSpacerItem(0, 5, QSizePolicy.Minimum, QSizePolicy.Minimum))
111    layout.addLayout(grid_wrap)
112    layout.addItem(QSpacerItem(0, 0, QSizePolicy.Minimum, QSizePolicy.Expanding))
113    layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
114    layout.addLayout(bottom_bar)
115
116    self.setLayout(layout)
117
118    self.lineedit_n.textChanged.connect(self.validate_n)
119    self.lineedit_m.textChanged.connect(self.validate_m)
120    self.lineedit_k.textChanged.connect(self.validate_k)
121
122    def validate_form(self):
123        if self.n_is_valid and self.m_is_valid and self.k_is_valid:
124            self.next_btn.setEnabled(True)
125        else:
126            self.next_btn.setEnabled(False)
127
128    def validate_n(self, value):

```

```

127     try:
128         typed_value = int(value)
129         if typed_value <= 0:
130             raise ValueError()
131
132         self.n_is_valid = True
133         self.n_valid.emit(typed_value)
134         self.msg_n.hide()
135
136     except ValueError:
137         self.n_is_valid = False
138         self.msg_n.show()
139
140     finally:
141         self.validate_form()
142
143 def validate_m(self, value):
144     try:
145         typed_value = int(value)
146         if typed_value <= 0:
147             raise ValueError()
148
149         self.m_is_valid = True
150         self.m_valid.emit(typed_value)
151         self.msg_m.hide()
152
153     except ValueError:
154         self.m_is_valid = False
155         self.msg_m.show()
156
157     finally:
158         self.validate_form()
159
160 def validate_k(self, value):
161     try:
162         typed_value = int(value)
163         if typed_value <= 0:
164             raise ValueError()
165
166         self.k_is_valid = True
167         self.k_valid.emit(typed_value)
168         self.msg_k.hide()
169
170     except ValueError:
171         self.k_is_valid = False
172         self.msg_k.show()
173
174     finally:
175         self.validate_form()

```

Listing 36: Step 2 of data input for linear systems

```

1 from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \

```



```

2   QVBoxLayout, QPushButton, QApplication, QStyle
3
4   from ui.main.error import ErrorWidget
5   from ui.main.info import InfoIcon
6   from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9   class Step2(QWidget):
10
11      def __init__(self, parent=None):
12          super(QWidget, self).__init__(parent)
13
14          self.errors = 0
15
16          header = QLabel("Setting of matrix A", parent=self)
17          font = header.font()
18          font.setPointSize(15)
19          header.setFont(font)
20
21          info = InfoIcon("Elements of matrix A are\n"
22                          "expected to be real values\n"
23                          "Size: n x n")
24
25          self.msg = ErrorWidget("Wrong values in matrix!")
26          self.msg.hide()
27
28          self.matrix = MatrixWidget(self)
29
30          self.next_btn = QPushButton("Next", self)
31          self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33          self.prev_btn = QPushButton("Back", self)
34          self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36          header_layout = QHBoxLayout()
37          header_layout.addWidget(info)
38          header_layout.addWidget(header)
39          header_layout.addWidget(self.msg)
40          header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42          bottom_bar = QHBoxLayout()
43          bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44          bottom_bar.addWidget(self.prev_btn)
45          bottom_bar.addWidget(self.next_btn)
46
47          layout = QVBoxLayout()
48          layout.addLayout(header_layout)
49          layout.addWidget(self.matrix)
50          layout.addLayout(bottom_bar)
51
52          self.setLayout(layout)
53
54          self.matrix.itemChanged.connect(self.validate_item)
55
56      def validate_item(self, item):

```

```

57
58     value = item.text()
59     try:
60         float(value)
61         if not item.valid:
62             item.valid = True
63             self.errors -= 1
64
65     except ValueError:
66         if item.valid:
67             item.valid = False
68             self.errors += 1
69
70     finally:
71         self.validate_form()
72
73     def validate_form(self):
74
75         if self.errors == 0:
76             self.msg.hide()
77             self.next_btn.setEnabled(True)
78         else:
79             self.msg.show()
80             self.next_btn.setEnabled(False)

```

Listing 37: Step 3 of data input for linear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QSpacerItem, QSizePolicy, \
2     QVBoxLayout, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step3(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of matrix B", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of matrix B are\n"
22             "expected to be real values\n"
23             "Size: n x k")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()

```

```
27
28     self.matrix = MatrixWidget(self)
29
30     self.next_btn = QPushButton("Next", self)
31     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33     self.prev_btn = QPushButton("Back", self)
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36     header_layout = QHBoxLayout()
37     header_layout.addWidget(info)
38     header_layout.addWidget(header)
39     header_layout.addWidget(self.msg)
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42     bottom_bar = QHBoxLayout()
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44     bottom_bar.addWidget(self.prev_btn)
45     bottom_bar.addWidget(self.next_btn)
46
47     layout = QVBoxLayout()
48     layout.addLayout(header_layout)
49     layout.addWidget(self.matrix)
50     layout.addLayout(bottom_bar)
51
52     self.setLayout(layout)
53
54     self.matrix.itemChanged.connect(self.validate_item)
55
56     def validate_item(self, item):
57
58         value = item.text()
59         try:
60             float(value)
61             if not item.valid:
62                 item.valid = True
63                 self.errors -= 1
64
65         except ValueError:
66             if item.valid:
67                 item.valid = False
68                 self.errors += 1
69
70         finally:
71             self.validate_form()
72
73     def validate_form(self):
74
75         if self.errors == 0:
76             self.msg.hide()
77             self.next_btn.setEnabled(True)
78         else:
79             self.msg.show()
80             self.next_btn.setEnabled(False)
```

Listing 38: Step 4 of data input for linear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
2      QSizePolicy, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step4(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of matrix F", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of matrix F are\n"
22             "expected to be real values\n"
23             "Size: n x m")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()
27
28         self.matrix = MatrixWidget(self)
29
30         self.next_btn = QPushButton("Next", self)
31         self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33         self.prev_btn = QPushButton("Back", self)
34         self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36         header_layout = QHBoxLayout()
37         header_layout.addWidget(info)
38         header_layout.addWidget(header)
39         header_layout.addWidget(self.msg)
40         header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42         bottom_bar = QHBoxLayout()
43         bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44         bottom_bar.addWidget(self.prev_btn)
45         bottom_bar.addWidget(self.next_btn)
46
47         layout = QVBoxLayout()
48         layout.addLayout(header_layout)
49         layout.addWidget(self.matrix)
50         layout.addLayout(bottom_bar)
51
52         self.setLayout(layout)
53

```

```

54     self.matrix.itemChanged.connect(self.validate_item)
55
56     def validate_item(self, item):
57
58         value = item.text()
59         try:
60             float(value)
61             if not item.valid:
62                 item.valid = True
63                 self.errors -= 1
64
65         except ValueError:
66             if item.valid:
67                 item.valid = False
68                 self.errors += 1
69
70         finally:
71             self.validate_form()
72
73     def validate_form(self):
74
75         if self.errors == 0:
76             self.msg.hide()
77             self.next_btn.setEnabled(True)
78         else:
79             self.msg.show()
80             self.next_btn.setEnabled(False)

```

Listing 39: Step 5 of data input for linear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
2     QSizePolicy, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step5(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of vector function u(t)", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of vector u(t) are expected to be functions\n"
22             "Size: k x 1\n"
23             "Functions must be set in python and SymPy notation")

```

```

24
25     self.msg = ErrorWidget("Wrong values in matrix!")
26     self.msg.hide()
27
28     self.matrix = MatrixWidget(self)
29
30     self.next_btn = QPushButton("Next", self)
31     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33     self.prev_btn = QPushButton("Back", self)
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36     header_layout = QHBoxLayout()
37     header_layout.addWidget(info)
38     header_layout.addWidget(header)
39     header_layout.addWidget(self.msg)
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42     bottom_bar = QHBoxLayout()
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44     bottom_bar.addWidget(self.prev_btn)
45     bottom_bar.addWidget(self.next_btn)
46
47     layout = QVBoxLayout()
48     layout.addLayout(header_layout)
49     layout.addWidget(self.matrix)
50     layout.addLayout(bottom_bar)
51
52     self.setLayout(layout)

```

Listing 40: Step 6 of data input for linear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
2     QSizePolicy, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step6(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of matrix H", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of matrix H are\n")

```

```
22         "expected to be int values\n"  
23         "Size: 1 x n")  
24  
25     self.msg = ErrorWidget("Wrong values in matrix!")  
26     self.msg.hide()  
27  
28     self.matrix = MatrixWidget(self)  
29  
30     self.next_btn = QPushButton("Next", self)  
31     self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))  
32  
33     self.prev_btn = QPushButton("Back", self)  
34     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))  
35  
36     header_layout = QHBoxLayout()  
37     header_layout.addWidget(info)  
38     header_layout.addWidget(header)  
39     header_layout.addWidget(self.msg)  
40     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))  
41  
42     bottom_bar = QHBoxLayout()  
43     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))  
44     bottom_bar.addWidget(self.prev_btn)  
45     bottom_bar.addWidget(self.next_btn)  
46  
47     layout = QVBoxLayout()  
48     layout.addLayout(header_layout)  
49     layout.addWidget(self.matrix)  
50     layout.addLayout(bottom_bar)  
51  
52     self.setLayout(layout)  
53  
54     self.matrix.itemChanged.connect(self.validate_item)  
55  
56     def validate_item(self, item):  
57  
58         value = item.text()  
59         try:  
60             float(value)  
61             if not item.valid:  
62                 item.valid = True  
63                 self.errors -= 1  
64  
65         except ValueError:  
66             if item.valid:  
67                 item.valid = False  
68                 self.errors += 1  
69  
70         finally:  
71             self.validate_form()  
72  
73     def validate_form(self):  
74  
75         if self.errors == 0:  
76             self.msg.hide()
```

```

77     self.next_btn.setEnabled(True)
78     else:
79         self.msg.show()
80         self.next_btn.setEnabled(False)

```

Listing 41: Step 7 of data input for linear systems

```

1  from PyQt5.QtWidgets import QWidget, QHBoxLayout, QVBoxLayout, QLabel, QSpacerItem, \
2     QSizePolicy, QPushButton, QApplication, QStyle
3
4  from ui.main.error import ErrorWidget
5  from ui.main.info import InfoIcon
6  from ui.main.modeling.matrix_widget import MatrixWidget
7
8
9  class Step7(QWidget):
10
11     def __init__(self, parent=None):
12         super(QWidget, self).__init__(parent)
13
14         self.errors = 0
15
16         header = QLabel("Setting of column x0", parent=self)
17         font = header.font()
18         font.setPointSize(15)
19         header.setFont(font)
20
21         info = InfoIcon("Elements of column x0 are\n"
22             "expected to be real values\n"
23             "Size: n x 1")
24
25         self.msg = ErrorWidget("Wrong values in matrix!")
26         self.msg.hide()
27
28         self.matrix = MatrixWidget(self)
29
30         self.next_btn = QPushButton("Next", self)
31         self.next_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
32
33         self.prev_btn = QPushButton("Back", self)
34         self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
35
36         header_layout = QHBoxLayout()
37         header_layout.addWidget(info)
38         header_layout.addWidget(header)
39         header_layout.addWidget(self.msg)
40         header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
41
42         bottom_bar = QHBoxLayout()
43         bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
44         bottom_bar.addWidget(self.prev_btn)
45         bottom_bar.addWidget(self.next_btn)
46

```



```

47 layout = QVBoxLayout()
48 layout.addLayout(header_layout)
49 layout.addWidget(self.matrix)
50 layout.addLayout(bottom_bar)
51
52 self.setLayout(layout)
53
54 self.matrix.itemChanged.connect(self.validate_item)
55
56 def validate_item(self, item):
57
58     value = item.text()
59     try:
60         float(value)
61         if not item.valid:
62             item.valid = True
63             self.errors -= 1
64
65     except ValueError:
66         if item.valid:
67             item.valid = False
68             self.errors += 1
69
70     finally:
71         self.validate_form()
72
73 def validate_form(self):
74
75     if self.errors == 0:
76         self.msg.hide()
77         self.next_btn.setEnabled(True)
78     else:
79         self.msg.show()
80         self.next_btn.setEnabled(False)

```

Listing 42: Step 8 of data input for linear systems

```

1 import sys
2
3 from PyQt5.QtWidgets import QWidget, QHBoxLayout, QLabel, QLineEdit, QGridLayout, \
4     QVBoxLayout, QSizePolicy, QSpacerItem, QPushButton, QApplication, QStyle
5
6 from ui.main.error import ErrorWidget
7 from ui.main.info import InfoIcon
8
9
10 class Step8(QWidget):
11
12     def __init__(self, parent=None):
13         super(QWidget, self).__init__(parent)
14
15         self.t0 = sys.float_info.min
16         self.dt = 0

```

```

17     self.t1 = sys.float_info.max
18     self.s = 0
19
20     self.t0_is_valid = False
21     self.dt_is_valid = False
22     self.t1_is_valid = False
23     self.s_is_valid = True
24
25     self.input_stack = self.parent()
26
27     info_t0 = InfoIcon("Start point of integration interval\n"
28                       "Must be in [0, t1) range")
29     info_dt = InfoIcon("Integration step\n"
30                       "Must be set in (0, 1) interval")
31     info_t1 = InfoIcon("Final point of integration interval\n"
32                       "Must be more then t0")
33     info_s = InfoIcon("This is random generator seed\n"
34                     "If You do not want to use specific\n"
35                     "seed just leave this field empty")
36
37     label_t0 = QLabel("t0")
38     label_dt = QLabel("dt")
39     label_t1 = QLabel("t1")
40     label_s = QLabel("seed")
41
42     self.lineedit_t0 = QLineEdit()
43     self.lineedit_dt = QLineEdit()
44     self.lineedit_t1 = QLineEdit()
45     self.lineedit_s = QLineEdit()
46
47     self.msg_t0 = ErrorWidget("Wrong value!")
48     self.msg_t0.hide()
49
50     self.msg_dt = ErrorWidget("Wrong value!")
51     self.msg_dt.hide()
52
53     self.msg_t1 = ErrorWidget("Wrong value!")
54     self.msg_t1.hide()
55
56     self.msg_s = ErrorWidget("Wrong value!")
57     self.msg_s.hide()
58
59     header = QLabel("Accuracy settings", parent=self)
60     font = header.font()
61     font.setPointSize(15)
62     header.setFont(font)
63
64     self.prev_btn = QPushButton("Back", self)
65     self.prev_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowBack))
66
67     self.run_btn = QPushButton("Perform modeling", self)
68     self.run_btn.setIcon(QApplication.style().standardIcon(QStyle.SP_ArrowForward))
69     self.run_btn.setEnabled(False)
70
71     header_layout = QHBoxLayout()

```

```
72     header_layout.addWidget(header)
73     header_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
74
75     bottom_bar = QHBoxLayout()
76     bottom_bar.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Minimum))
77     bottom_bar.addWidget(self.prev_btn)
78     bottom_bar.addWidget(self.run_btn)
79
80     grid_layout = QGridLayout()
81
82     grid_layout.addWidget(self.msg_t0, 0, 2)
83
84     grid_layout.addWidget(info_t0, 1, 0)
85     grid_layout.addWidget(label_t0, 1, 1)
86     grid_layout.addWidget(self.lineedit_t0, 1, 2)
87
88     grid_layout.addWidget(self.msg_dt, 0, 5)
89
90     grid_layout.addWidget(info_dt, 1, 3)
91     grid_layout.addWidget(label_dt, 1, 4)
92     grid_layout.addWidget(self.lineedit_dt, 1, 5)
93
94     grid_layout.addWidget(self.msg_t1, 0, 8)
95
96     grid_layout.addWidget(info_t1, 1, 6)
97     grid_layout.addWidget(label_t1, 1, 7)
98     grid_layout.addWidget(self.lineedit_t1, 1, 8)
99
100    grid_layout.addWidget(self.msg_s, 2, 2)
101
102    grid_layout.addWidget(info_s, 3, 0)
103    grid_layout.addWidget(label_s, 3, 1)
104    grid_layout.addWidget(self.lineedit_s, 3, 2)
105
106    column_layout = QVBoxLayout()
107    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
108    )
109    column_layout.addLayout(header_layout)
110    column_layout.addItem(QSpacerItem(0, 25, QSizePolicy.Minimum, QSizePolicy.Minimum))
111    column_layout.addLayout(grid_layout)
112    column_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
113    )
114
115    control_layout = QHBoxLayout()
116    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
117    )
118    control_layout.addLayout(column_layout)
119    control_layout.addItem(QSpacerItem(0, 0, QSizePolicy.Expanding, QSizePolicy.Expanding)
120    )
121
122    layout = QVBoxLayout()
123    layout.addLayout(control_layout)
124    layout.addLayout(bottom_bar)
125
126    self.setLayout(layout)
```

```
123
124     self.lineedit_t0.textChanged.connect(self.validate_t0)
125     self.lineedit_t0.textChanged.connect(self.validate_t1)
126     self.lineedit_t0.textChanged.connect(self.validate_dt)
127     self.lineedit_dt.textChanged.connect(self.validate_dt)
128     self.lineedit_t1.textChanged.connect(self.validate_t0)
129     self.lineedit_t1.textChanged.connect(self.validate_t1)
130     self.lineedit_t1.textChanged.connect(self.validate_dt)
131     self.lineedit_s.textChanged.connect(self.validate_s)
132
133     def count_c(self, flag: bool):
134         if flag:
135             self.info_c.show()
136             self.label_c.show()
137         else:
138             self.info_c.hide()
139             self.label_c.hide()
140
141         self.msg_c.hide()
142
143     def validate_form(self):
144         if self.t0_is_valid \
145             and self.dt_is_valid \
146             and self.t1_is_valid \
147             and self.s_is_valid:
148             self.run_btn.setEnabled(True)
149         else:
150             self.run_btn.setEnabled(False)
151
152     def validate_t0(self):
153         try:
154             typed_value = float(self.lineedit_t0.text())
155             if typed_value >= self.t1:
156                 raise ValueError()
157
158             self.t0_is_valid = True
159             self.t0 = typed_value
160             self.msg_t0.hide()
161
162         except ValueError:
163             self.t0_is_valid = False
164             self.msg_t0.show()
165
166     def validate_dt(self):
167         try:
168             typed_value = float(self.lineedit_dt.text())
169             if typed_value <= 0 \
170                 or (self.t1 - self.t0) / typed_value < 1 \
171                 or typed_value >= 1:
172                 raise ValueError()
173
174             self.dt_is_valid = True
175             self.dt = typed_value
176             self.msg_dt.hide()
177             self.validate_form()
```

```

178
179     except ValueError:
180         self.dt_is_valid = False
181         self.msg_dt.show()
182
183     def validate_t1(self):
184         try:
185             typed_value = float(self.lineedit_t1.text())
186             if typed_value <= self.t0:
187                 raise ValueError()
188
189             self.t1_is_valid = True
190             self.t1 = typed_value
191             self.msg_t1.hide()
192
193         except ValueError:
194             self.t1_is_valid = False
195             self.msg_t1.show()
196
197     def validate_s(self):
198         try:
199             if self.lineedit_s.text() == "":
200                 self.s_is_valid = True
201                 self.s = 0
202             else:
203                 typed_value = int(self.lineedit_s.text())
204                 if typed_value <= 0:
205                     raise ValueError()
206                 self.s = typed_value
207
208             self.s_is_valid = True
209             self.msg_s.hide()
210
211         except ValueError:
212             self.s_is_valid = False
213             self.msg_s.show()
214
215     finally:
216         self.validate_form()

```

6.2 Source Codes for Nonlinear Systems of Itô SDEs

6.2.1 Source Codes for Calculation of the Fourier–Legendre Coefficients

Listing 43: Symbolic function of the Legendre polinomial

```

1 from sympy import Rational, factorial, diff
2

```

```

3
4 def polynomial(n: int):
5     """
6     Returns the Legendre polynomial in symbolic format
7     Parameters
8     =====
9     n : int
10     degree of the Legendre polynomial
11     Returns
12     =====
13     sympy.Expr
14     """
15     from sympy.abc import x
16     return Rational(1, 2) ** n / factorial(n) * diff((x ** 2 - 1) ** n, x, n)

```

Listing 44: Symbolic function of the Fourier–Legendre coefficient calculation

```

1 from sympy import S, integrate
2
3 from mathematics.sde.nonlinear.legendre_polynomial import polynomial
4
5
6 def get_c(indices: tuple, weights: tuple):
7     """
8     Calculates the Fourier–Legendre coefficient depending on indices and weights
9     Parameters
10    =====
11    indices : tuple
12    indices of the Fourier–Legendre coefficient
13    weights : tuple
14    weights of the Fourier–Legendre coefficient
15    Returns
16    =====
17    sympy.Rational
18    """
19    from sympy.abc import x, y
20    # multiplicity of iterated integral which is the Fourier–Legendre coefficient
21    n = len(indices)
22    w = list(reversed(weights))
23    c = S.One
24
25    for i in reversed(range(1, n)):
26        c = integrate(polynomial(indices[i]) * (x + 1) ** w[i] * c, (x, -1, y)).subs(y, x)
27    c = integrate(polynomial(indices[0]) * (x + 1) ** w[0] * c, (x, -1, 1))
28
29    if sum(w) % 2 == 0:
30        return c
31    else:
32        return -c

```

Listing 45: Symbolic function of the Fourier–Legendre coefficient

```

1 import logging

```

```

2
3 from sympy import sympify, Function
4
5 import tools.database as db
6 from mathematics.sde.nonlinear.c import get_c
7
8
9 class C(Function):
10     """
11     Gives the Fourier–Legendre coefficient with requested indices and weights
12     """
13     _preloaded = dict()
14
15     def __new__(cls, indices: tuple, weights: tuple, to_float=True, **kwargs):
16         """
17         Creates the Fourier–Legendre coefficient object with needed indices and weights
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple
23             requested weights
24         Returns
25         =====
26         symbolic.Rational or C
27             calculated value or symbolic expression
28         """
29         if not len(indices) == len(weights):
30             return super(C, cls).__new__(cls, indices, weights, **kwargs)
31
32         index = f"{' ':'.join([str(i) for i in indices])}_{' ':'.join([str(i) for i in weights])}"
33
34         try:
35             return cls._value(index, to_float)
36
37         except KeyError:
38             respond = cls._download_one(index)
39             if len(respond) != 0:
40                 cls._preloaded[respond[0][0]] = respond[0][1]
41                 return cls._value(index, to_float)
42             else:
43                 new_c = cls._calculate(index, indices, weights)
44                 cls._upload_one(new_c)
45                 cls._preloaded[new_c[0]] = new_c[1]
46                 return cls._value(index, to_float)
47
48     @classmethod
49     def _calculate(cls, index, indices, weights):
50         new_c = get_c(indices, weights)
51         return index, (new_c, sympify(new_c).evalf())
52
53     @classmethod
54     def _upload_one(cls, c):
55         logging.info(f"C: ADDING NEW C_{c[0]} = {c[1][0]}")
56         db.execute(f"INSERT INTO 'C' ('index', 'value', 'value_f') VALUES ('{c[0]}', '{c[1][0]}')")

```

```

    [1][0]}', {c[1][1]})")
56
57 @classmethod
58 def _unpack(cls, rows):
59     return [(rows[i][0], (rows[i][1], rows[i][2])) for i in range(len(rows))]
60
61 @classmethod
62 def _value(cls, index, to_float):
63     c = cls._preloaded[index]
64     if to_float:
65         return c[1]
66     else:
67         return sympify(c[0])
68
69 @classmethod
70 def _download_one(cls, index):
71     logging.info(f"C: MISSING PRELOADED VERSION OF C-{index}")
72     respond = db.execute(
73         f"SELECT 'index', 'value', 'value_f' FROM 'C'"
74         f"WHERE REGEXP('index', '^{'index}$')")
75     )
76     return cls._unpack(respond)
77
78 @classmethod
79 def preload(cls, *args):
80     """
81     Updates dictionary of the preloaded Fourier–Legendre coefficients
82     Note: weights are not accepted, such coefficients are loaded
83     with all available weights
84     Parameters
85     =====
86     args
87     Indices for the Fourier–Legendre coefficients
88     to download them from database
89     """
90     logging.info(f"C: PRELOADING COEFFICIENTS {args}")
91
92     query = []
93     for q in range(len(args)):
94         numbers = [int(char) for char in str(args[q] + 1)]
95         pattern = []
96
97         for i in range(1, len(numbers)):
98             pattern.append("[0-9]" * i)
99
100        for i in range(len(numbers)):
101            p = []
102            for j in range(len(numbers)):
103                if j < i:
104                    p.append(str(numbers[j]))
105                elif i == j:
106                    p.append(f"[0-{{numbers[j] - 1}}]")
107                elif j > i:
108                    p.append("[0-9]")
109            pattern.append("".join(p))

```



```

110
111     regex = f" ^{':'.join(['|'.join(pattern) for _ in range(q + 2)])}-.*$"
112     query.append(
113         f"SELECT 'index', 'value', 'value_f' FROM 'C'"
114         f"WHERE REGEXP('index', '{regex}')"
115     )
116
117     result = db.execute("\nUNION\n".join(query))
118     cls._preloaded.update(cls._unpack(result))
119
120     def doit(self, **hints):
121         """
122         Tries to expand or calculate function
123         Returns
124         =====
125         C
126         """
127     return C(*self.args, **hints)

```

Listing 46: Calculation of the Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{000}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C000(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 4
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple
23             requested weights
24         Returns
25         =====
26         symbolic.Rational or C000
27             calculated value or symbolic expression
28         """
29         j3, j2, j1, dt = sympify(args)
30
31         if not (isinstance(j1, Number) and
32               isinstance(j2, Number) and

```

```

33     isinstance(j3, Number) and
34     isinstance(dt, Number)):
35     return super(C000, cls).__new__(cls, *args, **kwargs)
36
37     return sqrt(
38         (j1 * 2 + 1) *
39         (j2 * 2 + 1) *
40         (j3 * 2 + 1)) * \
41         dt ** 1.5 * \
42         C((j3, j2, j1), (0, 0, 0)) / 8
43
44     def doit(self, **hints):
45         """
46         Tries to expand or calculate function
47         Returns
48         =====
49         C000
50         """
51     return C000(*self.args, **hints)

```

Listing 47: Calculation of the Fourier–Legendre coefficients $C_{j_2 j_1}^{10}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C10(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 3
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights
24         Returns
25         =====
26         symbolic.Rational or C10
27         calculated value or symbolic expression
28         """
29         j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and

```

```

32     isinstance(j2, Number) and
33     isinstance(dt, Number)):
34     return super(C10, cls).__new__(cls, *args, **kwargs)
35
36     return sqrt(
37         (j1 * 2 + 1) *
38         (j2 * 2 + 1)) * \
39         dt ** 2 * \
40         C((j2, j1), (1, 0)) / 8
41
42 def doit(self, **hints):
43     """
44     Tries to expand or calculate function
45     Returns
46     =====
47     C10
48     """
49     return C10(*self.args, **hints)

```

Listing 48: Calculation of the Fourier–Legendre coefficients $C_{j_2 j_1}^{01}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C01(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 3
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights
24         Returns
25         =====
26         symbolic.Rational or C01
27         calculated value or symbolic expression
28         """
29         j2, j1, dt = sympify(args)
30
31         if not (isinstance(j1, Number) and
32                isinstance(j2, Number) and

```

```

33     isinstance(dt, Number)):
34     return super(C01, cls).__new__(cls, *args, **kwargs)
35
36     return sqrt(
37         (j1 * 2 + 1) *
38         (j2 * 2 + 1)) * \
39         dt ** 2 * \
40         C((j2, j1), (0, 1)) / 8
41
42 def doit(self, **hints):
43     """
44     Tries to expand or calculate function
45     Returns
46     =====
47     C01
48     """
49     return C01(*self.args, **hints)

```

Listing 49: Calculation of the Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{0000}$

```

1 from math import sqrt
2
3 from sympy import sympify, Number, Function
4
5 from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8 class C0000(Function):
9     """
10    Gives the Fourier–Legendre coefficient with requested indices and weights
11    """
12    nargs = 5
13
14    def __new__(cls, *args, **kwargs):
15        """
16        Creates the Fourier–Legendre coefficient object with needed
17        indices and weights and calculates it
18        Parameters
19        =====
20        indices: tuple
21        requested indices
22        weights: tuple
23        requested weights
24        Returns
25        =====
26        symbolic.Rational or C0000
27        calculated value or symbolic expression
28        """
29        j4, j3, j2, j1, dt = sympify(args)
30
31        if not (isinstance(j1, Number) and
32                isinstance(j2, Number) and
33                isinstance(j3, Number) and

```

```

34     isinstance(j4, Number) and
35     isinstance(dt, Number)):
36     return super(C0000, cls).__new__(cls, *args, **kwargs)
37
38     return sqrt(
39         (j1 * 2 + 1) *
40         (j2 * 2 + 1) *
41         (j3 * 2 + 1) *
42         (j4 * 2 + 1)) * \
43         dt ** 2 * \
44         C((j4, j3, j2, j1), (0, 0, 0, 0)) / 16
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         C0000
52         """
53     return C0000(*self.args, **hints)

```

Listing 50: Calculation of the Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{100}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C100(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 4
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple
23             requested weights
24         Returns
25         =====
26         symbolic.Rational or C100
27         calculated value or symbolic expression
28         """
29         j3, j2, j1, dt = sympify(args)
30

```

```

31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(dt, Number)):
35         return super(C100, cls).__new__(cls, *args, **kwargs)
36
37     return sqrt(
38         (j1 * 2 + 1) *
39         (j2 * 2 + 1) *
40         (j3 * 2 + 1)) * \
41         dt ** 2.5 * \
42         C((j3, j2, j1), (1, 0, 0)) / 16
43
44     def doit(self, **hints):
45         """
46         Tries to expand or calculate function
47         Returns
48         =====
49         C100
50         """
51         return C100(*self.args, **hints)

```

Listing 51: Calculation of the Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{010}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C010(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 4
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple
23             requested weights
24         Returns
25         =====
26         symbolic.Rational or C010
27         calculated value or symbolic expression
28         """
29         j3, j2, j1, dt = sympify(args)

```

```

30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(dt, Number)):
35         return super(C010, cls).__new__(cls, *args, **kwargs)
36
37     return sqrt(
38         (j1 * 2 + 1) *
39         (j2 * 2 + 1) *
40         (j3 * 2 + 1)) * \
41         dt ** 2.5 * \
42         C((j3, j2, j1), (0, 1, 0)) / 16
43
44     def doit(self, **hints):
45         """
46         Tries to expand or calculate function
47         Returns
48         =====
49         C010
50         """
51     return C010(*self.args, **hints)

```

Listing 52: Calculation of the Fourier–Legendre coefficients $C_{j_3 j_2 j_1}^{001}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C001(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 4
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights
24         Returns
25         =====
26         symbolic.Rational or C001
27         calculated value or symbolic expression
28         """

```

```

29     j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(dt, Number)):
35         return super(C001, cls).__new__(cls, *args, **kwargs)
36
37     return sqrt(
38         (j1 * 2 + 1) *
39         (j2 * 2 + 1) *
40         (j3 * 2 + 1)) * \
41         dt ** 2.5 * \
42         C((j3, j2, j1), (0, 0, 1)) / 16
43
44     def doit(self, **hints):
45         """
46         Tries to expand or calculate function
47         Returns
48         =====
49         C001
50         """
51         return C001(*self.args, **hints)

```

Listing 53: Calculation of the Fourier–Legendre coefficients $C_{j_5 j_4 j_3 j_2 j_1}^{00000}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C00000(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 6
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple
23             requested weights
24         Returns
25         =====
26         symbolic.Rational or C00000
27         calculated value or symbolic expression

```



```

28     """
29     j5, j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(j4, Number) and
35             isinstance(j5, Number) and
36             isinstance(dt, Number)):
37         return super(C00000, cls).__new__(cls, *args, **kwargs)
38
39     return sqrt(
40         (j1 * 2 + 1) *
41         (j2 * 2 + 1) *
42         (j3 * 2 + 1) *
43         (j4 * 2 + 1) *
44         (j5 * 2 + 1)) * \
45         dt ** 2.5 * \
46         C((j5, j4, j3, j2, j1), (0, 0, 0, 0, 0)) / 32
47
48     def doit(self, **hints):
49         """
50         Tries to expand or calculate function
51         Returns
52         =====
53         C00000
54         """
55         return C00000(*self.args, **hints)

```

Listing 54: Calculation of the Fourier–Legendre coefficients $C_{j_2 j_1}^{20}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C20(Function):
9     """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 3
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21             requested indices
22         weights: tuple

```

```

23     requested weights
24     Returns
25     =====
26     symbolic.Rational or C20
27     calculated value or symbolic expression
28     """
29     j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(dt, Number)):
34         return super(C20, cls).__new__(cls, *args, **kwargs)
35
36     return sqrt(
37         (j1 * 2 + 1) *
38         (j2 * 2 + 1)) * \
39         dt ** 3 * \
40         C((j2, j1), (2, 0)) / 16
41
42     def doit(self, **hints):
43         """
44         Tries to expand or calculate function
45         Returns
46         =====
47         C20
48         """
49         return C20(*self.args, **hints)

```

Listing 55: Calculation of the Fourier–Legendre coefficients $C_{j_2 j_1}^{11}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C11(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 3
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights

```

```

24     Returns
25     =====
26     symbolic.Rational or C11
27     calculated value or symbolic expression
28     """
29     j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(dt, Number)):
34         return super(C11, cls).__new__(cls, *args, **kwargs)
35
36     return sqrt(
37         (j1 * 2 + 1) *
38         (j2 * 2 + 1)) * \
39         dt ** 3 * \
40         C((j2, j1), (1, 1)) / 16
41
42     def doit(self, **hints):
43         """
44         Tries to expand or calculate function
45         Returns
46         =====
47         C11
48         """
49         return C11(*self.args, **hints)

```

Listing 56: Calculation of the Fourier–Legendre coefficients $C_{j_2 j_1}^{02}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C02(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 3
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights
24         Returns

```

```

25  =====
26  symbolic.Rational or C02
27  calculated value or symbolic expression
28  """
29  j2, j1, dt = sympify(args)
30
31  if not (isinstance(j1, Number) and
32         isinstance(j2, Number) and
33         isinstance(dt, Number)):
34      return super(C02, cls).__new__(cls, *args, **kwargs)
35
36  return sqrt(
37      (j1 * 2 + 1) *
38      (j2 * 2 + 1)) * \
39      dt ** 3 * \
40      C((j2, j1), (0, 2)) / 16
41
42  def doit(self, **hints):
43      """
44      Tries to expand or calculate function
45      Returns
46      =====
47      C02
48      """
49  return C02(*self.args, **hints)

```

Listing 57: Calculation of the Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{1000}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C1000(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 5
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple
23         requested weights
24         Returns
25         =====

```

```

26     symbolic.Rational or C1000
27     calculated value or symbolic expression
28     """
29     j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(j4, Number) and
35             isinstance(dt, Number)):
36         return super(C1000, cls).__new__(cls, *args, **kwargs)
37
38     return sqrt(
39         (j1 * 2 + 1) *
40         (j2 * 2 + 1) *
41         (j3 * 2 + 1) *
42         (j4 * 2 + 1)) * \
43         dt ** 3 * \
44         C((j4, j3, j2, j1), (1, 0, 0, 0)) / 32
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         C1000
52         """
53         return C1000(*self.args, **hints)

```

Listing 58: Calculation of the Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{0100}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C0100(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 5
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====
20         indices: tuple
21         requested indices
22         weights: tuple

```

```

23     requested weights
24     Returns
25     =====
26     symbolic.Rational or C0100
27     calculated value or symbolic expression
28     """
29     j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(j4, Number) and
35             isinstance(dt, Number)):
36         return super(C0100, cls).__new__(cls, *args, **kwargs)
37
38     return sqrt(
39         (j1 * 2 + 1) *
40         (j2 * 2 + 1) *
41         (j3 * 2 + 1) *
42         (j4 * 2 + 1)) * \
43         dt ** 3 * \
44         C((j4, j3, j2, j1), (0, 1, 0, 0)) / 32
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         C0100
52         """
53         return C0100(*self.args, **hints)

```

Listing 59: Calculation of the Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{0010}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C0010(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 5
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed
17         indices and weights and calculates it
18         Parameters
19         =====

```

```

20     indices: tuple
21     requested indices
22     weights: tuple
23     requested weights
24     Returns
25     =====
26     symbolic.Rational or C0010
27     calculated value or symbolic expression
28     """
29     j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(j4, Number) and
35             isinstance(dt, Number)):
36         return super(C0010, cls).__new__(cls, *args, **kwargs)
37
38     return sqrt(
39         (j1 * 2 + 1) *
40         (j2 * 2 + 1) *
41         (j3 * 2 + 1) *
42         (j4 * 2 + 1)) * \
43         dt ** 3 * \
44         C((j4, j3, j2, j1), (0, 0, 1, 0)) / 32
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         C0010
52         """
53         return C0010(*self.args, **hints)

```

Listing 60: Calculation of the Fourier–Legendre coefficients $C_{j_4 j_3 j_2 j_1}^{0001}$

```

1  from math import sqrt
2
3  from sympy import sympify, Function, Number
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C0001(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 5
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates the Fourier–Legendre coefficient object with needed

```

```

17     indices and weights and calculates it
18     Parameters
19     =====
20     indices: tuple
21     requested indices
22     weights: tuple
23     requested weights
24     Returns
25     =====
26     symbolic.Rational or C0001
27     calculated value or symbolic expression
28     """
29     j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32             isinstance(j2, Number) and
33             isinstance(j3, Number) and
34             isinstance(j4, Number) and
35             isinstance(dt, Number)):
36         return super(C0001, cls).__new__(cls, *args, **kwargs)
37
38     return sqrt(
39         (j1 * 2 + 1) *
40         (j2 * 2 + 1) *
41         (j3 * 2 + 1) *
42         (j4 * 2 + 1)) * \
43         dt ** 3 * \
44         C((j4, j3, j2, j1), (0, 0, 0, 1)) / 32
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         C0001
52         """
53         return C0001(*self.args, **hints)

```

Listing 61: Calculation of the Fourier–Legendre coefficients $C_{j_6 j_5 j_4 j_3 j_2 j_1}^{000000}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
6
7
8  class C000000(Function):
9      """
10     Gives the Fourier–Legendre coefficient with requested indices and weights
11     """
12     nargs = 7
13

```



```

14 def __new__(cls, *args, **kwargs):
15     """
16     Creates the Fourier–Legendre coefficient object with needed
17     indices and weights and calculates it
18     Parameters
19     =====
20     indices: tuple
21         requested indices
22     weights: tuple
23         requested weights
24     Returns
25     =====
26     symbolic.Rational or C000000
27         calculated value or symbolic expression
28     """
29     j6, j5, j4, j3, j2, j1, dt = sympify(args)
30
31     if not (isinstance(j1, Number) and
32            isinstance(j2, Number) and
33            isinstance(j3, Number) and
34            isinstance(j4, Number) and
35            isinstance(j5, Number) and
36            isinstance(dt, Number)):
37         return super(C000000, cls).__new__(cls, *args, **kwargs)
38
39     return sqrt(
40         (j1 * 2 + 1) *
41         (j2 * 2 + 1) *
42         (j3 * 2 + 1) *
43         (j4 * 2 + 1) *
44         (j5 * 2 + 1) *
45         (j6 * 2 + 1)) * \
46         dt ** 3 * \
47         C((j6, j5, j4, j3, j2, j1), (0, 0, 0, 0, 0, 0)) / 64
48
49     def doit(self, **hints):
50         """
51         Tries to expand or calculate function
52         Returns
53         =====
54         C000000
55         """
56     return C000000(*self.args, **hints)

```

Listing 62: Program entry for generation of new Fourier–Legendre coefficients

```

1  #!/usr/bin/env python
2  import logging
3  import os
4  from datetime import datetime
5  from multiprocessing import cpu_count, Pool
6  from pprint import pprint
7

```

```

8 from config import csv, new_c_portion_size
9 from mathematics.sde.nonlinear.new_c import thread_c, split_task
10
11
12 def main():
13     logging.basicConfig(
14         level=logging.INFO,
15         format="%(asctime)s - %(levelname)s - %(message)s",
16         datefmt="%H:%M:%S"
17     )
18     logger = logging.getLogger(__name__)
19
20     filename = os.path.join(csv, f"c_{datetime.now().strftime('%d-%m-%Y_%H-%M-%S')}.csv")
21     logging.info(f"Writing to file {filename}")
22
23     tasks = [
24         ((57, 58), (57, 58), (57, 58)), (0, 0, 0),
25         # ((0, 56), (0, 56), (0, 56)), (0, 0, 0),
26         #
27         # ((0, 15), (0, 15)), (0, 1)),
28         # ((0, 15), (0, 15)), (1, 0)),
29         # ((0, 15), (0, 15), (0, 15)), (0, 0, 0, 0)),
30         #
31         # ((0, 6), (0, 6), (0, 6)), (0, 0, 1)),
32         # ((0, 6), (0, 6), (0, 6)), (0, 1, 0)),
33         # ((0, 6), (0, 6), (0, 6)), (1, 0, 0)),
34         # ((0, 6), (0, 6), (0, 6), (0, 6), (0, 6)), (0, 0, 0, 0, 0)),
35         #
36         # ((0, 2), (0, 2)), (0, 2)),
37         # ((0, 2), (0, 2)), (1, 1)),
38         # ((0, 2), (0, 2)), (2, 0)),
39         # ((0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 1)),
40         # ((0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 1, 0)),
41         # ((0, 2), (0, 2), (0, 2), (0, 2)), (0, 1, 0, 0)),
42         # ((0, 2), (0, 2), (0, 2), (0, 2)), (1, 0, 0, 0)),
43         # ((0, 2), (0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 0, 0)),
44         # ((0, 2), (0, 2), (0, 2), (0, 2), (0, 2), (0, 2)), (0, 0, 0, 0, 0, 0)),
45     ]
46
47     with Pool(cpu_count()) as p:
48
49         for t in tasks:
50
51             logger.info(f"Running task {t}")
52
53             for chunk in chunks(split_task(t), new_c_portion_size):
54                 c = p.map(thread_c, chunk)
55                 c.append("")
56
57                 with open(filename, "a") as f:
58                     f.write("\n".join(c))
59                     f.close()
60
61             logger.info(f"The portion of C has been written {chunk[0]}–{chunk[-1]}")
62

```

```

63     logger.info("Generation has been done")
64
65
66 def chunks(lst, n):
67     for i in range(0, len(lst), n):
68         yield lst[i:i + n]
69
70
71 if __name__ == "__main__":
72     main()

```

Listing 63: Module for the Fourier–Legendre coefficients generation

```

1  from mathematics.sde.nonlinear.c import get_c
2
3
4  def split_task(tasks):
5      c = len(tasks[1])
6
7      if c == 2:
8          return split_2(tasks)
9
10     if c == 3:
11         return split_3(tasks)
12
13     if c == 4:
14         return split_4(tasks)
15
16     if c == 5:
17         return split_5(tasks)
18
19     if c == 6:
20         return split_6(tasks)
21
22
23 def split_2(ranges):
24     return [(i, j), ranges[1]]
25         for j in range(ranges[0][1][1])
26         for i in range(ranges[0][0][1])
27         if i >= ranges[0][1][0]
28         or j >= ranges[0][0][1]
29
30
31 def split_3(ranges):
32     return [(i, j, k), ranges[1]]
33         for k in range(ranges[0][2][1])
34         for j in range(ranges[0][1][1])
35         for i in range(ranges[0][0][1])
36         if k >= ranges[0][2][0]
37         or j >= ranges[0][1][0]
38         or i >= ranges[0][0][0]
39
40

```

```
41 def split_4(ranges):
42     return [(i, j, k, l), ranges[1]]
43         for j in range(*ranges[0][3])
44         for i in range(*ranges[0][2])
45         for k in range(*ranges[0][1])
46         for l in range(*ranges[0][0])
47         if l >= ranges[0][3][0]
48         or k >= ranges[0][2][0]
49         or j >= ranges[0][1][0]
50         or i >= ranges[0][0][0]
51
52
53 def split_5(ranges):
54     return [(i, j, k, l, m), ranges[1]]
55         for j in range(*ranges[0][4])
56         for i in range(*ranges[0][3])
57         for k in range(*ranges[0][2])
58         for l in range(*ranges[0][1])
59         for m in range(*ranges[0][0])
60         if m >= ranges[0][4][0]
61         if l >= ranges[0][3][0]
62         or k >= ranges[0][2][0]
63         or j >= ranges[0][1][0]
64         or i >= ranges[0][0][0]
65
66
67 def split_6(ranges):
68     return [(i, j, k, l, m, n), ranges[1]]
69         for j in range(*ranges[0][5])
70         for i in range(*ranges[0][4])
71         for k in range(*ranges[0][3])
72         for l in range(*ranges[0][2])
73         for m in range(*ranges[0][1])
74         for n in range(*ranges[0][0])
75         if n >= ranges[0][5][0]
76         if m >= ranges[0][4][0]
77         if l >= ranges[0][3][0]
78         or k >= ranges[0][2][0]
79         or j >= ranges[0][1][0]
80         or i >= ranges[0][0][0]
81
82
83 def thread_c(ranges):
84     c = len(ranges[1])
85
86     if c == 2:
87         return gen_2(*ranges[0], ranges[1])
88
89     if c == 3:
90         return gen_3(*ranges[0], ranges[1])
91
92     if c == 4:
93         return gen_4(*ranges[0], ranges[1])
94
95     if c == 5:
```

```

96     return gen_5(*ranges[0], ranges[1])
97
98     if c == 6:
99         return gen_6(*ranges[0], ranges[1])
100
101
102 def gen_2(i, j, w):
103     return f"\{i}:{j}-{w[0]}:{w[1]}\";\{get_c((i, j), w)}\"
104
105
106 def gen_3(i, j, k, w):
107     return f"\{i}:{j}:{k}-{w[0]}:{w[1]}:{w[2]}\";\{get_c((i, j, k), w)}\"
108
109
110 def gen_4(i, j, k, l, w):
111     return f"\{i}:{j}:{k}:{l}-{w[0]}:{w[1]}:{w[2]}:{w[3]}\";\{get_c((i, j, k, l), w)}\"
112
113
114 def gen_5(i, j, k, l, m, w):
115     return f"\{i}:{j}:{k}:{l}:{m}-{w[0]}:{w[1]}:{w[2]}:{w[3]}:{w[4]}\";\{get_c((i, j, k,
116         l, m), w)}\"
117
118 def gen_6(i, j, k, l, m, n, w):
119     return f"\{i}:{j}:{k}:{l}:{m}-{w[0]}:{w[1]}:{w[2]}:{w[3]}:{w[4]}:{w[5]}\";\{get_c((i,
120         j, k, l, m, n), w)}\"

```

6.2.2 Source Codes for Supplementary Differential Operators and Functions

Listing 64: Implementation of the differential operator L

```

1 from sympy import Add, sympify, Number, diff
2
3 from mathematics.sde.nonlinear.symbolic.operator import Operator
4
5
6 class L(Operator):
7     nargs = 4
8
9     def __new__(cls, *args, **kwargs):
10         """
11         Creates new L object with given args
12         Parameters
13         =====
14         args
15         bunch of necessary arguments
16         Returns
17         =====
18         sympy.Expr

```

```

19     formula to simplify and substitute
20     """
21     a, b, f, dxs = sympify(args)
22
23     if not ((isinstance(f, Number) or f.has(*dxs)) and
24             not f.has(Operator) and a.is_Matrix):
25         return super(L, cls).__new__(cls, *args, **kwargs)
26
27     n = b.shape[0]
28     m = b.shape[1]
29     from sympy.abc import t
30
31     return Add(
32         diff(f, t),
33         *[a[i, 0] * diff(f, dxs[i]) for i in range(n)],
34         *[0.5 * b[i, j] * b[k, j] * diff(f, dxs[i], dxs[k])
35           for j in range(m)
36           for i in range(n)
37           for k in range(n)]
38     )
39
40     def doit(self, **hints):
41         """
42         Tries to expand or calculate function
43         Returns
44         =====
45         L
46         """
47         return L(*self.args, **hints)

```

Listing 65: Implementation of the differential operator $G_0^{(i)}$

```

1 from sympy import Add, sympify, Number, diff
2
3 from mathematics.sde.nonlinear.symbolic.operator import Operator
4
5
6 class G(Operator):
7     nargs = 3
8
9     _dict = dict()
10
11     def __new__(cls, *args, **kwargs):
12         """
13         Creates new G object with given args
14         Parameters
15         =====
16         args
17         bunch of necessary arguments
18         Returns
19         =====
20         sympy.Expr
21         formula to simplify and substitute

```

```

22     """
23     c, f, dxs = sympify(args)
24
25     if not ((isinstance(f, Number) or f.has(*dxs)) and
26             not f.has(Operator)):
27         return super(G, cls).__new__(cls, *args, **kwargs)
28
29     return Add(*[c[i, 0] * diff(f, dxs[i])
30                  for i in range(len(dxs))])
31
32 def doit(self, **hints):
33     """
34     Tries to expand or calculate function
35     Returns
36     =====
37     G
38     """
39     return G(*self.args, **hints)

```

Listing 66: Implementation of the function $\bar{a}(x, t)$

```

1 from sympy import sympify, Matrix, Add
2
3 from mathematics.sde.nonlinear.symbolic.g import G
4 from mathematics.sde.nonlinear.symbolic.operator import Operator
5
6
7 class Aj(Operator):
8     nargs = 4
9
10    def __new__(cls, *args, **kwargs):
11        """
12        Creates new Aj object with given args
13        Parameters
14        =====
15        args
16        bunch of necessary arguments
17        Returns
18        =====
19        sympy.Expr
20        formula to simplify and substitute
21        """
22        i, a, b, dxs = sympify(args)
23        n = b.shape[0]
24        m = b.shape[1]
25
26        return Matrix([a[i, 0] -
27                       (Add(*[0.5 * G(b[:, j], b[i, j], dxs)
28                              for j in range(m)])
29                       for i in range(n))])

```

Listing 67: Implementation of the differential operator \bar{L}

```

1 from sympy import sympify, Number, diff, Add

```

```

2
3 from mathematics.sde.nonlinear.symbolic.operator import Operator
4
5
6 class Lj(Operator):
7     nargs = 3
8
9     def __new__(cls, *args, **kwargs):
10        """
11        Creates new Lj object with given args
12        Parameters
13        =====
14        args
15        bunch of necessary arguments
16        Returns
17        =====
18        sympy.Expr
19        formula to simplify and substitute
20        """
21        a, f, dxs = sympify(args)
22
23        if not (isinstance(f, Number) or (f.has(*dxs) and
24            not f.has(Operator))):
25            return super(Lj, cls).__new__(cls, *args, **kwargs)
26
27        n = a.shape[0]
28        from sympy.abc import t
29        return Add(diff(f, t), *[a[i, 0] * diff(f, dxs[i])
30            for i in range(n)])
31
32    def doit(self, **hints):
33        """
34        Tries to expand or calculate function
35        Returns
36        =====
37        Lj
38        """
39        return Lj(*self.args, **hints)

```

Listing 68: Implementation of the indicator function $\mathbf{1}_{\{i_1=i_2\}}$

```

1 from sympy import sympify, Number, Function
2
3
4 class Ind(Function):
5     """
6     Indicator function
7     """
8     nargs = 2
9
10    def __new__(cls, *args, **kwargs):
11        """
12        Creates new Ind object with given args

```



```

13     Parameters
14     =====
15     args
16     bunch of necessary arguments
17     Returns
18     =====
19     sympy.Expr
20     formula to simplify and substitute
21     """
22     i1, i2 = sympify(args)
23
24     if not (isinstance(i1, Number) and
25             isinstance(i2, Number)):
26         return super(Ind, cls).__new__(cls, *args, **kwargs)
27
28     if i1 == i2:
29         return 1
30     else:
31         return 0
32
33     def doit(self, **hints):
34         """
35         Tries to expand or calculate function
36         Returns
37         =====
38         Ind
39         """
40     return Ind(*self.args, **hints)

```

6.2.3 Source Codes for Iterated Itô Stochastic Integrals Approximations Subprograms

Listing 69: Approximation of Itô stochastic integral $I_{(0)\tau_{p+1};\tau_p}^{(i_1)}$

```

1  from math import sqrt
2
3  from sympy import Function, sympify, Number
4
5
6  class I0(Function):
7      """
8      Ito stochastic integral
9      """
10     nargs = 3
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new I0 object with given args
15         Parameters
16         =====

```

```

17     i1 : int
18         integral index
19     dt : float
20         delta time
21     ksi : numpy.ndarray
22         matrix of Gaussian random variables
23     Returns
24     =====
25     sympy.Expr
26         formula to simplify and substitute
27     """
28     i1, dt, ksi = sympify(args)
29
30     if not isinstance(i1, Number):
31         return super(I0, cls).__new__(cls, *args, **kwargs)
32
33     return ksi[0, i1] * sqrt(dt)
34
35     def doit(self, **hints):
36         """
37         Tries to expand or calculate function
38         Returns
39         =====
40         I0
41         """
42     return I0(*self.args, **hints)

```

Listing 70: Approximation of iterated Itô stochastic integral $I_{(00)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1 from math import sqrt
2
3 from sympy import sympify, Number, Function, Add
4
5 from mathematics.sde.nonlinear.symbolic.ind import Ind
6
7
8 class I00(Function):
9     """
10     Iterated Ito stochastic integral
11     """
12     nargs = 5
13
14     def __new__(cls, *args, **kwargs):
15         """
16         Creates new I00 object with given args
17         Parameters
18         =====
19         i1 : int
20             integral index
21         i2 : int
22             integral index
23         q : int
24             amount of terms in approximation of

```

```

25     iterated Ito stochastic integral
26     dt : float
27     delta time
28     ksi : numpy.ndarray
29     matrix of Gaussian random variables
30     Returns
31     =====
32     sympy.Expr
33     formula to simplify and substitute
34     """
35     i1, i2, q, dt, ksi = sympify(args)
36
37     if not (isinstance(i1, Number) and
38             isinstance(i2, Number) and
39             isinstance(q, Number)):
40         return super(I00, cls).__new__(cls, *args, **kwargs)
41
42     return \
43         (ksi[0, i1] * ksi[0, i2] +
44          Add(*[
45              (ksi[j1 - 1, i1] * ksi[j1, i2] -
46               ksi[j1, i1] * ksi[j1 - 1, i2]) /
47              sqrt(j1 ** 2 * 4 - 1)
48               for j1 in range(1, q + 1)] -
49          Ind(i1, i2)) * dt / 2
50
51     def doit(self, **hints):
52         """
53         Tries to expand or calculate function
54         Returns
55         =====
56         I00
57         """
58         return I00(*self.args, **hints)

```

Listing 71: Approximation of Itô stochastic integral $I_{(1)\tau_{p+1}, \tau_p}^{(i_1)}$

```

1  from math import sqrt
2
3  from sympy import Function, sympify, Number
4
5
6  class I1(Function):
7      """
8      Ito stochastic integral
9      """
10     nargs = 3
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new I1 object with given args
15         Parameters
16         =====

```

```

17     i1 : int
18         integral index
19     dt : float
20         delta time
21     ksi : numpy.ndarray
22         matrix of Gaussian random variables
23     Returns
24     =====
25     sympy.Expr
26         formula to simplify and substitute
27     """
28     i1, dt, ksi = sympify(args)
29
30     if not isinstance(i1, Number):
31         return super(I1, cls).__new__(cls, *args, **kwargs)
32
33     return -(ksi[0, i1] + ksi[1, i1] / sqrt(3)) * dt ** 1.5 / 2
34
35 def doit(self, **hints):
36     """
37     Tries to expand or calculate function
38     Returns
39     =====
40     I1
41     """
42     return I1(*self.args, **hints)

```

Listing 72: Approximation of iterated Itô stochastic integral $I_{(000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c000 import C000
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I000(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 6
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I000 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index
22        i3 : int
23            integral index
24        q : int

```

```

25     amount of terms in approximation of
26     iterated Ito stochastic integral
27     dt : float
28     delta time
29     ksi : numpy.ndarray
30     matrix of Gaussian random variables
31     Returns
32     =====
33     sympy.Expr
34     formula to simplify and substitute
35     """
36     i1, i2, i3, q, dt, ksi = sympify(args)
37
38     if not (isinstance(i1, Number) and
39             isinstance(i2, Number) and
40             isinstance(i3, Number) and
41             isinstance(q, Number)):
42         return super(I000, cls).__new__(cls, *args, **kwargs)
43
44     return Add(*[
45         C000(j3, j2, j1, dt) *
46         (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
47          Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
48          Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
49          Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         I000
60         """
61         return I000(*self.args, **hints)

```

Listing 73: Approximation of iterated Itô stochastic integral $I_{(10)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c10 import C10
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I10(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 5
12
13     def __new__(cls, *args, **kwargs):

```

```

14 """
15     Creates new I10 object with given args
16     Parameters
17     =====
18     i1 : int
19         integral index
20     i2 : int
21         integral index
22     q : int
23         amount of terms in approximation of
24         iterated Ito stochastic integral
25     dt : float
26         delta time
27     ksi : numpy.ndarray
28         matrix of Gaussian random variables
29     Returns
30     =====
31     sympy.Expr
32         formula to simplify and substitute
33     """
34     i1, i2, q, dt, ksi = sympify(args)
35
36     if not (isinstance(i1, Number) and
37             isinstance(i2, Number) and
38             isinstance(q, Number)):
39         return super(I10, cls).__new__(cls, *args, **kwargs)
40
41     return Add(*[
42         C10(j2, j1, dt) *
43         (ksi[j1, i1] * ksi[j2, i2] -
44          Ind(i1, i2) * Ind(j1, j2))
45         for j2 in range(q + 1)
46         for j1 in range(q + 1)])
47
48     def doit(self, **hints):
49         """
50         Tries to expand or calculate function
51         Returns
52         =====
53         I10
54         """
55         return I10(*self.args, **hints)

```

Listing 74: Approximation of iterated Itô stochastic integral $I_{(01)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c01 import C01
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I01(Function):
8     """

```

```

 9  Iterated Ito stochastic integral
10  """
11  nargs = 5
12
13  def __new__(cls, *args, **kwargs):
14      """
15      Creates new I01 object with given args
16      Parameters
17      =====
18      i1 : int
19          integral index
20      i2 : int
21          integral index
22      q : int
23          amount of terms in approximation of
24          iterated Ito stochastic integral
25      dt : float
26          delta time
27      ksi : numpy.ndarray
28          matrix of Gaussian random variables
29      Returns
30      =====
31      sympy.Expr
32          formula to simplify and substitute
33      """
34      i1, i2, q, dt, ksi = sympify(args)
35
36      if not (isinstance(i1, Number) and
37              isinstance(i2, Number) and
38              isinstance(q, Number)):
39          return super(I01, cls).__new__(cls, *args, **kwargs)
40
41      return Add(*[
42          C01(j2, j1, dt) *
43          (ksi[j1, i1] * ksi[j2, i2] -
44           Ind(i1, i2) * Ind(j1, j2))
45          for j2 in range(q + 1)
46          for j1 in range(q + 1)])
47
48  def doit(self, **hints):
49      """
50      Tries to expand or calculate function
51      Returns
52      =====
53      I01
54      """
55      return I01(*self.args, **hints)

```

Listing 75: Approximation of iterated Itô stochastic integral $I_{(0000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c0000 import C0000

```

```

4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I0000(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 7
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I0000 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index
22        i3 : int
23            integral index
24        i4 : int
25            integral index
26        q : int
27            amount of terms in approximation of
28            iterated Ito stochastic integral
29        dt : float
30            delta time
31        ksi : numpy.ndarray
32            matrix of Gaussian random variables
33        Returns
34        =====
35        sympy.Expr
36            formula to simplify and substitute
37        """
38        i1, i2, i3, i4, q, dt, ksi = sympify(args)
39
40        if not (isinstance(i1, Number) and
41                isinstance(i2, Number) and
42                isinstance(i3, Number) and
43                isinstance(i4, Number) and
44                isinstance(q, Number)):
45            return super(I0000, cls).__new__(cls, *args, **kwargs)
46
47        return Add(*[
48            C0000(j4, j3, j2, j1, dt) *
49            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
50             Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
51             Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
52             Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
53             Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
54             Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
55             Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
56             Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
57             Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
58             Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))

```



```

59     for j4 in range(q + 1)
60     for j3 in range(q + 1)
61     for j2 in range(q + 1)
62     for j1 in range(q + 1)]])
63
64     def doit(self, **hints):
65         """
66         Tries to expand or calculate function
67         Returns
68         =====
69         I0000
70         """
71     return I0000(*self.args, **hints)

```

Listing 76: Approximation of Itô stochastic integral $I_{(2)\tau_{p+1}, \tau_p}^{(i_1)}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5
6  class I2(Function):
7      """
8      Ito stochastic integral
9      """
10     nargs = 3
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new I2 object with given args
15         Parameters
16         =====
17         i1 : int
18             integral index
19         dt : float
20             delta time
21         ksi : numpy.ndarray
22             matrix of Gaussian random variables
23         Returns
24         =====
25         sympy.Expr
26             formula to simplify and substitute
27         """
28         i1, dt, ksi = sympify(args)
29
30         if not isinstance(i1, Number):
31             return super(I2, cls).__new__(cls, *args, **kwargs)
32
33         return (ksi[0, i1] + ksi[1, i1] * sqrt(3) / 2 +
34               ksi[2, i1] / sqrt(5) / 2) * dt ** 2.5 / 3
35
36     def doit(self, **hints):
37         """

```

```

38     Tries to expand or calculate function
39     Returns
40     =====
41     I2
42     """
43     return I2(*self.args, **hints)

```

Listing 77: Approximation of iterated Itô stochastic integral $I_{(100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c100 import C100
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I100(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 6
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I100 object with given args
16         Parameters
17         =====
18         i1 : int
19         integral index
20         i2 : int
21         integral index
22         i3 : int
23         integral index
24         q : int
25         amount of terms in approximation of
26         iterated Ito stochastic integral
27         dt : float
28         delta time
29         ksi : numpy.ndarray
30         matrix of Gaussian random variables
31         Returns
32         =====
33         sympy.Expr
34         formula to simplify and substitute
35         """
36     i1, i2, i3, q, dt, ksi = sympify(args)
37
38     if not (isinstance(i1, Number) and
39             isinstance(i2, Number) and
40             isinstance(i3, Number) and
41             isinstance(q, Number)):
42         return super(I100, cls).__new__(cls, *args, **kwargs)
43
44     return Add(*[

```

```

45     C100(j3, j2, j1, dt) *
46     (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
47     Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
48     Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
49     Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
50     for j3 in range(q + 1)
51     for j2 in range(q + 1)
52     for j1 in range(q + 1)]
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         I100
60         """
61     return I100(*self.args, **hints)

```

Listing 78: Approximation of iterated Itô stochastic integral $I_{(010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c010 import C010
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I010(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 6
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I010 object with given args
16         Parameters
17         =====
18         i1 : int
19             integral index
20         i2 : int
21             integral index
22         i3 : int
23             integral index
24         q : int
25             amount of terms in approximation of
26             iterated Ito stochastic integral
27         dt : float
28             delta time
29         ksi : numpy.ndarray
30             matrix of Gaussian random variables
31         Returns
32         =====
33         sympy.Expr

```

```

34     formula to simplify and substitute
35     """
36     i1, i2, i3, q, dt, ksi = sympify(args)
37
38     if not (isinstance(i1, Number) and
39             isinstance(i2, Number) and
40             isinstance(i3, Number) and
41             isinstance(q, Number)):
42         return super(I010, cls).__new__(cls, *args, **kwargs)
43
44     return Add(*[
45         C010(j3, j2, j1, dt) *
46         (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
47          Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
48          Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
49          Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         I010
60         """
61         return I010(*self.args, **hints)

```

Listing 79: Approximation of iterated Itô stochastic integral $I_{(001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c001 import C001
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I001(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 6
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I001 object with given args
16         Parameters
17         =====
18         i1 : int
19         integral index
20         i2 : int
21         integral index
22         i3 : int

```

```

23     integral index
24     q : int
25     amount of terms in approximation of
26     iterated Ito stochastic integral
27     dt : float
28     delta time
29     ksi : numpy.ndarray
30     matrix of Gaussian random variables
31     Returns
32     =====
33     sympy.Expr
34     formula to simplify and substitute
35     """
36     i1, i2, i3, q, dt, ksi = sympify(args)
37
38     if not (isinstance(i1, Number) and
39             isinstance(i2, Number) and
40             isinstance(i3, Number) and
41             isinstance(q, Number)):
42         return super(I001, cls).__new__(cls, *args, **kwargs)
43
44     return Add(*[
45         C001(j3, j2, j1, dt) *
46         (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] -
47          Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] -
48          Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] -
49          Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1])
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         I001
60         """
61     return I001(*self.args, **hints)

```

Listing 80: Approximation of iterated Itô stochastic integral $I_{(00000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4 i_5)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c00000 import C00000
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I00000(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 8

```

```

12
13 def __new__(cls, *args, **kwargs):
14     """
15     Creates new I00000 object with given args
16     Parameters
17     =====
18     i1 : int
19         integral index
20     i2 : int
21         integral index
22     i3 : int
23         integral index
24     i4 : int
25         integral index
26     i5 : int
27         integral index
28     q : int
29         amount of terms in approximation of
30         iterated Ito stochastic integral
31     dt : float
32         delta time
33     ksi : numpy.ndarray
34         matrix of Gaussian random variables
35     Returns
36     =====
37     sympy.Expr
38         formula to simplify and substitute
39     """
40     i1, i2, i3, i4, i5, q, dt, ksi = sympify(args)
41
42     if not (isinstance(i1, Number) and
43           isinstance(i2, Number) and
44           isinstance(i3, Number) and
45           isinstance(i4, Number) and
46           isinstance(i5, Number) and
47           isinstance(q, Number) and
48           isinstance(dt, Number)):
49         return super(I00000, cls).__new__(cls, *args, **kwargs)
50
51     return Add(*[
52         C00000(j5, j4, j3, j2, j1, dt) *
53         (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
54          Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
55          Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] * ksi[j5, i5] -
56          Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] * ksi[j5, i5] -
57          Ind(i1, i5) * Ind(j1, j5) * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
58          Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] * ksi[j5, i5] -
59          Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] * ksi[j5, i5] -
60          Ind(i2, i5) * Ind(j2, j5) * ksi[j1, i1] * ksi[j3, i3] * ksi[j4, i4] -
61          Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] * ksi[j5, i5] -
62          Ind(i3, i5) * Ind(j3, j5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j4, i4] -
63          Ind(i4, i5) * Ind(j4, j5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] +
64          Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) * ksi[j5, i5] +
65          Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i5) * Ind(j3, j5) * ksi[j4, i4] +
66          Ind(i1, i2) * Ind(j1, j2) * Ind(i4, i5) * Ind(j4, j5) * ksi[j3, i3] +

```

```

67     Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) * ksi[j5, i5] +
68     Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i5) * Ind(j2, j5) * ksi[j4, i4] +
69     Ind(i1, i3) * Ind(j1, j3) * Ind(i4, i5) * Ind(j4, j5) * ksi[j2, i2] +
70     Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3) * ksi[j5, i5] +
71     Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i5) * Ind(j2, j5) * ksi[j3, i3] +
72     Ind(i1, i4) * Ind(j1, j4) * Ind(i3, i5) * Ind(j3, j5) * ksi[j2, i2] +
73     Ind(i1, i5) * Ind(j1, j5) * Ind(i2, i3) * Ind(j2, j3) * ksi[j4, i4] +
74     Ind(i1, i5) * Ind(j1, j5) * Ind(i2, i4) * Ind(j2, j4) * ksi[j3, i3] +
75     Ind(i1, i5) * Ind(j1, j5) * Ind(i3, i4) * Ind(j3, j4) * ksi[j2, i2] +
76     Ind(i2, i3) * Ind(j2, j3) * Ind(i4, i5) * Ind(j4, j5) * ksi[j1, i1] +
77     Ind(i2, i4) * Ind(j2, j4) * Ind(i3, i5) * Ind(j3, j5) * ksi[j1, i1] +
78     Ind(i2, i5) * Ind(j2, j5) * Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1])
79     for j5 in range(q + 1)
80     for j4 in range(q + 1)
81     for j3 in range(q + 1)
82     for j2 in range(q + 1)
83     for j1 in range(q + 1)]
84
85     def doit(self, **hints):
86         """
87         Tries to expand or calculate function
88         Returns
89         =====
90         I00000
91         """
92     return I00000(*self.args, **hints)

```

Listing 81: Approximation of iterated Itô stochastic integral $I_{(20)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c20 import C20
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I20(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 5
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I20 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index
22        q : int
23            amount of terms in approximation of
24            iterated Ito stochastic integral

```

```

25     dt : float
26         delta time
27     ksi : numpy.ndarray
28         matrix of Gaussian random variables
29     Returns
30     =====
31     sympy.Expr
32         formula to simplify and substitute
33     """
34     i1, i2, q, dt, ksi = sympify(args)
35
36     if not (isinstance(i1, Number) and
37             isinstance(i2, Number)):
38         return super(I20, cls).__new__(cls, *args, **kwargs)
39
40     return Add(*[
41         C20(j2, j1, dt) *
42         (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
43         for j2 in range(q + 1)
44         for j1 in range(q + 1)])
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         I20
52         """
53         return I20(*self.args, **hints)

```

Listing 82: Approximation of iterated Itô stochastic integral $I_{(02)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c02 import C02
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I02(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 5
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I02 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index

```



```

22     q : int
23         amount of terms in approximation of
24         iterated Ito stochastic integral
25     dt : float
26         delta time
27     ksi : numpy.ndarray
28         matrix of Gaussian random variables
29     Returns
30     =====
31     sympy.Expr
32         formula to simplify and substitute
33     """
34     i1, i2, q, dt, ksi = sympify(args)
35
36     if not (isinstance(i1, Number) and
37             isinstance(i2, Number)):
38         return super(I02, cls).__new__(cls, *args, **kwargs)
39
40     return Add(*[
41         C02(j2, j1, dt) *
42         (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
43         for j2 in range(q + 1)
44         for j1 in range(q + 1)])
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         I02
52         """
53         return I02(*self.args, **hints)

```

Listing 83: Approximation of iterated Itô stochastic integral $I_{(11)\tau_{p+1}, \tau_p}^{(i_1 i_2)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c11 import C11
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I11(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 5
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I11 object with given args
16         Parameters
17         =====
18         i1 : int

```

```

19     integral index
20     i2 : int
21     integral index
22     q : int
23     amount of terms in approximation of
24     iterated Ito stochastic integral
25     dt : float
26     delta time
27     ksi : numpy.ndarray
28     matrix of Gaussian random variables
29     Returns
30     =====
31     sympy.Expr
32     formula to simplify and substitute
33     """
34     i1, i2, q, dt, ksi = sympify(args)
35
36     if not (isinstance(i1, Number) and
37           isinstance(i2, Number)):
38         return super(I11, cls).__new__(cls, *args, **kwargs)
39
40     return Add(*[
41         C11(j2, j1, dt) *
42         (ksi[j1, i1] * ksi[j2, i2] - Ind(i1, i2) * Ind(j1, j2))
43         for j2 in range(q + 1)
44         for j1 in range(q + 1)])
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         I11
52         """
53     return I11(*self.args, **hints)

```

Listing 84: Approximation of iterated Itô stochastic integral $I_{(1000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c1000 import C1000
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I1000(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 7
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I1000 object with given args

```

```

16     Parameters
17     =====
18     i1 : int
19         integral index
20     i2 : int
21         integral index
22     i3 : int
23         integral index
24     i4 : int
25         integral index
26     q : int
27         amount of terms in approximation of
28         iterated Ito stochastic integral
29     dt : float
30         delta time
31     ksi : numpy.ndarray
32         matrix of Gaussian random variables
33     Returns
34     =====
35     sympy.Expr
36         formula to simplify and substitute
37     """
38     i1, i2, i3, i4, q, dt, ksi = sympify(args)
39
40     if not (isinstance(i1, Number) and
41             isinstance(i2, Number) and
42             isinstance(i3, Number) and
43             isinstance(i4, Number) and
44             isinstance(q, Number)):
45         return super(I1000, cls).__new__(cls, *args, **kwargs)
46
47     return Add(*[
48         C1000(j4, j3, j2, j1, dt) *
49         (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
50          Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
51          Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
52          Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
53          Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
54          Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
55          Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
56          Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
57          Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
58          Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
59         for j4 in range(q + 1)
60         for j3 in range(q + 1)
61         for j2 in range(q + 1)
62         for j1 in range(q + 1)])
63
64     def doit(self, **hints):
65         """
66         Tries to expand or calculate function
67         Returns
68         =====
69         I1000
70         """

```

```
71 return I1000(*self.args, **hints)
```

Listing 85: Approximation of iterated Itô stochastic integral $I_{(0100)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)}$

```
1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c0100 import C0100
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I0100(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 7
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I0100 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index
22        i3 : int
23            integral index
24        i4 : int
25            integral index
26        q : int
27            amount of terms in approximation of
28            iterated Ito stochastic integral
29        dt : float
30            delta time
31        ksi : numpy.ndarray
32            matrix of Gaussian random variables
33        Returns
34        =====
35        sympy.Expr
36            formula to simplify and substitute
37        """
38        i1, i2, i3, i4, q, dt, ksi = sympify(args)
39
40        if not (isinstance(i1, Number) and
41                isinstance(i2, Number) and
42                isinstance(i3, Number) and
43                isinstance(i4, Number) and
44                isinstance(q, Number)):
45            return super(I0100, cls).__new__(cls, *args, **kwargs)
46
47        return Add(*[
48            C0100(j4, j3, j2, j1, dt) *
49            (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
```

```

50     Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
51     Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
52     Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
53     Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
54     Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
55     Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
56     Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
57     Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
58     Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
59     for j4 in range(q + 1)
60     for j3 in range(q + 1)
61     for j2 in range(q + 1)
62     for j1 in range(q + 1)]
63
64     def doit(self, **hints):
65         """
66         Tries to expand or calculate function
67         Returns
68         =====
69         I0100
70         """
71     return I0100(*self.args, **hints)

```

Listing 86: Approximation of iterated Itô stochastic integral $I_{(0010)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c0010 import C0010
4  from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7  class I0010(Function):
8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 7
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I0010 object with given args
16         Parameters
17         =====
18         i1 : int
19             integral index
20         i2 : int
21             integral index
22         i3 : int
23             integral index
24         i3 : int
25             integral index
26         q : int
27             amount of terms in approximation of
28             iterated Ito stochastic integral

```

```

29 dt : float
30     delta time
31 ksi : numpy.ndarray
32     matrix of Gaussian random variables
33 Returns
34 =====
35 sympy.Expr
36     formula to simplify and substitute
37 """
38 i1, i2, i3, i4, q, dt, ksi = sympify(args)
39
40 if not (isinstance(i1, Number) and
41         isinstance(i2, Number) and
42         isinstance(i3, Number) and
43         isinstance(i4, Number) and
44         isinstance(q, Number)):
45     return super(I0010, cls).__new__(cls, *args, **kwargs)
46
47 return Add(*[
48     C0010(j4, j3, j2, j1, dt) *
49     (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
50     Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
51     Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
52     Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
53     Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
54     Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
55     Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
56     Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
57     Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
58     Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
59     for j4 in range(q + 1)
60     for j3 in range(q + 1)
61     for j2 in range(q + 1)
62     for j1 in range(q + 1)])
63
64 def doit(self, **hints):
65     """
66     Tries to expand or calculate function
67     Returns
68     =====
69     I0010
70     """
71     return I0010(*self.args, **hints)

```

Listing 87: Approximation of iterated Itô stochastic integral $I_{(0001)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c0001 import C0001
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I0001(Function):

```

```

8      """
9      Iterated Ito stochastic integral
10     """
11     nargs = 7
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new I0001 object with given args
16         Parameters
17         =====
18         i1 : int
19             integral index
20         i2 : int
21             integral index
22         i3 : int
23             integral index
24         i4 : int
25             integral index
26         q : int
27             amount of terms in approximation of
28             iterated Ito stochastic integral
29         dt : float
30             delta time
31         ksi : numpy.ndarray
32             matrix of Gaussian random variables
33         Returns
34         =====
35         sympy.Expr
36             formula to simplify and substitute
37         """
38         i1, i2, i3, i4, q, dt, ksi = sympify(args)
39
40         if not (isinstance(i1, Number) and
41                 isinstance(i2, Number) and
42                 isinstance(i3, Number) and
43                 isinstance(i4, Number) and
44                 isinstance(q, Number)):
45             return super(I0001, cls).__new__(cls, *args, **kwargs)
46
47         return Add(*[
48             C0001(j4, j3, j2, j1, dt) *
49             (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
50              Ind(i1, i2) * Ind(j1, j2) * ksi[j3, i3] * ksi[j4, i4] -
51              Ind(i1, i3) * Ind(j1, j3) * ksi[j2, i2] * ksi[j4, i4] -
52              Ind(i1, i4) * Ind(j1, j4) * ksi[j2, i2] * ksi[j3, i3] -
53              Ind(i2, i3) * Ind(j2, j3) * ksi[j1, i1] * ksi[j4, i4] -
54              Ind(i2, i4) * Ind(j2, j4) * ksi[j1, i1] * ksi[j3, i3] -
55              Ind(i3, i4) * Ind(j3, j4) * ksi[j1, i1] * ksi[j2, i2] +
56              Ind(i1, i2) * Ind(j1, j2) * Ind(i3, i4) * Ind(j3, j4) +
57              Ind(i1, i3) * Ind(j1, j3) * Ind(i2, i4) * Ind(j2, j4) +
58              Ind(i1, i4) * Ind(j1, j4) * Ind(i2, i3) * Ind(j2, j3))
59             for j4 in range(q + 1)
60             for j3 in range(q + 1)
61             for j2 in range(q + 1)
62             for j1 in range(q + 1)])

```

```

63
64 def doit(self, **hints):
65     """
66     Tries to expand or calculate function
67     Returns
68     =====
69     I0001
70     """
71     return I0001(*self.args, **hints)

```

Listing 88: Approximation of iterated Itô stochastic integral $I_{(000000)\tau_{p+1}, \tau_p}^{(i_1 i_2 i_3 i_4 i_5 i_6)}$

```

1 from sympy import Function, Number, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c000000 import C000000
4 from mathematics.sde.nonlinear.symbolic.ind import Ind
5
6
7 class I000000(Function):
8     """
9     Iterated Ito stochastic integral
10    """
11    nargs = 9
12
13    def __new__(cls, *args, **kwargs):
14        """
15        Creates new I000000 object with given args
16        Parameters
17        =====
18        i1 : int
19            integral index
20        i2 : int
21            integral index
22        i3 : int
23            integral index
24        i4 : int
25            integral index
26        i5 : int
27            integral index
28        i6 : int
29            integral index
30        q : int
31            amount of terms in approximation of
32            iterated Ito stochastic integral
33        dt : float
34            delta time
35        ksi : numpy.ndarray
36            matrix of Gaussian random variables
37        Returns
38        =====
39        sympy.Expr
40            formula to simplify and substitute
41        """

```



```

42 i1, i2, i3, i4, i5, i6, q, dt, ksi = sympify(args)
43
44 if not (isinstance(i1, Number) and
45         isinstance(i2, Number) and
46         isinstance(i3, Number) and
47         isinstance(i4, Number) and
48         isinstance(i5, Number) and
49         isinstance(i6, Number) and
50         isinstance(q, Number) and
51         isinstance(dt, Number)):
52     return super(I000000, cls).__new__(cls, *args, **kwargs)
53
54 return Add(*[
55     C000000(j6, j5, j4, j3, j2, j1, dt) *
56     (ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] *
57     ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6] -
58     Ind(j1, j6) * Ind(i1, i6) * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
59     Ind(j2, j6) * Ind(i2, i6) * ksi[j1, i1] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] -
60     Ind(j3, j6) * Ind(i3, i6) * ksi[j1, i1] * ksi[j2, i2] * ksi[j4, i4] * ksi[j5, i5] -
61     Ind(j4, j6) * Ind(i4, i6) * ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j5, i5] -
62     Ind(j5, j6) * Ind(i5, i6) * ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] -
63     Ind(j1, j2) * Ind(i1, i2) * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6] -
64     Ind(j1, j3) * Ind(i1, i3) * ksi[j2, i2] * ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6] -
65     Ind(j1, j4) * Ind(i1, i4) * ksi[j2, i2] * ksi[j3, i3] * ksi[j5, i5] * ksi[j6, i6] -
66     Ind(j1, j5) * Ind(i1, i5) * ksi[j2, i2] * ksi[j4, i4] * ksi[j3, i3] * ksi[j6, i6] -
67     Ind(j2, j3) * Ind(i2, i3) * ksi[j1, i1] * ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6] -
68     Ind(j2, j4) * Ind(i2, i4) * ksi[j1, i1] * ksi[j3, i3] * ksi[j5, i5] * ksi[j6, i6] -
69     Ind(j2, j5) * Ind(i2, i5) * ksi[j1, i1] * ksi[j3, i3] * ksi[j4, i4] * ksi[j6, i6] -
70     Ind(j3, j4) * Ind(i3, i4) * ksi[j1, i1] * ksi[j2, i2] * ksi[j5, i5] * ksi[j6, i6] -
71     Ind(j3, j5) * Ind(i3, i5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j4, i4] * ksi[j6, i6] -
72     Ind(j4, j5) * Ind(i4, i5) * ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j6, i6] +
73     Ind(j1, j2) * Ind(i1, i2) * Ind(j3, j4) * Ind(i3, i4) * ksi[j5, i5] * ksi[j6, i6] +
74     Ind(j1, j2) * Ind(i1, i2) * Ind(j3, j5) * Ind(i3, i5) * ksi[j4, i4] * ksi[j6, i6] +
75     Ind(j1, j2) * Ind(i1, i2) * Ind(j4, j5) * Ind(i4, i5) * ksi[j3, i3] * ksi[j6, i6] +
76     Ind(j1, j3) * Ind(i1, i3) * Ind(j2, j4) * Ind(i2, i4) * ksi[j5, i5] * ksi[j6, i6] +
77     Ind(j1, j3) * Ind(i1, i3) * Ind(j2, j5) * Ind(i2, i5) * ksi[j4, i4] * ksi[j6, i6] +
78     Ind(j1, j3) * Ind(i1, i3) * Ind(j4, j5) * Ind(i4, i5) * ksi[j2, i2] * ksi[j6, i6] +
79     Ind(j1, j4) * Ind(i1, i4) * Ind(j2, j3) * Ind(i2, i3) * ksi[j5, i5] * ksi[j6, i6] +
80     Ind(j1, j4) * Ind(i1, i4) * Ind(j2, j5) * Ind(i2, i5) * ksi[j3, i3] * ksi[j6, i6] +
81     Ind(j1, j4) * Ind(i1, i4) * Ind(j3, j5) * Ind(i3, i5) * ksi[j2, i2] * ksi[j6, i6] +
82     Ind(j1, j5) * Ind(i1, i5) * Ind(j2, j3) * Ind(i2, i3) * ksi[j4, i4] * ksi[j6, i6] +
83     Ind(j1, j5) * Ind(i1, i5) * Ind(j2, j4) * Ind(i2, i4) * ksi[j3, i3] * ksi[j6, i6] +
84     Ind(j1, j5) * Ind(i1, i5) * Ind(j3, j4) * Ind(i3, i4) * ksi[j2, i2] * ksi[j6, i6] +
85     Ind(j2, j3) * Ind(i2, i3) * Ind(j4, j5) * Ind(i4, i5) * ksi[j1, i1] * ksi[j6, i6] +
86     Ind(j2, j4) * Ind(i2, i4) * Ind(j3, j5) * Ind(i3, i5) * ksi[j1, i1] * ksi[j6, i6] +
87     Ind(j2, j5) * Ind(i2, i5) * Ind(j3, j4) * Ind(i3, i4) * ksi[j1, i1] * ksi[j6, i6] +
88     Ind(j6, j1) * Ind(i6, i1) * Ind(j3, j4) * Ind(i3, i4) * ksi[j2, i2] * ksi[j5, i5] +
89     Ind(j6, j1) * Ind(i6, i1) * Ind(j3, j5) * Ind(i3, i5) * ksi[j2, i2] * ksi[j4, i4] +
90     Ind(j6, j1) * Ind(i6, i1) * Ind(j2, j5) * Ind(i2, i5) * ksi[j3, i3] * ksi[j4, i4] +
91     Ind(j6, j1) * Ind(i6, i1) * Ind(j2, j4) * Ind(i2, i4) * ksi[j3, i3] * ksi[j5, i5] +
92     Ind(j6, j1) * Ind(i6, i1) * Ind(j4, j5) * Ind(i4, i5) * ksi[j2, i2] * ksi[j3, i3] +
93     Ind(j6, j1) * Ind(i6, i1) * Ind(j2, j3) * Ind(i2, i3) * ksi[j4, i4] * ksi[j5, i5] +
94     Ind(j6, j2) * Ind(i6, i2) * Ind(j3, j5) * Ind(i3, i5) * ksi[j1, i1] * ksi[j4, i4] +
95     Ind(j6, j2) * Ind(i6, i2) * Ind(j4, j5) * Ind(i4, i5) * ksi[j1, i1] * ksi[j3, i3] +
96     Ind(j6, j2) * Ind(i6, i2) * Ind(j3, j4) * Ind(i3, i4) * ksi[j1, i1] * ksi[j5, i5] +

```

```

97     Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j5) * Ind(i1 , i5) * ksi[j3 , i3] * ksi[j4 , i4] +
98     Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j4) * Ind(i1 , i4) * ksi[j3 , i3] * ksi[j5 , i5] +
99     Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j3) * Ind(i1 , i3) * ksi[j4 , i4] * ksi[j5 , i5] +
100    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j2 , j5) * Ind(i2 , i5) * ksi[j1 , i1] * ksi[j4 , i4] +
101    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j4 , j5) * Ind(i4 , i5) * ksi[j1 , i1] * ksi[j2 , i2] +
102    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j2 , j4) * Ind(i2 , i4) * ksi[j1 , i1] * ksi[j5 , i5] +
103    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j1 , j5) * Ind(i1 , i5) * ksi[j2 , i2] * ksi[j4 , i4] +
104    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j1 , j4) * Ind(i1 , i4) * ksi[j2 , i2] * ksi[j5 , i5] +
105    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j1 , j2) * Ind(i1 , i2) * ksi[j4 , i4] * ksi[j5 , i5] +
106    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j3 , j5) * Ind(i3 , i5) * ksi[j1 , i1] * ksi[j2 , i2] +
107    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j2 , j5) * Ind(i2 , i5) * ksi[j1 , i1] * ksi[j3 , i3] +
108    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j2 , j3) * Ind(i2 , i3) * ksi[j1 , i1] * ksi[j5 , i5] +
109    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j5) * Ind(i1 , i5) * ksi[j2 , i2] * ksi[j3 , i3] +
110    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j3) * Ind(i1 , i3) * ksi[j2 , i2] * ksi[j5 , i5] +
111    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j2) * Ind(i1 , i2) * ksi[j3 , i3] * ksi[j5 , i5] +
112    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j3 , j4) * Ind(i3 , i4) * ksi[j1 , i1] * ksi[j2 , i2] +
113    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j2 , j4) * Ind(i2 , i4) * ksi[j1 , i1] * ksi[j3 , i3] +
114    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j2 , j3) * Ind(i2 , i3) * ksi[j1 , i1] * ksi[j1 , i4] +
115    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j4) * Ind(i1 , i4) * ksi[j2 , i2] * ksi[j3 , i3] +
116    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j3) * Ind(i1 , i3) * ksi[j2 , i2] * ksi[j4 , i4] +
117    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j2) * Ind(i1 , i2) * ksi[j3 , i3] * ksi[j4 , i4] -
118    Ind(j6 , j1) * Ind(i6 , i1) * Ind(j2 , j5) * Ind(i2 , i5) * Ind(j3 , j4) * Ind(i3 , i4) -
119    Ind(j6 , j1) * Ind(i6 , i1) * Ind(j2 , j4) * Ind(i2 , i4) * Ind(j3 , j5) * Ind(i3 , i5) -
120    Ind(j6 , j1) * Ind(i6 , i1) * Ind(j2 , j3) * Ind(i2 , i3) * Ind(j4 , j5) * Ind(i4 , i5) -
121    Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j5) * Ind(i1 , i5) * Ind(j3 , j4) * Ind(i3 , i4) -
122    Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j4) * Ind(i1 , i4) * Ind(j3 , j5) * Ind(i3 , i5) -
123    Ind(j6 , j2) * Ind(i6 , i2) * Ind(j1 , j3) * Ind(i1 , i3) * Ind(j4 , j5) * Ind(i4 , i5) -
124    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j1 , j5) * Ind(i1 , i5) * Ind(j2 , j4) * Ind(i2 , i4) -
125    Ind(j6 , j3) * Ind(i6 , i3) * Ind(j1 , j4) * Ind(i1 , i4) * Ind(j2 , j5) * Ind(i2 , i5) -
126    Ind(j3 , j6) * Ind(i3 , i6) * Ind(j1 , j2) * Ind(i1 , i2) * Ind(j4 , j5) * Ind(i4 , i5) -
127    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j5) * Ind(i1 , i5) * Ind(j2 , j3) * Ind(i2 , i3) -
128    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j3) * Ind(i1 , i3) * Ind(j2 , j5) * Ind(i2 , i5) -
129    Ind(j6 , j4) * Ind(i6 , i4) * Ind(j1 , j2) * Ind(i1 , i2) * Ind(j3 , j5) * Ind(i3 , i5) -
130    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j4) * Ind(i1 , i4) * Ind(j2 , j3) * Ind(i2 , i3) -
131    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j2) * Ind(i1 , i2) * Ind(j3 , j4) * Ind(i3 , i4) -
132    Ind(j6 , j5) * Ind(i6 , i5) * Ind(j1 , j3) * Ind(i1 , i3) * Ind(j2 , j4) * Ind(i2 , i4))
133     for j6 in range(q + 1)
134     for j5 in range(q + 1)
135     for j4 in range(q + 1)
136     for j3 in range(q + 1)
137     for j2 in range(q + 1)
138     for j1 in range(q + 1)]]
139
140     def doit(self , **hints):
141         """
142         Tries to expand or calculate function
143         Returns
144         =====
145         I000000
146         """
147     return I000000(*self.args , **hints)

```

6.2.4 Source Codes for Iterated Stratonovich Stochastic Integrals Approximations Subprograms

Listing 89: Approximation of Stratonovich stochastic integral $I_{(0)\tau_{p+1},\tau_p}^{*(i_1)}$

```

1 from math import sqrt
2
3 from sympy import Function, sympify, Number
4
5
6 class J0(Function):
7     """
8     Stratonovich stochastic integral
9     """
10    nargs = 3
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J0 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        dt : float
20            delta time
21        ksi : numpy.ndarray
22            matrix of Gaussian random variables
23        Returns
24        =====
25        sympy.Expr
26            formula to simplify and substitute
27        """
28        i1, dt, ksi = sympify(args)
29
30        if not isinstance(i1, Number):
31            return super(J0, cls).__new__(cls, *args, **kwargs)
32
33        return ksi[0, i1] * sqrt(dt)
34
35    def doit(self, **hints):
36        """
37        Tries to expand or calculate function
38        Returns
39        =====
40        J0
41        """
42        return J0(*self.args, **hints)

```

Listing 90: Approximation of iterated Stratonovich stochastic integral $I_{(00)\tau_{p+1},\tau_p}^{*(i_1 i_2)}$

```

1 from math import sqrt

```

```

2
3 from sympy import sympify, Number, Function, Add
4
5
6 class J00(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 5
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J00 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        q : int
22            amount of terms in approximation of
23            iterated Stratonovich stochastic integral
24        dt : float
25            delta time
26        ksi : numpy.ndarray
27            matrix of Gaussian random variables
28        Returns
29        =====
30        sympy.Expr
31            formula to simplify and substitute
32        """
33        i1, i2, q, dt, ksi = sympify(args)
34
35        if not (isinstance(i1, Number) and
36                isinstance(i2, Number) and
37                isinstance(q, Number)):
38            return super(J00, cls).__new__(cls, *args, **kwargs)
39
40        return \
41            (ksi[0, i1] * ksi[0, i2] +
42             Add(*[
43                 (ksi[j1 - 1, i1] * ksi[j1, i2] -
44                  ksi[j1, i1] * ksi[j1 - 1, i2]) /
45                 sqrt(j1 ** 2 * 4 - 1)
46                 for j1 in range(1, q + 1)])) * dt / 2
47
48    def doit(self, **hints):
49        """
50        Tries to expand or calculate function
51        Returns
52        =====
53        J00
54        """
55        return J00(*self.args, **hints)

```

Listing 91: Approximation of Stratonovich stochastic integral $I_{(1)T,t}^{*(i_1)}$

```

1  from math import sqrt
2
3  from sympy import Function, sympify, Number
4
5
6  class J1(Function):
7      """
8      Stratonovich stochastic integral
9      """
10     nargs = 3
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J1 object with given args
15         Parameters
16         =====
17         i1 : int
18             integral index
19         dt : float
20             delta time
21         ksi : numpy.ndarray
22             matrix of Gaussian random variables
23         Returns
24         =====
25         sympy.Expr
26             formula to simplify and substitute
27         """
28         i1, dt, ksi = sympify(args)
29
30         if not isinstance(i1, Number):
31             return super(J1, cls).__new__(cls, *args, **kwargs)
32
33         return -(ksi[0, i1] + ksi[1, i1] / sqrt(3)) * dt ** 1.5 / 2
34
35     def doit(self, **hints):
36         """
37         Tries to expand or calculate function
38         Returns
39         =====
40         J1
41         """
42         return J1(*self.args, **hints)

```

Listing 92: Approximation of iterated Stratonovich stochastic integral $I_{(000)\tau_{p+1},\tau_p}^{*(i_1 i_2 i_3)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c000 import C000
4
5
6  class J000(Function):
7      """

```

```

8  Iterated Stratonovich stochastic integral
9  """
10 nargs = 6
11
12 def __new__(cls, *args, **kwargs):
13     """
14     Creates new J000 object with given args
15     Parameters
16     =====
17     i1 : int
18         integral index
19     i2 : int
20         integral index
21     i3 : int
22         integral index
23     q : int
24         amount of terms in approximation of
25         iterated Stratonovich stochastic integral
26     dt : float
27         delta time
28     ksi : numpy.ndarray
29         matrix of Gaussian random variables
30     Returns
31     =====
32     sympy.Expr
33         formula to simplify and substitute
34     """
35     i1, i2, i3, q, dt, ksi = sympify(args)
36
37     if not (isinstance(i1, Number) and
38             isinstance(i2, Number) and
39             isinstance(i3, Number) and
40             isinstance(q, Number)):
41         return super(J000, cls).__new__(cls, *args, **kwargs)
42
43     return Add(*[
44         C000(j3, j2, j1, dt) *
45         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
46         for j3 in range(q + 1)
47         for j2 in range(q + 1)
48         for j1 in range(q + 1)])
49
50 def doit(self, **hints):
51     """
52     Tries to expand or calculate function
53     Returns
54     =====
55     J000
56     """
57     return J000(*self.args, **hints)

```

Listing 93: Approximation of iterated Stratonovich stochastic integral $I_{(10)\tau_{p+1}, \tau_p}^{*(i_1 i_2)}$

```

1  from sympy import Function, sympify, Number, Add

```

```

2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c10 import C10
4
5
6 class J10(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 5
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J10 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        q : int
22            amount of terms in approximation of
23            iterated Stratonovich stochastic integral
24        dt : float
25            delta time
26        ksi : numpy.ndarray
27            matrix of Gaussian random variables
28        Returns
29        =====
30        sympy.Expr
31            formula to simplify and substitute
32        """
33        i1, i2, q, dt, ksi = sympify(args)
34
35        if not (isinstance(i1, Number) and
36                isinstance(i2, Number) and
37                isinstance(q, Number)):
38            return super(J10, cls).__new__(cls, *args, **kwargs)
39
40        return Add(*[
41            C10(j2, j1, dt) *
42            ksi[j1, i1] * ksi[j2, i2]
43            for j2 in range(q + 1)
44            for j1 in range(q + 1)])
45
46    def doit(self, **hints):
47        """
48        Tries to expand or calculate function
49        Returns
50        =====
51        J10
52        """
53        return J10(*self.args, **hints)

```

Listing 94: Approximation of iterated Stratonovich stochastic integral $I_{(01)\tau_{p+1}, \tau_p}^{*(i_1 i_2)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c01 import C01
4
5
6  class J01(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10
11     nargs = 5
12
13     def __new__(cls, *args, **kwargs):
14         """
15         Creates new J01 object with given args
16         Parameters
17         =====
18         i1 : int
19             integral index
20         i2 : int
21             integral index
22         q : int
23             amount of terms in approximation of
24             iterated Stratonovich stochastic integral
25         dt : float
26             delta time
27         ksi : numpy.ndarray
28             matrix of Gaussian random variables
29         Returns
30         =====
31         sympy.Expr
32             formula to simplify and substitute
33         """
34         i1, i2, q, dt, ksi = sympify(args)
35
36         if not (isinstance(i1, Number) and
37                 isinstance(i2, Number) and
38                 isinstance(q, Number)):
39             return super(J01, cls).__new__(cls, *args, **kwargs)
40
41         return Add(*[
42             C01(j2, j1, dt) *
43             ksi[j1, i1] * ksi[j2, i2]
44             for j2 in range(q + 1)
45             for j1 in range(q + 1)])
46
47     def doit(self, **hints):
48         """
49         Tries to expand or calculate function
50         Returns
51         =====
52         J01
53         """

```



```
54 return J01(*self.args, **hints)
```

Listing 95: Approximation of iterated Stratonovich stochastic integral $I_{(0000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)}$

```
1 from sympy import Function, sympify, Number, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c0000 import C0000
4
5
6 class J0000(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 7
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J0000 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        i3 : int
22            integral index
23        i4 : int
24            integral index
25        q : int
26            amount of terms in approximation of
27            iterated Stratonovich stochastic integral
28        dt : float
29            delta time
30        ksi : numpy.ndarray
31            matrix of Gaussian random variables
32        Returns
33        =====
34        sympy.Expr
35            formula to simplify and substitute
36        """
37        i1, i2, i3, i4, q, dt, ksi = sympify(args)
38
39        if not (isinstance(i1, Number) and
40                isinstance(i2, Number) and
41                isinstance(i3, Number) and
42                isinstance(i4, Number) and
43                isinstance(q, Number)):
44            return super(J0000, cls).__new__(cls, *args, **kwargs)
45
46        return Add(*[
47            C0000(j4, j3, j2, j1, dt) *
48            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
49            for j4 in range(q + 1)
```

```

50     for j3 in range(q + 1)
51     for j2 in range(q + 1)
52     for j1 in range(q + 1)]])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         J0000
60         """
61     return J0000(*self.args, **hints)

```

Listing 96: Approximation of Stratonovich stochastic integral $I_{(2)\tau_{p+1}, \tau_p}^{*(i_1)}$

```

1  from math import sqrt
2
3  from sympy import sympify, Number, Function
4
5
6  class J2(Function):
7      """
8      Stratonovich stochastic integral
9      """
10     nargs = 3
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J2 object with given args
15         Parameters
16         =====
17         i1 : int
18             integral index
19         dt : float
20             delta time
21         ksi : numpy.ndarray
22             matrix of Gaussian random variables
23         Returns
24         =====
25         sympy.Expr
26             formula to simplify and substitute
27         """
28         i1, dt, ksi = sympify(args)
29
30         if not isinstance(i1, Number):
31             return super(J2, cls).__new__(cls, *args, **kwargs)
32
33         return (ksi[0, i1] + ksi[1, i1] * sqrt(3) / 2 +
34               ksi[2, i1] / sqrt(5) / 2) * dt ** 2.5 / 3
35
36     def doit(self, **hints):
37         """
38         Tries to expand or calculate function

```

```

39     Returns
40     =====
41     J2
42     """
43     return J2(*self.args, **hints)

```

Listing 97: Approximation of iterated Stratonovich stochastic integral $I_{(100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c100 import C100
4
5
6  class J100(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 6
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J100 object with given args
15
16         Parameters
17         =====
18         i1 : int
19             integral index
20         i2 : int
21             integral index
22         i3 : int
23             integral index
24         q : int
25             amount of terms in approximation of
26             iterated Stratonovich stochastic integral
27         dt : float
28             delta time
29         ksi : numpy.ndarray
30             matrix of Gaussian random variables
31         Returns
32         =====
33         sympy.Expr
34             formula to simplify and substitute
35         """
36     i1, i2, i3, q, dt, ksi = sympify(args)
37
38     if not (isinstance(i1, Number) and
39             isinstance(i2, Number) and
40             isinstance(i3, Number) and
41             isinstance(q, Number)):
42         return super(J100, cls).__new__(cls, *args, **kwargs)
43
44     return Add(*[
45         C100(j3, j2, j1, dt) *

```

```

46     ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
47     for j3 in range(q + 1)
48     for j2 in range(q + 1)
49     for j1 in range(q + 1)])
50
51     def doit(self, **hints):
52         """
53         Tries to expand or calculate function
54         Returns
55         =====
56         J100
57         """
58     return J100(*self.args, **hints)

```

Listing 98: Approximation of iterated Stratonovich stochastic integral $I_{(010)_{\tau_{p+1}, \tau_p}}^{*(i_1 i_2 i_3)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c010 import C010
4
5
6  class J010(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 6
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J010 object with given args
15         Parameters
16         =====
17         i1 : int
18             integral index
19         i2 : int
20             integral index
21         i3 : int
22             integral index
23         q : int
24             amount of terms in approximation of
25             iterated Stratonovich stochastic integral
26         dt : float
27             delta time
28         ksi : numpy.ndarray
29             matrix of Gaussian random variables
30         Returns
31         =====
32         sympy.Expr
33             formula to simplify and substitute
34         """
35     i1, i2, i3, q, dt, ksi = sympify(args)
36
37     if not (isinstance(i1, Number) and

```

```

38     isinstance(i2, Number) and
39     isinstance(i3, Number) and
40     isinstance(q, Number):
41     return super(J010, cls).__new__(cls, *args, **kwargs)
42
43     return Add(*[
44         C010(j3, j2, j1, dt) *
45         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
46         for j3 in range(q + 1)
47         for j2 in range(q + 1)
48         for j1 in range(q + 1)])
49
50 def doit(self, **hints):
51     """
52     Tries to expand or calculate function
53     Returns
54     =====
55     J010
56     """
57     return J010(*self.args, **hints)

```

Listing 99: Approximation of iterated Stratonovich stochastic integral $I_{(001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3)}$

```

1 from sympy import Function, sympify, Number, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c001 import C001
4
5
6 class J001(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 6
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J001 object with given args
15        Parameters
16        =====
17        i1 : int
18        integral index
19        i2 : int
20        integral index
21        i3 : int
22        integral index
23        q : int
24        amount of terms in approximation of
25        iterated Stratonovich stochastic integral
26        dt : float
27        delta time
28        ksi : numpy.ndarray
29        matrix of Gaussian random variables
30        Returns

```

```

31  =====
32  sympy.Expr
33  formula to simplify and substitute
34  """
35  i1, i2, i3, q, dt, ksi = sympify(args)
36
37  if not (isinstance(i1, Number) and
38          isinstance(i2, Number) and
39          isinstance(i3, Number) and
40          isinstance(q, Number)):
41      return super(J001, cls).__new__(cls, *args, **kwargs)
42
43  return Add(*[
44      C001(j3, j2, j1, dt) *
45      ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3]
46      for j3 in range(q + 1)
47      for j2 in range(q + 1)
48      for j1 in range(q + 1)])
49
50  def doit(self, **hints):
51      """
52      Tries to expand or calculate function
53      Returns
54      =====
55      J001
56      """
57  return J001(*self.args, **hints)

```

Listing 100: Approximation of iterated Stratonovich stochastic integral $I_{(00000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4 i_5)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c00000 import C00000
4
5
6  class J00000(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 8
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J00000 object with given args
15         Parameters
16         =====
17         i1 : int
18         integral index
19         i2 : int
20         integral index
21         i3 : int
22         integral index
23         i4 : int

```

```

24     integral index
25     i5 : int
26     integral index
27     q : int
28     amount of terms in approximation of
29     iterated Stratonovich stochastic integral
30     dt : float
31     delta time
32     ksi : numpy.ndarray
33     matrix of Gaussian random variables
34     Returns
35     =====
36     sympy.Expr
37     formula to simplify and substitute
38     """
39     i1, i2, i3, i4, i5, q, dt, ksi = sympify(args)
40
41     if not (isinstance(i1, Number) and
42             isinstance(i2, Number) and
43             isinstance(i3, Number) and
44             isinstance(i4, Number) and
45             isinstance(i5, Number) and
46             isinstance(q, Number) and
47             isinstance(dt, Number)):
48         return super(J00000, cls).__new__(cls, *args, **kwargs)
49
50     return Add(*[
51         C00000(j5, j4, j3, j2, j1, dt) *
52         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4] * ksi[j5, i5]
53         for j5 in range(q + 1)
54         for j4 in range(q + 1)
55         for j3 in range(q + 1)
56         for j2 in range(q + 1)
57         for j1 in range(q + 1)])
58
59     def doit(self, **hints):
60         """
61         Tries to expand or calculate function
62         Returns
63         =====
64         J00000
65         """
66         return J00000(*self.args, **hints)

```

Listing 101: Approximation of iterated Stratonovich stochastic integral $I_{(20)\tau_p+1, \tau_p}^{*(i_1 i_2)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c20 import C20
4
5
6  class J20(Function):
7      """

```

```

8  Iterated Stratonovich stochastic integral
9  """
10 nargs = 5
11
12 def __new__(cls, *args, **kwargs):
13     """
14     Creates new J20 object with given args
15     Parameters
16     =====
17     i1 : int
18         integral index
19     i2 : int
20         integral index
21     q : int
22         amount of terms in approximation of
23         iterated Stratonovich stochastic integral
24     dt : float
25         delta time
26     ksi : numpy.ndarray
27         matrix of Gaussian random variables
28     Returns
29     =====
30     sympy.Expr
31         formula to simplify and substitute
32     """
33     i1, i2, q, dt, ksi = sympify(args)
34
35     if not (isinstance(i1, Number) and
36            isinstance(i2, Number)):
37         return super(J20, cls).__new__(cls, *args, **kwargs)
38
39     return Add(*[
40         C20(j2, j1, dt) *
41         ksi[j1, i1] * ksi[j2, i2]
42         for j2 in range(q + 1)
43         for j1 in range(q + 1)])
44
45     def doit(self, **hints):
46         """
47         Tries to expand or calculate function
48         Returns
49         =====
50         J20
51         """
52         return J20(*self.args, **hints)

```

Listing 102: Approximation of iterated Stratonovich stochastic integral $I_{(02)\tau_p+1, \tau_p}^{*(i_1 i_2)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c02 import C02
4
5

```



```

6 class J02(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 5
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J02 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        q : int
22            amount of terms in approximation of
23            iterated Stratonovich stochastic integral
24        dt : float
25            delta time
26        ksi : numpy.ndarray
27            matrix of Gaussian random variables
28        Returns
29        =====
30        sympy.Expr
31            formula to simplify and substitute
32        """
33        i1, i2, q, dt, ksi = sympify(args)
34
35        if not (isinstance(i1, Number) and
36                isinstance(i2, Number)):
37            return super(J02, cls).__new__(cls, *args, **kwargs)
38
39        return Add(*[
40            C02(j2, j1, dt) *
41            ksi[j1, i1] * ksi[j2, i2]
42            for j2 in range(q + 1)
43            for j1 in range(q + 1)])
44
45    def doit(self, **hints):
46        """
47        Tries to expand or calculate function
48        Returns
49        =====
50        J02
51        """
52    return J02(*self.args, **hints)

```

Listing 103: Approximation of iterated Stratonovich stochastic integral $I_{(11)\tau_{p+1}, \tau_p}^{*(i_1 i_2)}$

```

1 from sympy import Function, sympify, Number, Add
2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c11 import C11

```

```

4
5
6 class J11(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 5
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J11 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        q : int
22            amount of terms in approximation of
23            iterated Stratonovich stochastic integral
24        dt : float
25            delta time
26        ksi : numpy.ndarray
27            matrix of Gaussian random variables
28        Returns
29        =====
30        sympy.Expr
31            formula to simplify and substitute
32        """
33        i1, i2, q, dt, ksi = sympify(args)
34
35        if not (isinstance(i1, Number) and
36                isinstance(i2, Number)):
37            return super(J11, cls).__new__(cls, *args, **kwargs)
38
39        return Add(*[
40            C11(j2, j1, dt) *
41            ksi[j1, i1] * ksi[j2, i2]
42            for j2 in range(q + 1)
43            for j1 in range(q + 1)])
44
45    def doit(self, **hints):
46        """
47        Tries to expand or calculate function
48        Returns
49        =====
50        J11
51        """
52    return J11(*self.args, **hints)

```

Listing 104: Approximation of iterated Stratonovich stochastic integral $I_{(1000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)}$

```

1 from sympy import Function, Number, sympify, Add

```

```

2
3 from mathematics.sde.nonlinear.symbolic.coefficients.c1000 import C1000
4
5
6 class J1000(Function):
7     """
8     Iterated Stratonovich stochastic integral
9     """
10    nargs = 7
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new J1000 object with given args
15        Parameters
16        =====
17        i1 : int
18            integral index
19        i2 : int
20            integral index
21        i3 : int
22            integral index
23        i4 : int
24            integral index
25        q : int
26            amount of terms in approximation of
27            iterated Stratonovich stochastic integral
28        dt : float
29            delta time
30        ksi : numpy.ndarray
31            matrix of Gaussian random variables
32        Returns
33        =====
34        sympy.Expr
35            formula to simplify and substitute
36        """
37        i1, i2, i3, i4, q, dt, ksi = sympify(args)
38
39        if not (isinstance(i1, Number) and
40                isinstance(i2, Number) and
41                isinstance(i3, Number) and
42                isinstance(i4, Number) and
43                isinstance(q, Number)):
44            return super(J1000, cls).__new__(cls, *args, **kwargs)
45
46        return Add(*[
47            C1000(j4, j3, j2, j1, dt) *
48            ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
49            for j4 in range(q + 1)
50            for j3 in range(q + 1)
51            for j2 in range(q + 1)
52            for j1 in range(q + 1)])
53
54    def doit(self, **hints):
55        """
56        Tries to expand or calculate function

```

```

57     Returns
58     =====
59     J1000
60     """
61     return J1000(*self.args, **hints)

```

Listing 105: Approximation of iterated Stratonovich stochastic integral $I_{(0100)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c0100 import C0100
4
5
6  class J0100(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 7
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J0100 object with given args
15         Parameters
16         =====
17         i1 : int
18         integral index
19         i2 : int
20         integral index
21         i3 : int
22         integral index
23         i4 : int
24         integral index
25         q : int
26         amount of terms in approximation of
27         iterated Stratonovich stochastic integral
28         dt : float
29         delta time
30         ksi : numpy.ndarray
31         matrix of Gaussian random variables
32         Returns
33         =====
34         sympy.Expr
35         formula to simplify and substitute
36         """
37     i1, i2, i3, i4, q, dt, ksi = sympify(args)
38
39     if not (isinstance(i1, Number) and
40             isinstance(i2, Number) and
41             isinstance(i3, Number) and
42             isinstance(i4, Number) and
43             isinstance(q, Number)):
44         return super(J0100, cls).__new__(cls, *args, **kwargs)
45

```

```

46     return Add(*[
47         C0100(j4, j3, j2, j1, dt) *
48         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
49         for j4 in range(q + 1)
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         J0100
60         """
61     return J0100(*self.args, **hints)

```

Listing 106: Approximation of iterated Stratonovich stochastic integral $I_{(0010)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)}$

```

1  from sympy import Function, sympify, Number, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c0010 import C0010
4
5
6  class J0010(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 7
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J0010 object with given args
15         Parameters
16         =====
17         i1 : int
18             integral index
19         i2 : int
20             integral index
21         i3 : int
22             integral index
23         i3 : int
24             integral index
25         q : int
26             amount of terms in approximation of
27             iterated Stratonovich stochastic integral
28         dt : float
29             delta time
30         ksi : numpy.ndarray
31             matrix of Gaussian random variables
32         Returns
33         =====
34         sympy.Expr

```

```

35     formula to simplify and substitute
36     """
37     i1, i2, i3, i4, q, dt, ksi = sympify(args)
38
39     if not (isinstance(i1, Number) and
40            isinstance(i2, Number) and
41            isinstance(i3, Number) and
42            isinstance(i4, Number) and
43            isinstance(q, Number)):
44         return super(J0010, cls).__new__(cls, *args, **kwargs)
45
46     return Add(*[
47         C0010(j4, j3, j2, j1, dt) *
48         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
49         for j4 in range(q + 1)
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         J0010
60         """
61         return J0010(*self.args, **hints)

```

Listing 107: Approximation of iterated Stratonovich stochastic integral $I_{(0001)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c0001 import C0001
4
5
6  class J0001(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 7
11
12     def __new__(cls, *args, **kwargs):
13         """
14         Creates new J0001 object with given args
15         Parameters
16         =====
17         i1 : int
18         integral index
19         i2 : int
20         integral index
21         i3 : int
22         integral index
23         i4 : int

```

```

24     integral index
25     q : int
26     amount of terms in approximation of
27     iterated Stratonovich stochastic integral
28     dt : float
29     delta time
30     ksi : numpy.ndarray
31     matrix of Gaussian random variables
32     Returns
33     =====
34     sympy.Expr
35     formula to simplify and substitute
36     """
37     i1, i2, i3, i4, q, dt, ksi = sympify(args)
38
39     if not (isinstance(i1, Number) and
40             isinstance(i2, Number) and
41             isinstance(i3, Number) and
42             isinstance(i4, Number) and
43             isinstance(q, Number)):
44         return super(J0001, cls).__new__(cls, *args, **kwargs)
45
46     return Add(*[
47         C0001(j4, j3, j2, j1, dt) *
48         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] * ksi[j4, i4]
49         for j4 in range(q + 1)
50         for j3 in range(q + 1)
51         for j2 in range(q + 1)
52         for j1 in range(q + 1)])
53
54     def doit(self, **hints):
55         """
56         Tries to expand or calculate function
57         Returns
58         =====
59         J0001
60         """
61         return J0001(*self.args, **hints)

```

Listing 108: Approximation of iterated Stratonovich stochastic integral $I_{(000000)\tau_{p+1}, \tau_p}^{*(i_1 i_2 i_3 i_4 i_5 i_6)}$

```

1  from sympy import Function, Number, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.coefficients.c000000 import C000000
4
5
6  class J000000(Function):
7      """
8      Iterated Stratonovich stochastic integral
9      """
10     nargs = 9
11
12     def __new__(cls, *args, **kwargs):

```

```

13     """
14     Creates new J000000 object with given args
15     Parameters
16     =====
17     i1 : int
18         integral index
19     i2 : int
20         integral index
21     i3 : int
22         integral index
23     i4 : int
24         integral index
25     i5 : int
26         integral index
27     i6 : int
28         integral index
29     q : int
30         amount of terms in approximation of
31         iterated Stratonovich stochastic integral
32     dt : float
33         delta time
34     ksi : numpy.ndarray
35         matrix of Gaussian random variables
36     Returns
37     =====
38     sympy.Expr
39         formula to simplify and substitute
40     """
41     i1, i2, i3, i4, i5, i6, q, dt, ksi = sympify(args)
42
43     if not (isinstance(i1, Number) and
44             isinstance(i2, Number) and
45             isinstance(i3, Number) and
46             isinstance(i4, Number) and
47             isinstance(i5, Number) and
48             isinstance(i6, Number) and
49             isinstance(q, Number) and
50             isinstance(dt, Number)):
51         return super(J000000, cls).__new__(cls, *args, **kwargs)
52
53     return Add(*[
54         C000000(j6, j5, j4, j3, j2, j1, dt) *
55         ksi[j1, i1] * ksi[j2, i2] * ksi[j3, i3] *
56         ksi[j4, i4] * ksi[j5, i5] * ksi[j6, i6]
57         for j6 in range(q + 1)
58         for j5 in range(q + 1)
59         for j4 in range(q + 1)
60         for j3 in range(q + 1)
61         for j2 in range(q + 1)
62         for j1 in range(q + 1)])
63
64     def doit(self, **hints):
65         """
66         Tries to expand or calculate function
67         Returns

```



```

68     =====
69     J000000
70     """
71     return J000000(*self.args, **hints)

```

6.2.5 Source Codes for Calculation of the Numbers q, q_1, \dots, q_{15}

Listing 109: Calculation of the numbers q, q_1, \dots, q_{15}

```

1  from mathematics.sde.nonlinear.symbolic.coefficients.c import C
2
3
4  def solve_q(i):
5      """
6      Calculates value for iteration of loop
7      Parameters
8      =====
9      i : int
10     amount of members
11     Returns
12     =====
13     values : float
14     value for iteration of loop that calculates amount of q
15     """
16     return 1 / 4 - 1 / 2 * sum([
17         1 / (4 * j ** 2 - 1)
18         for j in range(1, i + 1)
19     ])
20
21
22 def solve_q1(i):
23     """
24     Calculates value for iteration of loop
25     Parameters
26     =====
27     i : int
28     amount of members
29     Returns
30     =====
31     values : float
32     value for iteration of loop that calculates amount of q
33     """
34     return 1 / 6 - 1 / 64 * sum([
35         (2 * j1 + 1) *
36         (2 * j2 + 1) *
37         (2 * j3 + 1) *
38         C((j1, j2, j3), (0, 0, 0)) ** 2
39         for j1 in range(i + 1)
40         for j2 in range(i + 1)
41         for j3 in range(i + 1)
42     ])

```

```

43
44
45 def solve_q2(i):
46     """
47     Calculates value for iteration of loop
48     Parameters
49     =====
50     i : int
51     amount of members
52     Returns
53     =====
54     values : float
55     value for iteration of loop that calculates amount of q
56     """
57     return 1 / 4 - 1 / 64 * sum([
58         (2 * j1 + 1) *
59         (2 * j2 + 1) *
60         (C((j1, j2), (0, 1)) ** 2)
61         for j1 in range(i + 1)
62         for j2 in range(i + 1)
63     ])
64
65
66 def solve_q2_optional(i):
67     """
68     Calculates value for iteration of loop
69     Parameters
70     =====
71     i : int
72     amount of members
73     Returns
74     =====
75     values : float
76     value for iteration of loop that calculates amount of q
77     """
78     return 1 / 12 - 1 / 64 * sum([
79         (2 * j1 + 1) *
80         (2 * j2 + 1) *
81         (C((j1, j2), (1, 0)) ** 2)
82         for j1 in range(i + 1)
83         for j2 in range(i + 1)
84     ])
85
86
87 def solve_q3(i):
88     """
89     Calculates value for iteration of loop
90     Parameters
91     =====
92     i : int
93     amount of members
94     Returns
95     =====
96     values : float
97     value for iteration of loop that calculates amount of q

```

```

98     """
99     return 1 / 24 - 1 / 256 * sum([
100         (2 * j1 + 1) *
101         (2 * j2 + 1) *
102         (2 * j3 + 1) *
103         (2 * j4 + 1) *
104         (C((j1, j2, j3, j4), (0, 0, 0, 0)) ** 2)
105         for j1 in range(i + 1)
106         for j2 in range(i + 1)
107         for j3 in range(i + 1)
108         for j4 in range(i + 1)
109     ])
110
111
112 def solve_q4(i):
113     """
114     Calculates value for iteration of loop
115     Parameters
116     =====
117     i : int
118         amount of members
119     Returns
120     =====
121     values : float
122         value for iteration of loop that calculates amount of q
123     """
124     return 1 / 120 - 1 / (32 ** 2) * sum([
125         (2 * j1 + 1) *
126         (2 * j2 + 1) *
127         (2 * j3 + 1) *
128         (2 * j4 + 1) *
129         (2 * j5 + 1) *
130         (C((j1, j2, j3, j4, j5), (0, 0, 0, 0, 0)) ** 2)
131         for j1 in range(i + 1)
132         for j2 in range(i + 1)
133         for j3 in range(i + 1)
134         for j4 in range(i + 1)
135         for j5 in range(i + 1)
136     ])
137
138
139 def solve_q5(i):
140     """
141     Calculates value for iteration of loop
142     Parameters
143     =====
144     i : int
145         amount of members
146     Returns
147     =====
148     values : float
149         value for iteration of loop that calculates amount of q
150     """
151     return 1 / 60 - 1 / 256 * sum([
152         (2 * j1 + 1) *

```

```

153     (2 * j2 + 1) *
154     (C((j1, j2), (2, 0)) ** 2)
155     for j1 in range(i + 1)
156     for j2 in range(i + 1)
157 ])
158
159
160 def solve_q6(i):
161     """
162     Calculates value for iteration of loop
163     Parameters
164     =====
165     i : int
166         amount of members
167     Returns
168     =====
169     values : float
170         value for iteration of loop that calculates amount of q
171     """
172     return 1 / 18 - 1 / 256 * sum([
173         (2 * j1 + 1) *
174         (2 * j2 + 1) *
175         (C((j1, j2), (1, 1)) ** 2)
176         for j1 in range(i + 1)
177         for j2 in range(i + 1)
178     ])
179
180
181 def solve_q7(i):
182     """
183     Calculates value for iteration of loop
184     Parameters
185     =====
186     i : int
187         amount of members
188     Returns
189     =====
190     values : float
191         value for iteration of loop that calculates amount of q
192     """
193     return 1 / 6 - 1 / 256 * sum([
194         (2 * j1 + 1) *
195         (2 * j2 + 1) *
196         (C((j1, j2), (0, 2)) ** 2)
197         for j1 in range(i + 1)
198         for j2 in range(i + 1)
199     ])
200
201
202 def solve_q8(i):
203     """
204     Calculates value for iteration of loop
205     Parameters
206     =====
207     i : int

```

```

208     amount of members
209     Returns
210     =====
211     values : float
212     value for iteration of loop that calculates amount of q
213     """
214     return 1 / 10 - 1 / 256 * sum([
215         (2 * j1 + 1) *
216         (2 * j2 + 1) *
217         (2 * j3 + 1) *
218         (C((j1, j2, j3), (0, 0, 1)) ** 2)
219         for j1 in range(i + 1)
220         for j2 in range(i + 1)
221         for j3 in range(i + 1)
222     ])
223
224
225 def solve_q9(i):
226     """
227     Calculates value for iteration of loop
228     Parameters
229     =====
230     i : int
231     amount of members
232     Returns
233     =====
234     values : float
235     value for iteration of loop that calculates amount of q
236     """
237     return 1 / 20 - 1 / 256 * sum([
238         (2 * j1 + 1) *
239         (2 * j2 + 1) *
240         (2 * j3 + 1) *
241         (C((j1, j2, j3), (0, 1, 0)) ** 2)
242         for j1 in range(i + 1)
243         for j2 in range(i + 1)
244         for j3 in range(i + 1)
245     ])
246
247
248 def solve_q10(i):
249     """
250     Calculates value for iteration of loop
251     Parameters
252     =====
253     i : int
254     amount of members
255     Returns
256     =====
257     values : float
258     value for iteration of loop that calculates amount of q
259     """
260     return 1 / 60 - 1 / 256 * sum([
261         (2 * j1 + 1) *
262         (2 * j2 + 1) *

```

```

263     (2 * j3 + 1) *
264     (C((j1, j2, j3), (1, 0, 0)) ** 2)
265     for j1 in range(i + 1)
266     for j2 in range(i + 1)
267     for j3 in range(i + 1)
268 ])
269
270
271 def solve_q11(i):
272     """
273     Calculates value for iteration of loop
274     Parameters
275     =====
276     i : int
277         amount of members
278     Returns
279     =====
280     values : float
281         value for iteration of loop that calculates amount of q
282     """
283     return 1 / 36 - 1 / (32 ** 2) * sum([
284         (2 * j1 + 1) *
285         (2 * j2 + 1) *
286         (2 * j3 + 1) *
287         (2 * j4 + 1) *
288         (C((j1, j2, j3, j4), (0, 0, 0, 1)) ** 2)
289         for j1 in range(i + 1)
290         for j2 in range(i + 1)
291         for j3 in range(i + 1)
292         for j4 in range(i + 1)
293     ])
294
295
296 def solve_q12(i):
297     """
298     Calculates value for iteration of loop
299     Parameters
300     =====
301     i : int
302         amount of members
303     Returns
304     =====
305     values : float
306         value for iteration of loop that calculates amount of q
307     """
308     return 1 / 60 - 1 / (32 ** 2) * sum([
309         (2 * j1 + 1) *
310         (2 * j2 + 1) *
311         (2 * j3 + 1) *
312         (2 * j4 + 1) *
313         (C((j1, j2, j3, j4), (0, 0, 1, 0)) ** 2)
314         for j1 in range(i + 1)
315         for j2 in range(i + 1)
316         for j3 in range(i + 1)
317         for j4 in range(i + 1)

```

```

318     ])
319
320
321 def solve_q13(i):
322     """
323     Calculates value for iteration of loop
324     Parameters
325     =====
326     i : int
327     amount of members
328     Returns
329     =====
330     values : float
331     value for iteration of loop that calculates amount of q
332     """
333     return 1 / 120 - 1 / (32 ** 2) * sum([
334         (2 * j1 + 1) *
335         (2 * j2 + 1) *
336         (2 * j3 + 1) *
337         (2 * j4 + 1) *
338         (C((j1, j2, j3, j4), (0, 1, 0, 0)) ** 2)
339         for j1 in range(i + 1)
340         for j2 in range(i + 1)
341         for j3 in range(i + 1)
342         for j4 in range(i + 1)
343     ])
344
345
346 def solve_q14(i):
347     """
348     Calculates value for iteration of loop
349     Parameters
350     =====
351     i : int
352     amount of members
353     Returns
354     =====
355     values : float
356     value for iteration of loop that calculates amount of q
357     """
358     return 1 / 360 - 1 / (32 ** 2) * sum([
359         (2 * j1 + 1) *
360         (2 * j2 + 1) *
361         (2 * j3 + 1) *
362         (2 * j4 + 1) *
363         (C((j1, j2, j3, j4), (1, 0, 0, 0)) ** 2)
364         for j1 in range(i + 1)
365         for j2 in range(i + 1)
366         for j3 in range(i + 1)
367         for j4 in range(i + 1)
368     ])
369
370
371 def solve_q15(i):
372     """

```

```

373 Calculates value for iteration of loop
374 Parameters
375 =====
376 i : int
377 amount of members
378 Returns
379 =====
380 values : float
381 value for iteration of loop that calculates amount of q
382 """
383 return 1 / 720 - 1 / (64 ** 2) * sum([
384     (2 * j1 + 1) *
385     (2 * j2 + 1) *
386     (2 * j3 + 1) *
387     (2 * j4 + 1) *
388     (2 * j5 + 1) *
389     (2 * j6 + 1) *
390     (C((j1, j2, j3, j4, j5, j6), (0, 0, 0, 0, 0, 0)) ** 2)
391     for j1 in range(i + 1)
392     for j2 in range(i + 1)
393     for j3 in range(i + 1)
394     for j4 in range(i + 1)
395     for j5 in range(i + 1)
396     for j6 in range(i + 1)
397 ])
398
399
400 solvers = [
401     solve_q, solve_q1, solve_q2, solve_q3, solve_q8,
402     solve_q9, solve_q10, solve_q4, solve_q7, solve_q6,
403     solve_q5, solve_q11, solve_q12, solve_q13,
404     solve_q14, solve_q15,
405 ]
406
407 dt_degrees = [
408     [1],
409     [2, 1],
410     [3, 2, 1, 1],
411     [4, 3, 2, 2, 1, 1, 1, 1],
412     [5, 4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1],
413 ]
414
415 q_ranges = [
416     1, 2, 4, 8, 16
417 ]
418
419
420 def loop(dt: float, k: float, degree: int, solver):
421     """
422     Loop that chooses amount of q that provides necessary accuracy
423     Parameters
424     =====
425     dt : float
426     delta time
427     k : float

```



```

428     user chosen coefficient of accuracy
429     degree : int
430     degree of dt depending on q
431     solver : function
432     function that
433     Returns
434     =====
435     i : int
436     amount of q
437     """
438     i = 0
439     while True:
440         if solver(i) <= k * dt ** degree:
441             break
442         i += 1
443     return i
444
445
446 def get_q(dt: float, k: float, r: float):
447     """
448     Iterates solvers and get q values necessary to
449     achieve given accuracy
450     Parameters
451     =====
452     dt: float
453         integration step
454     k: float
455         user chosen coefficient of accuracy
456     r: float
457         strong numerical scheme order
458     Returns
459     =====
460     qs_result: tuple
461         q values
462     """
463     qs_result = []
464
465     degree = int(r * 2)
466     range_id = degree - 2
467
468     for q_id in range(q_ranges[range_id]):
469         qs_result.append(loop(dt, k, dt_degrees[range_id][q_id], solvers[q_id]))
470
471     return tuple(qs_result)

```

6.2.6 Source Codes for Strong Taylor–Itô Numerical Schemes with Convergence Orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs

Listing 110: Euler scheme modeling subprogram

```

1 import logging

```

```

2 from time import time
3
4 import numpy as np
5 from sympy import lambdify, Matrix, symbols, MatrixSymbol, Symbol
6
7 from mathematics.sde.nonlinear.symbolic.schemes.euler import Euler
8
9
10 def euler(y0: np.array, a: Matrix, b: Matrix, times: tuple):
11     """
12     Performs modeling of Euler scheme
13     Parameters
14     =====
15     y0 : numpy.ndarray
16         initial conditions
17     a : numpy.ndarray
18         vector function a
19     b : numpy.ndarray
20         matrix function b
21     times : tuple
22         integration limits and step
23     Returns
24     =====
25     y : numpy.ndarray
26         vector of solution
27     t : list
28         list of time moments
29     """
30     start_time = time()
31
32     logger = logging.getLogger(__name__)
33
34     logger.info(f"[{(time() - start_time):.3f} seconds] Euler start")
35
36     # Ranges
37     n = b.shape[0]
38     m = b.shape[1]
39     t1 = times[0]
40     dt = times[1]
41     t2 = times[2]
42
43     # Defining context
44     args = symbols(f"x1:{n + 1}")
45     ticks = int((t2 - t1) / dt)
46
47     # Symbols
48     sym_i, sym_t = Symbol("i"), Symbol("t")
49     sym_ksi = MatrixSymbol("ksi", 1, m)
50     sym_y = Euler(sym_i, Matrix(args), a, b, dt, sym_ksi)
51
52     args_extended = list()
53     args_extended.extend(args)
54     args_extended.extend([sym_t, sym_ksi])
55
56     # Compilation of formulas

```

```

57 y_compiled = list()
58 for tr in range(n):
59     y_compiled.append(lambdify(args_extended, sym.y.subs(sym.i, tr), "numpy"))
60
61 logger.info(f"[{(time() - start_time):.3f} seconds] Euler subs are finished")
62
63 # Substitution values
64 t = [t1 + i * dt for i in range(ticks)]
65 y = np.zeros((n, ticks))
66 y[:, 0] = y0[:, 0]
67
68 # Dynamic substitutions with integration
69 for p in range(ticks - 1):
70     values = [*y[:, p], t[p], np.random.randn(1, m)]
71     for tr in range(n):
72         y[tr, p + 1] = y_compiled[tr](*values)
73
74 logger.info(f"[{(time() - start_time):.3f} seconds] Euler calculations are finished")
75
76 return y, t

```

Listing 111: Euler scheme

```

1 from sympy import Function, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
4
5
6 class Euler(Function):
7     """
8     Euler scheme
9     """
10    nargs = 6
11
12    def __new__(cls, *args, **kwargs):
13        """
14        Creates new Euler object with given args
15        Parameters
16        =====
17        i : int
18            component of stochastic process
19        yp : numpy.ndarray
20            initial conditions
21        a : numpy.ndarray
22            algebraic, given in the variables x and t
23        b : numpy.ndarray
24            algebraic, given in the variables x and t
25        dt : float
26            integration step
27        ksi : numpy.ndarray
28            matrix of Gaussian random variables
29        Returns
30        =====

```

```

31     sympy.Expr
32     formula to simplify and substitute
33     """
34     i, yp, a, b, dt, ksi = sympify(args)
35     n, m = b.shape[0], b.shape[1]
36
37     return Add(
38
39         yp[i, 0], a[i, 0] * dt,
40
41         *[b[i, i1] * I0(i1, dt, ksi)
42           for i1 in range(m)]
43
44     )
45
46     def doit(self, **hints):
47         """
48         Tries to expand or calculate function
49         Returns
50         =====
51         sympy.Expr
52         """
53         return Euler(*self.args, **hints)

```

Listing 112: Milstein scheme modeling subprogram

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import symbols, MatrixSymbol, Matrix, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.milstein import Milstein
9
10
11 def milstein(y0: np.ndarray, a: Matrix, b: Matrix, k: float, times: tuple):
12     """
13     Performs modeling of Milstein scheme
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     q : tuple
23         amount of independent random variables
24     times : tuple
25         integration limits and step
26     Returns
27     =====

```

```

28  y : numpy.ndarray
29      vector of solution
30  t : list
31      list of time moments
32  """
33  start_time = time()
34
35  logger = logging.getLogger(__name__)
36
37  logger.info(f"[{(time() - start_time):.3f} seconds] Milstein start")
38
39  # Ranges
40  n = b.shape[0]
41  m = b.shape[1]
42  t1 = times[0]
43  dt = times[1]
44  t2 = times[2]
45
46  # Defining context
47  args = symbols(f"x1:{n + 1}")
48  ticks = int((t2 - t1) / dt)
49  q = get_q(dt, k, 1)
50  logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51  logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52  logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54  # Symbols
55  sym_i, sym_t = symbols("i t")
56  sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
57  sym_y = Milstein(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59  args_extended = list()
60  args_extended.extend(args)
61  args_extended.extend([sym_t, sym_ksi])
62
63  # Compilation of formulas
64  y_compiled = list()
65  for tr in range(n):
66      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68  logger.info(f"[{(time() - start_time):.3f} seconds] Milstein subs are finished")
69
70  # Substitution values
71  t = [t1 + i * dt for i in range(ticks)]
72  y = np.zeros((n, ticks))
73  y[:, 0] = y0[:, 0]
74
75  # Dynamic substitutions with integration
76  for p in range(ticks - 1):
77      values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
78      for tr in range(n):
79          y[tr, p + 1] = y_compiled[tr](*values)
80
81  logger.info(f"[{(time() - start_time):.3f} seconds] Milstein calculations are finished")
82  )

```

```

82
83 return y, t

```

Listing 113: Milstein scheme

```

1 from sympy import Function, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.g import G
4 from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
5 from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
6
7
8 class Milstein(Function):
9     """
10    Milstein scheme
11    """
12    nargs = 8
13
14    def __new__(cls, *args, **kwargs):
15        """
16        Creates new Milstein object with given args
17        Parameters
18        =====
19        i : int
20        component of stochastic process
21        yp : numpy.ndarray
22        initial conditions
23        a : numpy.ndarray
24        algebraic, given in the variables x and t
25        b : numpy.ndarray
26        algebraic, given in the variables x and t
27        dt : float
28        integration step
29        ksi : numpy.ndarray
30        matrix of Gaussian random variables
31        q : tuple
32        amounts of q for integrals approximations
33        Returns
34        =====
35        sympy.Expr
36        formula to simplify and substitute
37        """
38        i, yp, a, b, dt, ksi, dxs, q = sympify(args)
39        n, m = b.shape[0], b.shape[1]
40
41        return Add(
42
43            yp[i, 0], a[i, 0] * dt,
44
45            *[b[i, i1] * I0(i1, dt, ksi)
46              for i1 in range(m)],
47
48            *[G(b[:, i1], b[i, i2], dxs) *

```

```

49     I00(i1, i2, q[0], dt, ksi)
50     for i2 in range(m)
51     for i1 in range(m)]
52
53 )
54
55 def doit(self, **hints):
56     """
57     Tries to expand or calculate function
58     Returns
59     =====
60     sympy.Expr
61     """
62     return Milstein(*self.args, **hints)

```

Listing 114: Strong Taylor–Itô scheme with convergence order 1.5 modeling subprogram

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import Matrix, symbols, MatrixSymbol, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_1p5 import
   StrongTaylorIto1p5
9
10
11 def strong_taylor_ito_1p5(y0: np.ndarray, a: Matrix, b: Matrix, k: float, times: tuple):
12     """
13     Performs modeling of Strong Taylor–Ito scheme with convergence order 1.5
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple
25         integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29         vector of solution
30     t : list
31         list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)

```

```

36
37 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 1.5 start")
38
39 # Ranges
40 n = b.shape[0]
41 m = b.shape[1]
42 t1 = times[0]
43 dt = times[1]
44 t2 = times[2]
45
46 # Defining context
47 args = symbols(f"x1:{n + 1}")
48 ticks = int((t2 - t1) / dt)
49 q = get_q(dt, k, 1.5)
50 logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51 logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52 logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54 # Symbols
55 sym_i, sym_t = symbols("i t")
56 sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
57 sym_y = StrongTaylorIto1p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59 args_extended = list()
60 args_extended.extend(args)
61 args_extended.extend([sym_t, sym_ksi])
62
63 # Compilation of formulas
64 y_compiled = list()
65 for tr in range(n):
66     y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 1.5 subs are
69     finished")
70
71 # Substitution values
72 t = [t1 + i * dt for i in range(ticks)]
73 y = np.zeros((n, ticks))
74 y[:, 0] = y0[:, 0]
75
76 # Dynamic substitutions with integration
77 for p in range(ticks - 1):
78     values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
79     for tr in range(n):
80         y[tr, p + 1] = y_compiled[tr>(*values)
81
82 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 1.5 calculations
83     are finished")
84
85 return y, t

```

Listing 115: Strong Taylor–Itô scheme with convergence order 1.5

```

1 from sympy import Function, sympify, Add

```



```

2
3 from mathematics.sde.nonlinear.symbolic.g import G
4 from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
5 from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
6 from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
7 from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
8 from mathematics.sde.nonlinear.symbolic.l import L
9
10
11 class StrongTaylorIto1p5(Function):
12     """
13     Strong Taylor–Ito scheme with convergence order 1.5
14     """
15     nargs = 8
16
17     def __new__(cls, *args, **kwargs):
18         """
19         Creates new StrongTaylorIto1p5 object with given args
20         Parameters
21         =====
22         i : int
23             component of stochastic process
24         yp : numpy.ndarray
25             initial conditions
26         a : numpy.ndarray
27             algebraic, given in the variables x and t
28         b : numpy.ndarray
29             algebraic, given in the variables x and t
30         dt : float
31             integration step
32         ksi : numpy.ndarray
33             matrix of Gaussian random variables
34         q : tuple
35             amounts of q for stochastic integrals approximations
36         Returns
37         =====
38         sympy.Expr
39             formula to simplify and substitute
40         """
41         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
42         n, m = b.shape[0], b.shape[1]
43
44         return Add(
45
46             yp[i, 0], a[i, 0] * dt,
47
48             *[b[i, i1] * I0(i1, dt, ksi)
49               for i1 in range(m)],
50
51             *[G(b[:, i1], b[i, i2], dxs) *
52               I00(i1, i2, q[0], dt, ksi)
53                 for i2 in range(m)
54                 for i1 in range(m)],
55
56             *[G(b[:, i1], a[i, 0], dxs) *

```

```

57     (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
58     L(a, b, b[i, i1], dxs) *
59     I1(i1, dt, ksi)
60     for i1 in range(m)],
61
62     *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
63     I000(i1, i2, i3, q[1], dt, ksi)
64     for i3 in range(m)
65     for i2 in range(m)
66     for i1 in range(m)],
67
68     dt ** 2 / 2 * L(a, b, a[i, 0], dxs)
69
70 )
71
72 def doit(self, **hints):
73     """
74     Tries to expand or calculate function
75     Returns
76     =====
77     sympy.Expr
78     """
79     return StrongTaylorIto1p5(*self.args, **hints)

```

Listing 116: Strong Taylor–Itô scheme with convergence order 2.0 modeling subprogram

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import symbols, Matrix, MatrixSymbol, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_2p0 import
9      StrongTaylorIto2p0
10
11 def strong_taylor_ito_2p0(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
12     """
13     Performs modeling of Strong Taylor–Ito scheme with convergence order 2.0
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple
25         integration limits and step

```

```

26 Returns
27 =====
28 y : numpy.ndarray
29 vector of solution
30 t : list
31 list of time moments
32 """
33 start_time = time()
34
35 logger = logging.getLogger(__name__)
36
37 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.0 start")
38
39 # Ranges
40 n = b.shape[0]
41 m = b.shape[1]
42 t1 = times[0]
43 dt = times[1]
44 t2 = times[2]
45
46 # Defining context
47 args = symbols(f"x1:{n + 1}")
48 ticks = int((t2 - t1) / dt)
49 q = get_q(dt, k, 2)
50 logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51 logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52 logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54 # Symbols
55 sym_i, sym_t = symbols("i t")
56 sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
57 sym_y = StrongTaylorIto2p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59 args_extended = list()
60 args_extended.extend(args)
61 args_extended.extend([sym_t, sym_ksi])
62
63 # Compilation of formulas
64 y_compiled = list()
65 for tr in range(n):
66     y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.0 subs are
69     finished")
70
71 # Substitution values
72 t = [t1 + i * dt for i in range(ticks)]
73 y = np.zeros((n, ticks))
74 y[:, 0] = y0[:, 0]
75
76 # Dynamic substitutions with integration
77 for p in range(ticks - 1):
78     values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
79     for tr in range(n):
80         y[tr, p + 1] = y_compiled[tr>(*values)

```

```

80
81     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.0 calculations
      are finished")
82
83     return y, t

```

Listing 117: Strong Taylor–Itô scheme with convergence order 2.0

```

1  from sympy import Function, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.g import G
4  from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
5  from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
6  from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
7  from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
8  from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
9  from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
10 from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
11 from mathematics.sde.nonlinear.symbolic.l import L
12
13
14 class StrongTaylorIto2p0(Function):
15     """
16     Strong Taylor–Ito scheme with convergence order 2.0
17     """
18     nargs = 8
19
20     def __new__(cls, *args, **kwargs):
21         """
22         Creates new StrongTaylorIto2p0 object with given args
23         Parameters
24         =====
25         i : int
26             component of stochastic process
27         yp : numpy.ndarray
28             initial conditions
29         a : numpy.ndarray
30             algebraic, given in the variables x and t
31         b : numpy.ndarray
32             algebraic, given in the variables x and t
33         dt : float
34             integration step
35         ksi : numpy.ndarray
36             matrix of Gaussian random variables
37         q : tuple
38             amounts of q for stochastic integrals approximations
39         Returns
40         =====
41         sympy.Expr
42             formula to simplify and substitute
43         """
44         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
45         n, m = b.shape[0], b.shape[1]

```

```

46
47     return Add(
48
49         yp[i, 0], a[i, 0] * dt,
50
51         *[b[i, i1] * I0(i1, dt, ksi)
52           for i1 in range(m)],
53
54         *[G(b[:, i1], b[i, i2], dxs) *
55           I00(i1, i2, q[0], dt, ksi)
56           for i2 in range(m)
57           for i1 in range(m)],
58
59         *[G(b[:, i1], a[i, 0], dxs) *
60           (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
61           L(a, b, b[i, i1], dxs) *
62           I1(i1, dt, ksi)
63           for i1 in range(m)],
64
65         *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
66           I000(i1, i2, i3, q[1], dt, ksi)
67           for i3 in range(m)
68           for i2 in range(m)
69           for i1 in range(m)],
70
71         dt ** 2 / 2 * L(a, b, a[i, 0], dxs),
72
73         *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
74           (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
75           L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
76           G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
77           (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
78           for i2 in range(m)
79           for i1 in range(m)],
80
81         *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
82           I0000(i1, i2, i3, i4, q[3], dt, ksi)
83           for i4 in range(m)
84           for i3 in range(m)
85           for i2 in range(m)
86           for i1 in range(m)]
87
88     )
89
90     def doit(self, **hints):
91         """
92         Tries to expand or calculate function
93         Returns
94         =====
95         sympy.Expr
96         """
97         return StrongTaylorIto2p0(*self.args, **hints)

```

Listing 118: Strong Taylor–Itô scheme with convergence order 2.5 modeling subprogram

```

1 import logging
2 from time import time
3
4 import numpy as np
5 from sympy import symbols, Matrix, MatrixSymbol, lambdify
6
7 from mathematics.sde.nonlinear.q import get_q
8 from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_2p5 import
   StrongTaylorIto2p5
9
10
11 def strong_taylor_ito_2p5(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
12     """
13     Performs modeling of Strong Taylor–Ito scheme with convergence order 2.5
14     Parameters
15     =====
16     y0 : numpy.ndarray
17     initial conditions
18     a : numpy.ndarray
19     vector function a
20     b : numpy.ndarray
21     matrix function b
22     k : float
23     precision constant
24     times : tuple
25     integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29     vector of solution
30     t : list
31     list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.5 start")
38
39     # Ranges
40     n = b.shape[0]
41     m = b.shape[1]
42     t1 = times[0]
43     dt = times[1]
44     t2 = times[2]
45
46     # Defining context
47     args = symbols(f"x1:{n + 1}")
48     ticks = int((t2 - t1) / dt)
49     q = get_q(dt, k, 2.5)
50     logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")

```

```

51 logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52 logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54 # Symbols
55 sym_i, sym_t = symbols("i t")
56 sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
57 sym_y = StrongTaylorIto2p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59 args_extended = list()
60 args_extended.extend(args)
61 args_extended.extend([sym_t, sym_ksi])
62
63 # Compilation of formulas
64 y_compiled = list()
65 for tr in range(n):
66     y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.5 subs are
69     finished")
70
71 # Substitution values
72 t = [t1 + i * dt for i in range(ticks)]
73 y = np.zeros((n, ticks))
74 y[:, 0] = y0[:, 0]
75
76 # Dynamic substitutions with integration
77 for p in range(ticks - 1):
78     values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
79     for tr in range(n):
80         y[tr, p + 1] = y_compiled[tr>(*values)
81
82 logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 2.5 calculations
83     are finished")
84
85 return y, t

```

Listing 119: Strong Taylor–Itô scheme with convergence order 2.5

```

1 from sympy import Function, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.g import G
4 from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
5 from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
6 from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
7 from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
8 from mathematics.sde.nonlinear.symbolic.ito.i00000 import I00000
9 from mathematics.sde.nonlinear.symbolic.ito.i001 import I001
10 from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
11 from mathematics.sde.nonlinear.symbolic.ito.i010 import I010
12 from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
13 from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
14 from mathematics.sde.nonlinear.symbolic.ito.i100 import I100
15 from mathematics.sde.nonlinear.symbolic.ito.i2 import I2

```

```

16 from mathematics.sde.nonlinear.symbolic.l import L
17
18
19 class StrongTaylorIto2p5(Function):
20     """
21     Strong Taylor–Ito scheme with convergence order 2.5
22     """
23     nargs = 8
24
25     def __new__(cls, *args, **kwargs):
26         """
27         Creates new StrongTaylorIto2p5 object with given args
28         Parameters
29         =====
30         i : int
31             component of stochastic process
32         yp : numpy.ndarray
33             initial conditions
34         a : numpy.ndarray
35             algebraic, given in the variables x and t
36         b : numpy.ndarray
37             algebraic, given in the variables x and t
38         dt : float
39             integration step
40         ksi : numpy.ndarray
41             matrix of Gaussian random variables
42         q : tuple
43             amounts of q for stochastic integrals approximations
44         Returns
45         =====
46         sympy.Expr
47             formula to simplify and substitute
48         """
49         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
50         n, m = b.shape[0], b.shape[1]
51
52         return Add(
53
54             yp[i, 0], a[i, 0] * dt,
55
56             *[b[i, i1] * I0(i1, dt, ksi)
57               for i1 in range(m)],
58
59             *[G(b[:, i1], b[i, i2], dxs) *
60               I0(i1, i2, q[0], dt, ksi)
61               for i2 in range(m)
62               for i1 in range(m)],
63
64             *[G(b[:, i1], a[i, 0], dxs) *
65               (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
66               L(a, b, b[i, i1], dxs) *
67               I1(i1, dt, ksi)
68               for i1 in range(m)],
69
70             *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *

```



```

71     I000(i1, i2, i3, q[1], dt, ksi)
72     for i3 in range(m)
73     for i2 in range(m)
74     for i1 in range(m)],
75
76     dt ** 2 / 2 * L(a, b, a[i, 0], dxs),
77     *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
78       (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
79       L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
80       G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
81       (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
82     for i2 in range(m)
83     for i1 in range(m)],
84
85     *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
86       I0000(i1, i2, i3, i4, q[3], dt, ksi)
87     for i4 in range(m)
88     for i3 in range(m)
89     for i2 in range(m)
90     for i1 in range(m)],
91
92     *[G(b[:, i1], L(a, b, a[i, 0], dxs), dxs) *
93       (I2(i1, dt, ksi) / 2 + dt * I1(i1, dt, ksi) + dt ** 2 / 2 * I0(i1, dt, ksi)) +
94       L(a, b, L(a, b, b[i, i1], dxs), dxs) * I2(i1, dt, ksi) / 2 -
95       L(a, b, G(b[:, i1], a[i, 0], dxs), dxs) * (I2(i1, dt, ksi) + dt * I1(i1, dt, ksi))
96     for i1 in range(m)],
97
98     *[G(b[:, i1], L(a, b, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
99       (I100(i1, i2, i3, q[6], dt, ksi) - I010(i1, i2, i3, q[5], dt, ksi)) +
100     G(b[:, i1], G(b[:, i2], L(a, b, b[i, i3], dxs), dxs), dxs) *
101     (I010(i1, i2, i3, q[5], dt, ksi) - I001(i1, i2, i3, q[4], dt, ksi)) +
102     G(b[:, i1], G(b[:, i2], G(b[:, i3], a[i, 0], dxs), dxs), dxs) *
103     (dt * I000(i1, i2, i3, q[1], dt, ksi) + I001(i1, i2, i3, q[4], dt, ksi)) -
104     L(a, b, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
105     I100(i1, i2, i3, q[6], dt, ksi)
106     for i3 in range(m)
107     for i2 in range(m)
108     for i1 in range(m)],
109
110     *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
111     b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
112     I00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
113     for i5 in range(m)
114     for i4 in range(m)
115     for i3 in range(m)
116     for i2 in range(m)
117     for i1 in range(m)],
118
119     dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs)
120
121 )
122
123 def doit(self, **hints):
124     """
125     Tries to expand or calculate function

```

```

126     Returns
127     =====
128     sympy.Expr
129     """
130     return StrongTaylorIto2p5(*self.args, **hints)

```

Listing 120: **Strong Taylor–Itô** scheme with convergence order 3.0 modeling subprogram

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import Matrix, symbols, MatrixSymbol, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_ito_3p0 import
   StrongTaylorIto3p0
9
10
11 def strong_taylor_ito_3p0(y0: np.array, a: Matrix, b: Matrix, k: float, times: tuple):
12     """
13     Performs modeling of Strong Taylor–Ito scheme with convergence order 3.0
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple
25         integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29         vector of solution
30     t : list
31         list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 3.0 start")
38
39     # Ranges
40     n = b.shape[0]
41     m = b.shape[1]
42     t1 = times[0]
43     dt = times[1]

```

```

44     t2 = times[2]
45
46     # Defining context
47     args = symbols(f"x1:{n + 1}")
48     ticks = int((t2 - t1) / dt)
49     q = get_q(dt, k, 3)
50     logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51     logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52     logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54     # Symbols
55     sym_i, sym_t = symbols("i t")
56     sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
57     sym_y = StrongTaylorIto3p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59     args_extended = list()
60     args_extended.extend(args)
61     args_extended.extend([sym_t, sym_ksi])
62
63     # Compilation of formulas
64     y_compiled = list()
65     for tr in range(n):
66         y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 3.0 subs are
69         finished")
70
71     # Substitution values
72     t = [t1 + i * dt for i in range(ticks)]
73     y = np.zeros((n, ticks))
74     y[:, 0] = y0[:, 0]
75
76     # Dynamic substitutions with integration
77     for p in range(ticks - 1):
78         values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
79         for tr in range(n):
80             y[tr, p + 1] = y_compiled[tr](*values)
81
82     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Ito 3.0 calculations
83         are finished")
84
85     return y, t

```

Listing 121: Strong Taylor–Itô scheme with convergence order 3.0

```

1 from sympy import Function, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.g import G
4 from mathematics.sde.nonlinear.symbolic.ito.i0 import I0
5 from mathematics.sde.nonlinear.symbolic.ito.i00 import I00
6 from mathematics.sde.nonlinear.symbolic.ito.i000 import I000
7 from mathematics.sde.nonlinear.symbolic.ito.i0000 import I0000
8 from mathematics.sde.nonlinear.symbolic.ito.i00000 import I00000

```

```

9 from mathematics.sde.nonlinear.symbolic.ito.i000000 import I000000
10 from mathematics.sde.nonlinear.symbolic.ito.i0001 import I0001
11 from mathematics.sde.nonlinear.symbolic.ito.i001 import I001
12 from mathematics.sde.nonlinear.symbolic.ito.i0010 import I0010
13 from mathematics.sde.nonlinear.symbolic.ito.i01 import I01
14 from mathematics.sde.nonlinear.symbolic.ito.i010 import I010
15 from mathematics.sde.nonlinear.symbolic.ito.i0100 import I0100
16 from mathematics.sde.nonlinear.symbolic.ito.i02 import I02
17 from mathematics.sde.nonlinear.symbolic.ito.i1 import I1
18 from mathematics.sde.nonlinear.symbolic.ito.i10 import I10
19 from mathematics.sde.nonlinear.symbolic.ito.i100 import I100
20 from mathematics.sde.nonlinear.symbolic.ito.i1000 import I1000
21 from mathematics.sde.nonlinear.symbolic.ito.i11 import I11
22 from mathematics.sde.nonlinear.symbolic.ito.i2 import I2
23 from mathematics.sde.nonlinear.symbolic.ito.i20 import I20
24 from mathematics.sde.nonlinear.symbolic.l import L
25
26
27 class StrongTaylorIto3p0(Function):
28     """
29     Strong Taylor–Ito scheme with convergence order 3.0
30     """
31     nargs = 8
32
33     def __new__(cls, *args, **kwargs):
34         """
35         Creates new StrongTaylorIto3p0 object with given args
36         Parameters
37         =====
38         i : int
39         component of stochastic process
40         yp : numpy.ndarray
41         initial conditions
42         a : numpy.ndarray
43         algebraic, given in the variables x and t
44         b : numpy.ndarray
45         algebraic, given in the variables x and t
46         dt : float
47         integration step
48         ksi : numpy.ndarray
49         matrix of Gaussian random variables
50         q : tuple
51         amounts of q for stochastic integrals approximations
52         Returns
53         =====
54         sympy.Expr
55         formula to simplify and substitute
56         """
57         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
58         n, m = b.shape[0], b.shape[1]
59
60         return Add(
61
62             yp[i, 0], a[i, 0] * dt,
63

```

```

64      *[b[i, i1] * I0(i1, dt, ksi)
65         for i1 in range(m)],
66
67      *[G(b[:, i1], b[i, i2], dxs) *
68         I00(i1, i2, q[0], dt, ksi)
69         for i2 in range(m)
70         for i1 in range(m)],
71
72      *[G(b[:, i1], a[i, 0], dxs) *
73         (dt * I0(i1, dt, ksi) + I1(i1, dt, ksi)) -
74         L(a, b, b[i, i1], dxs) *
75         I1(i1, dt, ksi)
76         for i1 in range(m)],
77
78      *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
79         I000(i1, i2, i3, q[1], dt, ksi)
80         for i3 in range(m)
81         for i2 in range(m)
82         for i1 in range(m)],
83
84      dt ** 2 / 2 * L(a, b, a[i, 0], dxs),
85
86      *[G(b[:, i1], L(a, b, b[i, i2], dxs), dxs) *
87         (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi)) -
88         L(a, b, G(b[:, i1], b[i, i2], dxs), dxs) * I10(i1, i2, q[2], dt, ksi) +
89         G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs) *
90         (I01(i1, i2, q[2], dt, ksi) + dt * I00(i1, i2, q[0], dt, ksi))
91         for i2 in range(m)
92         for i1 in range(m)],
93
94      *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
95         I0000(i1, i2, i3, i4, q[3], dt, ksi)
96         for i4 in range(m)
97         for i3 in range(m)
98         for i2 in range(m)
99         for i1 in range(m)],
100
101      *[G(b[:, i1], L(a, b, a[i, 0], dxs), dxs) *
102         (I2(i1, dt, ksi) / 2 + dt * I1(i1, dt, ksi) + dt ** 2 / 2 * I0(i1, dt, ksi)) +
103         L(a, b, L(a, b, b[i, i1], dxs), dxs) * I2(i1, dt, ksi) / 2 -
104         L(a, b, G(b[:, i1], a[i, 0], dxs), dxs) * (I2(i1, dt, ksi) + dt * I1(i1, dt, ksi))
105         for i1 in range(m)],
106
107      *[G(b[:, i1], L(a, b, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
108         (I100(i1, i2, i3, q[6], dt, ksi) - I010(i1, i2, i3, q[5], dt, ksi)) +
109         G(b[:, i1], G(b[:, i2], L(a, b, b[i, i3], dxs), dxs), dxs) *
110         (I010(i1, i2, i3, q[5], dt, ksi) - I001(i1, i2, i3, q[4], dt, ksi)) +
111         G(b[:, i1], G(b[:, i2], G(b[:, i3], a[i, 0], dxs), dxs), dxs) *
112         (dt * I000(i1, i2, i3, q[1], dt, ksi) + I001(i1, i2, i3, q[4], dt, ksi)) -
113         L(a, b, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
114         I100(i1, i2, i3, q[6], dt, ksi)
115         for i3 in range(m)
116         for i2 in range(m)
117         for i1 in range(m)],
118

```

```

119 *G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
120 b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
121 I00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
122 for i5 in range(m)
123 for i4 in range(m)
124 for i3 in range(m)
125 for i2 in range(m)
126 for i1 in range(m)],
127
128 dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs),
129
130 *[G(b[:, i1], G(b[:, i2], L(a, b, a[i, 0], dxs), dxs), dxs) *
131 (I02(i1, i2, q[6], dt, ksi) / 2 + dt * I01(i1, i2, q[2], dt, ksi) +
132 dt ** 2 / 2 * I00(i1, i2, q[2], dt, ksi)) +
133 L(a, b, L(a, b, G(b[:, i1], b[i, i2], dxs), dxs), dxs) / 2 *
134 I20(i1, i2, q[10], dt, ksi) +
135 G(b[:, i1], L(a, b, G(b[:, i2], a[i, 0], dxs), dxs), dxs) *
136 (I11(i1, i2, q[9], dt, ksi) - I02(i1, i2, q[8], dt, ksi) +
137 dt * (I10(i1, i2, q[2], dt, ksi) - I01(i1, i2, q[2], dt, ksi))) +
138 L(a, b, G(b[:, i1], L(a, b, b[i, i2], dxs), dxs), dxs) *
139 (I11(i1, i2, q[9], dt, ksi) - I20(i1, i2, q[10], dt, ksi)) +
140 G(b[:, i1], L(a, b, L(a, b, b[i, i2], dxs), dxs), dxs) *
141 (I02(i1, i2, q[8], dt, ksi) / 2 + I20(i1, i2, q[10], dt, ksi) / 2 -
142 I11(i1, i2, q[9], dt, ksi)) -
143 L(a, b, G(b[:, i1], G(b[:, i2], a[i, 0], dxs), dxs), dxs) *
144 (dt * I10(i1, i2, q[2], dt, ksi) + I11(i1, i2, q[9], dt, ksi))
145 for i2 in range(m)
146 for i1 in range(m)],
147
148 *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], a[i, 0], dxs), dxs), dxs), dxs) *
149 (dt * I0000(i1, i2, i3, i4, q[3], dt, ksi) +
150 I0001(i1, i2, i3, i4, q[11], dt, ksi)) +
151 G(b[:, i1], G(b[:, i2], L(a, b, G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
152 (I0100(i1, i2, i3, i4, q[13], dt, ksi) - I0010(i1, i2, i3, i4, q[12], dt, ksi)) -
153 L(a, b, G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
154 I1000(i1, i2, i3, i4, q[14], dt, ksi) +
155 G(b[:, i1], L(a, b, G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
156 (I1000(i1, i2, i3, i4, q[14], dt, ksi) - I0100(i1, i2, i3, i4, q[13], dt, ksi)) +
157 G(b[:, i1], G(b[:, i2], G(b[:, i3], L(a, b, b[i, i4], dxs), dxs), dxs), dxs) *
158 (I0010(i1, i2, i3, i4, q[12], dt, ksi) - I0001(i1, i2, i3, i4, q[11], dt, ksi))
159 for i4 in range(m)
160 for i3 in range(m)
161 for i2 in range(m)
162 for i1 in range(m)],
163
164 *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], G(
165 b[:, i5], b[i, i6], dxs), dxs), dxs), dxs), dxs) *
166 I000000(i1, i2, i3, i4, i5, i6, q[15], dt, ksi)
167 for i6 in range(m)
168 for i5 in range(m)
169 for i4 in range(m)
170 for i3 in range(m)
171 for i2 in range(m)
172 for i1 in range(m)]
173

```

```

174 )
175
176 def doit(self, **hints):
177     """
178     Tries to expand or calculate function
179     Returns
180     =====
181     sympy.Expr
182     """
183     return StrongTaylorIto3p0(*self.args, **hints)

```

6.2.7 Source Codes for Strong Taylor–Stratonovich Numerical Schemes with Convergence Orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs

Listing 122: Strong Taylor–Stratonovich scheme with convergence order 1.0 modeling subprogram

```

1 import logging
2 from time import time
3
4 import numpy as np
5 from sympy import Matrix, symbols, MatrixSymbol, lambdify
6
7 from mathematics.sde.nonlinear.q import get_q
8 from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_1p0 import
   StrongTaylorStratonovich1p0
9
10
11 def strong_taylor_stratonovich_1p0(y0: np.ndarray, a: Matrix, b: Matrix, k: float, times:
   tuple):
12     """
13     Performs modeling of Strong Taylor–Stratonovich scheme with convergence order 1.0
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple
25         integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29         vector of solution

```

```

30     t : list
31     list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Taylor–Stratonovich 1.0 start")
38
39     # Ranges
40     n = b.shape[0]
41     m = b.shape[1]
42     t1 = times[0]
43     dt = times[1]
44     t2 = times[2]
45
46     # Defining context
47     args = symbols(f"x1:{n + 1}")
48     ticks = int((t2 - t1) / dt)
49     q = get_q(dt, k, 1)
50     logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51     logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52     logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54     # Symbols
55     sym_i, sym_t = symbols("i t")
56     sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
57     sym_y = StrongTaylorStratonovich1p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59     args_extended = list()
60     args_extended.extend(args)
61     args_extended.extend([sym_t, sym_ksi])
62
63     # Compilation of formulas
64     y_compiled = list()
65     for tr in range(n):
66         y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68     logger.info(f"[{(time() - start_time):.3f} seconds] Strong ")
69         f"Taylor–Stratonovich 1.0 subs are finished")
70
71     # Substitution values
72     t = [t1 + i * dt for i in range(ticks)]
73     y = np.zeros((n, ticks))
74     y[:, 0] = y0[:, 0]
75
76     # Dynamic substitutions with integration
77     for p in range(ticks - 1):
78         values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
79         for tr in range(n):
80             y[tr, p + 1] = y_compiled[tr](*values)
81
82     logger.info(f"[{(time() - start_time):.3f} seconds] Strong ")
83         f"Taylor–Stratonovich 1.0 calculations are finished")
84

```



```
85     return y, t
```

Listing 123: Strong Taylor–Stratonovich scheme with convergence order 1.0

```

1  from sympy import Function, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.aj import Aj
4  from mathematics.sde.nonlinear.symbolic.g import G
5  from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
6  from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
7
8
9  class StrongTaylorStratonovich1p0(Function):
10     """
11     Strong Taylor–Stratonovich scheme with convergence order 1.0
12     """
13     nargs = 8
14
15     def __new__(cls, *args, **kwargs):
16         """
17         Creates new StrongTaylorStratonovich1p0 object with given args
18         Parameters
19         =====
20         i : int
21             component of stochastic process
22         yp : numpy.ndarray
23             initial conditions
24         a : numpy.ndarray
25             algebraic, given in the variables x and t
26         b : numpy.ndarray
27             algebraic, given in the variables x and t
28         dt : float
29             integration step
30         ksi : numpy.ndarray
31             matrix of Gaussian random variables
32         q : tuple
33             amounts of q for stochastic integrals approximations
34         Returns
35         =====
36         sympy.Expr
37             formula to simplify and substitute
38         """
39         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
40         n, m = b.shape[0], b.shape[1]
41
42         aj = Aj(i, a, b, dxs)
43
44         return Add(
45
46             yp[i, 0], aj[i, 0] * dt,
47
48             *[b[i, il] * J0(il, dt, ksi)
49               for il in range(m)],

```

```

50
51     *[G(b[:, i1], b[i, i2], dxs) *
52       J00(i1, i2, q[0], dt, ksi)
53       for i2 in range(m)
54       for i1 in range(m)]
55
56     )
57
58     def doit(self, **hints):
59         """
60         Tries to expand or calculate function
61         Returns
62         =====
63         sympy.Expr
64         """
65         return StrongTaylorStratonovich1p0(*self.args, **hints)

```

Listing 124: Strong Taylor–Stratonovich scheme with convergence order 1.5 modeling subprogram

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import Matrix, MatrixSymbol, symbols, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_1p5 import
   StrongTaylorStratonovich1p5
9
10
11 def strong_taylor_stratonovich_1p5(y0: np.ndarray, a: Matrix, b: Matrix, k: float, times:
   tuple):
12     """
13     Performs modeling of Strong Taylor–Stratonovich scheme with convergence order 1.5
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple
25         integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29         vector of solution
30     t : list
31         list of time moments

```

```

32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Stratonovich 1.5
38         start")
39
40     # Ranges
41     n = b.shape[0]
42     m = b.shape[1]
43     t1 = times[0]
44     dt = times[1]
45     t2 = times[2]
46
47     # Defining context
48     args = symbols(f"x1:{n + 1}")
49     ticks = int((t2 - t1) / dt)
50     q = get_q(dt, k, 1.5)
51     logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
52     logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
53     logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
54
55     # Symbols
56     sym_i, sym_t = symbols("i t")
57     sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
58     sym_y = StrongTaylorStratonovich1p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
59
60     args_extended = list()
61     args_extended.extend(args)
62     args_extended.extend([sym_t, sym_ksi])
63
64     # Compilation of formulas
65     y_compiled = list()
66     for tr in range(n):
67         y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
68
69     logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
70         f"Taylor–Stratonovich 1.5 subs are finished")
71
72     # Substitution values
73     t = [t1 + i * dt for i in range(ticks)]
74     y = np.zeros((n, ticks))
75     y[:, 0] = y0[:, 0]
76
77     # Dynamic substitutions with integration
78     for p in range(ticks - 1):
79         values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
80         for tr in range(n):
81             y[tr, p + 1] = y_compiled[tr](*values)
82
83     logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
84         f"Taylor–Stratonovich 1.5 calculations are finished")
85
86     return y, t

```

Listing 125: Strong Taylor–Stratonovich scheme with convergence order 1.5

```

1 from sympy import Function, sympify, Add
2
3 from mathematics.sde.nonlinear.symbolic.aj import Aj
4 from mathematics.sde.nonlinear.symbolic.g import G
5 from mathematics.sde.nonlinear.symbolic.l import L
6 from mathematics.sde.nonlinear.symbolic.lj import Lj
7 from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
8 from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
9 from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
10 from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
11
12
13 class StrongTaylorStratonovich1p5(Function):
14     """
15     Strong Taylor–Stratonovich scheme with convergence order 1.5
16     """
17     nargs = 8
18
19     def __new__(cls, *args, **kwargs):
20         """
21         Creates new StrongTaylorStratonovich1p5 object with given args
22         Parameters
23         =====
24         i : int
25             component of stochastic process
26         yp : numpy.ndarray
27             initial conditions
28         a : numpy.ndarray
29             algebraic, given in the variables x and t
30         b : numpy.ndarray
31             algebraic, given in the variables x and t
32         dt : float
33             integration step
34         ksi : numpy.ndarray
35             matrix of Gaussian random variables
36         q : tuple
37             amounts of q for stochastic integrals approximations
38         Returns
39         =====
40         sympy.Expr
41             formula to simplify and substitute
42         """
43         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
44         n, m = b.shape[0], b.shape[1]
45
46         aj = Aj(i, a, b, dxs)
47
48         return Add(
49
50             yp[i, 0], aj[i, 0] * dt,
51
52             *[b[i, il] * J0(il, dt, ksi)
53               for il in range(m)],

```

```

54
55     *[G(b[:, i1], b[i, i2], dxs) *
56     J00(i1, i2, q[0], dt, ksi)
57     for i2 in range(m)
58     for i1 in range(m)],
59
60     *[G(b[:, i1], aj[i, 0], dxs) *
61     (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
62     Lj(a, b[i, i1], dxs) *
63     J1(i1, dt, ksi)
64     for i1 in range(m)],
65
66     *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
67     J000(i1, i2, i3, q[1], dt, ksi)
68     for i3 in range(m)
69     for i2 in range(m)
70     for i1 in range(m)],
71
72     dt ** 2 / 2 * L(a, b, a[i, 0], dxs)
73
74 )
75
76 def doit(self, **hints):
77     """
78     Tries to expand or calculate function
79     Returns
80     =====
81     sympy.Expr
82     """
83     return StrongTaylorStratonovich1p5(*self.args, **hints)

```

Listing 126: Strong Taylor–Stratonovich scheme with convergence order 2.0 modeling subprogram

```

1 import logging
2 from time import time
3
4 import numpy as np
5 from sympy import symbols, Matrix, MatrixSymbol, lambdify
6
7 from mathematics.sde.nonlinear.q import get_q
8 from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_2p0 import
   StrongTaylorStratonovich2p0
9
10
11 def strong_taylor_stratonovich_2p0(y0: np.array, a: Matrix, b: Matrix, k: float, times:
   tuple):
12     """
13     Performs modeling of Strong Taylor–Stratonovich scheme with convergence order 2.0
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions

```

```

18  a : numpy.ndarray
19  vector function a
20  b : numpy.ndarray
21  matrix function b
22  k : float
23  precision constant
24  times : tuple
25  integration limits and step
26  Returns
27  =====
28  y : numpy.ndarray
29  vector of solution
30  t : list
31  list of time moments
32  """
33  start_time = time()
34
35  logger = logging.getLogger(__name__)
36
37  logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Stratonovich 2.0
38  start")
39
40  # Ranges
41  n = b.shape[0]
42  m = b.shape[1]
43  t1 = times[0]
44  dt = times[1]
45  t2 = times[2]
46
47  # Defining context
48  args = symbols(f"x1:{n + 1}")
49  ticks = int((t2 - t1) / dt)
50  q = get_q(dt, k, 2)
51  logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
52  logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
53  logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
54
55  # Symbols
56  sym_i, sym_t = symbols("i t")
57  sym_ksi = MatrixSymbol("ksi", q[0] + 2, m)
58  sym_y = StrongTaylorStratonovich2p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
59
60  args_extended = list()
61  args_extended.extend(args)
62  args_extended.extend([sym_t, sym_ksi])
63
64  # Compilation of formulas
65  y_compiled = list()
66  for tr in range(n):
67      y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
68
69  logger.info(f"[{(time() - start_time):.3f} seconds] Strong
70  f"Taylor–Stratonovich 2.0 subs are finished")
71
72  # Substitution values

```

```

72     t = [t1 + i * dt for i in range(ticks)]
73     y = np.zeros((n, ticks))
74     y[:, 0] = y0[:, 0]
75
76     # Dynamic substitutions with integration
77     for p in range(ticks - 1):
78         values = [*y[:, p], t[p], np.random.randn(q[0] + 2, m)]
79         for tr in range(n):
80             y[tr, p + 1] = y_compiled[tr>(*values)
81
82     logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
83               f"Taylor–Stratonovich 2.0 calculations are finished")
84
85     return y, t

```

Listing 127: Strong Taylor–Stratonovich scheme with convergence order 2.0

```

1  from sympy import Function, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.aj import Aj
4  from mathematics.sde.nonlinear.symbolic.g import G
5  from mathematics.sde.nonlinear.symbolic.lj import Lj
6  from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
7  from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
8  from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
9  from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
10 from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
11 from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
12 from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10
13
14
15 class StrongTaylorStratonovich2p0(Function):
16     """
17     Strong Taylor–Stratonovich scheme with convergence order 2.0
18     """
19     nargs = 8
20
21     def __new__(cls, *args, **kwargs):
22         """
23         Creates new StrongTaylorStratonovich2p0 object with given args
24         Parameters
25         =====
26         i : int
27             component of stochastic process
28         yp : numpy.ndarray
29             initial conditions
30         a : numpy.ndarray
31             algebraic, given in the variables x and t
32         b : numpy.ndarray
33             algebraic, given in the variables x and t
34         dt : float
35             integration step
36         ksi : numpy.ndarray

```

```

37     matrix of Gaussian random variables
38     q : tuple
39     amounts of q for stochastic integrals approximations
40     Returns
41     =====
42     sympy.Expr
43     formula to simplify and substitute
44     """
45     i, yp, a, b, dt, ksi, dxs, q = sympify(args)
46     n, m = b.shape[0], b.shape[1]
47
48     aj = Aj(i, a, b, dxs)
49
50     return Add(
51
52         yp[i, 0], aj[i, 0] * dt,
53
54         *[b[i, i1] * J0(i1, dt, ksi)
55           for i1 in range(m)],
56
57         *[G(b[:, i1], b[i, i2], dxs) *
58           J00(i1, i2, q[0], dt, ksi)
59           for i2 in range(m)
60           for i1 in range(m)],
61
62         *[G(b[:, i1], aj[i, 0], dxs) *
63           (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
64           Lj(a, b[i, i1], dxs) *
65           J1(i1, dt, ksi)
66           for i1 in range(m)],
67
68         *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
69           J000(i1, i2, i3, q[1], dt, ksi)
70           for i3 in range(m)
71           for i2 in range(m)
72           for i1 in range(m)],
73
74         dt ** 2 / 2 * Lj(a, aj[i, 0], dxs),
75
76         *[G(b[:, i1], Lj(a, b[i, i2], dxs), dxs) *
77           (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi)) -
78           Lj(a, G(b[:, i1], b[i, i2], dxs), dxs) * J10(i1, i2, q[2], dt, ksi) +
79           G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs) *
80           (J01(i1, i2, q[2], dt, ksi) + dt * J00(i1, i2, q[0], dt, ksi))
81           for i2 in range(m)
82           for i1 in range(m)],
83
84         *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
85           J0000(i1, i2, i3, i4, q[3], dt, ksi)
86           for i4 in range(m)
87           for i3 in range(m)
88           for i2 in range(m)
89           for i1 in range(m)]
90
91     )

```



```

92
93 def doit(self, **hints):
94     """
95     Tries to expand or calculate function
96     Returns
97     =====
98     sympy.Expr
99     """
100    return StrongTaylorStratonovich2p0(*self.args, **hints)

```

Listing 128: **Strong Taylor–Stratonovich scheme with convergence order 2.5 modeling subprogram**

```

1  import logging
2  from time import time
3
4  import numpy as np
5  from sympy import Matrix, symbols, MatrixSymbol, lambdify
6
7  from mathematics.sde.nonlinear.q import get_q
8  from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_2p5 import
   StrongTaylorStratonovich2p5
9
10
11 def strong_taylor_stratonovich_2p5(y0: np.array, a: Matrix, b: Matrix, k: float, times:
   tuple):
12     """
13     Performs modeling of Strong Taylor–Stratonovich scheme with convergence order 2.5
14     Parameters
15     =====
16     y0 : numpy.ndarray
17     initial conditions
18     a : numpy.ndarray
19     vector function a
20     b : numpy.ndarray
21     matrix function b
22     k : float
23     precision constant
24     times : tuple
25     integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29     vector of solution
30     t : list
31     list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Stratonovich 2.5
   start")

```

```

38
39 # Ranges
40 n = b.shape[0]
41 m = b.shape[1]
42 t1 = times[0]
43 dt = times[1]
44 t2 = times[2]
45
46 # Defining context
47 args = symbols(f"x1:{n + 1}")
48 ticks = int((t2 - t1) / dt)
49 q = get_q(dt, k, 2.5)
50 logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
51 logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
52 logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
53
54 # Symbols
55 sym_i, sym_t = symbols("i t")
56 sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
57 sym_y = StrongTaylorStratonovich2p5(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
58
59 args_extended = list()
60 args_extended.extend(args)
61 args_extended.extend([sym_t, sym_ksi])
62
63 # Compilation of formulas
64 y_compiled = list()
65 for tr in range(n):
66     y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
67
68 logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
69           f"Taylor-Stratonovich 2.5 subs are finished")
70
71 # Substitution values
72 t = [t1 + i * dt for i in range(ticks)]
73 y = np.zeros((n, ticks))
74 y[:, 0] = y0[:, 0]
75
76 # Dynamic substitutions with integration
77 for p in range(ticks - 1):
78     values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]
79     for tr in range(n):
80         y[tr, p + 1] = y_compiled[tr>(*values)
81
82 logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
83           f"Taylor-Stratonovich 2.5 calculations are finished")
84
85 return y, t

```

Listing 129: Strong Taylor-Stratonovich scheme with convergence order 2.5

```

1 from sympy import Function, sympify, Add
2

```

```

3 from mathematics.sde.nonlinear.symbolic.aj import Aj
4 from mathematics.sde.nonlinear.symbolic.g import G
5 from mathematics.sde.nonlinear.symbolic.l import L
6 from mathematics.sde.nonlinear.symbolic.lj import Lj
7 from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
8 from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
9 from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
10 from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
11 from mathematics.sde.nonlinear.symbolic.stratonovich.j00000 import J00000
12 from mathematics.sde.nonlinear.symbolic.stratonovich.j001 import J001
13 from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
14 from mathematics.sde.nonlinear.symbolic.stratonovich.j010 import J010
15 from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
16 from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10
17 from mathematics.sde.nonlinear.symbolic.stratonovich.j100 import J100
18 from mathematics.sde.nonlinear.symbolic.stratonovich.j2 import J2
19
20
21 class StrongTaylorStratonovich2p5(Function):
22     """
23     Strong Taylor–Stratonovich scheme with convergence order 2.5
24     """
25     nargs = 8
26
27     def __new__(cls, *args, **kwargs):
28         """
29         Creates new StrongTaylorStratonovich2p5 object with given args
30         Parameters
31         =====
32         i : int
33         component of stochastic process
34         yp : numpy.ndarray
35         initial conditions
36         a : numpy.ndarray
37         algebraic, given in the variables x and t
38         b : numpy.ndarray
39         algebraic, given in the variables x and t
40         dt : float
41         integration step
42         ksi : numpy.ndarray
43         matrix of Gaussian random variables
44         q : tuple
45         amounts of q for stochastic integrals approximations
46         Returns
47         =====
48         sympy.Expr
49         formula to simplify and substitute
50         """
51         i, yp, a, b, dt, ksi, dxs, q = sympify(args)
52         n, m = b.shape[0], b.shape[1]
53
54         aj = Aj(i, a, b, dxs)
55
56         return Add(
57

```

```

58     yp[i, 0], aj[i, 0] * dt,
59
60     *[b[i, i1] * J0(i1, dt, ksi)
61       for i1 in range(m)],
62
63     *[G(b[:, i1], b[i, i2], dxs) *
64       J00(i1, i2, q[0], dt, ksi)
65       for i2 in range(m)
66       for i1 in range(m)],
67
68     *[G(b[:, i1], aj[i, 0], dxs) *
69       (dt * J0(i1, dt, ksi) + J1(i1, dt, ksi)) -
70       Lj(a, b[i, i1], dxs) *
71       J1(i1, dt, ksi)
72       for i1 in range(m)],
73
74     *[G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs) *
75       J000(i1, i2, i3, q[1], dt, ksi)
76       for i3 in range(m)
77       for i2 in range(m)
78       for i1 in range(m)],
79
80     dt ** 2 / 2 * Lj(a, aj[i, 0], dxs),
81
82     *[G(b[:, i1], Lj(a, b[i, i2], dxs), dxs) *
83       (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi)) -
84       Lj(a, G(b[:, i1], b[i, i2], dxs), dxs) * J10(i1, i2, q[2], dt, ksi) +
85       G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs) *
86       (J01(i1, i2, q[2], dt, ksi) + dt * J00(i1, i2, q[0], dt, ksi))
87       for i2 in range(m)
88       for i1 in range(m)],
89
90     *[G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs) *
91       J0000(i1, i2, i3, i4, q[3], dt, ksi)
92       for i4 in range(m)
93       for i3 in range(m)
94       for i2 in range(m)
95       for i1 in range(m)],
96
97     *[G(b[:, i1], Lj(a, aj[i, 0], dxs), dxs) *
98       (J2(i1, dt, ksi) / 2 + dt * J1(i1, dt, ksi) + dt ** 2 / 2 * J0(i1, dt, ksi)) +
99       Lj(a, Lj(a, b[i, i1], dxs), dxs) * J2(i1, dt, ksi) / 2 -
100      Lj(a, G(b[:, i1], aj[i, 0], dxs), dxs) * (J2(i1, dt, ksi) + dt * J1(i1, dt, ksi))
101      for i1 in range(m)],
102
103     *[G(b[:, i1], Lj(a, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
104       (J100(i1, i2, i3, q[6], dt, ksi) - J010(i1, i2, i3, q[5], dt, ksi)) +
105       G(b[:, i1], G(b[:, i2], Lj(a, b[i, i3], dxs), dxs), dxs) *
106       (J010(i1, i2, i3, q[5], dt, ksi) - J001(i1, i2, i3, q[4], dt, ksi)) +
107       G(b[:, i1], G(b[:, i2], G(b[:, i3], aj[i, 0], dxs), dxs), dxs) *
108       (dt * J000(i1, i2, i3, q[1], dt, ksi) + J001(i1, i2, i3, q[4], dt, ksi)) -
109       Lj(a, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
110       J100(i1, i2, i3, q[6], dt, ksi)
111       for i3 in range(m)
112       for i2 in range(m)

```

```

113     for i1 in range(m)],
114
115     *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
116     b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
117     J00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
118     for i5 in range(m)
119     for i4 in range(m)
120     for i3 in range(m)
121     for i2 in range(m)
122     for i1 in range(m)],
123
124     dt ** 3 / 6 * L(a, b, L(a, b, a[i, 0], dxs), dxs)
125
126 )
127
128 def doit(self, **hints):
129     """
130     Tries to expand or calculate function
131     Returns
132     =====
133     sympy.Expr
134     """
135     return StrongTaylorStratonovich2p5(*self.args, **hints)

```

Listing 130: Strong Taylor–Stratonovich scheme with convergence order 3.0 modeling subprogram

```

1 import logging
2 from time import time
3
4 import numpy as np
5 from sympy import Matrix, symbols, MatrixSymbol, lambdify
6
7 from mathematics.sde.nonlinear.q import get-q
8 from mathematics.sde.nonlinear.symbolic.schemes.strong_taylor_stratonovich_3p0 import
   StrongTaylorStratonovich3p0
9
10
11 def strong_taylor_stratonovich_3p0(y0: np.array, a: Matrix, b: Matrix, k: float, times:
   tuple):
12     """
13     Performs modeling of Strong Taylor–Stratonovich scheme with convergence order 3.0
14     Parameters
15     =====
16     y0 : numpy.ndarray
17         initial conditions
18     a : numpy.ndarray
19         vector function a
20     b : numpy.ndarray
21         matrix function b
22     k : float
23         precision constant
24     times : tuple

```

```

25     integration limits and step
26     Returns
27     =====
28     y : numpy.ndarray
29     vector of solution
30     t : list
31     list of time moments
32     """
33     start_time = time()
34
35     logger = logging.getLogger(__name__)
36
37     logger.info(f"[{(time() - start_time):.3f} seconds] Strong Taylor–Stratonovich 3.0
38         start")
39
40     # Ranges
41     n = b.shape[0]
42     m = b.shape[1]
43     t1 = times[0]
44     dt = times[1]
45     t2 = times[2]
46
47     # Defining context
48     args = symbols(f"x1:{n + 1}")
49     ticks = int((t2 - t1) / dt)
50     q = get_q(dt, k, 3)
51     logger.info(f"[{(time() - start_time):.3f} seconds] Using C = {k}")
52     logger.info(f"[{(time() - start_time):.3f} seconds] Using dt = {dt}")
53     logger.info(f"[{(time() - start_time):.3f} seconds] Using q = {q}")
54
55     # Symbols
56     sym_i, sym_t = symbols("i t")
57     sym_ksi = MatrixSymbol("ksi", q[0] + 3, m)
58     sym_y = StrongTaylorStratonovich3p0(sym_i, Matrix(args), a, b, dt, sym_ksi, args, q)
59
60     args_extended = list()
61     args_extended.extend(args)
62     args_extended.extend([sym_t, sym_ksi])
63
64     # Compilation of formulas
65     y_compiled = list()
66     for tr in range(n):
67         y_compiled.append(lambdify(args_extended, sym_y.subs(sym_i, tr), "numpy"))
68
69     logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
70         f"Taylor–Stratonovich 3.0 subs are finished")
71
72     # Substitution values
73     t = [t1 + i * dt for i in range(ticks)]
74     y = np.zeros((n, ticks))
75     y[:, 0] = y0[:, 0]
76
77     # Dynamic substitutions with integration
78     for p in range(ticks - 1):
79         values = [*y[:, p], t[p], np.random.randn(q[0] + 3, m)]

```

```

79     for tr in range(n):
80         y[tr, p + 1] = y_compiled[tr>(*values)
81
82     logger.info(f"[{(time() - start_time):.3f} seconds] Strong "
83               f"Taylor–Stratonovich 3.0 calculations are finished")
84
85     return y, t

```

Listing 131: Strong Taylor–Stratonovich scheme with convergence order 3.0

```

1  from sympy import Function, sympify, Add
2
3  from mathematics.sde.nonlinear.symbolic.aj import Aj
4  from mathematics.sde.nonlinear.symbolic.g import G
5  from mathematics.sde.nonlinear.symbolic.lj import Lj
6  from mathematics.sde.nonlinear.symbolic.stratonovich.j0 import J0
7  from mathematics.sde.nonlinear.symbolic.stratonovich.j00 import J00
8  from mathematics.sde.nonlinear.symbolic.stratonovich.j000 import J000
9  from mathematics.sde.nonlinear.symbolic.stratonovich.j0000 import J0000
10 from mathematics.sde.nonlinear.symbolic.stratonovich.j00000 import J00000
11 from mathematics.sde.nonlinear.symbolic.stratonovich.j000000 import J000000
12 from mathematics.sde.nonlinear.symbolic.stratonovich.j0001 import J0001
13 from mathematics.sde.nonlinear.symbolic.stratonovich.j001 import J001
14 from mathematics.sde.nonlinear.symbolic.stratonovich.j0010 import J0010
15 from mathematics.sde.nonlinear.symbolic.stratonovich.j01 import J01
16 from mathematics.sde.nonlinear.symbolic.stratonovich.j010 import J010
17 from mathematics.sde.nonlinear.symbolic.stratonovich.j0100 import J0100
18 from mathematics.sde.nonlinear.symbolic.stratonovich.j02 import J02
19 from mathematics.sde.nonlinear.symbolic.stratonovich.j1 import J1
20 from mathematics.sde.nonlinear.symbolic.stratonovich.j10 import J10
21 from mathematics.sde.nonlinear.symbolic.stratonovich.j100 import J100
22 from mathematics.sde.nonlinear.symbolic.stratonovich.j1000 import J1000
23 from mathematics.sde.nonlinear.symbolic.stratonovich.j11 import J11
24 from mathematics.sde.nonlinear.symbolic.stratonovich.j2 import J2
25 from mathematics.sde.nonlinear.symbolic.stratonovich.j20 import J20
26
27
28 class StrongTaylorStratonovich3p0(Function):
29     """
30     Strong Taylor–Stratonovich scheme with convergence order 3.0
31     """
32     nargs = 8
33
34     def __new__(cls, *args, **kwargs):
35         """
36         Creates new StrongTaylorStratonovich3p0 object with given args
37         Parameters
38         =====
39         i : int
40             component of stochastic process
41         yp : numpy.ndarray
42             initial conditions
43         a : numpy.ndarray

```

```

44     algebraic , given in the variables x and t
45     b : numpy.ndarray
46     algebraic , given in the variables x and t
47     dt : float
48     integration step
49     ksi : numpy.ndarray
50     matrix of Gaussian random variables
51     q : tuple
52     amounts of q for stochastic integrals approximations
53 Returns
54 =====
55 sympy.Expr
56 formula to simplify and substitute
57 """
58 i , yp , a , b , dt , ksi , dxs , q = sympify(args)
59 n , m = b.shape[0] , b.shape[1]
60
61 aj = Aj(i , a , b , dxs)
62
63 return Add(
64
65     yp[i , 0] , aj[i , 0] * dt ,
66
67     *[b[i , i1] * J0(i1 , dt , ksi)
68       for i1 in range(m)] ,
69
70     *[G(b[:, i1] , b[i , i2] , dxs) *
71       J00(i1 , i2 , q[0] , dt , ksi)
72       for i2 in range(m)
73       for i1 in range(m)] ,
74
75     *[G(b[:, i1] , aj[i , 0] , dxs) *
76       (dt * J0(i1 , dt , ksi) + J1(i1 , dt , ksi)) -
77       Lj(a , b[i , i1] , dxs) *
78       J1(i1 , dt , ksi)
79       for i1 in range(m)] ,
80
81     *[G(b[:, i1] , G(b[:, i2] , b[i , i3] , dxs) , dxs) *
82       J000(i1 , i2 , i3 , q[1] , dt , ksi)
83       for i3 in range(m)
84       for i2 in range(m)
85       for i1 in range(m)] ,
86
87     dt ** 2 / 2 * Lj(a , aj[i , 0] , dxs) ,
88
89     *[G(b[:, i1] , Lj(a , b[i , i2] , dxs) , dxs) *
90       (J10(i1 , i2 , q[2] , dt , ksi) - J01(i1 , i2 , q[2] , dt , ksi)) -
91       Lj(a , G(b[:, i1] , b[i , i2] , dxs) , dxs) * J10(i1 , i2 , q[2] , dt , ksi) +
92       G(b[:, i1] , G(b[:, i2] , aj[i , 0] , dxs) , dxs) *
93       (J01(i1 , i2 , q[2] , dt , ksi) + dt * J00(i1 , i2 , q[0] , dt , ksi))
94       for i2 in range(m)
95       for i1 in range(m)] ,
96
97     *[G(b[:, i1] , G(b[:, i2] , G(b[:, i3] , b[i , i4] , dxs) , dxs) , dxs) *
98       J0000(i1 , i2 , i3 , i4 , q[3] , dt , ksi)

```



```

99     for i4 in range(m)
100    for i3 in range(m)
101    for i2 in range(m)
102    for i1 in range(m)],
103
104    *[G(b[:, i1], Lj(a, aj[i, 0], dxs), dxs) *
105      (J2(i1, dt, ksi) / 2 + dt * J1(i1, dt, ksi) + dt ** 2 / 2 * J0(i1, dt, ksi)) +
106      Lj(a, Lj(a, b[i, i1], dxs), dxs) * J2(i1, dt, ksi) / 2 -
107      Lj(a, G(b[:, i1], aj[i, 0], dxs), dxs) * (J2(i1, dt, ksi) + dt * J1(i1, dt, ksi))
108      for i1 in range(m)],
109
110    *[G(b[:, i1], Lj(a, G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
111      (J100(i1, i2, i3, q[6], dt, ksi) - J010(i1, i2, i3, q[5], dt, ksi)) +
112      G(b[:, i1], G(b[:, i2], Lj(a, b[i, i3], dxs), dxs), dxs) *
113      (J010(i1, i2, i3, q[5], dt, ksi) - J001(i1, i2, i3, q[4], dt, ksi)) +
114      G(b[:, i1], G(b[:, i2], G(b[:, i3], aj[i, 0], dxs), dxs), dxs) *
115      (dt * J000(i1, i2, i3, q[1], dt, ksi) + J001(i1, i2, i3, q[4], dt, ksi)) -
116      Lj(a, G(b[:, i1], G(b[:, i2], b[i, i3], dxs), dxs), dxs) *
117      J100(i1, i2, i3, q[6], dt, ksi)
118      for i3 in range(m)
119      for i2 in range(m)
120      for i1 in range(m)],
121
122    *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
123      b[:, i4], b[i, i5], dxs), dxs), dxs), dxs) *
124      J00000(i1, i2, i3, i4, i5, q[7], dt, ksi)
125      for i5 in range(m)
126      for i4 in range(m)
127      for i3 in range(m)
128      for i2 in range(m)
129      for i1 in range(m)],
130
131    dt ** 3 / 6 * Lj(a, Lj(a, aj[i, 0], dxs), dxs),
132
133    *[G(b[:, i1], G(b[:, i2], Lj(a, aj[i, 0], dxs), dxs), dxs) *
134      (J02(i1, i2, q[6], dt, ksi) / 2 + dt * J01(i1, i2, q[2], dt, ksi) +
135      dt ** 2 / 2 * J00(i1, i2, q[2], dt, ksi)) +
136      Lj(a, Lj(a, G(b[:, i1], b[i, i2], dxs), dxs), dxs) / 2 *
137      J20(i1, i2, q[10], dt, ksi) +
138      G(b[:, i1], Lj(a, G(b[:, i2], aj[i, 0], dxs), dxs), dxs) *
139      (J11(i1, i2, q[9], dt, ksi) - J02(i1, i2, q[8], dt, ksi) +
140      dt * (J10(i1, i2, q[2], dt, ksi) - J01(i1, i2, q[2], dt, ksi))) +
141      Lj(a, G(b[:, i1], Lj(a, b[i, i2], dxs), dxs), dxs) *
142      (J11(i1, i2, q[9], dt, ksi) - J20(i1, i2, q[10], dt, ksi)) +
143      G(b[:, i1], Lj(a, Lj(a, b[i, i2], dxs), dxs), dxs) *
144      (J02(i1, i2, q[8], dt, ksi) / 2 + J20(i1, i2, q[10], dt, ksi) / 2 -
145      J11(i1, i2, q[9], dt, ksi)) -
146      Lj(a, G(b[:, i1], G(b[:, i2], aj[i, 0], dxs), dxs), dxs) *
147      (dt * J10(i1, i2, q[2], dt, ksi) + J11(i1, i2, q[9], dt, ksi))
148      for i2 in range(m)
149      for i1 in range(m)],
150
151    *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(
152      b[:, i4], aj[i, 0], dxs), dxs), dxs), dxs) *
153      (dt * J0000(i1, i2, i3, i4, q[3], dt, ksi) +

```

```

154     J0001(i1, i2, i3, i4, q[11], dt, ksi)) +
155     G(b[:, i1], G(b[:, i2], Lj(a, G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
156     (J0100(i1, i2, i3, i4, q[13], dt, ksi) - J0010(i1, i2, i3, i4, q[12], dt, ksi)) -
157     Lj(a, G(b[:, i1], G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
158     J1000(i1, i2, i3, i4, q[14], dt, ksi) +
159     G(b[:, i1], Lj(a, G(b[:, i2], G(b[:, i3], b[i, i4], dxs), dxs), dxs), dxs) *
160     (J1000(i1, i2, i3, i4, q[14], dt, ksi) - J0100(i1, i2, i3, i4, q[13], dt, ksi)) +
161     G(b[:, i1], G(b[:, i2], G(b[:, i3], Lj(a, b[i, i4], dxs), dxs), dxs), dxs) *
162     (J0010(i1, i2, i3, i4, q[12], dt, ksi) - J0001(i1, i2, i3, i4, q[11], dt, ksi))
163     for i4 in range(m)
164     for i3 in range(m)
165     for i2 in range(m)
166     for i1 in range(m)],
167
168     *[G(b[:, i1], G(b[:, i2], G(b[:, i3], G(b[:, i4], G(
169     b[:, i5], b[i, i6], dxs), dxs), dxs), dxs), dxs) *
170     J000000(i1, i2, i3, i4, i5, i6, q[15], dt, ksi)
171     for i6 in range(m)
172     for i5 in range(m)
173     for i4 in range(m)
174     for i3 in range(m)
175     for i2 in range(m)
176     for i1 in range(m)]
177
178     )
179
180     def doit(self, **hints):
181         """
182         Tries to expand or calculate function
183         Returns
184         =====
185         sympy.Expr
186         """
187         return StrongTaylorStratonovich3p0(*self.args, **hints)

```

6.3 Source Codes for Linear Stationary Systems of Itô SDEs

Listing 132: Implementation of supplementary functions

```

1  import numpy as np
2
3
4  class NotASquareMatrix(Exception):
5      pass
6
7
8  def vec_to_eye(vector):
9      """
10     Converts vector to eye matrix
11     Parameters
12     =====

```

```

13     vector : iterable
14     Returns
15     =====
16     numpy.ndarray
17     """
18     n = len(vector)
19     matrix = np.zeros((n, n))
20
21     for i in range(len(matrix)):
22         matrix[i][i] = vector[i]
23
24     return matrix
25
26
27 def diagonal_to_column(matrix):
28     """
29     Converts diagonal matrix to column vector
30     Parameters
31     =====
32     matrix : numpy.ndarray
33     Returns
34     =====
35     column : numpy.ndarray
36     """
37     height = np.shape(matrix)[0]
38     if height != np.shape(matrix)[1]:
39         raise NotASquareMatrix()
40
41     column = np.zeros((height, 1))
42     for i in range(height):
43         column[i][0] = matrix[i][i]
44
45     return column

```

Listing 133: Implementation of Algorithm 11.2 [61]

```

1 import numpy as np
2 import scipy.linalg as sci
3
4
5 def dindet(n: int, k: int, m_a: np.ndarray, m_b: np.ndarray, dt: float):
6     """
7     Algorithm 11.2 [61]
8     Parameters
9     =====
10    n : int
11    k : int
12    m_a : numpy.array
13    m_b : numpy.array
14    dt : float
15    Returns
16    =====
17    m_ad : numpy.array

```

```

18  m_bd : numpy.array
19  """
20  m_okn = np.zeros((k, n))
21  m_okk = np.zeros((k, k))
22  m_idt = np.eye(n + k) * dt
23  m_aa = np.vstack((np.hstack((m_a, m_b)),
24                    np.hstack((m_okn, m_okk))))
25  m_ex_aah = sci.expm(m_aa.dot(m_idt))
26  m_ad = m_ex_aah[:n, :n]
27  m_bd = m_ex_aah[:n, n:(n + k)]
28  return m_ad, m_bd

```

Listing 134: Implementation of Algorithm 11.6 [61]

```

1  import numpy as np
2
3  from mathematics.sde.linear.dindet import dindet
4  from mathematics.sde.linear.matrix import vec_to_eye
5
6
7  def stoch(n: int, m_a: np.ndarray, m_f: np.ndarray, dt: float):
8      """
9      Root function for set of algorithms implemented below
10     Parameters
11     =====
12     n : int
13     m_a : numpy.ndarray
14     m_f : numpy.ndarray
15     dt : float
16     Returns
17     =====
18     numpy.ndarray
19     """
20     v_l2, m_s, m_d1 = algorithm_11_2(n, m_a, m_f, dt)
21     mat_l = vec_to_eye(np.sqrt(v_l2))
22     return m_s.dot(mat_l)
23
24
25  def algorithm_11_2(n: int, m_a: np.ndarray, m_f: np.ndarray, dt: float):
26      """
27     Parameters
28     =====
29     n : int
30     m_a : numpy.ndarray
31     m_f : numpy.ndarray
32     dt : float
33     Returns
34     =====
35     eigenvalues : numpy.ndarray
36     eigenvectors : numpy.ndarray
37     m_d1 : numpy.ndarray
38     """
39     m_ac = algorithm_11_5(n, m_a)

```

```

40     m_g = m_f.dot(np.transpose(m_f))
41     m_gv = algorithm_11_3(n, m_g)
42     m_dd, m_dv = dindet(int(n * (n + 1) / 2), 1, m_ac, m_gv, dt)
43     m_d1 = algorithm_11_4(n, m_dv)
44     eigenvalues, eigenvectors = np.linalg.eig(m_d1)
45     return eigenvalues, eigenvectors, m_d1
46
47
48 def algorithm_11_3(n: int, m_g: np.ndarray):
49     """
50     Algorithm 11.3 [61]
51     Parameters
52     -----
53     n : int
54     m_g : numpy.ndarray
55     Returns
56     -----
57     m_vec : numpy.ndarray
58         column vector
59     """
60     i2 = 0
61     v_size = 0
62     for i in range(n):
63         n2 = n - i
64         for j in range(n2):
65             if v_size < j + i2:
66                 v_size = j + i2
67             i2 = i2 + n - i
68
69     m_vec = np.ndarray((v_size + 1, 1))
70
71     i2 = 0
72     for i in range(n):
73         n2 = n - i
74         for j in range(n2):
75             m_vec[j + i2][0] = m_g[j][j + i]
76             i2 = i2 + n - i
77
78     return m_vec
79
80
81 def algorithm_11_4(n: int, m_dv: np.ndarray):
82     """
83     Algorithm 11.4 [61]
84     Parameters
85     -----
86     n : int
87     m_dv : numpy.ndarray
88     Returns
89     -----
90     m_d1 : numpy.ndarray
91     """
92     i2 = 0
93     size = 0
94     for i in range(n):

```

```

95     n2 = n - i
96     for j in range(n2):
97         if size < j + i:
98             size = j + i
99         i2 = i2 + n - i
100
101     m.d1 = np.ndarray((size + 1, size + 1))
102
103     i2 = 0
104     for i in range(n):
105         n2 = n - i
106         for j in range(n2):
107             m.d1[j][j + i] = m.dv[j + i2][0]
108             m.d1[j + i][j] = m.dv[j + i2][0]
109         i2 = i2 + n - i
110
111     return m.d1
112
113
114 def algorithm_11_5(n: int, m_a: np.ndarray):
115     """
116     Algorithm 11.5 [61]
117     Parameters
118     -----
119     n : int
120     m_a : numpy.ndarray
121     Returns
122     -----
123     m_ac : numpy.ndarray
124     """
125     r = 0
126     v_size = 0
127     h_size = 0
128
129     for i in range(n):
130         n2 = n - i
131         for j in range(n2):
132             o = 0
133             for k in range(n):
134                 n3 = n - k
135                 for m in range(n3):
136                     if v_size < m + o:
137                         v_size = m + o
138                     if h_size < r:
139                         h_size = r
140                     o = o + n - k
141                 r = r + 1
142
143     m.ones = np.zeros((n, n))
144     m.ac = np.ndarray((v_size + 1, h_size + 1))
145
146     r = 0
147     for i in range(n):
148         n2 = n - i
149         for j in range(n2):

```

```

150     i2 = j + i
151     m_ones[j][i2] = 1
152     m_ones[i2][j] = 1
153     m_one_a = m_ones.dot(np.transpose(m_a)) + m_a.dot(m_ones)
154     o = 0
155     for k in range(n):
156         n3 = n - k
157         for m in range(n3):
158             m_ac[m + o][r] = m_one_a[m][m + k]
159         o = o + n - k
160     m_ones = np.zeros((n, n))
161     r = r + 1
162
163     return m_ac

```

Listing 135: Implementation of the vector function $u(t)$

```

1  import numpy as np
2  from sympy import lambdify, sympify
3
4
5  class AbstractDistortion:
6      def t(self, t: float):
7          raise NotImplementedError("Method t is not implemented")
8
9
10 class Symbolic(AbstractDistortion):
11
12     def __init__(self, fn: str):
13         from sympy.abc import t
14         self._u = lambdify(t, sympify(fn), "numpy")
15
16     def t(self, t):
17         return self._u(t)
18
19
20 class ComplexDistortion(AbstractDistortion):
21     """
22     Vector function  $u(t)$ 
23     """
24
25     def __init__(self, n: int, mat_u: np.ndarray):
26         self._mat_u = mat_u
27         self._mat_ut = np.ndarray(shape=(n, 1), dtype=float)
28
29     def t(self, t: float):
30         """
31         Provides vector function  $u(t)$  at moment  $t$ 
32
33         Parameters
34         -----
35         t : float
36             moment of time

```

```

37     Returns
38     -----
39     numpy.ndarray
40     column u(t)
41     """
42     for i in range(self._mat_u.shape[0]):
43         self._mat_ut[i][0] = self._mat_u[i][0].t(t)
44     return self._mat_ut

```

Listing 136: Modeling of linear system of Itô SDEs

```

1  import numpy as np
2  from numpy import transpose
3
4  from mathematics.sde.linear.matrix import diagonal_to_column
5
6
7  class Integral:
8      """
9      Provides numerical integration
10     """
11
12     def __init__(self, n: int):
13         self.n, self.t0, self.tk, self.dt, self.t = \
14             n, 0, 0, 0, 0
15         self.m_a, self.m_ad, self.m_bd, self.m_h, self.m_fd, self.distortion = \
16             None, None, None, None, None, None
17         self.m_x0, self.m_mx0, self.m_dx0, self.m_xt, self.m_mx, self.m_dx = \
18             None, None, None, np.ndarray((n, 0)), np.ndarray((n, 0)), np.ndarray((n, 0))
19         self.v_yt, self.v_my, self.v_dy, self.v_t = \
20             [], [], [], []
21         self.v_ry = []
22
23     def integrate(self):
24         """
25         Performs numerical integration
26         """
27         higher_limit = self.t + int((self.tk - self.t0) / self.dt + 1)
28         lower_limit = self.t
29
30         self.m_xt = np.hstack((self.m_xt, np.ndarray((self.n, higher_limit - lower_limit))))
31         self.m_mx = np.hstack((self.m_mx, np.ndarray((self.n, higher_limit - lower_limit))))
32         self.m_dx = np.hstack((self.m_dx, np.ndarray((self.n, higher_limit - lower_limit))))
33
34         for self.t in range(lower_limit, higher_limit):
35             t = self.t0 + self.t * self.dt
36             ft = np.random.randn(self.n, 1)
37             mat_ut = self.distortion.t(t)
38
39             # solution of sde
40             xt = self.m_ad.dot(self.m_x0) + self.m_bd.dot(mat_ut) + self.m_fd.dot(ft)
41             # exit process of stochastic system
42             self.m_xt[:, self.t] = xt[:, 0]

```



```

43     self.v_yt.append(self.m.h.dot(xt)[0][0])
44
45     # expectation of solution of sde
46     mx = self.m.ad.dot(self.m.mx0) + self.m.bd.dot(mat_ut)
47     # expectation of exit process
48     self.m.mx[:, self.t] = mx[:, 0]
49     self.v_my.append(self.m.h.dot(mx)[0][0])
50
51     # dispersion of solution of sde
52     dx = self.m.ad.dot(self.m_dx0).dot(np.transpose(self.m.ad)) + self.m.fd.dot(np.
transpose(self.m.fd))
53     # dispersion of exit process
54     self.m_dx[:, self.t] = diagonal_to_column(dx)[:, 0]
55     self.v_dy.append(self.m.h.dot(dx).dot(np.transpose(self.m.h))[0][0])
56
57     self.v_t.append(t)
58
59     self.m_x0, self.m_mx0, self.m_dx0 = xt, mx, dx

```

6.4 Source Codes for Utilities and Initialization

Listing 137: Initialization module

```

1  from config import database
2  from init.database import initdb
3  from tools.database import connect, disconnect
4
5
6  def initialization():
7      """
8      Initializes various components of application
9      """
10     connect(database)
11     initdb()
12     disconnect()

```

Listing 138: Module for database initialization

```

1  import csv
2  import logging
3  import os
4
5  import sympy as sp
6
7  import config as c
8  from tools import fsys
9  from tools.database import execute
10 from tools.fsys import get_files
11
12 logger = logging.getLogger(__name__)

```

```

13
14
15 def initdb():
16     """
17     Initializes database with necessary table drivers
18     """
19
20     if not fsys.is_locked(".db.lock"):
21         logger.info("Initializing database...")
22         create_files_table()
23         create_c_table()
24         fsys.lock(".db.lock")
25     else:
26         logger.info("Updating database...")
27         update_coefficients()
28
29
30 def create_files_table():
31     """
32     Initializes the Fourier–Legendre coefficients table
33     """
34     execute("DROP TABLE IF EXISTS 'files'")
35     execute(
36         "CREATE TABLE 'files' ("
37         " 'id' integer PRIMARY KEY AUTOINCREMENT,"
38         " 'name' text unique"
39         ")")
40 )
41
42
43 def create_c_table():
44     """
45     Initializes the Fourier–Legendre coefficients table
46     """
47     execute("DROP TABLE IF EXISTS 'C'")
48     execute(
49         "CREATE TABLE 'C' ("
50         " 'id' integer PRIMARY KEY AUTOINCREMENT,"
51         " 'index' text unique,"
52         " 'value' text,"
53         " 'value_f' double"
54         ")")
55 )
56
57     update_coefficients()
58
59
60 def update_coefficients():
61     """
62     Updates the Fourier–Legendre coefficients table
63     """
64     files = get_files(c.csv, r'c_.*\.csv')
65     loaded_files = [record[0] for record in execute("SELECT 'name' FROM 'files'")]
66     difference = [f for f in files if f not in loaded_files]
67

```

```

68 rows = []
69 for file in difference:
70
71     with open(os.path.join(file)) as f:
72         reader = csv.reader(f, delimiter=';', quotechar='')
73         for row in reader:
74             if len(rows) > c.read_buffer_size:
75                 execute(f"INSERT INTO 'C' ('index', 'value', 'value-f') VALUES {'','.join(rows)}")
76                 rows.clear()
77
78                 rows.append(f"('{row[0]}', '{row[1]}', {float(sp.sympify(row[1]).evalf())})")
79
80             execute(f"INSERT INTO 'files' ('name') VALUES ('{file}')")
81
82 if len(rows) > 0:
83     execute(f"INSERT INTO 'C' ('index', 'value', 'value-f') VALUES {'','.join(rows)}")

```

Listing 139: Database module

```

1 import logging
2 import re
3 import sqlite3
4
5 logger = logging.getLogger(__name__)
6
7 connection: sqlite3.Connection
8 cursor: sqlite3.Cursor
9
10
11 def is_connected():
12     """
13     Checks if application is connected to database
14     Returns
15     =====
16     True or False
17     """
18     global connection
19     if connection is None:
20         return False
21     else:
22         return True
23
24
25 def connect(db: str):
26     """
27     Connects application to database
28     Parameters
29     =====
30     db : str
31         path to database file
32     """
33     try:

```

```

34     global connection
35     global cursor
36
37     connection = sqlite3.connect(db)
38     connection.create_function("REGEXP", 2, regex)
39
40     cursor = connection.cursor()
41     logger.info(f"SQLite Database is successfully connected")
42
43     query = "select sqlite_version();"
44     cursor.execute(query)
45     record = cursor.fetchall()
46     logger.info(f"SQLite Database Version is: {record[0][0]}")
47
48     query = "PRAGMA foreign_keys = ON;"
49     cursor.execute(query)
50
51     except sqlite3.Error as error:
52         logger.error(f"Error while connecting to sqlite: {error}")
53
54
55 def disconnect():
56     """
57     Disconnects application from database
58     """
59     try:
60         global connection
61         global cursor
62
63         connection.close()
64         logger.info("The SQLite connection is closed")
65
66     except sqlite3.Error as error:
67         logger.error(f"Error while connecting to sqlite: {error}")
68
69
70 def execute(query: str):
71     """
72     Sends query to database and receives data
73     Parameters
74     =====
75     query : str
76         query to database
77     Returns
78     =====
79     list of tuples (rows)
80     """
81     try:
82         global connection
83         global cursor
84
85         cursor.execute(query)
86         records = cursor.fetchall()
87         connection.commit()
88     return records

```

```

89
90     except sqlite3.Error as error:
91         logger.error(f"Error while connecting to sqlite: {error}")
92
93
94 def regex(value, pattern):
95     """
96     Regular expression for search in database
97     Parameters
98     =====
99     value
100     column to apply
101     pattern
102     regular expression
103     Returns
104     =====
105     Search results
106     """
107     c_pattern = re.compile(r"\b" + pattern.lower() + r"\b")
108     return c_pattern.search(value) is not None

```

Listing 140: File system utilities

```

1  import os
2  import re
3
4  import config as c
5
6
7  def get_files(path, pattern):
8      """
9      Gives list of files containing the Fourier–Legendre coefficients
10     Returns
11     =====
12     list
13     list of available files
14     """
15     return [os.path.join(path, f)
16             for f in os.listdir(path) if re.match(pattern, f)]
17
18
19 def is_locked(filename: str):
20     """
21     Checks if lock is set
22     Parameters
23     =====
24     filename : str
25     name of lock file
26     Returns
27     =====
28     True of False
29     """
30     if os.path.isfile(os.path.join(c.resources, filename)):

```

```

31     return True
32 else:
33     return False
34
35
36 def lock(filename):
37     """
38     Performs locking
39     Parameters
40     =====
41     filename : str
42         name of lock file
43     """
44     f = open(os.path.join(c.resources, filename), "w")
45     f.close()
46
47
48 def unlock(filename):
49     """
50     Performs unlocking
51     Parameters
52     =====
53     filename : str
54         name of lock file
55     """
56     os.remove(os.path.join(c.resources, filename))

```

7 Future Work

Considering the future work, it is important to say that symbolic algebra gives a wide field for optimizations of modeling process. Symbolic operations is actually operations with strings. Such operations has relatively high complexity and slows down modeling process significantly. One of possible ways to improve modeling performance is to parallelize computations. Since strong numerical schemes for Itô SDEs have massive amount of terms this idea appears justified.

The strong numerical schemes for Itô SDEs seem to be easily optimizable, on the other hand, superpositions of the differential operators (4), (5), and (23) are not. They are called recursively during calculation process which is more difficult to parallelize than strong numerical schemes for Itô SDEs. Differential operators obviously include differentiating which is high cost and optimization of them is a dedicated issue.

In the future, it is possible to improve the SDE-MATH software package in a number of other directions. In particular, high-order strong numerical methods of the Runge-Kutta type [2], [7], [42], [61] (including multistep numerical

methods [2], [7], [42], [61]) for Itô SDEs can be implemented. In addition, software for solving the filtering problem and the problem of stochastic optimal control can also be developed. These improvements will lead to changes of the graphical user interface due to new features.

Bibliography

- [1] Ito, K. On Stochastic Differential Equations. *Memoirs of the American Mathematical Society*, 4 (1951), 1-51.
- [2] Kloeden, P.E., Platen, E. *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin, 1992, 632 pp.
- [3] Kloeden, P.E., Platen, E., Schurz, H. *Numerical Solution of SDE Through Computer Experiments*. Springer, Berlin, 1994, 292 pp.
- [4] Arato, M. *Linear Stochastic Systems with Constant Coefficients. A Statistical Approach*. Springer-Verlag, Berlin, Heidelberg, N.Y., 1982, 289 pp.
- [5] Shiryaev, A.N. *Essentials of Stochastic Finance: Facts, Models, Theory*. World Scientific Publishing Co United States, 1999, 852 pp.
- [6] Karatzas, I., Shreve, S. *Methods of Mathematical Finance*. Springer-Verlag, New York, 1998, 415 pp.
- [7] Platen, E., Bruti-Liberati, N. *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. Springer, Berlin, Heidelberg, 2010, 868 pp.
- [8] Han, X., Kloeden, P.E., *Random Ordinary Differential Equations and Their Numerical Solution*. Springer, Singapore, 2017, 250 pp.
- [9] Allen, E. *Modeling with Ito Stochastic Differential Equations*. Springer, Dordrecht, 2007, 230 pp.
- [10] Merton, R.C. *Continuous-Time Finance*. Blackwell, Oxford, 1992, 754 pp.
- [11] Heston, S.L. A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financial Studies*, 6, 2 (1993), 327–343.

- [12] Cox, J.C., Ingersoll, J.E., Ross, S.A. A Theory of the term structure of interest rates, *Econometrica*, 53 (1985), 385-408.
- [13] Kimura, M., Ohta, T. Theoretical aspects of Population genetics. Princeton Univ. Press, Boston, 1971, 232 pp.
- [14] Iacus, S.M. Simulation and Inference for Stochastic Differential Equations. With R Examples. Springer-Verlag, New York, 2008, 285 pp.
- [15] Ricciardi, L.M. Diffusion Processes and Related Topics in Biology, Lecture Notes in Biomathematics. Springer, New York, 1977, 202 pp.
- [16] Stratonovich, R.L., Selected Problems of Fluctuations Theory in Radio Engineering. [In Russian]. Sovetskoe Radio, Moscow, 1961, 556 pp.
- [17] Liptser, R.Sh., Shirjaev, A.N. Statistics of stochastic processes: nonlinear filtering and related problems. [In Russian]. Nauka, Moscow, 1974, 696 pp.
- [18] Nasyrov, F.S. Local times, symmetric integrals and stochastic analysis. [In Russian]. Fizmatlit Publ., Moscow, 2011, 212 pp.
- [19] Kagirowa, G.R., Nasyrov, F.S. On an optimal filtration problem for one-dimensional diffusion processes. *Siberian Adv. Math.*, 28, 3 (2018), 155-165.
- [20] Chugai, K.N., Kosachev, I.M., Rybakov, K.A. Approximate filtering methods in continuous-time stochastic systems. *Smart Innovation, Systems and Technologies*, vol. 173, Eds. Jain L.C., Favorskaya M.N., Nikitin I.S., Reviznikov D.L. Springer, 2020, pp. 351-371. DOI: http://doi.org/10.1007/978-981-15-2600-8_24
- [21] Averina, T.A., Rybakov, K.A. Using maximum cross section method for filtering jump-diffusion random processes. *Russian Journal of Numerical Analysis and Mathematical Modelling*. 35, 2 (2020), 55-67. DOI: <http://doi.org/10.1515/rnam-2020-0005>
- [22] Kloeden, P.E., Platen, E., Schurz, H., Sorensen, M. On effects of discretization on estimators of drift parameters for diffusion processes. *J. Appl. Probab.*, 33 (1996), 1061-1076.
- [23] Clark, J.M.C., Cameron, R.J. The maximum rate of convergence of discrete approximations for stochastic differential equations. *Stochastic Differential Systems Filtering and Control. Lecture Notes in Control and*

- Information Sciences, vol 25. Ed. Grigelionis B. Springer, Berlin, Heidelberg, 1980, pp. 162-171.
- [24] Kulchitskiy, O.Yu., Kuznetsov, D.F. The unified Taylor–Ito expansion. *Journal of Mathematical Sciences (New York)*, 99, 2 (2000), 1130-1140. DOI: <http://doi.org/10.1007/BF02673635>
- [25] Kuznetsov, D.F. New representations of the Taylor–Stratonovich expansion. *Journal of Mathematical Sciences (New York)*, 118, 6 (2003), 5586-5596. DOI: <http://doi.org/10.1023/A:1026138522239>
- [26] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. *Differentsialnie Uravnenia i Protsesy Upravlenia*, 4 (2020), A.1-A.606. Available at: <http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html>
- [27] Milstein, G.N. *Numerical Integration of Stochastic Differential Equations*. [In Russian]. Ural University Press, Sverdlovsk, 1988, 225 pp.
- [28] Kloeden, P.E., Platen, E., Wright, I.W. The approximation of multiple stochastic integrals. *Stochastic Analysis and Applications*, 10, 4 (1992), 431-441.
- [29] Averina, T.A., Prigarin, S.M. Calculation of stochastic integrals of Wiener processes. [In Russian]. Preprint 1048. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1995, 15 pp.
- [30] Prigarin, S.M., Belov, S.M. One application of series expansions of Wiener process. [In Russian]. Preprint 1107. Novosibirsk, Institute of Computational Mathematics and Mathematical Geophysics of Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp.
- [31] Wiktorsson, M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. *The Annals of Applied Probability*, 11, 2 (2001), 470-487.
- [32] Ryden, T., Wiktorsson, M. On the simulation of iterated Ito integrals. *Stochastic Processes and their Applications*, 91, 1 (2001), 151-168.
- [33] Gaines, J.G., Lyons, T.J. Random generation of stochastic area integrals. *SIAM J. Appl. Math.*, 54 (1994), 1132-1146.

- [34] Milstein, G.N., Tretyakov, M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
- [35] Allen, E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.
- [36] Rybakov, K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. [In Russian]. *Differencialnie Uravnenia i Protsesy Upravlenia*, 4 (2019), 1-31. Available at: <http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html>
- [37] Tang, X., Xiao, A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. *Advances in Computational Mathematics*, 45 (2019), 813-846.
- [38] Zahri, M. Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution. *Journal of Taibah University for Science*, 8, 2 (2014), 186-198.
- [39] Li, C.W., Liu, X.Q. Approximation of multiple stochastic integrals and its application to stochastic differential equations. *Nonlinear Anal. Theor. Meth. Appl.*, 30, 2 (1997), 697-708.
- [40] Rybakov K. Application of Walsh series to represent iterated Stratonovich stochastic integrals. IOP Conference Series: Materials science and engineering. 2020, vol. 927, id 012080. DOI: <http://doi.org/10.1088/1757-899X/927/1/012080>
- [41] Rybakov, K.A. Modeling and analysis of output processes of linear continuous stochastic systems based on orthogonal expansions of random functions. *J. of Computer and Systems Sci. Int.*, 59, 3 (2020), 322-337. DOI: <http://doi.org/10.1134/S1064230720030156>
- [42] Kuznetsov, D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: <http://doi.org/10.18720/SPBPU/2/s17-227>
- [43] Kuznetsov, D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. *Differencialnie Uravnenia i Protsesy*

- Upravlenia, 1 (1997), 18-77. Available at: <http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html>
- [44] Kuznetsov, D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. *Computational Mathematics and Mathematical Physics*, 59, 8 (2019), 1236-1250. DOI: <http://doi.org/10.1134/S0965542519080116>
- [45] Kuznetsov, D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. *Differentsialnie Uravnenia i Protsey Upravlenia*, 3 (2019), 18-62. Available at: <http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html>
- [46] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. *Differentsialnie Uravnenia i Protsey Upravlenia*, 3 (2020), 129-162. Available at: <http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html>
- [47] Jentzen, A., Röckner, M. A Milstein scheme for SPDEs. *Foundations Comp. Math.*, 15, 2 (2015), 313-362.
- [48] Becker, S., Jentzen, A., Kloeden, P.E. An exponential Wagner–Platen type scheme for SPDEs. *SIAM J. Numer. Anal.*, 54, 4 (2016), 2389-2426.
- [49] Mishura, Y.S., Shevchenko, G.M. Approximation schemes for stochastic differential equations in a Hilbert space. *Theor. Prob. Appl.*, 51, 3 (2007), 442-458.
- [50] Bao, J., Reisinger, C., Renz, P., Stockinger, W. First order convergence of Milstein schemes for McKean equations and interacting particle systems [arXiv:2004.03325v1](https://arxiv.org/abs/2004.03325) [math.PR], 2020, 27 pp.
- [51] Son, L.N., Tuan, A.H., Dung, T.N., Yin G. Milstein-type procedures for numerical solutions of stochastic differential equations with Markovian switching. *SIAM J. Numer. Anal.*, 55, 2 (2017), 953–979.
- [52] Sun, Y., Yang, J., Zhao W. Ito–Taylor schemes for solving mean-field stochastic differential equations. *Numer. Math. Theor. Meth. Appl.*, 10, 4 (2017), 798-828.

- [53] Higham, D.J. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations. *SIAM Rev.*, 43, 3 (2001), 525–546.
- [54] Cyganowski, S., Grune, L., Kloeden, P.E. Maple for stochastic differential equations. *Theory and Numerics of Differential Equations*. Eds. Blowey, J.F., Coleman, J.P., Craig, A.W. Universitext. Springer, Berlin, Heidelberg, 2001, pp. 127-177.
- [55] Higham, D.J., Kloeden, P.E. MAPLE and MATLAB for stochastic differential equations in finance. *Programming Languages and Systems in Computational Economics and Finance. Advances in Computational Economics*, vol 18, Ed. Nielsen, S.S. Springer, Boston, MA, 2002, pp. 233-269.
- [56] Cyganowski, S., Grune, L., Kloeden P.E. MAPLE for jump-diffusion stochastic differential equations in finance. *Programming Languages and Systems in Computational Economics and Finance. Advances in Computational Economics*, vol. 18, Ed. Nielsen, S.S. Springer, Boston, MA, 2002, pp. 441-460.
- [57] Gilling, H., Shardlow, T. SDELab: A package for solving stochastic differential equations in MATLAB. *Journal of Computational and Applied Mathematics*, 2, 205 (2007), 1002-1018.
- [58] Kuznetsov, D.F. *Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab programs, 3rd Edition. [In Russian].* Polytechnical University Publishing House, Saint-Petersburg, 2009, 768 pp. DOI: <http://doi.org/10.18720/SPBPU/2/s17-230>
- [59] Kiesewetter, S., Polkinghorne, R., Opanchuk, B., Drummond, P.D. xSP-DE: Extensible software for stochastic equations. *SoftwareX*, 5 (2016), 12-15.
- [60] Gevorkyan, M.N., Velieva, T.R., Korolkova, A.V., Kulyabov, D.S., Sevastyanov, L.A. Stochastic Runge–Kutta software package for stochastic differential equations. *Dependability Engineering and Complex Systems. DepCoS-RELCOMEX 2016. Advances in Intelligent Systems and Computing*, vol. 470, Eds. Zamojski, W., Mazurkiewicz, J., Sugier, J., Walkowiak, T., Kacprzyk, J. Springer, Cham, 2016, pp. 169-179.
- [61] Kuznetsov, D.F. *Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Edition. [In Russian].* *Differentsialnie Uravnenia i Protsesy Upravlenia*, 4 (2018), A.1-A.1073.

- Available at: <http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html>
- [62] Kulchitskiy, O.Yu., Kuznetsov, D.F. Numerical Simulation of Stochastic Systems of Linear Stationary Differential Equations. [In Russian]. *Differencialnie Uravnenia i Protsesy Upravlenia*, 1 (1998), 41-65. Available at: <http://diffjournal.spbu.ru/pdf/j010.pdf>
- [63] Gihman, I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. [In Russian]. Naukova Dumka, Kiev, 1982, 612 pp.
- [64] Kuznetsov, D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 1.5 and 2.0 Orders of Strong Convergence. *Automation and Remote Control*, 79, 7 (2018), 1240-1254. DOI: <http://doi.org/10.1134/S0005117918070056>
- [65] Kuznetsov, D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 2.5 Order of Strong Convergence. *Automation and Remote Control*, 80, 5 (2019), 867-881. DOI: <http://doi.org/10.1134/S0005117919050060>
- [66] Kuznetsov, D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor–Stratonovich expansion. *Computational Mathematics and Mathematical Physics*, 60, 3 (2020), 379-389. DOI: <http://doi.org/10.1134/S0965542520030100>
- [67] Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. [arXiv:2003.14184](https://arxiv.org/abs/2003.14184) [math.PR], 2020, 652 pp.
- [68] Kuznetsov, D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. Polytechnical University Publishing House: St.-Petersburg, 2013, 382 pp. DOI: <http://doi.org/10.18720/SPBPU/2/s17-234>
- [69] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. *Ufa Mathematical Journal*, 11, 4 (2019), 49-77. DOI: <http://doi.org/10.13108/2019-11-4-49> Available at: http://matem.anrb.ru/en/article?art_id=604

- [70] Kuznetsov, D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. *Computational Mathematics and Mathematical Physics*, 58, 7 (2018), 1058-1070. DOI: <http://doi.org/10.1134/S0965542518070096>
- [71] Kuznetsov, D.F. Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendres Polynomials. *Journal of Automation and Information Sciences (Begell House)*, 32, Issue 12, (2000), 69-86. DOI: <http://doi.org/10.1615/JAutomatInfScien.v32.i12.80>
- [72] Kuznetsov, D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier–Legendre and trigonometric expansions, approximations, formulas. *Differencialnie Uravnenia i Protsesy Upravlenia*, 1 (2017), A.1–A.385. Available at: <http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html>
- [73] Kuznetsov, M.D., Kuznetsov, D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series [arXiv:2010.13564](https://arxiv.org/abs/2010.13564) [math.PR], 2020, 59 pp.
- [74] Kuznetsov, M.D., Kuznetsov, D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. [arXiv:2009.14011](https://arxiv.org/abs/2009.14011) [math.PR], 2020, 188 pp.
- [75] Kuznetsov, D.F., Kuznetsov, M.D. A software package for implementation of strong numerical methods of convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative multi-dimensional noise. 19th International Conference "Aviation and Cosmonautics" (AviaSpace-2020). Abstracts (Moscow, MAI, 23-27 November, 2020), Publishing house "Pero", 2020, 569-570.
- [76] Kuznetsov, D.F., Kuznetsov, M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs based on multiple Fourier-Legendre series. *Springer Proceedings in Mathematics & Statistics. Recent Developments in Stochastic Methods and Applications*. Ed. Shiryaev, A.N. Samouylov, K.E, Kozyrev, D.V., 2021 (to appear).